Notes on homotopical algebra

Z.L. Low

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Preface

These notes are intended as a kind of annotated index to the various standard references in homotopical algebra: the focus is on definitions and statements of results, *not* proofs.

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Foundations

0.1 Set theory

In category theory it is often convenient to invoke a certain set-theoretic device commonly known as a 'Grothendieck universe', but we shall say simply 'universe', so as to simplify exposition and proofs by eliminating various circumlocutions involving cardinal bounds, proper classes etc.

Definition 0.1.1. A **pre-universe** is a set $\mathbf{U}$ satisfying these axioms:

1. If $x \in y$ and $y \in \mathbf{U}$, then $x \in \mathbf{U}$.
2. If $x \in \mathbf{U}$ and $y \in \mathbf{U}$ (but not necessarily distinct), then $\{x, y\} \in \mathbf{U}$.
3. If $x \in \mathbf{U}$, then $\mathscr{P}(x) \in \mathbf{U}$, where $\mathscr{P}(x)$ denotes the set of all subsets of x.
4. If $x \in \mathbf{U}$ and $f : x \to \mathbf{U}$ is a map, then $\bigcup_{i \in x} f(i) \in \mathbf{U}$.

A **universe** is a pre-universe $\mathbf{U}$ with this additional property:

5. $\omega \in \mathbf{U}$, where ω is the set of all finite (von Neumann) ordinals.

Example 0.1.2. The empty set is a pre-universe, and with very mild assumptions, so is the set $\mathbf{HF}$ of all hereditarily finite sets.

¶ 0.1.3. The notion of universe makes sense in any material set theory, but their existence must be postulated. We adopt the following:

- **Grothendieck–Verdier universe axiom.** For each set x, there exists a universe $\mathbf{U}$ with $x \in \mathbf{U}$.

For definiteness, we may take our base theory to be Mac Lane set theory, which is a weak subsystem of Zermelo–Fraenkel set theory with choice (ZFC). Readers interested in the details of Mac Lane set theory are referred to [Mathias, 2001], but in practice, as long as one is working at all times *inside some universe*, one may as well be working in ZFC. Indeed:

Proposition 0.1.4. *With the assumptions of Mac Lane set theory, any universe is a transitive model of ZFC.*

Proof. Let $\mathbf{U}$ be a universe. By definition, $\mathbf{U}$ is a transitive set containing pairs, power sets, unions, and ω, so the axioms of extensionality, empty set, pairs, power sets, unions, choice, and infinity are all automatically satisfied. We must show that the axiom schemas of separation and replacement are also satisfied, and in fact it is enough to check that replacement is valid; but this is straightforward using axioms 2 and 4. ■

Definition 0.1.5. Let $\mathbf{U}$ be a pre-universe. A **$\mathbf{U}$-set** is a member of $\mathbf{U}$, a **$\mathbf{U}$-class** is a subset of $\mathbf{U}$, and a **proper $\mathbf{U}$-class** is a $\mathbf{U}$-class that is not a $\mathbf{U}$-set.

Lemma 0.1.6. *A $\mathbf{U}$-class X is a $\mathbf{U}$-set if and only if there exists a $\mathbf{U}$-class Y such that $X \in Y$.* ■

Proposition 0.1.7. *If $\mathbf{U}$ is a universe in Mac Lane set theory, then the collection of all $\mathbf{U}$-classes is a transitive model of Morse–Kelley class–set theory (MK), and so is a transitive model of von Neumann–Bernays–Gödel class–set theory (NBG) in particular.* ■

Definition 0.1.8. A **$\mathbf{U}$-small category** is a category $\mathbb{C}$ such that $\operatorname{ob}\mathbb{C}$ and $\operatorname{mor}\mathbb{C}$ are $\mathbf{U}$-sets. A **locally $\mathbf{U}$-small category** is a category $\mathcal{D}$ satisfying these conditions:

- $\operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{D}$ are $\mathbf{U}$-classes, and
- for all objects x and y in $\mathcal{D}$, the hom-set $\mathcal{D}(x, y)$ is a $\mathbf{U}$-set.

An **essentially $\mathbf{U}$-small category** is a category $\mathcal{D}$ for which there exist a $\mathbf{U}$-small category $\mathbb{C}$ and a functor $\mathbb{C} \to \mathcal{D}$ that is fully faithful and essentially surjective on objects.

Proposition 0.1.9. *If $\mathbb{D}$ is a $\mathbf{U}$-small category and $\mathcal{C}$ is a locally $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is locally $\mathbf{U}$-small.*

Proof. Strictly speaking, this depends on the set-theoretic implementation of ordered pairs, categories, functors, etc., but at the very least $[\mathbb{D}, \mathcal{C}]$ should be isomorphic to a locally $\mathbf{U}$-small category.

In the context of $[\mathbb{D}, \mathcal{C}]$, we may regard functors $\mathbb{D} \to \mathcal{C}$ as being the pair consisting of the *graph* of the object map $\operatorname{ob} \mathbb{D} \to \operatorname{ob} \mathcal{C}$ and the *graph* of the morphism map $\operatorname{mor} \mathbb{D} \to \operatorname{mor} \mathcal{C}$, and these are $\mathbf{U}$-sets by the $\mathbf{U}$-replacement axiom. Similarly, if F and G are objects in $[\mathbb{D}, \mathcal{C}]$, then we may regard a natural transformation $\alpha : F \Rightarrow G$ as being the triple (F, G, A), where A is the set of all pairs (c, α_c). ■

One complication introduced by having multiple universes concerns the existence of (co)limits.

Theorem 0.1.10 (Freyd)**.** *Let $\mathcal{C}$ be a category and let κ be a cardinal such that $|\operatorname{mor} \mathcal{C}| \leq \kappa$. If $\mathcal{C}$ has products for families of size κ, then any two parallel morphisms in $\mathcal{C}$ must be equal.*

Proof. Suppose, for a contradiction, that $f, g : X \to Y$ are distinct morphisms in $\mathcal{C}$. Let Z be the product of κ-many copies of Y in $\mathcal{C}$. The universal property of products implies there are at least 2^κ-many distinct morphisms $X \to Z$; but $\mathcal{C}(X, Z) \subseteq \operatorname{mor} \mathcal{C}$, so this is an absurdity. ■

Definition 0.1.11. Let $\mathbf{U}$ be a pre-universe. A **$\mathbf{U}$-complete** (resp. **$\mathbf{U}$-cocomplete**) **category** is a category $\mathcal{C}$ with the following property:

- For all $\mathbf{U}$-small categories $\mathbb{D}$ and all diagrams $A : \mathbb{D} \to \mathcal{C}$, a limit (resp. colimit) of A exists in $\mathcal{C}$.

We may instead say $\mathcal{C}$ has all **finite limits** (resp. **finite colimits**) in the special case $\mathbf{U} = \mathbf{HF}$.

Proposition 0.1.12. *Let $\mathcal{C}$ be a category and let $\mathbf{U}$ be a non-empty pre-universe. The following are equivalent:*

(i) *$\mathcal{C}$ is $\mathbf{U}$-complete.*

(ii) *$\mathcal{C}$ has all finite limits and products for all families of objects indexed by a $\mathbf{U}$-set.*

(iii) *For each* **U**-*small category* $\mathbb{D}$, *there exists an adjunction*

$$\Delta \dashv \varprojlim_{\mathbb{D}} : [\mathbb{D}, C] \to C$$

where ΔX *is the constant functor with value* X.

Dually, the following are equivalent:

(i′) C *is* **U**-*cocomplete.*

(ii′) C *has all finite colimits and coproducts for all families of objects indexed by a* **U**-*set.*

(iii′) *For each* **U**-*small category* $\mathbb{D}$, *there exists an adjunction*

$$\varinjlim_{\mathbb{D}} \dashv \Delta : C \to [\mathbb{D}, C]$$

where ΔX *is the constant functor with value* X.

Proof. This is a standard result; but we remark that we do require a sufficiently powerful form of the axiom of choice to pass from (ii) to (iii). □

¶ 0.1.13. In the **explicit universe convention**, the words 'set', 'class', etc. have their usual meanings, and in the **one-universe convention**, these instead abbreviate '**U**-set', '**U**-class', etc. for a fixed (but arbitrary) universe **U**. However, the word 'category' always refers to a category that is contained in *some* universe, which may or may not be locally **U**-small, and we shall use the word 'ensemble' to refer to sets which may or may not be in **U**. In subsequent chapters, the implicit universe convention should be assumed *unless otherwise stated*.

We now recall some definitions and results about ordinal and cardinal numbers. Readers familiar with axiomatic set theory may wish to skip ahead.

Definition 0.1.14. A **von Neumann ordinal** is a set α with the following properties:

- If $x \in y$ and $y \in \alpha$, then $x \in \alpha$.
- The binary relation $\in$ is strict total ordering of α.
- If S is a subset of α such that
  - $\varnothing \in S$,

- If $\beta \in S$ and $\beta \cup \{\beta\} \in \alpha$, then $\beta \cup \{\beta\} \in S$.
- If $T \subseteq S$, then $\bigcup T \in S$.

then $S = \alpha$.

We identify 0 with the von Neumann ordinal $\varnothing$, and by induction, we identify the natural number $n + 1$ with the von Neumann ordinal $\{0, \dots, n\}$.

Proposition 0.1.15.

(i) *If α is a von Neumann ordinal, then every member of α is an initial segment of α and is in particular a von Neumann ordinal.*

(ii) *If α is a von Neumann ordinal, so is $\alpha \cup \{\alpha\}$. (This is usually denoted by $\alpha + 1$ and called the* **successor** *of α.)*

(iii) *The union of a set S of von Neumann ordinals is another von Neumann ordinal. (This is usually denoted by* $\sup S$ *and called the* **supremum** *of S.)*

(iv) *If $\mathbf{U}$ is a pre-universe and $\kappa(\mathbf{U})$ is the set of von Neumann ordinals in $\mathbf{U}$, then $\kappa(\mathbf{U})$ a von Neumann ordinal, but $\kappa(\mathbf{U}) \notin \mathbf{U}$.*

Proof. Claims (i) – (iii) are all easy, and claim (iv) is Burali-Forti's paradox. ♦

Theorem 0.1.16 (Classification of well-orderings)**.**

(i) *In Zermelo–Fraenkel set theory, every well-ordered set is isomorphic to a unique von Neumann ordinal.*

(ii) *In Mac Lane set theory, if $\mathbf{U}$ is a pre-universe and X is a well-ordered set in $\mathbf{U}$, then X is isomorphic to a unique von Neumann ordinal in $\mathbf{U}$.*

Proof. Claim (i) is a standard result in axiomatic set theory, and claim (ii) is an obvious corollary. □

Definition 0.1.17. A **transitive set** is a set T such that, given $x \in y$, if $y \in T$, then $x \in T$ as well. The **transitive closure** of a set X is a set $\operatorname{tcl}(X)$ such that, for all transitive sets T with $X \subseteq T$, we have $\operatorname{tcl}(X) \subseteq T$ as well.

Lemma 0.1.18. *In Mac Lane set theory, every set has a unique transitive closure.*

Proof. One of the axioms of Mac Lane set theory states that every set X is a member of some transitive set T, and so $X \subseteq T$. Clearly, the intersection of any family of transitive sets containing X is again a transitive set containing X, so $\operatorname{tcl}(X)$ exists and is unique so long as there is at least one transitive set containing X. ■

Definition 0.1.19. A **partial rank function** from a transitive set T to a well-ordered set W is a partial function $\rho : T \to W$ with these properties:

- If $\varnothing \in T$, then $\rho(\varnothing)$ is the least element of W.
- If $y \in T$ and $\rho(x)$ is defined for all $x \in y$, then

  $$\rho(y) = \min \{ w \in W \mid \forall x \in y.\ \rho(x) < w \}$$

  provided the RHS is defined.
- Otherwise $\rho(y)$ is undefined.

A **total rank function** is a partial rank function that is defined on its entire domain. The **rank** of a set X, if it exists, the least von Neumann ordinal $\operatorname{rank}(X)$ for which there exists a total rank function $\operatorname{tcl}(X) \to \operatorname{rank}(X)$.

Proposition 0.1.20. *In Mac Lane set theory:*

(i) *If T is a transitive set and W is a well-ordered set, then there is a unique partial rank function $\rho : T \to W$.*

(ii) *If $\mathbf{U}$ is a pre-universe and $x \in \mathbf{U}$, then* $\operatorname{rank}(x)$ *can be defined by a Δ_0-formula with $\mathbf{U}$ as a parameter, and for each von Neumann ordinal α in $\mathbf{U}$, the set*

$$\mathbf{V}_\alpha = \{ x \in \mathbf{U} \mid \operatorname{rank}(x) < \alpha \}$$

is a $\mathbf{U}$-set.

(iii) *Assuming the Grothendieck–Verdier universe axiom,* $\operatorname{rank}(x)$ *is defined for all x.*

Proof. (i). This is a straightforward application of well-founded induction.

(ii). $\mathbf{U}$ is a transitive set and the set $\kappa(\mathbf{U})$ of all von Neumann ordinals in $\mathbf{U}$ is well-ordered by inclusion, so by claim (i) there is a partial rank function ρ :

$\mathbf{U} \rightharpoonup \kappa(\mathbf{U})$. ZFC proves that every set has a rank, so ρ must in fact be a total rank function; hence, for any $x \in \mathbf{U}$, $\operatorname{rank}(x)$ is defined. It is clear that ρ can be defined by a Δ_0-formula with only $\mathbf{U}$ as a parameter, and the rest of the claim follows.

(iii). Obvious, assuming claim (ii). □

Definition 0.1.21. Two sets are **equinumerous** if there exists a bijection between them. A **cardinality class** in a pre-universe $\mathbf{U}$ is an equivalence class under the relation of equinumerosity.

Definition 0.1.22. An **$\aleph$-number** is an infinite von Neumann ordinal κ such that, for any von Neumann ordinal λ such that κ and λ are equinumerous, we have $\kappa \subseteq \lambda$.

Example 0.1.23. The first infinite von Neumann ordinal, i.e. $\omega = \{0, 1, 2, \ldots\}$, is the $\aleph$-number $\aleph_0$.

Lemma 0.1.24. *If κ is an $\aleph$-number, then there exists a unique $\aleph$-number κ^+ with the following property:*

- *For any $\aleph$-number λ such that $\kappa < \lambda$, we have $\kappa^+ \leq \lambda$.*

The **cardinal successor** *of κ is κ^+.*

Proof. The class of $\aleph$-numbers is well-ordered and unbounded, so the class of all $\aleph$-numbers $> \kappa$ has a minimal element κ^+, as required. ■

Theorem 0.1.25 (Classification of cardinalities)**.**

(i) *In Zermelo–Fraenkel set theory, for every well-ordered infinite set X, there exists a unique $\aleph$-number κ such that X and κ are equinumerous.*

(ii) *In Zermelo–Fraenkel set theory with the axiom of choice, the same is true for any infinite set whatsoever.*

(iii) *In Mac Lane set theory, if $\mathbf{U}$ is a universe and X is an infinite set in $\mathbf{U}$, then there exists a unique $\aleph$-number κ in the cardinality class of X.*

(iv) *In Mac Lane set theory with the Grothendieck–Verdier universe axiom, if $\mathbf{U}$ is a pre-universe and κ is an $\aleph$-number not in $\mathbf{U}$, then the cardinality of $\mathbf{U}$ is at most κ.*

Proof. Claim (i) is a standard fact, whence claims (ii) and (iii), by the well-ordering theorem. Claim (iv) can be proven using axiom 4 for pre-universes. □

¶ 0.1.26. Henceforth, we identify the cardinality class of a finite set with the unique von Neumann ordinal contained in that class, and similarly we identify the cardinality class of an infinite set with the unique $\aleph$-number in that class. These are the **cardinal numbers**.

Definition 0.1.27. A **cofinal subset** of a partially-ordered set X is a subset $Y \subseteq X$ such that, for all x in X, there exists some y in Y such that $x \leq y$. A **regular cardinal number** is an $\aleph$-number κ such that any cofinal subset of κ has cardinality equal to κ. A **singular cardinal number** is an $\aleph$-number that is not regular.

The following helps to motivate the definition of regular cardinal numbers.

Definition 0.1.28. Let $\mathbf{U}$ be a pre-universe. An **arity class** in $\mathbf{U}$ is a $\mathbf{U}$-class K of cardinal numbers satisfying the following conditions:

- $1 \in K$.
- If $\kappa \in K$ and $\lambda : \kappa \to K$ is a function, then the cardinal sum $\sum_{\alpha \in \kappa} \lambda(\alpha)$ is also in K.
- If $\kappa \in K$ and $\lambda : \kappa \to \mathbf{U}$ is a function such that each $\lambda(\alpha)$ is a cardinal number and $\sum_{\alpha \in \kappa} \lambda(\alpha) \in K$, then $\lambda(\alpha) \in K$ as well.

Theorem 0.1.29 (Classification of arity classes)**.** *In Mac Lane set theory, if K is an arity class in a pre-universe $\mathbf{U}$, then K must be either*

- $\{1\}$*, or*
- $\{0, 1\}$*, or*
- *of the form $\{\lambda \in \mathbf{U} \mid \lambda \text{ is a cardinal number and } \lambda < \kappa\}$ for some regular cardinal number κ (possibly not in $\mathbf{U}$).*

Proof. The notion of arity class and this result are due to Shulman [2012]. □

Definition 0.1.30. Let κ be a regular cardinal number. A **κ-small category** is a category $\mathbb{C}$ such that $\operatorname{mor}\mathbb{C}$ has cardinality $< \kappa$. A **finite category** is an $\aleph_0$-small category, i.e. a category $\mathbb{C}$ such that $\operatorname{mor}\mathbb{C}$ is finite. A **finite diagram**

(resp. **κ-small diagram**, **U-small diagram**) in a category C is a functor $\mathbb{D} \to C$ where $\mathbb{D}$ is a finite (resp. κ-small, $\mathbf{U}$-small) category.

Theorem 0.1.31. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{U}^+$ be a universe with $\mathbf{U} \in \mathbf{U}^+$, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathbf{Set}^+$ be the category of $\mathbf{U}^+$-sets.*

(i) *If $X : \mathbb{D} \to \mathbf{Set}$ is a $\mathbf{U}$-small diagram, then there exist a limit and a colimit for X in $\mathbf{Set}$.*

(ii) *The inclusion $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ is fully faithful and preserves limits and colimits for all $\mathbf{U}$-small diagrams.*

Proof. One can construct products, equalisers, coproducts, coequalisers, and hom-sets in a completely explicit way, making the preservation properties obvious. ♦

Corollary 0.1.32. *The inclusion $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ reflects limits and colimits for all $\mathbf{U}$-small diagrams.* ■

Corollary 0.1.33. *For any $\mathbf{U}$-small category $\mathbb{C}$:*

(i) *The functor category $[\mathbb{C}, \mathbf{Set}]$ is $\mathbf{U}$-complete and $\mathbf{U}$-cocomplete, with limits and colimits for $\mathbf{U}$-small diagrams computed componentwise in $\mathbf{Set}$.*

(ii) *The inclusion $[\mathbb{C}, \mathbf{Set}] \hookrightarrow [\mathbb{C}, \mathbf{Set}^+]$ is fully faithful and both preserves and reflects limits and colimits for all $\mathbf{U}$-small diagrams.* ■

Definition 0.1.34. An **strongly inaccessible cardinal number** is a regular cardinal number κ such that, for all sets X of cardinality less than κ, the power set $\mathscr{P}(X)$ is also of cardinality less than κ.

Example 0.1.35. $\aleph_0$ is a strongly inaccessible cardinal number and is the only one that can be proven to exist in ZFC. It is more conventional to exclude $\aleph_0$ from the definition of strongly inaccessible cardinal number by demanding that they be uncountable.

Proposition 0.1.36. *In Mac Lane set theory:*

(i) *If $\mathbf{U}$ is a non-empty pre-universe, then there exists a strongly inaccessible cardinal number κ such that the members of $\mathbf{U}$ are all the sets of rank less than κ. Moreover, this κ is the rank and the cardinality of $\mathbf{U}$.*

(ii) *If* $\mathbf{U}$ *is a universe and* κ *is a strongly inaccessible cardinal number such that* $\kappa \in \mathbf{U}$*, then there exists a* $\mathbf{U}$*-set* $\mathbf{V}_\kappa$ *whose members are all the sets of rank less than* κ*, and* $\mathbf{V}_\kappa$ *is a pre-universe.*

(iii) *If* $\mathbf{U}$ *and* $\mathbf{U}'$ *are pre-universes, then either* $\mathbf{U} \subseteq \mathbf{U}'$ *or* $\mathbf{U}' \subseteq \mathbf{U}$*; and if* $\mathbf{U} \subsetneqq \mathbf{U}'$*, then* $\mathbf{U} \in \mathbf{U}'$*.*

Proof. (i). Let κ be the set of all von Neumann ordinals in $\mathbf{U}$; this exists by Δ_0-separation applied to $\mathbf{U}$. Since $\mathbf{U}$ is closed under power sets and internally-indexed unions, κ must be a strongly inaccessible cardinal.

We can construct the set all of $\mathbf{U}$-sets of rank less than κ using transfinite recursion on κ as follows: starting with $\mathbf{V}_0 = \varnothing$, for each von Neumann ordinal α less than κ, we set $\mathbf{V}_{\alpha+1} = \mathscr{P}\left(\mathbf{V}_\alpha\right)$, and for each ordinal λ that is not a successor, we set $\mathbf{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha$. The well-foundedness of $\in$ (restricted to $\mathbf{U}$) implies that in fact this must be all of $\mathbf{U}$.

Clearly, every set of rank less than κ is in fact a $\mathbf{U}$-set, and $\mathbf{U}$ is itself a set of rank κ. The cardinality of $\mathbf{U}$ is also κ, since κ is a regular cardinal number and any cardinal number less than κ is a member of $\mathbf{U}$.

(ii). We may construct $\mathbf{V}_\kappa$ using the same method as in (i). By construction $\mathbf{V}_\kappa$ satisfies axiom 1; since κ is infinite, $\mathbf{V}_\kappa$ satisfies axioms 2 and 3; and since κ is strongly inaccessible, $\mathbf{V}_\kappa$ satisfies axiom 4. Thus $\mathbf{V}_\kappa$ is a pre-universe.

(iii). Again, let κ be the rank of $\mathbf{U}$. If $\kappa \in \mathbf{U}'$ then we can show by transfinite induction that $\mathbf{V}_\kappa \in \mathbf{U}'$ and so $\mathbf{U} \subsetneqq \mathbf{U}'$; else we must have $\mathbf{U}' \subseteq \mathbf{V}_\kappa = \mathbf{U}$. ■

0.2 Accessibility and ind-completions

Prerequisites. §0.1.

A classical technology for controlling size problems in category theory, due to Gabriel and Ulmer [1971], Grothendieck and Verdier [SGA 4a, Exposé I, §9], and Makkai and Paré [1989], is the notion of accessibility. Though we make use of universes, accessibility remains important and is a crucial tool in verifying the stability of various universal constructions when one passes from one universe to a larger one.

Definition 0.2.1. Let κ be a regular cardinal.

- A **κ-filtered category** is a category $\mathcal{J}$ with the following property:
  - For each κ-small diagram A in $\mathcal{J}$, there exist an object j and a cocone $A \Rightarrow \Delta j$.

  A **κ-filtered diagram** in a category $\mathcal{C}$ is a functor $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a κ-filtered category.

- A **κ-directed preorder** is a preordered set X that is κ-filtered when considered as a category, i.e. a preorder with the following property:
  - For each κ-small subset $Y \subseteq X$, there exists an element x of X such that $y \le x$ for all y in Y.

  A **κ-directed diagram** in a category $\mathcal{C}$ is a functor $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a κ-directed preorder (considered as a category).

In both cases, it is conventional to omit κ when $\kappa = \aleph_0$.

Remark 0.2.2. For any regular cardinal κ, the category with one object and only one non-trivial arrow f is κ-filtered if and only if $f = f \circ f$. In particular, any category that has colimits for small κ-filtered diagrams must also have splittings for idempotents.

Example 0.2.3. Let X be any set. The set of all finite subsets of X, partially ordered by inclusion, is a directed preorder. More generally, if κ is any regular cardinal, then the set of all subsets of X of cardinality $< \kappa$ is a κ-directed preorder.

Lemma 0.2.4. *Let $\mathcal{J}$ be a category. The following are equivalent:*

(i) *$\mathcal{J}$ is a filtered category.*

(ii) *$\mathcal{J}$ is inhabited; for any two objects j and j' in $\mathcal{J}$ there exist an object j'' and morphisms $j \to j''$ and $j' \to j''$ in $\mathcal{J}$; and for any parallel pair $f_0, f_1 : j \to j'$ in $\mathcal{J}$, there is a morphism $g : j' \to j''$ in $\mathcal{J}$ such that $g \circ f_0 = g \circ f_1$.*

Proof. (i) ⇒ (ii). The conditions say precisely that $\mathcal{J}$ has cocones for diagrams of shape $\varnothing$, $\{\bullet, \bullet\}$, and $\{\bullet \rightrightarrows \bullet\}$, respectively.

(ii) ⇒ (i). See Lemma 2.13.2 in [Borceux, 1994a]. □

Definition 0.2.5. Let α be an ordinal. An **α-chain** in a category $\mathcal{C}$ is a functor $\alpha \to \mathcal{C}$, where we have identified α with the well-ordered set of ordinals $< \alpha$.

Remark 0.2.6. If α is an ordinal with cofinality κ, then α is a κ-directed preorder. In particular, α-chains are κ-directed diagrams.

Lemma 0.2.7. *Let $\mathcal{I}$ be any category and let $\mathcal{J}$ be a filtered category. Given a full functor $F : \mathcal{I} \to \mathcal{J}$, the following are equivalent:*

(i) *$F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor.*[1]

(ii) *For each object j in $\mathcal{J}$, there exist an object i in $\mathcal{I}$ and a morphism $j \to Fi$ in $\mathcal{J}$.*

Proof. (i) ⇒ (ii). Since $F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor, the comma category $(j \downarrow F)$ is connected; in particular, it is inhabited.

(ii) ⇒ (i). The hypothesis says that the comma category $(j \downarrow F)$ is inhabited for all objects j in $\mathcal{J}$; it remains to be shown that each $(j \downarrow F)$ is connected. Suppose we have morphisms $f : j \to Fi$ and $f' : j \to Fi'$ in $\mathcal{J}$. Since $\mathcal{J}$ is a filtered category, there exist morphisms $g : Fi \to j'$ and $g' : Fi' \to j'$ such that $g \circ f = g' \circ f'$. By hypothesis, there is a morphism $h : j' \to Fi''$ in $\mathcal{J}$, and since $F : \mathcal{I} \to \mathcal{J}$ is full, there exist morphisms $k : i \to i''$ and $k' : i' \to i''$ in $\mathcal{I}$ such that $Fk = h \circ g$ and $Fk' = h \circ g'$. Thus, we have $Fk \circ f = Fk' \circ f'$, so $(j \downarrow F)$ is indeed connected. ■

Lemma 0.2.8. *Let $\mathcal{I}$ be a filtered category and let $\mathcal{J}$ be any preorder. Given a functor $F : \mathcal{I} \to \mathcal{J}$, the following are equivalent:*

(i) *$F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor.*

(ii) *For each object j in $\mathcal{J}$, there exist an object i in $\mathcal{I}$ such that $j \leq Fi$ in $\mathcal{J}$.*

[1] See definition A.5.31.

Proof. (i) ⇒ (ii). Since $F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor, the comma category $(j \downarrow F)$ is connected; in particular, it is inhabited.

(ii) ⇒ (i). The hypothesis says that the comma category $(j \downarrow F)$ is inhabited for all objects j in $\mathcal{J}$; it remains to be shown that each $(j \downarrow F)$ is connected. Suppose we have morphisms $j \le Fi$ and $j \le Fi'$ in $\mathcal{J}$. Since $\mathcal{I}$ is a filtered category, there exist an object i'' in $\mathcal{I}$ and morphisms $i \to i''$ and $i' \to i''$; thus, we have $j \le Fi \le Fi''$ and $j \le Fi' \le Fi''$, so $(j \downarrow F)$ is indeed connected. ■

Lemma 0.2.9. *Let $\mathcal{J}$ be a κ-filtered diagram. If $\mathcal{J}$ is also κ-small, then there exist an object j in $\mathcal{J}$ and an idempotent morphism $e : j \to j$ such that the subcategory of $\mathcal{J}$ generated by e is cofinal in $\mathcal{J}$.*

Proof. Since $\mathrm{id} : \mathcal{J} \to \mathcal{J}$ is a κ-small diagram in $\mathcal{J}$, there must exist an object j in $\mathcal{J}$ and a cocone $\lambda : \mathrm{id} \Rightarrow \Delta j$. Let $e = \lambda_j : j \to j$. Since λ is a cocone, we must have $e = e \circ e$, i.e. $e : j \to j$ is idempotent.

Let $\mathcal{I}$ be the subcategory of $\mathcal{J}$ generated by e and let j' be any object in $\mathcal{J}$. We must show that the comma category $(j' \downarrow \mathcal{I})$ is connected. It is inhabited: $\lambda_{j'} : j' \to j$ is an object in $(j' \downarrow \mathcal{I})$. Moreover, given any morphism $f : j' \to j$ in $\mathcal{J}$, we must have $\lambda_{j'} = \lambda_j \circ f = e \circ f$, so $(j' \downarrow \mathcal{I})$ is indeed connected. Thus, $\mathcal{I}$ is a cofinal subcategory of $\mathcal{J}$. ■

Lemma 0.2.10. *Let κ be a regular cardinal and let $\left(\mathcal{J}_i \,\middle|\, i \in I\right)$ be a set of κ-filtered categories.*

(i) *The product $\mathcal{J} = \prod_{i \in I} \mathcal{J}_i$ is a κ-filtered category.*

(ii) *Each projection $\pi_i : \mathcal{J} \to \mathcal{J}_i$ is a cofinal functor.*

Proof. (i). We may construct cones over κ-small diagrams in $\mathcal{J}$ componentwise.

(ii). Similarly, one can show that the comma categories $\left(j_i \downarrow \pi_i\right)$ are connected for all j_i in $\mathcal{J}_i$ and all i in I. ■

Theorem 0.2.11. *Let κ be a regular cardinal in a universe* **U**. *If $\mathcal{J}$ is a* **U**-*small κ-filtered category, then there exist a* **U**-*small κ-directed poset $\mathcal{I}$ and a cofinal functor $P : \mathcal{I} \to \mathcal{J}$.*

Proof. See Theorem 1.5 and Remark 1.21 in [LPAC]. □

Theorem 0.2.12. *Let* $\mathbf{U}$ *be a universe. The following are equivalent for a category* $\mathcal{C}$*:*

(i) $\mathcal{C}$ *has colimits for* $\mathbf{U}$*-small* $\aleph_0$*-filtered diagrams.*

(ii) $\mathcal{C}$ *has colimits for* $\mathbf{U}$*-small* $\aleph_0$*-directed diagrams.*

(iii) $\mathcal{C}$ *has colimits for* α*-chains for all infinite ordinals* α *in* $\mathbf{U}$.

Proof. (i) ⇔ (ii). This is implied by theorem 0.2.11.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (ii). See Corollary 1.7 in [LPAC]. □

Theorem 0.2.13. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* κ *be any regular cardinal in* $\mathbf{U}$*. Given a* $\mathbf{U}$*-small category* $\mathcal{J}$*, the following are equivalent:*

(i) $\mathcal{J}$ *is a* κ*-filtered category.*

(ii) *The functor* $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ *preserves limits for all* κ*-small diagrams.*

Proof. The claim (i) ⇒ (ii) is very well known, and the converse is an exercise in using the Yoneda lemma and manipulating limits and colimits for diagrams of representable functors; see Satz 5.2 in [Gabriel and Ulmer, 1971]. □

Definition 0.2.14. Let κ and λ be regular cardinals in a universe $\mathbf{U}$ and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets.

- A (κ, λ)**-compact object** in a locally $\mathbf{U}$-small category $\mathcal{C}$ is an object A such that the representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ preserves colimits for all λ-small κ-filtered diagrams.
- Let $\mathbf{U}'$ be a universe with $\mathbf{U}' \subseteq \mathbf{U}$. A $(\kappa, \mathbf{U}')$**-compact object** in a locally $\mathbf{U}$-small category is an object that is (κ, λ)-compact for all regular cardinals λ in $\mathbf{U}'$.

Though the above definition is stated using a universe $\mathbf{U}$, the following lemma shows there is in fact no dependence on $\mathbf{U}$.

Lemma 0.2.15. *Let A be an object in a locally* $\mathbf{U}$*-small category C. The following are equivalent:*

(i) *A is a (κ, λ)-compact object in C.*

(ii) *For all λ-small κ-filtered diagrams $B : \mathcal{J} \to C$, if $\varepsilon : B \Rightarrow \Delta C$ is a colimiting cocone, then for any morphism $f : A \to C$, there exist an object i in $\mathcal{J}$ and a morphism $f' : A \to Bi$ in C such that $f = \varepsilon_i \circ f'$; and moreover if $f = \varepsilon_j \circ f''$ for some morphism $f'' : A \to Bj$ in C, then there exists an object k and a pair of arrows $g : i \to k$, $h : i \to k$ in $\mathcal{J}$ such that $Bg \circ f' = Bh \circ f''$.*

Proof. Use the explicit description of $\varinjlim_{\mathcal{J}} C(A, B)$ as a filtered colimit of sets; see Definition 1.1 in [LPAC], or Proposition 5.1.3 in [Borceux, 1994b]. □

Corollary 0.2.16. *Let $B : \mathcal{J} \to C$ be a λ-small κ-filtered diagram, and let $\lambda : B \Rightarrow \Delta C$ be a colimiting cocone in C. If C is a (κ, λ)-compact object in C, then C is a retract of some vertex of B, i.e. there exists an object i in $\mathcal{J}$ such that $\lambda_i : Bi \to C$ is a* split *epimorphism.* ■

Lemma 0.2.17. *Let A be an object in a category C.*

(i) *If A is a (κ, λ)-compact object in C and λ' is any regular cardinal $\leq \lambda$, then A is (κ, λ')-compact as well.*

(ii) *If A is (κ, λ)-compact and μ is any regular cardinal $\geq \kappa$, then A is also (μ, λ)-compact.*

Proof. Obvious. ♦

Lemma 0.2.18. *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*. If $B : \mathbb{D} \to C$ is a κ-small diagram of (κ, λ)-compact objects in a locally* $\mathbf{U}$*-small category, then the colimit $\varinjlim_{\mathbb{D}} B$, if it exists, is also a (κ, λ)-compact object in C.*

Proof. Use theorem 0.2.13 and the fact that $C(-, C) : C^{\mathrm{op}} \to \mathbf{Set}^+$ maps colimits in C to limits in $\mathbf{Set}^+$. ■

Corollary 0.2.19. *A retract of a (κ, λ)-compact object is also a (κ, λ)-compact object.*

Proof. Suppose $r : A \to B$ and $s : B \to A$ are morphisms in $\mathcal{C}$ such that $r \circ s = \mathrm{id}_B$. Then $e = s \circ r$ is an idempotent morphism and the diagram below

$$A \underset{e}{\overset{\mathrm{id}_A}{\rightrightarrows}} A \xrightarrow{r} B$$

is a (split) coequaliser diagram in $\mathcal{C}$, so B is (κ, λ)-compact if A is. ■

Proposition 0.2.20. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. For any* $\mathbf{U}$*-set* A*, the following are equivalent:*

(i) *A has cardinality less than κ.*

(ii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-directed diagrams.*

Proof. The claim (i) ⇒ (ii) follows from theorem 0.2.13, and (ii) ⇒ (iii) is obvious. To see (iii) ⇒ (i), we may use corollary 0.2.16 and the fact that every set is the κ-directed union of its subsets of cardinality $< \kappa$. ■

Corollary 0.2.21. *A* $\mathbf{U}$*-set* X *is* $(\kappa, \mathbf{U})$*-compact if and only if* $|X| < \kappa$. ■

Definition 0.2.22. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible $\mathbf{U}$-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ satisfying the following conditions:

- $\mathcal{C}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams.
- There exists a $\mathbf{U}$-set $\mathcal{G}$ whose element are $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ such that, for each object B in $\mathcal{C}$, there exists a $\mathbf{U}$-small κ-filtered diagram in $\mathcal{C}$ whose vertices are in $\mathcal{G}$ and whose colimit is B.

We write $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the $(\kappa, \mathbf{U})$-compact objects.

Example 0.2.23. The category of $\mathbf{U}$-sets is a κ-accessible $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Theorem 0.2.24. *Let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category and let* κ *be a regular cardinal in* $\mathbf{U}$*. There exist a locally* $\mathbf{U}$*-small category* $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *and a functor* $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *with the following properties:*

(i) *The objects of* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *are* $\mathbf{U}$*-small* κ*-filtered diagrams* $B : \mathbb{D} \to \mathcal{C}$*, and* γ *sends an object* C *in* $\mathcal{C}$ *to the corresponding trivial diagram* $\mathbb{1} \to \mathcal{C}$ *with value* C.

(ii) *The functor* $\gamma : \mathcal{C} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is fully faithful, injective on objects, preserves all limits that exist in* $\mathcal{C}$*, and preserves all* κ*-small colimits that exist in* $\mathcal{C}$.

(iii) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *has colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iv) *For every object* C *in* $\mathcal{C}$*, the object* γC *is* $(\kappa, \mathbf{U})$*-compact in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$*, and for each* $\mathbf{U}$*-small* κ*-filtered diagram* $B : \mathbb{D} \to \mathcal{C}$*, there is a canonical colimiting cocone* $\gamma B \Rightarrow \Delta B$ *in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$.

(v) *If* $\mathcal{D}$ *is a category with colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams, then for each functor* $F : \mathcal{C} \to \mathcal{D}$*, there exists a functor* $\bar{F} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *that preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *such that* $\gamma \bar{F} = F$*, and given any functor* $\bar{G} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *whatsoever, the induced map* $\mathrm{Nat}\big(\bar{F}, \bar{G}\big) \to \mathrm{Nat}\big(F, \gamma \bar{G}\big)$ *is a bijection.*

The category $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is called the* **free** $(\kappa, \mathbf{U})$**-ind-completion** *of* $\mathcal{C}$*, or the* **category of** $(\kappa, \mathbf{U})$**-ind-objects** *in* $\mathcal{C}$.

Proof. If $B : \mathbb{D} \to \mathcal{C}$ and $B' : \mathbb{D}' \to \mathcal{C}$ are two $\mathbf{U}$-small κ-filtered diagrams, then properties (ii) and (iii) together imply that

$$\mathrm{Hom}(B', B) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and so, taking the RHS as the *definition* of the LHS, we need only find a suitable notion of composition to make $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ into a locally $\mathbf{U}$-small category. However, we observe that, if $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, then

$$\mathrm{Hom}\Big(\varinjlim_{\mathbb{D}'} \mathrm{N}B', \varinjlim_{\mathbb{D}} \mathrm{N}B\Big) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and, assuming property (v), the Yoneda embedding $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ must extend along γ to a functor $\bar{\mathrm{N}} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that preserves colimits for $\mathbf{U}$-small κ-filtered diagram, so, in consideration of properties (i) and (iv), we may as well *define* the composition in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ so that $\bar{\mathrm{N}}$ becomes fully faithful. This completes the definition of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ as a category.

It remains to be shown that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ actually has properties (ii), (iii), (iv), and (v); see Corollary 6.4.14 in [Borceux, 1994a] and Theorem 2.26 in [LPAC]. Note that the fact that γ preserves colimits for κ-small diagrams essentially follows from theorem 0.2.13. □

Proposition 0.2.25. *Let $\mathbb{B}$ be a $\mathbf{U}$-small category and let κ be a regular cardinal in $\mathbf{U}$.*

(i) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ *is a κ-accessible $\mathbf{U}$-category.*

(ii) *Every $(\kappa, \mathbf{U})$-compact object in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is a retract of an object of the form γB, where $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is the canonical embedding.*

(iii) $\mathbf{K}^{\mathbf{U}}_{\kappa}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big)$ *is an essentially $\mathbf{U}$-small category.*

Proof. (i). This claim more-or-less follows from the properties of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ explained in the previous theorem.

(ii). Use corollary 0.2.19.

(iii). Since $\mathbb{B}$ is $\mathbf{U}$-small and $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is locally $\mathbf{U}$-small, claim (ii) implies that $\mathbf{K}^{\mathbf{U}}_{\kappa}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big)$ must be essentially $\mathbf{U}$-small. ■

Proposition 0.2.26. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category and let C be an object in $\mathcal{C}$.*

(i) *The comma category $\big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \downarrow C\big)$ is an essentially $\mathbf{U}$-small κ-filtered category.*

(ii) *If $P^C : \big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \downarrow C\big) \to \mathcal{C}$ is the canonical diagram, then the tautological cocone*[2] *$P^C \Rightarrow \Delta C$ is a colimiting cocone in $\mathcal{C}$.*

Proof. See Proposition 2.1.5 in [Makkai and Paré, 1989] or Proposition 2.8 in [LPAC]. □

Corollary 0.2.27. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category. For any $\mathbf{U}$-small κ-filtered diagram $\mathbb{D}$, $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ preserves componentwise limits for κ-small diagrams.*

Proof. The claim is certainly true when $\mathcal{C} = [\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$, by theorem 0.2.13. In general, choose a fully faithful functor $R : \mathcal{C} \to [\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$ that preserves limits for all κ-small diagrams and colimits for all $\mathbf{U}$-small κ-filtered diagrams; then R

[2] See definition A.5.7.

reflects limits for κ-small diagrams and colimits for $\mathbf{U}$-small κ-filtered diagrams, so we may deduce the claim from the corresponding fact for $[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$. Note that such a functor exists: propositions 0.2.26 and A.5.25 imply we may take $\mathbb{B}$ to be $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and R to be the induced Yoneda representation. ■

Definition 0.2.28. Let κ be a regular cardinal in a universe $\mathbf{U}$. A $(\kappa, \mathbf{U})$**-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and
- F preserves all colimits for $\mathbf{U}$-small κ-filtered diagrams.

We write $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D})$ for the full subcategory of the functor category $[\mathcal{C}, \mathcal{D}]$ spanned by the $(\kappa, \mathbf{U})$-accessible functors. An **accessible functor** is a functor that is $(\kappa, \mathbf{U})$-accessible functor for some regular cardinal κ in some universe $\mathbf{U}$.

Theorem 0.2.29 (Classification of accessible categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category.*

(ii) *The inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ extends along $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}\big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})\big)$ to a $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}\big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})\big) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *There exist a $\mathbf{U}$-small category $\mathbb{B}$ and a functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

Proof. See Theorem 2.26 in [LPAC], or Theorem 5.3.5 in [Borceux, 1994b]. □

Corollary 0.2.30. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category and $\mathcal{D}$ is any category, then:*

(i) *The restriction $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D}) \to \big[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}), \mathcal{D}\big]$ is fully faithful and surjective on objects.*

(ii) *In particular, if $\mathcal{D}$ is also locally $\mathbf{U}$-small, then $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D})$ is equivalent to a locally $\mathbf{U}$-small category.*

(iii) *If $\mathcal{D}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams, then the inclusion $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D}) \hookrightarrow [\mathcal{C}, \mathcal{D}]$ has a left adjoint.* ■

Proposition 0.2.31. *Let $\mathcal{C}$ be a κ-accessible* $\mathbf{U}$*-category and let $\mathcal{D}$ be a locally* $\mathbf{U}$*-small category. Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, if G is fully faithful and preserves colimits for all* $\mathbf{U}$*-small κ-filtered diagrams, then $\mathcal{D}$ is also a κ-accessible* $\mathbf{U}$*-category.*

Proof. Under our hypotheses, given any $\mathbf{U}$-small κ-filtered diagram $A : \mathcal{J} \to \mathcal{D}$, we may take $F \varinjlim_{\mathcal{J}} GA$ as its colimit in $\mathcal{D}$. Our hypotheses also imply that F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U})$-compact objects in $\mathcal{D}$; thus if $\mathcal{G}$ is a $\mathbf{U}$-small set of objects that generates $\mathcal{C}$ under $\mathbf{U}$-small κ-filtered colimits, then $\{FX \mid X \in \mathcal{G}\}$ is a $\mathbf{U}$-small set of objects that generates $\mathcal{D}$ in the same sense. ■

Definition 0.2.32. Let κ and λ be regular cardinals and let $\mathscr{P}_\kappa(X)$ denote the set of all κ-small subsets of a set X. We say κ is **sharply less than** λ if

- $\kappa < \lambda$, and
- for all λ-small sets X, there exists a λ-small cofinal subposet of the poset $\mathscr{P}_\kappa(X)$.

We define $\kappa \lhd \lambda$ to mean that κ is sharply less than λ.

Example 0.2.33. Let κ be a regular cardinal and let κ^+ be its cardinal successor. Then $\kappa \lhd \kappa^+$: every κ^+-small set can be mapped bijectively onto an initial segment α of κ (but possibly all of κ), and it is clear that the subposet

$$\{\beta \mid \beta \leq \alpha\} \subseteq \mathscr{P}_\kappa(\alpha)$$

is a κ^+-small cofinal subposet of $\mathscr{P}_\kappa(\alpha)$: given any κ-small subset $X \subseteq \alpha$, we must have $\sup X \leq \alpha$, and $X \subseteq \sup X$ by definition.

Theorem 0.2.34. *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*, and suppose $\kappa < \lambda$. The following are equivalent:*

(i) $\kappa \lhd \lambda$.

(ii) *For any* $\mathbf{U}$*-small κ-directed poset X and any λ-small subset $Y \subseteq X$, there exists a λ-small κ-directed subposet $X' \subseteq X$ with $Y \subseteq X'$.*

(iii) *Any κ-accessible* $\mathbf{U}$*-category is also a λ-accessible* $\mathbf{U}$*-category.*

Proof. See Theorem 2.11 in [LPAC]. □

Proposition 0.2.35.

(i) *The binary relation $\lhd$ is transitive.*

(ii) *If $\kappa \leq \lambda$, then $\kappa \lhd \left(2^{<\lambda}\right)^+$, where $2^{<\lambda} = \sup\{2^\mu \mid \mu$ is a cardinal $< \lambda\}$ and $2^\mu = |\mathscr{P}(\mu)|$, and also $\kappa \lhd \left(2^\lambda\right)^+$.*

(iii) *For any set K of regular cardinals, there exists a regular cardinal λ such that $\kappa \lhd \lambda$ for all κ in K.*

Proof. (i). See Proposition 2.3.2 in [Makkai and Paré, 1989], or theorem 0.2.34.

(ii). See Proposition 2.3.5 in [Makkai and Paré, 1989], or Example 2.13(5) in [LPAC], or Proposition 5.4.7 in [Borceux, 1994b].

(iii). This follows from claim (ii). □

Definition 0.2.36. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **locally κ-presentable $\mathbf{U}$-category** is a κ-accessible $\mathbf{U}$-category that is also $\mathbf{U}$-cocomplete. A **locally presentable $\mathbf{U}$-category** is one that is a locally κ-presentable $\mathbf{U}$-category for some regular cardinal κ in $\mathbf{U}$, and we often say 'locally finitely presentable' instead of 'locally $\aleph_0$-presentable'.

Example 0.2.37. The category of $\mathbf{U}$-sets is a locally κ-presentable $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Lemma 0.2.38. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category.*

(i) *For any regular cardinal λ in $\mathbf{U}$, if $\kappa \leq \lambda$, then $\mathcal{C}$ is a locally λ-presentable $\mathbf{U}$-category.*

(ii) *With λ as above, if $F : \mathcal{C} \to \mathcal{D}$ is a $(\kappa, \mathbf{U})$-accessible functor, then it is also a $(\lambda, \mathbf{U})$-accessible functor.*

(iii) *If $\mathbf{U}^+$ is any universe with $\mathbf{U} \in \mathbf{U}^+$, and $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}^+$-category, then $\mathcal{C}$ must be a preorder.*

Proof. (i). See the remark after Theorem 1.20 in [LPAC], or Propositions 5.3.2 and 5.2.3 in [Borceux, 1994b].

(ii). A λ-filtered diagram is certainly κ-filtered, so if F preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$, it must also preserve colimits for all $\mathbf{U}$-small λ-filtered diagrams.

(iii). This is a corollary of theorem 0.1.10. ■

Corollary 0.2.39. *A category $\mathcal{C}$ is a locally presentable* $\mathbf{U}$*-category for* at most *one universe* $\mathbf{U}$*, provided $\mathcal{C}$ is not a preorder.*

Proof. Use proposition 0.1.36 together with the above lemma. ■

Theorem 0.2.40 (Classification of locally presentable categories)**.** *Let κ be a regular cardinal in a universe* $\mathbf{U}$*, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let $\mathcal{C}$ be a locally* $\mathbf{U}$*-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a locally κ-presentable* $\mathbf{U}$*-category.*

(ii) *There exist a* $\mathbf{U}$*-small category $\mathbb{B}$ that has colimits for κ-small diagrams and a functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *The restricted Yoneda embedding $\mathcal{C} \to \left[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ is fully faithful, $(\kappa, \mathbf{U})$-accessible, and has a left adjoint.*

(iv) *There exist a* $\mathbf{U}$*-small category $\mathbb{A}$ and a fully faithful $(\kappa, \mathbf{U})$-accessible functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ such that $\mathbb{A}$ has limits for all κ-small diagrams, R has a left adjoint, and R is essentially surjective onto the full subcategory of functors $\mathbb{A} \to \mathbf{Set}$ that preserve limits for all κ-small diagrams.*

(v) *There exist a* $\mathbf{U}$*-small category $\mathbb{A}$ and a fully faithful functor $\mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ that preserves colimits for small κ-filtered diagrams and has a left adjoint.*

(vi) *$\mathcal{C}$ is a κ-accessible* $\mathbf{U}$*-category and is* $\mathbf{U}$*-complete.*

Proof. See Proposition 1.27, Corollary 1.28, Theorem 1.46, and Corollary 2.47 in [LPAC], or Theorems 5.2.7 and 5.5.8 in [Borceux, 1994b]. □

REMARK 0.2.41. If $\mathcal{C}$ is equivalent to $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ for some $\mathbf{U}$-small category $\mathbb{B}$ that has colimits for all κ-small diagrams, then $\mathbb{B}$ must be equivalent to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ by proposition 0.2.25. In other words, every locally κ-presentable $\mathbf{U}$-category is, up to equivalence, the free $(\kappa, \mathbf{U})$-ind-completion of an essentially unique $\mathbf{U}$-small κ-cocomplete category.

Example 0.2.42. Obviously, for any $\mathbf{U}$-small category $\mathbb{A}$, the functor category $[\mathbb{A}, \mathbf{Set}]$ is locally finitely presentable. More generally, one may show that for any κ-ary algebraic theory T, possibly many-sorted, the category of T-algebras in $\mathbf{U}$ is a locally κ-presentable $\mathbf{U}$-category. The above theorem can also be used to show that $\mathbf{Cat}$, the category of $\mathbf{U}$-small categories, is a locally finitely presentable $\mathbf{U}$-small category.

Proposition 0.2.43. *If $\mathcal{C}$ is an accessible $\mathbf{U}$-category and $\mathbb{D}$ is any $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is also an accessible $\mathbf{U}$-category.*

Proof. See Theorem 2.39 in [LPAC]. ☐

Proposition 0.2.44. *If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category and $\mathbb{D}$ is any $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is also a locally κ-presentable $\mathbf{U}$-category.*

Proof. This can be proven using the classification theorem by noting that the 2-functor $[\mathbb{D}, -]$ preserves reflective subcategories, but see also Corollary 1.54 in [LPAC]. ☐

It is commonplace to say 'λ-presentable object' instead of 'λ-compact object', especially in algebraic contexts. The following propositions justify the alternative terminology.

Proposition 0.2.45. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category. If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \lhd \lambda$, then the following are equivalent for an object C in $\mathcal{C}$:*

(i) *C is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$.*

(ii) *There exists a λ-small κ-filtered diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} A$.*

(iii) *There exists a λ-small κ-directed diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and C is a retract of $\varinjlim_{\mathcal{J}} A$.*

Proof. (i) ⇔ (ii). See Proposition 2.3.11 in [Makkai and Paré, 1989].

(i) ⇔ (iii). See Remark 2.15 in [LPAC]. ☐

Proposition 0.2.46. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let λ be a regular cardinal in $\mathbf{U}$ with $\lambda \geq \kappa$. If $\mathcal{H}$ is a $\mathbf{U}$-small full subcategory of $\mathcal{C}$ such that*

- *every $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ is isomorphic to an object in $\mathcal{H}$, and*
- *$\mathcal{H}$ is closed in $\mathcal{C}$ under colimits for λ-small diagrams,*

then every $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$ is isomorphic to an object in $\mathcal{H}$. In particular, $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is the smallest replete full subcategory of $\mathcal{C}$ containing $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and closed in $\mathcal{C}$ under colimits for λ-small diagrams.

Proof. Let C be any $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. Clearly, the comma category $(\mathcal{H} \downarrow C)$ is a $\mathbf{U}$-small λ-filtered category. Let $\mathcal{G} = \mathcal{H} \cap \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$. One can show that $(\mathcal{G} \downarrow C)$ is a cofinal subcategory in $(\mathcal{H} \downarrow C)$, and the classification theorem (0.2.40) plus proposition A.5.25 implies that the tautological cocone on the diagram $(\mathcal{G} \downarrow C) \to \mathcal{C}$ is colimiting, so the tautological cocone on the diagram $(\mathcal{H} \downarrow C) \to \mathcal{C}$ is also colimiting. Now, by corollary 0.2.16, C is a retract of an object in $\mathcal{H}$, and hence C must be isomorphic to an object in $\mathcal{H}$, because $\mathcal{H}$ is closed under coequalisers.

TODO: Simplify this argument.

For the final claim, note that $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is certainly a replete full subcategory of $\mathcal{C}$ and contained in any replete full subcategory containing $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and closed in $\mathcal{C}$ under colimits for λ-small diagrams, so we just have to show that $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is also closed in $\mathcal{C}$ under colimits for λ-small diagrams; for this, we simply appeal to lemma 0.2.18. ■

Proposition 0.2.47. *Let $\mathcal{C}$ be a locally $\mathbf{U}$-small category and let $\mathbb{D}$ be a κ-small category in $\mathbf{U}$.*

(i) *If λ is a regular cardinal $\geq \kappa$, $\mathcal{C}$ has colimits for $\mathbf{U}$-small λ-filtered diagrams, and $A : \mathbb{D} \to \mathcal{C}$ is componentwise $(\lambda, \mathbf{U})$-compact, then A is a $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$.*

(ii) *If $\mathcal{C}$ is a λ-accessible $\mathbf{U}$-category and has products for κ-small families of objects, then every $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$ is componentwise $(\lambda, \mathbf{U})$-compact.*

Proof. (i). First, note that the Mac Lane subdivision category[3] $\mathbb{D}^{\S}$ is also κ-small, so $[\mathbb{D}, \mathcal{C}](A, B)$ is computed as the limit of a κ-small diagram of hom-sets. More precisely, using end notation,[4]

$$[\mathbb{D}, \mathcal{C}](A, B) \cong \int_{d:\mathbb{D}} \mathcal{C}(Ad, Bd)$$

[3] See definition A.6.7.
[4] See § A.6.

and so if $\kappa \le \lambda$ and A is componentwise $(\lambda, \mathbf{U})$-compact, then $[\mathbb{D}, \mathcal{C}](A, -)$ preserves colimits for $\mathbf{U}$-small λ-filtered diagrams, hence A is itself $(\lambda, \mathbf{U})$-compact.

(ii). Now, suppose A is a $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$. Let d be an object in $\mathbb{D}$, let $d^* : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ be evaluation at d, and let $d_* : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$ be the right adjoint, which is explicitly given by

$$\big(d_* C\big)(d') = \mathbb{D}(d', d) \pitchfork C$$

where $\pitchfork$ is defined by following adjunction:

$$\mathbf{Set}(X, \mathcal{C}(C, C')) \cong \mathcal{C}(C, X \pitchfork C')$$

The unit $\eta_A : A \to d_* d^* A$ is constructed using the universal property of $\pitchfork$ in the obvious way, and the counit $\epsilon_C : d^* d_* C \to C$ is the projection $\mathbb{D}(d, d) \pitchfork C \to C$ corresponding to $\mathrm{id}_d \in \mathbb{D}(d, d)$. Since $\mathcal{C}$ is a λ-accessible $\mathbf{U}$-category, there exist a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{C}$ consisting of $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$ and a colimiting cocone $\alpha : B \Rightarrow \Delta d^* A$, and since each $\mathbb{D}(d', d)$ has cardinality $< \kappa$, the cocone $d_* \alpha : d_* B \Rightarrow \Delta d_* d^* A$ is also colimiting, by corollary 0.2.27. Lemma 0.2.15 then implies $\eta_A : A \to d_* d^* A$ factors through $d_* \alpha_j : d_*(Bj) \to d_* d^* A$ for some j in $\mathcal{J}$, say

$$\eta_A = d_* \alpha_j \circ \sigma$$

for some $\sigma : A \to d_* Bj$. But then, by the triangle identity,

$$\mathrm{id}_{Ad} = \epsilon_{Ad} \circ d^* \eta_A = \epsilon_{Ad} \circ d^* d_* \alpha_j \circ d^* \sigma = \alpha_j \circ \epsilon_{Bj} \circ d^* \sigma$$

and so $\alpha_j : Bj \to Ad$ is a split epimorphism, hence Ad is a $(\lambda, \mathbf{U})$-compact object, by corollary 0.2.19. ■

Remark 0.2.48. The claim in the above proposition can fail if $\kappa > \lambda$. For example, we could take $\mathcal{C} = \mathbf{Set}$, with $\mathbb{D}$ being the set ω considered as a discrete category; then the terminal object in $[\mathbb{D}, \mathbf{Set}]$ is componentwise finite, but is not itself an $\aleph_0$-compact object in $\mathbf{Set}$.

Lemma 0.2.49. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, with $\kappa \le \lambda$.*

(i) *If $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and $G : \mathcal{D} \to \mathcal{C}$ is a $(\lambda, \mathbf{U})$-accessible functor that preserves limits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, then, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object.*

(ii) *If $\mathcal{C}$ is a locally κ-presentable* **U**-*category, $\mathcal{D}$ is a locally* **U**-*small category, and $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves colimits for all* **U**-*small diagrams in $\mathcal{C}$, then, for any object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object.*

Proof. (i). Let $\mathcal{F}$ be the full subcategory of $(C \downarrow G)$ spanned by those (D, g) where D is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$. G preserves colimits for all **U**-small λ-filtered diagrams, so, by lemma 0.2.15, $\mathcal{F}$ must be a weakly initial family in $(C \downarrow G)$. Proposition 0.2.25 implies $\mathcal{F}$ is an essentially **U**-small category, and since $\mathcal{D}$ has limits for all **U**-small diagrams and G preserves them, $(C \downarrow G)$ is also **U**-complete. Thus, the inclusion $\mathcal{F} \hookrightarrow (C \downarrow G)$ has a limit, and it can be shown that this is an initial object in $(C \downarrow G)$.[5]

(ii). Let $\mathcal{G}$ be the full subcategory of $(F \downarrow D)$ spanned by those (C, f) where C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$; note that proposition 0.2.25 implies $\mathcal{G}$ is an essentially **U**-small category. Since $\mathcal{C}$ has colimits for all **U**-small diagrams and F preserves them, $(F \downarrow D)$ is also **U**-cocomplete.[6] Let (C, f) be a colimit for the inclusion $\mathcal{G} \hookrightarrow (F \downarrow D)$. It is not hard to check that (C, f) is a weakly terminal object in $(F \downarrow D)$, so the formal dual of Freyd's initial object lemma[7] gives us a terminal object in $(F \downarrow D)$; explicitly, it may be constructed as the joint coequaliser of all the endomorphisms of (C, f). ■

Theorem 0.2.50 (Accessible adjoint functor theorem)**.** *Let κ and λ be regular cardinals in a universe* **U**, *with $\kappa \leq \lambda$, let $\mathcal{C}$ be a locally κ-presentable* **U**-*category, and let $\mathcal{D}$ be a locally λ-presentable* **U**-*category.*

Given a functor $F : \mathcal{C} \to \mathcal{D}$, the following are equivalent:

(i) *F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and G is a $(\lambda, \mathbf{U})$-accessible functor.*

(ii) *F preserves colimits for all* **U**-*small diagrams and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(iii) *F has a right adjoint and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

[5] See Theorem 1 in [CWM, Ch. X, §2].
[6] See the Lemma in [CWM, Ch. V, §6].
[7] See Theorem 1 in [CWM, Ch. V, §6].

On the other hand, given a functor $G : \mathcal{D} \to \mathcal{C}$, the following are equivalent:

(iv) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(v) *G is a $(\lambda, \mathbf{U})$-accessible functor and preserves limits for all $\mathbf{U}$-small diagrams.*

(vi) *G is a $(\lambda, \mathbf{U})$-accessible functor and there exist a functor $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ and hom-set bijections*

$$\mathcal{C}(C, GD) \cong \mathcal{D}\big(F_0 C, D\big)$$

natural in D for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, where D varies in $\mathcal{D}$.

Proof. We will need to refer back to the details of the proof of this theorem later, so here is a sketch of the constructions involved.

(i) ⇒ (ii). If F is a left adjoint, then F certainly preserves colimits for all $\mathbf{U}$-small diagrams. Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{D}\Big(FC, \varinjlim_{\mathcal{J}} B\Big) \cong \mathcal{C}\Big(C, G \varinjlim_{\mathcal{J}} B\Big) \cong \mathcal{C}\Big(C, \varinjlim_{\mathcal{J}} GB\Big) & \\ \cong \varinjlim_{\mathcal{J}} \mathcal{C}(C, GB) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(FC, B) &\end{aligned}$$

and thus FC is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$.

(ii) ⇒ (iii). It is enough to show that, for each object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object $\big(GD, \varepsilon_D\big)$;[8] but this was done in the previous lemma.

(iii) ⇒ (i). Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{C}\Big(C, G \varinjlim_{\mathcal{J}} B\Big) \cong \mathcal{D}\Big(FC, \varinjlim_{\mathcal{J}} B\Big) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(FC, B) & \\ \cong \varinjlim_{\mathcal{J}} \mathcal{C}(C, GB) \cong \mathcal{C}\Big(C, \varinjlim_{\mathcal{J}} GB\Big) &\end{aligned}$$

[8] See Theorem 2 in [CWM, Ch. IV, § 1].

because FC is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$; but theorem 0.2.40 says the restricted Yoneda embedding $\mathcal{C} \to \left[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ is fully faithful, so this is enough to conclude that G preserves colimits for $\mathbf{U}$-small λ-filtered diagrams.

(iv) ⇒ (v). If G is a right adjoint, then G certainly preserves limits for all $\mathbf{U}$-small diagrams; the rest of this implication is just (iii) ⇒ (i).

(v) ⇒ (vi). It is enough to show that, for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object $\left(F_0 C, \eta_C\right)$; but this was done in the previous lemma. It is clear how to make F_0 into a functor $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$.

(vi) ⇒ (iv). We use theorems 0.2.24 and 0.2.40 to extend $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ along the inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ to get $(\kappa, \mathbf{U})$-accessible functor $F : \mathcal{C} \to \mathcal{D}$. We then observe that, for any $\mathbf{U}$-small κ-filtered diagram $A : \mathbb{I} \to \mathcal{C}$ of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}\left(\varinjlim_{\mathbb{I}} A, GD\right) &\cong \varprojlim_{\mathbb{I}} \mathcal{C}(A, GD) \cong \varprojlim_{\mathbb{I}} \mathcal{C}\left(F_0 A, D\right) \\ &\cong \mathcal{C}\left(\varinjlim_{\mathbb{I}} FA, D\right) \cong \mathcal{C}\left(F \varinjlim_{\mathbb{I}} A, D\right) \end{aligned}$$

is a series of bijections natural in D, where D varies in $\mathcal{D}$; but $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, so this is enough to show that F is a left adjoint of G. The remainder of the claim is a corollary of (i) ⇒ (ii). ■

Corollary 0.2.51. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable $\mathbf{U}$-categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then there exists a regular cardinal μ in $\mathbf{U}$ such that G is a $(\mu, \mathbf{U})$-accessible functor.*

Proof. Suppose $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, and $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint for G. Since $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ is an essentially $\mathbf{U}$-small category, recalling lemma 0.2.17, there certainly exists a regular cardinal μ in $\mathbf{U}$ such that $\mu \geq \lambda$ and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U})$-compact objects in $\mathcal{D}$. The above theorem, plus lemma 0.2.38, implies G is an $(\mu, \mathbf{U})$-accessible functor. ■

0.3 Accessible constructions

Prerequisites. §§ 0.1, 0.2, A.5

Definition 0.3.1. Let $\mathbf{U}$ be a universe and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The **U-rank** of F is the smallest regular cardinal κ in $\mathbf{U}$ such that F preserves colimits for $\mathbf{U}$-small κ-filtered diagrams, provided any such cardinal exists.

Remark 0.3.2. The class of regular cardinals is well-ordered, so the definition above makes sense. Of course, every $(\kappa, \mathbf{U})$-accessible functor has $\mathbf{U}$-rank $\le \kappa$.

Definition 0.3.3. Let $\mathbf{U}$ be a universe and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. The **compactness U-rank** of an object A in $\mathcal{C}$ is the $\mathbf{U}$-rank of the hom-functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of $\mathbf{U}$-sets.

Remark 0.3.4. Lemma 0.2.18 implies that, for each object A in an accessible $\mathbf{U}$-category, there exists a regular cardinal λ in $\mathbf{U}$ such that A is $(\lambda, \mathbf{U})$-compact; in particular, every object in an accessible $\mathbf{U}$-category has a compactness $\mathbf{U}$-rank.

Definition 0.3.5. Let κ and λ be regular cardinals in a universe $\mathbf{U}$. A (κ, λ)**-compactly generated U-category** is an essentially $\mathbf{U}$-small category $\mathcal{C}$ that satisfies the following conditions:

- $\mathcal{C}$ has colimits for all λ-small κ-filtered diagrams.
- Every object in $\mathcal{C}$ is a colimit for some λ-small κ-filtered diagram of (κ, λ)-compact objects in $\mathcal{C}$.

We write $\mathbf{K}_\kappa^\lambda(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the (κ, λ)-compact objects.

Remark 0.3.6. Lemma 0.2.9 implies an essentially $\mathbf{U}$-small category is (κ, κ)-compactly generated if and only if it is Cauchy-complete, i.e. if and only if all idempotent endomorphisms in $\mathcal{C}$ are split.

Proposition 0.3.7. *Let $\mathcal{C}$ be a κ-accessible* $\mathbf{U}$*-category.*

(i) $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ *is a (κ, κ)-compactly generated* $\mathbf{U}$*-category, and every object in* $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ *is (κ, κ)-compact.*

(ii) *If λ is a regular cardinal in* $\mathbf{U}$ *and $\kappa \lhd \lambda$, then* $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ *is a (κ, λ)-compactly generated* $\mathbf{U}$*-category, and the (κ, λ)-compact objects in* $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ *are precisely the $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$.*

Proof. (i). This follows from lemma 0.2.17, corollary 0.2.19, and remark 0.3.6.

(ii). Combine corollary 0.2.16, lemma 0.2.18, and proposition 0.2.45. ■

Proposition 0.3.8. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, let $\mathbb{A}$ and $\mathbb{B}$ be $\mathbf{U}$-small categories, and let $F : \mathbb{A} \to \mathbb{B}$ be a fully faithful functor. Assume the following hypotheses:*

- *$\kappa \le \lambda$.*
- *$\mathbb{A}$ is a Cauchy-complete category and $\mathbb{B}$ has colimits for λ-small κ-filtered diagrams.*
- *Each $F A$ is a (κ, λ)-compact object in $\mathbb{B}$, and each object in $\mathbb{B}$ is a colimit for a λ-small κ-filtered diagram of objects in the image of F.*

Then:

(i) *Every (κ, λ)-compact object in $\mathbb{B}$ is isomorphic to an object in the image of $F : \mathbb{A} \to \mathbb{B}$.*

(ii) *There exists a functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ equipped with a natural bijection of the form below,*

$$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})(A, U B) \cong \mathbb{B}(F A, B)$$

and it is unique up to unique isomorphism.

(iii) *Moreover, the functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective onto the full subcategory of $(\lambda, \mathbf{U})$-compact objects in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$.*

(iv) *$F : \mathbb{A} \to \mathbb{B}$ is a dense functor.*

(v) *If $\kappa \lhd \lambda$, then the $(\lambda, \mathbf{U})$-accessible functor $\bar{U} : \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ induced by $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective on objects.*

Proof. (i). Let B be an object in $\mathbb{B}$. By hypothesis, there is a λ-small κ-filtered diagram $Y : \mathcal{J} \to \mathbb{B}$ such that each $Y j$ is in the image of F and $B \cong \varinjlim_{\mathcal{J}} Y$. Thus, if B is a (κ, λ)-compact object in $\mathbb{B}$, then B must be a retract of some $Y j$ (by corollary 0.2.16). But $\mathbb{A}$ is Cauchy-complete and $F : \mathbb{A} \to \mathbb{B}$ is fully faithful, so B must be isomorphic to some object in the image of F.

(ii). The assumptions imply each functor $\mathbb{B}(F-, B) : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ is a colimit for a λ-small κ-filtered diagram of functors of the form $\mathbb{A}(-, A')$ for various A'

in $\mathbb{A}$. Hence, for each object B in $\mathbb{B}$, there exist an object UB in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ and bijections

$$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})(A, UB) \cong \mathbb{B}(FA, B)$$

that are natural in A. Since the canonical embedding $\mathbb{A} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is dense, we thus obtain a functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ with the required property.

(iii). It is clear that U is a fully faithful functor that preserves colimits for λ-small κ-filtered diagrams. We may then apply proposition 0.2.45 to deduce that every $(\lambda, \mathbf{U})$-compact object in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is isomorphic to one in the image of U.

(iv). This follows from claim (iii) and the fact that the canonical embedding $\mathbb{A} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is dense.

(v). If $\kappa \lhd \lambda$, then theorem 0.2.34 says $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is a λ-accessible category, so we may apply the classification theorem (0.2.29) to deduce that $\bar{U} : \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective on objects. ■

Corollary 0.3.9 (Classification of compactly generated categories)**.** *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*. If either $\kappa = \lambda$ or $\kappa \lhd \lambda$, then the following are equivalent for a Cauchy-complete category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is a (κ, λ)-compactly generated* $\mathbf{U}$*-category.*

(ii) $\mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{C})$ *is a κ-accessible* $\mathbf{U}$*-category.*

(iii) *$\mathcal{C}$ is equivalent to $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{D})$ for some κ-accessible* $\mathbf{U}$*-category $\mathcal{D}$.*

Proof. (i) ⇒ (ii). See proposition 0.3.8.

(ii) ⇒ (iii). Apply proposition 0.2.25.

(iii) ⇒ (i). See proposition 0.3.7. ■

Definition 0.3.10. Let κ and λ be regular cardinals in a universe $\mathbf{U}$. A (κ, λ)**-compactly defined functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- $\mathcal{C}$ is a (κ, λ)-compactly generated $\mathbf{U}$-category.
- $F : \mathcal{C} \to \mathcal{D}$ preserves colimits for λ-small κ-filtered diagrams of (κ, λ)-compact objects in $\mathcal{C}$.

Lemma 0.3.11. *Let $\mathcal{C}$ be a (κ, λ)-compactly generated* **U***-category, let $\mathcal{D}$ be a locally* **U***-small category, and let* **Set** *be the category of* **U***-sets. If $F : \mathcal{C} \to \mathcal{D}$ is a (κ, λ)-compactly defined functor, then the natural maps*

$$\begin{aligned} \mathcal{D}(FC, D) &\to \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, C), \mathcal{D}(F-, D)) \\ f &\mapsto (c \mapsto f \circ Fc) \end{aligned}$$

are bijections.

Proof. Choose a λ-small κ-filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that each vertex is (κ, λ)-compact in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} X$. We then have a natural bijection

$$\mathcal{C}(A, C) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(A, X)$$

as A varies in $\mathbf{K}_\kappa^\lambda(\mathcal{C})$, so

$$\left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, C), \mathcal{D}(-, D)) \cong \varprojlim_{\mathcal{J}} \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, X), \mathcal{D}(F-, D))$$

and by applying the Yoneda lemma, we have

$$\varprojlim_{\mathcal{J}} \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, X), \mathcal{D}(F-, D)) \cong \varprojlim_{\mathcal{J}} \mathcal{D}(FX, D)$$

but $F : \mathcal{C} \to \mathcal{D}$ preserves colimits for λ-small κ-filtered diagrams of (κ, λ)-compact objects in $\mathcal{C}$, so:

$$\varprojlim_{\mathcal{J}} \mathcal{D}(FX, D) \cong \mathcal{D}\left(\varinjlim_{\mathcal{J}} FX, D\right) \cong \mathcal{D}(FC, D)$$

We may therefore deduce that the indicated maps are bijections. ■

Proposition 0.3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be (κ, λ)-compactly generated* **U***-categories. If $F : \mathcal{C} \to \mathcal{D}$ is a (κ, λ)-compactly defined functor, then the induced functor $\mathbf{Ind}_{\mathbf{U}}^\lambda(F) : \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C}) \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ is $(\kappa, \mathbf{U})$-accessible.*

Proof. Let $\mathcal{A} = \mathbf{K}_\kappa^\lambda(\mathcal{C})$, let $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ be the canonical embeddings and let $\bar{F} = \mathbf{Ind}_{\mathbf{U}}^\lambda(F)$. Theorems 0.2.24 and A.5.15 imply $\bar{F} : \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C}) \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ along $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$. By proposition 0.3.8, $\mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$ and $\mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ are κ-accessible **U**-categories, and to verify that $\bar{F}$ is a $(\kappa, \mathbf{U})$-accessible functor, it suffices to show that $\bar{F}$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F\big|_{\mathcal{A}}$ along $\gamma_{\mathcal{C}}\big|_{\mathcal{A}}$.

Since $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{D})$ preserves colimits for λ-small diagrams, the composite $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{D})$ is also a (κ, λ)-compactly defined functor, and so $\gamma_{\mathcal{D}} F$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F\big|_{\mathcal{A}}$ along the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ (by lemma 0.3.11). We may therefore apply theorem A.5.20 to deduce that $\bar{F}$ is indeed (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F\big|_{\mathcal{A}}$ along $\gamma_{\mathcal{C}}\big|_{\mathcal{A}}$. ■

Definition 0.3.13. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **strongly $(\kappa, \mathbf{U})$-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- Both $\mathcal{C}$ and $\mathcal{D}$ are κ-accessible $\mathbf{U}$-categories.
- F preserves colimits for $\mathbf{U}$-small κ-filtered diagrams.
- F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U})$-compact objects in $\mathcal{D}$.

Example 0.3.14. Given any functor $F : \mathbb{A} \to \mathbb{B}$, if $\mathcal{A}$ and $\mathcal{B}$ are small categories, then the induced functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(F) : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is strongly $(\kappa, \mathbf{U})$-accessible, by corollaries 0.2.16 and 0.2.19.

Proposition 0.3.15 (Products of accessible categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$. If $\left(\mathcal{C}_i \,\middle|\, i \in I\right)$ is a κ-small family of κ-accessible $\mathbf{U}$-categories, then:*

(i) *The product $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$ is also a κ-accessible $\mathbf{U}$-category.*

(ii) *Moreover, the projection functors $\mathcal{C} \to \mathcal{C}_i$ are strongly $(\kappa, \mathbf{U})$-accessible functors.*

Proof. It is clear that $\mathcal{C}$ has colimits for $\mathbf{U}$-small κ-filtered diagrams: indeed, they can be computed componentwise. Theorem 0.2.13 implies that an object in $\mathcal{C}$ is $(\kappa, \mathbf{U})$-compact as soon as its components are $(\kappa, \mathbf{U})$-compact objects in their respective categories. Recalling lemma 0.2.10, it follows that $\mathcal{C}$ is generated under $\mathbf{U}$-small κ-filtered colimits by a $\mathbf{U}$-small family of $(\kappa, \mathbf{U})$-compact objects, as required of a κ-accessible $\mathbf{U}$-category. ■

Lemma 0.3.16. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$, let $\mathcal{C}$ be an accessible $\mathbf{U}$-category, let $\mathcal{D}$ be an accesible $\mathbf{U}^+$-category, and let $F : \mathcal{C} \to \mathcal{D}$ be a $(\kappa, \mathbf{U})$-accessible functor.*

(i) *There is a regular cardinal λ in $\mathbf{U}^+$ such that F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U}^+)$-compact objects in $\mathcal{D}$.*

(ii) *Moreover, if μ is a regular cardinal in $\mathbf{U}^+$ such that $\kappa \lhd \mu$ and $\lambda \leq \mu$, then F sends $(\mu, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U}^+)$-compact objects in $\mathcal{D}$.*

Proof. (i). Such a regular cardinal exists by remark 0.3.4 and proposition 0.2.25.

(ii). If μ is not in $\mathbf{U}$, then the claim is trivial; otherwise, proposition 0.2.45 and lemma 0.2.18 imply that F sends $(\mu, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U}^+)$-compact objects in $\mathcal{D}$, as required. ■

Corollary 0.3.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be accessible $\mathbf{U}$-categories. If $F : \mathcal{C} \to \mathcal{D}$ is a $(\kappa, \mathbf{U})$-accessible functor, then:*

(i) *There exists a regular cardinal λ in $\mathbf{U}$ such that F is strongly $(\lambda, \mathbf{U})$-accessible.*

(ii) *Moreover, if μ is a regular cardinal in $\mathbf{U}$ and $\lambda \lhd \mu$, then F is also strongly $(\mu, \mathbf{U})$-accessible.*

Proof. Combine lemma 0.3.16, theorem 0.2.34, and proposition 0.2.35. ■

Lemma 0.3.18. *Let $\mathcal{J}$ be a κ-filtered category. If $\mathbb{A}$ is a κ-small category, then the functor category $[\mathbb{A}, \mathcal{J}]$ is also a κ-filtered category.*

Proof. There is a natural bijection between diagrams $\mathbb{D} \to [\mathbb{A}, \mathcal{J}]$ and diagrams $\mathbb{D} \times \mathbb{A} \to \mathcal{J}$; but if $\mathbb{D}$ is κ-small, then so is $\mathbb{D} \times \mathbb{A}$. Thus, every κ-small diagram in $[\mathbb{A}, \mathcal{J}]$ has a cocone, as required. ■

Lemma 0.3.19. *Let $\mathcal{J}$ be a κ-filtered category, let $A : \mathcal{I} \to \mathcal{J}$ be a κ-small diagram, let ${}^{A/}\mathcal{J}$ be the cocone category $(A \downarrow \Delta)$, and let $P : {}^{A/}\mathcal{J} \to \mathcal{J}$ be the projection functor.*

(i) *The cocone category ${}^{A/}\mathcal{J}$ is also a κ-filtered category.*

(ii) *$P : {}^{A/}\mathcal{J} \to \mathcal{J}$ is a cofinal functor.*[9]

Proof. (i). Let $\mathbb{D}$ be a κ-small category. There exists a κ-small category $\tilde{\mathbb{D}}$ equipped with a functor $L : \mathcal{I} \to \tilde{\mathbb{D}}$ and a natural bijection between diagrams $X : \mathbb{D} \to {}^{A/}\mathcal{J}$ and diagrams $\tilde{X} : \tilde{\mathbb{D}} \to \mathcal{J}$ such that $\tilde{X} L = A$, and moreover

[9] See definition A.5.31.

this construction is natural in $\mathbb{D}$. Thus, every κ-small diagram in ${}^{A}/\mathcal{J}$ admits a cocone, as required.

(ii). We must show that the comma category $(b \downarrow P)$ is connected for all objects b in $\mathcal{J}$. Since $\mathcal{J}$ is filtered, there must exist an object c, a cocone $A \Rightarrow \Delta c$, and a morphism $b \to c$ in $\mathcal{J}$; thus, $(b \downarrow P)$ is inhabited. Moreover, any diagram in $[\mathcal{I}, \mathcal{J}]$ of the form shown below on the left can be completed to one of the form shown below on the right,

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ \Delta c & & \Delta d \\ & \Delta b \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\ \Delta c \arrow[r, dashed] & \Delta e & \Delta d \arrow[l, dashed] \\ & \Delta b \arrow[ul] \arrow[ur] \arrow[u, dashed] & \end{tikzcd}$$

so we may conclude that $(b \downarrow P)$ is indeed connected. ■

Lemma 0.3.20. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $X : \mathcal{I} \to \mathcal{C}$ be a κ-small diagram, let $Y : \mathcal{J} \to \mathcal{C}$ be a $\mathbf{U}$-small κ-filtered diagram, and let $\varepsilon : Y \Rightarrow \Delta B$ be a colimiting cocone in $\mathcal{C}$. If each Xi is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, then every cocone $X \Rightarrow \Delta B$ must factor through $\varepsilon_j : Yj \to B$ for some j in $\mathcal{J}$.*

Proof. Let $\varphi : X \Rightarrow \Delta B$ be a cocone, and regard it as a morphism in the functor category $[\mathcal{I}, \mathcal{C}]$. By proposition 0.2.47, X is a $(\kappa, \mathbf{U})$-compact object in $[\mathcal{I}, \mathcal{C}]$; but $\Delta\varepsilon : \Delta Y \Rightarrow \Delta\Delta B$ is a colimiting cocone in $[\mathcal{I}, \mathcal{C}]$, so we may apply lemma 0.2.15. ■

Lemma 0.3.21. *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors that send $(\kappa, \mathbf{U})$-compact objects to $(\kappa, \mathbf{U})$-compact objects. Given an object (C, D, e) in the comma category $(F \downarrow G)$, if C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and D is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}$, then (C, D, e) is a $(\kappa, \mathbf{U})$-compact object in $(F \downarrow G)$.*

Proof. Let $\mathcal{B} = (F \downarrow G)$ and let $\varphi : FP \Rightarrow GQ$ be the canonical natural transformation. Then, given any two objects B and B' in $\mathcal{B}$, we have the following pullback diagram,

$$\begin{tikzcd} \mathcal{B}(B, B') \arrow[r] \arrow[d] & \mathcal{D}(QB, QB') \arrow[d] \\ \mathcal{C}(PB, PB') \arrow[r] & \mathcal{E}(FPB, GQB') \end{tikzcd}$$

where the map $\mathcal{C}(PB, PB') \to \mathcal{E}(FPB, GQB')$ is induced by the functor $F : \mathcal{C} \to \mathcal{E}$ and the morphism $\varphi_{B'} : FPB' \to GQB'$, and the map $\mathcal{D}(QB, QB') \to \mathcal{E}(FPB, GQB')$ is induced by the functor $G : \mathcal{D} \to \mathcal{E}$ and the morphism $\varphi_B : FPB \to GQB$. Thus, if PB and QB are $(\kappa, \mathbf{U})$-compact objects, then so are FPB and GQB, and therefore we may use theorem 0.2.13 deduce that B is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{B}$. ■

Theorem 0.3.22 (Accessibility of comma categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be $(\kappa, \mathbf{U})$-accessible functors.*

(i) *The comma category $(F \downarrow G)$ has colimits for $\mathbf{U}$-small κ-filtered diagrams, created by the projection functor $(F \downarrow G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *If F and G are strongly $(\kappa, \mathbf{U})$-accessible functors, then $(F \downarrow G)$ is a κ-accessible $\mathbf{U}$-category, and the projection functors $P : (F \downarrow G) \to \mathcal{C}$ and $Q : (F \downarrow G) \to \mathcal{D}$ are strongly $(\kappa, \mathbf{U})$-accessible.*

Proof. See Theorem 2.43 in [LPAC]. □

Corollary 0.3.23. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category and A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, then:*

- *The slice category ${}^{A/}\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and the projection functor ${}^{A/}\mathcal{C} \to \mathcal{C}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor.*
- *The slice category $\mathcal{C}_{/A}$ is a κ-accessible $\mathbf{U}$-category, and the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor.* ■

Corollary 0.3.24. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, then so is the functor category $[\mathbb{2}, \mathcal{C}]$, and moreover the $(\kappa, \mathbf{U})$-compact objects in $[\mathbb{2}, \mathcal{C}]$ are precisely the componentwise $(\kappa, \mathbf{U})$-compact objects.*

Proof. The functor category $[\mathbb{2}, \mathcal{C}]$ is isomorphic to the comma category $(\mathcal{C} \downarrow \mathcal{C})$, and $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is certainly a strongly $(\kappa, \mathbf{U})$-accessible functor. ■

Corollary 0.3.25. *If $\mathcal{C}$ is a (κ, λ)-compactly generated $\mathbf{U}$-category, then so is $[\mathbb{2}, \mathcal{C}]$.*

Proof. Combine lemma 0.3.21 and corollaries 0.3.9 and 0.3.24. ∎

Lemma 0.3.26. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, with $\kappa \le \lambda$, let $\mathcal{E}$ be a locally $\mathbf{U}$-small category with colimits for $\mathbf{U}$-small κ-filtered diagrams, let $X : \mathcal{I} \to \mathcal{E}$ and $Y : \mathcal{J} \to \mathcal{E}$ be $\mathbf{U}$-small λ-filtered diagrams that are componentwise $(\lambda, \mathbf{U})$-compact, let $C = \varinjlim_{\mathcal{I}} X$ and $D = \varinjlim_{\mathcal{J}} Y$, and let $c_i : X i \to C$ and $d_j : Y j \to D$ be the components of the respective colimiting cocones.*

(i) *Given any object i_0 in $\mathcal{I}$ and any morphism $e : C \to D$, there exist an object j_0 in $\mathcal{J}$ and a morphism $f_0 : X i_0 \to Y j_0$ such that the following diagram commutes:*

$$\begin{tikzcd} X i_0 \arrow[r, "c_{i_0}"] \arrow[d, dashed, "f_0"'] & C \arrow[d, "e"] \\ Y j_0 \arrow[r, "d_{j_0}"'] & D \end{tikzcd}$$

(ii) *Given any commutative diagram of the above form, if $e : C \to D$ is an isomorphism in $\mathcal{E}$, then there exist chains $I : \kappa \to \mathcal{I}$ and $J : \kappa \to \mathcal{J}$ and a factorisation of the form below,*

$$\begin{tikzcd} X i_0 \arrow[r] \arrow[d, "f_0"'] & C' \arrow[r] \arrow[d, "e'"] & C \arrow[d, "e"] \\ Y j_0 \arrow[r] & D' \arrow[r] & D \end{tikzcd}$$

where $I(0) = i_0$, $J(0) = j_0$, $C' = \varinjlim_{\alpha < \kappa} X I(\alpha)$, $D' = \varinjlim_{\alpha < \kappa} Y J(\alpha)$, $e' : C' \to D'$ is an isomorphism, and the morphisms $C' \to C$ and $D' \to D$ are the ones induced by the evident cocones.

Proof. (i). Since $X i_0$ is $(\lambda, \mathbf{U})$-compact and $Y : \mathcal{J} \to \mathcal{E}$ is a $\mathbf{U}$-small λ-filtered diagram, such a factorisation of $e \circ c_{i_0}$ must exist, by lemma 0.2.15.

(ii). We will construct I, J, and e' by transfinite induction on κ.

- Given j_α and f_α, choose a morphism $i_{\alpha\to\alpha+1} : i_\alpha \to i_{\alpha+1}$ in $\mathcal{I}$ and a morphism $g_\alpha : Y j_\alpha \to X i_{\alpha+1}$ in $\mathcal{E}$ such that the diagram below commutes:

$$\begin{tikzcd}
X i_\alpha \arrow[r, dashed, "X i_{\alpha\to\alpha+1}"] \arrow[d, "f_\alpha"'] & X i_{\alpha+1} \arrow[r, "c_{i_{\alpha+1}}"] & C \\
Y j_\alpha \arrow[ur, dashed, "g_\alpha"'] \arrow[rr, "d_{j_\alpha}"'] & & D \arrow[u, "e^{-1}"']
\end{tikzcd}$$

  Such $i_{\alpha\to\alpha+1}$ and g_α exist because $f_\alpha : X i_\alpha \to Y j_\alpha$ defines a $(\lambda, \mathbf{U})$-compact object in the slice category ${}^{X i_\alpha}/\mathcal{E}$ (by lemma 0.3.21) and there is an evident $\mathbf{U}$-small λ-filtered diagram ${}^{i_\alpha}/X : {}^{i_\alpha}/\mathcal{I} \to {}^{X i_\alpha}/\mathcal{E}$ with colimit defined by $c_{i_\alpha} : X i_\alpha \to C$ (by lemma 0.3.19).

- Given $i_{\alpha+1}$ and g_α, choose a morphism $j_{\alpha\to\alpha+1} : j_\alpha \to j_{\alpha+1}$ in $\mathcal{J}$ and a morphism $f_{\alpha+1} : X i_{\alpha+1} \to Y j_{\alpha+1}$ in $\mathcal{E}$ such that the diagram below commutes:

$$\begin{tikzcd}
Y j_\alpha \arrow[r, dashed, "Y j_{\alpha\to\alpha+1}"] \arrow[d, "g_\alpha"'] & Y j_{\alpha+1} \arrow[r, "d_{i_{\alpha+1}}"] & D \\
X i_{\alpha+1} \arrow[ur, dashed, "f_{\alpha+1}"'] \arrow[rr, "c_{i_{\alpha+1}}"'] & & C \arrow[u, "e"']
\end{tikzcd}$$

- Given a limit ordinal $\beta < \kappa$ and i_α for all ordinals $\alpha < \beta$, choose an object i_β in $\mathcal{I}$ and a cocone from the chain defined by $\left(i_\alpha \,\middle|\, \alpha < \beta\right)$ to i_β.

- Given i_β for a limit ordinal $\beta < \kappa$ and j_α for all ordinals $\alpha < \beta$, choose an object j_β in $\mathcal{J}$, a cocone from the chain defined by $\left(j_\alpha \,\middle|\, \alpha < \beta\right)$, and a morphism $f_\beta : X i_\beta \to Y j_\beta$ such that the following diagram commutes for all ordinals $\alpha < \beta$:

$$\begin{tikzcd}
Y j_\alpha \arrow[r, dashed, "Y j_{\alpha\to\beta}"] \arrow[d, "g_\alpha"'] & Y j_\beta \arrow[r, "d_{j_\beta}"] & D \\
X i_{\alpha+1} \arrow[r, "X i_{\alpha+1\to\beta}"'] & X i_\beta \arrow[u, dashed, "f_\beta"'] \arrow[r, "c_{i_\beta}"'] & C \arrow[u, "e"']
\end{tikzcd}$$

  Such data exist because the chains X' and Y' defined by $\left(X i_\alpha \,\middle|\, \alpha < \beta\right)$ and $\left(Y j_\alpha \,\middle|\, \alpha < \beta\right)$ are $(\lambda, \mathbf{U})$-compact objects in the category $[\beta, \mathcal{E}]$ (by proposition 0.2.47) and there is an evident $\mathbf{U}$-small λ-filtered diagram in ${}^{Y'}/[\beta, \mathcal{E}]$ with colimit ΔD (by lemma 0.3.19).

Now take $I : \kappa \to \mathcal{I}$ and $J : \kappa \to \mathcal{J}$ to be the chains defined by $I(\alpha) = i_\alpha$ and $J(\alpha) = j_\alpha$. Let $C' = \varinjlim_{\alpha<\kappa} X i_\alpha$ and $D' = \varinjlim_{\alpha<\kappa} Y j_\alpha$. The above construction yields commutative diagrams of the form below for all ordinals $\alpha < \beta < \kappa$,

$$\begin{tikzcd} X i_\alpha \arrow[r, "X i_{\alpha\to\beta}"] \arrow[d, "f_\alpha"'] & X i_\beta \arrow[d, "f_\beta"] \\ Y j_\alpha \arrow[r, "Y j_{\alpha\to\beta}"'] & Y j_\beta \end{tikzcd} \qquad \begin{tikzcd} Y j_\alpha \arrow[r, "Y j_{\alpha\to\beta}"] \arrow[d, "g_\alpha"'] & Y j_\beta \arrow[d, "g_\beta"] \\ X i_{\alpha+1} \arrow[r, "X i_{\alpha+1\to\beta+1}"'] & X i_{\beta+1} \end{tikzcd}$$

so there are induced morphisms $f : C' \to D'$ and $g : D' \to C'$; moreover, since $g_\alpha \circ f_\alpha = X i_{\alpha\to\alpha+1}$ and $f_{\alpha+1} \circ g_\alpha = Y j_{\alpha\to\alpha+1}$, we have $g \circ f = \mathrm{id}_{C'}$ and $f \circ g = \mathrm{id}_{D'}$. Thus, we have the required isomorphism $e : C' \to D'$. ■

Theorem 0.3.27 (Accessibility of iso-comma categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories with colimits for $\mathbf{U}$-small κ-filtered diagrams, and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be be functors of $\mathbf{U}$-rank $\leq \kappa$.*

(i) *The iso-comma category $(F \wr G)$ has colimits for $\mathbf{U}$-small κ-filtered diagrams, created by the projection functor $(F \wr G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *Assuming F and G are strongly λ-accessible functors, given an object (C, D, e) in $(F \wr G)$, if C is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$ and D is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$, then (C, D, e) is a $(\lambda, \mathbf{U})$-compact object in $(F \wr G)$.*

(iii) *If F and G are strongly $(\lambda, \mathbf{U})$-accessible functors and $\kappa < \lambda$, then $(F \wr G)$ is a λ-accessible $\mathbf{U}$-category, and the projection functors $P : (F \wr G) \to \mathcal{C}$ and $Q : (F \wr G) \to \mathcal{D}$ are strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). This is a straightforward consequence of the hypothesis that both $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve colimits for $\mathbf{U}$-small κ-filtered diagrams.

(ii). Since the iso-comma category $(F \wr G)$ is a full subcategory of the comma category $(F \downarrow G)$, the claim is an immediate corollary of lemma 0.3.21.

(iii). Let $\mathcal{B} = (F \wr G)$. First, we must show that there is a $\mathbf{U}$-small set of $(\lambda, \mathbf{U})$-compact objects in $\mathcal{B}$ that generate $\mathcal{B}$ under colimits for $\mathbf{U}$-small λ-filtered colimits. Let (C, D, e) be an object in $\mathcal{B}$. Since $\mathcal{C}$ and $\mathcal{D}$ are $(\lambda, \mathbf{U})$-accessible categories, we may choose $\mathbf{U}$-small skeletons $\mathcal{I}$ and $\mathcal{J}$ of the comma categories $\left(\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{C}) \downarrow C\right)$ and $\left(\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{D}) \downarrow D\right)$ and obtain $\mathbf{U}$-small λ-filtered diagrams $X : \mathcal{I} \to$

$\mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{D}$ that are componentwise $(\lambda, \mathbf{U})$-compact and have $C \cong \varinjlim_{\mathcal{I}} X$ and $D \cong \varinjlim_{\mathcal{J}} Y$ (by proposition 0.2.26 and theorem 0.2.34). Let $\mathcal{K}$ be full subcategory of the iso-comma category $(FX \wr GY)$ spanned by those objects (i, j, f) such that the following diagram commutes,

$$\begin{tikzcd} FXi \arrow[r, "Fc_i"] \arrow[d, "f"'] & FC \arrow[d, "e"] \\ GYj \arrow[r, "Gd_j"'] & GD \end{tikzcd}$$

where $c_i : Xi \to C$ and $d_j : Yj \to D$ are the components of the respective colimiting cocones. Let $P' : \mathcal{K} \to \mathcal{I}$ and $Q' : \mathcal{K} \to \mathcal{J}$ be the projection functors, and let $Z : \mathcal{K} \to \mathcal{B}$ be the evident diagram with $PZ = FXP'$ and $QZ = GYQ'$. It is clear that $\mathcal{K}$ is a $\mathbf{U}$-small category, and we claim $Z : \mathcal{K} \to \mathcal{B}$ is λ-filtered diagram with (C, D, e) as its colimit.

First, we verify that (C, D, e) is a colimit for the diagram $Z : \mathcal{K} \to \mathcal{B}$. Let i be any object in $\mathcal{I}$ and consider the comma category $(i \downarrow P')$. Lemma 0.3.26 implies it is inhabited. Suppose we have two objects in $(i \downarrow P')$, i.e. two objects (i_0, j_0, f_0) and (i_1, j_1, f_1) in $\mathcal{K}$ and two morphisms $h_0 : i \to i_0$ and $h_1 : i \to i_1$ in $\mathcal{I}$. Since $\mathcal{I}$ is a filtered category, there exist an object i' in $\mathcal{I}$ and morphisms $h'_0 : i_0 \to i'$ and $h'_1 : i_1 \to i'$ such that $h'_0 \circ h_0 = h'_1 \circ h_1$. Similarly, $\mathcal{J}$ is a filtered category, so there exist an object j_2 in $\mathcal{J}$ and morphisms $j_0 \to j_2$ and $j_1 \to j_2$. By considering a suitable diagram of shape ${}^{j_2}/\mathcal{J}$ in the category ${}^{(GYj_0, GYj_1)}/\mathcal{E} \times \mathcal{E}$ (using the fact that $f_0 : FXi_0 \to GYj_0$ and $f_1 : FXi_1 \to GYj_1$ are isomorphisms in $\mathcal{E}$) and applying lemmas 0.3.19 and 0.3.26, we see that there is a commutative diagram in $\mathcal{E}$ of the form shown below,

$$\begin{tikzcd}
& FXi_0 \arrow[drr, bend left, "Fc_{i_0}"] \arrow[dr, "FXh'_0"] \arrow[ddd, "f_0"'] & & \\
& & FXi' \arrow[r, "Fc_{i'}"] \arrow[ddd, dashed, "f'"] & FC \arrow[ddd, "e"] \\
FXi_1 \arrow[urr, "Fh'_1"] \arrow[urrr, bend right, "Fc_{i_1}"] \arrow[ddd, "f_1"'] & & & \\
& GYj_0 \arrow[drr, bend left, "Gd_{j_0}"] \arrow[dr, dashed, "GYk'_0"'] & & \\
& & GYj' \arrow[r, "Gd_{j'}"'] & GD \\
GYj_1 \arrow[urr, dashed, "GYk'_1"'] \arrow[urrr, bend right, "Gd_{j_1}"'] & & &
\end{tikzcd}$$

and recalling lemma 0.2.18, we may assume that $f' : FXi' \to GYj'$ is an isomorphism in $\mathcal{E}$. Thus, the comma category $(i \downarrow P')$ is connected, and therefore $P' : \mathcal{K} \to \mathcal{I}$ is a cofinal functor. The symmetric argument shows that $Q' : \mathcal{K} \to \mathcal{J}$ is also a cofinal functor, and since $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve colimits for $\mathbf{U}$-small λ-filtered diagrams, we may deduce that the canonical cocone from Z to (C, D, e) in $\mathcal{B}$ is a colimiting cocone.

It remains to be shown that $\mathcal{K}$ is a $\mathbf{U}$-small λ-filtered category. Indeed, suppose $K : \mathbb{A} \to \mathcal{K}$ is a λ-small diagram. Since $\mathcal{I}$ is a λ-filtered category, there is an object i_0 in $\mathcal{I}$ with a cocone $P'K \Rightarrow \Delta i_0$, and by considering a suitable λ-filtered diagram in the category ${}^{GQ'K/}[\mathbb{A}, \mathcal{E}]$, we obtain an object j_0 in $\mathcal{J}$ and a morphism $f_0 : FXi_0 \to GYj_0$ such that the diagram below commutes,

$$\begin{tikzcd}
FXi_0 \arrow[r, "Fc_{i_0}"] \arrow[d, "f_0"'] & FC \arrow[d, "e"] \\
GYj_0 \arrow[r, "Gd_{j_0}"'] & GD
\end{tikzcd}$$

as well as a cocone from K to $\left(Xi_0, Yj_0, f_0\right)$ in the comma category $(F \downarrow G)$ that is compatible with the colimiting cocone $GY \Rightarrow \Delta GD$. Combining lemmas 0.2.18 and 0.3.26, we then obtain a cocone under P in $\mathcal{K}$, as required. This shows that every object in $\mathcal{B}$ is a colimit for a $\mathbf{U}$-small λ-filtered diagram of componentwise $(\lambda, \mathbf{U})$-compact objects in $\mathcal{B}$, and since $\mathcal{C}$ and $\mathcal{D}$ are λ-accessible

$\mathbf{U}$-categories, proposition 0.2.25 implies the full subcategory of $\mathcal{B}$ spanned by such componentwise $(\lambda, \mathbf{U})$-compact objects is essentially $\mathbf{U}$-small.

Finally, observe that every $(\lambda, \mathbf{U})$-compact object in $\mathcal{B}$ is a retract of a componentwise $(\lambda, \mathbf{U})$-compact object (because the set of such objects generate $\mathcal{B}$ under colimits for $\mathbf{U}$-small λ-filtered diagrams), and thus we may apply corollary 0.2.19 to deduce that every $(\lambda, \mathbf{U})$-compact object in $\mathcal{B}$ is itself componentwise $(\lambda, \mathbf{U})$-compact. Thus the projection functors $P : \mathcal{B} \to \mathcal{C}$ and $Q : \mathcal{B} \to \mathcal{D}$ are strongly $(\lambda, \mathbf{U})$-accessible. ■

Definition 0.3.28. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible $\mathbf{U}$-subcategory** of a κ-accessible $\mathbf{U}$-category $\mathcal{C}$ is a subcategory $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}$ is a κ-accessible $\mathbf{U}$-category and the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a $(\kappa, \mathbf{U})$-accessible functor.

Proposition 0.3.29. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category and let $\mathcal{B}$ be a replete and full κ-accessible $\mathbf{U}$-subcategory of $\mathcal{C}$.*

(i) *If A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and A is in $\mathcal{B}$, then A is also a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$.*

(ii) *If the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly $(\kappa, \mathbf{U})$-accessible, then $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{B}) = \mathcal{B} \cap \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$.*

Proof. (i). This is clear, since hom-sets and colimits for $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{B}$ are computed as in $\mathcal{C}$.

(ii). Given claim (i), it suffices to show that every $(\kappa, \mathbf{U})$-compact object in $\mathcal{B}$ is also $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$, but this is precisely the hypothesis that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly $(\kappa, \mathbf{U})$-accessible. ■

Proposition 0.3.30. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ and $\mathcal{E}$ be categories with colimits for $\mathbf{U}$-small κ-filtered diagrams, let $\mathcal{D}$ be a replete and full subcategory of $\mathcal{E}$ that is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams, let $F : \mathcal{C} \to \mathcal{E}$ be a functor of $\mathbf{U}$-rank $\leq \kappa$, and let $\mathcal{B}$ be the preimage of $\mathcal{D}$ under F, so that we have the following strict pullback diagram:*

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d, hook] & \mathcal{D} \arrow[d, hook] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

(i) *$\mathcal{B}$ is a replete and full subcategory of $\mathcal{D}$ and is closed under colimits for* $\mathbf{U}$*-small κ-filtered diagrams in $\mathcal{D}$.*

(ii) *If $F : \mathcal{C} \to \mathcal{E}$ and the inclusion $\mathcal{D} \hookrightarrow \mathcal{E}$ are strongly $(\lambda, \mathbf{U})$-accessible functors and $\kappa < \lambda$, then $\mathcal{B}$ is a λ-accessible* $\mathbf{U}$*-subcategory of $\mathcal{C}$ and the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is also strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). This is a straightforward exercise.

(ii). Consider the iso-comma category $(F \wr \mathcal{D})$ and the induced comparison functor $K : \mathcal{B} \to (F \wr \mathcal{D})$. It is clear that $\mathcal{B}$ is fully faithful; but since $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$, for every object (C, D, e) in $(F \wr \mathcal{D})$, there is a canonical isomorphism $KC \to (C, D, e)$, namely the one corresponding to the following commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd} FC \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & FC \arrow[d, "e"] \\ FC \arrow[r, "e"'] & D \end{tikzcd}$$

Thus, $K : \mathcal{B} \to (F \wr \mathcal{D})$ is (half of) an equivalence of categories. Theorem 0.3.27 says the projection $P : (F \wr \mathcal{D}) \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor, so we may deduce that the same is true for the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$. ■

Proposition 0.3.31. *Let κ be a regular cardinal in a universe* $\mathbf{U}$*, let $F : \mathcal{C} \to \mathcal{D}$ be a strongly $(\kappa, \mathbf{U})$-accessible functor, and let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ spanned by the image of F.*

(i) *Every object in $\mathcal{D}'$ is a colimit for some* $\mathbf{U}$*-small κ-filtered diagram consisting of objects in $\mathcal{D}'$ that are $(\kappa, \mathbf{U})$-compact as objects in $\mathcal{D}$.*

(ii) *Every $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}'$ is also $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$.*

(iii) *If $\mathcal{D}'$ is closed under colimits for* $\mathbf{U}$*-small κ-filtered diagrams in $\mathcal{D}$, then $\mathcal{D}'$ is a κ-accessible* $\mathbf{U}$*-subcategory of $\mathcal{D}$.*

Proof. (i). Let D be any object in $\mathcal{D}'$. By definition, there is an object C in $\mathcal{C}$ such that $D = FC$, and since $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, there is a $\mathbf{U}$-small κ-filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that each Xj is a $(\kappa, \mathbf{U})$-compact object in

$\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} X$. Since $F : \mathcal{C} \to \mathcal{D}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor, each FXj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}$ and we have $D \cong \varinjlim_{\mathcal{J}} FX$.

(ii). Moreover, if D is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}'$, then D must be a retract of FXj for some object j in $\mathcal{J}$, and so D is also $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$.

(iii). Any object in $\mathcal{D}'$ that is $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$ must be $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}'$, because $\mathcal{D}'$ is a full subcategory of $\mathcal{D}$ that is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams. ■

Proposition 0.3.32. *Let $\mathbb{B}$ be a $\mathbf{U}$-small category and let $\mathbb{D}$ be a κ-small poset. If $\mathbb{D}$ is well-founded, then:*

(i) *The $(\kappa, \mathbf{U})$-accessible functor*

$$\mathbf{Ind}^{\kappa}([\mathbb{D}, \mathbb{B}]) \to \left[\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathcal{B})\right]$$

obtained by extending the canonical embedding $[\mathbb{D}, \mathbb{B}] \to \left[\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathcal{B})\right]$ is fully faithful and essentially surjective on objects.

(ii) *The evaluation functors $\left[\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathbb{B})\right] \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ are strongly $(\kappa, \mathbf{U})$-accessible functors.*

Proof. Let $Y : \mathbb{D} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ be a diagram, let $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ be the canonical embedding, and consider the following pullback diagram,

$$\begin{tikzcd}
\mathcal{J} \arrow[r] \arrow[d, "P"'] & \left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]_{/Y} \arrow[d] \\
{[\mathbb{D}, \mathbb{B}]} \arrow[r, "{[\mathbb{D}, \gamma]}"'] & \left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]
\end{tikzcd}$$

where the functor $\left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]_{/C} \to \left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]$ is the projection. The objects of the category $\mathcal{J}$ are diagrams $\mathbb{D} \to \mathbb{B}$ (regarded as diagrams $\mathbb{D} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$) equipped with a morphism $X \to Y$ in $\left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]$, so $\mathcal{J}$ is a $\mathbf{U}$-small category. Recalling corollaries 0.2.16 and 0.2.19 and proposition 0.2.47, to prove the claims, it suffices to show that $\mathcal{J}$ is a κ-filtered category and that the tautological cocone is a colimiting cocone.

Let $X : \mathcal{I} \to \mathcal{J}$ be a κ-small diagram. We can then build an object $\tilde{X}$ in $\mathcal{J}$ equipped with a cocone under X by well-founded induction over $\mathbb{D}$:

- Given $\tilde{X}(d')$ and the cocone components for all $d' < d$ in $\mathbb{D}$, by considering an appropriate diagram in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ and using lemma 0.3.20, we may choose an object $\tilde{X}(d)$ in $\mathbb{B}$ equipped with morphisms $\tilde{X}(d') \to \tilde{X}(d)$ for all $d' < d$, morphisms $X(i)(d) \to \tilde{X}(d)$ for all i in $\mathcal{I}$, and a morphism $\tilde{X}(d) \to Y(d)$, all these making the appropriate diagrams commute.

Thus $\mathcal{J}$ is indeed a κ-filtered category. To complete the proof, we must check that the tautological cocone to Y is a colimiting cocone in $\left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]$. Let d be an object in $\mathbb{D}$ and consider the comma category $(\gamma \downarrow Y d)$. There is an evident functor $P_d : \mathcal{J} \to (\gamma \downarrow Y d)$ induced by $\mathcal{J} \to \left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]_{/Y}$, and P_d is a cofinal functor: indeed, by modifying the construction above (at the stage where $\tilde{X}(d)$ is chosen) in the cases $\mathcal{I} = \emptyset$ and $\mathcal{I} = \operatorname{disc} 2$, one may verify that the comma category $\left((B, q) \downarrow P_d\right)$ is connected for each object (B, q) in $(\gamma \downarrow Y d)$. Thus, the tautological cocone under the canonical diagram $\mathcal{J} \to \left[\mathbb{D}, \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right]$ is a colimiting cocone, as required. ■

Corollary 0.3.33. *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $\mathbb{D}$ be a κ-small well-founded poset.*

(i) *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, then so is $[\mathbb{D}, \mathcal{C}]$, and the evaluation functors $[\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ are strongly $(\kappa, \mathbf{U})$-accessible.*

(ii) *If $\mathbb{B}$ is a $\mathbf{U}$-small category with colimits (resp. limits) of shape $\mathbb{D}$, then $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ has colimits (resp. limits) of shape $\mathbb{D}$.*

Proof. (i). The classification theorem for accessible categories (theorem 0.2.29) says $\mathcal{C}$ is equivalent to $\mathbf{Ind}^{\kappa}(\mathbb{B})$ for some small category $\mathbb{B}$, so we may apply proposition 0.3.32.

(ii). Recalling proposition 0.1.12, this follows from claim (i) and the fact that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(-)$ is pseudofunctorial (hence, preserves adjunctions). ■

Corollary 0.3.34. *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $\mathbb{B}$ be a $\mathbf{U}$-small Cauchy-complete category. The following are equivalent:*

(i) *$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ has colimits for $\mathbf{U}$-small $\aleph_0$-filtered diagrams.*

(ii) *$\mathbb{B}$ has colimits for κ-small $\aleph_0$-filtered diagrams.*

(iii) *$\mathbb{B}$ has colimits for α-chains for all ordinals α of cardinality $< \kappa$.*

Proof. (i) ⇒ (ii). Use lemma 0.2.18 and proposition 0.2.25.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). By corollary 0.3.33, $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ has colimits for α-chains for all ordinals α of cardinality $< \kappa$ if $\mathbb{B}$ has them; but since α-chains for ordinals α of cardinality $\geq \kappa$ are κ-filtered, it then follows that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ has colimits for all $\mathbf{U}$-small chains. We may then apply theorem 0.2.12 to deduce that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ has colimits for $\mathbf{U}$-small $\aleph_0$-filtered diagrams. ■

Theorem 0.3.35 (The category of algebras for an accessible monad)**.** *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$, and let $\mathcal{C}^{\mathsf{T}}$ be the category of algebras for T. If $T : \mathcal{C} \to \mathcal{C}$ is a $(\kappa, \mathbf{U})$-accessible functor, then:*

(i) *The forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ creates colimits for $\mathbf{U}$-small κ-filtered diagrams and creates limits for all $\mathbf{U}$-small diagrams.*

(ii) *$\mathcal{C}^{\mathsf{T}}$ is a locally κ-presentable $\mathbf{U}$-category.*

Proof. (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). See Theorem 2.78 and the following remark in [LPAC], or Theorem 5.5.9 in [Borceux, 1994b]. □

Lemma 0.3.36. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category and let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. If the forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\kappa, \mathbf{U})$-accessible, then so is the functor $T : \mathcal{C} \to \mathcal{C}$.*

Proof. The accessible adjoint functor theorem (0.2.50) says the free T-algebra functor $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ is strongly $(\kappa, \mathbf{U})$-accessible if the forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is $(\kappa, \mathbf{U})$-accessible; but $T = UF$, so T is strongly $(\kappa, \mathbf{U})$-accessible when U is. ■

Theorem 0.3.37 (The category of algebras for a strongly accessible monad)**.** *Let $\mathcal{C}$ be a locally λ-presentable $\mathbf{U}$-category, let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$ where $T : \mathcal{C} \to \mathcal{C}$ has $\mathbf{U}$-rank κ, and let $\mathcal{C}^{\mathsf{T}}$ be the category of algebras for T. If $T : \mathcal{C} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor and $\kappa < \lambda$, then:*

(i) *Given a coequaliser diagram in $\mathcal{C}^{\mathsf{T}}$ of the form below,*

$$(A, \alpha) \rightrightarrows (B, \beta) \longrightarrow (C, \gamma)$$

if A and B are $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$, then so is C.

(ii) *Given a λ-small family $\left((A_i, \alpha_i) \mid i \in I \right)$ of T-algebras, if each A_i is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$, then so is the underlying object of the T-algebra coproduct $\sum_{i \in I} (A_i, \alpha_i)$.*

(iii) *The forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). By referring to the explicit construction of coequalisers in $\mathcal{C}^{\mathsf{T}}$ given in the proof of Proposition 4.3.6 in [Borceux, 1994b] and applying lemma 0.2.18, we see that C is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$ when A and B are, provided $T : \mathcal{C} \to \mathcal{C}$ has $\mathbf{U}$-rank κ *and* is strongly $(\lambda, \mathbf{U})$-accessible.

(ii). Let $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ be a left adjoint for $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$. In the proof of Proposition 4.3.4 in [Borceux, 1994b], we find that the T-algebra coproduct $\sum_{i \in I} (A_i, \alpha_i)$ may be computed by a coequaliser diagram of the following form:

$$F\left(\textstyle\sum_{i \in I} T A_i\right) \rightrightarrows F\left(\textstyle\sum_{i \in I} A_i\right) \longrightarrow \textstyle\sum_{i \in I} (A_i, \alpha_i)$$

Since $T : \mathcal{C} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible, the underlying objects of the T-algebras $F\left(\sum_{i \in I} T A_i\right)$ and $F\left(\sum_{i \in I} A_i\right)$ are $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$. Thus, by claim (i), the underlying object of $\sum_{i \in I} (A_i, \alpha_i)$ must also be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$.

(iii). It is shown in the proof of Theorem 5.5.9 in [Borceux, 1994b] that the full subcategory $\mathcal{F}$ of $\mathcal{C}^{\mathsf{T}}$ spanned by the image of $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ under $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ is a dense subcategory. Let $\mathcal{G}$ be the smallest replete full subcategory of $\mathcal{C}^{\mathsf{T}}$ that is closed under colimits for λ-small diagrams in $\mathcal{C}$ and that contains $\mathcal{F}$. Observe that claims (i) and (ii) imply that the underlying object of every T-algebra that is in $\mathcal{G}$ must be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. To show that the forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible, it is enough to verify that every $(\lambda, \mathbf{U})$-compact object is in $\mathcal{G}$.

It is not hard to see that the comma category $(\mathcal{G} \downarrow (A, \alpha))$ is then an essentially $\mathbf{U}$-small λ-filtered category for any T-algebra (A, α), and moreover, it can be shown that the tautological cocone for the canonical diagram $(\mathcal{G} \downarrow (A, \alpha)) \to \mathcal{C}^{\mathsf{T}}$

is a colimiting cocone. Thus, if (A, α) is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}^{\mathsf{T}}$, it must be a retract of an object in $\mathcal{G}$. But $\mathcal{G}$ is closed under retracts, so (A, α) is indeed in $\mathcal{G}$. ∎

Definition 0.3.38. Let $\mathcal{C}$ be any category.

- A **pointed endofunctor** on $\mathcal{C}$ is a functor $J : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\iota : \mathrm{id}_{\mathcal{C}} \Rightarrow J$.
- An **algebra** for a pointed endofunctor (J, ι) on $\mathcal{C}$ is an object A in $\mathcal{C}$ equipped with a morphism $\alpha : JA \to A$ such that $\alpha \circ \iota_A = \mathrm{id}_A$.
- A **homomorphism of algebras** for a pointed endofunctor (J, ι) on $\mathcal{C}$, say $f : (A, \alpha) \to (B, \beta)$, is a morphism $f : A \to B$ making the following diagram commute:

$$\begin{tikzcd} JA \arrow[r, "Jf"] \arrow[d, "\alpha"'] & JB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

We write $\mathcal{C}^{(J,\iota)}$ for the category of algebras for a pointed endofunctor (J, ι) on $\mathcal{C}$.

The following result on the existence of free algebras for a pointed endofunctor is a special case of a general construction due to Kelly [1980].

Theorem 0.3.39 (Free algebras for a pointed endofunctor)**.** *Let κ be a regular cardinal, let $\mathcal{C}$ be a category with pushouts and colimits for chains of length $\leq \kappa$, and let (J, ι) be a pointed endofunctor on $\mathcal{C}$ such that $J : \mathcal{C} \to \mathcal{C}$ preserves colimits for κ-chains.*

(i) *The forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ has a left adjoint, say $F : \mathcal{C} \to \mathcal{C}^{(J,\iota)}$.*

(ii) *Let λ be a regular cardinal in a universe $\mathbf{U}$. If $J : \mathcal{C} \to \mathcal{C}$ sends $(\lambda, \mathbf{U})$-compact objects to $(\lambda, \mathbf{U})$-compact objects and $\kappa < \lambda$, then the functor $UF : \mathcal{C} \to \mathcal{C}$ has the same property.*

Proof. Let X be an object in $\mathcal{C}$. We now define a chain $X_\bullet : \kappa + 2 \to \mathcal{C}$ by transfinite induction:

- Let $X_0 = X$, let $X_1 = JX_0$, let $q_0 = \mathrm{id}_{JX_0}$, and let $X_{0\to 1} : X_0 \to X_1$ be ι_{X_0}.

- Given $q_\alpha : JX_\alpha \to X_{\alpha+1}$ for an ordinal $\alpha < \kappa$, define $X_{\alpha+2}$ by the following coequaliser diagram in $\mathcal{C}$:

$$\begin{tikzcd} JX_\alpha \arrow[r, shift left, "J q_\alpha \circ J \iota_{X_\alpha}"] \arrow[r, shift right, "J q_\alpha \circ \iota_{J X_\alpha}"'] & JX_{\alpha+1} \arrow[r, "q_{\alpha+1}"] & X_{\alpha+2} \end{tikzcd}$$

  Then, for all $\alpha' < \alpha + 2$, set $X_{\alpha' \to \alpha+2} = q_{\alpha+1} \circ \iota_{X_{\alpha+1}} \circ X_{\alpha' \to \alpha+1}$; note that the diagram below commutes:

$$\begin{tikzcd} JX_\alpha \arrow[r, "JX_{\alpha\to\alpha+1}"] \arrow[d, "q_\alpha"'] & JX_{\alpha+1} \arrow[d, "q_{\alpha+1}"] \\ X_{\alpha+1} \arrow[r, "X_{\alpha+1\to\alpha+2}"'] & X_{\alpha+2} \end{tikzcd}$$

- Given a limit ordinal $\beta \le \kappa$ and q_α for all ordinals $\alpha < \beta$, define $X_\beta = \varinjlim_{\alpha<\beta} X_\alpha$ and take $X_{\beta\to\alpha} : X_\beta \to X_\alpha$ to be the component of the colimiting cocone; then define $X_{\beta+1}$ to be the colimit of the following diagram,

$$\begin{tikzcd} JX_0 \arrow[r] \arrow[d, "q_0"'] & JX_1 \arrow[r] \arrow[d, "q_1"'] & JX_2 \arrow[r] \arrow[d, "q_2"'] & \cdots \arrow[rr] & & JX_\beta \\ X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_\beta & \end{tikzcd}$$

  and let $q_\beta : JX_\beta \to X_{\beta+1}$ and $X_{\beta\to\beta+1} : X_\beta \to X_{\beta+1}$ be the respective components of the colimiting cocone; note that the following diagram commutes,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\iota_{X_0}"'] & X_1 \arrow[r] \arrow[d, "\iota_{X_1}"'] & X_2 \arrow[r] \arrow[d, "\iota_{X_2}"'] & \cdots \arrow[rr] & & X_\beta \arrow[d, "\iota_{X_\beta}"] \\ JX_0 \arrow[r] \arrow[d, "q_0"'] & JX_1 \arrow[r] \arrow[d, "q_1"'] & JX_2 \arrow[r] \arrow[d, "q_2"'] & \cdots \arrow[rr] & & JX_\beta \arrow[d, "q_\beta"] \\ X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

  so we have $X_{\beta\to\beta+1} = q_\beta \circ \iota_{X_\beta}$.

Our hypothesis is that J preserves colimits for κ-chains, so the canonical comparison $\varinjlim_{\alpha<\kappa} JX_\alpha \to JX_\kappa$ is an isomorphism, as is $X_{\kappa\to\kappa+1}$. However, for all ordinals $\alpha < \beta < \kappa$, we have

$$X_{\alpha+1\to\beta+1} \circ q_\alpha = q_\beta \circ JX_{\alpha\to\beta}$$

so there is a unique morphism $\gamma_X : JX_\kappa \to X_\kappa$ such that

$$\gamma_X \circ JX_{\alpha\to\kappa} = X_{\alpha+1\to\kappa} \circ q_\alpha$$

for all ordinals $\alpha < \kappa$. Moreover, we have

$$\gamma_X \circ \iota_{X_\kappa} \circ X_{\alpha\to\kappa} = \gamma_X \circ JX_{\alpha\to\kappa} \circ \iota_{X_\alpha} = X_{\alpha+1\to\kappa} \circ q_\alpha \circ \iota_{X_\alpha} = X_{\alpha\to\kappa}$$

and $\left\{X_{\alpha\to\kappa} \,\middle|\, \alpha < \kappa\right\}$ is a jointly epimorphic family, so $\gamma_X \circ \iota_{X_\kappa} = \mathrm{id}_{X_\kappa}$, i.e. $\left(X_\kappa, \gamma_X\right)$ is a (J, ι)-algebra.

It remains to be shown that $\left(X_\kappa, \gamma_X\right)$ is a free (J, ι)-algebra generated by X. Let $\eta_X = X_{0\to\kappa}$, let (D, δ) be any (J, ι)-algebra, and let $f : X \to D$ be any morphism in $\mathcal{C}$. We construct a cocone $f_\bullet : X_\bullet \Rightarrow \Delta D$ by transfinite induction:

- Let $f_0 = f$, let $f_1 = \delta \circ Jf_0$, and note that $\delta \circ Jf_0 = f_1 \circ q_0$.

- Given $f_\alpha : X_\alpha \to D$ and $f_{\alpha+1} : X_{\alpha+1} \to D$ such that $f_{\alpha+1} \circ q_\alpha = \delta \circ Jf_\alpha$, let $f_{\alpha+2} : X_{\alpha+2} \to D$ be the unique morphism satisfying the following equation:

  $$f_{\alpha+2} \circ q_{\alpha+1} = \delta \circ Jf_{\alpha+1}$$

  Note that such a morphism exists because the diagrams below commute,

  $$\begin{tikzcd}
  JX_\alpha \arrow[r, "\iota_{JX_\alpha}"] \arrow[d, "q_\alpha"'] & JJX_\alpha \arrow[d, "Jq_\alpha"] \\
  X_{\alpha+1} \arrow[r, "\iota_{JX_{\alpha+1}}"] \arrow[d, "f_{\alpha+1}"'] & JX_{\alpha+1} \arrow[d, "Jf_{\alpha+1}"] \\
  D \arrow[r, "\iota_A"'] \arrow[dr, "\mathrm{id}"'] & JD \arrow[d, "\delta"] \\
  & D
  \end{tikzcd}
  \qquad
  \begin{tikzcd}
  JX_\alpha \arrow[r, "Jf_\alpha"] \arrow[d, "J\iota_{X_\alpha}"'] & JD \arrow[r, equal] \arrow[d, "J\iota_D"] & JD \arrow[dd, "\delta"] \\
  JJX_\alpha \arrow[r, "JJf_\alpha"] \arrow[d, "Jq_\alpha"'] & JJD \arrow[d, "J\delta"] & \\
  JX_{\alpha+1} \arrow[r, "Jf_{\alpha+1}"'] & JD \arrow[r, "\delta"'] & D
  \end{tikzcd}$$

  i.e. because the equation below holds,

  $$\left(\delta \circ Jf_{\alpha+1}\right) \circ \left(Jq_\alpha \circ \iota_{JX_\alpha}\right) = \left(\delta \circ Jf_{\alpha+1}\right) \circ \left(Jq_\alpha \circ J\iota_{X_\alpha}\right)$$

  and $q_{\alpha+1} : JX_{\alpha+1} \to X_{\alpha+2}$ is the coequaliser of $Jq_\alpha \circ \iota_{JX_\alpha}$ and $Jq_\alpha \circ J\iota_{X_\alpha}$.

- Given a limit ordinal $\beta \leq \kappa$, we define $f_\beta : X_\beta \to D$ be the unique morphism such that $f_\beta \circ X_{\alpha\to\beta} = f_\alpha$ for all ordinals $\alpha < \beta$; we may do this because the following equation holds:

  $$f_{\alpha+1} \circ X_{\alpha\to\alpha+1} = f_{\alpha+1} \circ q_\alpha \circ \iota_{X_{\alpha+1}} = \delta \circ Jf_\alpha \circ \iota_{X_{i+1}} = \delta \circ \iota_D \circ f_\alpha = f_\alpha$$

Furthermore,

$$\left(\delta \circ J f_\beta\right) \circ J X_{\alpha\to\beta} = \delta \circ J f_\alpha = f_{\alpha+1} \circ q_\alpha$$

so there exists a unique morphism $f_{\beta+1} : X_{\beta+1} \to D$ such that $f_{\beta+1} \circ q_\beta = \delta \circ J f_\beta$ and $f_{\beta+1} \circ X_{\alpha\to\beta+1} = f_\alpha$ for all ordinals $\alpha < \beta$.

Now observe that, for all ordinals $\alpha < \kappa$,

$$\begin{aligned}\delta \circ J f_\kappa \circ J X_{\alpha\to\kappa} &= \delta \circ J f_\alpha \\ &= f_{\alpha+1} \circ q_\alpha \\ &= f_\kappa \circ X_{\alpha+1\to\kappa} \circ q_\alpha \\ &= f_\kappa \circ \gamma_X \circ J X_{\alpha\to\kappa}\end{aligned}$$

and $\left\{ J X_{\alpha\to\kappa} \,\middle|\, \alpha < \kappa \right\}$ is a jointly epimorphic family, so $\delta \circ J f_\kappa = f_\kappa \circ \gamma_X$, i.e. f_κ is a $(J, \imath)$-algebra homomorphism $\left(X_\kappa, \gamma_X\right) \to (D, \delta)$. Finally, notice that, for any homomorphism $\bar{f} : \left(X_\kappa, \gamma_X\right) \to (D, \delta)$ such that $\bar{f} \circ \eta_X = f_0$, then,

$$\delta \circ J\left(\bar{f} \circ X_{\alpha\to\kappa}\right) = \bar{f} \circ \gamma_X \circ J X_{\alpha\to\kappa} = \left(\bar{f} \circ X_{\alpha+1\to\kappa}\right) \circ q_\alpha$$

hence we must have $\bar{f} = f_\kappa$, by transfinite induction.

The above argument shows that the comma category $(X \downarrow U)$ has an initial object, and it is well known that U has a left adjoint if and only if each comma category $(X \downarrow U)$ has an initial object, so this completes the proof of claim (i). For claim (ii), we simply observe that $\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{C})$ is closed under colimits for λ-small diagrams in $\mathcal{C}$ (by lemma 0.2.18), so the above construction can be carried out entirely in $\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{C})$. ■

Theorem 0.3.40 (The category of algebras for a accessible pointed endofunctor)**.** *Let $\mathcal{C}$ be a κ-accessible* **U***-category, let $J : \mathcal{C} \to \mathcal{C}$ be a $(\kappa, \mathbf{U})$-accessible functor, let $\imath : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, and let $\mathcal{C}^{(J,\imath)}$ be the category of algebras for the pointed endofunctor $(J, \imath)$.*

(i) *The forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ creates colimits for* **U***-small κ-filtered diagrams; and if $\mathcal{C}$ is* **U***-complete, then $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ also creates limits for all* **U***-small diagrams.*

(ii) *$\mathcal{C}^{(J,\imath)}$ is an accessible* **U***-category.*

(iii) *If $\mathcal{C}$ has pushouts and colimits for chains of length $\leq \kappa$, then $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ is a monadic functor.*

Proof. (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). We may construct $\mathcal{C}^{(J,\imath)}$ using inserters and equifiers, as in the proof of Theorem 2.78 in [LPAC].

(iii). Since κ-chains are $\mathbf{U}$-small κ-filtered diagrams, the hypotheses of theorem 0.3.39 are satisfied, and so the forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ has a left adjoint. It is not hard to check that the other hypotheses of Beck's monadicity theorem are satisfied, so U is indeed a monadic functor. □

Theorem 0.3.41 (The category of algebras for a strongly accessible pointed endofunctor)**.** *Let $\mathcal{C}$ be a locally λ-presentable $\mathbf{U}$-category, let $J : \mathcal{C} \to \mathcal{C}$ be a functor of $\mathbf{U}$-rank $\le \kappa$, let $\imath : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, let $\mathcal{C}^{(J,\imath)}$ be the category of algebras for the pointed endofunctor $(J,\imath)$, and let $\mathbb{T} = (T, \eta, \mu)$ be the induced monad on $\mathcal{C}$. If $J : \mathcal{C} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor and $\kappa < \lambda$, then:*

(i) *The functor $T : \mathcal{C} \to \mathcal{C}$ has $\mathbf{U}$-rank $\le \kappa$ and is strongly $(\lambda, \mathbf{U})$-accessible.*

(ii) *$\mathcal{C}^{(J,\imath)}$ is a locally κ-presentable $\mathbf{U}$-category.*

(iii) *The forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor.*

Proof. (i). We know that the forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ creates colimits for $\mathbf{U}$-small κ-filtered diagrams when $J : \mathcal{C} \to \mathcal{C}$ has $\mathbf{U}$-rank $\le \kappa$, so $T : \mathcal{C} \to \mathcal{C}$ must also have $\mathbf{U}$-rank $\le \kappa$. Moreover, theorem 0.3.39 implies $T : \mathcal{C} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible if $J : \mathcal{C} \to \mathcal{C}$ is.

(ii). Apply theorem 0.3.35.

(iii). Apply theorem 0.3.37. ■

0.4 Change of universe

Prerequisites. §§ 0.1, 0.2, A.1, A.5.

Having introduced universes into our ontology, it becomes necessary to ask whether an object with some universal property retains that property when we enlarge the universe. Though it sounds inconceivable, there do exist examples of badly-behaved constructions that are not stable under change-of-universe; for example, Waterhouse [1975] defined a functor $F : \mathbf{CRing} \to \mathbf{Set}^+$, where **CRing** is the category of commutative rings in a universe $\mathbf{U}$ and $\mathbf{Set}^+$ is the category of $\mathbf{U}^+$-sets for some universe $\mathbf{U}^+$ with $\mathbf{U} \in \mathbf{U}^+$, such that the value of F at any given commutative ring in $\mathbf{U}$ does not depend on $\mathbf{U}$, and yet the value of the fpqc sheaf associated with F at the field $\mathbb{Q}$ depends on the size of $\mathbf{U}$.

Definition 0.4.1. Let κ be a regular cardinal in a universe $\mathbf{U}$, and let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$. A $(\kappa, \mathbf{U}, \mathbf{U}^+)$**-accessible extension** is a $(\kappa, \mathbf{U})$-accessible functor $i : \mathcal{C} \to \mathcal{C}^+$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category,
- $\mathcal{C}^+$ is a κ-accessible $\mathbf{U}^+$-category,
- i sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U}^+)$-compact objects in $\mathcal{C}^+$, and
- the functor $\mathbf{K}^{\mathbf{U}}_\kappa(\mathcal{C}) \to \mathbf{K}^{\mathbf{U}^+}_\kappa(\mathcal{C}^+)$ so induced by i is fully faithful and essentially surjective on objects.

Remark 0.4.2. Let $\mathbb{B}$ be a $\mathbf{U}$-small category in which idempotents split. Then the $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}^\kappa_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^\kappa_{\mathbf{U}^+}(\mathbb{B})$ obtained by extending the embedding $\gamma^+ : \mathbb{B} \to \mathbf{Ind}^\kappa_{\mathbf{U}^+}(\mathbb{B})$ along $\gamma : \mathbb{B} \to \mathbf{Ind}^\kappa_{\mathbf{U}}(\mathbb{B})$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, by proposition 0.2.25. The classification theorem (0.2.29) implies all examples of $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions are essentially of this form.

Proposition 0.4.3. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

(i) *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category if and only if $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*

(ii) *The functor $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful.*

(iii) *If $B : \mathcal{J} \to \mathcal{C}$ is any diagram (not necessarily $\mathbf{U}$-small) and $\mathcal{C}$ has a limit for B, then i preserves this limit.*

Proof. (i). If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, then $\mathbf{K}^{\mathbf{U}}_\kappa(\mathcal{C})$ has colimits for all κ-small diagrams, so $\mathbf{K}^{\mathbf{U}^+}_\kappa(\mathcal{C}^+)$ also has colimits for all κ-small diagrams. The classification theorem (0.2.29) then implies $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category. Reversing this argument proves the converse.

(ii). Let $A : \mathbb{I} \to \mathcal{C}$ and $B : \mathbb{J} \to \mathcal{C}$ be two $\mathbf{U}$-small κ-filtered diagrams of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$. Then,

$$\begin{aligned} \mathcal{C}\left(\varinjlim_{\mathbb{I}} A, \varinjlim_{\mathbb{J}} B\right) &\cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} \mathcal{C}(A, B) \cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} \mathcal{C}^{+}(iA, iB) \\ &\cong \mathcal{C}^{+}\left(\varinjlim_{\mathbb{I}} iA, \varinjlim_{\mathbb{J}} iB\right) \cong \mathcal{C}^{+}\left(i \varinjlim_{\mathbb{I}} A, i \varinjlim_{\mathbb{J}} B\right) \end{aligned}$$

because i is $(\kappa, \mathbf{U})$-accessible and is fully faithful on the subcategory $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$, and therefore $i : \mathcal{C} \to \mathcal{C}^{+}$ itself is fully faithful. Note that this hinges crucially on theorem 0.1.31.

(iii). Let $B : \mathcal{J} \to \mathcal{C}$ be any diagram. We observe that, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}^{+}\left(iC, i \varprojlim_{\mathcal{J}} B\right) &\cong \mathcal{C}\left(C, \varprojlim_{\mathcal{J}} B\right) && \text{because } i \text{ is fully faithful} \\ &\cong \varprojlim_{\mathcal{J}} \mathcal{C}(C, B) && \text{by definition of limit} \\ &\cong \varprojlim_{\mathcal{J}} \mathcal{C}^{+}(iC, iB) && \text{because } i \text{ is fully faithful} \end{aligned}$$

but we know the restricted Yoneda embedding $\mathcal{C}^{+} \to \left[\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}^{+}\right]$ is fully faithful, so this is enough to conclude that $i \varprojlim_{\mathcal{J}} B$ is the limit of iB in $\mathcal{C}^{+}$. ■

Remark 0.4.4. Similar methods show that any fully faithful functor $\mathcal{C} \to \mathcal{C}^{+}$ satisfying the four bulleted conditions in the definition above is necessarily $(\kappa, \mathbf{U})$-accessible.

Lemma 0.4.5. *Let $\mathbf{U}$ and $\mathbf{U}^{+}$ be universes, with $\mathbf{U} \in \mathbf{U}^{+}$, and let κ be a regular cardinal in $\mathbf{U}$. Suppose:*

- *$\mathcal{C}$ and $\mathcal{D}$ are locally κ-presentable $\mathbf{U}$-categories.*
- *$\mathcal{C}^{+}$ and $\mathcal{D}^{+}$ are locally κ-presentable $\mathbf{U}^{+}$-categories.*
- *$i : \mathcal{C} \to \mathcal{C}^{+}$ and $j : \mathcal{D} \to \mathcal{D}^{+}$ are $(\kappa, \mathbf{U}, \mathbf{U}^{+})$-accessible extensions.*

Given a strictly commutative diagram of the form below,

$$\begin{tikzcd} \mathcal{D} \arrow[r, "j"] \arrow[d, "G"'] & \mathcal{D}^+ \arrow[d, "G^+"] \\ \mathcal{C} \arrow[r, "i"'] & \mathcal{C}^+ \end{tikzcd}$$

where G is $(\kappa, \mathbf{U})$-accessible, G^+ is $(\kappa, \mathbf{U}^+)$-accessible, if both have left adjoints, then the diagram satisfies the left Beck–Chevalley condition.

Proof. Let C be a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$. Inspecting the proof of theorem 0.2.50, we see that the functor $(C \downarrow G) \to (iC \downarrow G^+)$ induced by j preserves initial objects. Lemma A.1.10 says the component at C of the left Beck–Chevalley natural transformation $F^+ i \Rightarrow jF$ is an isomorphism; but $\mathcal{C}$ is generated by $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and the functors F, F^+, i, j all preserve colimits for $\mathbf{U}$-small κ-filtered diagrams, so in fact $F^+ i \Rightarrow jF$ is a natural isomorphism. ■

Proposition 0.4.6. *If $i : \mathcal{C} \to \mathcal{C}^+$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, then i preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$.*

Proof. It is well-known that a functor preserves colimits for all $\mathbf{U}$-small diagrams if and only if it preserves coequalisers for all parallel pairs and coproducts for all $\mathbf{U}$-small families, but coproducts for $\mathbf{U}$-small families can be constructed in a uniform way using coproducts for κ-small families and colimits for $\mathbf{U}$-small κ-filtered diagrams. It is therefore enough to show that $i : \mathcal{C} \to \mathcal{C}^+$ preserves all colimits for κ-small diagrams, since i is already $(\kappa, \mathbf{U})$-accessible.

Let $\mathbb{D}$ be a κ-small category. Recalling proposition 0.1.12, our problem amounts to showing that the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "\Delta"'] & \mathcal{C}^+ \arrow[d, "\Delta^+"] \\ {[\mathbb{D}, \mathcal{C}]} \arrow[r, "i_*"'] & {[\mathbb{D}, \mathcal{C}^+]} \end{tikzcd}$$

satisfies the left Beck–Chevalley condition. It is clear that i_* is fully faithful. Colimits for $\mathbf{U}$-small diagrams in $[\mathbb{D}, \mathcal{C}]$ and in $[\mathbb{D}, \mathcal{C}^+]$ are computed componentwise, so Δ and i_* are certainly $(\kappa, \mathbf{U})$-accessible, and Δ^+ is $(\kappa, \mathbf{U}^+)$-accessible. Using proposition 0.2.47, we see that i_* is also a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, so we apply the lemma above to conclude that the left Beck–Chevalley condition is satisfied. ■

Theorem 0.4.7 (Stability of accessible adjoint functors)**.** *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$*, and let* κ *and* λ *be regular cardinals in* $\mathbf{U}$*, with* $\kappa \leq \lambda$*. Suppose:*

- $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}$*-category.*
- $\mathcal{D}$ *is a locally* λ*-presentable* $\mathbf{U}$*-category.*
- $\mathcal{C}^+$ *is a locally* κ*-presentable* $\mathbf{U}^+$*-category.*
- $\mathcal{D}^+$ *is a locally* λ*-presentable* $\mathbf{U}^+$*-category.*

Let $i : \mathcal{C} \to \mathcal{C}^+$ *be a* $(\kappa, \mathbf{U}, \mathbf{U}^+)$*-accessible extension and let* $j : \mathcal{D} \to \mathcal{D}^+$ *be a fully faithful functor.*

(i) *Given a strictly commutative diagram of the form below,*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "j"] \arrow[d, "G"'] & \mathcal{D}^+ \arrow[d, "G^+"] \\ \mathcal{C} \arrow[r, "i"'] & \mathcal{C}^+ \end{tikzcd}$$

where G *is* $(\lambda, \mathbf{U})$*-accessible and* G^+ *is* $(\lambda, \mathbf{U}^+)$*-accessible, if both have left adjoints and* j *is a* $(\lambda, \mathbf{U}, \mathbf{U}^+)$*-accessible extension, then the diagram satisfies the left Beck–Chevalley condition.*

(ii) *Given a strictly commutative diagram of the form below,*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "F"'] & \mathcal{C}^+ \arrow[d, "F^+"] \\ \mathcal{D} \arrow[r, "j"'] & \mathcal{D}^+ \end{tikzcd}$$

if both F *and* F^+ *have right adjoints, then the diagram satisfies the right Beck–Chevalley condition.*

Proof. (i). The proof is essentially the same as lemma 0.4.5, though we have to use proposition 0.4.6 to ensure that j preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$.

(ii). Let D be any object in $\mathcal{D}$. Inspecting the proof of theorem 0.2.50, we see that our hypotheses, plus the fact that i preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, imply that the functor $(F \downarrow D) \to (F^+ \downarrow jD)$ induced by i preserves terminal objects. Thus, lemma A.1.10 implies that the diagram satisfies the right Beck–Chevalley condition. ■

Theorem 0.4.8. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category.*

(i) *If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \leq \lambda$, then $i : \mathcal{C} \to \mathcal{C}^+$ is also a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

(ii) *If μ is the cardinality of $\mathbf{U}$, then $i : \mathcal{C} \to \mathcal{C}^+$ factors through the inclusion $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+) \hookrightarrow \mathcal{C}^+$ as functor $\mathcal{C} \to \mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+)$ that is (fully faithful and) essentially surjective on objects.*

(iii) *The $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}^\mu_{\mathbf{U}^+}(\mathcal{C}) \to \mathcal{C}^+$ induced by $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful and essentially surjective on objects.*

Proof. (i). Since $i : \mathcal{C} \to \mathcal{C}^+$ is a $(\kappa, \mathbf{U})$-accessible functor, it is certainly also $(\lambda, \mathbf{U})$-accessible, by lemma 0.2.38. It is therefore enough to show that i restricts to a functor $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \to \mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$ that is (fully faithful and) essentially surjective on objects.

Proposition 0.2.46 says $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ is the smallest replete full subcategory of $\mathcal{C}$ that contains $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and is closed in $\mathcal{C}$ under colimits for λ-small diagrams, therefore the replete closure of the image of $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ must be the smallest replete full subcategory of $\mathcal{C}^+$ that contains $\mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$ and is closed in $\mathcal{C}^+$ under colimits for λ-small diagrams, since i is fully faithful and preserves colimits for all $\mathbf{U}$-small diagrams. This proves the claim.

(ii). Since every object in $\mathcal{C}$ is $(\lambda, \mathbf{U})$-compact for some regular cardinal $\lambda < \mu$, claim (i) implies that the image of $i : \mathcal{C} \to \mathcal{C}^+$ is contained in $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C})$. To show i is essentially surjective onto $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C})$, we simply have to observe that the inaccessibility of μ (proposition 0.1.36) and proposition 0.2.46 imply that, for C' any $(\mu, \mathbf{U}^+)$-compact object in $\mathcal{C}^+$, there exists a regular cardinal $\lambda < \mu$ such that C' is also a $(\lambda, \mathbf{U}^+)$-compact object, which reduces the question to claim (i).

(iii). This is an immediate corollary of claim (ii) and the classification theorem (0.2.29) applied to $\mathcal{C}^+$, considered as a $(\mu, \mathbf{U}^+)$-accessible category. ■

Remark 0.4.9. Although the fact $i : \mathcal{C} \to \mathcal{C}^+$ that preserves limits and colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$ is a formal consequence of the theorem above (via e.g. corollary A.5.30), it is not clear whether the theorem can be proved without already knowing this.

Corollary 0.4.10. *If $\mathbb{B}$ is a $\mathbf{U}$-small category and has colimits for all κ-small diagrams, and μ is the cardinality of $\mathbf{U}$, then the canonical $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}^{\mu}_{\mathbf{U}^+}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathbb{B})$ is fully faithful and essentially surjective on objects.* ■

Proposition 0.4.11. *Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, and let κ and λ be regular cardinals in $\mathbf{U}$. Suppose:*

- *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category.*
- *$\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.*
- *$\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*
- *$\mathcal{D}^+$ is a locally λ-presentable $\mathbf{U}^+$-category.*

Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ be functors, let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, and let $j : \mathcal{D} \to \mathcal{D}^+$ be a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension. Consider the following (not necessarily commutative) diagram:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{D} \arrow[r, "j"] & \mathcal{D}^+ \\
\mathcal{C} \arrow[ur, "H"'] \arrow[d, "i"'] & & \\
\mathcal{C}^+ \arrow[uurr, "H^+"'] & &
\end{tikzcd}$$

(i) *If H is a pointwise right Kan extension of G along F, then jH is a pointwise right Kan extension of jG along F, and if H^+ is a pointwise right Kan extension of jH along i, then H^+ is also a pointwise right Kan extension of jG along iF.*

(ii) *Assuming $\mathcal{A}$ is $\mathbf{U}$-small, if H is a pointwise left Kan extension of G along F, then jH is a pointwise left Kan extension of jG along F, and if H^+ is a pointwise left Kan extension of jH along i, then H^+ is also a pointwise left Kan extension of jG along iF.*

Proof. Use theorem A.5.20 and the fact that i and j preserve limits for *all* diagrams and colimits for $\mathbf{U}$-small diagrams. ■

0.5 Small object arguments

Prerequisites. §§0.1, 0.2, 0.3, 0.4, A.3, A.5.

The small object argument is a recurring construction in homotopical algebra, originally due to Quillen [1967, Ch. II, §3] but refined by many authors since—notably by Garner [2009]. Roughly speaking, the small object argument shows that, under certain hypotheses, starting from a small set $\mathcal{I}$ of morphisms in a cocomplete category $\mathcal{C}$, one can define the notions of 'relative $\mathcal{I}$-cell complex' and '$\mathcal{I}$-fibration' so that every morphism in $\mathcal{C}$ factors as a relative $\mathcal{I}$-cell complex followed by an $\mathcal{I}$-fibration.

In this section, we will study the small object argument with a view toward questions of stability under change-of-universe.

Definition 0.5.1. Let $\mathcal{C}$ be a category, and let $\mathcal{I}$ be a subset of $\operatorname{mor}\mathcal{C}$. A **presentation for a relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ consists of the following data:

- An ordinal α. (We say the presentation is **indexed over** α.)
- A colimit-preserving functor $X_\bullet : [\alpha] \to \mathcal{C}$, where $[\alpha]$ is the well-ordered set $\{0, \ldots, \alpha\}$ considered as a preorder category.
- For each ordinal $\beta < \alpha$, a (possibly empty) indexing set T_β; and for each element j of T_β, a commutative diagram of the form below,

$$\begin{tikzcd} U_{\beta,j} \arrow[r, "u_{\beta,j}"] \arrow[d, "e_{\beta,j}"'] & X_\beta \arrow[d, "X_{\beta\to\beta+1}"] \\ V_{\beta,j} \arrow[r, "v_{\beta,j}"'] & X_{\beta+1} \end{tikzcd}$$

  where $e_{\beta,j} : U_{\beta,j} \to V_{\beta,j}$ is a morphism in $\mathcal{I}$.

These data are moreover required to satisfy the following condition:

- For each ordinal $\beta < \gamma$, the coproducts $\coprod_{j\in T_\beta} S_{\beta,j}$ and $\coprod_{j\in T_\beta} D_{\beta,j}$ exist in $\mathcal{C}$, and the induced diagram

$$\begin{tikzcd} \coprod_{j\in T_\beta} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j\in T_\beta} e_{\beta,j}"'] & X_\beta \arrow[d, "X_{\beta\to\beta+1}"] \\ \coprod_{j\in T_\beta} V_{\beta,j} \arrow[r, "v_\beta"'] & X_{\beta+1} \end{tikzcd}$$

  is a pushout square in $\mathcal{C}$.

The presentation is said to be **U-small** (resp. **κ-small** for a regular cardinal κ) if α is an ordinal in **U** (resp. $|\alpha| < \kappa$) and the disjoint union $\coprod_{\beta<\alpha} T_\beta$ is in **U** (resp. has cardinality less than κ). A **sequential presentation** is one where each T_β is a singleton, in which case we suppress the index j in $e_{\beta,j}$, $u_{\beta,j}$, and $v_{\beta,j}$.

A **relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ is a morphism $f : X \to Y$ in $\mathcal{C}$ for which there exists a presentation as above with f equal to $X_0 \to X_\alpha$. Given an initial object 0 in $\mathcal{C}$, an **$\mathcal{I}$-cell complex** in $\mathcal{C}$ is an object Y for which the unique morphism $0 \to Y$ is a relative $\mathcal{I}$-cell complex.

Remark 0.5.2. For any object X in $\mathcal{C}$ and any subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, the morphism $\mathrm{id} : X \to X$ is a relative $\mathcal{I}$-cell complex in $\mathcal{C}$ (with the obvious presentation indexed over 0). More generally, every isomorphism in $\mathcal{C}$ is a relative $\mathcal{I}$-cell complex, with a presentation indexed over 1 (and $T_0 = \emptyset$); but in order to get a *sequential* presentation, one must assume that there is an isomorphism in $\mathcal{I}$.

Proposition 0.5.3. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$*, let κ be a regular cardinal, and let* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes in $\mathcal{C}$ that admit a κ-small presentation.*

(i) *Every morphism in $\mathcal{I}$ is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(ii) *For each object X in $\mathcal{C}$, the morphism* $\mathrm{id} : X \to X$ *is in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(iii) *If $f : X \to Y$ and $g : Y \to Z$ are both in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$*, then so is $g \circ f$.*

(iv) *Let α be an ordinal and let $X_\bullet : \alpha \to \mathcal{C}$ be a colimit-preserving functor. If $|\alpha| < \kappa$ and λ is a colimiting cocone from $X_\bullet$ to Y and, for $\beta \le \gamma < \alpha$, the morphism $X_{\beta\to\gamma} : X_\beta \to X_\gamma$ is in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$*, then each component $\lambda_\beta : X_\beta \to Y$ is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(v) *Given a pushout diagram of the form below in $\mathcal{C}$,*

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

if g is in $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$ *and $\mathcal{C}$ has colimits for all κ-small diagrams, then f is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

Proof. (i). Given any morphism $e : U \to V$ in $\mathcal{I}$, we have the following pushout diagram:

$$\begin{tikzcd} U \arrow[r, "\mathrm{id}"] \arrow[d, "e"'] & U \arrow[d, "e"] \\ V \arrow[r, "\mathrm{id}"'] & V \end{tikzcd}$$

Thus $e : U \to V$ is in $\mathrm{cell}_{\mathcal{I}}\, \mathcal{C}$.

(ii). See remark 0.5.2.

(iii). It is clear that appending any κ-small presentation for g to any κ-small presentation for f yields a κ-small presentation of $g \circ f$.

(iv). The case $\alpha = 0$ falls under claim (ii). If $\alpha = \gamma + 1$, then the component $\lambda_\gamma : X_\gamma \to Y$ must be an isomorphism, and thus $\lambda_\beta = \lambda_\gamma \circ X_{\beta \to \gamma}$ is also in $\mathrm{cell}_{\mathcal{I}}\, \mathcal{C}$; and if α is a positive limit ordinal, since every terminal segment of α is cofinal in α, it is clear that concatenating κ-small presentations for $X_{\gamma \to \gamma+1}$ for $\beta \le \gamma < \alpha$ yields a κ-small presentation for $\lambda_\beta : X_\beta \to Y$.

(v). Fix a κ-small presentation of $g : Z \to W$. By the pushout pasting lemma, given a commutative diagram of the form below,

$$\begin{tikzcd} \coprod_{j \in T_\beta} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j \in T_\beta} e_{\beta,j}"'] & Z_\beta \arrow[r] \arrow[d, "Z_{\beta \to \beta+1}"] & X_\beta \arrow[d, "X_{\beta \to \beta+1}"] \\ \coprod_{j \in T_\beta} V_{\beta,j} \arrow[r, "v_\beta"'] & Z_{\beta+1} \arrow[r] & X_{\beta+1} \end{tikzcd}$$

if both squares are pushout diagrams, then the outer rectangle is a pushout diagram as well. Since pushout along $z : Z \to X$ is the left adjoint of the evident functor $z^* : {}^{X/}\mathcal{C} \to {}^{Z/}\mathcal{C}$, it preserves all colimits, and thus we obtain a κ-small presentation of $f : X \to Y$. ■

Definition 0.5.4. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of $\mathrm{mor}\, \mathcal{C}$. An **$\mathcal{I}$-injective morphism** in $\mathcal{C}$ is a morphism that has the right lifting property with respect to every morphism in $\mathcal{I}$.[10] An **$\mathcal{I}$-cofibration** in $\mathcal{C}$ is a morphism that has the left lifting property with respect to every $\mathcal{I}$-injective morphism.

[10] Equivalently, it is a morphism $f : X \to Y$ in $\mathcal{C}$ that is an $\mathcal{I}$-injective object in the slice category $\mathcal{C}_{/Y}$.

Proposition 0.5.5. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$*, and let* $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$, $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$*, and* $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes, $\mathcal{I}$-injections, and $\mathcal{I}$-cofibrations in $\mathcal{C}$, respectively.*

(i) *We have $\mathcal{I} \subseteq \operatorname{cell}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}$.*

(ii) *A morphism is in $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ if and only if it has the right lifting property with respect to every $\mathcal{I}$-cofibration.*

(iii) *In particular, a morphism is in $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ if and only if it has the right lifting property with respect to every relative $\mathcal{I}$-cell complex.*

Proof. (i). Follows immediately from the definition of 'relative $\mathcal{I}$-cell complex' and proposition A.3.17.

(ii) and (iii). See proposition A.3.3. ■

Some authors define 'relative $\mathcal{I}$-cell complex' so that every such morphism admits a *sequential* presentation. The following lemma and its corollary show that there is no loss of generality in doing so.

Lemma 0.5.6. *Let κ be a regular cardinal, let $\mathcal{C}$ be a category with colimits for all κ-small diagrams, and let α be an ordinal of cardinality less than κ. For each ordinal $\beta < \alpha$, let $e_\beta : U_\beta \to V_\beta$ be a morphism in $\mathcal{C}$, and for each ordinal $\beta \le \alpha$, let*

$$C_\beta = \left(\coprod_{\gamma < \beta} V_\gamma \right) \amalg \left(\coprod_{\beta \le \gamma < \alpha} U_\gamma \right)$$

be a coproduct in $\mathcal{C}$ with coproduct insertions $u_{\gamma,\beta} : U_\gamma \to C_\beta$ (for $\beta \le \gamma < \alpha$) and $v_{\gamma,\beta} : V_\gamma \to C_\beta$ (for $\gamma < \beta$).

Given ordinals $\beta < \beta' \le \alpha$, there is a unique morphism $C_\beta \to C_{\beta'}$ such that, for $\zeta < \beta \le \zeta' < \beta' \le \zeta''$, the following diagrams commute:

$$\begin{tikzcd} V_\zeta \arrow[r, "v_{\zeta,\beta}"] \arrow[d, "\mathrm{id}"'] & C_\beta \arrow[d, dashed] \\ V_\zeta \arrow[r, "v_{\zeta,\beta'}"'] & C_{\beta'} \end{tikzcd} \qquad \begin{tikzcd} U_{\zeta'} \arrow[r, "u_{\zeta',\beta}"] \arrow[d, "e_{\zeta'}"'] & C_\beta \arrow[d, dashed] \\ V_{\zeta'} \arrow[r, "v_{\zeta',\beta'}"'] & C_{\beta'} \end{tikzcd} \qquad \begin{tikzcd} U_{\zeta''} \arrow[r, "u_{\zeta'',\beta}"] \arrow[d, "\mathrm{id}"'] & C_\beta \arrow[d, dashed] \\ U_{\zeta''} \arrow[r, "u_{\zeta'',\beta'}"'] & C_{\beta'} \end{tikzcd}$$

This yields a functor $C_\bullet : [\alpha] \to C$, and it preserves colimits. Moreover, the diagrams below are pushout squares for all ordinals $\beta < \alpha$:

$$\begin{tikzcd} U_\beta \arrow[r, "u_{\beta,\beta}"] \arrow[d, "e_\beta"'] & C_\beta \arrow[d] \\ V_\beta \arrow[r, "v_{\beta,\beta+1}"'] & C_{\beta+1} \end{tikzcd}$$

Proof. This is a straightforward exercise. See Proposition 10.2.7 in [Hirschhorn, 2003]. □

Corollary 0.5.7. *Let κ be a regular cardinal, let C be a category with colimits for κ-small diagrams, and let $\mathcal{I}$ be a subset of* $\operatorname{mor} C$. *If $f : X \to Y$ is a relative $\mathcal{I}$-cell complex in C that admits a κ-small presentation, and either*

- *$X = Y$ and $f = \mathrm{id}_X$, or*
- *f is an isomorphism and $\mathcal{I}$ contains an isomorphism, or*
- *f is not an isomorphism,*

then f also admits a κ-small sequential presentation.

Proof. We have already commented on the first two cases in remark 0.5.2. The third case is proven by transfinite induction, where in the induction step we may assume that f is presented by just one pushout diagram:

$$\begin{tikzcd} \coprod_{j\in T} U_j \arrow[r, "u"] \arrow[d, "\coprod_{j\in T} e_j"'] & X \arrow[d, "f"] \\ \coprod_{j\in T} V_j \arrow[r, "v"'] & Y \end{tikzcd}$$

By decomposing the morphism $\coprod_{j\in T} e_j : \coprod_{j\in T} U_j \to \coprod_{j\in T} V_j$ as in the earlier lemma and applying the pushout pasting lemma, we obtain a sequential presentation of f, which is κ-small precisely if $|T| < \kappa$. ■

Definition 0.5.8. Let $\mathbf{U}$ be a universe, let C be a category, let $\mathcal{I}$ be a subset of $\operatorname{mor} C$, and let $\operatorname{cell}_{\mathcal{I},\mathbf{U}} C$ be the set of relative $\mathcal{I}$-cell complexes in C that have a $\mathbf{U}$-small presentation. We say $(\mathcal{I}, C)$ is **admissible for the U-small object argument** when the following conditions are satisfied:

- $\mathcal{I}$ is a **U**-set.
- $\mathcal{C}$ be a locally **U**-small category with colimits for all **U**-small diagrams.
- There is a regular cardinal κ in **U** such that, for every morphism $e : U \to V$ in $\mathcal{I}$, every ordinal α in **U**, and every functor $X_\bullet : \alpha \to \mathcal{C}$, if $|\alpha| \geq \kappa$, and the morphism $X_{\beta\to\gamma} : X_\beta \to X_\gamma$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$ for all ordinals $\beta \leq \gamma < \alpha$, then the canonical comparison map $\varinjlim_{\beta<\alpha} \mathcal{C}(U, X_\beta) \to \mathcal{C}\big(U, \varinjlim_{\beta<\alpha} X_\beta\big)$ is a bijection.

The **sequential U-rank** of $\mathcal{I}$ in $\mathcal{C}$ is the least cardinal κ with the above property.

Remark 0.5.9. Notice that, if $|\alpha| \geq \kappa$, then α is a κ-directed preorder. Thus, for any locally presentable **U**-category $\mathcal{C}$ and any **U**-subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ whatsoever, $(\mathcal{I}, \mathcal{C})$ is admissible for the **U**-small object argument.

Definition 0.5.10. Let **U** be a universe. A **U-cofibrantly generated factorisation system** on a category $\mathcal{C}$ on is a weak factorisation system on $\mathcal{C}$ that is cofibrantly generated by some **U**-subset of $\operatorname{mor} \mathcal{C}$.

Lemma 0.5.11. *Let $\mathcal{C}$ be a κ-accessible **U**-category, let A be a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, and let B be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. If the hom-set $\mathcal{C}(A, A')$ is μ-small for all $(\kappa, \mathbf{U})$-compact objects A' in $\mathcal{C}$ and $\kappa \lhd \lambda$, then the hom-set $\mathcal{C}(A, B)$ has cardinality $< \max\{\lambda, \mu\}$.*

Proof. By proposition 0.2.45, there is a λ-small κ-filtered diagram $Y : \mathcal{J} \to \mathcal{C}$ with each vertex $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$ and $B \cong \varinjlim_{\mathcal{J}} Y$. Since A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, we have

$$\mathcal{C}(A, B) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(A, Y)$$

and the RHS is a set of cardinality $< \max\{\lambda, \mu\}$ by lemma 0.2.18. ■

Theorem 0.5.12 (Quillen's small object argument)**.** *Let **U** be a universe, let $\mathcal{C}$ be a locally **U**-small category with colimits for all **U**-small diagrams, and let $\mathcal{I}$ be a **U**-subset of* $\operatorname{mor} \mathcal{C}$.

(i) *There exist a functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ and two natural transformations $i : \mathrm{dom} \Rightarrow M$, $p : M \Rightarrow \mathrm{codom}$ such that, for all morphisms $f : X \to Y$ in $\mathcal{C}$, the morphism $i_f : X \to M(f)$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$, and we have $f = p_f \circ i_f$.*

(ii) *If* $(\mathcal{I}, \mathcal{C})$ *is moreover admissible for the* $\mathbf{U}$*-small object argument, then we may choose* M*,* i*, and* p *so that, for all morphisms* $f : X \to Y$ *in* $\mathcal{C}$*, the morphism* $p_f : M(f) \to Y$ *in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$.

(iii) *In particular, if* $(\mathcal{I}, \mathcal{C})$ *is admissible for the* $\mathbf{U}$*-small object argument, then* $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ *is a* $\mathbf{U}$*-cofibrantly generated factorisation system on* $\mathcal{C}$ *and extends to a functorial weak factorisation system.*

Proof. (i). Let κ be any regular cardinal, and let α be the least ordinal of cardinality κ.[11] For each morphism $f : X \to Y$ in $\mathcal{C}$, we construct by transfinite recursion a colimit-preserving functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ and a cocone $p_{f;\bullet} : M_\bullet(f) \to Y$ satisfying the following conditions:

- $M_0(f) = X$, $p_{f;0} = p$.
- For each ordinal $\beta < \alpha$, if $T_\beta(f)$ is the set of all commutative diagrams in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} U_{\beta,j} \arrow[r, "u_{\beta,j}"] \arrow[d, "e_{\beta,j}"'] & M_\beta(f) \arrow[d, "p_{f;\beta}"] \\ V_{\beta,j} \arrow[r, "v_{\beta,j}"'] & Y \end{tikzcd}$$

  where $e_{\beta,j} : U_{\beta,j} \to V_{\beta,j}$ is in $\mathcal{I}$, then $T_\beta(f)$ is a $\mathbf{U}$-set (because $\mathcal{I}$ is a $\mathbf{U}$-set and $\mathcal{C}$ is a locally $\mathbf{U}$-small category), and we have a pushout square of the following form,

$$\begin{tikzcd} \coprod_{j \in T_\beta(f)} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j \in T_\beta(f)} e_{\beta,j}"'] & M_\beta(f) \arrow[d, "x_{\beta \to \beta+1}"] \\ \coprod_{j \in T_\beta(f)} V_{\beta,j} \arrow[r, "\bar{v}_\beta"'] & M_{\beta+1}(f) \end{tikzcd}$$

  where $u_\beta : \coprod_{j \in T_\beta(f)} U_{\beta,j} \to M_\beta(f)$ is the evident morphism induced by the universal property of coproducts. Observe that there is then a unique morphism $p_{f;\beta+1} : M_{\beta+1}(f) \to Y$ such that

$$p_{f;\beta+1} \circ M_{\beta \to \beta+1}(f) = p_\beta$$

  and

$$p_{f;\beta+1} \circ \bar{v}_{\beta,j} = v_{\beta,j}$$

  for all j in $T_\beta(f)$, where $\bar{v}_{\beta,j} : V_{\beta,j} \to M_{\beta+1}(f)$ is the evident component of $\bar{v}_\beta : \coprod_{j \in T_\beta(f)} V_{\beta,j} \to M_{\beta+1}(f)$.

[11] In particular, we could take $\kappa = 0$, but then the factorisation so obtained is trivial.

- For limit ordinals $\gamma \leq \alpha$, $M_\gamma(f) = \varinjlim_{\beta<\gamma} M_\beta(f)$, and $p_\gamma : M_\gamma(f) \to Y$ is defined by the universal property of X_γ.

It is not hard to see that the functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ so defined is itself functorial in f; in particular, defining $M(f) = M_\alpha(f)$, $i_f = M_{0\to\alpha}(f)$, $p_f = p_{f;\alpha}$, we obtain a functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ with two natural transformations $i : M \Rightarrow \mathrm{dom}$ and $p : M \Rightarrow \mathrm{codom}$; by construction, we have $f = p_f \circ i_f$, and $i_f : X \to M(f)$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$.

(ii). Now, take κ to be a regular cardinal as in definition 0.5.8. We wish to show that the morphism p_f constructed above has the right lifting property with respect to all morphisms in $\mathcal{I}$. Consider a lifting problem of the form below,

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "e"'] & M(f) \arrow[d, "p_f"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

where $e : U \to V$ is in $\mathcal{I}$. Since $\mathcal{I}$ is admissible, there must exist an ordinal $\beta < \alpha$ and a morphism $u' : U \to M_\beta(f)$ such that $u = M_{\beta\to\alpha}(f) \circ u'$. We then obtain the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "u'"] \arrow[d, "e"'] & M_\beta(f) \arrow[d, "p_{f;\beta}"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

Since this is one of the diagrams in the set $T_\beta(f)$, it must embed in a commutative diagram of the form below,

$$\begin{tikzcd} U \arrow[r, "u'"] \arrow[d, "e"'] & M_\beta(f) \arrow[r] \arrow[d] & M_\alpha(f) \arrow[dd, "p_{f;\alpha}"] \\ V \arrow[r] \arrow[d, equal] & M_{\beta+1}(f) \arrow[ur] & \\ V \arrow[rr, "v"'] & & Y \end{tikzcd}$$

and thus we have the required lift $V \to M(f)$.

(iii). Finally, apply proposition 0.5.5 and theorem A.3.35. ■

Corollary 0.5.13. *With other notation in the theorem, a morphism $g : Z \to W$ is in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$ if and only if there exists a commutative diagram of the following form in $\mathcal{C}$,*

$$\begin{tikzcd}
Z \arrow[r, "i"] \arrow[d, "g"'] & W' \arrow[d] \\
W \arrow[r, "\mathrm{id}"'] \arrow[ur] & W
\end{tikzcd}$$

where $i : Z \to W'$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$.

Proof. (i). If $g : Z \to W$ is in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$, then g has the left lifting property with respect to $p_g : M(g) \to W$, and so there exists a commutative diagram of the required form. Conversely, suppose we have $g = p \circ i$, $i = j \circ g$, and $\mathrm{id}_W = p \circ j$ for some $i : Z \to W'$ in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$ and some $j : W \to W'$ in $\mathcal{C}$. Then g is a retract of i,

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Z \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & Z \arrow[d, "g"] \\
W \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & W' \arrow[r, "p"] & W
\end{tikzcd}$$

but proposition 0.5.5 says i is in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$, so by proposition A.3.17, g is also in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$. ■

Corollary 0.5.14. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let $\mathcal{I}$ be a $\mathbf{U}$-small subset of $\mathrm{mor}\, \mathcal{C}$. If the morphisms that are in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact as objects in $[\mathbb{2}, \mathcal{C}]$, then there exist a $(\kappa, \mathbf{U})$-accessible functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ and two natural transformations $i : \mathrm{dom} \Rightarrow M$ and $p : M \Rightarrow \mathrm{codom}$ such that, for all objects f in $[\mathbb{2}, \mathcal{C}]$:*

- $f = p_f \circ i_f$.
- *i_f is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$.*
- *p_f is in $\mathrm{inj}^{\mathcal{I}}\, \mathcal{C}$.*

Moreover, if λ is a regular cardinal in $\mathbf{U}$ such that every hom-set of $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ is λ-small, $\mathcal{I}$ is λ-small, and $\kappa \lhd \lambda$, then $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ is also strongly $(\lambda, \mathbf{U})$-accessible.

Proof. As observed in remark 0.5.9, under these hypotheses, $(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument and the sequential $\mathbf{U}$-rank of $\mathcal{I}$ is $\leq \kappa$. By tracing the construction of the functor M in theorem 0.5.12, we see that M preserves colimits for κ-filtered $\mathbf{U}$-small diagrams, so we are done. Similarly, applying proposition 0.2.47 and lemmas 0.2.18 and 0.5.11 shows that M is strongly $(\lambda, \mathbf{U})$-accessible. ∎

Corollary 0.5.15. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let $\mathcal{I}$ be a $\mathbf{U}$-small subset of* $\operatorname{mor} \mathcal{C}$. *If the morphisms that are in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact as objects in $[\mathbb{2}, \mathcal{C}]$, then there exists a $(\kappa, \mathbf{U})$-accessible functor $L : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ such that $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ is the closure of the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by the image of L under the splitting of idempotent endomorphisms.*

Proof. Take L to be the functor that sends a morphism in $\mathcal{C}$ (considered as an object in $[\mathbb{2}, \mathcal{C}]$) to the left half of its $\left(\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$-factorisation, and then apply theorem A.3.35. ∎

Lemma 0.5.16. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$, *and let κ be a regular cardinal. If $\mathcal{C}$ is closed in $\mathcal{C}^+$ under colimits for all κ-small diagrams, then $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C} = \operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}^+ \cap \operatorname{mor} \mathcal{C}$.*

Proof. Obvious. ◆

Theorem 0.5.17 (Stability of cofibrantly generated factorisation systems)**.** *Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$. Suppose:*

- *$\mathcal{C}$ is a locally $\mathbf{U}$-small and $\mathbf{U}$-cocomplete category.*
- *$\mathcal{C}^+$ is a locally $\mathbf{U}^+$-small and $\mathbf{U}^+$-cocomplete category.*
- *The inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ preserves colimits for all $\mathbf{U}$-small diagrams.*
- *$\mathcal{I}$ is a $\mathbf{U}$-subset of* $\operatorname{mor} \mathcal{C}$.
- *$(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument, and (L, R) is the functorial factorisation system on $\mathcal{C}$ constructed by Quillen's small object argument argument.*
- *$(\mathcal{I}, \mathcal{C}^+)$ is admissible for the $\mathbf{U}^+$-small object argument, and (L^+, R^+) is the functorial factorisation system on $\mathcal{C}^+$ constructed by Quillen's small object argument argument.*

Under these hypotheses, if the sequential $\mathbf{U}$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}$ *is equal to the sequential* $\mathbf{U}^+$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}^+$*, then:*

(i) *For each morphism* $f : X \to Y$ *in* $\mathcal{C}$*, we have a commutative diagram of the following form in* $\mathcal{C}^+$*,*

$$\begin{tikzcd}
& M^+(f) \arrow[dd, "\cong"] \arrow[dr, "R^+ f"] & \\
X \arrow[ur, "L^+ f"] \arrow[dr, "L f"'] & & Y \\
& M(f) \arrow[ur, "R f"'] &
\end{tikzcd}$$

and the isomorphism $M^+(f) \to M(f)$ *is moreover canonical and natural in* f*.*

(ii) *We have* $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C} \subseteq \operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}^+ \subseteq \operatorname{cell}_{\mathcal{I},\mathbf{U}^+} \mathcal{C}^+$.

(iii) $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}^+, \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+\right)$ *is an extension of* $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$.

Proof. (i). This can be seen by examining the explicit construction in the proof of theorem 0.5.12.

(ii). This is implied by the lemma.

(iii). Since $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ and $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}^+, \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+\right)$ are both cofibrantly generated by $\mathcal{I}$, by proposition A.3.25, we have $\operatorname{inj}^{\mathcal{I}} \mathcal{C} \subseteq \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+$ and so $\operatorname{cof}_{\mathcal{I}} \mathcal{C} \supseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}^+ \cap \operatorname{mor} \mathcal{C}$. It remains to be shown that $\operatorname{cof}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}^+$, but this is implied by corollary 0.5.13 applied to claim (ii). ■

Remark 0.5.18. Let κ be a regular cardinal in $\mathbf{U}$, let $\mathcal{B}$ be a $\mathbf{U}$-small category with colimits for all κ-small diagrams, let $\mathcal{C} = \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{B})$, and let $\mathcal{C}^+ = \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathcal{B})$. Then $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is an accessible $(\kappa, \mathbf{U}, \mathbf{U}^+)$ extension, and any $\mathbf{U}$-subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ whatsoever will satisfy the hypotheses of the theorem.

Proposition 0.5.19. *Let* $F \dashv U : \mathcal{D} \to \mathcal{C}$ *be an adjunction of categories, let* $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$*, and let* $\mathcal{J} = \{ F f \mid f \in \mathcal{I} \}$*.*

(i) F *sends relative* $\mathcal{I}$*-cell complexes in* $\mathcal{C}$ *to relative* $\mathcal{J}$*-cell complexes in* $\mathcal{D}$*.*

(ii) U *sends* $\mathcal{J}$*-injective morphisms in* $\mathcal{D}$ *to* $\mathcal{I}$*-injective morphisms in* $\mathcal{C}$*.*

(iii) *F sends $\mathcal{I}$-cofibrations in $\mathcal{C}$ to $\mathcal{J}$-cofibrations in $\mathcal{D}$.*

Proof. (i). This is a corollary of the fact that F preserves all colimits.

(ii). As in the proof of proposition A.3.26, a morphism $f : X \to Y$ in $\mathcal{D}$ has the right lifting property with respect to all morphisms in $\mathcal{J}$ if and only if $Uf : UX \to UY$ has the right lifting property with respect to all morphisms in $\mathcal{I}$.

(iii). Similarly, a morphism $g : Z \to W$ in $\mathcal{C}$ has the left lifting property with respect to all morphisms of the form $Uf : UX \to UY$ where $f : X \to Y$ is a $\mathcal{J}$-injective morphism $f : X \to Y$ in $\mathcal{D}$ if and only if $Fg : FZ \to FW$ is a $\mathcal{J}$-cofibration in $\mathcal{D}$; but we know that U sends $\mathcal{J}$-injective morphisms in $\mathcal{D}$ to $\mathcal{I}$-injective morphisms in $\mathcal{C}$, so F must send $\mathcal{I}$-cofibrations in $\mathcal{C}$ to $\mathcal{J}$-cofibrations in $\mathcal{D}$. ■

Proposition 0.5.20. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, let* $\mathbb{B}$ *be a* $\mathbf{U}$*-small category, let* $\mathcal{C} = [\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$*, and let* $\mathcal{I}$ *be the subset of* $\operatorname{mor} \mathcal{C}$ *consisting of all* monomorphisms $e : U \to V$ *in* $\mathcal{C}$ *where* V *is a quotient of a representable presheaf.*

(i) $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ *is a* $\mathbf{U}$*-cofibrantly generated weak factorisation system.*

(ii) $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$ *is precisely the class of all monomorphisms in* $\mathcal{C}$*.*

(iii) $\operatorname{cof}_{\mathcal{I}} \mathcal{C} = \operatorname{cell}_{\mathcal{I}} \mathcal{C}$*.*

Proof. (i). Since $\mathbb{B}$ is small and $\mathcal{C}$ is well-powered and well-copowered, the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by $\mathcal{I}$ is essentially $\mathbf{U}$-small. We know that $\mathcal{C}$ is locally finitely presentable, thus, taking a $\mathbf{U}$-set of representatives of the isomorphism classes in $\mathcal{I}$, and recalling remark 0.5.9, Quillen's small object argument (theorem 0.5.12) implies $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ is indeed a $\mathbf{U}$-cofibrantly generated weak factorisation system.

(ii). It is clear that the class of injective maps is closed under pushout and transfinite composition in $\mathbf{Set}$, so the same must be true of monomorphisms in $\mathcal{C}$, since colimits in $\mathcal{C}$ are computed componentwise. Thus every morphism in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$ is a monomorphism.

Conversely, suppose $f : X \to Y$ is a monomorphism. Fix an ordinal α and a bijection $y_\bullet : \alpha \to \coprod_{B \in \operatorname{ob} \mathbb{B}} Y(B)$, and write B_β for the object in $\mathbb{B}$ such that $y_\beta \in Y\left(B_\beta\right)$. We will construct a $\mathbf{U}$-small presentation for f by transfinite recursion on α.

- To begin, put $X_0 = X$ and $f_0 = f$.

- For each ordinal $\beta < \alpha$, the Yoneda lemma implies there is a unique morphism $a_\beta : \hbar_{B_\beta} \to Y$ in $\mathcal{C}$ such that $a_\beta\left(\mathrm{id}_{B_\beta}\right) = y_\beta$; let $\bar{v}_\beta : V_\beta \to Y$ be the image of a_β, and let $e_\beta : U_\beta \to V_\beta$ and $u_\beta : U_\beta \to V_\beta$ be defined by the pullback square shown below:

$$\begin{tikzcd} U_\beta \arrow[r, "u_\beta"] \arrow[d, "e_\beta"'] & X_\beta \arrow[d, "f_\beta"] \\ V_\beta \arrow[r, "\bar{v}_\beta"'] & Y \end{tikzcd}$$

  Since f_β is a monomorphism, e_β must also be a monomorphism and hence is in $\mathcal{I}$. There is then a commutative diagram in $\mathcal{C}$ of the following form,

$$\begin{tikzcd} U_\beta \arrow[r, "u_\beta"] \arrow[d, "e_\beta"'] & X_\beta \arrow[d] \arrow[ddr, bend left, "f_\beta"] & \\ V_\beta \arrow[r, "v_\beta"'] \arrow[drr, bend right, "\bar{v}_\beta"'] & X_{\beta+1} \arrow[dr, "f_{\beta+1}"] & \\ & & Y \end{tikzcd}$$

  where $f_{\beta+1} : X_{\beta+1} \to Y$ is the union of $f_\beta : X_\beta \to Y$ and $\bar{v}_\beta : V_\beta \to Y$ considered as subobjects of Y; note that the inner square of the diagram is then a pushout square.

- Finally, for limit ordinals $\gamma < \alpha$, we take $f_\gamma : X_\gamma \to Y$ to be the union $\bigcup_{\beta<\gamma} f_\beta$.

This completes the presentation of $f : X \to Y$ as a relative $\mathcal{I}$-cell complex in $\mathcal{C}$, and it is clearly $\mathbf{U}$-small.

(iii). Corollary 0.5.13 implies that each morphism in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$ is a retract of some morphism in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$, but the class of monomorphisms is closed under retracts, so in this case we must have $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C} = \mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$. Since $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C} \subseteq \mathrm{cell}_{\mathcal{I}}\, \mathcal{C} \subseteq \mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$, we also deduce that $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C} = \mathrm{cell}_{\mathcal{I}}\, \mathcal{C}$. ■

We now turn our attention to Garner's small object argument.

Lemma 0.5.21. *Let κ be a regular cardinal in a universe* $\mathbf{U}$*, let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category, let* $F : \mathcal{A} \to \mathcal{C}$ *be a functor, and let* $G : \mathcal{C} \to \mathcal{C}$ *be (the functor part of) a pointwise left Kan extension of* F *along itself. If each* FA *is a* $(\kappa, \mathbf{U})$*-compact object in* $\mathcal{C}$*, then:*

(i) $G : \mathcal{C} \to \mathcal{C}$ *preserves colimits for* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(ii) *In addition, if* $\mathcal{C}$ *is a* κ*-accessible* $\mathbf{U}$*-category,* λ *is a regular cardinal in* $\mathbf{U}$ *such that every hom-set of* $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ *is* λ*-small,* $\mathcal{A}$ *is a* λ*-small category, and* $\kappa \lhd \lambda$*, then* $G : \mathcal{C} \to \mathcal{C}$ *is strongly* $(\lambda, \mathbf{U})$*-accessible.*

Proof. (i). Theorem A.5.15 says there is a natural bijection of the form below:

$$\mathcal{C}(GX, C) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](\mathcal{C}(F-, X), \mathcal{C}(F-, C))$$

Since colimits are computed componentwise in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, the hypothesis implies $\mathcal{C}(F, -) : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves colimits for $\mathbf{U}$-small κ-filtered diagrams. By the Yoneda lemma, the functors $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ jointly reflect limits, so it follows that $G : \mathcal{C} \to \mathcal{C}$ preserves colimits for $\mathbf{U}$-small κ-filtered diagrams.

(ii). Now suppose X is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. Lemma 0.5.11 then says each hom-set $\mathcal{C}(FA, X)$ is λ-small, and since $\mathcal{A}$ is a λ-small category, this shows that the comma category $(F \downarrow X)$ is also λ-small. Thus, GX is a colimit for a λ-small diagram of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$, and so we may use lemma 0.2.18 to deduce that it is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. ■

Proposition 0.5.22. *Let* $\mathcal{C}$ *be a category with pushouts and let* $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ *be a functor. Suppose a pointwise left Kan extension of* U *along itself exists.*

(i) $\mathbf{RLP}(U)$ *is isomorphic as a category over* $[\mathbb{2}, \mathcal{C}]$ *to the category of algebras for a pointed endofunctor* (J, ι) *on* $[\mathbb{2}, \mathcal{C}]$*.*

(ii) *Moreover, if (the functor part of) the pointwise left Kan extension of* U *along itself is a* $(\kappa, \mathbf{U})$*-accessible functor (resp. strongly* $(\kappa, \mathbf{U})$*-accessible functor), then so is* J*.*

Proof. Let $G : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ be (the functor part of) a pointwise left Kan extension of U along itself and let $\alpha : U \Rightarrow GU$ be the unit. Then there is a unique natural transformation $\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{E}}$ such that $\varepsilon U \bullet \alpha = \mathrm{id}_{[\mathbb{2}, \mathcal{C}]}$. Let

$f : X \to Y$ be a morphism in $\mathcal{C}$. By theorem A.5.15, there is a natural bijection of the form below:

$$[\mathbb{2}, \mathcal{C}](Gf, g) \cong [\mathcal{I}^{\mathrm{op}}, \mathbf{Set}]([\mathbb{2}, \mathcal{C}](U-, f), [\mathbb{2}, \mathcal{C}](U-, g))$$

It is not hard to see that a coherent choice Φ of right liftings for f with respect to $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ is the same thing as a natural transformation $[\mathbb{2}, \mathcal{C}](U-, f) \Rightarrow [\mathbb{2}, \mathcal{C}]\big(U-, \mathrm{id}_X\big)$ making the following diagram commute for all objects e in $\mathcal{I}$,

$$\begin{tikzcd} & {[\mathbb{2}, \mathcal{C}]\big(Ue, \mathrm{id}_X\big)} \arrow[d] \\ {[\mathbb{2}, \mathcal{C}](Ue, f)} \arrow[ur, dashed] \arrow[r, "\mathrm{id}"'] & {[\mathbb{2}, \mathcal{C}](Ue, f)} \end{tikzcd}$$

where the map $[\mathbb{2}, \mathcal{C}]\big(Ue, \mathrm{id}_X\big) \to [\mathbb{2}, \mathcal{C}](Ue, f)$ is the one induced by the morphism $\big(\mathrm{id}_X, f\big) : \mathrm{id}_X \to f$ in $[\mathbb{2}, \mathcal{C}]$. We may therefore identity choices Φ with morphisms $l : d_0(Gf) \to X$ in $\mathcal{C}$ making the diagram below commute:

(∗)
$$\begin{tikzcd} \bullet \arrow[r, "d_1(\varepsilon_f)"] \arrow[d, "Gf"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\ \bullet \arrow[r, dashed, "l"] \arrow[rr, bend right, "d_0(\varepsilon_f)"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Now, define functors $J, K : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ so the square in the following diagram is a natural pushout square in $\mathcal{C}$:

$$\begin{tikzcd} \bullet \arrow[r, "d_1(\varepsilon_f)"] \arrow[d, "Gf"'] & X \arrow[d, "Kf"] \arrow[ddr, bend left, "f"] & \\ \bullet \arrow[r] \arrow[drr, bend right, "d_0(\varepsilon_f)"'] & Mf \arrow[dr, "Jf"] & \\ & & Y \end{tikzcd}$$

We then have a natural transformation $\iota : \mathrm{id}_{[\mathbb{2}, \mathcal{C}]} \Rightarrow J$ where $\iota_f = \big(\mathrm{id}_X, Kf\big)$, and the universal property of pushouts yields a natural bijection between morphisms $l : d_0(Gf) \to X$ making the diagram (∗) commute and morphisms $\tilde{l} : Mf \to X$ such that $\tilde{l} \circ Kf = \mathrm{id}_X$ and $Jf = f \circ \tilde{l}$, i.e. coalgebra structures on f for the pointed endofunctor $(J, \imath)$. The naturality of these identifications then ensures that $\mathbf{RLP}(U)$ is indeed isomorphic to $[\mathbb{2}, \mathcal{C}]^{(J,\imath)}$ as categories over $[\mathbb{2}, \mathcal{C}]$. This proves claim (i).

For claim (ii), simply observe that pushouts preserve all colimits, so $J : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible if $G : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ is, and lemmas 0.2.18 and 0.3.21 imply J is strongly $(\kappa, \mathbf{U})$-accessible if G is. ■

Proposition 0.5.23. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, let $\mathcal{I}$ be a $\mathbf{U}$-small category, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. If each $U e$ is a $(\kappa, \mathbf{U})$-compact object in $[\mathbb{2}, \mathcal{C}]$, then:*

(i) *The forgetful functor $\mathbf{RLP}(U) \to [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible and monadic.*

(ii) *In addition, if λ is a regular cardinal in $\mathbf{U}$ such that each hom-set in $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$ is λ-small, $\mathcal{I}$ is a λ-small category, and $\kappa \lhd \lambda$, then the forgetful functor $\mathbf{RLP}(U) \to [\mathbb{2}, \mathcal{C}]$ is strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. Use theorems 0.3.40 and 0.3.41, lemma 0.5.21, and proposition 0.5.22. ■

Theorem 0.5.24 (Garner's small object argument)**.** *Let $\mathcal{C}$ be a locally presentable $\mathbf{U}$-category, let $\mathcal{I}$ be a $\mathbf{U}$-small category, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor.*

(i) *There exists a free algebraic factorisation system (L, R) on $\mathcal{C}$ cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$.*

(ii) *(L, R) is (part of) an algebraically free natural weak factorisation system on $\mathcal{C}$ cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$.*

(iii) *In particular, if $\mathcal{I}$ is discrete, then there exists a functorial weak factorisation system on $\mathcal{C}$ cofibrantly generated by the image of* $\operatorname{ob} \mathcal{I} \to \operatorname{mor} \mathcal{C}$.

Proof. (i). See Theorem 4.4 in [Garner, 2009].

(ii). See Theorem 5.4 in [Garner, 2009].

(iii). This is proposition A.3.49. □

Lemma 0.5.25. *Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of* $\operatorname{mor}\mathcal{C}$. *If κ is a regular cardinal in a universe $\mathbf{U}$ such that the domains of morphisms in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$, then the class of $\mathcal{I}$-injective objects in $\mathcal{C}$ is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$.*

Proof. Let $\mathbb{D}$ be a $\mathbf{U}$-small κ-filtered category and let $X : \mathbb{D} \to \mathcal{C}$ be a diagram such that each Xd is an $\mathcal{I}$-injective object in $\mathcal{C}$. Suppose $\bar{X}$ is a colimit for X in $\mathcal{C}$ with colimiting cocone $\lambda : X \Rightarrow \Delta\bar{X}$. Let $g : Z \to W$ be in $\mathcal{I}$, and consider the induced hom-set map $g^* : \mathcal{C}\big(W, \bar{X}\big) \to \mathcal{C}\big(Z, \bar{X}\big)$; we must show that it is surjective. Since Z is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, the canonical comparison $\varinjlim_{\mathbb{D}} \mathcal{C}(Z, X) \to \mathcal{C}\big(Z, \bar{X}\big)$ is a bijection, and so every morphism $Z \to \bar{X}$ factors through $\lambda_d : Xd \to X$ for some d in $\mathbb{D}$. By hypothesis Xd is $\mathcal{I}$-injective, so we obtain an extension of $Z \to Xd$ along $g : Z \to W$, and hence, an extension of $Z \to \bar{X}$ along g. Thus X is also $\mathcal{I}$-injective. ■

Lemma 0.5.26. *Let $\mathcal{C}$ be a category and let $g : Z \to W$ be a morphism in $\mathcal{C}$. A morphism $f : X \to Y$ has the left lifting property with respect to g if and only if f is injective as an object in $[\mathbb{2}, \mathcal{C}]$ with respect to the singleton set* $\big\{\big(g, \mathrm{id}_W\big) : g \to \mathrm{id}_W\big\}$. ■

Corollary 0.5.27. *Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of* $\operatorname{mor}\mathcal{C}$. *If the domains and codomains of morphisms in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$, then* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ *is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams in* $[\mathbb{2}, \mathcal{C}]$.

Proof. Apply proposition 0.2.47 and the two lemmas above. ■

Proposition 0.5.28. *Let $\mathcal{C}$ be a locally presentable $\mathbf{U}$-category, let (L, R) be a functorial weak factorisation system on $\mathcal{C}$, and let $\lambda : \mathrm{id}_{[\mathbb{2},\mathcal{C}]} \Rightarrow R$ be the natural transformation whose component at an object f in $[\mathbb{2}, \mathcal{C}]$ corresponds to the following commutative square in $\mathcal{C}$:*

$$\begin{tikzcd} \bullet \arrow[r, "Lf"] \arrow[d, "f"'] & \bullet \arrow[d, "Rf"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

Let $\mathcal{R}$ be the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by the morphisms in $\mathcal{C}$ that are in the right class of the induced weak factorisation system.

(i) *$\mathcal{R}$ is also the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by the image of the forgetful functor $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbb{2}, \mathcal{C}]$, where $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)}$ is the category of algebras for the pointed endofunctor (R, λ).*

(ii) *If* $R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ *is an accessible functor, then* $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)}$ *is a locally presentable* $\mathbf{U}$*-category, and the forgetful functor* $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbb{2}, \mathcal{C}]$ *is monadic.*

(iii) *If* $R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ *is strongly* $(\pi, \mathbf{U})$*-accessible and has* $\mathbf{U}$*-rank* $\kappa < \pi$*, and* $\mathcal{R}$ *is closed under colimits for* $\mathbf{U}$*-small* π*-filtered diagrams in* $[\mathbb{2}, \mathcal{C}]$*, then* $\mathcal{R}$ *is a* π*-accessible* $\mathbf{U}$*-subcategory of* $[\mathbb{2}, \mathcal{C}]$.

Proof. (i). This is proposition A.3.37.

(ii). Apply theorem 0.3.40.

(iii). By theorem 0.3.41, $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)}$ is a locally π-presentable $\mathbf{U}$-category, and the forgetful functor $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbb{2}, \mathcal{C}]$ is moreover strongly $(\pi, \mathbf{U})$-accessible. Thus, we may apply proposition 0.3.31 to claim (i) and deduce that $\mathcal{R}$ is a π-accessible $\mathbf{U}$-subcategory. ■

Proposition 0.5.29. *Let* $\mathcal{C}$ *be a locally presentable* $\mathbf{U}$*-category, and let* $\mathcal{I}$ *be a* $\mathbf{U}$*-subset of* $\operatorname{mor} \mathcal{C}$*. Then* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$*, considered as a full subcategory of* $[\mathbb{2}, \mathcal{C}]$*, is an accessible* $\mathbf{U}$*-subcategory.*

Proof. Combine corollary 0.5.14 and proposition 0.5.28. ■

Lemma 0.5.30. *Let* $\mathcal{C}$ *be a* κ*-accessible* $\mathbf{U}$*-category and let* $\mathcal{R}$ *be a* κ*-accessible full subcategory of* $[\mathbb{2}, \mathcal{C}]$*. If* $g : Z \to W$ *is a morphism in* $\mathcal{C}$ *and* Z *and* W *are* $(\kappa, \mathbf{U})$*-compact objects in* $\mathcal{C}$*, then:*

(i) *Given a morphism* $f : X \to Y$ *in* $\mathcal{C}$ *that is in* $\mathcal{R}$*, any morphism* $g \to f$ *in* $[\mathbb{2}, \mathcal{C}]$ *admits a factorisation of the form* $g \to f' \to f$ *where* f' *is in* $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{R})$.

(ii) *The morphism* $g : Z \to W$ *has the left lifting property with respect to* $\mathcal{R}$ *if and only if it has the left lifting property with respect to* $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{R})$.

Proof. (i). Proposition 0.2.47 says that g is a $(\kappa, \mathbf{U})$-compact object in $[\mathbb{2}, \mathcal{C}]$; but every object in $\mathcal{R}$ is the colimit of a $\mathbf{U}$-small κ-filtered diagram of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{R}$, and the inclusion $\mathcal{R} \hookrightarrow [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible, so any morphism $g \to f$ must factor through some $(\kappa, \mathbf{U})$-compact object in $\mathcal{R}$.

(ii). If g has the left lifting property with respect to $\mathcal{R}$, then it certainly has the left lifting property with respect to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{R})$. Conversely, by factorising morphisms

$g \to f$ as in claim (i), we see that g has the left lifting property with respect to $\mathcal{R}$ as soon as it has the left lifting property with respect to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{R})$. ∎

Lemma 0.5.31. *Let C be a category, let $g : Z \to W$ be a morphism in C, and suppose we have a pushout diagram in C of the form below:*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "g"'] & W \arrow[d, "j_1"] \\ W \arrow[r, "j_0"'] & W \cup^Z W \end{tikzcd}$$

Let $e : W \cup^Z W \to W$ be the unique morphism such that $e \circ j_0 = e \circ j_1 = \mathrm{id}_W$. The following are equivalent for a morphism $f : X \to Y$ in C:

(i) *$f : X \to Y$ is right orthogonal to $g : Z \to W$.*

(ii) *$f : X \to Y$ has the right lifting property with respect to $g : Z \to W$ and $e : W \cup^Z W \to W$.*

Proof. Let $\mathcal{R} = \{g\}^{\perp}$ and let $\mathcal{L} = {}^{\perp}\mathcal{R}$.

(i) ⇒ (ii). By proposition A.3.17, $j_1 : W \to W \cup^Z W$ and $\mathrm{id} : W \to W$ are in $\mathcal{L}$; so by proposition A.3.18, $e : W \cup^Z W \to W$ is also in $\mathcal{L}$. But proposition A.3.3 says that $\mathcal{R} = \mathcal{L}^{\perp}$ and $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash}$, so if $f : X \to Y$ is right orthogonal to $g : Z \to W$, then $f : X \to Y$ indeed has the right lifting property with respect to $g : Z \to W$ and $e : W \cup^Z W \to W$.

(ii) ⇒ (i). Suppose $f : X \to Y$ has the right lifting property with respect to $g : Z \to W$ and $e : W \cup^Z W \to W$. Consider a lifting problem in C of the form below:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

By hypothesis, there is at least one $h : W \to X$ in C such that $h \circ g = z$ and $f \circ h = w$. Suppose $k : W \to X$ is another. Then there is a unique morphism $l : W \cup^Z W \to X$ such that $l \circ j_0 = h$ and $l \circ j_1 = k$, and by construction, $f \circ l = w \circ e$, so there is at least one morphism $m : W \to X$ such that $m \circ e = l$ (and $f \circ m = w$). But that implies $m = h = k$, so $f : X \to Y$ is indeed right orthogonal to $g : W \to Z$. ∎

Theorem 0.5.32. *Let* $\mathbf{U}$ *be a universe, let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category with colimits for all* $\mathbf{U}$*-small diagrams, and let* $\mathcal{J}$ *be a* $\mathbf{U}$*-subset of* $\operatorname{mor} \mathcal{C}$.

(i) *There is a* $\mathbf{U}$*-subset* $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ *such that* $\mathcal{I}^{\boxslash} = \mathcal{J}^{\perp}$.

(ii) *If* $(\mathcal{I}, \mathcal{C})$ *is admissible for the* $\mathbf{U}$*-small object argument, then* $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \mathcal{J}^{\perp}\right)$ *is a* $\mathbf{U}$*-cofibrantly generated orthogonal factorisation system on* $\mathcal{C}$.

Proof. (i). Apply lemma 0.5.31.

(ii). This is a special case of Quillen's small object argument (theorem 0.5.12). ■

Corollary 0.5.33. *Let* $\mathbf{U}$ *be a universe, let* $\mathcal{C}$ *be a locally presentable* $\mathbf{U}$*-category, let* $\mathcal{J}$ *be a* $\mathbf{U}$*-subset of* $\operatorname{mor} \mathcal{C}$*, and let* $\mathcal{D}$ *is the full subcategory of* $\mathcal{C}$ *spanned by those objects* X *such that the unique morphism* $X \to 1$ *is right orthogonal to* $\mathcal{J}$.

(i) $\mathcal{D}$ *is a reflective subcategory of* $\mathcal{C}$.

(ii) $\mathcal{D}$ *is a locally presentable* $\mathbf{U}$*-category and the inclusion* $\mathcal{D} \hookrightarrow \mathcal{C}$ *is an accessible functor.*

(iii) *If* κ *is a regular cardinal in* $\mathbf{U}$ *such that* $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}$*-category and every morphism in* $\mathcal{J}$ *has* $(\kappa, \mathbf{U})$*-compact domain and codomain, then* $\mathcal{D}$ *is also a locally* κ*-presentable* $\mathbf{U}$*-category and the inclusion* $\mathcal{D} \hookrightarrow \mathcal{C}$ *is a* $(\kappa, \mathbf{U})$*-accessible functor.*

Proof. (i). We must show that the inclusion $\mathcal{C} \hookrightarrow \mathcal{D}$ admits a left adjoint, so it suffices to verify the following: for every object X in $\mathcal{C}$, the functor

$$\mathcal{C}(X, -) : \mathcal{D} \to \mathbf{Set}$$

is representable in $\mathcal{D}$. Let $\mathcal{R} = \mathcal{J}^{\perp}$ and $\mathcal{L} = {}^{\perp}\mathcal{R}$. By theorem 0.5.12, there exists a morphism $\eta_X : X \to \hat{X}$ such that $\hat{X}$ is in $\mathcal{D}$ and $\eta_X : X \to \hat{X}$ is in $\mathcal{L}$; but if D in an object in $\mathcal{D}$ and $g : Z \to W$ is in $\mathcal{L}$, then

$$\mathcal{C}(g, D) : \mathcal{C}(W, D) \to \mathcal{C}(Z, D)$$

is a bijection, so we deduce that $\hat{X}$ represents $\mathcal{C}(X, -) : \mathcal{D} \to \mathbf{Set}$.

(ii) and (iii). By corollary 0.5.14, the endofunctor $X \mapsto \hat{X}$ is $(\kappa, \mathbf{U})$-accessible, so $\mathcal{D}$ is isomorphic to the category of algebras for a monad on $\mathcal{C}$ whose underlying endofunctor is $(\kappa, \mathbf{U})$-accessible. We may then apply theorem 0.3.35. ■

Theorem 0.5.34. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, let* $\mathcal{A}$ *be a* $\mathbf{U}$*-small category, let* κ *be a regular cardinal in* $\mathbf{U}$*, let* $\mathcal{K}$ *be a* $\mathbf{U}$*-set of cocones under* κ*-small diagrams in* $\mathcal{A}$*, and let* $\mathcal{C}$ *be the full subcategory of* $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *spanned by those* $M : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ *that send the cocones that are in* $\mathcal{K}$ *to limiting cones in* $\mathbf{Set}$.

(i) $\mathcal{C}$ *is a reflective subcategory of* $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

(ii) $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}$*-category.*

(iii) *For each object* a *in* $\mathcal{A}$*, the functor* $\mathcal{C} \to \mathbf{Set}$ *defined by* $M \mapsto Ma$ *is representable, say by* Fa*, and the resulting functor* $F : \mathcal{A} \to \mathcal{C}$ *sends cocones that are in* $\mathcal{K}$ *to colimiting cocones.*

Proof. (i) and (ii). For each cocone $k : A \Rightarrow \Delta a$ that is in $\mathcal{K}$, let $f_k : \varinjlim \hbar_A \to \hbar_a$ be the induced morphism in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. The Yoneda lemma then implies that a functor $M : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ sends the cocone $k : A \Rightarrow \Delta a$ to a limiting cone in $\mathbf{Set}$ if and only if the induced map

$$[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](f_k, M) : [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](\hbar_a, M) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]\Big(\varinjlim \hbar_A, M\Big)$$

is a bijection, and by lemma A.3.2, this happens if and only if the unique morphism $M \to 1$ is right orthogonal with respect to $f_k : \varinjlim \hbar_A \to \hbar_a$. Moreover, each A is a κ-small diagram, so by proposition 0.2.46, $\varinjlim \hbar_A$ is a $(\kappa, \mathbf{U})$-compact object in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Thus we may apply corollary 0.5.33.

(iii). By the Yoneda lemma, we may take Fa to be the reflection of $\hbar_a$ in $\mathcal{C}$. Let $k : A \Rightarrow \Delta a$ be a cocone that is in $\mathcal{K}$ and let M be any object in $\mathcal{C}$. Then,

$$\mathcal{C}(Fa, M) \cong Ma \cong \varprojlim MA \cong \varprojlim \mathcal{C}(FA, M)$$

so $Fk : FA \Rightarrow \Delta Fa$ is indeed a colimiting cocone in $\mathcal{C}$. ■

— I —

Simplicial sets

Simplicial sets, like simplicial complexes, are combinatorial models for spaces built up by gluing standard n-simplices together; unlike simplicial complexes, an n-simplex in a simplicial set need not be uniquely determined by its vertices. It is for this reason that simplicial sets were once known by the unwieldy name 'complete semi-simplicial (c.s.s.) complex'.

In the 1960s, it was discovered that one can mimic the definitions and constructions of classical homotopy theory by combinatorial means using simplicial sets, and that the resulting theory is moreover equivalent to the classical theory in a natural, functorial way. More recently, it has been shown that the homotopy theory of simplicial sets is *universal* in a precise sense,[1] so it seems fitting that we begin here.

1.1 Basics

Definition 1.1.1. The **simplex category** is the category $\mathbf{\Delta}$ whose objects are the positive finite ordinals and whose morphisms are the monotone maps. We use the geometer's convention: $[n]$ denotes the ordinal $\{0, 1, \ldots, n\}$.

Definition 1.1.2. A **simplicial object** in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and a **morphism of simplicial objects** in $\mathcal{C}$ is a natural transformation of such functors. The **category of simplicial objects** in $\mathcal{C}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$ and is denoted by $\mathbf{s}\mathcal{C}$.

[1] See [Dugger, 2001a].

Definition 1.1.3. The **coface maps** in $\mathbf{\Delta}$ are the morphisms $\delta_n^i : [n-1] \to [n]$, where δ_n^i is the unique injective monotone map that misses i; and the **codegeneracy maps** in $\mathbf{\Delta}$ are the morphisms $\sigma_n^i : [n+1] \to [n]$, where σ_n^i is the unique surjective monotone map with $\sigma_n^i(i) = \sigma_n^i(i+1) = i$.

Theorem 1.1.4 (Cosimplicial identities)**.** *The following equations hold in* $\mathbf{\Delta}$*:*

$$\begin{aligned}
\delta_{n+1}^{j+1} \circ \delta_n^i &= \delta_{n+1}^i \circ \delta_n^j && \textit{if } 0 \le i \le j \le n \\
\sigma_n^j \circ \sigma_{n+1}^i &= \sigma_n^i \circ \sigma_{n+1}^{j+1} && \textit{if } 0 \le i \le j \le n \\
\sigma_{n+1}^{j+1} \circ \delta_{n+1}^i &= \delta_n^i \circ \sigma_n^j && \textit{if } 0 \le i \le j \le n \\
\delta_n^{j+1} \circ \sigma_n^i &= \sigma_{n+1}^i \circ \delta_{n+1}^{j+2} && \textit{if } 0 \le i < j < n \\
\sigma_n^i \circ \delta_n^i &= \mathrm{id} && \textit{if } 0 \le i \le n \\
\sigma_n^{i+1} \circ \delta_n^i &= \mathrm{id} && \textit{if } 0 \le i < n
\end{aligned}$$

Equivalently, the following diagrams commute:

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\delta^i"] \arrow[d, "\delta^j"'] & {[n]} \arrow[d, "\delta^{j+1}"] \\
{[n]} \arrow[r, "\delta^i"'] & {[n+1]}
\end{tikzcd} \quad \textit{for } 0 \le i \le j \le n$$

$$\begin{tikzcd}
{[n+1]} \arrow[r, "\sigma^i"] \arrow[d, "\sigma^{j+1}"'] & {[n]} \arrow[d, "\sigma^j"] \\
{[n]} \arrow[r, "\sigma^i"'] & {[n-1]}
\end{tikzcd} \quad \textit{for } 0 \le i \le j \le n$$

$$\begin{tikzcd}
{[n]} \arrow[r, "\delta^i"] \arrow[d, "\sigma^j"'] & {[n+1]} \arrow[d, "\sigma^{j+1}"] \\
{[n-1]} \arrow[r, "\delta^i"'] & {[n]}
\end{tikzcd} \quad \textit{for } 0 \le i \le j \le n$$

$$\begin{tikzcd}
{[n]} \arrow[r, "\sigma^i"] \arrow[d, "\delta^{j+2}"'] & {[n-1]} \arrow[d, "\delta^{j+1}"] \\
{[n+1]} \arrow[r, "\sigma^i"'] & {[n]}
\end{tikzcd} \quad \textit{for } 0 \le i < j < n$$

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\delta^i"] \arrow[d, "\delta^i"'] \arrow[dr, "\mathrm{id}"] & {[n]} \arrow[d, "\sigma^{i+1}"] \\
{[n]} \arrow[r, "\sigma^i"'] & {[n-1]}
\end{tikzcd} \quad \textit{for } 0 \le i \le n$$

Moreover, every morphism $[n] \to [m]$ *in* $\mathbf{\Delta}$ *is* uniquely *a composite of the form*

$$\delta_m^{j_1} \circ \cdots \circ \delta_k^{j_{m-k}} \circ \sigma_k^{i_{n-k}} \circ \cdots \circ \sigma_n^{i_1}$$

where $k \leq \min\{n, m\}$, *and*

$$0 \leq i_{n-k} \leq \cdots \leq i_1 \leq n$$
$$0 \leq j_{m-k} \leq \cdots \leq j_1 \leq m$$

The category $\mathbf{\Delta}$ *is uniquely characterised by these properties.*

Proof. See [May, 1967, §2], [GZ, Ch. II, §2], or [Weibel, 1994, §8.1]. □

Definition 1.1.5. Let A be a simplicial object in a category $\mathcal{C}$. A **face operator** for A is a morphism of the form $A(\delta_n^i) : A([n]) \to A([n-1])$, and a **degeneracy operator** for A is a morphism of the form $A(\sigma_n^i) : A([n]) \to A([n+1])$. For brevity, we will usually write A_n instead of $A([n])$, d_i^n instead of $A(\delta_n^i)$, and s_i^n instead of $A(\sigma_n^i)$.

Corollary 1.1.6 (Simplicial identities)**.** *The face and degeneracy operators of a simplicial object satisfy the formal duals of the equations in theorem 1.1.4.* ■

Corollary 1.1.7. *A simplicial object A is uniquely determined by the sequence of objects* $A_0, A_1, A_2, \ldots$ *together with the face and degeneracy operators. Conversely, any sequence of objects equipped with face and degeneracy operators satisfying the simplicial identities defined a simplicial object.* ■

Observe that there is an identity-on-objects automorphism $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$ that sends coface maps $\delta_n^i : [n-1] \to [n]$ to $\delta_n^{n-i} : [n-1] \to [n]$ and codegeneracy maps $\sigma_n^i : [n] \to [n+1]$ to $\sigma_n^{n-i} : [n] \to [n+1]$ for all $n \geq 0$ and $0 \leq i \leq n$. This in turn induces an automorphism on the category of simplicial objects.

Definition 1.1.8. The **opposite** of a simplicial object A in a category $\mathcal{C}$ is the simplicial object A^{op} obtained by composing $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ with $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$.

Remark 1.1.9. Although $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$ acts as the identity on objects, the functor $(-)^{\mathrm{op}}$ is *not* isomorphic to $\mathrm{id}_{\mathbf{\Delta}}$. More generally, a simplicial object A *may* be isomorphic to its opposite A^{op}, but the functor $(-)^{\mathrm{op}} : \mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}$ is usually *not* isomorphic to $\mathrm{id} : \mathbf{s}\mathcal{C} \to \mathbf{s}\mathcal{C}$.

Definition 1.1.10. A **simplicial set** is a simplicial object in **Set**, and the **category of simplicial sets** is denoted by **sSet**.

Lemma 1.1.11.

(i) *Limits (resp. colimits) in* **sSet** *are constructed degreewise: a cone (resp. cocone) in* **sSet** *over a diagram is limiting (resp. colimiting) if and only if it is so in every degree.*

(ii) *A morphism of* **sSet** *is monic (resp. epic) if and only if it is degreewise injective (resp. surjective).*

Proof. These are standard facts about functor categories. □

Definition 1.1.12. The **standard n-simplex** in **sSet**, denoted by Δ^n, is the representable presheaf $\mathbf{\Delta}(-, [n])$.

Theorem 1.1.13. *Let* $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ *be the functor* $[n] \mapsto \Delta^n$.

(i) *For any simplicial set X, the map* $\mathbf{sSet}(\Delta^n, X) \to X_n$ *defined by* $f \mapsto f_n(\mathrm{id}_{[n]})$ *is a bijection and is moreover natural in $[n]$ and X.*

(ii) **sSet** *has limits and colimits for all small diagrams, every epimorphism is effective, and for all morphisms $f : X \to Y$ in* **sSet**, *the pullback functor* $f^* : \mathbf{sSet}_{/Y} \to \mathbf{sSet}_{/X}$ *preserves colimits.*

(iii) $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ *is a dense functor, i.e. for any simplicial set X, the tautological cocone*[2] *from the canonical diagram* $(\Delta^\bullet \downarrow X) \to \mathbf{sSet}$ *to X is colimiting.*

(iv) *Let $\mathcal{E}$ be a locally small category with colimits for all small diagrams. If* $F : \mathbf{sSet} \to \mathcal{E}$ *is a functor that preserves small colimits, then it is left adjoint to the functor* $\mathcal{E} \to \mathbf{sSet}$ *defined by* $E \mapsto \mathcal{E}(F\Delta^\bullet, E)$.

(v) *With $\mathcal{E}$ as above, the functor* $F \mapsto F\Delta^\bullet$ *from the category of colimit-preserving functors* $\mathbf{sSet} \to \mathcal{E}$ *to the category of all functors* $\mathbf{\Delta} \to \mathcal{E}$ *is fully faithful and essentially surjective on objects.*

Proof. Claim (i) is just the Yoneda lemma, claim (ii) follows from the lemma above, and claims (iii)–(v) are just facts about dense functors, pointwise left Kan extensions, weighted colimits: see proposition A.5.25, theorem A.5.15, and proposition A.6.15. ■

[2] See definition A.5.7.

Definition 1.1.14. Let X be a simplicial set. An ***n*-simplex** of X is an element of X_n; a **vertex** is a 0-simplex, and an **edge** is a 1-simplex. This is justified by statement (i) in the above theorem. Given an edge f of X, the **source** of f is the vertex $d_1(f)$, and the **target** of f is the vertex $d_0(f)$; we write $f : x \to y$ to mean $d_1(f) = x$ and $d_0(f) = y$.

Definition 1.1.15. A **degenerate *n*-simplex** of a simplicial set X is an n-simplex α for which there exist an $(n-1)$-simplex β and $0 \leq i < n$ such that $s_i(\beta) = \alpha$. A **non-degenerate *n*-simplex** of X is an n-simplex that is not degenerate.

Remark 1.1.16. An n-simplex of X can be non-degenerate even when the corresponding morphism $\Delta^n \to X$ is not a monomorphism! Similarly, it is possible for all the proper faces of a non-degenerate simplex to be degenerate.

Definition 1.1.17. A **finite simplicial set** is a simplicial set that has only finitely many *non-degenerate* simplices.

Proposition 1.1.18. *Let X be a simplicial set. The following are equivalent:*

(i) *X is a finite simplicial set.*

(ii) *X is an $\aleph_0$-compact object in* **sSet**.[3]

(iii) *X is in the smallest full subcategory of* **sSet** *that contains the standard simplices and is closed in* **sSet** *under (isomorphisms and) colimits for finite diagrams.*

Proof. (i) $\Rightarrow$ (ii). A morphism $f : X \to Y$ is determined uniquely by the images of the non-degenerate simplices of X, and the faces of any particular simplex can only satisfy finitely many equations, so if X is a finite simplicial set and Y is a colimit for a small filtered diagram of simplicial sets, then f must factor through one of the components of the colimiting cocone. It is straightforward to check that the factorisation of f is unique up to the appropriate equivalence relation, and we may then deduce that X is an $\aleph_0$-compact object.

(ii) $\Rightarrow$ (iii). Let $\mathcal{K}$ be the indicated full subcategory of **sSet**, and consider the comma category $(\mathcal{K} \downarrow X)$. Let $P : (\mathcal{K} \downarrow X) \to$ **sSet** be the projection, and let $\lambda : P \Rightarrow \Delta X$ be the tautological cocone.[4] It is not hard to check that λ is a

[3] See definition 0.2.14.
[4] See definition A.5.7.

colimiting cocone. Since $\mathcal{K}$ has colimits for finite diagrams, $(\mathcal{K} \downarrow X)$ is filtered; and it is clear that $\mathcal{K}$ is essentially small, so we deduce that X is a retract of an object in $\mathcal{K}$ if X is $\aleph_0$-compact. Noting that $\mathcal{K}$ is closed under retracts, we conclude that X is in $\mathcal{K}$ if it is $\aleph_0$-compact.

(iii) ⇒ (i). Now, let $\mathcal{K}'$ be the full subcategory of **sSet** spanned by the finite simplicial sets. It is easy to see that $\mathcal{K}'$ is closed in **sSet** under (isomorphisms and) finite colimits, and the standard simplices are all in $\mathcal{K}'$, so we must have $\mathcal{K} \subseteq \mathcal{K}'$, as required. ■

Definition 1.1.19. The **standard n-simplex** in **Top**, denoted by $|\Delta^n|$, is the topological space

$$|\Delta^n| = \left\{ \left(x_0, \dots, x_n\right) \in [0,1]^{n+1} \,\middle|\, x_0 + \cdots + x_n = 1 \right\}$$

where $[0,1]$ is the closed unit interval with the standard metric. The functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ sends $[n]$ to $|\Delta^n|$ and is defined on morphisms by linearly interpolating the obvious map of vertices.

Corollary 1.1.20. *There exists an adjunction*

$$|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$$

extending the functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ *defined above, and this adjunction is unique up to unique isomorphism. Explicitly, we may take*

$$\mathrm{S}(Y)_n = \mathbf{Top}\left(|\Delta^n|, Y\right)$$

with the evident face and degeneracy operators induced by the coface and codegeneracy maps in $\mathbf{\Delta}$. ■

Definition 1.1.21. The **geometric realisation** of a simplicial set X is the topological space $|X|$, and the **singular set** of a topological space Y is the simplicial set $\mathrm{S}(Y)$.

Remark 1.1.22. The geometric realisation $|X|$ is stable under universe enlargement, by theorem A.5.20.

Theorem 1.1.23. *Let* **CGHaus** *be the category of compactly generated Hausdorff spaces*[5] *and continuous maps.*

[5] See definition A.2.26.

(i) *The topological standard n-simplex* $|\Delta^n|$ *is a compact Hausdorff space.*

(ii) *For any simplicial set* X*, the geometric realisation* $|X|$ *is a compactly generated Hausdorff space.*

(iii) *The previously-constructed adjunction* $|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$ *restricts to an adjunction between* **CGHaus** *and* **sSet***, and moreover the functor* $|-| : \mathbf{sSet} \to \mathbf{CGHaus}$ *preserves finite limits and reflects isomorphisms.*

Proof. Claim (i) is a standard fact, while claims (ii) and (iii) are proven in [GZ, Ch. III, § 3]. □

1.2 Nerves, skeletons, and coskeletons

Prerequisites. §§ 1.1, A.2.

Proposition 1.2.1. *Let* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{C})_n = \mathrm{Fun}([n], \mathbb{C})$$

where $[n]$ *here denotes the preorder category* $\{0 \to \cdots \to n\}$.

(i) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *has a left adjoint* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *such that* $\tau_1 \Delta^n = [n]$.

(ii) *The functor* N *is fully faithful and exhibits* **Cat** *as a reflective subcategory of* **sSet**.

(iii) $\mathrm{N}(-)^{\mathrm{op}}$ *and* $\mathrm{N}((-)^{\mathrm{op}})$ *are isomorphic as functors* $\mathbf{Cat} \to \mathbf{sSet}$.

(iv) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(v) *The functor* τ_1 *preserves finite products.*

Proof. (i). Apply theorem 1.1.13.

(ii). A functor is entirely determined by its action on objects, arrows, and composable strings of arrows, so N is fully faithful.

(iii). It is clear that there is a canonical isomorphism $\mathrm{N}(\mathbb{C})^{\mathrm{op}}$ and $\mathrm{N}(\mathbb{C}^{\mathrm{op}})$ for all small categories $\mathbb{C}$, and it is straightforward to verify naturality.

(iv). N preserves binary products, so we have the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}(\Delta^n, \mathrm{N}([\mathbb{C}, \mathbb{D}])) &\cong \mathrm{Fun}([n], [\mathbb{C}, \mathbb{D}]) \\
&\cong \mathrm{Fun}([n] \times \mathbb{C}, \mathbb{D}) \\
&\cong \mathbf{sSet}(\mathrm{N}([n] \times \mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]) \times \mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]), [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]) \\
&\cong \mathbf{sSet}(\Delta^n, [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})])
\end{aligned}$$

Thus, by the Yoneda lemma, the canonical morphism $\mathrm{N}([\mathbb{C}, \mathbb{D}]) \to [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]$ is an isomorphism.

(v). It is clear that τ_1 preserves terminal objects. Let X and Y be simplicial sets. We wish to show that the canonical morphism $\tau_1(X \times Y) \to \tau_1 X \times \tau_1 Y$ is an isomorphism; but since τ_1 is a left adjoint and both **sSet** and **Cat** are cartesian closed, it is enough to check the claim for $Y = \Delta^n$, because **sSet** is generated under colimits by $\{\Delta^n \mid n \in \mathbb{N}\}$. We have the following natural bijections:

$$\begin{aligned}
\mathrm{Fun}\big(\tau_1(X \times \Delta^n), \mathbb{C}\big) &\cong \mathbf{sSet}(X \times \Delta^n, \mathrm{N}(\mathbb{C})) \\
&\cong \mathbf{sSet}\big(X, \mathrm{N}(\mathbb{C})^{\Delta^n}\big) \\
&\cong \mathbf{sSet}(X, \mathrm{N}([[n], \mathbb{C}])) \\
&\cong \mathrm{Fun}\big(\tau_1 X, [[n], \mathbb{C}]\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times [n], \mathbb{C}\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times \tau_1 \Delta^n, \mathbb{C}\big)
\end{aligned}$$

The claim follows by the Yoneda lemma. ■

Definition 1.2.2. The **fundamental category** of a simplicial set X is the small category $\tau_1 X$, and the **nerve** of a small category $\mathbb{C}$ is the simplicial set $\mathrm{N}(\mathbb{C})$.

Remark 1.2.3. Given a simplicial set X, the fundamental category $\tau_1 X$ admits the following presentation by generators and relations: the objects are the vertices of X, and the arrows are generated by the edges of X, modulo the relation $d_0(\alpha) \bullet d_2(\alpha) = d_1(\alpha)$ for all 2-simplices α in X. This shows that $\tau_1 X$ is stable under universe enlargement.

Proposition 1.2.4. *Let* disc : **Set** $\to$ **sSet** *be the functor defined by the formula*

$$(\operatorname{disc} Y)_n = Y$$

with id_Y *for all the face and degeneracy maps.*

(i) $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ *has a left adjoint* $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ *such that* $\pi_0 \Delta^n = 1$.

(ii) *The functor* disc *is fully faithful and exhibits* **Set** *as a reflective subcategory of* **sSet**.

(iii) *The functor* π_0 *preserves products.*

(iv) $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ *is a cartesian closed functor.*

Proof. (i). We could apply theorem 1.1.13, but it is also fairly straightforward to check that this explicit construction works: for each simplicial set X, we define $\pi_0 X$ by the coequaliser diagram in **Set** shown below,

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X$$

and for each morphism $f : X \to Y$ in **sSet**, we define $\pi_0 f$ to be the unique morphism making the evident diagram commute.

(ii). It is clear that disc is fully faithful.

(iii). By remark A.5.35, $\mathbf{\Delta}^{\mathrm{op}}$ is a sifted category, and $\pi_0 \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$, so we may apply theorem A.5.36.

(iv). Use proposition A.2.13. ■

Definition 1.2.5. The **set of connected components** of a simplicial set X is the set $\pi_0 X$, and a **discrete simplicial set** is one that is isomorphic to $\mathrm{disc}\, Y$ for some set Y.

¶ 1.2.6. We will usually not distinguish between Y and $\mathrm{disc}\, Y$ notationally.

Proposition 1.2.7. *Let* $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{G})_n = \mathrm{Fun}(\mathbf{I}[n], \mathbb{G})$$

where $\mathbf{I}[n]$ *here denotes the groupoid obtained by freely inverting the arrows in the preorder category* $[n]$.

(i) *For any groupoid* $\mathbb{G}$*, the nerve* $\mathrm{N}(\mathbb{G})$ *is the same (up to isomorphism) whether computed for* $\mathbb{G}$ *as a groupoid or* $\mathbb{G}$ *as a category.*

(ii) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *has a left adjoint* $\pi_1 : \mathbf{sSet} \to \mathbf{Grpd}$ *such that* $\pi_1 \Delta^n = \mathbf{I}[n]$.

(iii) *The functor* N *is fully faithful and exhibits* **Grpd** *as a reflective subcategory of* **sSet**.

(iv) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(v) *The functor* π_1 *preserves finite products.*

Proof. (i). By the universal property of $\mathbf{I}[n]$, there is a natural bijection

$$\mathrm{Fun}(\mathbf{I}[n], \mathbb{G}) \cong \mathrm{Fun}([n], \mathbb{G})$$

for all groupoids $\mathbb{G}$, so the two nerve constructions do indeed agree.

(ii) and (iii). These are proven in exactly the same way as in proposition 1.2.1.

(iv) and (v). These are proven in exactly the same way as in proposition 1.2.4. ■

Definition 1.2.8. The **fundamental groupoid** of a simplicial set X is the small groupoid $\pi_1 X$.

Remark 1.2.9. Given a simplicial set X, the fundamental groupoid $\pi_1 X$ admits a presentation of the same kind as the fundamental category $\tau_1 X$, and in fact $\pi_1 X$ is isomorphic to the groupoid obtained by freely inverting the arrows in $\tau_1 X$:

$$\mathrm{Fun}\big(\pi_1 X, \mathbb{G}\big) \cong \mathbf{sSet}(X, \mathrm{N}(\mathbb{G})) \cong \mathrm{Fun}\big(\tau_1 X, \mathbb{G}\big)$$

This shows that $\pi_1 X$ is stable under universe enlargement.

Definition 1.2.10. Let n be a natural number, and let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0], \dots, [n]$. An ***n*-truncated simplicial set** is a functor ${\mathbf{\Delta}_{\leq n}}^{\mathrm{op}} \to \mathbf{Set}$, and we write $\mathbf{sSet}_{\leq n}$ for the category of n-truncated simplicial sets. The **brutal *n*-truncation** of a simplicial set X is the n-truncated simplicial set $X_{\leq n}$ defined by the evident reduct:

$$X_{\leq n}([m]) = X([m])$$

Proposition 1.2.11. *Let n be a natural number, and let $j : \mathbf{\Delta}_{\le n} \to \mathbf{\Delta}$ be the inclusion.*

(i) *The functor* $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\le n}$ *has a left adjoint* $\operatorname{Lan}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$.

(ii) *The unit* $\mathrm{id} \Rightarrow j^* \operatorname{Lan}_j$ *is a natural isomorphism.*

(iii) $\operatorname{Lan}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$ *is a fully faithful functor.*

(i′) *The functor* $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\le n}$ *has a right adjoint* $\operatorname{Ran}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$.

(ii′) *The counit* $j^* \operatorname{Ran}_j \Rightarrow \mathrm{id}$ *is a natural isomorphism.*

(iii′) $\operatorname{Ran}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$ *is a fully faithful functor.*

Proof. (i) and (i′). Use theorem A.5.15.

(ii) and (ii′). The inclusion $j : \mathbf{\Delta}_{\le n} \to \mathbf{\Delta}$ is fully faithful, so the unit $\mathrm{id} \Rightarrow j^* \operatorname{Lan}_j$ and the counit $j^* \operatorname{Ran}_j \Rightarrow \mathrm{id}$ are natural isomorphisms, by corollary A.5.19.

(iii) and (iii′). Use proposition A.1.3. ∎

Definition 1.2.12. For each natural number n, with notation as above, let $\mathrm{sk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Lan}_j j^*$, and let $\mathrm{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Ran}_j j^*$. The ***n*-skeleton** of a simplicial set X is the simplicial set $\mathrm{sk}_n(X)$, and the ***n*-coskeleton** of a simplicial set is the simplicial set $\mathrm{cosk}_n(X)$. A ***n*-skeletal simplicial set** is one that is isomorphic to the n-skeleton of some simplicial set, and an ***n*-coskeletal simplicial set** is one that is isomorphic to the n-coskeleton of some simplicial set.

Remark 1.2.13. In the special case $n = 0$, Lan_j may be identified with the functor $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ defined in proposition 1.2.4. Thus, 0-skeletal simplicial sets are precisely the discrete simplicial sets. On the other hand, given a set X, $\operatorname{Ran}_j X$ can be identified with the simplicial set whose m-simplices are $(m + 1)$-tuples of elements of X, with face and degeneracy maps induced by the appropriate projections.

Proposition 1.2.14. *Let n be a natural number.*

(i) *The full subcategory of n-skeletal simplicial sets is a coreflective subcategory of* **sSet**, *with coreflector* sk_n.

(ii) sk_n *is the underlying endofunctor of an idempotent comonad on* **sSet**.

(iii) *A simplicial set X is n-skeletal if and only if the counit* $\mathrm{sk}_n(X) \to X$ *is an isomorphism.*

(iv) *If $m \geq n$, then any n-skeletal simplicial set is also m-skeletal.*

(i′) *The full subcategory of n-coskeletal simplicial sets is a reflective subcategory of* **sSet**, *with reflector* cosk_n.

(ii′) cosk_n *is the underlying endofunctor of an idempotent monad on* **sSet**.

(iii′) *A simplicial set X is n-coskeletal if and only if the unit* $X \to \mathrm{cosk}_n(X)$ *is an isomorphism.*

(iv′) *If $m \geq n$, then any n-coskeletal simplicial set is also m-coskeletal.*

Proof. All straightforward from the definitions. ◆

Proposition 1.2.15. *Let n be a natural number, and let X be a simplicial set.*

(i) *We have the following adjunction:*

$$\mathrm{sk}_n \dashv \mathrm{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$$

(ii) *The counit* $\mathrm{sk}_n(X) \to X$ *is a monomorphism, and X is n-skeletal if and only if all m-simplices of X are degenerate for $m > n$.*

(iii) *X is n-coskeletal if and only if, for all natural numbers m, the map*

$$X_m \cong \mathbf{sSet}(\Delta^m, X) \to \mathbf{sSet}\big(\mathrm{sk}_n(\Delta^m), X\big)$$

induced by the counit $\mathrm{sk}_n(\Delta^m) \to \Delta^m$ *is a bijection.*

Proof. (i). Immediate from the definition of sk_n and cosk_n.

(ii). The most straightforward way of seeing this is to construct $\mathrm{sk}_n(X)$ explicitly as the smallest simplicial subset of X containing all of its n-simplices.

(iii). Apply the Yoneda lemma in conjunction with claim (i). ■

Example 1.2.16. For any small category $\mathbb{C}$, the nerve $\mathrm{N}(\mathbb{C})$ is a 2-coskeletal simplicial set: by definition, an m-simplex of $\mathrm{N}(\mathbb{C})$ is just a functor $[m] \to \mathbb{C}$, but the property of being a functor can be detected by only inspecting the vertices, edges, and 2-cells.

Proposition 1.2.17. *The following full subcategories are exponential ideals of* **sSet***:*

(i) *Discrete simplicial sets.*

(ii) *Simplicial sets isomorphic to the nerve of some category.*

(iii) *Simplicial sets isomorphic to the nerve of some groupoid.*

(iv) *n-coskeletal simplicial sets for some natural number n.*

Proof. Apply proposition A.2.13 to propositions 1.2.4, 1.2.1, 1.2.7, and 1.2.14. ■

Definition 1.2.18. The **boundary** of Δ^n is the simplicial subset $\partial\Delta^n \subseteq \Delta^n$ generated by the images of $\delta_n^0, \ldots, \delta_n^n : \Delta^{n-1} \to \Delta^n$.

Remark 1.2.19. The boundary $\partial\Delta^n$ may be identified with $\mathrm{sk}_{n-1}\Delta^n$.

Proposition 1.2.20 (Relative skeletal filtration)**.** *Let $f : X \to Y$ be a monomorphism in* **sSet***. There exist simplicial sets $Y^{(0)}, Y^{(1)}, Y^{(2)}, \ldots$ and a chain of monomorphisms*

$$X = Y^{(-1)} \xrightarrow{i^{(0)}} Y^{(0)} \xrightarrow{i^{(1)}} Y^{(1)} \xrightarrow{i^{(2)}} Y^{(2)} \longrightarrow \cdots$$

such that the following conditions are satisfied:

- *There is a colimiting cocone from the above chain to Y where the component $Y^{(-1)} \to Y$ is $f : X \to Y$.*

- *For each natural number n, there is a pushout diagram of the form below,*

$$\begin{tikzcd} I_n \odot \partial\Delta^n \arrow[r, hook] \arrow[d] & I_n \odot \Delta^n \arrow[d] \\ Y^{(n-1)} \arrow[r, "i^{(n)}"'] & Y^{(n)} \end{tikzcd}$$

> *where $I_n \subseteq Y_n$ is the set of non-degenerate n-simplices of Y that are* not *in the image of $f : X \to Y$, $I_n \odot \partial\Delta^n \hookrightarrow I_n \odot \Delta^n$ is induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, and $I_n \odot \Delta^n \to Y^{(n)}$ is the tautological morphism induced by the inclusion $I_n \hookrightarrow Y_n$.*

In particular, if Y is a finite simplicial set, then there is a natural number d such that $i^{(n)} : Y^{(n-1)} \to Y^{(n)}$ is an isomorphism for all $n > d$.

Proof. We may assume without loss of generality that $f : X \to Y$ is the inclusion of a simplicial subset of Y. Let $Y^{(n)}$ be the union of X and the image of counit $\mathrm{sk}_n(Y) \to Y$, i.e. the smallest simplicial subset of Y containing X and all the n-simplices of Y, and let $i^{(n)} : Y^{(n-1)} \to Y^{(n)}$ be the inclusion. Then I_n is precisely the set of n-simplices of $Y^{(n)}$ that are *not* in $Y^{(n-1)}$, so we have the desired pushout diagram for each n. It is clear that the inclusions $Y^{(n-1)} \hookrightarrow Y$ define the required colimiting cocone. ■

In the language of § 0.5, what we have shown is that every monomorphism in **sSet** is a relative $\mathcal{I}$-cell complex, where $\mathcal{I} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$. Since the class of monomorphisms is closed under retracts, the following definition is justified:

Definition 1.2.21. A **cofibration of simplicial sets** is a monomorphism in **sSet**.

Remark 1.2.22. Cofibrations of simplicial sets have a homotopy extension property, albeit one that is weaker than what one might expect from the homotopy theory of topological spaces: see theorem 1.3.25.

1.3 Intrinsic homotopy

Prerequisites. §§ 1.2, 3.1, A.4.

Definition 1.3.1. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in **sSet**. An **intrinsic homotopy** $\alpha : f_0 \Rightarrow f_1$ is an edge of the exponential object $[X, Y]$ such that $d_1(\alpha) = f_0$ and $d_0(\alpha) = f_1$. (Note the subscripts!) We say f_0 and f_1 are **intrisically homotopic** if there is a zigzag of intrinsic homotopies connecting f_0 and f_1, and we write $f_0 \sim f_1$ in this case.

Remark 1.3.2. By the Yoneda lemma,

$$[X, Y]_1 \cong \mathbf{sSet}\left(\Delta^1, [X, Y]\right) \cong \mathbf{sSet}\left(\Delta^1 \times X, Y\right)$$

so an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$ is essentially the same thing as a morphism $\tilde{\alpha} : \Delta^1 \times X \to Y$ such that $\tilde{\alpha} \circ \left(\delta^1 \times \mathrm{id}_Y\right) = f_0$ and $\tilde{\alpha} \circ \left(\delta^0 \times \mathrm{id}_Y\right) = f_1$ (where we have suppressed the canonical isomorphism $X \cong \Delta^0 \times X$), just as in classical homotopy theory. Also,

$$\mathbf{sSet}\left(\Delta^1 \times X, Y\right) \cong \mathbf{sSet}\left(X, \left[\Delta^1, Y\right]\right)$$

so intrinsic homotopies $\alpha : f_0 \Rightarrow f_1$ correspond to morphisms $\hat{\alpha} : X \to \left[\Delta^1, Y\right]$ such that $\left[\delta^1, Y\right] \circ \hat{\alpha} = f_0$ and $\left[\delta^0, Y\right] \circ \hat{\alpha} = f_1$ (where we have suppressed the canonical isomorphism $\left[\Delta^0, Y\right] \cong Y$).

Remark 1.3.3. The notion of intrinsic homotopy is not well behaved for general simplicial sets Y. For example, the existence of an intrinsic homotopy $f_0 \Rightarrow f_1$ does not guarantee the existence of an "inverse" intrinsic homotopy $f_1 \Rightarrow f_0$, and even if we have intrinsic homotopies $f_0 \Rightarrow f_1$ and $f_1 \Rightarrow f_2$, there need not be an intrinsic homotopy $f_0 \Rightarrow f_2$.

¶ 1.3.4. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms and let $\alpha : f_0 \Rightarrow f_1$ be an intrinsic homotopy.

- Given a morphism $g : W \to X$, the intrinsic homotopy $\alpha g : f_0 \circ g \Rightarrow f_1 \circ g$ is the image of α under the induced morphism $[g, Y] : [X, Y] \to [W, X]$.
- Given a morphism $g : Y \to Z$, the intrinsic homotopy $g\alpha : g \circ f_0 \Rightarrow g \circ f_1$ is the image of α under the induced morphism $[X, g] : [X, Y] \to [X, Z]$.

Lemma 1.3.5. *The relation of intrinsic homotopy is a congruence on* **sSet**, *i.e. given morphisms $f_0, f_1 : X \to Y$ and $g_0, g_1 : Y \to Z$, if $f_0 \sim f_1$ and $g_0 \sim g_1$, then $g_0 \circ f_0 \sim g_1 \circ f_1$.* ■

Definition 1.3.6. The **intrinsic homotopy category of simplicial sets** is the category $\mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$ obtained by taking the quotient of **sSet** with respect to the congruence of intrinsic homotopy.

Remark 1.3.7. A parallel pair $f_0, f_1 : X \to Y$ in **sSet** are intrinsically homotopic if and only if they are in the same connected component of $[X, Y]$. In particular, we have a bijection of the form below,

$$\mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}(X, Y) \cong \pi_0[X, Y]$$

and it is natural as X and Y vary in **sSet**.

Remark 1.3.8. The set $\pi_0[X, Y]$ can be very far from what one expects geometrically. For instance, $\pi_0\left[\partial\Delta^2, \partial\Delta^2\right]$ contains only two elements, while the set of homotopy classes of continuous endomaps of the circle is countably infinite!

Lemma 1.3.9. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in* **sSet**. *Given an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$, for each simplicial set Z, there is an induced intrinsic homotopy $[\alpha, Z] : \left[f_0, Z\right] \Rightarrow \left[f_1, Z\right]$.*

Proof. Let $\tilde{\alpha} : \Delta^1 \times X \to Y$ be the morphism corresponding to $\alpha : f_0 \Rightarrow f_1$. Then we have a morphism $[\tilde{\alpha}, Z] : [Y, Z] \to \left[\Delta^1 \times X, Z\right]$. Proposition A.2.11 says there is a natural isomorphism

$$\left[\Delta^1 \times X, Z\right] \cong \left[\Delta^1, [X, Z]\right]$$

so $[\tilde{\alpha}, Z]$ corresponds to an intrinsic homotopy $[\alpha, Z]$ between two morphisms of type $[Y, Z] \to [X, Z]$; it is not hard to check that it is an intrinsic homotopy of type $\left[f_0, Z\right] \Rightarrow \left[f_1, Z\right]$. ■

Lemma 1.3.10. *Let X be a simplicial set, let $\mathbb{D}$ be a small category, let $f_0, f_1 : X \to \mathrm{N}(D)$ be a parallel pair of morphisms, and let $F_0, F_1 : \tau_1 X \to \mathbb{D}$ be their left adjoint transposes. Then there is a natural bijection between intrinsic homotopies $f_0 \Rightarrow f_1$ and natural transformations $F_0 \Rightarrow F_1$.*

Proof. Propositions 1.2.1 and A.2.13 give a natural isomorphism $[X, \mathrm{N}(\mathbb{D})] \cong \mathrm{N}\left(\left[\tau_1 X, \mathbb{D}\right]\right)$, and the claim is an immediate consequence. ■

Corollary 1.3.11. *If $F \dashv G : \mathbb{C} \to \mathbb{D}$ is an adjunction of small categories, then the induced morphisms in* Ho_{Δ^1} **sSet** *are mutually inverse.* ■

Definition 1.3.12. Let $f : X \to Y$ be a morphism in **sSet**.

- An **intrinsic homotopy left inverse** for f is a morphism $g : Y \to X$ such that $g \circ f$ and id_X are intrinsically homotopic.
- An **intrinsic homotopy right inverse** for f is a morphism $g : Y \to X$ such that $f \circ g$ and id_Y are intrinsically homotopic.

Definition 1.3.13. An **intrinsic homotopy equivalence** in **sSet** is a pair (f, g) where g (resp. f) is both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse for f (resp. g).

Example 1.3.14. The pair (σ_0^0, δ_1^0) is an intrinsic homotopy equivalence between the standard simplices Δ^0 and Δ^1. Multiplying by id_X, we deduce that X and $\Delta^1 \times X$ are naturally isomorphic in $\mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$.

Proposition 1.3.15. *Let $\gamma : \mathbf{sSet} \to \mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$ be the functor that sends each morphism to its intrinsic homotopy class. For any functor $F : \mathbf{sSet} \to \mathcal{C}$, the following are equivalent:*

(i) *For all simplicial sets X, $F(\delta_1^0 \times \mathrm{id}_X) : F(\Delta^0 \times X) \to F(\Delta^1 \times X)$ is an isomorphism in $\mathcal{C}$.*

(ii) *For all simplicial sets X, $F(\delta_1^1 \times \mathrm{id}_X) : F(\Delta^0 \times X) \to F(\Delta^1 \times X)$ is an isomorphism in $\mathcal{C}$.*

(iii) *For all simplicial sets X, $F(\sigma_0^0 \times \mathrm{id}_X) : F(\Delta^1 \times X) \to F(\Delta^0 \times X)$ is an isomorphism in $\mathcal{C}$.*

(iv) *For all parallel pairs $f_0, f_1 : X \to Y$ in $\mathbf{sSet}$, if $f_0 \sim f_1$, then $F f_0 = F f_1$.*

(v) *$F : \mathbf{sSet} \to \mathcal{C}$ factors through $\gamma : \mathbf{sSet} \to \mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$.*

Moreover, the factorisation is unique if it exists.

Proof. (i) $\Leftrightarrow$ (iii), (ii) $\Leftrightarrow$ (iii). Let $e \in \{0, 1\}$. The simplicial identity $\sigma_0^0 \circ \delta_1^e = \mathrm{id}$ implies that $F(\sigma_0^0 \times \mathrm{id}_X)$ is an isomorphism in $\mathcal{C}$ if and only if $F(\delta_0^e \times \mathrm{id}_X)$ is an isomorphism in $\mathcal{C}$.

(iii) $\Rightarrow$ (iv). It suffices to show that $F f_0 = F f_1$ whenever there is an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$, where $f_0, f_1 : X \to Y$ are an arbitrary parallel pair of morphisms in $\mathbf{sSet}$. Let $\tilde{\alpha} : \Delta^1 \times X \to Y$ be the morphism corresponding to $\alpha : f_0 \Rightarrow f_1$. Since $F(\sigma_0^0 \times \mathrm{id}_X)$ is an isomorphism, the uniqueness of inverses implies $F(\delta_1^0 \times \mathrm{id}_X) = F(\delta_1^1 \times \mathrm{id}_X)$; so, suppressing the canonical isomorphism $\Delta^0 \times X \cong X$, we obtain the required equation:

$$F f_0 = F\tilde{\alpha} \circ F(\delta_1^1 \times \mathrm{id}_X) = F\tilde{\alpha} \circ F(\delta_1^0 \times \mathrm{id}_X) = F f_1$$

(iv) $\Leftrightarrow$ (v). This is the universal property of the quotient by the congruence of intrinsic homotopy.

(v) $\Rightarrow$ (iii). Since $\sigma_0^0 \times \mathrm{id}_X : \Delta^1 \times X \to \Delta^0 \times X$ is (half of) an intrinsic homotopy equivalence, $\gamma(\sigma_0^0 \times \mathrm{id}_X)$ is an isomorphism in $\mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$. Hence, if the functor

$F : \mathbf{sSet} \to \mathcal{C}$ factors through $\gamma : \mathbf{sSet} \to \mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$, $F(\sigma_0^0 \times \mathrm{id}_X)$ must be an isomorphism in $\mathcal{C}$. ■

Corollary 1.3.16. *The functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ factors through $\gamma : \mathbf{sSet} \to \mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$.*

Proof. By proposition 1.2.4, $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ preserves finite products; but $\pi_0\Delta^1 \cong \pi_0\Delta^0 \cong 1$, so $\pi_0(\sigma_0^0 \times \mathrm{id}_X) : \pi_0(\Delta^1 \times X) \to \pi_0(\Delta^0 \times X)$ is a bijection for any simplicial set X. ■

Definition 1.3.17. A **contractible simplicial set** is a simplicial set that is isomorphic to Δ^0 in $\mathrm{Ho}_{\Delta^1}\,\mathbf{sSet}$.

Example 1.3.18. It is not hard to verify that each Δ^n is a contractible simplicial set: indeed, we may apply corollary 1.3.11.

Definition 1.3.19. Let X be a simplicial set.

- A **forward contracting homotopy** for X consists of a set X_{-1} and maps $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$ satisfying these identities:

$$\begin{aligned}
r \circ d_1^1 &= r \circ d_0^1 \\
r \circ s &= \mathrm{id} \\
d_0^1 \circ h^0 &= s \circ r \\
d_1^1 \circ h^0 &= \mathrm{id} \\
d_i^{n+1} \circ h^n &= h^{n-1} \circ d_i^n && \text{if } 0 \le i \le n \\
d_{n+1}^{n+1} \circ h^n &= \mathrm{id} \\
h^{n+1} \circ s_i^n &= s_i^{n+1} \circ h^n && \text{if } 0 \le i \le n \\
h^{n+1} \circ h^n &= s_{n+1}^{n+1} \circ h^n
\end{aligned}$$

- A **backward contracting homotopy** for X consists of a set X_{-1} and maps $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$ satisfying these identities:

$$\begin{aligned}
r \circ d_1^1 &= r \circ d_0^1 \\
r \circ s &= \mathrm{id} \\
d_0^1 \circ h^0 &= \mathrm{id}
\end{aligned}$$

$$\begin{aligned}
d_1^1 \circ h^0 &= s \circ r \\
d_0^{n+1} \circ h^n &= \mathrm{id} \\
d_{i+1}^{n+1} \circ h^n &= h^{n-1} \circ d_i^n && \text{if } 0 \le i \le n \\
h^{n+1} \circ h^n &= s_0^{n+1} \circ h^n \\
h^{n+1} \circ s_i^n &= s_{i+1}^{n+1} \circ h^n && \text{if } 0 \le i \le n
\end{aligned}$$

Proposition 1.3.20. *Let X be a simplicial set.*

- *Given a forward contracting homotopy for X, say $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$, there are unique morphisms $\tilde{r} : X \to \operatorname{disc} X_{-1}$ and $\tilde{s} : \operatorname{disc} X_{-1} \to X$ defined in degree 0 by r and s respectively, and we have $\tilde{r} \circ \tilde{s} = \mathrm{id}_{\operatorname{disc} X_{-1}}$ and an intrinsic homotopy $\mathrm{id}_X \Rightarrow \tilde{s} \circ \tilde{r}$; moreover, the canonical map $\pi_0 X \to X_{-1}$ is a bijection.*
- *Given a backward contracting homotopy for X, say $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$, there are unique morphisms $\tilde{r} : X \to \operatorname{disc} X_{-1}$ and $\tilde{s} : \operatorname{disc} X_{-1} \to X$ defined in degree 0 by r and s respectively, and we have $\tilde{r} \circ \tilde{s} = \mathrm{id}_{\operatorname{disc} X_{-1}}$ and an intrinsic homotopy $\tilde{s} \circ \tilde{r} \Rightarrow \mathrm{id}_X$; moreover, the canonical map $\pi_0 X \to X_{-1}$ is a bijection.*

Proof. The two claims are formally dual; we will prove the first version.

Observe that the definition implies that we have the following absolute coequaliser diagram:

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "d_1^1"] \arrow[r, shift right, "d_0^1"'] & X_0 \arrow[l, bend left=50, "h^0"] \arrow[r, "r"] & X_{-1} \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

Thus, as remarked in the proof of proposition 1.2.4, $\pi_0 X \cong X_{-1}$. As always, there is a unique morphism $\tilde{s} : \operatorname{disc} X_{-1} \to X$ whose degree 0 component is $s : X_{-1} \to X_0$, and the above observation ensures that there also exists a unique morphism $\tilde{r} : X \to \operatorname{disc} X_{-1}$ whose degree 0 component is $r : X_0 \to X_{-1}$.

Clearly, $\tilde{r} \circ \tilde{s} = \mathrm{id}_{\operatorname{disc} X_{-1}}$; we must show that $\tilde{s} \circ \tilde{r} \sim \mathrm{id}_X$. Let $\chi_n^i : [n] \to [1]$ denote the map in $\mathbf{\Delta}$ defined below:

$$\chi_n^i(j) = \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \ge i \end{cases}$$

It is not hard to see that $\mathbf{\Delta}([n], [1]) = \left\{ \chi_n^i \,\middle|\, 0 \le i \le n+1 \right\}$, and moreover we have the following identities:

$$\chi_{n+1}^i \circ \delta_{n+1}^j = \chi_n^i \qquad \text{if } 0 \le i \le j \le n+1$$

$$\begin{aligned}
\chi_{n+1}^{j} \circ \delta_{n+1}^{i} &= \chi_{n}^{j-1} && \text{if } 0 \leq i < j \leq n+2 \\
\chi_{n}^{i} \circ \sigma_{n}^{j} &= \chi_{n+1}^{i} && \text{if } 0 \leq i \leq j \leq n \\
\chi_{n}^{j} \circ \sigma_{n}^{i} &= \chi_{n+1}^{j+1} && \text{if } 0 \leq i < j \leq n+1
\end{aligned}$$

We construct by recursion a sequence of maps $H_n : X_n \times \mathbf{\Delta}([n],[1]) \to X_n$:

- For all x in X_0:

$$\begin{aligned}
H_0\left(x, \chi_0^1\right) &= x \\
H_0\left(x, \chi_0^0\right) &= s(r(x))
\end{aligned}$$

- For each x in X_{n+1}:

$$\begin{aligned}
H_{n+1}\left(x, \chi_{n+1}^{n+2}\right) &= x \\
H_{n+1}\left(x, \chi_{n+1}^{n+1}\right) &= h^n\left(d_{n+1}^{n+1}(x)\right) \\
H_{n+1}\left(x, \chi_{n+1}^{j}\right) &= s_n^n\left(H_n\left(d_{n+1}^{n+1}(x), \chi_n^j\right)\right) && \text{for } 0 \leq j \leq n
\end{aligned}$$

It is straightforward to check that these equations hold,

$$d_0^1 \circ H_1 = H_0 \circ d_0^1 \qquad d_1^1 \circ H_1 = H_0 \circ d_1^1 \qquad s_0^0 \circ H_0 = H_1 \circ s_0^0$$

so we assume for induction that these identities hold for some $n > 0$:

$$\begin{aligned}
d_i^k \circ H_k &= H_{k-1} \circ d_i^k && \text{for } 0 < k \leq n, 0 \leq i \leq k \\
s_i^k \circ H_k &= H_{k+1} \circ s_i^k && \text{for } 0 \leq k < n, 0 \leq i \leq k
\end{aligned}$$

Then, for $0 \leq i \leq n+1$,

$$\begin{aligned}
d_i^{n+1}\left(H_{n+1}\left(x, \chi_{n+1}^{n+2}\right)\right) &= d_i^{n+1}(x) \\
&= H_n\left(d_i^{n+1}(x), \chi_n^{n+1}\right) = H_n\left(d_i^{n+1}(x), \chi_{n+1}^{n+2} \circ \delta_{n+1}^{i}\right)
\end{aligned}$$

and, for $0 \leq i \leq n$,

$$\begin{aligned}
d_i^{n+1}\left(H_{n+1}\left(x, \chi_{n+1}^{n+1}\right)\right) &= d_i^{n+1}\left(h^n\left(d_{n+1}^{n+1}(x)\right)\right) \\
&= h^{n-1}\left(d_i^n\left(d_{n+1}^{n+1}(x)\right)\right) = h^{n-1}\left(d_n^n\left(d_i^{n+1}(x)\right)\right) \\
&= H_n\left(d_i^{n+1}(x), \chi_n^n\right) = H_n\left(d_i^{n+1}(x), \chi_n^n \circ \delta_{n+1}^{i}\right)
\end{aligned}$$

while, for $i = n+1$:

$$\begin{aligned} d_{n+1}^{n+1}\big(H_{n+1}\big(x,\chi_{n+1}^{n+1}\big)\big) &= d_{n+1}^{n+1}\big(h^n\big(d_{n+1}^{n+1}(x)\big)\big) \\ &= d_{n+1}^{n+1}(x) = H_n\big(d_{n+1}^{n+1}(x),\chi_n^{n+1}\big) = H_n\big(d_{n+1}^{n+1}(x),\chi_{n+1}^{n+1}\circ\delta_{n+1}^{n+1}\big) \end{aligned}$$

Similarly, for $0 \le j < n$,

$$\begin{aligned} d_{n+1}^{n+1}\Big(H_{n+1}\Big(x,\chi_{n+1}^{j}\Big)\Big) &= d_{n+1}^{n+1}\Big(s_n^n\Big(H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big)\Big)\Big) \\ &= H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big)H_n\Big(d_{n+1}^{n+1}(x),\chi_{n+1}^j\circ\delta_{n+1}^{n+1}\Big) \end{aligned}$$

$$\begin{aligned} d_n^{n+1}\Big(H_{n+1}\Big(x,\chi_{n+1}^{j}\Big)\Big) &= d_n^{n+1}\Big(s_n^n\Big(H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big)\Big)\Big) = H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big) \\ &= s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n\big(d_{n+1}^{n+1}(x)\big),\chi_{n-1}^j\Big)\Big) = s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n\big(d_n^{n+1}(x)\big),\chi_{n-1}^j\Big)\Big) \\ &= H_n\Big(d_n^{n+1}(x),\chi_n^j\Big) = H_n\Big(d_n^{n+1}(x),\chi_{n+1}^j\circ\delta_{n+1}^n\Big) \end{aligned}$$

and for $0 \le i < n$, we have:

$$\begin{aligned} d_i^{n+1}\Big(H_{n+1}\Big(x,\chi_{n+1}^{j}\Big)\Big) &= d_i^{n+1}\Big(s_n^n\Big(H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big)\Big)\Big) \\ &= s_{n-1}^{n-1}\Big(d_i^n\Big(H_n\Big(d_{n+1}^{n+1}(x),\chi_n^j\Big)\Big)\Big) = s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_i^n\big(d_{n+1}^{n+1}(x)\big),\chi_n^j\circ\delta_n^i\Big)\Big) \\ &= s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n\big(d_i^{n+1}(x)\big),\chi_n^j\circ\delta_n^i\Big)\Big) = H_n\Big(d_i^{n+1}(x),\chi_{n+1}^j\circ\delta_{n+1}^i\Big) \end{aligned}$$

On the other hand, for $0 \le i \le n$,

$$s_i^n\big(H_n\big(x,\chi_n^{n+1}\big)\big) = s_i^n(x) = H_{n+1}\big(s_i^n(x),\chi_{n+1}^{n+2}\big) = H_{n+1}\big(s_i^n(x),\chi_n^{n+1}\circ\sigma_n^i\big)$$

and for $0 \le i < n$,

$$\begin{aligned} s_i^n\big(H_n\big(x,\chi_n^n\big)\big) &= s_i^n\big(h^{n-1}\big(d_n^n(x)\big)\big) \\ &= h^n\big(s_i^{n-1}\big(d_n^n(x)\big)\big) = h^n\big(d_{n+1}^{n+1}\big(s_i^n(x)\big)\big) \\ &= H_{n+1}\big(s_i^n(x),\chi_{n+1}^{n+1}\big) = H_{n+1}\big(s_i^n(x),\chi_n^n\circ\sigma_n^i\big) \end{aligned}$$

while for $i = n$, we have:

$$\begin{aligned} s_n^n\big(H_n\big(x,\chi_n^n\big)\big) &= s_n^n\big(H_n\big(d_{n+1}^{n+1}\big(s_n^n(x)\big),\chi_n^n\big)\big) \\ &= H_{n+1}\big(s_n^n(x),\chi_{n+1}^n\big) = H_{n+1}\big(s_n^n(x),\chi_n^n\circ\sigma_n^n\big) \end{aligned}$$

Finally, for $0 \le i \le n$ and $0 \le j < n$:

$$\begin{aligned}
s_i^n\Big(H_n\Big(x, \chi_n^j\Big)\Big) &= s_i^n\Big(s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n(x), \chi_{n-1}^j\Big)\Big)\Big) \\
&= s_n^n\Big(s_i^{n-1}\Big(H_{n-1}\Big(d_n^n(x), \chi_{n-1}^j\Big)\Big)\Big) = s_n^n\Big(H_n\Big(s_i^{n-1}\big(d_n^n(x)\big), \chi_{n-1}^j \circ \sigma_{n-1}^i\Big)\Big) \\
&= s_n^n\Big(H_n\Big(d_{n+1}^{n+1}\big(s_i^n(x)\big), \chi_{n-1}^j \circ \sigma_{n-1}^i\Big)\Big) = H_{n+1}\Big(s_i^n(x), \chi_n^j \circ \sigma_n^i\Big)
\end{aligned}$$

We therefore have a morphism $H : X \times \Delta^1 \to X$ such that $H \circ \left(\mathrm{id}_X \times \delta_1^0\right) = \tilde{s} \circ \tilde{r}$ and $H \circ \left(\mathrm{id}_X \times \delta_1^1\right) = \mathrm{id}_X$. By remark 1.3.2, this is the required intrinsic homotopy. ■

Corollary 1.3.21. *A simplicial set X is contractible if the unique morphism $X \to \Delta^0$ admits a forward or backward contracting homotopy.* ■

Definition 1.3.22. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in **sSet**.

- f has the **forward homotopy lifting property** with respect to g if, for every commutative diagram of the following form,

$$\begin{tikzcd} Z \arrow[r, "z_0"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w_0"'] \arrow[ur, "h_0"] & Y \end{tikzcd}$$

  given intrinsic homotopies $\alpha : w_0 \Rightarrow w_1$ and $\beta : z_0 \Rightarrow z_1$ such that $\alpha g = f\beta$, there exist a morphism $h_1 : W \to X$ and an intrinsic homotopy $\gamma : h_0 \Rightarrow h_1$ such that $f \circ h_1 = w_1$, $h_1 \circ g = z_1$, $f\gamma = \alpha$, and $\gamma g = \beta$.

- f has the **backward homotopy lifting property** with respect to g if, for every commutative diagram of the following form,

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w_1"'] \arrow[ur, "h_1"] & Y \end{tikzcd}$$

  given intrinsic homotopies $\alpha : w_0 \Rightarrow w_1$ and $\beta : z_0 \Rightarrow z_1$ such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \beta$, there exist a morphism $h_0 : W \to X$ and an intrinsic homotopy $\gamma : h_0 \Rightarrow h_1$ such that $f \circ h_0 = w_0$, $h_0 \circ g = z_0$, $f\gamma = \alpha$, and $\gamma g = \beta$.

- f has the **intrinsic homotopy lifting property** with respect to g if f has *both* the forward and backward homotopy lifting properties with respect to g.

- f has the **forward** (resp. **backward**, **intrinsic**) **homotopy lifting property** with respect to the object W if f has the forward (resp. backward, intrinsic) homotopy lifting property with respect to the unique morphism $0 \to W$.

- g has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to f if f has the forward (resp. backward, intrinsic) homotopy lifting property with respect to g.

- g has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to the object X if g has the forward (resp. backward, intrinsic) homotopy extension property with respect to the unique morphism $X \to 1$.

Proposition 1.3.23. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in* **sSet***, and suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[g,X]}"] \arrow[ddr, bend right, "{[W,f]}"'] & & \\
& L(g,f) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,f]}"] \\
& {[W,Y]} \arrow[r, "{[g,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square. The following are equivalent:

(i) *f has the forward homotopy lifting property with respect to g.*

(ii) *The morphism $q : [W,X] \to L(g,f)$ has the right lifting property with respect to the horn inclusion $\Lambda^1_0 \hookrightarrow \Delta^1$.*

(iii) *Suppose we have a commutative diagram*

$$\begin{tikzcd}
\Lambda^1_0 \times Z \arrow[r, hook] \arrow[d, "\mathrm{id}_{\Lambda^1_0}\times g"'] & \Delta^1 \times Z \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_{\Delta^1}\times g"] & \\
\Lambda^1_0 \times W \arrow[r] \arrow[drr, hook, bend right] & V_0(g) \arrow[dr, dashed, "j"] & \\
& & \Delta^1 \times W
\end{tikzcd}$$

where the square in the upper left is a pushout square. Then the morphism $j : V_0(g) \to \Delta^1 \times W$ has the left lifting property with respect to $f : X \to Y$.

Symmetrically, the following are equivalent:

(i′) *f has the backward homotopy lifting property with respect to g.*

(ii′) *The morphism $q : [W, X] \to L(g, f)$ has the right lifting property with respect to the horn inclusion $\Lambda^1_1 \hookrightarrow \Delta^1$.*

(iii′) *Suppose we have a commutative diagram*

$$\begin{tikzcd}
\Lambda^1_1 \times Z \arrow[r, hook] \arrow[d, "\mathrm{id}_{\Lambda^1_1} \times g"'] & \Delta^1 \times Z \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_{\Delta^1} \times g"] & \\
\Lambda^1_1 \times W \arrow[r] \arrow[drr, hook, bend right] & V_1(g) \arrow[dr, dashed, "j"] & \\
& & \Delta^1 \times W
\end{tikzcd}$$

where the square in the upper left is a pushout square. Then the morphism $j : V_1(g) \to \Delta^1 \times W$ has the left lifting property with respect to $f : X \to Y$.

Proof. This is a special case of proposition 5.5.1: use remark 1.3.2 and the exponential adjunction. ■

Definition 1.3.24. A **horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where Λ^n_k is the union of the images of $\delta^0_n, \ldots, \delta^{k-1}_n, \delta^{k+1}_n, \ldots, \delta^n_n : \Delta^{n-1} \to \Delta^n$ in $\mathbf{sSet}$. In other words, Λ^n_k is the union of all the faces of Δ^n that include the k-th vertex.

Theorem 1.3.25. *Let $p : X \to Y$ be a morphism in* **sSet**. *The following are equivalent:*

(i) *$p : X \to Y$ has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ (for all $n \geq 1$ and $0 \leq k \leq n$).*

(ii) *$p : X \to Y$ has the intrinsic homotopy lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ (for all $n \geq 0$).*

(iii) *$p : X \to Y$ has the intrinsic homotopy lifting property with respect to any monomorphism in* **sSet**.

Proof. Combine propositions 1.3.23 and A.3.17 with either Theorem 2.1 in [GZ, Ch. IV] or Proposition 4.2 in [GJ, Ch. I]. □

Remark. The analogous theorem for cubical sets was announced as Theorem 2 in [Kan, 1955].

Remark 1.3.26. Let B^n be the closed unit ball in the euclidean space $\mathbb{R}^n$, let ∂B^n be its boundary, and let I be the closed unit interval $[0, 1]$. It is not hard to see that the inclusion $B^n \times \{0\} \hookrightarrow B^n \times I$ is isomorphic to the inclusion $B^n \times \{0\} \cup \partial B^n \times I \hookrightarrow B^n \times I$. Thus, a continuous map $p : X \to Y$ has the homotopy lifting property with respect to all B^n if and only if it has the homotopy lifting property with respect to all boundary inclusions $\partial B^n \hookrightarrow B^n$.

Unfortunately, **sSet** does not have the analogous property. Indeed, for any simplicial set X, the unique morphism $X \to 1$ has the intrinsic homotopy lifting property with respect to the n-simplices Δ^n, but it need not have the intrinsic right lifting property with respect to all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$.

Lemma 1.3.27. *Let $p : X \to Y$ be a morphism in* **sSet**.

(i) *If $p : X \to Y$ has the right lifting property with respect to the boundary inclusion $\partial\Delta^0 \hookrightarrow \Delta^0$, then $p_0 : X_0 \to Y_0$ is surjective.*

(ii) *If $p : X \to Y$ has the right lifting property with respect to the boundary inclusions $\partial\Delta^0 \hookrightarrow \Delta^0$ and $\partial\Delta^1 \hookrightarrow \Delta^1$, then $p : X \to Y$ has the intrinsic homotopy lifting property with respect to Δ^0, and $\pi_0 p : \pi_0 X \to \pi_0 Y$ is a bijection.*

Proof. (i). Let y be a vertex of Y. Then the right lifting property of $p : X \to Y$ with respect to the boundary inclusion $\partial\Delta^0 \hookrightarrow \Delta^0$ yields a vertex x of X such that $p_0(x) = y$, as required.

(ii). By proposition 1.3.23, $p : X \to Y$ has the intrinsic homotopy lifting property with respect to Δ^0 if and only if it has the right lifting property with respect to the horn inclusions $\Lambda^1_0 \hookrightarrow \Delta^1$ and $\Lambda^1_1 \hookrightarrow \Delta^1$. Since Δ^1 is 1-skeletal, we may apply propositions 1.2.20 and A.3.17 to deduce that $p : X \to Y$ does indeed have the aforementioned right lifting properties.

It remains to be shown that $\pi_0 : \pi_0 X \to \pi_0 Y$ is a bijection. We already know that $\pi_0 p : \pi_0 X \to \pi_0 Y$ is a surjection, so it suffices to show that it is also injective. Let x_0 and x_1 be vertices of X such that $y_0 = p(x_0)$ and $y_1 = p(x_1)$

are in the same connected component. We proceed by induction on the length of the shortest path (i.e. zigzag of edges) in Y connecting y_0 and y_1.

If y_0 and y_1 are connected by an edge of Y, then we may use the right lifting property of $p : X \to Y$ with respect to the boundary inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$ to find an edge of X connecting x_0 and x_1. Otherwise, we use the intrinsic homotopy lifting property of $p : X \to Y$ with respect to Δ^0 to reduce to the case where y_0 and y_1 are connected by a strictly shorter path. ■

Definition 1.3.28. An **anodyne extension of simplicial sets** is a member of the smallest class $\mathcal{A} \subset \mathbf{sSet}$ satisfying the following conditions:

- Every horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is in $\mathcal{A}$.

- $\mathcal{A}$ is closed under pushouts, i.e. given a pushout diagram in **sSet** of the form below,

$$\begin{tikzcd} Z' \arrow[r] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ W' \arrow[r] & W \end{tikzcd}$$

  if $g' : Z' \to W'$ is in $\mathcal{A}$, then so is $g : Z \to W$.

- $\mathcal{A}$ is closed under (finite and) transfinite composition, i.e. given an ordinal α and a colimit-preserving functor $X : \alpha \to \mathbf{sSet}$ such that the morphisms $X(\beta) \to X(\gamma)$ are in $\mathcal{A}$, the induced morphism $X(0) \to \varinjlim_{\beta<\alpha} X(\beta)$ is also in $\mathcal{A}$.

- $\mathcal{A}$ is closed under retracts, i.e. given a commutative diagram in **sSet** of the form below,

$$\begin{tikzcd} Z' \arrow[r, "i_Z"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "g'"'] & Z \arrow[r, "r_Z"'] \arrow[d, "g"'] & Z' \arrow[d, "g'"] \\ W' \arrow[r, "i_W"] \arrow[rr, bend right, "\mathrm{id}"'] & W \arrow[r, "r_W"] & W' \end{tikzcd}$$

  if $g : Z \to W$ is in $\mathcal{A}$, then so is $g' : Z' \to W'$.

Lemma 1.3.29. *Let $f : X \to Y$ and $g : Z \to W$ be monomorphisms in* **sSet**. *Suppose the square in the diagram below is a pushout square in* **sSet***:*

$$\begin{tikzcd}
X \times Z \arrow[r, "\mathrm{id}_X \times g"] \arrow[d, "f \times \mathrm{id}_Z"'] & X \times W \arrow[d] \arrow[ddr, bend left, "f \times \mathrm{id}_W"] & \\
Y \times Z \arrow[r] \arrow[drr, bend right, "\mathrm{id}_Y \times g"'] & (X \times W) \cup^{X \times Z} (Y \times Z) \arrow[dr, dashed, "f \mathbin{\Box} g"] & \\
 & & Y \times W
\end{tikzcd}$$

(i) *The morphism $f \mathbin{\Box} g : (X \times W) \cup^{X \times Z} (Y \times Z) \to Y \times W$ is a monomorphism.*

(ii) *If at least one of $f : X \to Y$ or $g : Z \to W$ is an anodyne extension, then so is $f \mathbin{\Box} g : (X \times W) \cup^{X \times Z} (Y \times Z) \to Y \times W$.*

Proof. (i). Using the fact that limits and colimits in **sSet** can be calculated degreewise, this reduces to a well-known fact about **Set**.

(ii). See Proposition 2.2 in [GZ], or Corollary 4.6 in [GJ, Ch. I]. □

Definition 1.3.30. Let L be any simplicial set, let K be a simplicial subset of L, and let $f_0, f_1 : L \to Y$ be a parallel pair of morphisms in **sSet**. An **intrinsic homotopy** $f_0 \Rightarrow f_1$ relative to K is an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$ such that the image of α under morphism $[L, Y] \to [K, Y]$ (induced by the inclusion $K \hookrightarrow L$) is a degenerate edge. (In particular, the restrictions of f_0 and f_1 along $K \hookrightarrow L$ must be equal.) We write $\pi_{(L,K)}(Y, y)$ for the set of morphisms $L \to Y$ whose restriction along $K \hookrightarrow L$ is $y : K \to Y$, *modulo* the equivalence relation generated by intrinsic homotopy relative to K.

Remark 1.3.31. For fixed L and K, the assignment $(Y, y) \mapsto \pi_{(L,K)}(Y, y)$ is clearly the object part of a functor $^{K/}\mathbf{sSet} \to \mathbf{Set}$. Indeed, we may construct it as follows: given $y : K \to Y$, form the following pullback square in **sSet**,

$$\begin{tikzcd}
{[L, Y]_y} \arrow[r] \arrow[d] & {[L, Y]} \arrow[d] \\
\Delta^0 \arrow[r] & {[K, Y]}
\end{tikzcd}$$

where the morphism $[L, Y] \to [K, Y]$ is induced by the inclusion $K \hookrightarrow L$ and the morphism $\Delta^0 \to [K, Y]$ corresponds to y (considered as a vertex of $[K, Y]$); then $\pi_{(L,K)}(Y, y)$ can be identified with $\pi_0[L, Y]_y$.

Definition 1.3.32. Let L be any simplicial set and let K be a simplicial subset of L. The **relative cylinder** on (L, K) is the simplicial set $C(L, K)$ defined by the following pushout diagram,

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, hook] \arrow[d] & L \times \Delta^1 \arrow[d, "q"] \\ K \arrow[r] & C(L, K) \end{tikzcd}$$

where $K \times \Delta^1 \to K$ is the projection and $K \times \Delta^1 \hookrightarrow L \times \Delta^1$ is induced by the inclusion $K \hookrightarrow L$.

Remark 1.3.33. Let $j_0, j_1 : L \to C(L, K)$ be the morphisms obtained by composing with $q : L \times \Delta^1 \to C(L, K)$ the two morphisms $L \to L \times \Delta^1$ induced by the two vertex inclusions $\Delta^0 \to \Delta^1$. There is a natural bijection between the set of intrinsic homotopies $f_0 \Rightarrow f_1$ relative to K and the set of morphisms $h : C(L, K) \to Y$ such that $h \circ j_0 = f_0$ and $h \circ j_1 = f_1$.

Definition 1.3.34. Let $f : X \to Y$ be a morphism in **sSet**. With other notation as above, we say that f has the **homotopical right lifting property** with respect to $K \hookrightarrow L$ if, for each commutative diagram the form below,

$$\begin{tikzcd} K \arrow[r, "\partial x"] \arrow[d, hook] & X \arrow[d, "f"] \\ L \arrow[r, "y"'] & Y \end{tikzcd}$$

there exist a morphism $x : L \to X$ and a homotopy $\alpha : y \Rightarrow f \circ x$ relative to K, or equivalently, morphisms $x : L \to X$ and $h : C(L, K) \to Y$ making the following diagram commute:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] \arrow[rr, bend left, "\partial x"] & L \arrow[r, dashed, "x"] \arrow[d, "j_1"] & X \arrow[d, "f"] \\ L \arrow[r, "j_0"'] \arrow[rr, bend right, "y"'] & C(L, K) \arrow[r, dashed, "h"'] & Y \end{tikzcd}$$

Proposition 1.3.35. *Let $f : X \to Y$ be a morphism in* **sSet** *and let $\mathcal{A}$ be the class of pairs (L, K) such that f has the homotopical right lifting property with respect to $K \hookrightarrow L$.*

(i) *$\mathcal{A}$ is closed under coproducts for small families.*

(ii) *$\mathcal{A}$ is closed under pushout.*

(iii) *$\mathcal{A}$ is closed under retracts.*

Proof. See Lemma 3.4 in [Dugger and Isaksen, 2004]. □

1.4 Kan complexes

Prerequisites. §§1.3, 3.1, 3.7, A.4.

We have seen in the previous section that the notion of intrinsic homotopy is not well behaved for general simplicial sets. To remedy this, we shall (temporarily) restrict our attention to Kan complexes. These are simplicial sets with the so-called "extension property", and they are named in honour of Kan [1955], who first observed the importance of the aforementioned property.

Definition 1.4.1. A **Kan fibration** is a morphism $f : X \to Y$ in **sSet** that has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, where $n \geq 1$ and $0 \leq k \leq n$. A **Kan complex** is a simplicial set X such that the unique morphism $X \to 1$ is a Kan fibration.

Remark 1.4.2. In other words, a Kan complex is a simplicial set X satisfying the **Kan condition**: every horn $\alpha' : \Lambda^n_k \to X$ has a **filler**, i.e. a morphism $\alpha : \Delta^n \to X$ (equivalently, an n-simplex of X) such that α' is the restriction along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

Proposition 1.4.3. *Let X be a simplicial set. The following are equivalent:*

(i) *X is a Kan complex.*

(ii) *X has the intrinsic homotopy extension property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$.*

(iii) *X has the intrinsic homotopy extension property with respect to any monomorphism in* **sSet**.

Proof. This is a special case of theorem 1.3.25. ∎

Lemma 1.4.4. *If X is a Kan complex, then the fundamental category $\tau_1 X$ is a groupoid, and the unit $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is an epimorphism.*

Proof. Let x, y, and z be vertices in X, and let $f : x \to y$ and $g : y \to z$ be edges in X.[6] Then the pair (f, g) defines a horn $\Lambda_1^2 \to X$, and so by the Kan condition, there exists a 2-simplex α of X such that $d_2(\alpha) = f$ and $d_0(\alpha) = g$. By remark remark 1.2.3, the composite $g \bullet f$ defined in $\tau_1 X$ must correspond to the edge $d_1(\alpha)$. Since the arrows in $\tau_1 X$ are generated by the edges of X, we conclude by induction that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on vertices and edges.

Similarly, given an edge $f : x \to y$, the Kan condition ensures that there exist two 2-simplices β and γ such that

$$\begin{aligned} d_2(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_x \\ d_0(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_y \end{aligned}$$

where $\mathrm{id}_x : x \to x$ is the edge $s_0(x)$, and $\mathrm{id}_y : y \to y$ is the edge $s_0(y)$. Together with the argument in the previous paragraph, this shows that $\tau_1 X$ is a groupoid.

Finally, to show that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on n-simplices for $n \geq 2$, we simply observe that an n-simplex of $\mathrm{N}(\tau_1 X)$ is just a string of n composable edges of X, so we may appeal to the Kan condition again to obtain the corresponding n-simplex of X. ∎

Corollary 1.4.5. *If X is a Kan complex, then the unit $\eta_X : X \to \mathrm{N}(\pi_1 X)$ is an epimorphism.*

Proof. Since $\tau_1 X$ is already a groupoid, the canonical functor $\tau_1 X \to \pi_1 X$ must be an isomorphism. (See remark 1.2.9.) ∎

Proposition 1.4.6. *Let X be a Kan complex and let $\alpha_0, \alpha_1 : x_0 \to x_1$ be edges in X. The following are equivalent:*

(i) *$\alpha_0 = \alpha_1$ in the fundamental groupoid $\pi_1 X$.*

(ii) *There exists a 2-simplex σ of X such that $d_0(\sigma) = s_0(x_1)$, $d_1(\sigma) = \alpha_1$, and $d_2(\sigma) = \alpha_0$.*

[6] Recall definition 1.1.14.

(iii) *There exists an edge* $\beta : \alpha_0 \to \alpha_1$ *in the exponential object* $[\Delta^1, X]$ *such that* $[\delta^1, X](\beta) = s_0(x_0)$ *and* $[\delta^0, X](\beta) = s_0(x_1)$.

Proof. (i) ⇔ (ii). See Proposition 1.2.3.9 in [HTT].

(i) ⇔ (iii). See paragraph 5.2 in [GZ]. □

Proposition 1.4.7. *Let* $\mathcal{I}$ *and* $\mathcal{I}'$ *be the following subsets of* mor **sSet***:*

$$\mathcal{I} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$$
$$\mathcal{I}' = \left\{\Lambda^n_k \hookrightarrow \Delta^n \,\middle|\, n \geq 1, 0 \leq k \leq n\right\}$$

(i) *There exist a pair of functorial factorisation systems on* **sSet**, *one inducing a weak factorisation system cofibrantly generated by* $\mathcal{I}$, *and the other inducing a weak factorisation system cofibrantly generated by* $\mathcal{I}'$.

(ii) *A morphism is* $\mathcal{I}'$*-injective if and only if it is a Kan fibration, and every* $\mathcal{I}'$*-cofibration is a monomorphism (but not vice versa).*

(iii) *A morphism is a* $\mathcal{I}$*-cofibration if and only if it is a monomorphism, and every* $\mathcal{I}$*-injective morphism is a Kan fibration (but not vice versa).*

Proof. (i). Since **sSet** is a locally finitely presentable category, we may apply Quillen's small object argument (theorem 0.5.12).

(ii). The definition of 'Kan fibration' is exactly the definition of '$\mathcal{I}$-injective morphism'; on the other hand, the class of monomorphisms is closed under pushout, transfinite composition, and retracts in **Set**, so the same is true for **sSet**, and thus, by corollary 0.5.13, every $\mathcal{I}$-cofibration must be a monomorphism.

(iii). To prove that $\mathrm{inj}^{\mathcal{I}}\, \mathcal{C} \supseteq \mathrm{inj}^{\mathcal{I}'}\, \mathcal{C}$, it is enough to check that $\mathcal{I} \subseteq \mathrm{cof}_{\mathcal{I}'}\, \mathcal{C}$. Since every morphism in $\mathcal{I}$ is a monomorphism, it will suffice to show that $\mathrm{cell}_{\mathcal{I}'}\, \mathcal{C}$ contains all monomorphisms; but this is an immediate corollary of proposition 1.2.20. ■

Corollary 1.4.8. *Let* $i : Z \to W$ *be a morphism in* **sSet**. *The following are equivalent:*

(i) $i : Z \to W$ *is an anodyne extension.*

(ii) $i : Z \to W$ *has the left lifting property with respect to any Kan fibration.*

(iii) $i : Z \to W$ *is a retract of a relative* $\mathcal{I}'$*-cell complex.*

Proof. (i) ⇒ (ii). Apply proposition A.3.17.

(ii) ⇒ (iii). This is a special case of corollary 0.5.13.

(iii) ⇒ (i). By definition, the class of anodyne extensions is closed under pushout, transfinite composition, and retracts. ∎

Definition 1.4.9. A **trivial Kan fibration** is a morphism in **sSet** that has the right lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, where $n \geq 0$.

Remark 1.4.10. Proposition 1.4.7 implies that a trivial Kan fibration is the same thing as as morphism in **sSet** that has the right lifting property with respect to any monomorphism. In particular, trivial Kan fibrations are Kan fibrations.

Proposition 1.4.11. *If $p : X \to Y$ is a trivial Kan fibration, then $p : X \to Y$ is fibrewise contractible, i.e. there exist a morphism $s : Y \to X$ and an intrinsic homotopy $\alpha : \mathrm{id}_X \Rightarrow s \circ p$ satisfying the following conditions:*

- $p \circ s = \mathrm{id}_Y$.
- *αs is the trivial homotopy $s \Rightarrow s$.*
- *$p\alpha$ is the trivial homotopy $p \Rightarrow p$.*

Moreover, given a monomorphism $i : Y' \to Y$ and any morphism $s' : Y' \to Y$ such that $p \circ s' = i'$, the morphism $s : Y \to X$ given above may be chosen so that $s \circ i = s'$.

Proof. Since $i : Y' \to Y$ is a monomorphism, the right lifting property of $p : X \to Y$ yields a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$ and $s \circ i = s'$. We then obtain a commutative diagram of the form below,

$$\begin{array}{ccc} X \cup^Y X & \xrightarrow{(\!(\mathrm{id}_X, s\circ p)\!)} & X \\ {\scriptstyle (\!(j_0, j_1)\!)}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ C(X,Y) & \xrightarrow[p\circ r]{} & Y \end{array}$$

where $C(X,Y)$ is the relative cylinder, $X \cup^Y X$ is the pushout of $s : Y \to X$ along itself, the morphisms $j_0, j_1 : X \to C(X,Y)$ are defined as in remark 1.3.33, and $r : C(X,Y) \to X$ is defined by the following commutative diagram:

$$\begin{tikzcd}
Y \times \Delta^1 \arrow[r, "s \times \mathrm{id}_{\Delta^1}"] \arrow[d, "\mathrm{proj}"'] & X \times \Delta^1 \arrow[d] \arrow[ddr, bend left, "\mathrm{proj}"] & \\
Y \arrow[r] \arrow[drr, bend right, "s"'] & C(X,Y) \arrow[dr, dashed, "r"] & \\
& & X
\end{tikzcd}$$

It is not hard to see that $(\!(j_0, j_1)\!) : X \cup^Y X \to C(X,Y)$ is a monomorphism, so there must exist $h : C(X,Y) \to Y$ making the evident triangles commute. The corresponding intrinsic homotopy $\mathrm{id}_X \Rightarrow s \circ p$ is then the desired α. ■

Proposition 1.4.12. *Let $\mathcal{K}$ be the full subcategory of* **sSet** *spanned by the finite simplicial sets.*

(i) *The class of monomorphisms that are in $\mathcal{K}$ is the smallest class containing the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ that is closed under composition and pushouts.*

(ii) *The class of anodyne extensions that are in $\mathcal{K}$ is the smallest class containing the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ that is closed under composition, pushouts, and retracts.*

Proof. (i). This is a corollary of proposition 1.2.20.

(ii). Corollary 1.4.8 says that every anodyne extension in **sSet** is a retract of a relative $\mathcal{I}'$-cell complex, where $\mathcal{I}'$ is the set of all horn inclusions. More precisely, if $g : Z \to W$ is an anodyne extension, then there is a commutative diagram in **sSet** of the form below,

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Z \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & Z \arrow[d, "g"] \\
W \arrow[r, "i_W"] \arrow[rr, bend right, "\mathrm{id}"'] & W' \arrow[r, "r_W"] & W
\end{tikzcd}$$

where $i : Z \to W'$ is a relative $\mathcal{I}'$-cell complex. Suppose W is a finite simplicial set. Proposition 1.1.18 says that finite simplicial sets are $\aleph_0$-compact objects in **sSet**, so by considering a sequential presentation for $i : Z \to W'$, we see that $g : Z \to W$ is a retract of some relative $\mathcal{I}'$-cell complex that admits an $\aleph_0$-small presentation. In particular, if Z is a finite simplicial set, then so is W' (by lemma 0.2.18). Hence, the class of anodyne extensions in $\mathcal{K}$ is the smallest class containing $\mathcal{I}'$ that is closed under composition, pushouts, and retracts. ■

Proposition 1.4.13. *Let $f : X \to Y$ be a morphism in* **sSet** *and, for each n-simplex $\alpha : \Delta^n \to Y$, let $f_\alpha : X_\alpha \to \Delta^n$ be defined by the pullback diagram in* **sSet** *shown below:*

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d, "f_\alpha"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

(i) *$f : X \to Y$ is a Kan fibration if and only if each $f_\alpha : X_\alpha \to \Delta^n$ is a Kan fibration.*

(ii) *$f : X \to Y$ is a trivial Kan fibration if and only if each $f_\alpha : X_\alpha \to \Delta^n$ is a trivial Kan fibration.*

Proof. This is a straightforward exercise. ◊

Corollary 1.4.14.

(i) *Let $(X_i \mid i \in I)$ be a small family of simplicial sets. The coproduct $\coprod_{i\in I} X_i$ is a Kan complex if and only if each X_i is a Kan complex.*

(ii) *Let $(f_i : X_i \to Y_i \mid i \in I)$ be a small family of morphisms of simplicial sets. The coproduct $\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is a Kan fibration if and only if each $f_i : X_i \to Y_i$ is a Kan fibration.*

(iii) *Let $(f_i : X_i \to Y_i \mid i \in I)$ be a small family of morphisms of simplicial sets. The coproduct $\coprod_{i\in I} f_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is a trivial Kan fibration if and only if each $f_i : X_i \to Y_i$ is a trivial Kan fibration.*

Proof. Given the previous proposition and the fact that coproducts in **sSet** are disjoint and stable under pullback, it suffices to observe that any $\Delta^n \to \coprod_{i\in I} Y_i$ must factor through one of the coproduct insertions $Y_j \to \coprod_{i\in I} Y_i$. ■

Proposition 1.4.15. *Let $i : Z \to W$ be a cofibration in* **sSet** *and let $p : X \to Y$ be a Kan fibration. Suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[i,X]}"] \arrow[ddr, bend right, "{[W,p]}"'] & & \\
& L(i,p) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,p]}"] \\
& {[W,Y]} \arrow[r, "{[i,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *The unique morphism $q : [W,X] \to L(i,p)$ making the diagram commute is a Kan fibration.*

(ii) *If $i : Z \to W$ is an anodyne extension, then $q : [W,X] \to L(i,p)$ is a trivial Kan fibration.*

(iii) *If $p : Z \to W$ is a trivial Kan fibration, then so is $q : [W,X] \to L(i,p)$.*

Proof. This is a special case of proposition 5.5.1: use lemma 1.3.29 and the exponential adjunction. ■

Corollary 1.4.16.

(i) *If $p : X \to Y$ is a Kan fibration (resp. trivial Kan fibration), then for all simplicial sets W, the morphism $[W,p] : [W,X] \to [W,Y]$ is also a Kan fibration (resp. trivial Kan fibration).*

(ii) *If $i : Z \to W$ is a monomorphism (resp. anodyne extension) of simplicial sets and X is a Kan complex, then the morphism $[i,X] : [W,X] \to [Z,X]$ is a Kan fibration (resp. trivial Kan fibration).*

(iii) *If W is any simplicial set and X is a Kan complex, then $[W,X]$ is also a Kan complex.*

Proof. (i). Take $Z = \varnothing$; noting that the canonical morphism $\varnothing \to W$ is a cofibration, and that $[\varnothing, p] : [\varnothing, X] \to [\varnothing, Y]$ is an isomorphism, the proposition above then implies $[W,p] : [W,X] \to [W,Y]$ is a Kan fibration (resp. trivial Kan fibration).

(ii). Take $Y = 1$; since $[W, 1] \to [Z, 1]$ is an isomorphism, the proposition above implies $[i, X] : [W, X] \to [Z, X]$ is a Kan fibration (resp. trivial Kan fibration).

(iii). Noting that $[\varnothing, X]$ is a terminal object in **sSet**, we apply claim (ii) to the case $Z = \varnothing$ to obtain the desired conclusion. ■

Proposition 1.4.17. *For any simplicial set X and any Kan complex Y, the relation $\rightsquigarrow$ on $\mathbf{sSet}(X, Y)$ defined by*

$$f_0 \rightsquigarrow f_1 \text{ if and only if there exists an intrinsic homotopy } f_0 \Rightarrow f_1$$

is an equivalence relation.

Proof. The relation $\rightsquigarrow$ is certainly reflexive whether or not Y is a Kan complex. By corollary 1.4.16, the exponential object $[X, Y]$ is a Kan complex; so recalling lemma 1.4.4, the transitivity of $\rightsquigarrow$ may be deduced from the fact that the unit $\eta_{[X,Y]} : [X, Y] \to \mathrm{N}\big(\tau_1[X, Y]\big)$ is an epimorphism, and the symmetry of $\rightsquigarrow$ corresponds to the fact that $\tau_1[X, Y]$ is a groupoid. ■

Proposition 1.4.18. *Let X and Y be Kan complexes. If $i : X \to Y$ is an anodyne extension, then there exist a morphism $r : Y \to X$ and an intrisic homotopy $\alpha : \mathrm{id}_Y \Rightarrow i \circ r$ satisfying the the following conditions:*

- $r \circ i = \mathrm{id}_X$.
- *αi is the trivial homotopy $i \Rightarrow i$.*

Proof. By hypothesis, the unique morphism $X \to 1$ is a Kan fibration, so corollary 1.4.8 implies there is a morphism $r : Y \to X$ such that $r \circ i = \mathrm{id}_X$. We then obtain the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r, "c \circ i"] \arrow[d, "i"'] & \left[\Delta^1, Y\right] \arrow[d, "\langle p_0, p_1 \rangle"] \\
Y \arrow[r, "\langle \mathrm{id}_Y, i \circ r \rangle"'] & Y \times Y
\end{tikzcd}$$

where $p_0, p_1 : \left[\Delta^1, Y\right] \to Y$ are the morphisms induced by the coface morphisms $\delta^1_0, \delta^0_0 : \Delta^0 \to \Delta^1$ (respectively) and $c : Y \to \left[\Delta^1, Y\right]$ is induced by the codegeneracy morphism $\sigma^0_0 : \Delta^1 \to \Delta^0$. Supressing a canonical isomorphism

$\left[\partial\Delta^1, Y\right] \cong Y \times Y$, we see that corollary 1.4.16 implies $\langle p_0, p_1\rangle : \left[\Delta^1, Y\right] \to Y \times Y$ is a Kan fibration. Thus, there exists a morphism $h : Y \to \left[\Delta^1, Y\right]$ making the evident triangles commute, and the corresponding intrinsic homotopy $\mathrm{id}_Y \Rightarrow i \circ r$ is then the desired α. ■

We will now define the homotopy groups of a Kan complex.

Definition 1.4.19. Let n be a positive integer and let X be a Kan complex.

- The **based n-loop fibration** on X is the Kan fibration $\Omega^n(X) \to X$ defined by the following pullback diagram in **sSet**,

$$\begin{tikzcd} \Omega^n(X) \arrow[r] \arrow[d] & {[\Delta^n, X]} \arrow[d] \\ X \arrow[r] & {[\partial\Delta^n, X]} \end{tikzcd}$$

  where $[\Delta^n, X] \to [\partial\Delta^n, X]$ is the Kan fibration induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ and $X \to [\partial\Delta^n, X]$ is the morphism induced by $\partial\Delta^n \to \Delta^0$.

- Let x be a vertex of X. The **based n-loop space** of (X, x) is the Kan complex $\Omega^n(X, x)$ defined by the following pullback diagram in **sSet**,

$$\begin{tikzcd} \Omega^n(X, x) \arrow[r] \arrow[d] & \Omega^n(X) \arrow[d] \\ \Delta^0 \arrow[r] & X \end{tikzcd}$$

  where $\Delta^0 \to X$ is the morphism corresponding to the vertex x. The **n-th homotopy group** of (X, x) is defined by $\pi_n(X, x) = \pi_0\Omega^n(X, x)$.

Remark 1.4.20. In other words, $\pi_n(X, x)$ is the set of morphisms $\Delta^n \to X$ whose restriction along $\partial\Delta^n \hookrightarrow \Delta^n$ factors through the morphism $\Delta^0 \to X$ corresponding to x, modulo the equivalence relation that identifies two morphisms $\Delta^n \to X$ if they are intrinsically homotopic relative to $\partial\Delta^n$.

Proposition 1.4.21. *Let n be a positive integer.*

(i) *The assignment $(X, x) \mapsto \pi_n(X, x)$ extends to a functor $\pi_n : {}^{\Delta^0/}\mathbf{Kan} \to \mathbf{Grp}$, and $\pi_n(X, x)$ is abelian for $n > 1$.*

(ii) *The functor* $\pi_n : {}^{\Delta^0/}\mathbf{Kan} \to \mathbf{Grp}$ *preserves finite products and colimits for small filtered diagrams.*

(iii) *Let* (X, x) *and* (Y, y) *be pointed Kan complexes. If* $f_0, f_1 : (X, x) \to (Y, y)$ *are a parallel pair of morphisms for which there exists an intrinsic homotopy* $f_0 \Rightarrow f_1$ *relative to* x *(considered as a subcomplex of* X*), then* $\pi_n f_0 = \pi_n f_1$.

Proof. (i). See Lemma 7.1 and Theorem 7.2 in [GJ, Ch. I]. Functoriality is straightforward.

(ii). It is not hard to check that the functor $\Omega^n : {}^{\Delta^0/}\mathbf{Kan} \to \mathbf{sSet}$ preserves all limits and colimits for small filtered diagrams, and $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ preserves finite products and all colimits by proposition 1.2.4, so $\pi_n : {}^{\Delta^0/}\mathbf{Kan} \to \mathbf{Set}$ preserves finite products and colimits for small filtered diagrams. But the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ creates finite products and colimits for small filtered diagrams, so the claim follows.

(iii). Use paragraph 1.3.4 and remark 1.4.20. □

Definition 1.4.22. The **homotopy category of Kan complexes** is the full subcategory $\mathbf{H} \subseteq \mathrm{Ho}_{\Delta^1} \mathbf{sSet}$ spanned by the Kan complexes.

Proposition 1.4.23. *Let* $\mathbf{Kan}$ *be the full subcategory of* $\mathbf{sSet}$ *spanned by the Kan complexes and let* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ *be the functor that sends each morphism to its intrinsic homotopy class. For any functor* $F : \mathbf{Kan} \to \mathcal{C}$*, the following are equivalent:*

(i) *For all Kan complexes* X, $F\left[\delta_1^0, X\right] : F\left[\Delta^1, X\right] \to F\left[\Delta^0, X\right]$ *is an isomorphism in* $\mathcal{C}$.

(ii) *For all Kan complexes* X, $F\left[\delta_1^1, X\right] : F\left[\Delta^1, X\right] \to F\left[\Delta^0, X\right]$ *is an isomorphism in* $\mathcal{C}$.

(iii) *For all simplicial sets* X, $F\left[\sigma_0^0, X\right] : F\left[\Delta^1, X\right] \to F\left[\Delta^0, X\right]$ *is an isomorphism in* $\mathcal{C}$.

(iv) *For all parallel pairs* $f_0, f_1 : X \to Y$ *in* $\mathbf{Kan}$*, if* $f_0 \sim f_1$*, then* $F f_0 = F f_1$.

(v) $F : \mathbf{Kan} \to \mathcal{C}$ *factors through* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$.

Moreover, the factorisation is unique if it exists.

Proof. The proof is similar to that of proposition 1.3.15. (Use corollary 1.4.16 to deduce that $\left[\Delta^1, X\right]$ is a Kan complex if X is.) ◊

Proposition 1.4.24. *Let* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ *be the functor that sends a morphism of Kan complexes to its intrinsic homotopy class.*

(i) *The functor* $\boldsymbol{\pi}$ *is full, surjective on objects, and preserves finite products and finite coproducts.*

(ii) **Kan** *is closed under products for all small families in* **sSet**, *and* **H** *has products for finite families.*

(iii) **Kan** *and* **H** *are cartesian closed categories, and* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ *is a cartesian closed functor.*

(iv) *A morphism* $f : X \to Y$ *in* **Kan** *admits an intrinsic homotopy inverse if and only if* $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ *is an isomorphism in* **H**.

Proof. (i). The construction of **H** as $\pi_0[\underline{\mathbf{Kan}}]$ ensures that $\boldsymbol{\pi}$ is indeed a functor.

(ii). It is clear from the construction of $\pi_0 Z$ as a coequaliser that $\chi_Z : Z_0 \to \pi_0 Z$ is a surjection; thus $\boldsymbol{\pi}$ is a full functor. It is obviously surjective on objects, and it preserves finite products and finite coproducts because π_0 preserves finite products.

(iii). By proposition A.3.17, the class of Kan fibrations is closed under products for small families, so **Kan** is as well. By claim (ii), **H** inherits finite products from **Kan**.

(iv). By proposition 1.4.15, $[Y, K]$ is a Kan complex whenever K is, which combined with claim (iii) implies **Kan** is cartesian closed. Proposition A.2.11 says we have natural isomorphisms $[X \times Y, K] \cong [X, [Y, K]]$, so it follows that we have natural bijections

$$\pi_0[X \times Y, K] \cong \pi_0[X, [Y, K]]$$

for all X, Y, and K in **Kan**, and this descends along $\boldsymbol{\pi}$ to make **H** cartesian closed.

(v). The definition says $f : X \to Y$ is a weak homotopy equivalence if and only if $\pi_0[f, K] : \pi_0[Y, K] \to \pi_0[X, K]$ is a bijection for all Kan complexes K; but this is natural in K, so the Yoneda lemma implies this happens if and only if $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ is an isomorphism in $\mathbf{H}$. ∎

Definition 1.4.25. Let n be an integer, $n \ge -2$. An **n-connected morphism of Kan complexes** is a morphism $f : X \to Y$ in $\mathbf{sSet}$, where X and Y are Kan complexes, such that the following conditions are satisfied:

- If $n \ge -1$, then $\pi_0 f : \pi_0 X \to \pi_0 Y$ is a surjection.
- If $n \ge 0$, then $\pi_0 f : \pi_0 X \to \pi_0 Y$ is a bijection and, for all vertices x of X, the homomorphism $\pi_1 f : \pi_1(X, x) \to \pi_1(Y, f(x))$ is a surjection.
- If $n \ge 1$, then for all $1 \le m \le n$ and all vertices x of X, the homomorphism $\pi_m f : \pi_m(X, x) \to \pi_m(Y, f(x))$ is an isomorphism, and $\pi_n f : \pi_{n+1}(X, x) \to \pi_{n+1}(Y, f(x))$ is a surjection.

An ∞-connected morphism of Kan complexes is one that is n-connected for all $n \ge -2$.

Proposition 1.4.26. *The class of ∞-connected morphisms of Kan complexes has the 2-out-of-3 and 2-out-of-6 properties.*[7]

Proof. This is a straightforward check (using lemma A.4.14). ◆

Theorem 1.4.27. *Let $p : X \to Y$ be a Kan fibration. If X and Y are Kan complexes, then the following are equivalent:*

(i) *$p : X \to Y$ is a trivial Kan fibration.*

(ii) *$p : X \to Y$ is an ∞-connected morphism of Kan complexes.*

Proof. (i) ⇒ (ii). Lemma 1.3.27 says $\pi_0 f : \pi_0 X \to \pi_0 Y$ is a bijection. Fix a positive integer n and a vertex x of X. Then proposition 1.4.11 implies that there exist a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$ and an intrinsic homotopy $\alpha : \mathrm{id}_X \Rightarrow s \circ p$ relative to x (considered as a subcomplex of X), so we may apply proposition 1.4.21 to deduce that $\pi_n p : \pi_n(X, x) \to \pi_n(Y, p(x))$ is an isomorphism.

(ii) ⇒ (i). See Theorem 7.10 in [GJ, Ch. I]. □

[7] See definition A.4.13.

Corollary 1.4.28. *Let X and Y be Kan complexes. If $f_0, f_1 : X \to Y$ are intrinsically homotopic, then for all positive integers n and all vertices x, there exists a commutative diagram of the form below:*

$$\begin{tikzcd}
\pi_n(X,x) \arrow[r, "\pi_n f_0"] \arrow[d, equal] & \pi_n\big(Y, f_0(x)\big) \arrow[d, "\cong"] \\
\pi_n(X,x) \arrow[r, "\pi_n f_1"'] & \pi_n\big(Y, f_1(x)\big)
\end{tikzcd}$$

Proof. We may assume without loss of generality that there is an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$. Let $h : X \to \left[\Delta^1, Y\right]$ be the corresponding morphism. It is clear that the coface morphisms $\delta_0^e : \Delta^0 \to \Delta^1$ are isomorphic to the horn inclusions $\Lambda_k^1 \hookrightarrow \Delta^1$ (where $k = 0$ if $e = 1$ and $k = 1$ if $e = 0$), so by corollary 1.4.16, the morphisms $\left[\delta_0^e, X\right] : \left[\Delta^1, Y\right] \to \left[\Delta^0, Y\right]$ are trivial Kan fibrations. Thus, we have the following commutative diagram,

$$\begin{tikzcd}
\pi_n(X,x) \arrow[r, "\pi_n f_0"] \arrow[d, equal] & \pi_n\big(Y, f_0(x)\big) \\
\pi_n(X,x) \arrow[r, "\pi_n h"] \arrow[d, equal] & \pi_n\big(\left[\Delta^1, Y\right], h(x)\big) \arrow[u, "\pi_n p_0"'] \arrow[d, "\pi_n p_1"] \\
\pi_n(X,x) \arrow[r, "\pi_n f_1"'] & \pi_n\big(Y, f_1(x)\big)
\end{tikzcd}$$

where $p_0, p_1 : \left[\Delta^1, Y\right] \to Y$ are the morphisms induced by $\delta_0^1, \delta_0^0 : \Delta^0 \to \Delta^1$ (respectively). But theorem 1.4.27 implies that $\pi_n p_0$ and $\pi_n p_1$ are isomorphisms, so we are done. ■

The homotopy groups of a Kan complex are a complete homotopy invariant. More precisely, we have the following analogue of a theorem of Whitehead [1949]:

Theorem 1.4.29 (Whitehead)**.** *Let X and Y be Kan complexes. For any morphism $f : X \to Y$, the following are equivalent:*

(i) *$f : X \to Y$ admits an intrinsic homotopy inverse.*

(ii) *$f : X \to Y$ is an ∞-connected morphism of Kan complexes.*

(iii) *$f : X \to Y$ admits a factorisation of the form $q \circ j$, where j is an anodyne extension and q is a trivial Kan fibration.*

Proof. (i) ⇒ (ii). By corollary 1.3.16, $\pi_0 f : \pi_0 X \to \pi_0 Y$ is a bijection, and using corollary 1.4.28, it is not hard to see that $\pi_n f : \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism for all positive integers n and all vertices x of X.

(ii) ⇒ (iii). Proposition 1.4.7 says we may factor f as $p \circ j$, where j is an anodyne extension and p is a Kan fibration; note that the domain of p is automatically a Kan complex. By proposition 1.4.18, anodyne extensions of Kan complexes admit homotopy inverses, so i is an ∞-connected morphism of Kan complexes; hence, applying proposition 1.4.26, we may deduce that p is ∞-connected morphism if (and only if) $f : X \to Y$ is ∞-connected. But theorem 1.4.27 says p is ∞-connected if and only if it is a trivial Kan fibration, so we are done.

(iii) ⇒ (i). Propositions 1.4.11 and 1.4.18 say that both p and i admits intrinsic homotopy inverses, so the same is true for $f = p \circ i$. ■

Corollary 1.4.30. *Let $i : X \to Y$ be a monomorphism. If X and Y are Kan complexes, then the following are equivalent:*

(i) *$i : X \to Y$ is an anodyne extension.*

(ii) *$i : X \to Y$ is an ∞-connected morphism of Kan complexes.*

Proof. (i) ⇒ (ii). Apply theorem 1.4.29.

(ii) ⇒ (i). If i is an ∞-connected morphism of Kan complexes, then i admits a factorisation of the form $q \circ j$, where j is an anodyne extension and q is a trivial Kan fibration. The right lifting property of q implies there is a morphism h such that $p \circ h = \mathrm{id}_Y$ and $h \circ i = j$; in particular, i is a retract of j. Thus, i is an anodyne extension. ■

Theorem 1.4.31. *Let* **Kan** *be the category of Kan complexes. Then* **Kan** *is a category of fibrant objects, where*

- *the weak equivalences are the ∞-connected morphisms,*
- *the fibrations are the Kan fibrations, and*
- *the trivial fibrations are the trivial Kan fibrations.*

Moreover, this makes **Kan** *a saturated homotopical category.*

Proof. First, note that theorem 1.4.27 and Whitehead's theorem (1.4.29) imply that the fibrations that are weak equivalences are precisely the trivial Kan fibrations. Thus, we may apply proposition A.3.17 to deduce that axioms B and C are satisfied. Axiom E is satisfied by definition. Axiom A is proposition 1.4.26; moreover, **Kan** is a saturated homotopical category, by proposition 1.4.24 and lemma 3.1.8. Finally, using corollary 1.4.16, it is not hard to see that $\left[\Delta^1, X\right]$ is (the object part of) a path object for X (provided X is a Kan complex), so axiom D is also satisfied. ■

Proposition 1.4.32. *Let $p : X \to Y$ and $p' : X' \to Y'$ be Kan fibrations. Given a pullback diagram in* **sSet** *of the form below,*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

if $g : Y' \to Y$ is an ∞-connected morphism of Kan complexes, then so is $f : X' \to X$.

Proof. In view of theorem 1.4.31, this is a special case of proposition 3.7.14. ■

Lemma 1.4.33. *Let $i : Z \to W$ be a monomorphism of simplicial sets and let $f : X \to Y$ be a morphism of Kan complexes. Consider the following commutative diagram in* **sSet**,

$$\begin{tikzcd} {[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[i,X]}"] \arrow[ddr, bend right, "{[W,f]}"'] & & \\ & L(i,f) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,f]}"] \\ & {[W,Y]} \arrow[r, "{[i,Y]}"'] & {[Z,Y]} \end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *If $f : X \to Y$ is an ∞-connected morphism of Kan complexes, then so is $q : [W,X] \to L(i,f)$.*

(ii) *If $i : Z \to W$ is an anodyne extension of simplicial sets, then $q : [W, X] \to L(i, f)$ is an ∞-connected morphism of Kan complexes.*

Proof. Since X and Y are Kan complexes, proposition 1.4.15 (plus proposition A.3.17) implies that every object in the commutative diagram is a Kan complex and that $[i, X] : [W, X] \to [Z, X]$ and $[i, Y] : [W, Y] \to [Z, Y]$ are Kan fibrations.

(i). Suppose $f : X \to Y$ is an ∞-connected morphism of Kan complexes. Recalling paragraph 1.3.4, we see that theorem 1.4.27 and Whitehead's theorem (1.4.29) imply that $[W, f] : [W, X] \to [W, Y]$ and $[Z, f] : [Z, X] \to [Z, Y]$ are also ∞-connected. Proposition 1.4.32 then says that the morphism $L(i, f) \to [W, Y]$ is also ∞-connected, so we may use the 2-out-of-3 property (proposition 1.4.26) to deduce that $q : [W, X] \to L(i, f)$ is indeed ∞-connected.

(ii). Suppose $i : Z \to W$ is an anodyne extension of simplicial sets. Then proposition 1.4.15 says $[i, X] : [W, X] \to [Z, X]$ and $[i, Y] : [W, Y] \to [Z, Y]$ are trivial Kan fibrations, and proposition A.3.17 says that the morphism $L(i, f) \to [Z, X]$ is also a trivial Kan fibration. Thus, theorem 1.4.27 and proposition 1.4.26 imply that $q : [W, X] \to L(i, f)$ is indeed ∞-connected. ■

Lemma 1.4.34. *Let $f : X \to Y$ be a morphism be a morphism in* **sSet**, *let L be a simplicial set, and let $K \subseteq L$ and $J \subseteq K$ be simplicial subsets. If Y is a Kan complex and $f : X \to Y$ has the homotopical right lifting property with respect to both $J \hookrightarrow K$ and $K \hookrightarrow L$, then $f : X \to Y$ also has the homotopical right lifting property with respect to $J \hookrightarrow L$.*

Proof. See Lemma 3.4 in [Dugger and Isaksen, 2004]. □

Theorem 1.4.35. *Let $f : X \to Y$ be a morphism of Kan complexes. The following are equivalent:*

(i) *$f : X \to Y$ is an ∞-connected morphism of Kan complexes.*

(ii) *$f : X \to Y$ has the homotopical right lifting property with respect to all monomorphisms between finite simplicial sets.*

(iii) *$f : X \to Y$ has the homotopical right lifting property with respect to all boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$.*

Proof. See Proposition 4.1 in [Dugger and Isaksen, 2004]. □

1.5 The Kan–Quillen model structure

Prerequisites. §§1.3, 1.4, 3.7, 4.1, 5.2, A.3.

In [1967], Quillen constructed an axiomatic framework for doing homotopy theory in abstract categories, which he called 'closed model categories', and showed that **sSet** can be endowed with a model structure such that the resulting homotopy theory is equivalent in a strong sense to the homotopy theory of topological spaces.

The following characterisation of weak homotopy equivalences appears in [Quillen, 1967, Ch. II, §3]; we follow Joyal and Tierney [2008] in taking it as our definition. Recalling that $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ from proposition 1.2.4 is the functor sending a simplicial set X to the set π_0 of its connected components,

Definition 1.5.1. A **weak homotopy equivalence of simplicial sets** is a morphism $f : W \to Z$ such that, for every Kan complex K, the induced map

$$\pi_0[f, K] : \pi_0[Z, K] \to \pi_0[W, K]$$

is a bijection of sets.

Lemma 1.5.2. **sSet**, *with the class of weak homotopy equivalences, is a saturated homotopical category. In particular, the class of weak homotopy equivalences of simplicial sets has the 2-out-of-3 property and is closed under retracts.*

Proof. Apply lemma 3.1.8. ∎

Proposition 1.5.3 (Formal Whitehead theorem)**.**

(i) *If a morphism in* **sSet** *admits an intrinsic homotopy left inverse and an intrinsic homotopy right inverse, then it is a weak homotopy equivalence.*

(ii) *A morphism in* **Kan** *is a weak homotopy equivalence if and only if it admits an intrinsic homotopy inverse.*

Proof. (i). If $f : X \to Y$ admits an intrinsic homotopy left inverse (resp. an intrinsic homotopy right inverse), then $\pi_0[f, K] : \pi_0[Y, K] \to \pi_0[X, K]$ is injective (resp. surjective) for all simplicial sets K. In particular, $f : X \to Y$ is a weak homotopy equivalence as soon as it has both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse.

(ii). Let $f : X \to Y$ be a weak homotopy equivalence of Kan complexes. The definition says $f : X \to Y$ is a weak homotopy equivalence if and only if $\pi_0[f, K] : \pi_0[Y, K] \to \pi_0[X, K]$ is a bijection for all Kan complexes K, so (recalling remark 1.3.7) we may obtain an intrinsic homotopy left inverse for $f : X \to Y$ by taking $K = X$, say $g : Y \to X$. By naturality, the following diagram commutes:

$$\begin{tikzcd}
\pi_0[Y, Y] \arrow[r, "{\pi_0[f,Y]}"] \arrow[d, "{\pi_0[Y,g]}"'] & \pi_0[X, Y] \arrow[d, "{\pi_0[Y,g]}"] \\
\pi_0[Y, X] \arrow[r, "{\pi_0[f,X]}"'] \arrow[d, "{\pi_0[Y,f]}"'] & \pi_0[X, X] \arrow[d, "{\pi_0[X,f]}"] \\
\pi_0[Y, Y] \arrow[r, "{\pi_0[f,Y]}"'] & \pi_0[X, Y]
\end{tikzcd}$$

Thus, by chasing the homotopy class of id_Y, we deduce that $g : Y \to X$ is also an intrinsic homotopy right inverse for $f : X \to Y$, as required. ■

Lemma 1.5.4. *Anodyne extensions are weak homotopy equivalences.*

Proof. If $i : X \to Y$ is an anodyne extension, then $[i, K] : [Y, K] \to [X, K]$ is a trivial Kan fibration for all Kan complexes K, by corollary 1.4.16. Applying lemma 1.3.27, we then deduce that $i : X \to Y$ is a weak homotopy equivalence. ■

Proposition 1.5.5. *There exist a functor $R : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ satisfying the following condition:*

- *For all simplicial sets X, RX is a Kan complex and $i_X : X \to RX$ is an anodyne extension.*

Moreover, any such functor R preserves and reflects weak homotopy equivalences.

Proof. Such (R, i) can be constructed using Quillen's small object argument (theorem 0.5.12); see also proposition 4.1.24. Given any such (R, i), consider the following commutative diagram in **sSet**:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\
Y \arrow[r, "i_Y"'] & RY
\end{tikzcd}$$

Using proposition 1.5.10 and the 2-out-of-3 property of weak homotopy equivalences, we see that $f : X \to Y$ is a weak homotopy equivalence if and only if $Rf : RX \to RY$ is a weak homotopy equivalence. ■

Definition 1.5.6. A **weakly contractible simplicial set** is a simplicial set X for which the unique morphism $X \to \Delta^0$ in **sSet** is a weak homotopy equivalence.

Remark 1.5.7. Proposition 1.5.3 implies that every contractible simplicial set is also weakly contractible.

Proposition 1.5.8. *Let X be a Kan complex. The following are equivalent:*

(i) *X is contractible (as a simplicial set).*

(ii) *X is weakly contractible (as a simplicial set).*

(iii) *$X \to \Delta^0$ is a trivial Kan fibration.*

Proof. (i) ⇒ (ii). Apply proposition 1.5.3.

(ii) ⇒ (iii). Use theorem 1.4.27.

(iii) ⇒ (i). This is a special case of proposition 1.4.11. ■

Remark 1.5.9. Not all weak homotopy equivalences admit an intrinsic homotopy inverse. For instance, if X is the nerve of the following category,

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots$$

then every morphism $\Delta^0 \to X$ is an anodyne extension (because the class of anodyne extensions is closed under pushout and transfinite composition), but none of them admit an intrinsic homotopy right inverse. In particular, X is weakly contractible but not contractible.

Proposition 1.5.10.

(i) *A Kan fibration $p : X \to Y$ is trivial if and only if it is a weak homotopy equivalence.*

(ii) *A monomorphism $i : Z \to W$ is an anodyne extension if and only if it is a weak homotopy equivalence.*

Proof. (i). See Proposition 3.4.1 in [Joyal and Tierney, 2008].

(ii). See Lemma 7 in [Quillen, 1967, Ch. II, §3] or Proposition 3.4.2 in [Joyal and Tierney, 2008]. □

Theorem 1.5.11. **sSet**, *regarded as a* **sSet***-enriched category via its cartesian closed structure, is a simplicial*[8] *strongly* $(\aleph_0, \aleph_1)$*-combinatorial model category where*

- *the cofibrations are the monomorphisms in* **sSet**,
- *the fibrations are the Kan fibrations, and*
- *the weak equivalences are the weak homotopy equivalences.*

This is the **Kan–Quillen model structure on simplicial sets**.

Proof. It is clear that there exist countable sets of generating cofibrations and generating trivial cofibrations whose domains and codomains are finite simplicial sets, and it is not hard to see that there are only finitely many morphisms between any two finite simplicial sets. Thus it suffices to verify that **sSet** is a simplicial model category.

We know **sSet** has limits and colimits for all small diagrams and is a cartesian closed category, so it satisfies axioms CM1 and SM0. Using the definition of weak homotopy equivalence given above, lemma 1.5.2 implies axiom CM2 is satisfied. Proposition 1.4.7 plus theorem 4.1.12 then shows that the announced cofibrations, fibrations, and weak equivalences do indeed constitute a model structure on **sSet**. Finally, we note that proposition 1.4.15 is precisely the condition required by axiom SM7. ■

Proposition 1.5.12. *Let* $\mathcal{W}$ *be the full subcategory of* $[\mathbb{2}, \mathbf{sSet}]$ *spanned by the weak homotopy equivalences. Then* $\mathcal{W}$ *is closed under colimits for small filtered diagrams in* $[\mathbb{2}, \mathbf{sSet}]$.

Proof. Since **sSet** is a strongly $(\aleph_0, \aleph_1)$-combinatorial model category, we may apply corollary 5.2.16. ■

Corollary 1.5.13. *Let* $\mathcal{A}$ *be the full subcategory of* $[\mathbb{2}, \mathbf{sSet}]$ *spanned by the anodyne extensions. Then* $\mathcal{A}$ *is closed under colimits for small filtered diagrams in* $[\mathbb{2}, \mathbf{sSet}]$.

[8] See definition 2.4.1.

Proof. Theorem 0.2.13 implies that the full subcategory of $[\mathbb{2}, \mathbf{sSet}]$ spanned by the monomorphisms is closed under colimits for small filtered diagrams, so the claim is a consequence of propositions 1.5.10 and 1.5.12. ■

Proposition 1.5.14. *Let $\left(f_i : X_i \to Y_i \,\middle|\, i \in I\right)$ be a small family of morphisms of simplicial sets. The following are equivalent:*

(i) *Each $f_i : X_i \to Y_i$ is a weak homotopy equivalence.*

(ii) *The coproduct $\coprod_{i \in I} f_i : \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i$ is a weak homotopy equivalence.*

Proof. Proposition 1.4.7 says we can factor each $f_i : X_i \to Y_i$ as an anodyne extension followed by a Kan fibration, and since the class of anodyne extensions is closed under coproducts, by lemma 1.5.2 and proposition 1.5.10, it suffices to prove the claim in the special case where each $f_i : X_i \to Y_i$ is a Kan fibration; but this was shown by corollary 1.4.14. ■

Proposition 1.5.15. *Let $f : W \to Z$ be a weak homotopy equivalence of simplicial sets and let X be any simplicial set.*

(i) *The morphism $f \times \mathrm{id}_X : W \times X \to Z \times X$ is a weak homotopy equivalence.*

(ii) *If X is a Kan complex, then $[f, X] : [Z, X] \to [W, X]$ is a weak homotopy equivalence.*

(iii) *If W and Z are Kan complexes, then $[X, f] : [X, W] \to [X, Z]$ is a weak homotopy equivalence.*

Proof. (i). We must show that, for all Kan complexes K, the induced map

$$\pi_0\left[f \times \mathrm{id}_X, K\right] : \pi_0[Z \times X, K] \to \pi_0[W \times X, K]$$

is a bijection. However, we have a commutative diagram

$$\begin{tikzcd}
\pi_0[Z \times X, K] \arrow[r, "{\pi_0\left[\mathrm{id}_X \times f, K\right]}"] \arrow[d, "\cong"'] & \pi_0[W \times X, K] \arrow[d, "\cong"] \\
\pi_0[Z, [X, K]] \arrow[r, "{\pi_0[f, [X, K]]}"'] & \pi_0[W, [X, K]]
\end{tikzcd}$$

and (by corollary 1.4.16) $[X, K]$ is a Kan complex, so $\pi_0[f, [X, K]]$ is a bijection; hence, $\pi_0\left[f \times \mathrm{id}_X, K\right]$ is indeed a bijection for all Kan complexes K.

(ii). If X is a Kan complex, then corollary 1.4.16 says that $[-, X]$ is a right Quillen functor; but every simplicial set is cofibrant, so Ken Brown's lemma (4.3.6) implies $[-, X]$ preserves weak homotopy equivalences.

(iii). Similarly, for any simplicial set X, $[X, -]$ is a right Quillen functor, and so Ken Brown's lemma implies $[X, -]$ preserves weak homotopy equivalences between Kan complexes. ■

Theorem 1.5.16. $\mathbf{sSet}^{\mathrm{op}}$ *is a category of fibrant objects, where*

- *the weak equivalences are the weak homotopy equivalences,*
- *the fibrations are the monomorphisms in* **sSet**, *and*
- *the trivial fibrations are anodyne extensions.*

Proof. Recall that proposition 1.5.3 says the anodyne extensions are precisely the monomorphisms (in **sSet**) that are weak homotopy equivalences. Thus, we may apply proposition A.3.17 to deduce that axioms B and C are satisfied. It is easy to verify axiom E. Axiom A is lemma 1.5.2. Finally, using proposition 1.5.15, it is not hard to see that $\Delta^1 \times X$ (in **sSet**) is (the object part of) a path object for X (in $\mathbf{sSet}^{\mathrm{op}}$), so axiom D is also satisfied. ■

Lemma 1.5.17. *Given a commutative diagram in* **sSet** *of the form below,*

$$\begin{tikzcd} Y_0 \arrow[d, "g"'] & X_0 \arrow[l, "i_0"'] \arrow[r, "t_0"] \arrow[d, "f"] & T_0 \arrow[d, "h"] \\ Y_1 & X_1 \arrow[l, "i_1"] \arrow[r, "t_1"'] & T_1 \end{tikzcd}$$

if $i_0 : X_0 \to Y_0$ and $i_1 : X_1 \to Y_1$ are monomorphisms, $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$ are anodyne extensions, and $h : T_0 \to T_1$ is a weak homotopy equivalence, then the induced morphism

$$T_0 \cup^{X_0} Y_0 \to T_1 \cup^{X_1} Y_1$$

is a weak homotopy equivalence.

Proof. In view of theorem 1.5.16, this is (the formal dual of) lemma 3.7.28. ■

Proposition 1.5.18.

(i) *Equipping* **Set** *with the discrete model structure,*[9] *the adjunction*

$$\pi_0 \dashv \operatorname{disc} : \mathbf{Set} \to \mathbf{sSet}$$

is a Quillen adjunction.[10]

(ii) *For every map* $f : X \to Y$*, the morphism* $\operatorname{disc} f : \operatorname{disc} X \to \operatorname{disc} Y$ *is a Kan fibration.*

(iii) *The functor* $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ *sends weak homotopy equivalences to bijections.*

Proof. (i). Since every map is a cofibration in the discrete model structure on **Set**, it is enough (by proposition 4.3.2) to show that $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ sends anodyne extensions in **sSet** to bijections; and by proposition 1.4.12, it suffices to show that the maps $\pi_0 \Lambda^n_k \to \pi_0 \Delta^n$ induced by the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$. But this is an immediate consequence of the fact that each Λ^n_k and Δ^n is connected.

(ii). Every map is a fibration in the discrete model structure on **Set**, and $\operatorname{disc} : \mathbf{Set} \to \mathbf{sSet}$ is a right Quillen functor, so each $\operatorname{disc} f : \operatorname{disc} X \to \operatorname{disc} Y$ is indeed a Kan fibration.

(iii). Every simplicial set is cofibrant, so this is a consequence of Ken Brown's lemma (4.3.6). ∎

Proposition 1.5.19. *Let* $\mathcal{W}$ *be a subcategory of* **sSet** *that satisfies these conditions:*

- *Every identity morphism in* **sSet** *is in* $\mathcal{W}$*.*
- $\mathcal{W}$ *has the 2-out-of-3 property in* **sSet**.
- *For every simplicial set* X*, the projection* $p_X : X \times \Delta^1 \to X$ *is in* $\mathcal{W}$*.*

Then:

(i) *Given a parallel pair* $f_0, f_1 : X \to Y$ *in* **sSet** *and an intrinsic homotopy* $\alpha : f_0 \Rightarrow f_1$*, the morphism* f_0 *is in* $\mathcal{W}$ *if and only if* f_1 *is in* $\mathcal{W}$*.*

[9] See example 4.1.5.
[10] See definition 4.3.1.

(ii) *If $\mathcal{W}$ has the special 2-out-of-4 property, then every trivial Kan fibration is in $\mathcal{W}$.*

(iii) *If $\mathcal{W}$ is closed under retracts* or *has the 2-out-of-6 property in* **sSet**, *then every trivial Kan fibration is in $\mathcal{W}$.*

Proof. (i). This follows from remark 1.3.2.

(ii). This is a special case of proposition 5.4.34.

(iii). Apply lemma A.4.17. ∎

Lemma 1.5.20. *Let $\mathcal{W}$ be a subcategory of* **sSet** *that satisfies these conditions:*

(a) *The class of monomorphisms that are in $\mathcal{W}$ is closed under pushout, composition, and retracts.*

(b) *$\mathcal{W}$ has the 2-out-of-3 property in* **sSet**, *and for all finite simplicial sets X, the morphism* $\mathrm{id} : X \to X$ *is in $\mathcal{W}$.*

(c) *For all natural numbers n, the unique morphism $\Delta^n \to \Delta^0$ is in $\mathcal{W}$.*

Then every horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is in $\mathcal{W}$.

Proof. We proceed by induction on n. For $n = 1$, observe that conditions (a) and (b) together imply that every isomorphism of finite simplicial sets is in $\mathcal{W}$, and so we may use the 2-out-of-3 property to deduce that the horn inclusions $\Lambda^1_0 \hookrightarrow \Delta^1$ and $\Lambda^1_1 \hookrightarrow \Delta^1$ are in $\mathcal{W}$.

Now, suppose that the horn inclusions $\Lambda^m_k \hookrightarrow \Delta^m$ are in $\mathcal{W}$ for all $m < n$. It is not hard to see that the horn Λ^n_l can be constructed by adjoining m copies of Δ^m along various horn inclusions (for $0 < m < n$), so conditions (a) and (b) imply that the l-th vertex $\Delta^0 \to \Lambda^n_l$ is in $\mathcal{W}$. Condition (c) says that the unique morphism $\Delta^n \to \Delta^0$ is in $\mathcal{W}$, so we can then use the 2-out-of-3 property to deduce that the horn inclusion $\Lambda^n_l \hookrightarrow \Delta^n$ is in $\mathcal{W}$. ∎

Proposition 1.5.21. *Let $\mathcal{W}$ be a subcategory of* **sSet** *that satisfies these conditions:*

(a) *The class of monomorphisms that are in $\mathcal{W}$ is closed under pushout, transfinite composition, and retracts.*

(b) *$\mathcal{W}$ has the 2-out-of-3 property in* **sSet**, *and for all simplicial sets X, the morphism* $\mathrm{id} : X \to X$ *is in $\mathcal{W}$.*

(c) *For all natural numbers n, the unique morphism $\Delta^n \to \Delta^0$ is in $\mathcal{W}$.*

Then every weak homotopy equivalence is in $\mathcal{W}$.

Proof. Lemma 1.5.20 says that the horn inclusions are in $\mathcal{W}$, so condition (a) implies that all anodyne extensions are in $\mathcal{W}$. Notice that, if $p : X \to Y$ is a trivial Kan fibration, then there is a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$, and by proposition 1.5.10, $s : Y \to X$ is an anodyne extension. Hence, condition (b) implies that all trivial Kan fibrations are in $\mathcal{W}$ as well. But every weak homotopy equivalence factors as an anodyne extension followed by a trivial Kan fibration (by proposition 1.5.10), so every weak homotopy equivalence is in $\mathcal{W}$. ■

Corollary 1.5.22. *The subcategory of weak homotopy equivalences in* **sSet** *is the smallest subcategory satisfying the conditions in the proposition.*

Proof. Proposition 1.5.10 says that the class of monomorphisms that are weak homotopy equivalences is precisely the class of anodyne extensions, which has the required closure properties by definition. Thus, the class of weak homotopy equivalences satisfies condition (a), and the remaining conditions are easily verified. ■

Corollary 1.5.23. *Let $\mathcal{M}$ be a derivable category. If F :* **sSet** *$\to \mathcal{M}$ is a functor that preserves cofibrations and colimits for small diagrams, then the following are equivalent:*

(i) *F :* **sSet** *$\to \mathcal{M}$ preserves trivial cofibrations.*

(ii) *F :* **sSet** *$\to \mathcal{M}$ preserves weak equivalences.*

(iii) *For each natural number n, the morphism $F(\Delta^n) \to F\left(\Delta^0\right)$ is a weak equivalence in $\mathcal{M}$.*

Proof. (i) ⇒ (ii). This is Ken Brown's lemma (4.3.6).

(ii) ⇒ (iii). The unique morphism $\Delta^n \to \Delta^0$ is a weak homotopy equivalence, so its image under F : **sSet** $\to \mathcal{M}$ must be a weak equivalence in $\mathcal{M}$.

(iii) ⇒ (i). Let $\mathcal{W}$ be the subcategory of **sSet** consisting of those morphisms that are sent to weak equivalences by F : **sSet** $\to \mathcal{M}$. Since monomorphisms are sent to cofibrations in **sSet**, proposition A.3.17 implies that the class of monomorphisms that are in $\mathcal{W}$ is closed under pushout, transfinite composition, and retracts. Axiom CM2 (for $\mathcal{M}$) and lemma A.4.14 imply that $\mathcal{W}$ has the 2-out-of-3 property, and it is clear that every isomorphism in **sSet** is also in $\mathcal{W}$. Thus, the conditions of proposition 1.5.21 are satisfied. ■

Proposition 1.5.24. *Let* $\mathbf{H}$ *be the homotopy category of Kan complexes.*

(i) *For each simplicial set* X*, the functor* $\pi_0[X, -] : \mathbf{Kan} \to \mathbf{Set}$ *factors through* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ *as a representable functor on* $\mathbf{H}$*.*

(ii) *The functor* $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ *extends to a functor* $\boldsymbol{\pi} : \mathbf{sSet} \to \mathbf{H}$ *that sends weak homotopy equivalences to isomorphisms, and this extension is unique up to unique isomorphism.*

Proof. (i). Given $i : X \to RX$ as in proposition 1.5.5, the maps $\pi_0[i, K] : \pi_0[RX, K] \to \pi_0[X, K]$ are bijections (natural in K), so we may as well assume X is a Kan complex. Proposition 1.4.23 and remark 1.3.7 then imply that the functor $\pi_0[X, -] : \mathbf{Kan} \to \mathbf{Set}$ factors through $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ and the resulting functor $\mathbf{H} \to \mathbf{Set}$ is isomorphic to $\mathbf{H}(\boldsymbol{\pi} X, -)$.

(ii). Formally, what we seek is a functor $F : \mathbf{sSet} \to \mathbf{H}$ such that, for all Kan complexes Y and K,

$$\mathbf{H}(FY, \boldsymbol{\pi} K) = \pi_0[Y, K]$$

and, for all weak homotopy equivalences $f : X \to Y$ in **sSet**, the induced hom-set map $\mathbf{H}(Ff, \boldsymbol{\pi} K) : \mathbf{H}(FY, \boldsymbol{\pi} K) \to \mathbf{H}(FX, \boldsymbol{\pi} K)$ is a bijection for all Kan complexes K. Clearly, for any such F and any simplicial set X, there must be bijections

$$\mathbf{H}(FX, \boldsymbol{\pi} K) \cong \pi_0[X, K]$$

that are natural in K, but by claim (i), this is representable as a functor $\mathbf{H} \to \mathbf{Set}$ for each X, so we can certainly construct such a functor F, and it is unique up to unique isomorphism.

(iii). This is a special case of proposition 5.5.16; but see also proposition 1.7.16. ■

Corollary 1.5.25. *The inclusion* $\mathbf{H} \hookrightarrow \mathrm{Ho}_{\Delta^1} \mathbf{sSet}$ *admits a left adjoint.* ■

Remark 1.5.26. Fixing a fibrant replacement functor $R : \mathbf{sSet} \to \mathbf{sSet}$ as in proposition 1.5.5, we have the following explicit construction of $\operatorname{Ho}\mathbf{sSet}$ (i.e. the localisation of $\mathbf{sSet}$ with respect to weak homotopy equivalences):

- The objects are simplicial sets.
- For any two simplicial sets X and Y, $\operatorname{Ho}\mathbf{sSet}(X, Y) = \pi_0[RX, RY]$.
- Composition and identity morphisms are constructed as in $\mathbf{H}$.
- The localising functor $\gamma : \mathbf{sSet} \to \operatorname{Ho}\mathbf{sSet}$ inverting weak homotopy equivalences is the one sending $f : X \to Y$ to the homotopy class of $Rf : RX \to RY$.

The **homotopy category of simplicial sets** is the category $\operatorname{Ho}\mathbf{sSet}$. Of course, it is equivalent to $\mathbf{H}$.

Definition 1.5.27. Two simplicial sets have the **same weak homotopy type** if they are isomorphic in $\operatorname{Ho}\mathbf{sSet}$.

Remark 1.5.28. Freyd [1970] proved that $\mathbf{H}$ is not a concrete category, i.e. that there does not exist a faithful functor $\mathbf{H} \to \mathbf{Set}$; in particular, $\mathbf{H}$ cannot be an accessible category. Nonetheless, the notion of weak homotopy type is stable under universe enlargement in the following sense:

(i) The property of being a weak homotopy equivalence is universe-independent: indeed, it is clear that the property of being a trivial Kan fibration is universe-independent, so we may apply remark 0.5.18 to the (trivial cofibration, Kan fibration) factorisation system to test whether or not a morphism is a weak homotopy equivalence in a universe-independent way.

(ii) Moreover, the property of being a Kan complex is universe-independent, and $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ is a left adjoint between locally presentable categories, so the hom-set $\mathbf{H}(K, L)$ depends only on the choice of Kan complexes K and L and does not depend on the choice of universe. Similarly, whether or not K and L have the same homotopy type is universe-independent.

(iii) Thus, for any two simplicial sets X and Y, the hom-set $\operatorname{Ho}\mathbf{sSet}(X, Y)$ is well-defined up to natural bijection independently of the choice of universe, and whether or not X and Y have the same weak homotopy type is also universe-indepdent.

1.6 Bisimplicial sets and cosimplicial simplicial sets

Prerequisites. §§1.1, 1.3, 1.5, 4.3, 4.6, 5.2, A.5, A.6.

Definition 1.6.1. A **bisimplicial set** is a simplicial object in **sSet**, i.e. a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, and a **morphism of bisimplicial sets** is a natural transformation of such functors. We write **ssSet** for the **category of bisimplicial sets**.

Definition 1.6.2. Let $X_\bullet$ be a bisimplicial set and let n be a natural number. The **n-th column** of $X_\bullet$ is the simplicial set $(X_n)_\bullet$, and the **m-th row** of $X_\bullet$ is the simplicial set $(X_\bullet)_m$.

Definition 1.6.3. A **Reedy weak homotopy equivalence of bisimplicial sets** is a morphism in **ssSet** that is a weak homotopy equivalence in each column, i.e. $f_\bullet : X_\bullet \to Y_\bullet$ such that each $f_n : X_n \to Y_n$ is a weak homotopy equivalence of simplicial sets.

Theorem 1.6.4. **ssSet** *is a combinatorial model category where*

- *the cofibrations are the monomorphisms in* **ssSet***,*
- *the fibrations are the Reedy fibrations, and*
- *the weak equivalences are the Reedy weak homotopy equivalences.*

This is the **Reedy model structure on bisimplicial sets**.

Proof. Given theorem 4.6.15, it suffices to verify the following:

(i) The Reedy model structure is cofibrantly generated.

(ii) The Reedy cofibrations are precisely the monomorphisms in **ssSet**.

For this, see Theorems 15.7.6 and 15.8.7 in [Hirschhorn, 2003]. □

Corollary 1.6.5. *The Reedy model structure on* **ssSet** *is the injective model structure on the functor category* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$. ■

Definition 1.6.6. The **realisation** of a bisimplicial set $X_\bullet$ is the simplicial set $|X_\bullet|$ defined by the following coend in **sSet**:

$$|X_\bullet| = \int^{[n]:\mathbf{\Delta}} \Delta^n \times X_n$$

Lemma 1.6.7. *Let $X_\bullet$ be a bisimplicial set. There is a canonical comparison morphism*

$$\left|X_\bullet\right| \to \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet$$

and it is natural in $X_\bullet$.

Proof. The unique natural transformation $\Delta^\bullet \Rightarrow \Delta 1$ induces a natural morphism

$$\int^{[n]:\mathbf{\Delta}} \Delta^n \times X_n \to \int^{[n]:\mathbf{\Delta}} 1 \times X_n$$

and it is not hard to verify that there is a natural isomorphism

$$\int^{[n]:\mathbf{\Delta}} 1 \times X_n \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} X_\bullet$$

so we are done. ■

Lemma 1.6.8. *Let $X_\bullet$ be a bisimplicial set.*

(i) *There is an isomorphism*

$$\left|X_\bullet\right| \cong \operatorname{diag} X$$

where $\operatorname{diag} X$ *is the simplicial set defined by* $(\operatorname{diag} X)_n = \left(X_n\right)_n$, *and this isomorphism is natural in $X_\bullet$.*

(ii) *In particular, there is a canonical morphism*

$$X_0 \to \left|X_\bullet\right|$$

and this is natural in $X_\bullet$.

Proof. The Yoneda lemma for coends (proposition A.6.18) yields natural bijections of the form below:

$$\int^{[n]:\mathbf{\Delta}} \mathbf{\Delta}([m],[n]) \times \left(X_n\right)_m \cong \left(X_m\right)_m$$

Thus, $\left|X_\bullet\right| \cong \operatorname{diag} X$. ■

Corollary 1.6.9.

(i) *If $X_\bullet$ is a bisimplicial set whose columns are discrete,*[11] *then the realisation $\left|X_\bullet\right|$ is naturally isomorphic to the simplicial set $\left(X_\bullet\right)_0$.*

[11] Recall definition 1.2.5.

(ii) *If $X_\bullet$ is a bisimplicial set whose rows are discrete, then the realisation $\left|X_\bullet\right|$ is naturally isomorphic to the simplicial set $\left(X_0\right)_\bullet$.* ■

Theorem 1.6.10.

(i) *The functor $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ has left and right adjoints.*

(ii) *$|-|$ sends Reedy weak homotopy equivalences in $\mathbf{ssSet}$ to weak homotopy equivalences in $\mathbf{sSet}$.*

(iii) *Equipping $\mathbf{ssSet}$ with the Reedy model structure and $\mathbf{sSet}$ with the Kan–Quillen model structure, $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ is a left Quillen functor.*

Proof. (i). Using the isomorphism $\mathbf{ssSet} \cong [\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and lemma 1.6.8, we may identify $|-|$ as the functor δ^* induced by the diagonal embedding $\delta : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$, and corollary A.5.17 says δ^* has left and right adjoints.

(ii). See Theorem 15.11.11 in [Hirschhorn, 2003], or Proposition 1.7 in [GJ, Ch. IV].

(iii). From claims (i) and (ii) it follows that $|-|$ is a left Quillen functor; alternatively, see Proposition 3.6 in [GJ, Ch. VII]. □

Corollary 1.6.11. *If $X_\bullet$ is a bisimplicial set such that every face and degeneracy operator is a weak homotopy equivalence, then the canonical morphism $X_0 \to \left|X_\bullet\right|$ is a weak homotopy equivalence.*

Proof. Let $T_\bullet$ be the bisimplicial set defined by $T_\bullet = X_0$, so that the rows of $T_\bullet$ are discrete simplicial sets. Then there is a unique morphism $T_\bullet \to X_\bullet$ whose component in degree 0 is $\mathrm{id} : X_0 \to X_0$, and the hypothesis (plus the 2-out-of-3 property) implies that it is a weak homotopy equivalence. We then apply corollary 1.6.9 and theorem 1.6.10. ■

The following result is useful for constructing subdivision functors.

Proposition 1.6.12. *Let $D^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ be a diagram, let $\rho^\bullet : D^\bullet \Rightarrow \Delta^\bullet$ be a natural transformation, let $E : \mathbf{sSet} \to \mathbf{sSet}$ be the functor defined by $E(X)_n = \mathbf{sSet}(D^n, X)$, and let $i_X : X \to E(X)$ be the natural morphism defined by $\left(i_X\right)_n = \mathbf{sSet}(\rho^n, X)$ (where we have identified $\mathbf{sSet}(\Delta^n, X)$ with X_n via the Yoneda lemma).*

(i) *Given a parallel pair $f_0, f_1 : X \to Y$ of morphisms in* **sSet**, *if $f_0 \sim f_1$, then $E(f_0) \sim E(f_1)$ as well.*

(ii) *If each D^n is a contractible simplicial set, then $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow E$ is a natural weak homotopy equivalence.*

Proof. (i). We may assume (by induction) that we have an intrinsic homotopy $f_0 \Rightarrow f_1$: let $h : \Delta^1 \times X \to Y$ be any morphism such that $h \circ (\delta_1^1 \times \mathrm{id}_X) = f_0$ and $h \circ (\delta_1^0 \times \mathrm{id}_X) = f_1$ (suppressing comparison isomorphisms). Since $\rho^\bullet : D^\bullet \Rightarrow \Delta^\bullet$ is a natural transformation, the following diagram commutes:

$$\begin{tikzcd}
D^0 \arrow[r, "\rho^0"] \arrow[d, "\delta_1^1"'] & \Delta^0 \arrow[d, "\delta_1^1"] \\
D^1 \arrow[r, "\rho^1"] & \Delta^1 \\
D^0 \arrow[r, "\rho^0"'] \arrow[u, "\delta_1^0"] & \Delta^0 \arrow[u, "\delta_1^0"']
\end{tikzcd}$$

Thus, $E(\Delta^1)$ has an edge connecting the vertices $\delta_1^1 \circ \rho^0 : D^0 \to \Delta^1$ and $\delta_1^0 \circ \rho^0 : D^0 \to \Delta^1$. It is not hard to see that $E : \mathbf{sSet} \to \mathbf{sSet}$ preserves products, so by considering $E(h) : E(\Delta^1) \times E(X) \to E(Y)$, we see that there is an intrinsic homotopy $E(f_0) \Rightarrow E(f_1)$, as required.

(ii). The following is a generalisation of the proof of Proposition 2.3.19 in [Cisinski, 2006].

Consider the following commutative diagram in **Set**,

$$(*) \qquad \begin{tikzcd}
\mathbf{sSet}(\Delta^n, X) \arrow[r] \arrow[d, equal] & \mathbf{sSet}(\Delta^n \times \Delta^m, X) \arrow[d, "\mathbf{sSet}(\mathrm{id}_{\Delta^n} \times \rho^m)"'] & \mathbf{sSet}(\Delta^m, X) \arrow[l] \arrow[d, "\mathbf{sSet}(\rho^m, X)"] \\
\mathbf{sSet}(\Delta^n, X) \arrow[r] & \mathbf{sSet}(\Delta^n \times D^m, X) & \mathbf{sSet}(D^m, X) \arrow[l]
\end{tikzcd}$$

where the horizontal arrows are induced by the evident projections. The diagram is natural in n and m, so defines a commutative diagram in **ssSet**, which (by the Yoneda lemma) in the n-th column can be identified with the commutative diagram in **sSet** shown below,

$$\begin{tikzcd}
\operatorname{disc} X_n \arrow[r] \arrow[d, equal] & {[\Delta^n, X]} \arrow[d, "i_{[\Delta^n, X]}"'] & X \arrow[l] \arrow[d, "i_X"] \\
\operatorname{disc} X_n \arrow[r] & E([\Delta^n, X]) & E(X) \arrow[l]
\end{tikzcd}$$

and in the m-th row can be identified with the following commutative diagram in **sSet**:

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & {[\Delta^m, X]} \arrow[d, "{[\rho^m, X]}"'] & \operatorname{disc} X_m \arrow[l] \arrow[d, "\operatorname{disc}(i_X)_m"] \\ X \arrow[r] & {[D^m, X]} & \operatorname{disc} E(X)_m \arrow[l] \end{tikzcd}$$

Since Δ^m (resp. D^m) is contractible by corollary 1.3.11 (resp. by hypothesis) and the functor $[-, X] : \mathbf{sSet} \to \mathbf{sSet}$ preserves intrinsic homotopy equivalences, the horizontal arrows in the left half of $(*)$ define row-wise weak homotopy equivalences of bisimplicial sets. Similarly, since $E : \mathbf{sSet} \to \mathbf{sSet}$ respects intrinsic homotopy, the horizontal arrows in the right half of $(*)$ are column-wise weak homotopy equivalences of bisimplicial sets.

Now, apply the realisation functor $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ to the diagram in **ssSet** defined by $(*)$. By lemma 1.6.8, we obtain a commutative diagram in **sSet** of the form below,

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[d] & X \arrow[l] \arrow[d, "i_X"] \\ X \arrow[r] & Z & E(X) \arrow[l] \end{tikzcd}$$

and by theorem 1.6.10, every horizontal arrow in the above diagram is a weak homotopy equivalence. We may then use the 2-out-of-3 property of weak homotopy equivalences to deduce that $i_X : X \to E(X)$ is a weak homotopy equivalence. ∎

Definition 1.6.13. A **cosimplicial simplicial set** is a cosimplicial object in **sSet**, i.e. a functor $\mathbf{\Delta} \to \mathbf{sSet}$, and a morphism of cosimplicial simplicial sets is a natural transformation of such functors. We write **csSet** for the **category of cosimplicial simplicial sets**.

Definition 1.6.14. Let $X^\bullet$ be a cosimplicial simplicial set and let n be a natural number. The ***n*-th column** of $X^\bullet$ is the simplicial set $(X^n)_\bullet$, and the ***n*-th row** of $X^\bullet$ is the cosimplicial set $(X^\bullet)_n$.

Definition 1.6.15. A **Reedy weak homotopy equivalence of cosimplicial simplicial sets** is a morphism in **csSet** that is a weak homotopy equivalence of simplicial sets in each column, i.e. a morphism $f^\bullet : X^\bullet \to Y^\bullet$ such that each $f^n : X^n \to Y^n$ is a weak homotopy equivalence.

Lemma 1.6.16. *Let $X^\bullet$ be a cosimplicial simplicial set. The limit $\varprojlim_{\mathbf{\Delta}} X^\bullet$ in* **sSet** *can be computed as the equaliser of the coface operators $\delta^0, \delta^1 : X^0 \to X^1$.*

Proof. This is a straightforward exercise. ◊

Definition 1.6.17. The **maximal augmentation** of a cosimplicial simplicial set $X^\bullet$ is the limit $\varprojlim_{\mathbf{\Delta}} X^\bullet$.

Theorem 1.6.18. **csSet** *is a combinatorial model category where*

- *the cofibrations are the monomorphisms in* **csSet** *that induce isomorphisms of maximal augmentations,*
- *the fibrations are the Reedy fibrations, and*
- *the weak equivalences are the Reedy weak homotopy equivalences.*

This is the **Reedy model structure on cosimplicial simplicial sets***.*

Proof. Given theorem 4.6.15, it suffices to verify the following:

(i) The Reedy model structure is cofibrantly generated.

(ii) The Reedy cofibrations are precisely the announced ones.

For this, see Theorems 15.7.6 and 15.9.9 in [Hirschhorn, 2003]. □

Corollary 1.6.19. *The standard simplex functor $\Delta^\bullet : \mathbf{\Delta} \to$* **sSet** *is a Reedy-cofibrant cosimplicial simplicial set.*

Proof. The maximal augmentation of $\Delta^\bullet$ is empty, so by theorem 1.6.18, $\Delta^\bullet$ is Reedy-cofibrant. ■

Definition 1.6.20. The **totalisation** of a cosimplicial simplicial set $X^\bullet$ is the simplicial set $\operatorname{Tot} X^\bullet$ defined by the following end in **sSet**:

$$\operatorname{Tot} X^\bullet = \int_{[n]:\mathbf{\Delta}} [\Delta^n, X^n]$$

Lemma 1.6.21. *Let $X^\bullet$ be a cosimplicial simplicial set. There is a canonical comparison morphism*

$$\varprojlim_{\mathbf{\Delta}} X^\bullet \to \operatorname{Tot} X^\bullet$$

and it is natural in $X^\bullet$.

Proof. The unique natural transformation $\Delta^\bullet \Rightarrow \Delta 1$ induces a natural morphism

$$\int_{[n]:\mathbf{\Delta}} [1, X^n] \to \int_{[n]:\mathbf{\Delta}} [\Delta^n, X^n]$$

and it is not hard to verify that there is a natural isomorphism

$$\int^{[n]:\mathbf{\Delta}} 1 \times X^n \cong \varprojlim_{\mathbf{\Delta}} X^\bullet$$

so we are done. ∎

Lemma 1.6.22. *Let $Y^\bullet$ be a cosimplicial simplicial set. There is a bijection*

$$\mathbf{sSet}(X, \operatorname{Tot} Y^\bullet) \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, (Y^m)_m\big)$$

for each simplicial set X, and this bijection is natural in X and Y.

Proof. Using remark A.6.5, the interchange law for ends (theorem A.6.17), and Yoneda lemma for ends (proposition A.6.18), we obtain the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}\left(X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, Y^n]\right) &\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X, [\Delta^n, Y^n]) \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X \times \Delta^n, Y^n) \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m \times \mathbf{\Delta}([m],[n]), (Y^n)_m\big) \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), (Y^n)_m\big)\big) \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), (Y^n)_m\big)\right) \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, (Y^m)_m\big)
\end{aligned}$$

∎

Lemma 1.6.23. *Let $Y^\bullet$ be a cosimplicial simplicial set. If the coface and codegeneracy operators of $Y^\bullet$ are isomorphisms (of simplicial sets), then*

$$\operatorname{Tot} Y^\bullet \cong Y^0$$

naturally in $Y^\bullet$.

Proof. Recalling remark A.6.5,

$$(\operatorname{Tot} Y^\bullet)_n \cong \int_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, (Y^m)_m\right)$$

and since the coface and codegeneracy operators of $Y^\bullet$ are isomorphisms, we may as well replace $(Y^m)_m$ with $\left(Y^0\right)_m$; but then the Yoneda lemma for ends (proposition A.6.18) gives a natural bijection

$$\int_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, \left(Y^0\right)_m\right) \cong \left(Y^0\right)_n$$

so we are done. ∎

Lemma 1.6.24. *Let $Y^\bullet$ be a cosimplicial simplicial set. If each Y^n is discrete as a simplicial set, then* $\operatorname{Tot} Y^\bullet$ *is also discrete.*

Proof. Recalling remark A.6.5, it suffices to verify that the sets

$$H_n = \int_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, (Y^m)_m\right)$$

do not depend on n (in the evident sense). Since each Y^m is discrete, we may as well replace $(Y^m)_m$ with $(Y^m)_0$; but

$$\int_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, (Y^m)_0\right) \cong \varprojlim_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, (Y^m)_0\right)$$

and $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted (by remark A.5.35), so theorem A.5.36 implies that the diagonal functor $\Delta : \boldsymbol{\Delta} \to \boldsymbol{\Delta} \times \boldsymbol{\Delta}$ is coinitial, thus:

$$\begin{aligned} H_n &\cong \varprojlim_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, (Y^m)_0\right) \\ &\cong \varprojlim_{[l]:\boldsymbol{\Delta}} \varprojlim_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m, \left(Y^l\right)_0\right) \\ &\cong \varprojlim_{[l]:\boldsymbol{\Delta}} \mathbf{Set}\left(\varinjlim_{[m]:\boldsymbol{\Delta}^{\mathrm{op}}} \Delta^n_m, \left(Y^l\right)_0\right) \end{aligned}$$

Hence, by proposition 1.2.4,

$$H_n \cong \varprojlim_{[l]:\boldsymbol{\Delta}} \left(Y^l\right)_0$$

and this is natural in n, so $\operatorname{Tot} Y^\bullet$ is indeed discrete. ∎

Lemma 1.6.25. *Let $X_\bullet$ be a bisimplicial set and let Y be a simplicial set. Then there is a canonical isomorphism*

$$\left[\left|X_\bullet\right|, Y\right] \cong \operatorname{Tot}\left[X_\bullet, Y\right]$$

and it is natural in $X_\bullet$ and Y.

Proof. By proposition A.6.11, we have the following natural isomorphisms:

$$\begin{aligned}
\left[\left|X_\bullet\right|, Y\right] &= \left[\int^{[n]:\mathbf{\Delta}} \Delta^n \times X_n, Y\right] \\
&\cong \int_{[n]:\mathbf{\Delta}} \left[\Delta^n \times X_n, Y\right] \\
&\cong \int_{[n]:\mathbf{\Delta}} \left[\Delta^n, \left[X_n, Y\right]\right] \\
&= \operatorname{Tot}\left[X_\bullet, Y\right]
\end{aligned}$$

■

Theorem 1.6.26.

(i) *The functor* Tot : **csSet** → **sSet** *has a left adjoint.*

(ii) *For each simplicial set X and each cosimplicial simplicial set $Y^\bullet$, the canonical comparison morphism* $\operatorname{Tot}[X, Y^\bullet] \to [X, \operatorname{Tot} Y^\bullet]$ *is an isomorphism.*

(iii) *Equipping* **csSet** *with the Reedy model structure and* **sSet** *with the Kan–Quillen model structure,* Tot : **csSet** → **sSet** *is a right Quillen functor.*

Proof. (i). It is straightforward to check that the functor sending a simplicial set X to the cosimplicial simplicial set $\Delta^\bullet \times X$ is a left adjoint for Tot; see also proposition A.6.15.

(ii). By proposition A.6.11, we have the following natural isomorphisms:

$$\begin{aligned}
[X, \operatorname{Tot} Y^\bullet] &= \left[X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, Y^n]\right] \\
&\cong \int_{[n]:\mathbf{\Delta}} [X, [\Delta^n, Y^n]] \\
&\cong \int_{[n]:\mathbf{\Delta}} [\Delta^n, [X, Y^n]] \\
&= \operatorname{Tot}[X, Y^\bullet]
\end{aligned}$$

(iii). See Theorem 18.6.7 in [Hirschhorn, 2003]. □

1.7 Subdivision and extension

Prerequisites. §§1.1, 1.2, 1.3, 1.4, 1.5, 1.6.

¶ 1.7.1. Let P^n be the partially ordered set of non-empty subsets of $[n]$ and, for each monotone map $f : [n] \to [m]$, let $f_* : P^n \to P^m$ be the map induced by taking images. Taking nerves, this defines a functor $\mathrm{N}(P^\bullet) : \mathbf{\Delta} \to \mathbf{sSet}$. Note that there is a natural surjective monotone map $\max : P^n \to [n]$, each with a canonical (but not natural!) splitting, so we get a natural transformation $\mathrm{N}(\max) : \mathrm{N}(P^\bullet) \Rightarrow \Delta^\bullet$ whose components are split epimorphisms.

Definition 1.7.2. The **extension** of a simplicial set X is the simplicial set $\mathrm{Ex}(X)$ defined by the formula below:

$$\mathrm{Ex}(X)_n = \mathbf{sSet}(\mathrm{N}(P^n), X)$$

The **canonical embedding** is the morphism $i_X : X \to \mathrm{Ex}(X)$ induced by $\mathrm{N}(\max) : \mathrm{N}(P^\bullet) \Rightarrow \Delta^\bullet$; note that it is a split monomorphism in $\mathbf{sSet}$.

Remark 1.7.3. Every simplex of $\mathrm{N}(P^n)$ is uniquely determined by its vertices and P^n has only finitely many elements, so $\mathrm{N}(P^n)$ is a finite simplicial set. In particular, each $\mathrm{Ex}(X)_n$ is a finite weighted limit of the diagram $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$.

¶ 1.7.4. Let $\mathrm{Sd} : \mathbf{sSet} \to \mathbf{sSet}$ be (the functor part of) a left Kan extension of $\mathrm{N}(\max) : \mathbf{\Delta} \to \mathbf{sSet}$ along $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$. Using the formulas of theorem A.5.15, we see there is a natural bijection of the form below:

$$\mathbf{sSet}(\mathrm{Sd}(X), Y) \cong \mathbf{sSet}(X, \mathrm{Ex}(Y))$$

In other words, we have the following adjunction:

$$\mathrm{Sd} \dashv \mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$$

Definition 1.7.5. The **subdivision** of a simplicial set X is the simplicial set $\mathrm{Sd}(X)$ defined above. The **last vertex projection** is the left adjoint transpose $\lambda_X : \mathrm{Sd}(X) \to X$ of the canonical embedding $i_X : X \to \mathrm{Ex}(X)$.

Lemma 1.7.6. *Let X be a simplicial set. For each morphism $z : \Lambda^n_k \to \mathrm{Ex}(X)$, there exists a morphism $w : \Delta^n_k \to \mathrm{Ex}(\mathrm{Ex}(X))$ making the diagram below commute:*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "z"] \arrow[d, hook] & \mathrm{Ex}(X) \arrow[d, "i_{\mathrm{Ex}(X)}"] \\
\Delta^n \arrow[r, dashed, "w"'] & \mathrm{Ex}(\mathrm{Ex}(X))
\end{tikzcd}$$

Proof. See Lemma 3.2 in [Kan, 1957], or Lemma 4.7 in [GJ, Ch. III]. □

Lemma 1.7.7. *The functor* Ex : **sSet** → **sSet** *preserves Kan fibrations. In particular, if X is a Kan complex, then so is* $\mathrm{Ex}(X)$.

Proof. See Lemma 3.4 in [Kan, 1957], or Lemma 4.5 in [GJ, Ch. III], or Corollary 2.1.27 in [Cisinski, 2006]. □

Lemma 1.7.8. *For any simplicial set X, the canonical embedding $i_X : X \to \mathrm{Ex}(X)$ is bijective on vertices.*

Proof. It is clear that $\max : P^0 \to [0]$ is an isomorphism of partially ordered sets; thus $i_X : X \to \mathrm{Ex}(X)$ is bijective on vertices. ■

Lemma 1.7.9. *For any simplicial set X, the canonical embedding $i_X : X \to \mathrm{Ex}(X)$ is a weak homotopy equivalence.*

Proof. By corollary 1.3.11, each $\mathrm{Sd}(\Delta^n)$ is contractible, so the claim is a special case of proposition 1.6.12. ■

Corollary 1.7.10. *The functor* Ex : **sSet** → **sSet** *preserves trivial Kan fibrations.*

Proof. Combine proposition 1.5.10 with lemmas 1.7.7 and 1.7.9. ■

Corollary 1.7.11. *We have the following Quillen equivalence:*

$$\mathrm{Sd} \dashv \mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$$

Proof. Lemma 1.7.7 and corollary 1.7.10 say $\mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$ is a right Quillen functor, so (by proposition 4.3.2) the indicated adjunction is indeed a Quillen adjunction. Consider the derived adjunction:

$$\mathbf{L}\mathrm{Sd} \dashv \mathbf{R}\mathrm{Ex} : \mathrm{Ho}\,\mathbf{sSet} \to \mathrm{Ho}\,\mathbf{sSet}$$

By proposition 1.5.10 and lemma 1.7.9, $\mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$ is a homotopical functor, so $\mathrm{Ho}\,\mathrm{Ex} : \mathrm{Ho}\,\mathbf{sSet} \to \mathrm{Ho}\,\mathbf{sSet}$ is well defined and isomorphic to both id and $\mathbf{R}\mathrm{Ex}$. Hence, $\mathbf{L}\mathrm{Sd}$ is also isomorphic to id, and (recalling lemma 1.5.2) we may apply theorem 4.3.13 to deduce that we have a Quillen equivalence. ■

Proposition 1.7.12.

(i) *There is a unique natural isomorphism* $\mathrm{Sd}(\Delta^\bullet) \cong \mathrm{Sd}((\Delta^\bullet)^{\mathrm{op}})$.

(ii) *There is a unique natural isomorphism* $\mathrm{Sd}(-) \cong \mathrm{Sd}((-)^{\mathrm{op}})$.

(iii) *For each simplicial set X, there is a diagram of the form below,*

$$X \longleftarrow \mathrm{Sd}(X) \longrightarrow X^{\mathrm{op}}$$

where the arrows are weak homotopy equivalences that are natural in X.

(iv) *There is a Quillen equivalence of the following form:*

$$(-)^{\mathrm{op}} \dashv (-)^{\mathrm{op}} : \mathbf{sSet} \to \mathbf{sSet}$$

(v) *The induced automorphism* $\mathrm{Ho}\,(-)^{\mathrm{op}} : \mathrm{Ho}\,\mathbf{sSet} \to \mathrm{Ho}\,\mathbf{sSet}$ *is isomorphic to* $\mathrm{id}_{\mathrm{Ho}\,\mathbf{sSet}}$.

Proof. (i). It is not hard to see that there is a unique isomorphism $\Delta^n \cong (\Delta^n)^{\mathrm{op}}$, namely the one that sends the k-th vertex to the $(n-k)$-th vertex. These isomorphisms are *not* natural, in the sense that they are incompatible with the coface and codegeneracy maps; nonetheless, these isomorphisms enable us to identify each $\mathrm{Sd}((\Delta^n)^{\mathrm{op}})$ with $\mathrm{N}(P^n)$ as objects. In turn, we may identify each $\mathrm{Sd}\left(\left(\delta_n^i\right)^{\mathrm{op}}\right) : \mathrm{Sd}\left(\left(\Delta^{n-1}\right)^{\mathrm{op}}\right) \to \mathrm{Sd}((\Delta^n)^{\mathrm{op}})$ with the morphism $\delta_n^{n-i} : \mathrm{N}\left(P^{n-1}\right) \to \mathrm{N}(P^n)$, and similarly for the codegeneracy maps. It is then clear that there is a unique natural isomorphism $\mathrm{Sd}(\Delta^\bullet) \cong \mathrm{Sd}((\Delta^\bullet)^{\mathrm{op}})$.

(ii). Since $(-)^{\mathrm{op}} : \mathbf{sSet} \to \mathbf{sSet}$ and $\mathrm{Sd} : \mathbf{sSet} \to \mathbf{sSet}$ both preserve colimits, theorem 1.1.13 implies that there is a unique natural isomorphism $\mathrm{Sd}(-) \cong \mathrm{Sd}((-)^{\mathrm{op}})$ extending the (unique) natural isomorphism $\mathrm{Sd}(\Delta^\bullet) \cong \mathrm{Sd}((\Delta^\bullet)^{\mathrm{op}})$ discussed above.

(iii). Given the (unique) natural isomorphism $\mathrm{Sd}(-) \cong \mathrm{Sd}((-)^{\mathrm{op}})$, it suffices to give a natural weak homotopy equivalence $\mathrm{Sd} \Rightarrow \mathrm{id}_{\mathbf{sSet}}$. But lemma 1.7.9 says that $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}$ is a natural weak homotopy equivalence, so by corollary 1.7.11, its left adjoint transpose is a natural weak homotopy equivalence $r : \mathrm{Sd} \Rightarrow \mathrm{id}_{\mathbf{sSet}}$, as desired.

(iv). Since $(-)^{\mathrm{op}} : \mathbf{sSet} \to \mathbf{sSet}$ is an automorphism, we have an adjunction of the required form. It is clear that $(-)^{\mathrm{op}}$ preserves monomorphisms, anodyne extensions, Kan fibrations, and trivial Kan fibrations, so the adjunction is a Quillen adjunction. We may also deduce that $(-)^{\mathrm{op}}$ preserves weak homotopy equivalences, and hence that the Quillen equivalence condition is satisfied.

(v). We have a zigzag of natural weak homotopy equivalences connecting $\mathrm{id}_{\mathbf{sSet}}$ to $(-)^{\mathrm{op}}$, and it immediately follows that $\mathrm{id}_{\mathrm{Ho}\,\mathbf{sSet}}$ is isomorphic to $\mathrm{Ho}\,(-)^{\mathrm{op}}$. ■

¶ 1.7.13. For each simplicial set X, we define $\mathrm{Ex}^\infty(X)$ to be the colimit of the diagram below:

$$X \xrightarrow{i_X} \mathrm{Ex}(X) \xrightarrow{i_{\mathrm{Ex}(X)}} \mathrm{Ex}^2(X) \xrightarrow{i_{\mathrm{Ex}^2(X)}} \mathrm{Ex}^3(X) \longrightarrow \cdots$$

The above defines a functor $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $i^\infty : \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}^\infty$.

Theorem 1.7.14.

(i) *For all simplicial sets X, the morphism $i^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is an anodyne extension and bijective on vertices.*

(ii) *For all simplicial sets X, the simplicial set $\mathrm{Ex}^\infty(X)$ is a Kan complex.*

(iii) *The functor $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ preserves Kan fibrations, trivial Kan fibrations, and finite limits.*

Proof. (i). Recalling proposition 1.5.10 and lemma 1.7.9, we see that the canonical embedding $i_X : X \to \mathrm{Ex}(X)$ is an anodyne extension for all simplicial sets X; but proposition A.3.17 implies that the class of anodyne extensions is closed under transfinite composition, and $i^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is a transfinite composite of these canonical embeddings, so i^∞_X is also an anodyne extension. A similar argument using lemma 1.7.8 shows that $i^\infty_X : X \to \mathrm{Ex}^\infty(X)$ is bijective on vertices.

(ii). Since horns are finite simplicial sets, any horn $\Lambda^n_k \to \mathrm{Ex}^\infty(X)$ must factor as $\Lambda^n_k \to \mathrm{Ex}^m(X) \to \mathrm{Ex}^\infty(X)$ for some m. We then apply lemma 1.7.6 to deduce that $\mathrm{Ex}^\infty(X)$ is a Kan complex.

(iii). Similar reasoning applied to lemma 1.7.7 (resp. corollary 1.7.10) shows that $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ preserves Kan fibrations (resp. trivial Kan fibrations). On the other hand, since $\mathrm{Ex} : \mathbf{sSet} \to \mathbf{sSet}$ preserves finite limits and Ex^∞ is a filtered colimit of iterations of Ex, corollary 0.2.27 implies Ex^∞ also preserves finite limits. ■

Remark 1.7.15. Neither $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ nor $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ preserve infinite products. Indeed, let X the simplicial set defined in remark 1.5.9. We know X is weakly contractible, so the unique morphism $\mathrm{Ex}^\infty(X) \to \Delta^0$ must be a trivial

Kan fibration (by theorem 1.4.27). However, for any infinite set I, the simplicial set X^I is *not* connected, i.e. $\pi_0(X^I)$ is *not* a singleton. Nonetheless, $\mathrm{Ex}^\infty(X)^I \to \Delta^0$ is a trivial Kan fibration (because the class of trivial Kan fibrations is closed under products); so the canonical morphism $\mathrm{Ex}^\infty(X^I) \to \mathrm{Ex}^\infty(X)^I$ cannot be a weak homotopy equivalence, let alone an isomorphism!

Proposition 1.7.16. *There exist a functor $R : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ satisfying the following conditions:*

- *For all simplicial sets X, RX is a Kan complex and $i_X : X \to RX$ is an anodyne extension.*
- *$R : \mathbf{sSet} \to \mathbf{sSet}$ preserves Kan fibrations and trivial Kan fibrations.*
- *$R : \mathbf{sSet} \to \mathbf{sSet}$ preserves finite limits.*

Moreover, any such functor R preserves and reflects weak homotopy equivalences.

Proof. By theorem 1.7.14, we may take (R, i) to be $(\mathrm{Ex}^\infty, i^\infty)$. Given any such (R, i), consider the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ Y \arrow[r, "i_Y"'] & RY \end{tikzcd}$$

Using proposition 1.5.10 and the 2-out-of-3 property of weak homotopy equivalences, we see that $f : X \to Y$ is a weak homotopy equivalence if and only if $Rf : RX \to RY$ is a weak homotopy equivalence. ■

Remark 1.7.17. We may construct a different functor satisfying the conditions of the above proposition by using an appropriate geometric realisation functor: see Proposition 2.4 and Proposition 10.10 in [GJ, Ch. I].

Theorem 1.7.18. *The Kan–Quillen model structure on* $\mathbf{sSet}$ *is proper.*

Proof. Since every simplicial set is cofibrant, we may apply proposition 5.1.8 to deduce that $\mathbf{sSet}$ is a left proper model category. On the other hand, by proposition 1.7.16, the right properness of $\mathbf{sSet}$ can be reduced to the right properness of $\mathbf{Kan}$, which was established by proposition 1.4.32. ■

Proposition 1.7.19. *Let $p : X \to Y$ be a Kan fibration. The following are equivalent:*

(i) *The morphism $p : X \to Y$ is a trivial Kan fibration.*

(ii) *For every n-simplex $\alpha : \Delta^n \to Y$ and any pullback diagram in* **sSet** *of the form below,*

$$\begin{tikzcd} X_\alpha \arrow[r] \arrow[d, "p_\alpha"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "\alpha"'] & Y \end{tikzcd}$$

the simplicial set X_α is weakly contractible.

(iii) *For every vertex y of Y, the fibre of $p : X \to Y$ over y is a contractible Kan complex.*

Proof. (i) $\Leftrightarrow$ (ii). Recalling lemma 1.5.2 and proposition 1.5.10, this is just proposition 1.4.13.

(ii) $\Rightarrow$ (iii). The class of Kan fibrations is closed under pullback (by proposition A.3.17), so the fibre of a Kan fibration over a vertex of the base is indeed a Kan complex. Thus, we may apply proposition 1.5.8.

(iii) $\Rightarrow$ (ii). Fix an n-simplex $\alpha : \Delta^n \to Y$, a pullback diagram as above, and a vertex y of Y that is contained in α. We then have the following pullback square in **sSet**,

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X_\alpha \arrow[d, "p_\alpha"] \\ \Delta^0 \arrow[r] & \Delta^n \end{tikzcd}$$

where $p_\alpha : X_\alpha \to \Delta^n$ is a Kan fibration. Since Δ^0 and Δ^n are both contractible, the 2-out-of-3 property implies that every morphism $\Delta^0 \to \Delta^n$ is a weak homotopy equivalence; thus, by theorem 1.7.18, the top horizontal arrow in the diagram above is also a weak homotopy equivalence. Hence, X_α is a weakly contractible simplicial set. ■

1.8 Bar and cobar complexes

Prerequisites. §§1.1, 1.3, 1.6, 4.5, A.5, A.6.

Definition 1.8.1. Let $\mathbb{C}$ be a small category.

The **bar complex** for a diagram $F : \mathbb{C} \to \mathbf{Set}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is the simplicial set $\mathrm{B}_\bullet(G, \mathbb{C}, F)$, where

$$\mathrm{B}_n(G, \mathbb{C}, F) = \coprod_{(c_0, \ldots, c_n)} \left(Gc_n \times \mathbb{C}\left(c_{n-1}, c_n\right) \times \cdots \times \mathbb{C}\left(c_0, c_1\right) \times Fc_0\right)$$

with $\left(c_0, \ldots, c_n\right)$ ranging over $(n+1)$-tuples of objects in $\mathbb{C}$, face maps defined by the following formulae,

$$\begin{aligned} d_0^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_2, F\left(f_1\right)(x)\right) \\ d_i^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1, x\right) \\ d_n^n\left(y, f_n, \ldots, f_1, x\right) &= \left(G\left(f_n\right)(y), f_{n-1}, \ldots, f_1, x\right) \end{aligned}$$

and degeneracy maps defined as below:

$$\begin{aligned} s_0^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_1, \mathrm{id}_{c_0}, x\right) \\ s_i^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_{i+1}, \mathrm{id}_{c_i}, f_i, \ldots, f_1, x\right) \\ s_n^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, \mathrm{id}_{c_n}, f_n, \ldots, f_1, x\right) \end{aligned}$$

The **cobar complex** for a diagram $F : \mathbb{C} \to \mathbf{Set}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ is the cosimplicial set $\mathrm{C}^\bullet(G, \mathbb{C}, F)$, where

$$\mathrm{C}^n(G, \mathbb{C}, F) = \prod_{(c_0, \ldots, c_n)} \left[Gc_n \times \mathbb{C}\left(c_n, c_{n-1}\right) \times \cdots \times \mathbb{C}\left(c_1, c_0\right), Fc_0\right]$$

with $\left(c_0, \ldots, c_n\right)$ ranging over $(n+1)$-tuples of objects in $\mathbb{C}$, coface maps defined by the following formulae,

$$\begin{aligned} \delta_n^0(x)_{(c_0, \ldots, c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto F\left(f_1\right)\left(x_{(c_1, \ldots, c_n)}\left(y, f_n, \ldots, f_2\right)\right)\right) \\ \delta_n^i(x)_{(c_0, \ldots, c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto x_{(\ldots, \widehat{c_i}, \ldots)}\left(y, f_n, \ldots, f_i \circ f_{i+1}, \ldots, f_1\right)\right) \\ \delta_n^n(x)_{(c_0, \ldots, c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto x_{(c_0, \ldots, c_{n-1})}\left(G\left(f_n\right)(y), f_{n-1}, \ldots, f_1\right)\right) \end{aligned}$$

and codegeneracy maps defined as below:

$$\sigma_n^0(x)_{(c_0, \ldots, c_n)} = \left(\left(y, f_n, \ldots, f_1\right) \mapsto x_{c_0, c_0, \ldots, c_n}\left(y, f_n, \ldots, f_1, \mathrm{id}_{c_0}\right)\right)$$

$$\sigma_n^i(x)_{(c_0,\dots,c_n)} = \big((y, f_n, \dots, f_1) \mapsto x_{\dots,c_i,c_i,\dots}(y, f_n, \dots, f_{i+1}, \mathrm{id}_{c_i}, f_i, \dots, f_1)\big)$$
$$\sigma_n^n(x)_{(c_0,\dots,c_n)} = \big((y, f_n, \dots, f_1) \mapsto x_{c_0,\dots,c_n,c_n}(y, \mathrm{id}_{c_n}, f_n, \dots, f_1)\big)$$

Remark 1.8.2. It is clear that $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is covariantly functorial in both F and G, while $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is contravariantly functorial in G and covariantly functorial in F. One may also verify that there are bijections

$$\mathbf{Set}\big(\mathrm{B}_n(G, \mathbb{C}, F), X\big) \cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}(F, X))$$

that are natural in n, F, G, and X: this is one sense in which the bar complex and cobar complex are formally dual.

Remark 1.8.3. There is another duality principle for bar complexes, namely the following natural isomorphism:

$$\mathrm{B}_\bullet(G, \mathbb{C}, F)^{\mathrm{op}} \cong \mathrm{B}_\bullet(F, \mathbb{C}^{\mathrm{op}}, G)$$

Unfortunately, there is no such statement for cobar complexes.

Remark 1.8.4. The nerve $\mathrm{N}(\mathbb{C})$ of a small category $\mathbb{C}$ is isomorphic to the bar complex $\mathrm{B}_\bullet(\Delta 1, \mathbb{C}, \Delta 1)$, so there is a canonical morphism $\mathrm{B}_\bullet(G, \mathbb{C}, F) \to \mathrm{N}(\mathbb{C})$ for any $F : \mathbb{C} \to \mathbf{Set}$ and any $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$.

Remark 1.8.5. More generally, the bar complex $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is isomorphic to the nerve of the following category $\mathbf{G}(G, \mathbb{C}, F)$:

- The objects are tuples (y, c, x), where c is an object in $\mathbb{C}$, x is an element of Fc, and y is an element of Gc.
- The morphisms $(y, c, x) \to (y', c', x')$ are morphisms $f : c \to c'$ in $\mathbb{C}$ such that $F(f)(x) = x'$ and $G(f)(y') = y$.
- Composition and identities are inherited from $\mathbb{C}$.

In particular, given a functor $U : \mathbb{C} \to \mathbb{D}$, $\mathrm{B}_\bullet\big(\Delta 1, \mathbb{C}, U^* \hbar^d\big)$ may be identified with the nerve of the comma category $(d \downarrow U)$, and $\mathrm{B}_\bullet\big(U^* \hbar_d, \mathbb{C}, \Delta 1\big)$ with the nerve of the comma category $(U \downarrow d)$.

Definition 1.8.6. Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category.

- A **bar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is a simplicial object $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ in $\mathcal{M}$ with bijections

$$\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) \cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M))$$

  that are natural in both n and M.

- A **cobar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ is a cosimplicial object $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ in $\mathcal{M}$ with bijections

$$\mathcal{M}(M, \mathrm{C}^n(G, \mathbb{C}, F)) \cong \mathrm{C}^n(G, \mathbb{C}, \mathcal{M}(M, F))$$

that are natural in both n and M.

Remark 1.8.7. Of course, this definition agrees with the previous one (up to isomorphism) in the special case $\mathcal{M} = \mathbf{Set}$, and it is clear that a cobar complex in $\mathcal{M}$ for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ becomes a bar complex in $\mathcal{M}^{\mathrm{op}}$ for $F^{\mathrm{op}} : \mathbb{C}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ weighted by the same $G : \mathbb{C} \to \mathbf{Set}$, and vice versa.

Remark 1.8.8. By general considerations about the representability of limits, we see that bar complexes exist for all small diagrams and weights if $\mathcal{M}$ has coproducts for small families of objects, while cobar complexes exist for all small diagrams and weights if $\mathcal{M}$ has products for small families of objects.

Lemma 1.8.9. *Let $\mathbb{C}$ be a small category. For each diagram $F : \mathbb{C} \to \mathbf{Set}$ and each weight $G : \mathbb{C} \to \mathbf{Set}$, we have a bijection*

$$[\mathbb{C}, \mathbf{Set}](G, F) \cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F)$$

that is natural in both F and G.

Proof. It is not hard to see that the (non-full) subcategory $\{[0] \rightrightarrows [1]\}$ is coinitial in $\mathbf{\Delta}$, so it suffices to show that there is an equaliser diagram of the following form,

$$[\mathbb{C}, \mathbf{Set}](G, F) \longrightarrow \mathrm{C}^0(G, \mathbb{C}, F) \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \mathrm{C}^1(G, \mathbb{C}, F)$$

However, if we take the map $[\mathbb{C}, \mathbf{Set}](G, F) \to \mathrm{C}^0(G, \mathbb{C}, F)$ to be the one sending a natural transformation $\alpha : G \Rightarrow F$ to its underlying family of maps $\left(\alpha_c : Gc \to Fc \,\middle|\, c \in \operatorname{ob} \mathbb{C}\right)$, then it is clear that the diagram is indeed an equaliser. ■

Proposition 1.8.10. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category.*

- *If $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is a bar complex in $\mathcal{M}$, then the colimit $\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$ exists in $\mathcal{M}$ if and only if the weighted colimit $G \star_{\mathbb{C}} F$ exists in $\mathcal{M}$, and the two are isomorphic:*

$$G \star_{\mathbb{C}} F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

- *If $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is a cobar complex in $\mathcal{M}$, then the limit $\varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F)$ exists in $\mathcal{M}$ if and only if the weighted limit $\{G, F\}^{\mathbb{C}}$ exists in $\mathcal{M}$, and the two are isomorphic:*

$$\{G, F\}^{\mathbb{C}} \cong \varprojlim_{\mathbf{\Delta}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

Proof. The two claims are formally dual; we will prove the first version.

Let M be any object in $\mathcal{M}$. Using lemma A.5.12, proposition A.5.13, and lemma 1.8.9, we obtain the following natural bijections:

$$\begin{aligned} \{G, \mathcal{M}(F, M)\}^{\mathbb{C}^{\mathrm{op}}} &\cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](G, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\mathbf{\Delta}} \mathcal{M}\big(\mathrm{B}_\bullet(G, \mathbb{C}, F), M\big) \end{aligned}$$

It follows by the Yoneda lemma that $G \star_{\mathbb{C}} F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$. ■

Lemma 1.8.11. *Let $\mathbb{C}$ be a small category.*

(i) *For each natural number n and each weight $G : \mathbb{C} \to$ **Set**, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \to$ **Set** preserves limits, weighted limits, and ends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to$ **Set**, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to$ **Set** sends colimits to limits, weighted colimits to weighted limits, and coends to ends.*

Proof. Obvious. ◆

Proposition 1.8.12. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category. If $\mathcal{M}$ has coproducts for small families of objects, then:*

(i) *For each natural number n and each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, the functor $\mathrm{B}_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ preserves colimits, weighted colimits, and coends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{B}_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \to \mathcal{M}$ preserves colimits, weighted colimits, and coends.*

Dually, if $\mathcal{M}$ has products for small families of objects, then:

(i) *For each natural number n and each weight $G : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ preserves limits, weighted limits, and ends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to \mathbf{Set}$ sends colimits to limits, weighted colimits to weighted limits, and coends to ends.*

Proof. We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

Lemma 1.8.13. *Let $\mathbb{C}$ be a small category.*

- *Let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram and let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a weight. For all sets X, we have bijections*

$$\mathrm{B}_n(G \times X, \mathbb{C}, F) \cong X \times \mathrm{B}_n(G, \mathbb{C}, F) \cong \mathrm{B}_n(G, \mathbb{C}, X \times F)$$

  that are natural in X, F, and G.

- *Let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram and let $G : \mathbb{C} \to \mathbf{Set}$ be a weight. For all sets X, we have bijections*

$$\mathrm{C}^n(X \times G, \mathbb{C}, F) \cong [X, \mathrm{C}^n(G, \mathbb{C}, F)] \cong \mathrm{C}^n(G, \mathbb{C}, [X, F])$$

  that are natural in X, F, and G.

Proof. Obvious. ♦

Proposition 1.8.14. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category. If $\mathcal{M}$ has coproducts for small families of objects, then:*

(i) *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a weight and let M be any object in $\mathcal{M}$. We then have bijections*

$$\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(Gc' \odot Fc, M)\big)$$

that are natural in n, F, G, and M.

(ii) *If $\mathcal{M}$ is cotensored, then for each natural number n and each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, the functor $\mathrm{B}_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ has a right adjoint, namely the functor that sends an object M to the diagram $c \mapsto \mathrm{B}_n\big(G, \mathbb{C}^{\mathrm{op}}, \hbar_c\big) \pitchfork M$.*

(iii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{B}_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \to \mathcal{M}$ has a right adjoint, namely the functor that sends an object M to the weight $c \mapsto \mathrm{C}^n\big(\hbar_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big)$.*

Dually, if $\mathcal{M}$ has products for small families of objects, then:

(i′) *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{C} \to \mathbf{Set}$ be a weight, and let M be an object in $\mathcal{M}$. We then have bijections*

$$\mathcal{M}(M, \mathrm{C}^n(G, \mathbb{C}, F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(M, Gc' \pitchfork Fc)\big)$$

that are natural in n, F, G, and M.

(ii′) *If $\mathcal{M}$ is tensored, then for each natural number n and each weight $G : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ has a left adjoint, namely the functor that sends an object M to the diagram $c \mapsto \mathrm{B}_n\big(G, \mathbb{C}^{\mathrm{op}}, \hbar_c\big) \odot M$.*

(iii′) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to \mathcal{M}$ has a left adjoint, namely the functor that sends an object M to the weight $c \mapsto \mathrm{C}^n\big(\hbar^c, \mathbb{C}, \mathcal{M}(M, F)\big)$.*

Proof. The two sets of claims are formally dual; we will prove the first version.

(i). Using the interchange law for ends (theorem A.6.17), the Yoneda lemma for ends (proposition A.6.18), and proposition 1.8.12, we obtain the following natural bijections:

$$
\begin{aligned}
&\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\text{よ}_{c'},\mathbb{C},\text{よ}^{c}\big),\mathcal{M}(Gc'\odot Fc,M)\big)\\
&\qquad\cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathrm{C}^n\big(\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\mathbf{Set}\big(\text{よ}^{c},\mathcal{M}(Gc'\odot Fc,M)\big)\big)\\
&\qquad\cong \int_{c':\mathbb{C}}\int_{c:\mathbb{C}} \mathrm{C}^n\big(\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\mathbf{Set}\big(\text{よ}^{c},\mathcal{M}(Gc'\odot Fc,M)\big)\big)\\
&\qquad\cong \int_{c':\mathbb{C}} \mathrm{C}^n\Big(\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\int_{c:\mathbb{C}}\mathbf{Set}\big(\text{よ}^{c},\mathcal{M}(Gc'\odot Fc,M)\big)\Big)\\
&\qquad\cong \int_{c':\mathbb{C}} \mathrm{C}^n\big(\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\mathcal{M}(Gc'\odot F,M)\big)\\
&\qquad\cong \int_{c':\mathbb{C}} \mathrm{C}^n\big(Gc'\times\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M)\big)\\
&\qquad\cong \mathrm{C}^n\Big(\int^{c':\mathbb{C}} Gc'\times\text{よ}_{c'},\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M)\Big)\\
&\qquad\cong \mathrm{C}^n(G,\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M))\\
&\qquad\cong \mathcal{M}\big(\mathrm{B}_n(G,\mathbb{C},F),M\big)
\end{aligned}
$$

(ii). Similarly, we have the following natural bijections:

$$
\begin{aligned}
\mathcal{M}\big(\mathrm{B}_n(G,\mathbb{C},F),M\big) &\cong \mathrm{C}^n(G,\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M))\\
&\cong \mathrm{C}^n\Big(G,\mathbb{C}^{\mathrm{op}},\int_{c:\mathbb{C}}\mathbf{Set}\big(\text{よ}^{c},\mathcal{M}(Fc,M)\big)\Big)\\
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(G,\mathbb{C}^{\mathrm{op}},\mathbf{Set}\big(\text{よ}^{c},\mathcal{M}(Fc,M)\big)\big)\\
&\cong \int_{c:\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(G,\mathbb{C},\text{よ}^{c}\big),\mathcal{M}(Fc,M)\big)\\
&\cong \int_{c:\mathbb{C}} \mathcal{M}\big(Fc,\mathrm{B}_n\big(G,\mathbb{C},\text{よ}^{c}\big)\pitchfork M\big)
\end{aligned}
$$

Now apply remark A.6.5.

(iii). Along the same lines:

$$
\begin{aligned}
\mathcal{M}\big(\mathrm{B}_n(G,\mathbb{C},F),M\big) &\cong \mathrm{C}^n(G,\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M))\\
&\cong \mathrm{C}^n\Big(\int^{c:\mathbb{C}} Gc\times\text{よ}_{c},\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(Gc \times \hbar_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big) \\
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(Gc \times \hbar_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big) \\
&\cong \int_{c:\mathbb{C}} \mathbf{Set}\big(Gc, \mathrm{C}^n\big(\hbar_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big)\big)
\end{aligned}
$$

Note that in the last step we are appealing to lemma 1.8.13. ■

Remark 1.8.15. The above proposition shows that bar complexes are a certain kind of weighted colimit, while cobar complexes are a certain kind of weighted limit.

Definition 1.8.16. Let $\mathbb{C}$ be a small category, let $\mathcal{A}$ be any category and let $\mathcal{M}$ be a locally small category.

- Given $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, a **bar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$ is a simplicial object $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ equipped with bijections

$$\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{M}(Gc' \odot Fc, M)\big)$$

  that are natural in both n and M.

- Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, a **cobar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathcal{A}$ is a cosimplicial object $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ equipped with bijections

$$\mathcal{M}(M, \mathrm{C}^n(G, \mathbb{C}, F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{M}(M, Gc' \pitchfork Fc)\big)$$

  that are natural in both n and M.

Remark 1.8.17. Although the definition given here is stated using an end, one can also state a version that only uses products. Thus these generalised bar (resp. cobar) complexes exist in a locally small category $\mathcal{M}$ as soon as $\mathcal{M}$ has coproducts (resp. products) for small families of objects.

Remark 1.8.18. In the case where $\mathcal{A} = \mathcal{M} = \mathbf{sSet}$, we will almost always take $A \odot M = A \times M$ and $A \pitchfork M = [A, M]$. With this choice, the formulae of definition 1.8.1 (understood appropriately) can be applied verbatim.

Proposition 1.8.19. *Let $\mathbb{C}$ be a small category, let $\mathcal{A}$ be any category, and let $\mathcal{M}$ be a locally small category.*

- *Given $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, a weight $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$, and a diagram $F : \mathbb{C} \to \mathcal{M}$, if $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is a bar complex in $\mathcal{M}$, then*

$$\int^{c:\mathbb{C}} Gc \odot Fc \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

where the LHS coend exists in $\mathcal{M}$ if and only if the RHS colimit exists in $\mathcal{M}$.

- *Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, a weight $G : \mathbb{C} \to \mathcal{A}$, and a diagram $F : \mathbb{C} \to \mathcal{M}$, if $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is a cobar complex in $\mathcal{M}$, then*

$$\int_{c:\mathbb{C}} Gc \pitchfork Fc \cong \varprojlim_{\mathbf{\Delta}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

where the LHS end exists in $\mathcal{M}$ if and only if the RHS limit exists in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

Let M be an object in $\mathcal{M}$. Then,

$$\begin{aligned}\varprojlim_{\mathbf{\Delta}} \mathcal{M}\big(\mathrm{B}_\bullet(G, \mathbb{C}, F), M\big) &\cong \varprojlim_{\mathbf{\Delta}} \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_\bullet\big(\text{ʎ}_{c'}, \mathbb{C}, \text{ʎ}^c\big), \mathcal{M}(Gc' \odot Fc, M)\big) \\ &\cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\Big(\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet\big(\text{ʎ}_{c'}, \mathbb{C}, \text{ʎ}^c\big), \mathcal{M}(Gc' \odot Fc, M)\Big)\end{aligned}$$

where in the last step we have used proposition A.6.11. By proposition 1.8.10,

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet\big(\text{ʎ}_{c'}, \mathbb{C}, \text{ʎ}^c\big) \cong \mathbb{C}(c, c')$$

and the interchange law for ends (theorem A.6.17) implies

$$\begin{aligned}&\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}(\mathbb{C}(c', c), \mathcal{M}(Gc' \odot Fc, M)) \\ &\qquad\qquad \cong \int_{c:\mathbb{C}} \int_{c':\mathbb{C}} \mathbf{Set}(\mathbb{C}(c', c), \mathcal{M}(Gc' \odot Fc, M))\end{aligned}$$

but by the Yoneda lemma for ends (proposition A.6.18),

$$\int_{c':\mathbb{C}} \mathbf{Set}(\mathbb{C}(c', c), \mathcal{M}(Gc' \odot Fc, M)) \cong \mathcal{M}(Gc \odot Fc, M)$$

so we deduce that

$$\varprojlim_{\boldsymbol{\Delta}} \mathcal{M}\big(\mathrm{B}_\bullet(G, \mathbb{C}, F), M\big) \cong \int_{c:\mathbb{C}} \mathcal{M}(Gc \odot Fc, M)$$

naturally in M. Hence,

$$\int^{c:\mathbb{C}} Gc \odot Fc \cong \varinjlim_{\boldsymbol{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

where the LHS exists in $\mathcal{M}$ if and only if the RHS exists in $\mathcal{M}$. ∎

Lemma 1.8.20. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories. There is a natural transformation $\mathrm{C}^\bullet(-, \mathbb{D}, -) \Rightarrow \mathrm{C}^\bullet(-U, \mathbb{C}, -U)$ such that the following diagram of cosimplicial sets commutes for all weights $G : \mathbb{D} \to \mathbf{Set}$ and all diagrams $F : \mathbb{D} \to \mathbf{Set}$,*

$$\begin{tikzcd}
\operatorname{disc}[\mathbb{D}, \mathbf{Set}](G, F) \arrow[r] \arrow[d, "U^*"'] & \mathrm{C}^\bullet(G, \mathbb{D}, F) \arrow[d] \\
\operatorname{disc}[\mathbb{C}, \mathbf{Set}](GU, FU) \arrow[r] & \mathrm{C}^\bullet(GU, \mathbb{C}, FU)
\end{tikzcd}$$

where the horizontal arrows are the maximal augmentations.[12]

Proof. By definition,

$$\mathrm{C}^n(G, \mathbb{D}, F) = \prod_{(d_0, \ldots, d_n)} \big[Gd_n \times \mathbb{D}(d_n, d_{n-1}) \times \cdots \times \mathbb{D}(d_1, d_0), Fd_0\big]$$
$$\mathrm{C}^n(GU, \mathbb{C}, FU) = \prod_{(c_0, \ldots, c_n)} \big[GUc_n \times \mathbb{C}(c_n, c_{n-1}) \times \cdots \times \mathbb{C}(c_1, c_0), FUc_0\big]$$

so the maps $U : \mathbb{C}(c, c') \to \mathbb{D}(Uc, Uc')$ induce a morphism of cobar complexes with the required properties. ∎

Proposition 1.8.21. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories and let $\mathcal{M}$ be a locally small category.*

- *Let $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ be given and assume $\mathcal{M}$ has products for small families of objects. There is a natural transformation*

$$\mathrm{B}_\bullet(-U, \mathbb{C}, -U) \Rightarrow \mathrm{B}_\bullet(-, \mathbb{D}, -)$$

[12] Recall definition 1.6.17 and lemma 1.8.9.

of functors $[\mathbb{D}^{\mathrm{op}}, \mathcal{A}] \times [\mathbb{D}, \mathcal{M}] \to \mathbf{c}\mathcal{M}$*, and when* $\mathcal{M}$ *is cocomplete, the following diagram in* $\mathcal{M}$ *commutes for all weights* $G : \mathbb{D} \to \mathcal{A}$ *and all diagrams* $F : \mathbb{D} \to \mathcal{M}$*,*

$$\begin{tikzcd}
\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(GU, \mathbb{C}, FU) \arrow[r, "\cong"] \arrow[d] & \int^{c:\mathbb{C}} GUc \odot FUc \arrow[d] \\
\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{D}, F) \arrow[r, "\cong"'] & \int^{d:\mathbb{D}} Gd \odot Fd
\end{tikzcd}$$

where the horizontal arrows are the canonical isomorphisms[13] *and the right vertical arrow is the canonical comparison morphism.*

- *Let* $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ *be given and assume* $\mathcal{M}$ *has coproducts for small families of objects. There is a natural transformation*

$$\mathrm{C}^\bullet(-, \mathbb{D}, -) \Rightarrow \mathrm{C}^\bullet(-U, \mathbb{C}, -U)$$

of functors $[\mathbb{D}, \mathcal{A}]^{\mathrm{op}} \times [\mathbb{D}, \mathcal{M}] \to \mathbf{c}\mathcal{M}$*, and when* $\mathcal{M}$ *is complete, the following diagram in* $\mathcal{M}$ *commutes for all weights* $G : \mathbb{D} \to \mathcal{A}$ *and all diagrams* $F : \mathbb{D} \to \mathcal{M}$*,*

$$\begin{tikzcd}
\int_{d:\mathbb{D}} Gd \pitchfork Fd \arrow[r, "\cong"] \arrow[d] & \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{D}, F) \arrow[d] \\
\int_{c:\mathbb{C}} GUc \pitchfork FUc \arrow[r, "\cong"'] & \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(GU, \mathbb{C}, FU)
\end{tikzcd}$$

where the horizontal arrows are the canonical isomorphisms and the left vertical arrow is the canonical comparison morphism.

Proof. The two claims are formally dual; we will prove the first version.

Recalling proposition 1.8.14, it not hard to see that

$$\mathrm{B}_\bullet\left(U^* \text{\textit{ћ}}_{d'}, \mathbb{C}, U^* \text{\textit{ћ}}^d\right) \cong \int^{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbb{D}(Uc', d') \times \mathrm{B}_\bullet\left(\text{\textit{ћ}}_{c'}, \mathbb{C}, \text{\textit{ћ}}^c\right) \times \mathbb{D}(d, Uc)$$

and thus, applying the interchange law (theorem A.6.17) and the Yoneda lemma for coends (proposition A.6.18):

[13] See proposition 1.8.19.

$$\begin{aligned}&\mathcal{M}\big(\mathrm{B}_\bullet(GU, \mathbb{C}, FU), M\big)\\ &\qquad\cong \int_{(d',d):\mathbb{D}^{\mathrm{op}}\times\mathbb{D}} \mathbf{Set}\big(\mathrm{B}_\bullet\big(U^*\mathcal{k}_{d'}, \mathbb{C}, U^*\mathcal{k}^d\big), \mathcal{M}(Gd' \odot Fd, M)\big)\end{aligned}$$

Similarly,

$$\begin{aligned}&\mathcal{M}\Big(\int^{c:\mathbb{C}} GUc \odot FUc, M\Big)\\ &\qquad\cong \int_{(d',d):\mathbb{D}^{\mathrm{op}}\times\mathbb{D}} \mathbf{Set}\big(U^*\mathcal{k}_{d'} \star_{\mathbb{C}} U^*\mathcal{k}^d, \mathcal{M}(Gd' \odot Fd, M)\big)\end{aligned}$$

so it suffices to verify that there is a natural commutative diagram of the form below in **Set**:

$$\begin{tikzcd}
\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet\big(U^*\mathcal{k}_{d'}, \mathbb{C}, U^*\mathcal{k}^d\big) \arrow[r, "\cong"] \arrow[d] & U^*\mathcal{k}_{d'} \star_{\mathbb{C}} U^*\mathcal{k}^d \arrow[d] \\
\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet\big(\mathcal{k}_{d'}, \mathbb{D}, \mathcal{k}^d\big) \arrow[r, "\cong"'] & \mathcal{k}_{d'} \star_{\mathbb{D}} \mathcal{k}^d
\end{tikzcd}$$

In particular, it is enough to prove the original claim in the special case where $\mathcal{A} = \mathcal{M} = \mathbf{Set}$ and $\odot = \times$; but this can in turn be reduced to lemma 1.8.20 by considering the Yoneda embedding $\mathbf{Set}^{\mathrm{op}} \to [\mathbf{Set}, \mathbf{Set}]$, so we are done. ■

Lemma 1.8.22. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, let $F : \mathbb{D} \to \mathbf{Set}$ be a diagram, and let $G : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ be a weight. Then we have the following pullback diagram in* **sSet**,

$$\begin{tikzcd}
\mathrm{B}_\bullet(G \circ U, \mathbb{C}, F \circ U) \arrow[r] \arrow[d] & \mathrm{B}_\bullet(G, \mathbb{D}, F) \arrow[d] \\
\mathrm{B}_\bullet(\Delta 1, \mathbb{C}, \Delta 1) \arrow[r] & \mathrm{B}_\bullet(\Delta 1, \mathbb{D}, \Delta 1)
\end{tikzcd}$$

where the vertical arrows are induced by the unique natural transformations $F \Rightarrow \Delta 1$ and $G \Rightarrow \Delta 1$ and the horizontal arrows are the canonical comparison morphisms of proposition 1.8.21.

Proof. This is a straightforward exercise. ◊

Corollary 1.8.23. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, let $F : \mathbb{D} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{D}^{\mathrm{op}} \to \mathbf{sSet}$ be a weight. Then we have the following pullback diagram in* **sSet**,

$$\begin{tikzcd} \mathrm{B}(G \circ U, \mathbb{C}, F \circ U) \arrow[r] \arrow[d] & \mathrm{B}(G, \mathbb{D}, F) \arrow[d] \\ \mathrm{B}(\Delta 1, \mathbb{C}, \Delta 1) \arrow[r] & \mathrm{B}(\Delta 1, \mathbb{D}, \Delta 1) \end{tikzcd}$$

where the vertical arrows are induced by the unique natural transformations $F \Rightarrow \Delta 1$ and $G \Rightarrow \Delta 1$ and the horizontal arrows are the canonical comparison morphisms of proposition 1.8.21.

Proof. By lemma 1.6.8, it suffices to verify that the corresponding diagram of bar complexes is a pullback square in **sSet**, but that follows by applying lemma 1.8.22 degreewise. ∎

Theorem 1.8.24. *Let $\mathbb{C}$ and $\mathbb{D}$ be two small categories, let $\mathcal{A}$ and $\mathcal{M}$ be two locally small categories, and let $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, and $\underline{\mathcal{M}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ be functors. Suppose $\mathcal{A}$ has coproducts for small families of objects, that there are bijections*

$$\mathcal{M}(A \odot M, N) \cong \mathcal{A}(A, \underline{\mathcal{M}}(M, N)) \cong \mathcal{M}(M, A \pitchfork N)$$

that are natural in A, M, and N, and that there are isomorphisms

$$(A \otimes B) \odot M \cong A \odot (B \odot M)$$
$$(A \otimes B) \pitchfork M \cong A \pitchfork (B \pitchfork M)$$

that are natural in A, B, and M.

- *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{D}^{\mathrm{op}} \to \mathcal{A}$ be a weight, and let $H : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{A}$ be a functor. If $\mathcal{M}$ has coproducts for small families of objects, then there is an isomorphism*

  $$\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big) \cong \mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big)$$

  that is natural in m, n, F, G, and H.

- *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{D} \to \mathcal{A}$ be a weight, and let $H : \mathbb{D}^{\mathrm{op}} \times \mathbb{C} \to \mathcal{A}$ be a functor. If $\mathcal{M}$ has products for small families of objects, then there is an isomorphism*

$$\mathrm{C}^m\big(\mathrm{B}_n(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F\big) \cong \mathrm{C}^n(G, \mathbb{D}, \mathrm{C}^m(H, \mathbb{C}, F))$$

that is natural in m, n, F, G, and H.

Proof. The two claims are formally dual; we will prove the first version.

Let M be any object in $\mathcal{M}$ and let $K : \mathbb{D}^{\mathrm{op}} \times \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \times \mathbb{C} \to \mathbf{Set}$ be the functor defined below:

$$K(d', c', d, c) = \mathcal{A}(Gd' \otimes H(c', d), \underline{\mathcal{M}}(Fc, M))$$

Notice that we have the following natural bijections:

$$\begin{aligned} K(d', c', d, c) &\cong \mathcal{M}((Gd' \otimes H(c', d)) \odot Fc, M) \\ &\cong \mathcal{M}(Gd' \odot (H(c', d) \odot Fc), M) \\ &\cong \mathcal{M}(H(c', d) \odot Fc, Gd' \pitchfork M) \end{aligned}$$

Now, using the definition of the generalised bar complex, we obtain the natural bijections shown below:

$$\begin{aligned} &\mathcal{M}\big(\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big), M\big) \\ &\quad\cong \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{M}\big(\mathrm{B}_m(G, \mathbb{D}, H(c', -)) \odot Fc, M\big)\big) \\ &\quad\cong \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{A}\big(\mathrm{B}_m(G, \mathbb{D}, H(c', -)), \underline{\mathcal{M}}(Fc, M)\big)\big) \\ &\quad\cong \int_{(c',c)} \mathbf{Set}\left(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_m\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), K(d', c', d, c)\big)\right) \\ &\quad\cong \int_{(c',c)} \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big) \times \mathrm{B}_m\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), K(d', c', d, c)\big) \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathcal{M}\big(\mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big), M\big) \\ &\quad\cong \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), \mathcal{M}\big(Gd' \odot \mathrm{B}_m(H(-, d), \mathbb{C}, F), M\big)\big) \\ &\quad\cong \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), \mathcal{M}\big(\mathrm{B}_m(H(-, d), \mathbb{C}, F), Gd' \pitchfork M\big)\big) \end{aligned}$$

$$\cong \int_{(d',d)} \mathbf{Set}\left(\mathrm{B}_n\left(\text{ネ}_{d'}, \mathbb{D}, \text{ネ}^d\right), \int_{(c',c)} \mathbf{Set}\left(\mathrm{B}_m\left(\text{ネ}_{c'}, \mathbb{C}, \text{ネ}^c\right), K(d', c', d, c)\right)\right)$$
$$\cong \int_{(d',d)} \int_{(c',c)} \mathbf{Set}\left(\mathrm{B}_n\left(\text{ネ}_{d'}, \mathbb{D}, \text{ネ}^d\right) \times \mathrm{B}_m\left(\text{ネ}_{c'}, \mathbb{C}, \text{ネ}^c\right), K(d', c', d, c)\right)$$

and so, applying the interchange law for ends (theorem A.6.17), we obtain a natural bijection

$$\mathcal{M}\left(\mathrm{B}_m\left(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\right), M\right) \cong \mathcal{M}\left(\mathrm{B}_n\left(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\right), M\right)$$

and the claim follows by the Yoneda lemma. ■

Definition 1.8.25. Let $\mathbb{C}$ be a small category.

- Given $\odot : \mathcal{A} \times \mathbf{sSet} \to \mathbf{sSet}$, the **bar construction** for a diagram $F : \mathbb{C} \to \mathbf{sSet}$ weighted by a functor $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$ is the following coend:

  $$\mathrm{B}(G, \mathbb{C}, F) = \int^{[n]:\mathbf{\Delta}} \Delta^n \times \mathrm{B}_n(G, \mathbb{C}, F)$$

  In other words, $\mathrm{B}(G, \mathbb{C}, F)$ is the realisation $\left|\mathrm{B}_\bullet(G, \mathbb{C}, F)\right|$.

- Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathbf{sSet} \to \mathbf{sSet}$, the **cobar construction** for a diagram $F : \mathbb{C} \to \mathbf{sSet}$ weighted by a functor $G : \mathbb{C} \to \mathcal{A}$ is the following end:

  $$\mathrm{C}(G, \mathbb{C}, F) = \int_{[n]:\mathbf{\Delta}} \left[\Delta^n, \mathrm{B}_n(G, \mathbb{C}, F)\right]$$

  In other words, $\mathrm{C}(G, \mathbb{C}, F)$ is the totalisation $\operatorname{Tot} \mathrm{C}^\bullet(G, \mathbb{C}, F)$.

Lemma 1.8.26. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ be a weight. We then have bijections*

$$(\mathrm{B}(G, \mathbb{C}, F))_n \cong \mathrm{B}_n\left(G_n, \mathbb{C}, F_n\right)$$

that are natural in n.

Proof. Apply lemma 1.6.8 to remark 1.8.18. ■

Corollary 1.8.27. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram, and let $G : \mathbb{C} \to \mathbf{Set}$ be weight. Then the bar construction $\mathrm{B}(\operatorname{disc} G, \mathbb{C}, \operatorname{disc} F)$ is isomorphic to the bar complex $\mathrm{B}_\bullet(G, \mathbb{C}, F)$.* ■

Corollary 1.8.28. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{C} \to \mathbf{sSet}$ be weight. Then the bar construction $\mathrm{B}(F, \mathbb{C}^{\mathrm{op}}, G)$ is isomorphic to $\mathrm{B}(G, \mathbb{C}, F)^{\mathrm{op}}$.* ■

Lemma 1.8.29. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{C} \to \mathbf{sSet}$ be a weight. We then have bijections*

$$(\mathrm{C}^n(G, \mathbb{C}, F))_m \cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^m_l, \mathrm{C}^n\big(G_l, \mathbb{C}, F_l\big)\big)$$

that are natural in n, m, F, and G.

Proof. By remark 1.8.18,

$$\mathrm{C}^n(G, \mathbb{C}, F) \cong \prod_{(c_0,\ldots,c_n)} \big[Gc_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big]$$

so (by the Yoneda lemma) we have the following natural bijection in degree m:

$$(\mathrm{C}^n(G, \mathbb{C}, F))_m \cong \prod_{(c_0,\ldots,c_n)} \mathbf{sSet}\big(\Delta^m \times Gc_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big)$$

Moreover, by remark A.6.5,

$$\begin{aligned}
&\mathbf{sSet}\big(\Delta^m \times Gc_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big) \\
&\qquad \cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^m_l \times G_l c_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), F_l c_0\big) \\
&\qquad\quad \cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^m_l, \mathbf{Set}\big(G_l c_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), F_l c_0\big)\big)
\end{aligned}$$

and the claim follows. ■

Lemma 1.8.30. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{C} \to \mathbf{sSet}$ be a weight. We then have bijections*

$$\mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}, F)) \cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}, F_n\big)\big)$$

that are natural in X.

Proof. Lemma 1.6.22 says,

$$\mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}, F)) \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \mathrm{C}^m(G, \mathbb{C}, F)_m\big)$$

and by lemma 1.8.29,

$$\mathrm{C}^m(G, \mathbb{C}, F)_m \cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_n, \mathrm{C}^m\left(G_n, \mathbb{C}, F_n\right)\right)$$

so the interchange law for ends (theorem A.6.17) and the Yoneda lemma for ends (proposition A.6.18), we obtain the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}, F)) &\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_n, \mathrm{C}^m\left(G_n, \mathbb{C}, F_n\right)\right)\right) \\
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \mathbf{Set}\left(\Delta^m_n, \mathrm{C}^m\left(G_n, \mathbb{C}, F_n\right)\right)\right) \\
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_n, \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_n, \mathbb{C}, F_n\right)\right)\right) \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_n, \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_n, \mathbb{C}, F_n\right)\right)\right) \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(X_n, \mathrm{C}^n\left(G_n, \mathbb{C}, F_n\right)\right) \qquad \blacksquare
\end{aligned}$$

Lemma 1.8.31. *Let $\mathbb{C}$ be a small category. For any diagram $F : \mathbb{C} \to \mathbf{sSet}$, any weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, and any simplicial set Y, there is an isomorphism*

$$[\mathrm{B}(G, \mathbb{C}, F), Y] \cong \mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [F, Y])$$

and it is natural in F, G, and Y.

Proof. The Yoneda lemma implies it is enough to show that there is a bijection

$$\mathbf{sSet}(X, [\mathrm{B}(G, \mathbb{C}, F), Y]) \cong \mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [F, Y]))$$

that is natural in F, G, X, and Y. Now,

$$\mathbf{sSet}(X, [\mathrm{B}(G, \mathbb{C}, F), Y]) \cong \mathbf{sSet}(X \times \mathrm{B}(G, \mathbb{C}, F), Y)$$

and by remark A.6.5 and lemma 1.8.26:

$$\mathbf{sSet}(X \times \mathrm{B}(G, \mathbb{C}, F), Y) \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m \times \mathrm{B}_m\left(G_m, \mathbb{C}, F_m\right), Y_m\right)$$

On the other hand, by lemma 1.8.30:

$$\mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [F, Y])) \cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(X_n, \mathrm{C}^n\left(G_n, \mathbb{C}^{\mathrm{op}}, [F, Y]_n\right)\right)$$

and by the Yoneda lemma,

$$[Fc, Y]_n \cong \mathbf{sSet}(\Delta^n \times Fc, Y) \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m \times F_m c, Y_m\big)$$

thus,

$$\begin{aligned}&\mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, [F, Y]_n\big)\big)\\ &\qquad\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\Delta^n_m \times F_m, Y_m\big)\big)\big)\end{aligned}$$

but we know that

$$\begin{aligned}&\mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\Delta^n_m \times F_m, Y_m\big)\big)\big)\\ &\qquad\cong \mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(F_m, \mathbf{Set}\big(\Delta^n_m, Y_m\big)\big)\big)\big)\\ &\qquad\cong \mathbf{Set}\big(X_n, \mathbf{Set}\big(\mathrm{B}_n\big(G_n, \mathbb{C}, F_m\big), \mathbf{Set}\big(\Delta^n_m, Y_m\big)\big)\big)\\ &\qquad\cong \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(X_n \times \mathrm{B}_n\big(G_n, \mathbb{C}, F_m\big), Y_m\big)\big)\end{aligned}$$

and so, by the Yoneda lemma for ends (proposition A.6.18),

$$\begin{aligned}&\int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(X_n, \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\Delta^n_m \times F_m, Y_m\big)\big)\big)\\ &\qquad\cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(X_n \times \mathrm{B}_n\big(G_n, \mathbb{C}, F_m\big), Y_m\big)\big)\\ &\qquad\qquad\cong \mathbf{Set}\big(X_m \times \mathrm{B}_n\big(G_m, \mathbb{C}, F_m\big), Y_m\big)\end{aligned}$$

thus an application of the interchange law for ends (theorem A.6.17) completes the proof. ∎

Corollary 1.8.32. *Let $\mathbb{C}$ be a small category. For any diagram $F : \mathbb{C} \to \mathbf{sSet}$, any weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, and any simplicial set Y, there is an isomorphism*

$$\mathbf{sSet}(\mathrm{B}(G, \mathbb{C}, F), Y) \cong \int_{[n]:\mathbf{\Delta}} \mathrm{C}^n\big(G_n, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(F_n, Y_n\big)\big)$$

and it is natural in F, G, and Y.

Proof. The Yoneda lemma implies

$$\mathbf{sSet}(\mathrm{B}(G, \mathbb{C}, F), Y) \cong [\mathrm{B}(G, \mathbb{C}, F), Y]_0$$

and by lemma 1.8.31,

$$[\mathrm{B}(G,\mathbb{C},F),Y]_0 \cong (\mathrm{C}(G,\mathbb{C}^{\mathrm{op}},[F,Y]))_0$$

but lemma 1.8.30 implies

$$(\mathrm{C}(G,\mathbb{C}^{\mathrm{op}},[F,Y]))_0 \cong \int_{[m]:\boldsymbol{\Delta}} \mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},[F,Y]_m\right)$$

and using remark A.6.5 and the fact that $\mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},-\right)$ preserves limits, we obtain:

$$\begin{aligned}
\int_{[m]:\boldsymbol{\Delta}} \mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},[F,Y]_m\right) &\cong \int_{[m]:\boldsymbol{\Delta}} \mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},\mathbf{sSet}(\Delta^m\times F,Y)\right)\\
&\cong \int_{[m]:\boldsymbol{\Delta}}\int_{[n]:\boldsymbol{\Delta}} \mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},\mathbf{Set}\left(\Delta^m_n\times F_n,Y_n\right)\right)\\
&\cong \int_{[m]:\boldsymbol{\Delta}}\int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^n_m,\mathrm{C}^m\left(G_m,\mathbb{C}^{\mathrm{op}},\mathbf{Set}\left(F_n,Y_n\right)\right)\right)
\end{aligned}$$

Applying the interchange law (theorem A.6.17) and the Yoneda lemma for ends (proposition A.6.18) then yields the required natural bijection. ■

Proposition 1.8.33. *Let $\mathbb{C}$ be a small category and let $\mathcal{A}$ be any category.*

- *Let $F:\mathbb{C}\to\mathbf{sSet}$ be a diagram, let $G:\mathbb{C}^{\mathrm{op}}\to\mathcal{A}$ be a weight, and let M be a simplicial set. Given $\odot:\mathcal{A}\times\mathbf{sSet}\to\mathbf{sSet}$, we have bijections*

$$\mathbf{sSet}(\mathrm{B}(G,\mathbb{C},F),M) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}_\bullet\left(\hbar_{c'},\mathbb{C},\hbar^c\right),\left[Gc'\odot Fc,M\right]\right)$$

  that are natural in F, G, and M.

- *Let $F:\mathbb{C}\to\mathbf{sSet}$ be a diagram, let $G:\mathbb{C}\to\mathcal{A}$ be a weight, and let M be a simplicial set. Given $\pitchfork:\mathcal{A}^{\mathrm{op}}\times\mathbf{sSet}\to\mathbf{sSet}$, we have bijections*

$$\mathbf{sSet}(M,\mathrm{C}(G,\mathbb{C},F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}_\bullet\left(\hbar_{c'},\mathbb{C},\hbar^c\right),\left[M,Gc'\pitchfork Fc\right]\right)$$

  that are natural in F, G, and M.

Proof. We will prove the first claim; the second can be proved in a similar way.

By definition, we have the natural bijection

$$\mathbf{sSet}(\mathrm{B}(G,\mathbb{C},F),M) \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{sSet}\left(\Delta^n\times\mathrm{B}_n(G,\mathbb{C},F),M\right)$$

and furthermore, we also have the following:

$$\begin{aligned}
&\mathbf{sSet}\left(\Delta^n \times \mathrm{B}_n(G, \mathbb{C}, F), M\right) \\
&\qquad \cong \mathbf{sSet}\left(\mathrm{B}_n(G, \mathbb{C}, F), [\Delta^n, M]\right) \\
&\qquad \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\left(\mathrm{B}_n\left(\text{ℏ}_{c'}, \mathbb{C}, \text{ℏ}^c\right), \mathbf{sSet}(Gc' \odot Fc, [\Delta^n, M])\right) \\
&\qquad \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\left(\mathrm{B}_n\left(\text{ℏ}_{c'}, \mathbb{C}, \text{ℏ}^c\right), \mathbf{sSet}\left(\Delta^n, \left[Gc' \odot Fc, M\right]\right)\right) \\
&\qquad \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\left(\operatorname{disc} \mathrm{B}_n\left(\text{ℏ}_{c'}, \mathbb{C}, \text{ℏ}^c\right) \times \Delta^n, \left[Gc' \odot Fc, M\right]\right)
\end{aligned}$$

Thus, applying the interchange law for ends (theorem A.6.17) and corollary 1.6.9, we obtain

$$\mathbf{sSet}(\mathrm{B}(G, \mathbb{C}, F), M) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}_\bullet\left(\text{ℏ}_{c'}, \mathbb{C}, \text{ℏ}^c\right), \left[Gc' \odot Fc, M\right]\right)$$

as required. ■

Proposition 1.8.34. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

- *There is a natural transformation*

$$\mathrm{B}(-U, \mathbb{C}, -U) \Rightarrow \mathrm{B}(-, \mathbb{D}, -)$$

of functors $[\mathbb{D}^{\mathrm{op}}, \mathbf{sSet}] \times [\mathbb{D}, \mathbf{sSet}] \to \mathbf{sSet}$ such that the following diagram in $\mathbf{sSet}$ commutes for all weights $G : \mathbb{D}^{\mathrm{op}} \to \mathbf{sSet}$ and all diagrams $F : \mathbb{D} \to \mathbf{sSet}$,

$$\begin{tikzcd}
\mathrm{B}(GU, \mathbb{C}, FU) \arrow[r] \arrow[d] & \int^{c:\mathbb{C}} GUc \times FUc \arrow[d] \\
\mathrm{B}(G, \mathbb{D}, F) \arrow[r] & \int^{d:\mathbb{D}} Gd \times Fd
\end{tikzcd}$$

where the horizontal arrows and the right vertical arrow are the canonical comparison morphisms.[14]

- *There is a natural transformation*

$$\mathrm{C}(-, \mathbb{D}, -) \Rightarrow \mathrm{C}(-U, \mathbb{C}, -U)$$

[14] See lemma 1.6.7.

of functors $[\mathbb{D}, \mathbf{sSet}]^{\mathrm{op}} \times [\mathbb{D}, \mathbf{sSet}] \to \mathbf{sSet}$ *such that the following diagram in* $\mathbf{sSet}$ *commutes for all weights* $G : \mathbb{D} \to \mathbf{sSet}$ *and all diagrams* $F : \mathbb{D} \to \mathbf{sSet}$*,*

$$\begin{tikzcd} \int_{d:\mathbb{D}} [Gd, Fd] \arrow[r] \arrow[d] & \mathrm{C}(G, \mathbb{D}, F) \arrow[d] \\ \int_{c:\mathbb{C}} [GUc, FUc] \arrow[r] & \mathrm{C}(GU, \mathbb{C}, FU) \end{tikzcd}$$

where the horizontal arrows and the right vertical arrow are the canonical comparison morphisms.[15]

Proof. In view of the functoriality of $|-|$ (resp. $\mathrm{Tot}\,(-)$), this is an immediate consequence of proposition 1.8.19. ■

¶ 1.8.35. Let $\mathbb{C}$ be a small category. Extending the notation used previously, we make the following definitions:

- Given a functor $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, $\mathrm{B}(G, \mathbb{C}, \mathbb{C}) : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ is the functor defined by $c \mapsto \mathrm{B}\big(G, \mathbb{C}, \mathrm{disc}\, \hbar^c\big)$.
- Given a functor $F : \mathbb{C} \to \mathbf{sSet}$, $\mathrm{B}(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \to \mathbf{sSet}$ is the functor defined by $c \mapsto \mathrm{B}\big(\mathrm{disc}\, \hbar_c, \mathbb{C}, F\big)$.
- Given a functor $G : \mathbb{C} \to \mathbf{sSet}$, $\mathrm{C}(G, \mathbb{C}, \mathbb{C}) : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ is the functor defined by $c \mapsto \mathrm{C}\big(G, \mathbb{C}, \mathrm{disc}\, \hbar^c\big)$.
- Given a functor $F : \mathbb{C} \to \mathbf{sSet}$, $\mathrm{C}(\mathbb{C}, \mathbb{C}, F) : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ is the functor defined by $c \mapsto \mathrm{C}\big(\mathrm{disc}\, \hbar^c, \mathbb{C}, F\big)$.

Proposition 1.8.36. *Let* $\mathbb{C}$ *be a small category.*

(i) *For each weight* $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$*, we have an adjunction of the form below:*

$$\mathrm{B}(G, \mathbb{C}, -) \dashv [\mathrm{B}(G, \mathbb{C}, \mathbb{C}), -] : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$$

(ii) *For each diagram* $F : \mathbb{C} \to \mathbf{sSet}$*, we have an adjunction of the form below:*

$$\mathrm{B}(-, \mathbb{C}, F) \dashv \mathrm{C}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, [F, -]) : \mathbf{sSet} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}]$$

[15] See lemma 1.6.21.

(iii) *For each simplicial set X, there are isomorphisms*

$$\mathrm{B}(X \times G, \mathbb{C}, F) \cong X \times \mathrm{B}(G, \mathbb{C}, F) \cong \mathrm{B}(G, \mathbb{C}, X \times F)$$

that are natural in X, F, and G.

Dually:

(i′) *For each weight $G : \mathbb{C} \to \mathbf{sSet}$, we have an adjunction of the form below:*

$$\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times (-) \dashv \mathrm{C}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$$

(ii′) *For each diagram $F : \mathbb{C} \to \mathbf{sSet}$, we have an adjunction of the form below:*

$$\mathrm{C}(\mathbb{C}, \mathbb{C}, [-, F]) \dashv \mathrm{C}(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{sSet}]^{\mathrm{op}} \to \mathbf{sSet}$$

(iii′) *For each simplicial set X, there are isomorphisms*

$$\mathrm{C}(X \times G, \mathbb{C}, F) \cong [X, \mathrm{C}(G, \mathbb{C}, F)] \cong \mathrm{C}(G, \mathbb{C}, [X, F])$$

that are natural in X, F, and G.

Proof. (i). Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram and let Y be a simplicial set. By remark A.6.5, we have the following natural bijection,

$$[\mathbb{C}, \mathbf{sSet}](F, [\mathrm{B}(G, \mathbb{C}, \mathbb{C}), Y]) \cong \int_{c:\mathbb{C}} \mathbf{sSet}\big(Fc, \big[\mathrm{B}\big(G, \mathbb{C}, \operatorname{disc} \hbar^c\big), Y\big]\big)$$

and by definition,

$$\mathbf{sSet}\big(Fc, \big[\mathrm{B}\big(G, \mathbb{C}, \operatorname{disc} \hbar^c\big), Y\big]\big) \cong \mathbf{sSet}\big(Fc \times \mathrm{B}\big(G, \mathbb{C}, \operatorname{disc} \hbar^c\big), Y\big)$$

so it suffices to show that there is a natural isomorphism of the form below:

$$\mathrm{B}(G, \mathbb{C}, F) \cong \int^{c:\mathbb{C}} Fc \times \mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \hbar^c\big)$$

Since limits and colimits in $\mathbf{sSet}$ can be computed degreewise, by lemma 1.8.26, this amounts to showing that there are natural bijections

$$\mathrm{B}_n\big(G_n, \mathbb{C}, F_n\big) \cong \int^{c:\mathbb{C}} F_n c \times \mathrm{B}_n\big(G_n, \mathbb{C}^{\mathrm{op}}, \hbar_c\big)$$

and (after expanding the definition of $\mathrm{B}_n(G_n, \mathbb{C}^{\mathrm{op}}, \mathcal{K}_c)$) this is a straightforward consequence of the Yoneda lemma for coends (proposition A.6.18).

(ii). Corollary 1.8.28 then implies we have an adjunction of the form below,

$$\mathrm{B}(-, \mathbb{C}, F) \dashv [\mathrm{B}(\mathbb{C}, \mathbb{C}, F), -] : \mathbf{sSet} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}]$$

but lemma 1.8.31 says there is a natural isomorphism

$$[\mathrm{B}(\mathbb{C}, \mathbb{C}, F), Y] \cong \mathrm{C}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, [F, Y])$$

so we are done.

(iii). This is an immediate consequence of lemmas 1.8.13 and 1.8.26.

(i′). Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram and let X be a simplicial set. By remark A.6.5, we have the following natural bijection,

$$[\mathbb{C}, \mathbf{sSet}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times X, F) \cong \int_{c:\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \mathcal{K}_c\right) \times X, Fc\right)$$

and furthermore, by lemma 1.8.26:

$$\mathbf{sSet}\left(\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \mathcal{K}_c\right) \times X, Fc\right) \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(\mathrm{B}_n\left(G_n, \mathbb{C}^{\mathrm{op}}, \mathcal{K}_c\right) \times X_n, F_n c\right)$$

Now, we have

$$\mathbf{Set}\left(\mathrm{B}_n\left(G_n, \mathbb{C}^{\mathrm{op}}, \mathcal{K}_c\right), F_n c\right) \cong \mathrm{C}^n\left(G_n, \mathbb{C}, \mathbf{Set}\left(\mathcal{K}_c, F_n c\right)\right)$$

and since $\mathrm{C}^n\left(G_n, \mathbb{C}, -\right)$ preserves limits,

$$\int_{c:\mathbb{C}} \mathrm{C}^n\left(G_n, \mathbb{C}, \mathbf{Set}\left(\mathcal{K}_c, F_n c\right)\right) \cong \mathrm{C}^n\left(G_n, \mathbb{C}, \int_{c:\mathbb{C}} \mathbf{Set}\left(\mathcal{K}_c, F_n C\right)\right) \cong \mathrm{C}^n\left(G_n, \mathbb{C}, F_n\right)$$

where in the last step we used the Yoneda lemma for ends (proposition A.6.18). Thus, by the interchange law for ends (theorem A.6.17),

$$\begin{aligned} &[\mathbb{C}, \mathbf{sSet}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times X, F) \\ &\qquad \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_n, \int_{c:\mathbb{C}} \mathrm{C}^n\left(G_n, \mathbb{C}, \mathbf{Set}\left(\mathcal{K}_c, F_n c\right)\right)\right) \\ &\qquad\qquad \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_n, \mathrm{C}^n\left(G_n, \mathbb{C}, F_n\right)\right) \end{aligned}$$

so lemma 1.8.30 yields the required natural bijection:

$$[\mathbb{C}, \mathbf{sSet}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times X, F) \cong \mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}, F))$$

(ii′). Let $G : \mathbb{C} \to \mathbf{sSet}$ be a weight and let X be a simplicial set. We wish to construct a natural bijection of the following form:

$$[\mathbb{C}, \mathbf{sSet}](G, \mathrm{C}(\mathbb{C}, \mathbb{C}, [X, F])) \cong \mathbf{sSet}(X, \mathrm{C}(G, \mathbb{C}, F))$$

To begin, by remark A.6.5,

$$[\mathbb{C}, \mathbf{sSet}](G, \mathrm{C}(\mathbb{C}, \mathbb{C}, [X, F])) \cong \int_{c:\mathbb{C}} \mathbf{sSet}\left(Gc, \mathrm{C}\left(\operatorname{disc} \hbar^c, \mathbb{C}, [X, F]\right)\right)$$

and by lemma 1.8.30,

$$\mathbf{sSet}\left(Gc, \mathrm{C}\left(\operatorname{disc} \hbar^c, \mathbb{C}, [X, F]\right)\right) \cong \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(G_n c, \mathrm{C}^n\left(\hbar^c, \mathbb{C}, [X, F]_n\right)\right)$$

but clearly,

$$\mathbf{Set}\left(G_n c, \mathrm{C}^n\left(\hbar^c, \mathbb{C}, [X, F]_n\right)\right) \cong \mathrm{C}^n\left(G_n c \times \hbar^c, \mathbb{C}, [X, F]_n\right)$$

and since $\mathrm{C}\left(-, \mathbb{C}, [X, F]_n\right)$ takes colimits to limits, the Yoneda lemma for coends (proposition A.6.18) implies

$$\int_{c:\mathbb{C}} \mathrm{C}^n\left(G_n c \times \hbar^c, \mathbb{C}, [X, F]_n\right) \cong \mathrm{C}^n\left(G_n, \mathbb{C}, [X, F]_n\right)$$

so by using the interchange law for ends (theorem A.6.17):

$$\begin{aligned}
&[\mathbb{C}, \mathbf{sSet}](G, \mathrm{C}(\mathbb{C}, \mathbb{C}, [X, F])) \\
&\qquad \cong \int_{[n]:\mathbf{\Delta}} \int_{c:\mathbb{C}} \mathrm{C}^n\left(G_n c \times \hbar^c, \mathbb{C}, [X, F]_n\right) \\
&\qquad\qquad \cong \int_{[n]:\mathbf{\Delta}} \mathrm{C}^n\left(G_n, \mathbb{C}, [X, F]_n\right)
\end{aligned}$$

On the other hand, the Yoneda lemma implies $[X, Fc]_n \cong \mathbf{sSet}(\Delta^n \times X, Fc)$, so

$$\begin{aligned}
\mathrm{C}^n\left(G_n, \mathbb{C}, [X, F]_n\right) &\cong \mathrm{C}^n\left(G_n, \mathbb{C}, \mathbf{sSet}(\Delta^n \times X, F)\right) \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathrm{C}^n\left(G_n, \mathbb{C}, \mathbf{Set}\left(\Delta^n_m \times X_m, F_m\right)\right)
\end{aligned}$$

$$\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(X_m, \mathrm{C}^n\big(G_n, \mathbb{C}, F_m\big)\big)\big)$$

and using the interchange law and Yoneda lemma for ends again,

$$\begin{aligned}\int_{[n]:\mathbf{\Delta}} &\mathrm{C}^n\big(G_n, \mathbb{C}, [X, F]_n\big) \\ &\cong \int_{[m]:\mathbf{\Delta}} \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(X_m, \mathrm{C}^n\big(G_n, \mathbb{C}, F_m\big)\big)\big) \\ &\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \mathrm{C}^m\big(G_m, \mathbb{C}, F_m\big)\big)\end{aligned}$$

which completes the proof.

(iii′). It is not hard to see that we have the following natural isomorphisms of cosimplicial simplicial sets:

$$\mathrm{C}^\bullet(X \times G, \mathbb{C}, F) \cong [X, \mathrm{C}^\bullet(G, \mathbb{C}, F)] \cong \mathrm{C}^\bullet(G, \mathbb{C}, [X, F])$$

We then apply theorem 1.6.26 to obtain the corresponding natural isomorphisms of simplicial sets. ■

Theorem 1.8.37. *Let $\mathbb{C}$ and $\mathbb{D}$ be two small categories.*

- *Let $F : \mathbb{C} \to$ **sSet** be a diagram, let $G : \mathbb{D}^{\mathrm{op}} \to$ **sSet** be a weight, and let $H : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to$ **sSet** be a functor. There is then an isomorphism*

$$\mathrm{B}(\mathrm{B}(G, \mathbb{D}, H), \mathbb{C}, F) \cong \mathrm{B}(G, \mathbb{D}, \mathrm{B}(H, \mathbb{C}, F))$$

  that is natural in F, G, and H.

- *Let $F : \mathbb{C} \to$ **sSet** be a diagram, let $G : \mathbb{D} \to$ **sSet** be a weight, and let $H : \mathbb{D}^{\mathrm{op}} \times \mathbb{C} \to$ **sSet** be a functor. There is then an isomorphism*

$$\mathrm{C}(\mathrm{B}(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F) \cong \mathrm{C}(G, \mathbb{D}, \mathrm{C}(H, \mathbb{C}, F))$$

  that is natural in F, G, and H.

Proof. The first claim is a straightforward consequence of theorem 1.8.24 and lemma 1.8.26. We will now prove the second claim.

To prove the claim, the Yoneda lemma implies it is enough to construct a bijection

$$\mathbf{sSet}(X, \mathrm{C}(\mathrm{B}(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F)) \cong \mathbf{sSet}(X, \mathrm{C}(G, \mathbb{D}, \mathrm{C}^m(H, \mathbb{C}, F)))$$

that is natural in X, F, G, and H. By lemma 1.8.26 and lemma 1.8.30,

$$\begin{aligned}
&\mathbf{sSet}(X, \mathrm{C}(\mathrm{B}(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F)) \\
&\qquad \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_n, \mathrm{C}^n\left(\mathrm{B}_n\left(G_n, \mathbb{D}^{\mathrm{op}}, H_n\right), \mathbb{C}, F_n\right)\right)
\end{aligned}$$

and similarly,

$$\begin{aligned}
&\mathbf{sSet}(X, \mathrm{C}(G, \mathbb{D}, \mathrm{C}(H, \mathbb{C}, F))) \\
&\qquad \cong \int_{[m]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_m, \mathbb{D}, (\mathrm{C}(H, \mathbb{C}, F))_m\right)\right) \\
&\qquad \cong \int_{[m]:\boldsymbol{\Delta}} \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_m, \mathbb{D}, \mathbf{Set}\left(\Delta^m_n, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right)\right)\right)
\end{aligned}$$

where in the last step we used the Yoneda lemma and the fact that $\mathrm{C}^m\left(G_m, \mathbb{D}, -\right)$ preserves limits. Furthermore,

$$\begin{aligned}
&\mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_m, \mathbb{D}, \mathbf{Set}\left(\Delta^m_n, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right)\right)\right) \\
&\qquad \cong \mathbf{Set}\left(\Delta^m_n, \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_m, \mathbb{D}, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right)\right)\right)
\end{aligned}$$

and by using the interchange law for ends (theorem A.6.17) and the Yoneda lemma for ends (proposition A.6.18),

$$\begin{aligned}
&\int_{[m]:\boldsymbol{\Delta}} \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(\Delta^m_n, \mathbf{Set}\left(X_m, \mathrm{C}^m\left(G_m, \mathbb{D}, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right)\right)\right) \\
&\qquad \cong \int_{[n]:\boldsymbol{\Delta}} \mathbf{Set}\left(X_n, \mathrm{C}^n\left(G_n, \mathbb{D}, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right)\right)
\end{aligned}$$

but (by theorem 1.8.24 again),

$$\mathrm{C}^n\left(G_n, \mathbb{D}, \mathrm{C}^n\left(H_n, \mathbb{C}, F_n\right)\right) \cong \mathrm{C}^n\left(\mathrm{B}_n\left(G_n, \mathbb{D}^{\mathrm{op}}, H_n\right), \mathbb{C}, F_n\right)$$

so we are done. ∎

Proposition 1.8.38. *Let $\mathbb{C}$ be a small category.*

- *For each diagram $F : \mathbb{C} \to \mathbf{sSet}$ and each functor $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, there is a morphism $\mathrm{B}(G, \mathbb{C}, F) \to G \star_{\mathbb{C}} F$, and it is natural in both F and G.*
- *For each diagram $F : \mathbb{C} \to \mathbf{sSet}$ and each functor $G : \mathbb{C} \to \mathbf{Set}$, there is a morphism $\{G, F\}^{\mathbb{C}} \to \mathrm{C}(G, \mathbb{C}, F)$, and it is natural in both F and G.*

Proof. By theorem A.6.14 and proposition 1.8.10, we have the following natural isomorphisms:

$$\int^{[n]:\mathbf{\Delta}} \mathrm{B}_n(G, \mathbb{C}, F) \cong \Delta 1 \star_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F) \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F) \cong G \star_{\mathbb{C}} F$$

$$\int_{[n]:\mathbf{\Delta}} \mathrm{C}^n(G, \mathbb{C}, F) \cong \{\Delta 1, \mathrm{C}^\bullet(G, \mathbb{C}, F)\}^{\mathbf{\Delta}} \cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F) \cong \{G, F\}^{\mathbb{C}}$$

The claim then follows from the existence of a (unique) natural transformation $\Delta^\bullet \Rightarrow \Delta 1$. ■

Definition 1.8.39. Let $\mathbb{C}$ be a small category, let $\mathcal{M}$ be a locally small category, and let $F : \mathbb{C} \to \mathcal{M}$ be a diagram.

- The **bar resolution** of F is the diagram $\mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]$ defined by the following formula,

  $$c \mapsto \mathrm{B}_\bullet\left(\hbar_c, \mathbb{C}, F\right)$$

  where $\hbar_c : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is the representable functor $\mathbb{C}(-, c)$.
- The **cobar resolution** of F is the diagram $\mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \to [\mathbf{\Delta}, \mathcal{M}]$ defined by the following formula,

  $$c \mapsto \mathrm{C}^\bullet\left(\hbar^c, \mathbb{C}, F\right)$$

  where $\hbar^c : \mathbb{C} \to \mathbf{Set}$ is the representable functor $\mathbb{C}(c, -)$.

Lemma 1.8.40. *Let $\mathbb{C}$ be a small category and let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram.*

(i) *There is an isomorphism*

$$F \cong \varprojlim_{\mathbf{\Delta}} \circ\, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in F.

(ii) *For each weight* $G : \mathbb{C} \to \mathbf{Set}$*, there is an isomorphism*

$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathrm{C}^\bullet(G, \mathbb{C}, F)$$

and it is natural in both F *and* G.

(iii) *For each object* c *in* $\mathbb{C}$*, there exist maps* $\eta_c : Fc \to \mathrm{C}^0\big(\hbar^c, \mathbb{C}, F\big)$, $\varepsilon_c : \mathrm{C}^0\big(\hbar^c, \mathbb{C}, F\big) \to Fc$*, and* $h_{n,c} : \mathrm{C}^{n+1}\big(\hbar^c, \mathbb{C}, F\big) \to \mathrm{C}^n\big(\hbar^c, \mathbb{C}, F\big)$ *satisfying these identities:*

$$\begin{aligned}
\delta_1^1 \circ \eta_c &= \delta_1^0 \circ \eta_c \\
\varepsilon_c \circ \eta_c &= \mathrm{id} \\
h_{0,c} \circ \delta_1^0 &= \eta_c \circ \varepsilon_c \\
h_{n,c} \circ \delta_{n+1}^i &= \delta_n^i \circ h_{n-1,c} && \text{if } 0 \le i \le n \\
h_{n,c} \circ \delta_{n+1}^{n+1} &= \mathrm{id} \\
\sigma_n^i \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^i && \text{if } 0 \le i \le n \\
h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^{n+1}
\end{aligned}$$

These maps are moreover natural in F*, and* η_c *is also natural in* c.

Proof. (i). By lemma 1.8.9, there are bijections

$$[\mathbb{C}, \mathbf{Set}]\big(\hbar^c, F\big) \cong \varprojlim_{\boldsymbol{\Delta}} \mathrm{C}^\bullet\big(\hbar^c, \mathbb{C}, F\big)$$

that are natural in c and F, so the Yoneda lemma implies $F \cong \varprojlim_{\boldsymbol{\Delta}} \circ \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$, naturally in F.

(ii). Applying the Yoneda lemma for ends (proposition A.6.18), we obtain the following natural bijections:

$$\begin{aligned}
&\int_{c:\mathbb{C}} \big[Gc, \big[\mathbb{C}\big(c, c_n\big) \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big]\big] \\
&\qquad \cong \int_{c:\mathbb{C}} \big[\mathbb{C}\big(c, c_n\big), \big[Gc \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big]\big] \\
&\qquad\qquad \cong \big[Gc_n \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), Fc_0\big]
\end{aligned}$$

Theorem A.6.14 implies that there is a natural isomorphism

$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \int_{c:\mathbb{C}} \big[Gc, \mathrm{C}^\bullet\big(\hbar^c, \mathbb{C}, F\big)\big]$$

and it is now clear that the claim holds.

(iii). Let η_c, ε_c, and $h_{n,c}$ be the maps defined below:

$$\eta_c(x)_{(c_0)} = (y \mapsto F(y)(x))$$
$$\varepsilon_c(x) = x_{(c)}\big(\mathrm{id}_c\big)$$
$$h_{n,c}(x)_{(c_0,\ldots,c_n)} = \Big(\big(y, f_n, \ldots, f_1\big) \mapsto x_{(c_0,\ldots,c_n,c)}\big(\mathrm{id}_c, y, f_n, \ldots, f_1\big)\Big)$$

By construction, we have $\varepsilon_c \circ \eta_c = \mathrm{id}_{F_c}$, and it is not hard to check that the other identities are satisfied. For naturality of η_c in c, observe that, given $f : c \to c'$ in $\mathbb{C}$, we have

$$\begin{aligned}
\eta_{c'}(F(f)(x))_{(c_0)} &= (y \mapsto F(y)(F(f)(x))) \\
&= (y \mapsto F(y \circ f)(x)) \\
&= \big(y \mapsto F\big(\hbar^f(y)\big)(x)\big) \\
&= \mathrm{C}^0\big(\hbar^f, \mathbb{C}, F\big)\big(\eta_c(x)\big)_{(c_0)}
\end{aligned}$$

and so $\eta_{c'} \circ F(f) = \mathrm{C}^0\big(\hbar^f, \mathbb{C}, F\big) \circ \eta_c$, as required. ■

Proposition 1.8.41. *Let $\mathbb{C}$ be a small category, let $\mathcal{M}$ be a locally small category, and let $F : \mathbb{C} \to \mathcal{M}$ be a diagram. If the bar resolution $\mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F)$ exists, then:*

(i) *There is an isomorphism*

$$F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \circ\, \mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in F.

(ii) *For each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, there is an isomorphism*

$$G \star_{\mathbb{C}} \mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F) \cong \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

and it is natural in both F and G.

(iii) *For each object c in $\mathbb{C}$, there exist morphisms $\eta_c : Fc \to \mathrm{B}_0\big(\hbar_c, \mathbb{C}, F\big)$, $\varepsilon_c : \mathrm{B}_0\big(\hbar_c, \mathbb{C}, F\big) \to Fc$, and $h^n_c : \mathrm{B}_n\big(\hbar_c, \mathbb{C}, F\big) \to \mathrm{B}_{n+1}\big(\hbar_c, \mathbb{C}, F\big)$ satisfying these identities:*

$$\varepsilon_c \circ d^1_1 = \varepsilon_c \circ d^1_0$$

$$\begin{aligned}
\varepsilon_c \circ \eta_c &= \mathrm{id} \\
d_0^1 \circ h_c^0 &= s \circ r \\
d_i^{n+1} \circ h_c^n &= h_c^{n-1} \circ d_i^n && \textit{if } 0 \le i \le n \\
d_{n+1}^{n+1} \circ h_c^n &= \mathrm{id} \\
h_c^{n+1} \circ s_i^n &= s_i^{n+1} \circ h_c^n && \textit{if } 0 \le i \le n \\
h_c^{n+1} \circ h_c^n &= s_{n+1}^{n+1} \circ h_c^n
\end{aligned}$$

These morphisms are moreover natural in F, and ε_c is also natural in c.

Dually, if the cobar resolution $\mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$ exists, then:

(i) *There is an isomorphism*

$$F \cong \varprojlim_{\Delta} \circ \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in F.

(ii) *For each weight $G : \mathbb{C} \to \mathbf{Set}$, there is an isomorphism*

$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathrm{C}^\bullet(G, \mathbb{C}, F)$$

and it is natural in both F and G.

(iii) *For each object c in $\mathbb{C}$, there exist morphisms $\eta_c : Fc \to \mathrm{C}^0\left(\mathcal{k}^c, \mathbb{C}, F\right)$, $\varepsilon_c : \mathrm{C}^0\left(\mathcal{k}^c, \mathbb{C}, F\right) \to Fc$, and $h_{n,c} : \mathrm{C}^{n+1}\left(\mathcal{k}^c, \mathbb{C}, F\right) \to \mathrm{C}^n\left(\mathcal{k}^c, \mathbb{C}, F\right)$ satisfying these identities:*

$$\begin{aligned}
\delta_1^1 \circ \eta_c &= \delta_1^0 \circ \eta_c \\
\varepsilon_c \circ \eta_c &= \mathrm{id} \\
h_{0,c} \circ \delta_1^0 &= \eta_c \circ \varepsilon_c \\
h_{n,c} \circ \delta_{n+1}^i &= \delta_n^i \circ h_{n-1,c} && \textit{if } 0 \le i \le n \\
h_{n,c} \circ \delta_{n+1}^{n+1} &= \mathrm{id} \\
\sigma_n^i \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^i && \textit{if } 0 \le i \le n \\
h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^{n+1}
\end{aligned}$$

These morphisms are moreover natural in F, and η_c is also natural in c.

Proof. We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

Lemma 1.8.42. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to$ **sSet** be a diagram, and let $G : C \to$ **sSet** be a weight. If each Fc is a Kan complex, then $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is a Reedy-fibrant cosimplicial simplicial set.*

Proof. We must show that, for each natural number n, the matching morphism $\mathrm{C}^n(G, \mathbb{C}, F) \to \mathrm{M}_n(\mathrm{C}^\bullet(G, \mathbb{C}, F))$ is a Kan fibration. Consider the matching category $\partial\big([n] \downarrow \mathbf{\Delta}_{\leftarrow}\big)$: it is (isomorphic to) the full subcategory of the slice category $[n]_{/\mathbf{\Delta}}$ spanned by the non-trivial quotients of $[n]$. If we make the identification

$$\mathrm{C}^m(G, \mathbb{C}, F) \cong \prod_{c_m \to \cdots \to c_0} \big[Gc_m, Fc_0\big]$$

where the product is taken over the set of all m-simplices of $\mathrm{N}(\mathbb{C})$, then it is not hard to see that every codegeneracy operator is the evident product projection. One may then directly verify that

$$\mathrm{M}_n(\mathrm{C}^\bullet(G, \mathbb{C}, F)) \cong \prod_{c_m \to \cdots \to c_0} \big[Gc_m, Fc_0\big]$$

where now the product is taken over the set of *degenerate* m-simplices of $\mathrm{N}(\mathbb{C})$, and that the n-matching morphism is again the evident product projection. But corollary 1.4.16 implies that every product projection in question is a Kan fibration, so $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is indeed Reedy-fibrant. ■

Remark 1.8.43. The lemma is true in greater generality: see Example 23.8 in [Shulman, 2009].

1.9 Bousfield–Kan limits and colimits

Prerequisites. §§1.5, 1.6, 1.8, 2.4, 3.3, 3.4, 4.3, A.6

There are many definitions of 'homotopy limit/colimit', of varying abstractness and complexity. In this section, we will study the theory of Bousfield and Kan [1972] and compare it with some of the other definitions of 'homotopy limit/colimit'.

Remark 1.9.1. It is important to stress that there is an asymmetry between the theory of homotopy colimits and the theory of homotopy limits in **sSet** because not all simplicial sets are fibrant. As such, it will often be necessary to restrict our attention to Kan complexes when working with homotopy limits.

Definition 1.9.2. Let $\mathbb{C}$ be a small category.

- The **Bousfield–Kan limit** of F is defined by the following end in **sSet**:

$$\varprojlim_{\mathbb{C}}{}^{\mathrm{BK}} F = \int_{c:\mathbb{C}} \left[\mathrm{B}\left(\Delta 1, \mathbb{C}^{\mathrm{op}}, \hbar_c\right), Fc\right]$$

- The **Bousfield–Kan colimit** of F is defined by the following coend in **sSet**:

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} F = \int^{c:\mathbb{C}} \mathrm{B}\left(\Delta 1, \mathbb{C}, \hbar^c\right) \times Fc$$

- The **reversed Bousfield–Kan limit** of F is defined by the following end in **sSet**:

$$\varprojlim_{\mathbb{C}}{}^{\mathrm{KB}} F = \int_{c:\mathbb{C}} \left[\mathrm{B}\left(\hbar_c, \mathbb{C}, \Delta 1\right), Fc\right]$$

- The **reversed Bousfield–Kan colimit** of F is defined by the following coend in **sSet**:

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{KB}} F = \int^{c:\mathbb{C}} \mathrm{B}\left(\hbar^c, \mathbb{C}^{\mathrm{op}}, \Delta 1\right) \times Fc$$

Remark 1.9.3. In other words:

- The Bousfield–Kan limit of F is the simplicially enriched limit of F weighted by $\mathrm{B}(\Delta 1, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}})$.
- The Bousfield–Kan colimit of F is the simplicially enriched colimit of F weighted by $\mathrm{B}(\Delta 1, \mathbb{C}, \mathbb{C})$.
- The reversed Bousfield–Kan limit of F is the simplicially enriched limit of F weighted by $\mathrm{B}(\mathbb{C}, \mathbb{C}, \Delta 1)$.
- The reversed Bousfield–Kan colimit of F is the simplicially enriched colimit of F weighted by $\mathrm{B}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, \Delta 1)$.

Remark. There are various definitions of 'homotopy (co)limit' in the literature:

- The definition of 'homotopy limit' (resp. 'homotopy colimit') appearing in [Bousfield and Kan, 1972, Ch. XI, resp. Ch. XII] is what we call the 'Bousfield–Kan limit' (resp. 'reversed Bousfield–Kan colimit'): but beware that what they call the 'underlying space of $\mathbb{C}$' is actually $\mathrm{N}(\mathbb{C}^{\mathrm{op}})$.

- The definition of 'homotopy limit' (resp. 'homotopy colimit') appearing in [Hirschhorn, 2003, Ch. 18] is what we call 'reversed Bousfield–Kan limit' (resp. 'reversed Bousfield–Kan colimit').

Our conventions have been chosen to make both remark 1.9.4 and corollary 1.9.6 true.

Remark 1.9.4. Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram. By remark 1.8.3, we have the following natural isomorphisms,

$$\varprojlim_{\mathbb{C}}{}^{\mathrm{BK}} F^{\mathrm{op}} \cong \left(\varprojlim_{\mathbb{C}}{}^{\mathrm{KB}} F\right)^{\mathrm{op}}$$
$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} F^{\mathrm{op}} \cong \left(\varinjlim_{\mathbb{C}}{}^{\mathrm{KB}} F\right)^{\mathrm{op}}$$

and for every simplicial set Y, we have the following natural isomorphisms:

$$\varprojlim_{\mathbb{C}^{\mathrm{op}}}{}^{\mathrm{BK}} [F, Y] \cong \left[\varinjlim_{\mathbb{C}}{}^{\mathrm{KB}} F, Y\right]$$
$$\varprojlim_{\mathbb{C}^{\mathrm{op}}}{}^{\mathrm{BK}} [F, Y] \cong \left[\varinjlim_{\mathbb{C}}{}^{\mathrm{KB}} F, Y\right]$$

These should be regarded as duality principles.

Lemma 1.9.5. *Let X be a simplicial set and let $\mathbb{C}$ be a small category. Then the Bousfield–Kan limit of the constant diagram $\Delta X : \mathbb{C} \to \mathbf{sSet}$ is (isomorphic to) the simplicial set $[\mathrm{N}(\mathbb{C}^{\mathrm{op}}), X]$ (naturally in X).*

Proof. By definition,

$$\varprojlim_{\mathbb{C}}{}^{\mathrm{BK}} \Delta X \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(\Delta 1, \mathbb{C}^{\mathrm{op}}, h_c\right), X\right]$$

and it is not hard to verify that

$$\int_{c:\mathbb{C}} \left[\mathrm{B}\left(\Delta 1, \mathbb{C}^{\mathrm{op}}, h_c\right), X\right] \cong \varprojlim_{\mathbb{C}} [\mathrm{B}(\Delta 1, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), X]$$

but $[-, X]$ sends colimits in $\mathbf{sSet}$ to limits in $\mathbf{sSet}$, and

$$\varinjlim_{\mathbb{C}} \mathbb{C}(c', -) \cong 1$$

for all objects c' in $\mathbb{C}$, so by proposition 1.8.36,

$$\varinjlim_{\mathbb{C}} \mathrm{B}(\Delta 1, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \cong \mathrm{B}(\Delta 1, \mathbb{C}^{\mathrm{op}}, \Delta 1)$$

which can be identified with $\mathrm{N}(\mathbb{C}^{\mathrm{op}})$, by remark 1.8.4. ∎

Corollary 1.9.6. *Let X be a simplicial set and let $\mathbb{C}$ be a small category. Then the Bousfield–Kan colimit of the constant diagram $\Delta X : \mathbb{C} \to \mathbf{sSet}$ is (isomorphic to) the simplicial set $\mathrm{N}(\mathbb{C}) \times X$ (naturally in X).*

Proof. By remark 1.9.4,

$$\left[\varinjlim\nolimits_{\mathbb{C}}^{\mathrm{BK}} \Delta X, Y\right] \cong \varprojlim\nolimits_{\mathbb{C}^{\mathrm{op}}}^{\mathrm{BK}} [\Delta X, Y]$$

and by lemma 1.9.5,

$$\varprojlim\nolimits_{\mathbb{C}^{\mathrm{op}}}^{\mathrm{BK}} [\Delta X, Y] \cong [\mathrm{N}(\mathbb{C}), [X, Y]] \cong [\mathrm{N}(\mathbb{C}) \times X, Y]$$

so an application of the Yoneda lemma yields the claim. ∎

Proposition 1.9.7. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *For each diagram $F : \mathbb{D} \to \mathbf{sSet}$ and each weight $G : \mathbb{C} \to \mathbf{sSet}$, there is an isomorphism*

$$\mathrm{C}(G, \mathbb{C}, FU) \cong \int_{d:\mathbb{D}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} U^* \hbar_d\right), Fd\right]$$

and it is natural in both F and G.

(ii) *For each diagram $F : \mathbb{C} \to \mathbf{sSet}$ and each weight $G : \mathbb{D} \to \mathbf{sSet}$, there is an isomorphism*

$$\mathrm{C}(GU, \mathbb{C}, F) \cong \int_{d:\mathbb{D}} \left[Gd, \mathrm{C}\left(\operatorname{disc} U^* \hbar^d, \mathbb{C}, F\right)\right]$$

and it is natural in both F and G.

Dually:

(i′) *For each diagram $F : \mathbb{D} \to \mathbf{sSet}$ and each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, there is an isomorphism*

$$\mathrm{B}(G, \mathbb{C}, FU) \cong \int^{d:\mathbb{D}} \mathrm{B}\left(G, \mathbb{D}, \operatorname{disc} U^* \hbar^d\right) \times Fd$$

and it is natural in both F and G.

(ii′) *For each diagram* $F : \mathbb{C} \to \mathbf{sSet}$ *and each weight* $G : \mathbb{D}^{\mathrm{op}} \to \mathbf{sSet}$, *there is an isomorphism*

$$\mathrm{B}(GU, \mathbb{C}, F) \cong \int^{d:\mathbb{D}} Gd \times \mathrm{B}\big(\operatorname{disc} U^* \hbar_d, \mathbb{C}, F\big)$$

and it is natural in both F *and* G.

Proof. We will prove the first set of claims; the second can be proved in a similar way.

(i). By lemma 1.8.31,

$$\big[\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} U^* \hbar_d\big), Fd\big] \cong \mathrm{C}\big(G, \mathbb{C}, \big[\operatorname{disc} U^* \hbar_d, Fd\big]\big)$$

and since $\mathrm{C}(G, \mathbb{C}, -)$ preserves limits (by proposition 1.8.36), it is enough to construct a natural isomorphism of the following form:

$$FUc \cong \int_{d:\mathbb{D}} [\operatorname{disc} \mathbb{D}(Uc, d), Fd]$$

However, it is clear that

$$[\operatorname{disc} \mathbb{D}(Uc, d), Fd] \cong \mathbb{D}(Uc, d) \pitchfork Fd$$

so we may apply the Yoneda lemma for ends (proposition A.6.18) to complete the proof.

(ii). By proposition 1.8.36,

$$\big[Gd, \mathrm{C}\big(\operatorname{disc} U^* \hbar^d, \mathbb{C}, F\big)\big] \cong \mathrm{C}\big(Gd \times \operatorname{disc} U^* \hbar^d, \mathbb{C}, F\big)$$

and since $\mathrm{C}(-, \mathbb{C}, F)$ sends colimits to limits, it is enough to construct a natural isomorphism of the following form:

$$GUc \cong \int^{d:\mathbb{D}} Gd \times \operatorname{disc} \mathbb{D}(d, Uc)$$

However, it is clear that

$$Gd \times \operatorname{disc} \mathbb{D}(d, Uc) \cong \mathbb{D}(d, Uc) \odot Gd$$

so we may apply the Yoneda lemma for coends (proposition A.6.18) to complete the proof. ■

Remark 1.9.8. Thus, we should regard the cobar construction $\mathrm{C}(G, \mathbb{C}, F)$ (resp. the bar construction $\mathrm{B}(G, \mathbb{C}, F)$) as being the Bousfield–Kan analogue of the simplicially enriched weighted limit $\{G, F\}^{\mathbb{C}}$ (resp. the simplicially enriched colimit $G \star_{\mathbb{C}} F$).

Proposition 1.9.9. *Let $\mathbb{C}$ and $\mathbb{D}$ be small categories.*

- *Given weights $G : \mathbb{C} \to$ **sSet** and $H : \mathbb{D} \to$ **sSet** and a diagram $F : \mathbb{C} \times \mathbb{D} \to$ **sSet**,*

$$\mathrm{C}(G, \mathbb{C}, \mathrm{C}(H, \mathbb{D}, F)) \cong \mathrm{C}(G \boxtimes H, \mathbb{C} \times \mathbb{D}, F)$$

  *naturally in F, G, and H, where $G \boxtimes H : \mathbb{C} \times \mathbb{D} \to$ **sSet** is the functor defined by $(c, d) \mapsto Gc \times Hc$.*

- *Given weights $G : \mathbb{C}^{\mathrm{op}} \to$ **sSet** and $H : \mathbb{D}^{\mathrm{op}} \to$ **sSet** and a diagram $F : \mathbb{C} \times \mathbb{D} \to$ **sSet**,*

$$\mathrm{B}(G, \mathbb{C}, \mathrm{B}(H, \mathbb{D}, F)) \cong \mathrm{B}(G \boxtimes H, \mathbb{C} \times \mathbb{D}, F)$$

  *naturally in F, G, and H, where $G \boxtimes H : \mathbb{C}^{\mathrm{op}} \times \mathbb{D}^{\mathrm{op}} \to$ **sSet** is the functor defined by $(c, d) \mapsto Gc \times Hc$.*

Proof. We will prove the first claim; the second can be proved in a similar way.

By proposition 1.9.7,

$$\mathrm{C}(G, \mathbb{C}, \mathrm{C}(H, \mathbb{D}, F)) \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_c\right), \mathrm{C}(G, \mathbb{D}, Fc)\right]$$

and by propositions 1.8.36 and A.6.11,

$$\begin{aligned}&\left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_c\right), \mathrm{C}(G, \mathbb{D}, Fc)\right] \\ &\qquad \cong \int_{d:\mathbb{D}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_c\right) \times \mathrm{B}\left(H, \mathbb{D}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_d\right), F(c, d)\right]\end{aligned}$$

but it is easy to see that

$$\mathrm{B}_\bullet\left(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\right) \times \mathrm{B}_\bullet\left(H, \mathbb{D}^{\mathrm{op}}, \mathcal{h}_d\right) \cong \mathrm{B}_\bullet\left(G \boxtimes H, \mathbb{C}^{\mathrm{op}} \times \mathbb{D}^{\mathrm{op}}, \mathcal{h}_c \boxtimes \mathcal{h}_d\right)$$

so by the interchange law for ends (A.6.17),

$$\mathrm{C}(G, \mathbb{C}, \mathrm{C}(H, \mathbb{D}, F)) \cong \mathrm{C}(G \boxtimes H, \mathbb{C} \times \mathbb{D}, F)$$

as claimed. ∎

The homotopical significance of Bousfield–Kan limits/colimits (and more generally, bar/cobar constructions) is best expressed in terms of certain model structures on $[\mathbb{C}, \mathbf{sSet}]$.

Definition 1.9.10. Let $\mathcal{C}$ be a category and let $F, F' : \mathcal{C} \to \mathbf{sSet}$ be functors. A **natural weak homotopy equivalence** $F \Rightarrow F'$ is a natural transformation whose components are weak homotopy equivalences of simplicial sets.

Definition 1.9.11. Let $\mathcal{C}$ be a category.

- An **injective cofibration** in $[\mathcal{C}, \mathbf{sSet}]$ is a natural transformation of functors $\mathcal{C} \to \mathbf{sSet}$ whose components are monomorphisms.
- An **injective trivial cofibration** in $[\mathcal{C}, \mathbf{sSet}]$ is an injective cofibration that is also a natural weak homotopy equivalence.
- A **projective fibration** in $[\mathcal{C}, \mathbf{sSet}]$ is a natural transformation of functors $\mathcal{C} \to \mathbf{sSet}$ whose components are Kan fibrations.
- A **projective trivial fibration** in $[\mathcal{C}, \mathbf{sSet}]$ is a projective fibration that is also a natural weak homotopy equivalence.

Definition 1.9.12. Let $\mathbb{C}$ be a small category.

- An **injective fibration** in $[\mathbb{C}, \mathbf{sSet}]$ is a morphism that has the right lifting property with respect to all injective trivial cofibrations.
- An **injective trivial fibration** in $[\mathbb{C}, \mathbf{sSet}]$ is a morphism that has the right lifting property with respect to all injective cofibrations.
- A **projective cofibration** in $[\mathbb{C}, \mathbf{sSet}]$ is a morphism with the left lifting property with respect to all projective trivial fibrations.
- A **projective trivial cofibration** in $[\mathbb{C}, \mathbf{sSet}]$ is a morphism with the left lifting property with respect to all projective fibrations.

Theorem 1.9.13 (Bousfield and Kan)**.** *Let $\mathbb{C}$ be a small category. The following data constitute a a cofibrantly generated simplicial model structure on* $[\mathbb{C}, \mathbf{sSet}]$*:*

- *The weak equivalences are the natural weak homotopy equivalences.*
- *The fibrations are the projective fibrations, i.e. the componentwise Kan fibrations.*

- *The cofibrations are the projective cofibrations, i.e. the morphisms with the left lifting property with respect to componentwise trivial Kan fibrations.*

This model structure is called the **Bousfield–Kan model structure** *or the* **projective model structure** *on* $[\mathbb{C}, \mathbf{sSet}]$.

Proof. See the proof of Proposition 8.1 in [Bousfield and Kan, 1972, Ch. XI], or apply theorem 5.2.7 and proposition 2.4.17. ■

Theorem 1.9.14 (Heller)**.** *Let* $\mathbb{C}$ *be a small category. The following data constitute a cofibrantly generated simplicial model structure on* $[\mathbb{C}, \mathbf{sSet}]$*:*

- *The weak equivalences are the natural weak homotopy equivalences.*
- *The fibrations are the injective cofibrations, i.e. the (componentwise) monomorphisms.*
- *The cofibrations are the injective fibrations, i.e. the morphisms with the right lifting property with respect to componentwise anodyne extensions.*

This model structure is called the **Heller model structure** *or the* **injective model structure** *on* $[\mathbb{C}, \mathbf{sSet}]$.

Proof. See Theorem 4.5 in [Heller, 1988, Ch. II], or apply theorem 8.4.9 and proposition 2.4.17. ■

Proposition 1.9.15. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories.*

- *For any functor* $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$*, the bar construction* $\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{h}^\bullet\right)$ *is a cofibrant object in the Bousfield–Kan model structure on* $[\mathbb{D}^{\mathrm{op}}, \mathbf{sSet}]$*, where* $U^* : [\mathbb{D}, \mathbf{Set}] \to [\mathbb{C}, \mathbf{Set}]$ *is the functor induced by composition.*
- *For any functor* $F : \mathbb{C} \to \mathbf{sSet}$*, the bar construction* $\mathrm{B}\left(\operatorname{disc} U^* \mathscr{h}_\bullet, \mathbb{C}, F\right)$ *is a cofibrant object in the Bousfield–Kan model structure on* $[\mathbb{D}, \mathbf{sSet}]$*, where* $U^* : [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ *is the functor induced by composition.*
- *For any functor* $G : \mathbb{C} \to \mathbf{sSet}$*, the bar construction* $\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} U^* \mathscr{h}_\bullet\right)$ *is a cofibrant object in the Bousfield–Kan model structure on* $[\mathbb{D}, \mathbf{sSet}]$*, where* $U^* : [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ *is the functor induced by composition.*
- *For any functor* $F : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$*, the bar construction* $\mathrm{B}\left(\operatorname{disc} U^* \mathscr{h}^\bullet, \mathbb{C}^{\mathrm{op}}, F\right)$ *is a cofibrant object in the Bousfield–Kan model structure on* $[\mathbb{D}^{\mathrm{op}}, \mathbf{sSet}]$*, where* $U^* : [\mathbb{D}, \mathbf{Set}] \to [\mathbb{C}, \mathbf{Set}]$ *is the functor induced by composition.*

Proof. The four claims are formally dual; we will prove the first version.

Let $\mathcal{I} = \left\{ \partial\Delta^n \odot \mathscr{k}_d \hookrightarrow \Delta^n \odot \mathscr{k}_d \,\middle|\, n \geq 0, d \in \operatorname{ob}\mathbb{D} \right\}$. Using the Yoneda lemma and proposition A.3.26, we see that each element of $\mathcal{I}$ is a projective cofibration; so by proposition A.3.17, it suffices to prove that $\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^\bullet\right)$ is a $\mathcal{I}$-cell complex in the sense of § 0.5.

We proceed inductively. As usual, we define $\operatorname{sk}_{-1}(Y) = \varnothing$ for all simplicial sets Y. Suppose we have shown that the (componentwise) $(n-1)$-skeleton of $\mathrm{B}_\bullet\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^\bullet\right)$ is an $\mathcal{I}$-cell complex. Let $I_n(d) \subseteq \left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^d\right)\right)_n$ be the subset of non-degenerate n-simplices of $\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^d\right)$. By proposition 1.2.20, there is a canonical pushout diagram in **sSet** of the form below:

$$\begin{tikzcd}
\partial\Delta^n \odot I_n(d) \arrow[r, hook] \arrow[d] & \Delta^n \odot I_n(d) \arrow[d] \\
\operatorname{sk}_{n-1}\left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} F^* \mathscr{k}^d\right)\right) \arrow[r] & \operatorname{sk}_n\left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} F^* \mathscr{k}^d\right)\right)
\end{tikzcd}$$

By lemma 1.8.26, an n-simplex of $\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^d\right)$ is a tuple

$$\left(y, f_n, \ldots, f_1, x\right) \in \coprod_{(c_0, \ldots, c_n)} G_n\left(c_n\right) \times \mathbb{C}\left(c_{n-1}, c_n\right) \times \cdots \times \mathbb{C}\left(c_0, c_1\right) \times \mathbb{D}\left(d, U c_0\right)$$

where $\left(c_0, \ldots, c_n\right)$ ranges over $(n+1)$-tuples of objects in $\mathbb{C}$, and this n-simplex is degenerate if and only if at least one $f_i : c_{i-1} \to c_i$ is an identity morphism in $\mathbb{C}$. Thus, I_n is a coproduct of representable functors $\mathbb{D}^{\mathrm{op}} \to$ **Set** and is a subfunctor of $\left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} U^* \mathscr{k}^d\right)\right)_n$, so we have a pushout diagram of the form below in $[\mathbb{D}^{\mathrm{op}}, \mathbf{sSet}]$:

$$\begin{tikzcd}
\partial\Delta^n \odot I_n \arrow[r, hook] \arrow[d] & \Delta^n \odot I_n \arrow[d] \\
\operatorname{sk}_{n-1}\left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} F^* \mathscr{k}^\bullet\right)\right) \arrow[r] & \operatorname{sk}_n\left(\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} F^* \mathscr{k}^\bullet\right)\right)
\end{tikzcd}$$

We may now conclude that $\mathrm{B}\left(G, \mathbb{C}, \operatorname{disc} F^* \mathscr{k}^\bullet\right)$ is an $\mathcal{I}$-cell complex. ■

Proposition 1.9.16. *Let $\mathbb{C}$ be a small category.*

- *For each functor $G : \mathbb{C}^{\mathrm{op}} \to$ **sSet**, there is a natural weak homotopy equivalence $\mathrm{B}(G, \mathbb{C}, \mathbb{C}) \Rightarrow G$, and it is also natural in G.*

- *For each functor $F : \mathbb{C} \to$ **sSet**, there is a natural weak homotopy equivalence $\mathrm{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$, and it is also natural in F.*

- *For each functor $G : \mathbb{C} \to$ **sSet**, there is a natural weak homotopy equivalence* $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \Rightarrow G$*, and it is also natural in* G.

- *For each functor $F : \mathbb{C}^{\mathrm{op}} \to$ **sSet**, there is a natural weak homotopy equivalence* $\mathrm{B}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, F) \Rightarrow F$*, and it is also natural in* F.

Proof. The four claims are formally dual; we will prove the first version.

Let $K_\bullet, L_\bullet : \mathbb{C}^{\mathrm{op}} \to \mathbf{ssSet}$ be the functors defined below:

$$K_{n,m}(c) = G_n(c)$$
$$L_{n,m}(c) = \mathrm{B}_m\big(G_n, \mathbb{C}, \hbar^c\big)$$

Observe that, by lemma 1.6.8, we have a natural isomorphism $\left|K_\bullet(c)\right| \cong G(c)$; and by lemma 1.8.26, $\left|L_\bullet(c)\right| \cong \mathrm{B}\big(G, \mathbb{C}, \hbar^c\big)$. On the other hand, recalling proposition 1.7.12 and corollary 1.8.28, we see that proposition 1.8.41 implies that there is a natural Reedy weak equivalence $L_\bullet(c) \to K_\bullet(c)$. Thus, by theorem 1.6.10, the induced natural transformation $\mathrm{B}(G, \mathbb{C}, \mathbb{C}) \Rightarrow G$ is a natural weak homotopy equivalence, and it is clearly also natural in G. ■

Corollary 1.9.17. *Let $\mathbb{C}$ be a small category. For any weight $G : \mathbb{C} \to$ **sSet**,* $\mathrm{C}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *sends natural weak homotopy equivalences between projective-fibrant diagrams to weak homotopy equivalences of Kan complexes.*

Proof. Apply Ken Brown's lemma (4.3.6) to proposition 1.9.19. ■

Proposition 1.9.18. *Let $\mathbb{C}$ be a small category. For any weight $G : \mathbb{C} \to$ **sSet**, there is an adjunction of the form below,*

$$\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times (-) \dashv \mathrm{C}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$$

and it is a Quillen adjunction with respect to both the Heller and Bousfield–Kan model structures on $[\mathbb{C}, \mathbf{sSet}]$

Proof. The existence of the adjunction has been shown in proposition 1.8.36, so by proposition 4.3.2, it suffices to show that

$$\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times (-) : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$$

is a left Quillen functor with respect to the Heller model structure and that

$$\mathrm{C}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$$

is a right Quillen functor with respect to the Bousfield–Kan model structure.

It is clear that the induced natural transformation $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times Z \Rightarrow \mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times W$ is an injective cofibration in $[\mathbb{C}, \mathbf{sSet}]$ if $Z \to W$ is a monomorphism in $\mathbf{sSet}$. Moreover, the 2-out-of-3 property and proposition 1.5.15 imply that $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times (-)$ preserves weak equivalences. Thus $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \times (-)$ is indeed a left Quillen functor with respect to the Heller model structure.

Now, proposition 1.9.7 says that

$$\mathrm{C}(G, \mathbb{C}, F) \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\right), Fc\right]$$

naturally in both F and G; but by remark 2.1.24,

$$[\mathbb{C}, \underline{\mathbf{sSet}}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), F) \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\right), Fc\right]$$

and proposition 1.9.15 says $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}})$ is cofibrant in the Bousfield–Kan model structure on $[\mathbb{C}, \mathbf{sSet}]$, so by (theorem 1.9.13 and) proposition 2.4.7, $\mathrm{C}(G, \mathbb{C}, -)$ is indeed a right Quillen functor with respect to the Bousfield–Kan model structure. ■

Proposition 1.9.19. *Let $\mathbb{C}$ be a small category.*

- *For any weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, there is an adjunction of the form below,*

  $$\mathrm{B}(G, \mathbb{C}, -) \dashv [\mathrm{B}(G, \mathbb{C}, \mathbb{C}), -] : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$$

  and it is a Quillen adjunction with respect to both the Bousfield–Kan and Heller model structures on $[\mathbb{C}, \mathbf{sSet}]$.

- *For any diagram $F : \mathbb{C} \to \mathbf{sSet}$, there is an adjunction of the form below,*

  $$\mathrm{B}(-, \mathbb{C}, F) \dashv \mathrm{C}(\mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, [F, -]) : \mathbf{sSet} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}]$$

  and it is a Quillen adjunction with respect to both the Bousfield–Kan and Heller model structures on $[\mathbb{C}, \mathbf{sSet}]$.

Proof. The two claims are formally dual;[16] we will prove the first version.

The existence of the adjunction has been shown in proposition 1.8.36, so by proposition 4.3.2, it suffices to show that

$$[\mathrm{B}(G, \mathbb{C}, \mathbb{C}), -] : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$$

[16] Recall proposition 1.7.12 and corollary 1.8.28.

is a right Quillen functor with respect to the Bousfield–Kan model structure and that

$$\mathrm{B}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$$

is a left Quillen functor with respect to the Heller model structure.

By corollary 1.4.16, the induced natural transformation $[\mathrm{B}(G, \mathbb{C}, \mathbb{C}), X] \Rightarrow [\mathrm{B}(G, \mathbb{C}, \mathbb{C}), Y]$ is a projective fibration (resp. projective trivial fibration) if $X \to Y$ is a Kan fibration (resp. trivial Kan fibration). Thus $[\mathrm{B}(G, \mathbb{C}, \mathbb{C}), -]$ is indeed a right Quillen functor with respect to the Bousfield–Kan model structure.

On the other hand, the cofibrations in the Heller model structure are the (componentwise) monomorphisms, lemma 1.8.26 implies that $\mathrm{B}(G, \mathbb{C}, -)$ preserves cofibrations. To complete the proof, it is enough to show that $\mathrm{B}(G, \mathbb{C}, -)$ preserves weak equivalences. Let $\varphi : X \Rightarrow Y$ be a natural weak homotopy equivalence of diagrams $\mathbb{C} \to \mathbf{sSet}$. To show that $\mathrm{B}(G, \mathbb{C}, \varphi) : \mathrm{B}(G, \mathbb{C}, X) \to \mathrm{B}(G, \mathbb{C}, Y)$ is a weak homotopy equivalence, it is enough to verify that the induced morphism

$$[\mathrm{B}(G, \mathbb{C}, \varphi), K] : [\mathrm{B}(G, \mathbb{C}, Y), K] \to [\mathrm{B}(G, \mathbb{C}, X), K]$$

is a weak homotopy equivalence for all Kan complexes K. But by lemma 1.8.31, this is the same as showing that

$$\mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [\varphi, K]) : \mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [Y, K]) \to \mathrm{C}(G, \mathbb{C}^{\mathrm{op}}, [X, K])$$

is a weak homotopy equivalence for all Kan complexes K; and corollary 1.4.16 implies that $[\varphi, K]$ is a natural weak homotopy equivalence between projective-fibrant diagrams, so we may apply corollary 1.9.17. ■

Corollary 1.9.20. *Let $\mathbb{C}$ be a small category.*

- *For any weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, $\mathrm{B}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ preserves weak equivalences.*
- *For any diagram $F : \mathbb{C} \to \mathbf{sSet}$, $\mathrm{B}(-, \mathbb{C}, F) : [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$ preserves weak equivalences.*

Proof. Apply Ken Brown's lemma (4.3.6) to proposition 1.9.19 and the fact that every object is cofibrant in the Heller model structure. ■

Proposition 1.9.21. *Let $\mathbb{C}$ be a small category. Then*

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} \simeq \varinjlim_{\mathbb{C}}{}^{\mathrm{KB}}$$

as functors $[\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$, i.e. there is a zigzag of natural weak homotopy equivalences connecting them.

Proof. By proposition 1.7.12 and corollary 1.9.20, we have the following zigzag of natural weak homotopy equivalences,

$$\begin{tikzcd}
& \varinjlim_{\mathbb{C}}^{\mathrm{BK}} \mathrm{Sd}(-) \arrow[dl] \arrow[dr] & & \mathrm{Sd}\left(\varinjlim_{\mathbb{C}}^{\mathrm{BK}} (-)^{\mathrm{op}}\right) \arrow[dl] \arrow[dr] & \\
\varinjlim_{\mathbb{C}}^{\mathrm{BK}} (-) & & \varinjlim_{\mathbb{C}}^{\mathrm{BK}} (-)^{\mathrm{op}} & & \left(\varinjlim_{\mathbb{C}}^{\mathrm{BK}} (-)^{\mathrm{op}}\right)^{\mathrm{op}}
\end{tikzcd}$$

and by remark 1.9.4, the claim follows. ■

Proposition 1.9.22. *Let $\mathbb{C}$ be a small category. For any diagram $F : \mathbb{C} \to \mathbf{sSet}$, there is an adjunction of the form below,*

$$\mathrm{C}(\mathbb{C}, \mathbb{C}, [-, F]) \dashv \mathrm{C}(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{sSet}]^{\mathrm{op}} \to \mathbf{sSet}$$

and moreover:

(i) *If F is projective-fibrant, then the adjunction is a Quillen adjunction with respect to the* Bousfield–Kan *model structure on* $[\mathbb{C}, \mathbf{sSet}]$.

(ii) *If F is injective-fibrant, then the adjunction is a Quillen adjunction with respect to the* Heller *model structure on* $[\mathbb{C}, \mathbf{sSet}]$.

Proof. The existence of the adjunction was shown in proposition 1.8.36; it remains to be shown that it is a Quillen adjunction under the appropriate hypotheses.

(i). Suppose F is projective-fibrant. Proposition 1.9.7 says that

$$\mathrm{C}(G, \mathbb{C}, F) \cong \int_{c:\mathbb{C}} \left[Gc, \mathrm{C}\left(\mathrm{disc}\, \hbar^c, \mathbb{C}, F\right)\right]$$

naturally in both F and G; but by remark 2.1.24,

$$[\mathbb{C}, \underline{\mathbf{sSet}}](G, \mathrm{C}(\mathbb{C}, \mathbb{C}, F)) \cong \int_{c:\mathbb{C}} \left[Gc, \mathrm{C}\left(\mathrm{disc}\, \hbar^c, \mathbb{C}, F\right)\right]$$

and proposition 1.9.18 implies that $\mathrm{C}(\mathbb{C}, \mathbb{C}, F)$ is projective-fibrant if F is, so by (theorem 1.9.13 and) proposition 2.4.7,

$$[\mathbb{C}, \underline{\mathbf{sSet}}](-, F) : [\mathbb{C}, \mathbf{sSet}]^{\mathrm{op}} \to \mathbf{sSet}$$

is indeed a right Quillen functor with respect to the Bousfield–Kan model structure.

(ii). Suppose F is injective-fibrant. Proposition 1.9.7 says that

$$\mathrm{C}(G, \mathbb{C}, F) \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \operatorname{disc} \hbar_c\right), Fc\right]$$

naturally in both F and G; but by remark 2.1.24,

$$[\mathbb{C}, \underline{\mathbf{sSet}}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), F) \cong \int_{c:\mathbb{C}} \left[\mathrm{B}\left(G, \mathbb{C}^{\mathrm{op}}, \hbar_c\right), Fc\right]$$

and (theorem 1.9.14 plus) proposition 2.4.7 says

$$[\mathbb{C}, \underline{\mathbf{sSet}}](-, F) : [\mathbb{C}, \mathbf{sSet}]^{\mathrm{op}} \to \mathbf{sSet}$$

is a right Quillen functor with respect to the Heller model structure; on the other hand, proposition 1.9.19 says

$$\mathrm{B}(-, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

is a left Quillen functor with respect to the Heller model structure, so (by propositions 4.3.2 and 4.3.5) $\mathrm{C}(-, \mathbb{C}, F)$ is indeed a right Quillen functor with respect to the Heller model structure. ■

The homotopical universal property of the bar/cobar constructions is traditionally stated in terms of derived functors.

Definition 1.9.23. Let $\mathbb{C}$ be a small category.

- A **homotopy limit functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ is a homotopical right approximation for the functor $\varprojlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.
- A **homotopy colimit functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ is a homotopical left approximation for the functor $\varinjlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.

Remark 1.9.24. Homotopy limit/colimit functors are *not* well defined up to isomorphism, but by remark 3.4.7, they are *homotopically* unique. By definition, each homotopy limit functor (resp. homotopy colimit functor) is equipped with a natural transformation from $\varprojlim_{\mathbb{C}}$ (resp. to $\varinjlim_{\mathbb{C}}$); and for general reasons (cf. proposition 4.3.17), the component at an injective-fibrant (resp. projective-cofibrant) diagram is a weak homotopy equivalence. However, we can do slightly better with homotopy limit functors.

Since there is no more difficulty in doing so, we will also consider generalised homotopy limits/colimits:

Definition 1.9.25. Let $\mathbb{C}$ be a small category.

- Let $G : \mathbb{C} \to \mathbf{sSet}$ be a weight and let $\{G, -\}^{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ be the functor defined below,

  $$\{G, F\}^{\mathbb{C}} = \int_{c:\mathbb{C}} [Gc, Fc]$$

  A **homotopy G-weighted limit functor** is a homotopical right approximation for $\{G, -\}^{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.

- Let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ be a weight and let $G \star_{\mathbb{C}} (-) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ be the functor defined below:

  $$G \star_{\mathbb{C}} F = \int^{c:\mathbb{C}} Gc \times Fc$$

  A **homotopy G-weighted colimit functor** is a homotopical left approximation for $G \star_{\mathbb{C}} (-) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.

Theorem 1.9.26. *Let $\mathbb{C}$ be a small category, let $G : \mathbb{C} \to \mathbf{sSet}$ be a weight, and let $R : \mathbf{sSet} \to \mathbf{sSet}$ be (the functor part of) any functorial fibrant replacement in* $\mathbf{sSet}$.

(i) $\{G, -\}^{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *sends natural weak homotopy equivalences between diagrams of the form* $\mathrm{C}(\mathbb{C}, \mathbb{C}, F)$ *where every Fc is a Kan complex to weak homotopy equivalences of simplicial sets.*

(ii) $\mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ *is (the functor part of) a functorial right deformation retract for* $\{G, -\}^{\mathbb{C}}$.

(iii) $\mathrm{C}(G, \mathbb{C}, R \circ -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *is (the functor part of) a homotopy G-weighted limit functor.*

Proof. (i). Let $X, Y : \mathbb{C} \to \mathbf{Kan}$ be diagrams and let $\varphi : \mathrm{C}(\mathbb{C}, \mathbb{C}, X) \Rightarrow \mathrm{C}(\mathbb{C}, \mathbb{C}, Y)$ be a natural weak homotopy equivalence. We have the following commutative diagram,

$$\begin{array}{ccc} \{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, X)\}^{\mathbb{C}} & \longrightarrow & \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, X)\}^{\mathbb{C}} \\ \downarrow & & \downarrow \\ \{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, Y)\}^{\mathbb{C}} & \longrightarrow & \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, Y)\}^{\mathbb{C}} \end{array}$$

where the horizontal arrows are induced by the natural weak homotopy equivalence of proposition 1.9.16 and the vertical arrows are induced by φ; but by remark 2.1.24,

$$\{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), -\}^{\mathbb{C}} \cong [\mathbb{C}, \underline{\mathbf{sSet}}](\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), -)$$

and $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}})$ is projective-cofibrant by proposition 1.9.15, and since both $\mathrm{C}(\mathbb{C}, \mathbb{C}, X)$ and $\mathrm{C}(\mathbb{C}, \mathbb{C}, Y)$ are projective-fibrant by proposition 1.9.18, we may use corollary 1.9.17 to deduce that

$$\{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, X)\}^{\mathbb{C}} \to \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, Y)\}^{\mathbb{C}}$$

is a weak homotopy equivalence of Kan complexes. On the other hand, the following diagram commutes for all diagrams $F : \mathbb{C} \to \mathbf{sSet}$,

$$\begin{array}{ccc} \{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} & \longrightarrow & \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), F\}^{\mathbb{C}} & \longrightarrow & \{\mathrm{B}(\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), F\}^{\mathbb{C}} \end{array}$$

where the horizontal arrows are induced by the natural weak homotopy equivalence $\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \Rightarrow G$ (from proposition 1.9.16), and the vertical arrows are the isomorphisms given by proposition 1.9.7. Moreover, by corollary 1.9.20,

$$\mathrm{B}(\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \to \mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}})$$

is a natural weak homotopy equivalence, so if F is projective-fibrant, then

$$\{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \to \{\mathrm{B}(G, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}), \mathrm{C}(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}}$$

is a weak homotopy equivalence of Kan complexes; hence, by the 2-out-of-3 property,

$$\{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, X)\}^{\mathbb{C}} \to \{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, Y)\}^{\mathbb{C}}$$

is also a weak homotopy equivalence of Kan complexes.

(ii). It remains to be shown that there is a natural weak equivalence $\mathrm{id}_{[\mathbb{C},\mathbf{sSet}]} \Rightarrow \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -)$. Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram. By (the Yoneda lemma for ends (proposition A.6.18 and) the arguments above, we have a natural weak homotopy equivalence

$$RFc \cong \left\{\operatorname{disc} \hbar^{c}, RF\right\}^{\mathbb{C}} \to \left\{\mathrm{B}\left(\operatorname{disc} \hbar^{c}, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}\right), F\right\}^{\mathbb{C}} \cong \mathrm{C}\left(\operatorname{disc} \hbar^{c}, \mathbb{C}, RF\right)$$

and we have a natural weak homotopy equivalence $Fc \to RFc$ by definition, so do indeed have a natural weak homotopy equivalence $F \Rightarrow \mathrm{C}(\mathbb{C}, \mathbb{C}, F)$, as required.

(iii). Thus, by theorem 3.4.11, $\{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -)\}^{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ is (the functor part of) a homotopical right approximation for $\{G, -\}^{\mathbb{C}}$, and by proposition 1.9.7,

$$\mathrm{C}(G, \mathbb{C}, R \circ -) \cong \{G, \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -)\}^{\mathbb{C}}$$

so we are done. ■

Theorem 1.9.27. *Let $\mathbb{C}$ be a small category and let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ be a weight.*

(i) $G \star_{\mathbb{C}} (-) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *sends natural weak homotopy equivalences between diagrams of the form* $\mathrm{B}(\mathbb{C}, \mathbb{C}, F)$ *to weak homotopy equivalences of simplicial sets.*

(ii) $\mathrm{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ *is (the functor part of) a functorial left deformation retract for* $G \star_{\mathbb{C}} (-)$.

(iii) $\mathrm{B}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *is (the functor part of) a homotopy G-weighted colimit functor.*

Proof. (i). An analogue of proposition A.6.15 says that we have an adjunction of the following form:

$$G \star_{\mathbb{C}} (-) \dashv [G, -] : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$$

By corollary 1.4.16, $[G, -] : \mathbf{sSet} \to [\mathbb{C}, \mathbf{sSet}]$ is a right Quillen functor with respect to the Bousfield–Kan model structure, so by proposition 4.3.2, $G \star_{\mathbb{C}} (-) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ is a left Quillen functor with respect to the Bousfield–Kan model structure. But proposition 1.9.15 says diagrams of the form $\mathrm{B}(\mathbb{C}, \mathbb{C}, F)$ are projective-cofibrant, so the claim is a consequence of Ken Brown's lemma (4.3.6).

(ii). It remains to be shown that there is a natural weak equivalence $\mathrm{B}(\mathbb{C}, \mathbb{C}, -) \Rightarrow \mathrm{id}_{[\mathbb{C}, \mathbf{sSet}]}$, but this was done in proposition 1.9.16.

(iii). Thus, by theorem 3.4.11, $G \star_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ is (the functor part of) a homotopical left approximation for $G \star_{\mathbb{C}} (-)$, and by proposition 1.9.7,

$$\mathrm{B}(G, \mathbb{C}, -) \cong G \star_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, -)$$

so we are done. ∎

The following comparison results are often useful.

Lemma 1.9.28. *There is a morphism* $\mathrm{N}\left(\mathbf{\Delta}_{/\bullet}\right)^{\mathrm{op}} \to \Delta^{\bullet}$ *in* $[\mathbf{\Delta}, \mathbf{sSet}]$.

Proof. Apply $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ to the functors $\left(\mathbf{\Delta}_{/[n]}\right)^{\mathrm{op}} \to [n]$ that send an object $\alpha : [m] \to [n]$ in $\left(\mathbf{\Delta}_{/[n]}\right)^{\mathrm{op}}$ to $\alpha(0)$ in $[n]$. ∎

Theorem 1.9.29.

(i) *There is an adjunction of the form below,*

$$|-| \dashv [\Delta^{\bullet}, -] : \mathbf{sSet} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$$

and it is a Quillen adjunction with respect to both the Bousfield–Kan and Heller model structures on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$.

(ii) *There is a conjugate pair of natural transformations*

$$\varphi : |-| \Rightarrow \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \qquad \psi : \Delta(-) \Rightarrow [\Delta^{\bullet}, -]$$

where ψ *is induced by the unique natural transformation* $\Delta^{\bullet} \Rightarrow \Delta 1$, *and the derived natural transformations*

$$\mathbf{L}\varphi : \mathbf{L}|-| \Rightarrow \mathbf{L}\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \qquad \mathbf{R}\psi : \mathbf{R}\Delta(-) \Rightarrow \mathbf{R}[\Delta^{\bullet}, -]$$

constitute a conjugate pair of natural isomorphisms.

(iii) *For any projective-cofibrant diagram* $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$*, the natural morphism* $\varphi_F : |F| \to \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} F$ *is a weak homotopy equivalence. In particular, the realisation functor* $|-| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$ *is (the functor part of) a homotopy colimit functor for diagrams* $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$.

Proof. (i). The existence of the adjunction is a special case of theorem B.3.19. Theorem 1.6.4 says that the Reedy model structure on $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$ coincides with the Heller model structure, so by theorem 1.6.10, the indicated adjunction is a Quillen adjunction with respect to the Heller model structure.

It remains to be shown that the adjunction in question is a Quillen adjunction with respect to the Bousfield–Kan model structure; by proposition 4.3.2, it suffices to show that

$$[\Delta^{\bullet}, -] : \mathbf{sSet} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$$

is a right Quillen functor (with respect to the Bousfield–Kan model structure); but this is an immediate consequence of corollary 1.4.16, so we are done.

(ii). Since the standard simplices Δ^n are contractible, the unique natural transformation $\Delta^{\bullet} \Rightarrow \Delta 1$ is a natural weak homotopy equivalence. Thus, for any Kan complex K, the natural morphism $\psi_K : \Delta K \to [\Delta^{\bullet}, K]$ is a weak homotopy equivalence (by proposition 1.5.15). Thus, considering the explicit description of $\mathbf{R}\psi$ afforded by theorems 3.3.17 and 4.3.12, we see that $\mathbf{R}\psi$ is a natural isomorphism; but $\mathbf{L}\varphi$ and $\mathbf{R}\psi$ are conjugate by theorem 3.3.24, so we deduce that $\mathbf{L}\varphi$ is also a natural isomorphism.

(iii). Since $\mathbf{L}\varphi : \mathbf{L}|-| \Rightarrow \mathbf{L}\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$ is a natural isomorphism, the natural morphism $\varphi_F : |F| \to \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} F$ must be a weak homotopy equivalence for every projective-cofibrant diagram $F : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$.

We claim that $|-| : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$ and $\varphi : |-| \Rightarrow \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$ constitute a homotopical left approximation for $\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$. Let (Q, p) be a functorial projective-cofibrant replacement for $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$; such exists, by Quillen's small object argument (theorem 0.5.12) and theorem 1.9.13. Then theorem 3.4.11 says that $\left(\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \circ Q, \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} p\right)$ homotopical left approximation for $\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$. But the following diagram commutes,

$$\begin{tikzcd}
{|-|} \arrow[d, "\varphi"'] & {|Q-|} \arrow[l, "|p|"'] \arrow[r, "\varphi Q"] \arrow[d] & \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \circ Q \arrow[d, "\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} p"] \\
\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, equal] & \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r, equal] & \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}
\end{tikzcd}$$

and by Ken Brown's lemma (4.3.6), both arrows in the top row are natural weak homotopy equivalences, so by proposition 3.2.2, $(|-|, \varphi)$ is also a homotopical left approximation for $\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$, as claimed. ∎

Corollary 1.9.30. *Given any morphism $\theta_\bullet : \mathrm{N}(\mathbf{\Delta}_{/\bullet})^{\mathrm{op}} \to \Delta^\bullet$ in $[\mathbf{\Delta}, \mathbf{sSet}]$:*

(i) *There is an induced natural transformation $\theta_* : \varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}^{\mathrm{op}}} \Rightarrow |-|$ making the diagram below commute,*

$$\begin{tikzcd} \varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}^{\mathrm{op}}} \arrow[r] \arrow[d, "\theta_*"'] & \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \arrow[d, equal] \\ |-| \arrow[r] & \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \end{tikzcd}$$

where the horizontal arrows are the counits of the respective homotopical right Kan extensions.

(ii) *For any diagram $X_\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, the morphism*

$$\theta_* : \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} X_\bullet \to |X_\bullet|$$

is a weak homotopy equivalence.

Proof. (i). Each $\mathrm{N}(\mathbf{\Delta}_{/[n]})$ and each Δ^n is contractible, by corollary 1.3.11, so $\theta_\bullet : \mathrm{N}(\mathbf{\Delta}_{/\bullet})^{\mathrm{op}} \to \Delta^\bullet$ is a natural weak homotopy equivalence. Remark 1.8.4 says that $\mathrm{B}(\Delta 1, \mathbf{\Delta}^{\mathrm{op}}, \text{\textit{ホ}}_{[n]}) \cong \mathrm{N}(\mathbf{\Delta}_{/[n]})^{\mathrm{op}}$, so proposition 1.9.7 implies that $\theta : \mathrm{N}(\mathbf{\Delta}_{/\bullet})^{\mathrm{op}} \to \Delta^\bullet$ induces a natural transformation

$$\int^{[n]:\mathbf{\Delta}} \mathrm{B}(\Delta 1, \mathbf{\Delta}^{\mathrm{op}}, \text{\textit{ホ}}_{[n]}) \times (-)_n \Rightarrow \int^{[n]:\mathbf{\Delta}} \Delta^n \times (-)_n$$

i.e. a natural transformation $\theta_* : \varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}^{\mathrm{op}}} \Rightarrow |-|$. Similarly, the unique natural transformation $\mathrm{N}(\mathbf{\Delta}_{/\bullet})^{\mathrm{op}} \Rightarrow \Delta 1$ (resp. $\Delta^\bullet \Rightarrow \Delta 1$) induces the canonical comparison $\varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}^{\mathrm{op}}} \Rightarrow \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$ (resp. $|-| \Rightarrow \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}$, so we have a commutative diagram of the required form.

(ii). This is a corollary of lemma 3.2.5. ∎

Corollary 1.9.31. *Let $\mathcal{C}$ be a locally small category.*

- *For each cosimplicial object $A^\bullet$ in $\mathcal{C}$ and each object B in $\mathcal{C}$, there is a weak homotopy equivalence*

$$\mathrm{N}((A \downarrow B)) \to \mathcal{C}(A^\bullet, B)$$

and it is natural in $A^\bullet$ and B.

- *For each object A in $\mathcal{C}$ and each simplicial object $B_\bullet$ in $\mathcal{C}$, there is a weak homotopy equivalence*

$$\mathrm{N}((A \downarrow B)) \to \mathcal{C}\big(A, B_\bullet\big)$$

and it is natural in A and $B_\bullet$.

Proof. The two claims are formally dual; we will prove the first version.

By remark 1.8.5 and proposition 1.9.7, we have natural isomorphisms

$$\mathrm{N}((A \downarrow B)) \cong \mathrm{B}\big(\Delta 1, \mathbf{\Delta}^{\mathrm{op}}, A^* \hbar^B\big) \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{C}(A^\bullet, B)$$

and by (lemma 1.9.28 and) corollary 1.9.30, we have a natural weak homotopy equivalence

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{C}(A^\bullet, B) \to |\operatorname{disc} \mathcal{C}(A^\bullet, B)|$$

but by corollary 1.6.9,

$$|\operatorname{disc} \mathcal{C}(A^\bullet, B)| \cong \mathcal{C}(A^\bullet, B)$$

so we are done. ■

Theorem 1.9.32.

(i) *There is an adjunction of the form below,*

$$\Delta^\bullet \times (-) \dashv \mathrm{Tot} : [\mathbf{\Delta}, \mathbf{sSet}] \to \mathbf{sSet}$$

and it is a Quillen adjunction with respect to both the Reedy and Heller model structures on $[\mathbf{\Delta}, \mathbf{sSet}]$.

(ii) *There is a conjugate pair of natural transformations*

$$\varphi : \Delta^{\bullet} \times (-) \Rightarrow \Delta(-) \qquad\qquad \psi : \varprojlim_{\mathbf{\Delta}} \Rightarrow \operatorname{Tot}$$

where ψ is induced by the unique natural transformation $\Delta^{\bullet} \Rightarrow \Delta 1$, and the derived natural transformations

$$\mathbf{L}\varphi : \mathbf{L}(\Delta^{\bullet} \times (-)) \Rightarrow \mathbf{L}\Delta(-) \qquad\qquad \mathbf{R}\psi : \mathbf{R}\varprojlim_{\mathbf{\Delta}} \Rightarrow \mathbf{R}\operatorname{Tot}$$

constitute a conjugate pair of natural isomorphisms.

(iii) *For any injective-fibrant diagram $F : \mathbf{\Delta} \to \mathbf{sSet}$, the natural morphism $\psi_F : \varprojlim_{\mathbf{\Delta}} F \Rightarrow \operatorname{Tot} F$ is a weak homotopy equivalence. In particular, given any* Reedy-*fibrant replacement functor $R : [\mathbf{\Delta}, \mathbf{sSet}] \to [\mathbf{\Delta}, \mathbf{sSet}]$, the composite* $\operatorname{Tot} \circ R : [\mathbf{\Delta}, \mathbf{sSet}] \to \mathbf{sSet}$ *is (the functor part of) a homotopy limit functor for diagrams $\mathbf{\Delta} \to \mathbf{sSet}$.*

Proof. (i). The existence of the adjunction is a special case of theorem B.3.18, and it is a Quillen adjunction with respect to the Reedy model structure by theorem 1.6.26.

It remains to be shown that the adjunction in question is a Quillen adjunction with respect to the injective model structure; by proposition 4.3.2, it suffices to show that

$$\Delta^{\bullet} \times (-) : \mathbf{sSet} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$$

is a left Quillen functor (with respect to the injective model structure). Clearly, each $\Delta^n \times (-)$ preserves monomorphisms, and by proposition 1.5.15, it also preserves weak homotopy equivalences; thus, $\Delta^{\bullet} \times (-)$ sends monomorphisms to injective cofibrations and (by proposition 1.5.10) anodyne extensions to injective trivial cofibrations, as required.

(ii). Since the standard simplices Δ^n are contractible, the unique natural transformation $\Delta^{\bullet} \Rightarrow \Delta 1$ is a natural weak homotopy equivalence. Thus, for any simplicial set X, the natural morphism $\varphi_X : \Delta^{\bullet} \times X \to \Delta X$ is a weak homotopy equivalence (by proposition 1.5.15). Thus, considering the explicit description of $\mathbf{L}\varphi$ afforded by theorems 3.3.17 and 4.3.12, we see that $\mathbf{L}\varphi$ is a natural isomorphism; but $\mathbf{L}\varphi$ and $\mathbf{R}\psi$ are conjugate by theorem 3.3.24, so we deduce that $\mathbf{R}\psi$ is also a natural isomorphism.

(iii). Since $\mathbf{R}\psi : \mathbf{R}\varprojlim_{\mathbf{\Delta}} \Rightarrow \mathbf{R}\mathrm{Tot}$ is a natural isomorphism, the natural morphism $\psi_F : \varprojlim_{\mathbf{\Delta}} F \to \mathrm{Tot}\, F$ must be a weak homotopy equivalence for every injective-fibrant diagram $F : \mathbf{\Delta} \to \mathbf{sSet}$.

Let (R, i) be any functorial Reedy-fibrant replacement for $[\mathbf{\Delta}, \mathbf{sSet}]$. We claim that $\mathrm{Tot} \circ R : [\mathbf{\Delta}, \mathbf{sSet}] \to \mathbf{sSet}$ and $\psi \circ i : \varprojlim_{\mathbf{\Delta}} \Rightarrow \mathrm{Tot} \circ R$ constitute a homotopical right approximation for $\varprojlim_{\mathbf{\Delta}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$. Let $\left(\hat{R}, \hat{\imath}\right)$ be a functorial injective-fibrant replacement for $[\mathbf{\Delta}, \mathbf{sSet}]$; such exists, by Quillen's small object argument (theorem 0.5.12) and theorem 1.9.14. Then theorem 3.4.11 says that $\left(\varprojlim_{\mathbf{\Delta}} \circ \hat{R}, \varprojlim_{\mathbf{\Delta}} \hat{\imath}\right)$ homotopical right approximation for $\varprojlim_{\mathbf{\Delta}}$. But the following diagram commutes,

$$\begin{tikzcd}
\varprojlim_{\mathbf{\Delta}} \arrow[r, equal] \arrow[d, "\psi\circ i"'] & \varprojlim_{\mathbf{\Delta}} \arrow[r, equal] \arrow[d] & \varprojlim_{\mathbf{\Delta}} \arrow[d, "\varprojlim_{\mathbf{\Delta}} \hat{\imath}"] \\
\mathrm{Tot} \circ R \arrow[r, "\mathrm{Tot}\, R\hat{\imath}"'] & \mathrm{Tot} \circ R \circ \hat{R} & \varprojlim_{\mathbf{\Delta}} \circ R \arrow[l, "(\psi\circ i)R"]
\end{tikzcd}$$

and by Ken Brown's lemma (4.3.6), both arrows in the bottom row are natural weak homotopy equivalences, so by proposition 3.2.2, $(\mathrm{Tot} \circ R, \psi \circ i)$ is also a homotopical right approximation for $\varprojlim_{\mathbf{\Delta}}$, as claimed. ■

Corollary 1.9.33. *Given any morphism $\theta_\bullet : \mathrm{N}\left(\mathbf{\Delta}_{/\bullet}\right)^{\mathrm{op}} \to \Delta^\bullet$ in $[\mathbf{\Delta}, \mathbf{sSet}]$:*

(i) *There is an induced natural transformation $\theta^* : \mathrm{Tot} \Rightarrow \varprojlim_{\mathbf{\Delta}}^{\mathrm{BK}}$ making the diagram below commute,*

$$\begin{tikzcd}
\varprojlim_{\mathbf{\Delta}} \arrow[r] \arrow[d, equal] & \mathrm{Tot} \arrow[d, "\theta^*"] \\
\varprojlim_{\mathbf{\Delta}} \arrow[r] & \varprojlim_{\mathbf{\Delta}}^{\mathrm{BK}}
\end{tikzcd}$$

where the horizontal arrows are the canonical comparisons.

(ii) *For any* Reedy-fibrant *diagram $X^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$, the morphism*

$$\theta^* : \mathrm{Tot}\, X^\bullet \to \varprojlim_{\mathbf{\Delta}}^{\mathrm{BK}} X^\bullet$$

is a weak homotopy equivalence.

Proof. (i). Each $\mathrm{N}(\boldsymbol{\Delta}_{/[n]})$ and each Δ^n is contractible, by corollary 1.3.11, so $\theta_\bullet : \mathrm{N}(\boldsymbol{\Delta}_{/\bullet})^{\mathrm{op}} \to \Delta^\bullet$ is a natural weak homotopy equivalence. Remark 1.8.4 says that $\mathrm{B}(\Delta 1, \boldsymbol{\Delta}^{\mathrm{op}}, \text{ぁ}_{[n]}) \cong \mathrm{N}(\boldsymbol{\Delta}_{/[n]})^{\mathrm{op}}$, so proposition 1.9.7 implies that $\theta : \mathrm{N}(\boldsymbol{\Delta}_{/\bullet})^{\mathrm{op}} \to \Delta^\bullet$ induces a natural transformation

$$\int_{[n]:\boldsymbol{\Delta}} [\Delta^n, (-)^n] \Rightarrow \int_{[n]:\boldsymbol{\Delta}} \left[\mathrm{B}(\Delta 1, \boldsymbol{\Delta}^{\mathrm{op}}, \text{ぁ}_{[n]}), (-)^n\right]$$

i.e. a natural transformation $\theta^* : \mathrm{Tot} \Rightarrow \varprojlim_{\boldsymbol{\Delta}}^{\mathrm{BK}}$. Similarly, the unique natural transformation $\mathrm{N}(\boldsymbol{\Delta}_{/\bullet})^{\mathrm{op}} \Rightarrow \Delta 1$ (resp. $\Delta^\bullet \Rightarrow \Delta 1$) induces the canonical comparison $\varprojlim_{\boldsymbol{\Delta}} \Rightarrow \varprojlim_{\boldsymbol{\Delta}}^{\mathrm{BK}}$ (resp. $\varprojlim_{\boldsymbol{\Delta}} \Rightarrow \mathrm{Tot}$, so we have a commutative diagram of the required form.

(ii). Let $X^\bullet : \boldsymbol{\Delta} \to \mathbf{sSet}$ be a Reedy-fibrant diagram. If $X^\bullet$ is injective-fibrant, then by applying theorem 4.3.12 to propositions 1.9.18 and 4.3.17 and using the 2-out-of-3 property, we may deduce that that $\theta^* : \mathrm{Tot}\, X^\bullet \to \varprojlim_{\boldsymbol{\Delta}}^{\mathrm{BK}} X^\bullet$ is a weak homotopy equivalence. However:

- By theorem 1.9.14 and proposition 4.1.24, we may replace an arbitrary $X^\bullet$ with a naturally weakly homotopy equivalent injective-fibrant diagram.
- By proposition 4.6.17, every injective-fibrant diagram is Reedy-fibrant, and every Reedy-fibrant diagram is projective-fibrant.
- By theorem 1.6.26 (resp. proposition 1.9.18) and Ken Brown's lemma (4.3.6), the functor Tot (resp. $\varprojlim_{\boldsymbol{\Delta}}^{\mathrm{BK}}$) sends natural weak homotopy equivalences between Reedy-fibrant (resp. projective-fibrant) diagrams $\boldsymbol{\Delta} \to \mathbf{sSet}$ to weak homotopy equivalences.

Thus, applying the 2-out-of-3 property again, $\theta^* : \mathrm{Tot}\, X^\bullet \to \varprojlim_{\boldsymbol{\Delta}}^{\mathrm{BK}} X^\bullet$ is indeed a weak homotopy equivalence for all Reedy-fibrant diagrams $X^\bullet$. ■

The following result is essentially due to Quillen [1973, § 1].

Proposition 1.9.34. *Let $\mathbb{C}$ be a small category and let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram.*

(i) *There is a natural pullback diagram of the form below,*

$$\begin{tikzcd} Fc \arrow[r] \arrow[d] & \varinjlim_{\mathbb{C}}^{\mathrm{BK}} F \arrow[d, "p"] \\ \Delta^0 \arrow[r] & \mathrm{N}(\mathbb{C}) \end{tikzcd}$$

where the bottom horizontal arrow is the morphism corresponding to the vertex c of $\mathrm{N}(\mathbb{C})$ and $p : \varinjlim^{\mathrm{BK}}_{\mathbb{C}} F \to \mathrm{N}(\mathbb{C})$ is the morphism induced by remark 1.8.5 and the unique natural transformation $F \Rightarrow \Delta 1$.

(ii) *Assuming $Ff : Fc' \to Fc$ is a weak homotopy equivalence for every morphism $f : c' \to c$ in $\mathbb{C}$, for any commutative diagram in **sSet** of the form below,*

$$\begin{tikzcd} X' \arrow[r, "u'"] \arrow[d] & Y' \arrow[r, "v'"] \arrow[d] & \varinjlim^{\mathrm{BK}}_{\mathbb{C}} F \arrow[d, "p"] \\ X \arrow[r, "u"'] & Y \arrow[r, "v"'] & \mathrm{N}(\mathbb{C}) \end{tikzcd}$$

*if $u : X \to Y$ is a weak homotopy equivalence and the two squares are pullback squares in **sSet**, then $u' : X' \to Y'$ is also a weak homotopy equivalence.*

Proof. We follow the proof of Lemma 5.7 in [GJ, Ch. IV].

(i). The diagram in question is obtained by applying $\mathrm{diag} : \mathbf{ssSet} \to \mathbf{sSet}$ to the following diagram in **ssSet**,

$$\begin{tikzcd} Fc \arrow[r] \arrow[d] & \mathrm{B}_\bullet(\Delta 1, \mathbb{C}, F) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{disc}\, \mathrm{B}_\bullet(\Delta 1, \mathbb{C}, \Delta 1) \end{tikzcd}$$

where the horizontal arrows are defined by the evident coproduct inclusions. Recalling remark 1.8.18, it is not hard to see that this is a pullback diagram in **ssSet**, so the same is true of its image under $\mathrm{diag} : \mathbf{ssSet} \to \mathbf{sSet}$.

(ii). Now suppose $Ff : Fc' \to Fc$ is a weak homotopy equivalence for every morphism $f : c' \to c$ in $\mathbb{C}$. The pullback functor $p^* : \mathbf{sSet}_{/\mathrm{N}(\mathbb{C})} \to \mathbf{sSet}_{/\varinjlim^{\mathrm{BK}}_{\mathbb{C}} F}$ has a right adjoint (by theorem A.2.22), so in particular it preserves transfinite compositions. Thus, in view of proposition 1.5.10, lemma 4.1.10, and proposition A.3.17, it suffices to prove the claim in the special case where $u : X \to Y$ is a horn inclusion, say $\Lambda^n_k \hookrightarrow \Delta^n$.

Identifying Δ^n with $\mathrm{N}([n])$, by proposition 1.2.1, there is a unique functor $V : [n] \to \mathbb{C}$ such that $\mathrm{N}(V) = v$. We then have a commutative diagram in **ssSet** of the form below,

$$\begin{tikzcd} \mathrm{B}_\bullet(\Delta 1, [n], FV) \arrow[r] \arrow[d] & \mathrm{B}_\bullet(\Delta 1, \mathbb{C}, F) \arrow[d] \\ \mathrm{B}_\bullet(\Delta 1, [n], \Delta 1) \arrow[r] & \mathrm{B}_\bullet(\Delta 1, \mathbb{C}, \Delta 1) \end{tikzcd}$$

where the horizontal arrows are the canonical comparison morphisms of proposition 1.8.21 and the vertical arrows are induced by the unique natural transformation to $\Delta 1$, and by lemma 1.8.22, it is a pullback square in **ssSet**. Furthermore, there is an evident natural transformation $\Delta FV[0] \Rightarrow FV$ of diagrams $[n] \to \mathbf{sSet}$, so we have the following diagram,

$$\begin{tikzcd} (\Lambda^n_k)_\bullet \times FV[0] \arrow[r] \arrow[d] & (\Delta^n)_\bullet \times FV[0] \arrow[d] \\ (\Lambda^n_k)_\bullet \times_{(\Delta^n)_\bullet} \mathrm{B}_\bullet(\Delta 1, [n], FV) \arrow[r] \arrow[d] & \mathrm{B}_\bullet(\Delta 1, [n], FV) \arrow[d] \\ (\Lambda^n_k)_\bullet \arrow[r, hook] & (\Delta^n)_\bullet \end{tikzcd}$$

where both squares are pullback squares in **ssSet**, and (recalling proposition 1.5.14) the hypothesis implies that the vertical arrows in the upper square are Reedy weak homotopy equivalences. Thus, by lemma 1.6.8 and theorem 1.6.10, in the induced diagram in **sSet**,

$$\begin{tikzcd} \Lambda^n_k \times FV[0] \arrow[r] \arrow[d] & \Delta^n \times FV[0] \arrow[d] \\ \Lambda^n_k \times_{\Delta^n} \varinjlim^{\mathrm{BK}}_{[n]} FV \arrow[r] & \varinjlim^{\mathrm{BK}}_{[n]} FV \end{tikzcd}$$

both vertical arrows are weak homotopy equivalences; but the top horizontal arrow is a weak homotopy equivalence by proposition 1.5.15, so the bottom horizontal arrow must also be a weak homotopy equivalence, by the 2-out-of-3 property. ■

1.10 Bousfield–Kan extensions

Prerequisites. §§1.5, 1.6, 1.8, 2.4, 3.3, 3.4, 4.3, A.5, A.6.

In this section, we study a homotopy-theoretic version of Kan extensions.

Definition 1.10.1. Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.

- The **left Bousfield–Kan extension** of a diagram $F : \mathbb{C} \to \mathbf{sSet}$ along $U : \mathbb{C} \to \mathbb{D}$ is the diagram $\operatorname{Lan}_U^{\mathrm{BK}} F : \mathbb{D} \to \mathbf{sSet}$ defined by the following formula:
$$\left(\operatorname{Lan}_U^{\mathrm{BK}} F\right) d = \mathrm{B}\left(U^* \text{\char"0127}_d, \mathbb{C}, F\right)$$
- The **right Bousfield–Kan extension** of a diagram $F : \mathbb{C} \to \mathbf{sSet}$ along $U : \mathbb{C} \to \mathbb{D}$ is the diagram $\operatorname{Ran}_U^{\mathrm{BK}} F : \mathbb{D} \to \mathbf{sSet}$ defined by the following formula:
$$\left(\operatorname{Ran}_U^{\mathrm{BK}} F\right) d = \mathrm{C}\left(U^* \hbar^d, \mathbb{C}, F\right)$$

Remark 1.10.2. Let $\mathbb{C}$ be a small category. If $U : \mathbb{C} \to \mathbb{1}$ is the unique functor, then $\operatorname{Lan}_U^{\mathrm{BK}}$ (resp. $\operatorname{Ran}_U^{\mathrm{BK}}$) can be identified with $\varinjlim_{\mathbb{C}}^{\mathrm{BK}}$ (resp. $\varprojlim_{\mathbb{C}}^{\mathrm{BK}}$).

Lemma 1.10.3. *Let $\mathbb{C}$ and $\mathbb{D}$ be small categories and let $W : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbf{sSet}$ be a functor.*

(i) *There is an adjunction of the form below:*
$$\mathrm{B}(W, \mathbb{C}, -) \dashv [\mathbb{D}, \underline{\mathbf{sSet}}](\mathrm{B}(W, \mathbb{C}, \mathbb{C}), -) : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

(ii) *There is an adjunction of the form below:*
$$\mathrm{B}(W, \mathbb{D}^{\mathrm{op}}, \mathbb{D}^{\mathrm{op}}) \star_{\mathbb{C}} (-) \dashv \mathrm{C}(W, \mathbb{D}, -) : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

Proof. Let $F : \mathbb{C} \to \mathbf{sSet}$ and $G : \mathbb{D} \to \mathbf{sSet}$ be diagrams. For convenience, we write W_d^c for the value of $W : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbf{sSet}$ at (c, d).

(i). By proposition 1.9.7,
$$\mathrm{B}\left(W_d, \mathbb{C}, F\right) \cong \int^{c:\mathbb{C}} \mathrm{B}\left(W_d, \mathbb{C}, \operatorname{disc} \hbar^c\right) \times Fc$$

and thus, by remark A.6.5, proposition A.6.11, and the interchange law for ends (theorem A.6.17), we have the following natural bijections,
$$\begin{aligned}
[\mathbb{D}, \mathbf{sSet}](\mathrm{B}(W, \mathbb{C}, F), G) &\cong \int_{d:\mathbb{D}} \int_{c:\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}\left(W_d, \mathbb{C}, \operatorname{disc} \hbar^c\right) \times Fc, Gd\right) \\
&\cong \int_{c:\mathbb{C}} \int_{d:\mathbb{D}} \mathbf{sSet}\left(Fc, \left[\mathrm{B}\left(W_d, \mathbb{C}, \operatorname{disc} \hbar^c\right), Gd\right]\right)
\end{aligned}$$

$$\cong [\mathbb{C}, \mathbf{sSet}](F, [\mathbb{D}, \underline{\mathbf{sSet}}](\mathrm{B}(W, \mathbb{C}, \mathbb{C}), G))$$

where in the last step we have used remark 2.1.24.

(ii). Similarly,

$$\mathrm{C}(W^c, \mathbb{D}, G) \cong \int_{d:\mathbb{D}} \left[\mathrm{B}\left(W_c, \mathbb{D}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_d\right), Gd\right]$$

and thus, we have the following natural bijections,

$$\begin{aligned}
[\mathbb{C}, \mathbf{sSet}](F, \mathrm{C}(W, \mathbb{D}, G)) &\cong \int_{c:\mathbb{C}} \int_{d:\mathbb{D}} \mathbf{sSet}\left(Fc, \left[\mathrm{B}\left(W^c, \mathbb{D}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_d\right), Gd\right]\right) \\
&\cong \int_{d:\mathbb{D}} \int_{c:\mathbb{C}} \mathbf{sSet}\left(\mathrm{B}\left(W^c, \mathbb{D}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_d\right) \times Fc, Gd\right) \\
&\cong [\mathbb{D}, \mathbf{sSet}]\left(\mathrm{B}(W, \mathbb{D}^{\mathrm{op}}, \mathbb{D}^{\mathrm{op}}) \star_{\mathbb{C}} F, G\right)
\end{aligned}$$

where in the last step we have used the coend formula for simplicially enriched weighted colimits:

TODO: Add a reference for this.

$$\mathrm{B}(W, \mathbb{D}^{\mathrm{op}}, \mathbb{D}^{\mathrm{op}}) \star_{\mathbb{C}} F \cong \int^{c:\mathbb{C}} \mathrm{B}(W^c, \mathbb{D}^{\mathrm{op}}, \mathbb{D}^{\mathrm{op}}) \times Fc$$ ■

Proposition 1.10.4. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *There is an adjunction of the form below:*

$$\mathrm{Lan}_U^{\mathrm{BK}} \dashv [\mathbb{D}, \underline{\mathbf{sSet}}](\mathrm{B}(U^*\mathbb{D}, \mathbb{C}, \mathbb{C}), -) : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

(ii) *The adjunction is a Quillen adjunction with respect to the Bousfield–Kan model structures on* $[\mathbb{C}, \mathbf{sSet}]$ *and* $[\mathbb{D}, \mathbf{sSet}]$.

(iii) *The adjunction is a Quillen adjunction with respect to the Heller model structures on* $[\mathbb{C}, \mathbf{sSet}]$ *and* $[\mathbb{D}, \mathbf{sSet}]$.

Proof. (i). Apply lemma 1.10.3 with $W = \mathbb{D}(U-, -)$.

(ii). By proposition 1.9.15, for each object c in $\mathbb{C}$, $\mathrm{B}\left(U^*\mathbb{D}, \mathbb{C}, \operatorname{disc} \mathcal{h}^c\right)$ is a cofibrant object in the Bousfield–Kan model structure on $[\mathbb{D}, \mathbf{sSet}]$. The Bousfield–Kan model structure is a simplicial model structure (by theorem 1.9.13), thus each $[\mathbb{D}, \underline{\mathbf{sSet}}]\left(\mathrm{B}\left(U^*\mathbb{D}, \mathbb{C}, \operatorname{disc} \mathcal{h}^c\right), -\right) : [\mathbb{D}, \mathbf{sSet}] \to \mathbf{sSet}$ is a right Quillen functor with respect to the Bousfield–Kan model structure; but fibrations and trivial fibrations are componentwise in the Bousfield–Kan model structure, so we conclude that $[\mathbb{D}, \underline{\mathbf{sSet}}](\mathrm{B}(U^*\mathbb{D}, \mathbb{C}, \mathbb{C}), -) : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ is a right Quillen functor. The claim is then a consequence of proposition 4.3.2.

(iii). Proposition 1.9.19 says that each $\mathrm{B}\big(U^* \mathcal{h}_d, \mathbb{C}, -\big) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ is a left Quillen functor with respect to the Heller model structure on $[\mathbb{C}, \mathbf{sSet}]$, and since cofibrations and trivial cofibrations in the Heller model structure are componentwise, we conclude that $\mathrm{B}(U^*\mathbb{D}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ is a left Quillen functor. As before, it follows that we have a Quillen adjunction. ■

Corollary 1.10.5. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories. Then $\mathrm{Lan}_U^{\mathrm{BK}} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ preserves natural weak homotopy equivalences.*

Proof. Apply Ken Brown's lemma (4.3.6) to proposition 1.10.4. ■

Proposition 1.10.6. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *There is an adjunction of the form below:*

$$\mathrm{B}(U^*(-), \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \dashv \mathrm{Ran}_U^{\mathrm{BK}} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$$

(ii) *The adjunction is a Quillen adjunction with respect to the Bousfield–Kan model structures on $[\mathbb{C}, \mathbf{sSet}]$ and $[\mathbb{D}, \mathbf{sSet}]$.*

(iii) *The adjunction is a Quillen adjunction with respect to the Heller model structures on $[\mathbb{C}, \mathbf{sSet}]$ and $[\mathbb{D}, \mathbf{sSet}]$.*

Proof. (i). By lemma 1.10.3 (with $W = \mathbb{D}(-, U-)$), we obtain the following adjunction:

$$\mathrm{B}(U^*\mathbb{D}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \star_{\mathbb{D}} (-) \dashv \mathrm{C}(U^*\mathbb{D}^{\mathrm{op}}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$$

The right adjoint is (by definition) $\mathrm{Ran}^{\mathrm{BK}} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$, and by proposition 1.9.7,

$$\mathrm{B}(U^*\mathbb{D}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) \star_{\mathbb{D}} (-) \cong \mathrm{B}(U^*(-), \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}})$$

so we indeed have an adjunction of the desired form.

(ii). Proposition 1.9.18 says that each $\mathrm{C}\big(U^* \mathcal{h}^d, \mathbb{C}, -\big) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ is a right Quillen functor with respect to the Bousfield–Kan model structure on $[\mathbb{C}, \mathbf{sSet}]$, and since fibrations and trivial fibrations are componentwise in the Bousfield–Kan model structure are componentwise, we conclude that $\mathrm{C}(U^*\mathbb{D}^{\mathrm{op}}, \mathbb{C}, \mathbb{C}) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ is a left Quillen functor. The claim is then a consequence of proposition 4.3.2.

(iii). Since cofibrations and trivial cofibrations are componentwise in the Heller model structure, $U^* : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ is a left Quillen functor (with respect to the Heller model structure), and by proposition 1.9.19, so is $\mathrm{B}(-, \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$. Thus, $\mathrm{B}(U^*(-), \mathbb{C}^{\mathrm{op}}, \mathbb{C}^{\mathrm{op}}) : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ is a left Quillen functor (by proposition 4.3.5), and as before, it follows that we have a Quillen adjunction. ■

Corollary 1.10.7. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories. Then* $\mathrm{Ran}_U^{\mathrm{BK}} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ *preserves natural weak homotopy equivalences between projective-fibrant diagrams.*

Proof. Apply Ken Brown's lemma (4.3.6) to proposition 1.10.6. ■

The homotopical universal property of Bousfield–Kan extensions is traditionally stated in terms of derived functors.

Definition 1.10.8. Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.

- A **homotopy left Kan extension functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ along $U : \mathbb{C} \to \mathbb{D}$ is a homotopical left approximation for the functor $\mathrm{Lan}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$.

- A **homotopy right Kan extension functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ along $U : \mathbb{C} \to \mathbb{D}$ is a homotopical right approximation for the functor $\mathrm{Ran}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$.

Theorem 1.10.9. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) $\mathrm{Lan}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ *sends natural weak homotopy equivalences between diagrams of the form* $\mathrm{B}(\mathbb{C}, \mathbb{C}, F)$ *to natural weak homotopy equivalences of diagrams* $\mathbb{D} \to \mathbf{sSet}$.

(ii) $\mathrm{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ *is (the functor part) of a functorial left deformation retract for* $\mathrm{Lan}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$.

(iii) $\operatorname{Lan}_U^{\mathrm{BK}} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ *is (the functor part of) a homotopy left Kan extension functor for diagrams* $\mathbb{C} \to \mathbf{sSet}$ *along* $U : \mathbb{C} \to \mathbb{D}$.

Proof. (i) and (ii). Recalling theorem A.5.15, this is a straightforward consequence of theorem 1.9.27.

(iii). Thus, by theorem 3.4.11, $\operatorname{Lan}_U \mathrm{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ is (the functor part of) a homotopical left approximation for Lan_U, and by proposition 1.9.7,

$$\operatorname{Lan}_U \mathrm{B}(\mathbb{C}, \mathbb{C}, -) \cong U^*\mathbb{D} \star_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, -) \cong \mathrm{B}(U^*\mathbb{D}, \mathbb{C}, -) = \operatorname{Lan}_U^{\mathrm{BK}}(-)$$

so we are done. ■

Theorem 1.10.10. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories and let* $R : \mathbf{sSet} \to \mathbf{sSet}$ *be (the functor part of) any functorial fibrant replacement in* $\mathbf{sSet}$.

(i) $\operatorname{Ran}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ *sends natural weak homotopy equivalences between diagrams of the form* $\mathrm{C}(\mathbb{C}, \mathbb{C}, F)$ *where every* Fc *is a Kan complex to natural weak homotopy equivalences of diagrams* $\mathbb{D} \to \mathbf{sSet}$.

(ii) $\mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ *is (the functor part) of a functorial right deformation retract for* $\operatorname{Ran}_U : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$.

(iii) $\operatorname{Ran}_U^{\mathrm{BK}}(R \circ -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ *is (the functor part of) a homotopy right Kan extension functor for diagrams* $\mathbb{C} \to \mathbf{sSet}$ *along* $U : \mathbb{C} \to \mathbb{D}$.

Proof. (i) and (ii). Recalling theorem A.5.15, this is a straightforward consequence of theorem 1.9.26.

(iii). Thus, by theorem 3.4.11, $\operatorname{Ran}_U \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ is (the functor part of) a homotopical right approximation for Lan_U, and by proposition 1.9.7,

$$\begin{aligned} \operatorname{Ran}_U \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -) &\cong \{U^*\mathbb{D}^{\mathrm{op}}, \mathrm{C}(\mathbb{C}, \mathbb{C}, R \circ -)\}^{\mathbb{C}} \\ &\cong \mathrm{C}(U^*\mathbb{D}^{\mathrm{op}}, \mathbb{C}, R \circ -) = \operatorname{Ran}_U^{\mathrm{BK}}(R \circ -) \end{aligned}$$

so we are done. ■

Lemma 1.10.11. *Let $U : \mathbb{C} \to \mathbb{D}$ and $V : \mathbb{D} \to \mathbb{E}$ be functors between small categories.*

(i) *There is a natural weak homotopy equivalence*

$$\mathrm{B}(U^*\mathbb{D}, \mathbb{C}, \mathbb{C}) \Rightarrow \operatorname{disc} \mathbb{D}(U-, -)$$

of functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathbf{Set}$.

(ii) *There is a natural weak homotopy equivalence*

$$U^*\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathbb{D}) \Rightarrow \operatorname{disc} \mathbb{E}(VU-, -)$$

of functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{E} \to \mathbf{Set}$.

(iii) *There is a natural weak homotopy equivalence*

$$\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathrm{B}(U^*\mathbb{D}, \mathbb{C}, \mathbb{C})) \Rightarrow U^*\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathbb{D})$$

of functors $\mathbb{C}^{\mathrm{op}} \times \mathbb{E} \to \mathbf{Set}$.

Proof. (i). This is a special case of proposition 1.9.16.

(ii). As above, there is a natural weak homotopy equivalence

$$\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathbb{D}) \Rightarrow \operatorname{disc} \mathbb{E}(V-, -)$$

of functors $\mathbb{D}^{\mathrm{op}} \times \mathbb{E} \to \mathbf{Set}$, and $U^* : [\mathbb{D}^{\mathrm{op}}, [\mathbb{E}, \mathbf{sSet}]] \to [\mathbb{C}^{\mathrm{op}}, [\mathbb{E}, \mathbf{sSet}]]$ preserves weak equivalences, so the claim follows.

(iii). We may apply corollary 1.10.5 to obtain a natural weak homotopy equivalence of the required form. ■

Proposition 1.10.12.

(i) *Let $\mathbb{C}$ be a small category. There is a natural weak equivalence* $\mathrm{Lan}^{\mathrm{BK}}_{\mathrm{id}_{\mathbb{C}}} \Rightarrow \mathrm{id}_{[\mathbb{C}, \mathbf{sSet}]}$.

(ii) *Let $U : \mathbb{C} \to \mathbb{D}$ and $V : \mathbb{D} \to \mathbb{E}$ be functors between small categories. There is a natural weak equivalence* $\mathrm{Lan}^{\mathrm{BK}}_V \circ \mathrm{Lan}^{\mathrm{BK}}_U \Rightarrow \mathrm{Lan}^{\mathrm{BK}}_{VU}$.

Dually:

(i′) *Let* $\mathbb{C}$ *be a small category. There is a natural weak equivalence* $\mathrm{id}_{[\mathbb{C},\mathbf{sSet}]} \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{\mathrm{id}_{\mathbb{C}}}$.

(ii′) *Let* $U : \mathbb{C} \to \mathbb{D}$ *and* $V : \mathbb{D} \to \mathbb{E}$ *be functors between small categories. There is a natural weak equivalence* $\mathrm{Ran}^{\mathrm{BK}}_{VU} \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{V} \circ \mathrm{Ran}^{\mathrm{BK}}_{U}$.

Proof. (i). By definition, $\mathrm{Lan}^{\mathrm{BK}}_{\mathrm{id}_{\mathbb{C}}} = \mathrm{B}(\mathbb{C}, \mathbb{C}, -)$, so this is a consequence of proposition 1.9.16.

(ii). By definition,

$$\mathrm{Lan}^{\mathrm{BK}}_{V} \circ \mathrm{Lan}^{\mathrm{BK}}_{U} = \mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathrm{B}(U^*\mathbb{D}, \mathbb{C}, -))$$

and by theorem 1.8.37,

$$\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathrm{B}(U^*\mathbb{D}, \mathbb{C}, -)) \cong \mathrm{B}(U^*\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathbb{D}), \mathbb{C}, -)$$

so by corollary 1.9.20, to prove the claim, it is enough to produce a natural weak homotopy equivalence of the following form,

$$U^*\mathrm{B}(V^*\mathbb{E}, \mathbb{D}, \mathbb{D}) \Rightarrow \mathrm{disc}(VU)^*\mathbb{E} = \mathrm{disc}\,\mathbb{E}(VU-, -)$$

but this was done in lemma 1.10.11. ∎

¶ 1.10.13. Henceforth, for any functor $U : \mathbb{C} \to \mathbb{D}$ between small categories, we write

$$\mathbb{R}^{\mathrm{BK}}U^* : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

for the right adjoint of $\mathrm{Lan}^{\mathrm{BK}}_{U} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$ and

$$\mathbb{L}^{\mathrm{BK}}U^* : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$$

for the left adjoint of $\mathrm{Ran}^{\mathrm{BK}}_{U} : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{D}, \mathbf{sSet}]$.

Proposition 1.10.14. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories.*

- *There is an adjunction of the form*

$$\mathbf{L}\mathrm{Lan}_U \dashv \mathrm{Ho}\, U^* : \mathrm{Ho}\,[\mathbb{D}, \mathbf{sSet}] \to \mathrm{Ho}\,[\mathbb{C}, \mathbf{sSet}]$$

where $\mathbf{L}\mathrm{Lan}_U\, F = \mathrm{Lan}^{\mathrm{BK}}_{U}\, F$ *for all diagrams* $F : \mathbb{C} \to \mathbf{sSet}$.

- *There is an adjunction of the form*

$$\operatorname{Ho} U^* \dashv \mathbf{R}\mathrm{Ran}_U : \operatorname{Ho}[\mathbb{C}, \mathbf{sSet}] \to \operatorname{Ho}[\mathbb{D}, \mathbf{sSet}]$$

  where $\mathbf{R}\mathrm{Ran}_U\, F = \mathrm{Ran}_U^{\mathrm{BK}}\, F$ *for all diagrams* $F : \mathbb{C} \to \mathbf{Kan}$.

Proof. Apply theorem 3.3.24 to theorems 1.10.9 and 1.10.10. ∎

Proposition 1.10.15. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories.*

- *There exist a conjugate pair of natural transformations*

$$\mathrm{Lan}_U^{\mathrm{BK}} \Rightarrow \mathrm{Lan}_U \qquad\qquad U^* \Rightarrow \mathbb{R}^{\mathrm{BK}} U^*$$

  that satisfy the following conditions:

  - $\mathrm{Lan}_U^{\mathrm{BK}}\, F \Rightarrow \mathrm{Lan}_U\, F$ *is a natural weak homotopy equivalence for every projective-cofibrant diagram* $F : \mathbb{C} \to \mathbf{sSet}$, *and in particular, for every diagram of the form* $F = \mathrm{B}(\mathbb{C}, \mathbb{C}, F')$ *for any diagram* $F' : \mathbb{C} \to \mathbf{sSet}$.
  - $U^* G \Rightarrow \mathbb{R}^{\mathrm{BK}} U^* G$ *is a natural weak homotopy equivalence for every projective-fibrant diagram* $G : \mathbb{D} \to \mathbf{sSet}$.

- *There exist a conjugate pair of natural transformations*

$$\mathbb{L}^{\mathrm{BK}} U^* \Rightarrow U^* \qquad\qquad \mathrm{Ran}_U \Rightarrow \mathrm{Ran}_U^{\mathrm{BK}}$$

  that satisfy the following conditions:

  - $\mathbb{L}^{\mathrm{BK}} U^* G \Rightarrow U^* G$ *is a natural weak homotopy equivalence for every (injective-cofibrant) diagram* $G : \mathbb{D} \to \mathbf{sSet}$.
  - $\mathrm{Ran}_U\, F \Rightarrow \mathrm{Ran}_U^{\mathrm{BK}}\, F$ *is a is natural weak homotopy equivalence for every injective-fibrant diagram* $F : \mathbb{C} \to \mathbf{sSet}$ *and also for every diagram of the form* $F = \mathrm{C}(\mathbb{C}, \mathbb{C}, F')$ *for any projective-fibrant diagram* $F' : \mathbb{C} \to \mathbf{sSet}$.

Proof. The two claims are formally dual; we will prove the first version.

By theorem 1.10.9,[17] there is a natural transformation $\mathrm{Lan}_U^{\mathrm{BK}} \Rightarrow \mathrm{Lan}_U$ whose components at diagrams of the form $\mathrm{B}(\mathbb{C}, \mathbb{C}, F')$ are natural weak homotopy equivalences (of diagrams $\mathbb{D} \to \mathbf{sSet}$).

[17] In the dual version, use theorem 1.10.10 instead.

On the other hand, it is not hard to see that $U^* : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ is a right Quillen functor with respect to the Bousfield–Kan model structure,[18] so by combining lemmas 1.5.2 and 3.1.11, propositions 3.3.10, 3.3.13, and 4.3.2, and theorem 4.3.12, we see that the components of the natural transformation at projective-cofibrant diagrams are also natural weak homotopy equivalences.

Finally, we may apply theorem 3.3.24 to deduce that the conjugate natural transformation $U^* \Rightarrow \mathbb{R}^{\mathrm{BK}} U^*$ has the property that its components at projective-fibrant diagrams $\mathbb{D} \to \mathbf{sSet}$ are natural weak homotopy equivalences (of diagrams $\mathbb{C} \to \mathbf{sSet}$). ∎

Lemma 1.10.16. *Let $U : \mathbb{C} \to \mathbb{D}$ and $V : \mathbb{D} \to \mathbb{E}$ be functors between small categories and let $G : \mathbb{D} \to \mathbf{sSet}$ be a functor. Consider the following diagram in $[\mathbb{E}, \mathbf{sSet}]$,*

$$\begin{tikzcd}
\operatorname{Lan}^{\mathrm{BK}}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \arrow[r] \arrow[d, "\operatorname{Lan}^{\mathrm{BK}}_{VU} p_{GU}"'] & \operatorname{Lan}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \arrow[d, "\operatorname{Lan}_{VU} p_{GU}"] \\
\operatorname{Lan}^{\mathrm{BK}}_{VU} GU \arrow[r] \arrow[d] & \operatorname{Lan}_{VU} GU \arrow[d, "\operatorname{Lan}_{V} \varepsilon_G"] \\
\operatorname{Lan}^{\mathrm{BK}}_{V} G \arrow[r] & \operatorname{Lan}_{V} G
\end{tikzcd}$$

where the horizontal arrows are the canonical comparisons of proposition 1.10.15, $p_{GU} : \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow GU$ is the natural weak homotopy equivalence of proposition 1.9.16, $\varepsilon_G : \operatorname{Lan}_U GU \Rightarrow G$ is the counit, and $\operatorname{Lan}^{\mathrm{BK}}_{VU} GU \Rightarrow \operatorname{Lan}^{\mathrm{BK}}_{V} G$ is the canonical comparison of proposition 1.8.34.

(i) *The diagram commutes and is functorial in G.*

(ii) *The canonical comparison*

$$\operatorname{Lan}^{\mathrm{BK}}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow \operatorname{Lan}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU)$$

and the natural transformation

$$\operatorname{Lan}^{\mathrm{BK}}_{VU} p_{GU} : \operatorname{Lan}^{\mathrm{BK}}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow \operatorname{Lan}^{\mathrm{BK}}_{VU} GU$$

[18] In the dual version, use the fact that $U^* : [\mathbb{D}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ is a left Quillen functor with respect to the Heller model structure.

are natural weak homotopy equivalences, and the image in $\operatorname{Ho}[\mathbb{D}, \mathbf{sSet}]$ *of*

$$\varepsilon_G \circ \operatorname{Lan}_U p_{GU} : \operatorname{Lan}_U \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow G$$

can be identified with the derived counit $\mathbf{L}\operatorname{Lan}_U (\operatorname{Ho} U^*) G \to G$.

(iii) *If* $G : \mathbb{D} \to \mathbf{sSet}$ *is projective-cofibrant, then the canonical comparison* $\operatorname{Lan}_V^{\mathrm{BK}} G \Rightarrow \operatorname{Lan}_V G$ *is also a natural weak homotopy equivalence, so the canonical comparison*

$$\operatorname{Lan}_{VU}^{\mathrm{BK}} GU \Rightarrow \operatorname{Lan}_V^{\mathrm{BK}} G$$

is a natural weak homotopy equivalence if and only if

$$\operatorname{Lan}_V \varepsilon_G \circ \operatorname{Lan}_{VU} p_{GU} : \operatorname{Lan}_{VU} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow \operatorname{Lan}_V G$$

is a natural weak homotopy equivalence.

Proof. (i). The top square commutes by naturality, and the bottom square commutes by proposition 1.8.34 (applied componentwise). Every arrow appearing in the diagram is natural in G, so the diagram itself is functorial in G.

(ii). The first subclaim was proved in proposition 1.10.15, the second subclaim is a consequence of corollary 1.10.5, and the third subclaim is an application of theorem 3.3.24 to theorem 1.10.9.

(iii). We know that $\operatorname{Lan}_V^{\mathrm{BK}} G \Rightarrow \operatorname{Lan}_V G$ is a natural weak homotopy equivalence when $G : \mathbb{D} \to \mathbf{sSet}$ is projective-cofibrant, and the rest of the claim is a simple application of the 2-out-of-3 property. ■

Corollary 1.10.17. *Let* $U : \mathbb{C} \to \mathbb{D}$ *and* $V : \mathbb{D} \to \mathbb{E}$ *be functors between small categories. For any diagram* $G : \mathbb{D} \to \mathbf{sSet}$*, the following are equivalent:*

(i) *The canonical comparison* $\operatorname{Lan}_{VU}^{\mathrm{BK}} GU \Rightarrow \operatorname{Lan}_V^{\mathrm{BK}} G$ *of proposition 1.8.34 is a natural weak homotopy equivalence of diagrams* $\mathbb{E} \to \mathbf{sSet}$.

(ii) *The morphism* $\mathbf{L}\operatorname{Lan}_V \varepsilon_G : \mathbf{L}\operatorname{Lan}_V \mathbf{L}\operatorname{Lan}_U (\operatorname{Ho} U^*) G \to \mathbf{L}\operatorname{Lan}_V G$ *is an isomorphism (in* $\operatorname{Ho}[\mathbb{E}, \mathbf{sSet}]$*).*

Proof. By theorem 1.9.13 and proposition 4.1.17, there is a projective-cofibrant replacement $(\tilde{G}, q)$ for G; but the following diagram in $[\mathbb{E}, \mathbf{sSet}]$ commutes,

$$\begin{tikzcd} \mathrm{Lan}^{\mathrm{BK}}_{VU} \tilde{G}U \arrow[r] \arrow[d, "\mathrm{Lan}^{\mathrm{BK}}_{VU} qU"'] & \mathrm{Lan}^{\mathrm{BK}}_{V} \tilde{G} \arrow[d, "\mathrm{Lan}^{\mathrm{BK}}_{V} q"] \\ \mathrm{Lan}^{\mathrm{BK}}_{VU} GU \arrow[r] & \mathrm{Lan}^{\mathrm{BK}}_{V} G \end{tikzcd}$$

and the vertical arrows are natural weak homotopy equivalences by corollary 1.10.5, so the claim is a consequence of lemma 1.10.16 (plus lemmas 1.5.2 and 3.1.11). ■

Lemma 1.10.18. *Let $U : \mathbb{C} \to \mathbb{D}$ and $V : \mathbb{D} \to \mathbb{E}$ be functors between small categories and let $G : \mathbb{D} \to \mathbf{sSet}$ be a diagram. Consider the following diagram in* $[\mathbb{E}, \mathbf{sSet}]$*,*

$$\begin{tikzcd} \mathrm{Ran}_V G \arrow[r] \arrow[d, "\mathrm{Ran}_V \eta_G"'] & \mathrm{Ran}^{\mathrm{BK}}_{V} G \arrow[d] \\ \mathrm{Ran}_{VU} GU \arrow[r] \arrow[d, "\mathrm{Ran}_{VU} i_{GU}"'] & \mathrm{Ran}^{\mathrm{BK}}_{VU} GU \arrow[d, "\mathrm{Ran}^{\mathrm{BK}}_{VU} i_{GU}"] \\ \mathrm{Ran}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU) \arrow[r] & \mathrm{Ran}^{\mathrm{BK}}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU) \end{tikzcd}$$

where the horizontal arrows are the canonical comparisons of proposition 1.10.15, $i_{GU} : GU \Rightarrow \mathrm{C}(\mathbb{C}, \mathbb{C}, GU)$ is the natural weak homotopy equivalence of proposition 1.9.16, $\eta_G : G \Rightarrow \mathrm{Ran}_U GU$ is the unit, and $\mathrm{Ran}^{\mathrm{BK}}_{V} G \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{VU} GU$ is the canonical comparison of proposition 1.8.34.

(i) *The diagram commutes and is functorial in G.*

(ii) *If $G : \mathbb{D} \to \mathbf{sSet}$ is projective-fibrant, then the canonical comparison*

$$\mathrm{Ran}^{\mathrm{BK}}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU) \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU)$$

and the natural transformation

$$\mathrm{Ran}^{\mathrm{BK}}_{VU} i_{GU} : \mathrm{Ran}^{\mathrm{BK}}_{VU} GU \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU)$$

are natural weak homotopy equivalences, and the image in $\mathrm{Ho}\,[\mathbb{D}, \mathbf{sSet}]$ *of*

$$\mathrm{Ran}_U i_{GU} \circ \eta_G : G \Rightarrow \mathrm{Lan}_U \mathrm{B}(\mathbb{C}, \mathbb{C}, GU)$$

can be identified with the derived unit $G \to \mathbf{R}\mathrm{Ran}_U (\mathrm{Ho}\, U^*) G$.

(iii) *If* $G : \mathbb{D} \to \mathbf{sSet}$ *is injective-fibrant, then the canonical comparison* $\operatorname{Ran}_V G \Rightarrow \operatorname{Ran}_V^{\mathrm{BK}} G$ *is also a natural weak homotopy equivalence, so the canonical comparison*

$$\operatorname{Ran}_V^{\mathrm{BK}} G \Rightarrow \operatorname{Ran}_{VU}^{\mathrm{BK}} GU$$

is a natural weak homotopy equivalence if and only if

$$\operatorname{Ran}_{VU} i_{GU} \circ \operatorname{Ran}_V \eta_G : \operatorname{Ran}_V G \Rightarrow \operatorname{Ran}_{VU} \mathrm{C}(\mathbb{C}, \mathbb{C}, GU)$$

is a natural weak homotopy equivalence.

Proof. The proof is essentially the same as that of lemma 1.10.16. ■

Corollary 1.10.19. *Let* $U : \mathbb{C} \to \mathbb{D}$ *and* $V : \mathbb{D} \to \mathbb{E}$ *be functors between small categories. For any diagram* $G : \mathbb{D} \to \mathbf{Kan}$*, the following are equivalent:*

(i) *The canonical comparison* $\operatorname{Ran}_V^{\mathrm{BK}} G \Rightarrow \operatorname{Ran}_{VU}^{\mathrm{BK}} GU$ *of proposition 1.8.34 is a natural weak homotopy equivalence of diagrams* $\mathbb{E} \to \mathbf{sSet}$.

(ii) *The morphism* $\mathbf{R}\operatorname{Ran}_V \eta_G : \mathbf{R}\operatorname{Ran}_V G \to \mathbf{R}\operatorname{Ran}_V \mathbf{R}\operatorname{Ran}_U (\operatorname{Ho} U^*) G$ *is an isomorphism (in* $\operatorname{Ho}[\mathbb{E}, \mathbf{sSet}]$*).*

Proof. By theorem 1.9.14 and proposition 4.1.17, there is an injective-fibrant replacement $(\hat{G}, j)$ for G; but the following diagram in $[\mathbb{E}, \mathbf{sSet}]$ commutes,

$$\begin{tikzcd}
\operatorname{Lan}_{VU}^{\mathrm{BK}} \tilde{G}U \arrow[r] \arrow[d, "\operatorname{Lan}_{VU}^{\mathrm{BK}} qU"'] & \operatorname{Lan}_V^{\mathrm{BK}} \tilde{G} \arrow[d, "\operatorname{Lan}_V^{\mathrm{BK}} q"] \\
\operatorname{Lan}_{VU}^{\mathrm{BK}} GU \arrow[r] & \operatorname{Lan}_U^{\mathrm{BK}} G
\end{tikzcd}$$

and since (by corollary 4.3.21) both $\hat{G}$ and G are projective-fibrant diagrams, the vertical arrows are natural weak homotopy equivalences by corollary 1.10.5; thus the claim is a consequence of lemma 1.10.18 (plus lemmas 1.5.2 and 3.1.11). ■

Lemma 1.10.20. *Let* $U : \mathbb{C} \to \mathbb{D}$ *and* $V : \mathbb{D} \to \mathbb{E}$ *be functors between small categories. The following are equivalent:*

(i) *For every diagram* $G : \mathbb{D} \to \mathbf{sSet}$*, the canonical comparison (of proposition 1.8.34)*

$$\operatorname{Lan}_{VU}^{\mathrm{BK}} GU \Rightarrow \operatorname{Lan}_V^{\mathrm{BK}} G$$

is a natural weak homotopy equivalence of diagrams $\mathbb{E} \to \mathbf{sSet}$.

(ii) *For every diagram* $G : \mathbb{D} \to \mathbf{sSet}$*, the morphism*

$$\mathbf{L}\mathrm{Lan}_V\, \varepsilon_G : \mathbf{L}\mathrm{Lan}_V\, \mathbf{L}\mathrm{Lan}_U(\mathrm{Ho}\, U^*)G \to \mathbf{L}\mathrm{Lan}_V\, G$$

is an isomorphism in $\mathrm{Ho}\,[\mathbb{E}, \mathbf{sSet}]$.

(iii) *For every diagram* $H : \mathbb{E} \to \mathbf{sSet}$*, the morphism*

$$\eta_{(\mathrm{Ho}\, V^*)H} : (\mathrm{Ho}\, V^*)H \to \mathbf{R}\mathrm{Ran}_U(\mathrm{Ho}\, U^*)(\mathrm{Ho}\, V^*)H$$

is an isomorphism in $\mathrm{Ho}\,[\mathbb{D}, \mathbf{sSet}]$.

Dually, the following are equivalent:

(i′) *For every diagram* $G : \mathbb{D} \to \mathbf{Kan}$*, the canonical comparison (of proposition 1.8.34)*

$$\mathrm{Ran}^{\mathrm{BK}}_V\, G \Rightarrow \mathrm{Ran}^{\mathrm{BK}}_{VU}\, GU$$

is a natural weak homotopy equivalence of diagrams $\mathbb{E} \to \mathbf{sSet}$.

(ii′) *For every diagram* $G : \mathbb{D} \to \mathbf{sSet}$*, the morphism*

$$\mathbf{R}\mathrm{Ran}_V\, \eta_G : \mathbf{R}\mathrm{Ran}_V\, G \Rightarrow \mathbf{R}\mathrm{Ran}_V\, \mathbf{R}\mathrm{Ran}_U(\mathrm{Ho}\, U^*)G$$

is an isomorphism in $\mathrm{Ho}\,[\mathbb{E}, \mathbf{sSet}]$.

(iii′) *For every diagram* $H : \mathbb{E} \to \mathbf{sSet}$*, the morphism*

$$\varepsilon_{(\mathrm{Ho}\, V^*)H} : \mathbf{L}\mathrm{Lan}_U(\mathrm{Ho}\, U^*)(\mathrm{Ho}\, V^*)H \Rightarrow (\mathrm{Ho}\, V^*)H$$

is an isomorphism in $\mathrm{Ho}\,[\mathbb{D}, \mathbf{sSet}]$.

Proof. (i) ⇔ (ii), (i′) ⇔ (ii′). See corollaries 1.10.17 and 1.10.19

(ii) ⇔ (iii), (ii′) ⇔ (iii ′). This is a special case of proposition A.1.12. ■

Corollary 1.10.21. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories and let* $G : \mathbb{D} \to \mathbf{sSet}$ *be a projective-cofibrant diagram. If* $U : \mathbb{C} \to \mathbb{D}$ *is cofinal, then the following are equivalent:*

(i) *The canonical comparison* $\varinjlim^{\mathrm{BK}}_{\mathbb{C}} GU \to \varinjlim^{\mathrm{BK}}_{\mathbb{D}} G$ *of proposition 1.8.34 is a weak homotopy equivalence of simplicial sets.*

(ii) *The canonical comparison* $\varinjlim_{\mathbb{C}}^{\mathrm{BK}} GU \to \varinjlim_{\mathbb{C}} GU$ *of* proposition 1.10.15 *is a weak homotopy equivalence of simplicial sets.*

(iii) *The canonical comparison* $\mathbf{L}\varinjlim_{\mathbb{C}} (\operatorname{Ho} U^*) G \to \mathbf{L}\varinjlim_{\mathbb{D}} G$ *is an isomorphism (in* $\operatorname{Ho}\mathbf{sSet}$*).*

Proof. With notation as in lemma 1.10.16, consider the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd}
\varinjlim_{\mathbb{C}}^{\mathrm{BK}} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \arrow[r, "\simeq"] \arrow[d, "\varinjlim_{\mathbb{C}}^{\mathrm{BK}} p_{GU}"', "\simeq"] & \varinjlim_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, GU) \arrow[d, "\operatorname{Lan}_{VU} p_{GU}"] \\
\varinjlim_{\mathbb{C}}^{\mathrm{BK}} GU \arrow[r] \arrow[d] & \varinjlim_{\mathbb{C}} GU \arrow[d, "\cong"', "\varinjlim_{\mathbb{D}} \varepsilon_G"] \\
\varinjlim_{\mathbb{D}}^{\mathrm{BK}} G \arrow[r, "\simeq"'] & \varinjlim_{\mathbb{D}} G
\end{tikzcd}$$

The lemma says that the marked arrows are weak homotopy equivalences of simplicial sets, and the cofinality hypothesis says $\varinjlim_{\mathbb{D}} \varepsilon_G : \varinjlim_{\mathbb{C}} GU \to \varinjlim_{\mathbb{D}} G$ is an isomorphism of simplicial sets. Thus, equivalence of (i) and (ii) is a consequence of the 2-out-of-3 property, and the equivalence of (i) and (iii) is a special case of corollary 1.10.17. ■

Corollary 1.10.22. *Let* $U : \mathbb{C} \to \mathbb{D}$ *be a functor between small categories and let* $G : \mathbb{D} \to \mathbf{sSet}$ *be an injective-fibrant diagram. If* $U : \mathbb{C} \to \mathbb{D}$ *is coinitial, then the following are equivalent:*

(i) *The canonical comparison* $\varprojlim_{\mathbb{D}}^{\mathrm{BK}} G \to \varprojlim_{\mathbb{C}}^{\mathrm{BK}} GU$ *of* proposition 1.8.34 *is a weak homotopy equivalence of simplicial sets.*

(ii) *The canonical comparison* $\varinjlim_{\mathbb{C}} GU \to \varprojlim_{\mathbb{C}}^{\mathrm{BK}} GU$ *of* proposition 1.10.15 *is a weak homotopy equivalence of simplicial sets.*

(iii) *The canonical comparison* $\mathbf{R}\varprojlim_{\mathbb{D}} G \to \mathbf{R}\varprojlim_{\mathbb{C}} (\operatorname{Ho} U^*) G$ *is an isomorphism (in* $\operatorname{Ho}\mathbf{sSet}$*).*

Proof. The proof is essentially the same as that of corollary 1.10.21. ■

Definition 1.10.23. Let $\mathbb{C}$ and $\mathbb{D}$ be small categories.

- A **homotopy coinitial functor** $U : \mathbb{C} \to \mathbb{D}$ is a functor such that, for all objects d in $\mathbb{D}$, the nerve $\mathrm{N}((U \downarrow d))$ is a weakly contractible simplicial set.
- A **homotopy cofinal functor** $U : \mathbb{C} \to \mathbb{D}$ is a functor such that, for all objects d in $\mathbb{D}$, the nerve $\mathrm{N}((d \downarrow U))$ is a weakly contractible simplicial set.

Remark 1.10.24. By proposition 1.7.12, $U : \mathbb{C} \to \mathbb{D}$ is a homotopy coinitial functor if and only if $U^{\mathrm{op}} : \mathbb{C}^{\mathrm{op}} \to \mathbb{D}^{\mathrm{op}}$ is a homotopy cofinal functor.

Remark 1.10.25. Every homotopy coinitial (resp. homotopy cofinal) functor is a coinitial (resp. cofinal) functor, but the converse is false.

Theorem 1.10.26. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories. The following are equivalent:*

(i) *$U : \mathbb{C} \to \mathbb{D}$ is a homotopy coinitial functor.*

(ii) *For every diagram $G : \mathbb{D} \to \mathbf{sSet}$, the canonical comparison morphism*

$$\mathbf{R}\varprojlim_{\mathbb{D}} G \to \mathbf{R}\varprojlim_{\mathbb{C}} GU$$

is an isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$.

(iii) *For every diagram $G : \mathbb{D} \to \mathbf{Kan}$, the canonical comparison morphism*

$$\varinjlim_{\mathbb{D}}{}^{\mathrm{BK}} G \to \varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} GU$$

is a weak homotopy equivalence of simplicial sets.

Proof. (i) $\Leftrightarrow$ (ii). By lemmas 1.10.20 and lemma 1.10.16 and corollary 1.10.17, it is equivalent to show that $U : \mathbb{C} \to \mathbb{D}$ is a homotopy coinitial functor if and only if

$$\mathrm{Lan}_U^{\mathrm{BK}} \Delta X \Rightarrow \Delta X$$

is a weak homotopy equivalence for every simplicial set X. By definition,

$$\left(\mathrm{Lan}_U^{\mathrm{BK}} \Delta X\right)d = \mathrm{B}\left(U^* \hbar_d, \mathbb{C}, \Delta X\right)$$

and by proposition 1.8.36,

$$\mathrm{B}\big(U^* \hbar_d, \mathbb{C}, \Delta X\big) \cong \mathrm{B}\big(U^* \hbar_d, \mathbb{C}^{\mathrm{op}}, \Delta 1\big) \times X$$

but it is not hard to see that the diagram below commutes,

$$\begin{tikzcd}
\operatorname{Ran}_U^{\mathrm{BK}} \Delta X \arrow[r] \arrow[d, "\cong"'] & \Delta X \arrow[d, "\cong"] \\
\mathrm{B}(U^* \mathbb{D}, \mathbb{C}, \Delta 1) \times X \arrow[r] & \Delta 1 \times X
\end{tikzcd}$$

where the bottom arrow is the morphism induced by the unique natural transformation $\mathrm{B}(U^* \mathbb{D}, \mathbb{C}^{\mathrm{op}}, \Delta 1) \to \Delta 1$, so by proposition 1.5.15, $\mathrm{B}\big(U^* \hbar^d, \mathbb{C}^{\mathrm{op}}, \Delta 1\big)$ is a weakly contractible simplicial set if and only if every $\operatorname{Ran}_U^{\mathrm{BK}} \Delta X \Rightarrow \Delta X$ is a natural weak homotopy equivalence. Remark 1.8.5 says,

$$\mathrm{B}\big(U^* \hbar_d, \mathbb{C}, \Delta 1\big) \cong \mathrm{N}((U \downarrow d))$$

so the claim follows.

(ii) $\Rightarrow$ (iii). This is corollary 1.10.17. ■

Theorem 1.10.27. *Let $U : \mathbb{C} \to \mathbb{D}$ be a functor between small categories. The following are equivalent:*

(i) *$U : \mathbb{C} \to \mathbb{D}$ is a homotopy cofinal functor.*

(ii) *For every diagram $G : \mathbb{D} \to \mathbf{sSet}$, the canonical comparison morphism*

$$\mathbf{L}\varinjlim_{\mathbb{C}} GU \to \mathbf{L}\varinjlim_{\mathbb{D}} G$$

is an isomorphism in $\operatorname{Ho} \mathbf{sSet}$.

(iii) *For every diagram $G : \mathbb{D} \to \mathbf{sSet}$, the canonical comparison morphism*

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} GU \to \varinjlim_{\mathbb{D}}{}^{\mathrm{BK}} G$$

is a weak homotopy equivalence of simplicial sets.

Proof. (i) ⇔ (ii). By lemmas 1.10.20 and 1.10.18 and corollary 1.10.19, it is equivalent to show that $U : \mathbb{C} \to \mathbb{D}$ if and only if the canonical comparison

$$\Delta X \Rightarrow \operatorname{Ran}_U^{\mathrm{BK}} \Delta X$$

is a weak homotopy equivalence for every Kan complex X. By definition,

$$\left(\operatorname{Ran}_U^{\mathrm{BK}} \Delta X\right) d = \mathrm{C}\left(U^* \hbar^d, \mathbb{C}, \Delta X\right)$$

and by proposition 1.8.36,

$$\mathrm{C}\left(U^* \hbar^d, \mathbb{C}, \Delta X\right) \cong \left[\mathrm{B}\left(U^* \hbar^d, \mathbb{C}^{\mathrm{op}}, \Delta 1\right), X\right]$$

but it is not hard to see that the diagram below commutes,

$$\begin{tikzcd}
\Delta X \arrow[r] \arrow[d, "\cong"'] & \operatorname{Ran}_U^{\mathrm{BK}} \Delta X \arrow[d, "\cong"] \\
{[\Delta 1, X]} \arrow[r] & \left[\mathrm{B}(U^* \mathbb{D}, \mathbb{C}, \Delta 1), X\right]
\end{tikzcd}$$

where the bottom arrow is the morphism induced by the unique natural transformation $\mathrm{B}(U^* \mathbb{D}, \mathbb{C}^{\mathrm{op}}, \Delta 1) \to \Delta 1$, so by proposition 1.5.15, $\Delta X \Rightarrow \operatorname{Ran}_U^{\mathrm{BK}} \Delta X$ is a weak homotopy equivalence for all Kan complexes X if and only if every $\mathrm{B}\left(U^* \hbar^d, \mathbb{C}^{\mathrm{op}}, \Delta 1\right)$ is a weakly contractible simplicial set. Remark 1.8.5 says,

$$\mathrm{B}\left(U^* \hbar^d, \mathbb{C}^{\mathrm{op}}, \Delta 1\right) \cong \mathrm{N}((d \downarrow U)^{\mathrm{op}}) \cong \mathrm{N}((d \downarrow U))^{\mathrm{op}}$$

so (recalling proposition 1.7.12) the claim follows.

(ii) ⇔ (iii). This is corollary 1.10.17. ■

1.11 Homotopy theory of nerves

Prerequisites. §§1.2, 1.3, 1.5, 1.7, 1.9, 3.1, 4.3, 5.1, B.5.

Although nerves of categories are not usually Kan complexes, they still possesses enough structure to have a good theory of weak homotopy equivalences: a surprising number of category-theoretic constructions have homotopical meaning when interpreted through the lens of the nerve functor. Most of these ideas were introduced by Quillen [1973] for the purpose of studying higher algebraic K-theory.

¶ 1.11.1. In this section, categories are small unless otherwise stated.

Definition 1.11.2. A **weak homotopy equivalence of categories** is a functor $f : \mathbb{A} \to \mathbb{B}$ such that the induced morphism $\mathrm{N}(f) : \mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathbb{B})$ is a weak homotopy equivalence of simplicial sets.

Remark. Weak homotopy equivalences of categories are also called **∞-equivalences**, but we should avoid this term as it conflicts with the terminology of higher category theory.

Lemma 1.11.3. **Cat**, *with the class of weak homotopy equivalences, is a saturated homotopical category. In particular, the class of weak homotopy equivalences of categories has the 2-out-of-3 property and is closed under retracts.*

Proof. Apply lemma 3.1.8 to lemma 1.5.2. ■

Remark 1.11.4. A functor $f : \mathbb{A} \to \mathbb{B}$ is a weak homotopy equivalence if and only if $f^{\mathrm{op}} : \mathbb{A}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is a weak homotopy equivalence, by propositions 1.2.1 and 1.7.12.

Definition 1.11.5. An **aspherical category** is a category whose nerve is weakly contractible, i.e. a category $\mathbb{A}$ such that the unique functor $\mathbb{A} \to \mathbb{1}$ is a weak homotopy equivalence.

Remark 1.11.6. If $\mathbb{A}$ has an initial object (resp. terminal object), then $\mathrm{N}(\mathbb{A})$ is contractible: indeed, then the unique functor $\mathbb{A} \to \mathbb{1}$ has a left adjoint (resp. right adjoint), and by corollary 1.3.11, we deduce that $\mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathbb{1})$ is an intrinsic homotopy equivalence. In particular, such an $\mathbb{A}$ is aspherical.

Lemma 1.11.7. *Let $p : \mathbb{A} \to \mathbb{C}$ be a functor and let $P : \mathbb{C} \to$ **sSet** be the diagram defined by $P(c) = \mathrm{N}((p \downarrow c))$.*

(i) *The projections $(p \downarrow c) \to \mathbb{A}$ induce a colimiting cocone $P \Rightarrow \Delta\mathrm{N}(\mathbb{A})$.*

(ii) *The canonical comparison morphism*[19]

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} P \to \varinjlim_{\mathbb{C}} P \cong \mathrm{N}(\mathbb{A})$$

is a weak homotopy equivalence.

*Dually, let $q : \mathbb{A} \to \mathbb{C}$ be a functor and let $Q : \mathbb{C}^{\mathrm{op}} \to$ **sSet** be the diagram defined by $Q(c) = \mathrm{N}((c \downarrow q))$.*

[19] See proposition 1.8.38.

(i′) *The projections* $(c \downarrow q) \to \mathbb{A}$ *induce a colimiting cocone* $Q \Rightarrow \Delta \mathrm{N}(\mathbb{A})$.

(ii′) *The canonical comparison morphism*

$$\varinjlim_{\mathbb{C}^{\mathrm{op}}}^{\mathrm{KB}} Q \to \varinjlim_{\mathbb{C}^{\mathrm{op}}} Q \cong \mathrm{N}(\mathbb{A})$$

is a weak homotopy equivalence.

Proof. (i). It is clear that the projections $(p \downarrow c) \to \mathbb{A}$ define a cocone, i.e.

$$\begin{tikzcd} (p \downarrow c_0) \arrow[r] \arrow[d] & \mathbb{A} \arrow[d, equal] \\ (p \downarrow c_1) \arrow[r] & \mathbb{A} \end{tikzcd}$$

commutes for every morphism $c_0 \to c_1$; we must show that the corresponding cocone $P \Rightarrow \Delta \mathrm{N}(\mathbb{A})$ is a colimiting cocone.

By remark 1.8.5, $\mathrm{N}((f \downarrow b)) \cong \mathrm{B}\left(p^* \not{h}_c, \mathbb{A}, \Delta 1\right)$, and under this identification, the forgetful functor $(p \downarrow c) \to \mathbb{A}$ corresponds to the morphism

$$\mathrm{B}\left(p^* \not{h}_c, \mathbb{A}, \Delta 1\right) \to \mathrm{B}(\Delta 1, \mathbb{A}, \Delta 1)$$

induced by the unique natural transformation $p^* \not{h}_c \Rightarrow \Delta 1$. Proposition 1.8.36 implies that $\mathrm{B}(-, \mathbb{A}, \Delta 1)$ preserves colimiting cocones, so it suffices to show that $\varinjlim_{\mathbb{B}} p^* \not{h}_\bullet \cong \Delta 1$; and since colimits in $[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}]$ can be calculated componentwise, it is enough to verify that $\varinjlim_{\mathbb{C}} \not{h}^c \cong 1$ for all objects c in $\mathbb{C}$. But the Yoneda lemma yields a bijection

$$[\mathbb{C}, \mathbf{Set}]\left(\not{h}^c, \Delta X\right) \cong \mathbf{Set}(1, X)$$

that is natural in X, so we are done.

(ii). Let $H : \mathbb{A}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{sSet}$ be the functor given by $H(a, c) = \operatorname{disc} \mathbb{B}(p(a), c)$. Then $P \cong \mathrm{B}(H, \mathbb{A}, \Delta 1)$, and by theorem 1.8.37 (and proposition 1.9.7),

$$\varinjlim_{\mathbb{C}}^{\mathrm{BK}} P \cong \mathrm{B}(\Delta 1, \mathbb{C}, \mathrm{B}(H, \mathbb{A}, \Delta 1)) \cong \mathrm{B}(\mathrm{B}(\Delta 1, \mathbb{C}, H), \mathbb{A}, \Delta 1)$$

but $\mathrm{B}(\Delta 1, \mathbb{C}, H) \cong \mathrm{N}\left({}^{p(\bullet)}/\mathbb{C}\right)$, so (by remark 1.11.6) the unique natural transformation $\mathrm{B}(\Delta 1, \mathbb{C}, H) \Rightarrow \Delta 1$ is a natural weak homotopy equivalence; moreover, the diagram below commutes,

$$\begin{tikzcd}
\mathrm{B}(\Delta 1, \mathbb{C}, \mathrm{B}(H, \mathbb{A}, \Delta 1)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{N}(\mathbb{A}) \arrow[d, "\cong"] \\
\mathrm{B}(\mathrm{B}(\Delta 1, \mathbb{C}, H), \mathbb{A}, \Delta 1) \arrow[r] & \mathrm{B}(\Delta 1, \mathbb{A}, \Delta 1)
\end{tikzcd}$$

so theorem 1.9.27 (plus the 2-out-of-3 property) implies that the horizontal arrows in the diagram are weak homotopy equivalences. In particular, the morphism $\mathrm{B}(\Delta 1, \mathbb{C}, P) \to \mathrm{N}(\mathbb{A})$ in question is a weak homotopy equivalence. ■

Lemma 1.11.8. *Let $p : \mathbb{A} \to \mathbb{C}$ be a functor and let $\mathbb{F} : \mathbb{C} \to$ **Cat** be the diagram defined by $\mathbb{F}(c) = (p \downarrow c)$.*

(i) *There is a natural transformation fitting into the following diagram in* **Cat**,

$$\begin{tikzcd}
\mathbf{G}(\Delta \mathbb{1}, \mathbb{C}, \mathbb{F}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathbb{A} \arrow[d, "p"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}$$

where the left vertical arrow is the canonical projection and the top horizontal arrow is the functor $\mathbf{G}(\Delta \mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{A}$ defined by $(c, (a, u)) \mapsto a$.

(ii) *The induced comparison functor $\mathbf{G}(\Delta \mathbb{1}, \mathbb{C}, \mathbb{F}) \to (p \downarrow \mathbb{C})$ is an isomorphism.*

(iii) *The projection $d_1 : (p \downarrow \mathbb{C}) \to \mathbb{A}$ is a weak homotopy equivalence of categories.*

*Dually, let $q : \mathbb{A} \to \mathbb{C}$ be a functor and let $\mathbb{E} : \mathbb{C}^{\mathrm{op}} \to$ **Cat** be the diagram defined by $\mathbb{E}(c) = (c \downarrow q)$.*

(i) *There is a natural transformation fitting into the following diagram in* **Cat**,

$$\begin{tikzcd}
\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta \mathbb{1}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbb{A} \arrow[d, "q"] \\
\mathbb{C} \arrow[r, equal] & \mathbb{C}
\end{tikzcd}$$

where the left vertical arrow is the canonical projection and the top horizontal arrow is the functor $\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta \mathbb{1}) \to \mathbb{A}$ defined by $((a, u), c) \mapsto a$.

(ii) *The induced comparison functor* $\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}) \to (\mathbb{C} \downarrow q)$ *is an isomorphism.*

(iii) *The projection* $d_0 : (\mathbb{C} \downarrow q) \to \mathbb{A}$ *is a weak homotopy equivalence of categories.*

Proof. (i). The required natural transformation is the one whose component at an object $((a, u), c)$ in $\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1})$ is the morphism $u : pa \to c$ in $\mathbb{C}$.

(ii). By unfolding the definitions, it is easy to see that the induced comparison functor $\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}) \to (\mathbb{C} \downarrow q)$ is bijective on objects and fully faithful.

(iii). The projection $d_1 : (p \downarrow \mathbb{C}) \to \mathbb{A}$ has an evident section $s : \mathbb{A} \to (p \downarrow \mathbb{C})$, namely the functor defined by $a \mapsto \big(a, pa, \mathrm{id}_{pa}\big)$, and there is an evident natural transformation $s \circ d_1 \Rightarrow \mathrm{id}_{(p\downarrow\mathbb{C})}$; so (by applying lemma 1.3.10) $\mathrm{N}\big(d_1\big) : \mathrm{N}((p \downarrow \mathbb{C})) \to \mathrm{N}(\mathbb{A})$ is half of an intrinsic homotopy equivalence, hence is a weak homotopy equivalence (by proposition 1.5.3). ■

Definition 1.11.9.

- A **right aspherical functor** is a functor $f : \mathbb{A} \to \mathbb{B}$ with the following property: for all objects b in $\mathbb{B}$, the comma category $(f \downarrow b)$ is aspherical.
- A **left aspherical functor** is a functor $g : \mathbb{B} \to \mathbb{A}$ with the following property: for all objects a in $\mathbb{A}$, the comma category $(a \downarrow g)$ is aspherical.

Remark 1.11.10. A right aspherical (resp. left aspherical) functor is the same thing as a homotopy coinitial (resp. homotopy cofinal) functor: see definition 1.10.23. In particular, right apsherical (resp. left aspherical) functors are coinitial (resp. cofinal) functors, but not vice versa.

Lemma 1.11.11.

- *If* $f : \mathbb{A} \to \mathbb{B}$ *is a functor that admits a right adjoint, then* $f : \mathbb{A} \to \mathbb{B}$ *is right aspherical.*
- *If* $g : \mathbb{B} \to \mathbb{A}$ *is a functor that admits a left adjoint, then* $g : \mathbb{B} \to \mathbb{A}$ *is left aspherical.*

Proof. The two claims are formally dual; we will prove the first version.

Let $f : \mathbb{A} \to \mathbb{B}$ be a functor. It is well known that $f : \mathbb{A} \to \mathbb{B}$ admits a right adjoint if and only if each comma category $(f \downarrow b)$ admits a terminal object; but by remark 1.11.6, a category that has a terminal object is aspherical, so $f : \mathbb{A} \to \mathbb{B}$ is right aspherical if it admits a right adjoint. ∎

Lemma 1.11.12.

- *If $f : \mathbb{A} \to \mathbb{A}$ is a functor for which there exists a natural transformation $\varepsilon : f \Rightarrow \mathrm{id}_{\mathbb{A}}$, then $f : \mathbb{A} \to \mathbb{A}$ is a right aspherical functor.*
- *If $g : \mathbb{A} \to \mathbb{A}$ is a functor for which there exists a natural transformation $\eta : \mathrm{id}_{\mathbb{A}} \Rightarrow g$, then $g : \mathbb{A} \to \mathbb{A}$ is a left aspherical functor.*

Proof. The two claims are formally dual; we will prove the first version.

Let $c : fa \to b$ be a morphism in $\mathbb{A}$. By naturality, the following diagram commutes:

$$\begin{tikzcd} fa \arrow[d, "c"'] & ffa \arrow[l, "f\varepsilon_a"'] \arrow[r, "fc"] \arrow[d] & fb \arrow[d, "\varepsilon_b"] \\ b \arrow[r, equal] & b \arrow[r, equal] & b \end{tikzcd}$$

Thus, the morphism $\Delta^0 \to \mathrm{N}((f \downarrow b))$ corresponding to the object $\left(b, \varepsilon_b\right)$ is half of an intrinsic homotopy equivalence and hence (by proposition 1.5.3) a weak homotopy equivalence *a fortiori*. ∎

Proposition 1.11.13. *Let $u : \mathbb{A} \to \mathbb{B}$ and $v : \mathbb{B} \to \mathbb{C}$ be functors.*

- *If $v \circ u : \mathbb{A} \to \mathbb{C}$ is right aspherical and $v : \mathbb{B} \to \mathbb{C}$ is fully faithful, then $v : \mathbb{A} \to \mathbb{B}$ is also right aspherical.*
- *If $v \circ u : \mathbb{A} \to \mathbb{C}$ is left aspherical and $v : \mathbb{B} \to \mathbb{C}$ is fully faithful, then $v : \mathbb{A} \to \mathbb{B}$ is also left aspherical.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $v \circ u : \mathbb{A} \to \mathbb{C}$ is right aspherical and $v : \mathbb{B} \to \mathbb{C}$ is fully faithful. Then, for any object b in $\mathbb{B}$, the comma category $(u \downarrow b)$ is naturally isomorphic to the comma category $(v \circ u \downarrow v(c))$, so $(u \downarrow b)$ is an aspherical category. Thus, $u : \mathbb{A} \to \mathbb{B}$ is indeed right aspherical. ∎

The following result is due to Grothendieck [1983, §40].

Theorem 1.11.14. *Consider a commutative triangle of categories and functors:*

$$\begin{tikzcd} \mathbb{A} \arrow[rr, "u"] \arrow[dr, "p"'] & & \mathbb{B} \arrow[dl, "q"] \\ & \mathbb{C} & \end{tikzcd}$$

- *If, for every object c in $\mathbb{C}$, the functor $u_c : (p \downarrow c) \to (q \downarrow c)$ induced by $u : \mathbb{A} \to \mathbb{B}$ is a weak homotopy equivalence, then the functor $u : \mathbb{A} \to \mathbb{B}$ itself is a weak homotopy equivalence.*

- *If, for every object c in $\mathbb{C}$, the functor ${}^c u : (c \downarrow p) \to (c \downarrow q)$ induced by $u : \mathbb{A} \to \mathbb{B}$ is a weak homotopy equivalence, then the functor $u : \mathbb{A} \to \mathbb{B}$ itself is a weak homotopy equivalence.*

Proof. The two claims are formally dual; we will prove the first version, following the proof of Théorème 2.1.13 in [Cisinski, 2004].

Let $P, Q : \mathbb{C} \to \mathbf{sSet}$ be the diagrams defined by $P(c) = \mathrm{N}((p \downarrow c))$ and $Q(c) = \mathrm{N}((q \downarrow c))$, respectively. Then $u : \mathbb{A} \to \mathbb{B}$ induces a natural transformation $\theta : P \Rightarrow Q$ with components $\theta_u = \mathrm{N}\left(u_c\right)$, and by hypothesis, $\theta : P \Rightarrow Q$ is a natural weak homotopy equivalence. Lemma 1.11.7 says we have a commutative diagram of the form below,

$$\begin{tikzcd}
\varinjlim_{\mathbb{C}}^{\mathrm{BK}} P \arrow[r] \arrow[d, "\varinjlim_{\mathbb{C}}^{\mathrm{BK}} \theta"'] & \mathrm{N}(\mathbb{A}) \arrow[d, "\mathrm{N}(u)"] \\
\varinjlim_{\mathbb{C}}^{\mathrm{BK}} Q \arrow[r] & \mathrm{N}(\mathbb{B})
\end{tikzcd}$$

where the horizontal arrows are weak homotopy equivalences; but corollary 1.9.20 implies $\varinjlim_{\mathbb{C}}^{\mathrm{BK}} \theta$ is also a weak homotopy equivalence, so (using the 2-out-of-3 property) we may deduce that $u : \mathbb{A} \to \mathbb{B}$ is indeed a weak homotopy equivalence of categories. ■

As a corollary, we obtain a famous result of Quillen [1973, § 1]:

Corollary 1.11.15 (Quillen's Theorem A)**.**

- *Right aspherical functors are weak homotopy equivalences of categories.*

- *Left aspherical functors are weak homotopy equivalences of categories.*

Proof. The two claims are formally dual; we will prove the first version.

Let $f : \mathbb{A} \to \mathbb{B}$ be a right aspherical functor. Consider the following commutative triangle:

$$\begin{tikzcd}
\mathbb{A} \arrow[rr, "f"] \arrow[dr, "f"'] & & \mathbb{B} \arrow[dl, "\mathrm{id}"] \\
& \mathbb{B} &
\end{tikzcd}$$

Remark 1.11.6 implies that the slice categories $\mathbb{B}_{/b}$ are aspherical, so the 2-out-of-3 property (lemma 1.11.3) plus right asphericity implies that the functors $f_b : (f \downarrow b) \to \mathbb{B}_{/b}$ are weak homotopy equivalences for all objects b in $\mathbb{B}$. Thus, by theorem 1.11.14, $f : \mathbb{A} \to \mathbb{B}$ itself is a weak homotopy equivalence. ■

Remark 1.11.16. In view of remarks 1.8.5 and 1.11.10, Quillen's Theorem A is also a corollary of theorem 1.10.27.

We may now prove a useful result of Thomason [1977, 1979].

Theorem 1.11.17 (Thomason's homotopy colimit theorem). *Let $\mathbb{C}$ be a category, let $\mathbb{F} : \mathbb{C} \to \mathbf{Cat}$ be a diagram, let $\mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F})$ be the Grothendieck construction, and let $p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{C}$ be the canonical projection.*

(i) *There is a natural weak homotopy equivalence*

$$\mathbb{F}(-) \Rightarrow (p \downarrow -)$$

and it is natural in $\mathbb{F}$.

(ii) *There is a weak homotopy equivalence*

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}}(\mathrm{N} \circ \mathbb{F}) \to \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}))$$

and it is natural in $\mathbb{F}$.

Dually, let $\mathbb{E} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a diagram, let $\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1})$ be the Grothendieck construction and let $q : \mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}) \to \mathbb{C}$ be the canonical projection.

(i) *There is a natural weak homotopy equivalence*

$$\mathbb{E}(-) \Rightarrow (- \downarrow q)$$

and it is natural in $\mathbb{E}$.

(ii) *There is a weak homotopy equivalence*

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{KB}}(\mathrm{N} \circ \mathbb{E}) \to \mathrm{N}(\mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}))$$

and it is natural in $\mathbb{E}$.

Proof. (i). By construction, the fibre $p^{-1}\{c\}$ is naturally isomorphic to $\mathbb{F}(c)$, and $p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{C}$ is a Grothendieck (pre-)opfibration by proposition B.5.31 (and proposition B.5.29), so proposition B.5.21 says that the canonical comparison functor $p^{-1}\{c\} \to (p \downarrow c)$ has a left adjoint; but lemma 1.11.11 says that such functors are left aspherical, and hence by Quillen's Theorem A (corollary 1.11.15), weak homotopy equivalences *a fortiori*. Thus, we have a natural weak homotopy equivalence $\mathbb{F}(-) \Rightarrow (p \downarrow -)$.

(ii). Lemma 1.11.7 says that there exist a canonical isomorphism

$$\varinjlim_{\mathbb{C}} \mathrm{N}((p \downarrow -)) \cong \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}))$$

and a canonical weak homotopy equivalence

$$\varinjlim_{\mathbb{C}}{}^{\mathrm{BK}} \mathrm{N}((p \downarrow -)) \to \varinjlim_{\mathbb{C}} \mathrm{N}((p \downarrow -))$$

so the claim is a consequence of the fact that $\varinjlim_{\mathbb{C}}^{\mathrm{BK}}$ preserves natural weak homotopy equivalences (corollary 1.9.20). ■

Corollary 1.11.18. *Let $\mathbb{C}$ be a category. Then the functor*

$$\mathbf{G}(-, \mathbb{C}, -) : [\mathbb{C}^{\mathrm{op}}, \mathbf{Cat}] \times [\mathbb{C}, \mathbf{Cat}] \to \mathbf{Cat}$$

defined by the Grothendieck construction sends natural weak homotopy equivalences to weak homotopy equivalences (in each variable and jointly).

Proof. Let $\varphi : \mathbb{E}' \to \mathbb{E}$ be a natural weak homotopy equivalence of functors $\mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ and let $\psi : \mathbb{F}' \to \mathbb{F}$ be a natural weak homotopy equivalence of functors $\mathbb{C} \to \mathbf{Cat}$. Clearly,

$$\mathbf{G}(\varphi, \mathbb{C}, \psi) = \mathbf{G}(\varphi, \mathbb{C}, \mathbb{F}) \circ \mathbf{G}(\mathbb{E}, \mathbb{C}, \psi)$$

and since the class of weak homotopy equivalences is closed under composition (by lemma 1.11.3), so it suffices to prove the claim for each variable separately; and by duality, it suffices to prove it for just one variable, say the second variable. But by proposition B.5.34, there is a commutative diagram of the form below,

$$\begin{tikzcd}
\mathbf{G}(\mathbb{E}, \mathbb{C}, \mathbb{F}') \arrow[r, "\cong"] \arrow[d, "\mathbf{G}(\mathbb{E},\mathbb{C},\psi)"'] & \mathbf{G}(\Delta\mathbb{1}, \mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}), \mathbb{F}') \arrow[d, "\mathbf{G}(\Delta\mathbb{1},\mathbf{G}(\mathbb{E},\mathbb{C},\Delta\mathbb{1}),\psi)"] \\
\mathbf{G}(\mathbb{E}, \mathbb{C}, \mathbb{F}) \arrow[r, "\cong"'] & \mathbf{G}(\Delta\mathbb{1}, \mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1}), \mathbb{F})
\end{tikzcd}$$

so (by the 2-out-of-3 property) the claim is a consequence of corollary 1.9.20 and Thomason's homotopy colimit theorem (1.11.17). ■

Definition 1.11.19.

- A **right-locally weakly constant functor** is a functor $p : \mathbb{A} \to \mathbb{B}$ with the following property: for every morphism $b' \to b$ in $\mathbb{B}$, the induced functor $(p \downarrow b') \to (p \downarrow b)$ is a weak homotopy equivalence.

- A **left-locally weakly constant functor** is a functor $q : \mathbb{A} \to \mathbb{B}$ with the following property: for every morphism $b \to b'$ in $\mathbb{B}$, the induced functor $(b' \downarrow q) \to (b \downarrow q)$ is a weak homotopy equivalence.

Lemma 1.11.20. *Let $\mathbb{B}$ be a category.*

- *Let $\mathbb{F} : \mathbb{B} \to$ **Cat** be a diagram. The induced Grothendieck opfibration*

$$p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F}) \to \mathbb{B}$$

is a right-locally weakly constant functor if and only if the reindexing functor $\mathbb{F}(b') \to \mathbb{F}(b)$ is a weak homotopy equivalence for every morphism $b' \to b$ in $\mathbb{B}$.

- *Let $\mathbb{E} : \mathbb{B}^{\mathrm{op}} \to$ **Cat** be a diagram. The induced Grothendieck fibration*

$$q : \mathbf{G}(\mathbb{E}, \mathbb{B}, \Delta\mathbb{1}) \to \mathbb{B}$$

is a left-locally weakly constant functor if and only if the reindexing functor $\mathbb{E}(b') \to \mathbb{E}(b)$ is a weak homotopy equivalence for every morphism $b \to b'$ in $\mathbb{B}$.

Proof. The two claims are formally dual; we will prove the first version.

Consider a morphism $b' \to b$ in $\mathbb{B}$. We then have the following commutative diagram in **Cat**,

$$\begin{tikzcd} \mathbb{F}(b') \arrow[r] \arrow[d] & (p \downarrow b') \arrow[d] \\ \mathbb{F}(b) \arrow[r] & (p \downarrow b) \end{tikzcd}$$

where the horizontal arrows are the right adjoint functors of (the formal dual of) proposition B.5.21 and the vertical arrows are induced by $b' \to b$. Thus, by lemma 1.11.11 and Quillen's Theorem A (corollary 1.11.15), the horizontal arrows are weak homotopy equivalences, so the 2-out-of-3 property (lemma 1.11.3) implies that $\mathbb{F}(b') \to \mathbb{F}(b)$ is a weak homotopy equivalence if and only if $(p \downarrow b') \to (p \downarrow b)$ is a weak homotopy equivalence. ∎

Definition 1.11.21. A **homotopy derived pullback diagram** in **Cat** is a commutative square in **Cat** whose image under $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is a derived pullback diagram in **sSet**.

Lemma 1.11.22. *Let $u : \mathbb{A} \to \mathbb{B}$ be a functor.*

- *Let $\mathbb{F} : \mathbb{B} \to \mathbf{Cat}$ be a diagram. If the reindexing functor $\mathbb{F}(b') \to \mathbb{F}(b)$ is a weak homotopy equivalence for every morphism $b' \to b$ in $\mathbb{B}$, then the pullback diagram in **Cat** of lemma B.5.33*

$$\begin{tikzcd} \mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ u) \arrow[r] \arrow[d] & \mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F}) \arrow[d] \\ \mathbb{A} \arrow[r, "u"'] & \mathbb{B} \end{tikzcd}$$

  is a homotopy derived pullback diagram.

- *Let $\mathbb{E} : \mathbb{B}^{\mathrm{op}} \to \mathbf{Cat}$ be a diagram. If the reindexing functor $\mathbb{E}(b') \to \mathbb{E}(b)$ is a weak homotopy equivalence for every morphism $b \to b'$ in $\mathbb{B}$, then the pullback diagram in **Cat** of lemma B.5.33*

$$\begin{tikzcd} \mathbf{G}(\mathbb{E} \circ u, \mathbb{A}, \Delta\mathbb{1}) \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{E}, \mathbb{B}, \Delta\mathbb{1}) \arrow[d] \\ \mathbb{A} \arrow[r, "u"'] & \mathbb{B} \end{tikzcd}$$

  is a homotopy derived pullback diagram.

Proof. The two claims are formally dual; we will prove the first version.

We have the following commutative cube in **sSet**,

$$\begin{tikzcd}[row sep=small, column sep=small]
\varinjlim^{\mathrm{BK}}_{\mathbb{A}} \mathrm{N} \circ \mathbb{F} \circ u \arrow[rr] \arrow[dd] \arrow[dr] & & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ u)) \arrow[dd] \arrow[dr] & \\
& \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \mathrm{N} \circ \mathbb{F} \arrow[rr] \arrow[dd] & & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F})) \arrow[dd] \\
\varinjlim^{\mathrm{BK}}_{\mathbb{A}} \Delta 1 \arrow[rr] \arrow[dr] & & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \Delta\mathbb{1})) \arrow[dr] & \\
& \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \Delta 1 \arrow[rr] & & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \Delta\mathbb{1}))
\end{tikzcd}$$

where the vertical arrows are induced by the unique natural transformation $\mathbb{F} \Rightarrow \Delta\mathbb{1}$, the horizontal arrows are the weak homotopy equivalences of Thomason's homotopy colimit theorem (1.11.17), and the diagonal arrows are as in corollary 1.8.23 and lemma B.5.33; note that the projections $\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \Delta\mathbb{1}) \to \mathbb{A}$ and $\mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \Delta\mathbb{1}) \to \mathbb{B}$ are isomorphisms. In particular, we have the following pullback diagram in **sSet**,

$$\begin{tikzcd}
\varinjlim^{\mathrm{BK}}_{\mathbb{A}} \mathrm{N} \circ \mathbb{F} \circ u \arrow[r] \arrow[d] & \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \mathrm{N} \circ \mathbb{F} \arrow[d] \\
\varinjlim^{\mathrm{BK}}_{\mathbb{A}} \Delta 1 \arrow[r] & \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \Delta 1
\end{tikzcd}$$

but the right vertical arrow is homotopically quadrable by proposition 1.9.34, so by by theorem 1.7.18 and proposition 5.1.23, the pullback diagram is a derived pullback diagram. The claim is then a consequence of proposition 5.1.20. ■

The following is also due to Quillen [1973, § 1].

Theorem 1.11.23 (Quillen's Theorem B)**.**

- *If $p : \mathbb{A} \to \mathbb{C}$ is a right-locally weakly constant functor, then for every object c in $\mathbb{C}$, we have the following homotopy derived pullback square in* **Cat**,

$$\begin{tikzcd}
(p \downarrow c) \arrow[r] \arrow[d] & \mathbb{A} \arrow[d, "p"] \\
\mathbb{C}_{/c} \arrow[r] & \mathbb{C}
\end{tikzcd}$$

where the horizontal arrows are induced by the evident projection functors.

- *If $q : \mathbb{A} \to \mathbb{B}$ is a left-locally weakly constant functor, then for every object c in $\mathbb{C}$, we have the following derived pullback square in* **sSet**,

$$\begin{tikzcd} (c \downarrow q) \arrow[r] \arrow[d] & \mathbb{A} \arrow[d, "q"] \\ {}^{c/}\mathbb{C} \arrow[r] & \mathbb{C} \end{tikzcd}$$

where the horizontal arrows are induced by the evident projection functors.

Proof. The two claims are formally dual; we will prove the first version.

Consider the following commutative diagram in **Cat**,

$$\begin{tikzcd} (p \downarrow c) \arrow[r] \arrow[d] & (p \downarrow \mathbb{C}) \arrow[r, "d_1"] \arrow[d] & \mathbb{A} \arrow[d, "p"] \\ \mathbb{C}_{/c} \arrow[r] \arrow[d] & {[2, \mathbb{C}]} \arrow[r, "d_1"'] \arrow[d, "d_0"] & \mathbb{C} \\ \mathbb{1} \arrow[r, "\ulcorner c \urcorner"'] & \mathbb{C} & \end{tikzcd}$$

where $d_1 : (p \downarrow \mathbb{C}) \to \mathbb{A}$ and $d_1 : [2, \mathbb{C}] \to \mathbb{C}$ are the domain projections, $d_0 : [2, \mathbb{C}] \to \mathbb{C}$ is the codomain projection, $(p \downarrow \mathbb{C}) \to [2, \mathbb{C}]$ is the functor defined by $(a, c, u) \mapsto u$, $\ulcorner c \urcorner : \mathbb{1} \to \mathbb{C}$ is the functor corresponding to the object c in $\mathbb{C}$, and every square is a pullback diagram. By lemmas 1.11.8 and 1.11.22,

$$\begin{tikzcd} (p \downarrow c) \arrow[r] \arrow[d] & (p \downarrow \mathbb{C}) \arrow[d, "d_0"] \\ \mathbb{1} \arrow[r, "\ulcorner c \urcorner"'] & \mathbb{C} \end{tikzcd}$$

is a homotopy derived pullback diagram, and by lemma 5.1.18,

$$\begin{tikzcd} \mathbb{C}_{/c} \arrow[r] \arrow[d] & {[2, \mathbb{C}]} \arrow[d, "d_0"] \\ \mathbb{1} \arrow[r, "\ulcorner c \urcorner"'] & \mathbb{C} \end{tikzcd} \qquad \begin{tikzcd} (p \downarrow \mathbb{C}) \arrow[r, "d_1"] \arrow[d] & \mathbb{A} \arrow[d, "p"] \\ {[2, \mathbb{C}]} \arrow[r, "d_1"'] & \mathbb{C} \end{tikzcd}$$

are homotopy derived pullback diagrams, so by applying lemma 5.1.19 (twice), the claim follows. ■

Proposition 1.11.24. *Let $\mathbb{C}$ be a category.*

- *Let $\mathbb{F} : \mathbb{C} \to \mathbf{Cat}$ be a functor. If the induced Grothendieck opfibration*

$$p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{C}$$

  is a right-locally weakly constant functor, then it is a homotopically quadrable morphism in $\mathbf{Cat}$ (with respect to weak homotopy equivalences).

- *Let $\mathbb{E} : \mathbb{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a functor. If the induced Grothendieck fibration*

$$q : \mathbf{G}(\mathbb{E}, \mathbb{C}, \Delta\mathbb{1})$$

  is a left-locally weakly constant functor, then it is a homotopically quadrable morphism in $\mathbf{Cat}$ (with respect to weak homotopy equivalences).

Proof. The two claims are formally dual; we will prove the first version.

Let $u : \mathbb{A} \to \mathbb{B}$ and $v : \mathbb{B} \to \mathbb{C}$ be functors. By lemma B.5.33, we have the following commutative diagram in $\mathbf{Cat}$,

$$\begin{tikzcd}
\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ v \circ u) \arrow[r, "\bar{u}"] \arrow[d] & \mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F} \circ v) \arrow[r, "\bar{v}"] \arrow[d] & \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \arrow[d, "p"] \\
\mathbb{A} \arrow[r, "u"'] & \mathbb{B} \arrow[r, "v"'] & \mathbb{C}
\end{tikzcd}$$

where both squares are pullback squares. Moreover, by lemma 1.11.20, if $p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{C}$ is right-locally weakly constant, then so are the other vertical arrows in the above diagram. We wish to show that the functor

$$\bar{u} : \mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ v \circ u) \to \mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F} \circ v)$$

is a weak homotopy equivalence if $u : \mathbb{A} \to \mathbb{B}$ is a weak homotopy equivalence and $p : \mathbf{G}(\Delta\mathbb{1}, \mathbb{C}, \mathbb{F}) \to \mathbb{C}$ is right-locally weakly constant.

Recalling Thomason's homotopy colimit theorem (1.11.17), we have a commutative diagram in $\mathbf{sSet}$ of the form below,

$$\begin{tikzcd}
\varinjlim^{\mathrm{BK}}_{\mathbb{A}} \mathrm{N} \circ \mathbb{F} \circ v \circ u \arrow[r, "\tilde{v}"] \arrow[d, "\simeq"'] & \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \mathrm{N} \circ \mathbb{F} \circ v \arrow[r, "\tilde{u}"] \arrow[d, "\simeq"'] & \varinjlim^{\mathrm{BK}}_{\mathbb{C}} \mathrm{N} \circ \mathbb{F} \arrow[d, "\simeq"] \\
\mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ v \circ u)) \arrow[r, "\mathrm{N}(\bar{u})"'] & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F} \circ v)) \arrow[r, "\mathrm{N}(\bar{v})"'] & \mathrm{N}(\mathbf{G}(\Delta\mathbb{1}, \mathbb{A}, \mathbb{F}))
\end{tikzcd}$$

where the vertical arrows are weak homotopy equivalences, and by corollary 1.8.23, we have a commutative diagram in **sSet** of form below,

$$\begin{tikzcd} \varinjlim^{\mathrm{BK}}_{\mathbb{A}} \mathrm{N} \circ \mathbb{F} \circ v \circ u \arrow[r, "\tilde{u}"] \arrow[d] & \varinjlim^{\mathrm{BK}}_{\mathbb{B}} \mathrm{N} \circ \mathbb{F} \circ v \arrow[r, "\tilde{v}"] \arrow[d] & \varinjlim^{\mathrm{BK}}_{\mathbb{C}} \mathrm{N} \circ \mathbb{F} \arrow[d] \\ \mathrm{N}(\mathbb{A}) \arrow[r, "\mathrm{N}(u)"'] & \mathrm{N}(\mathbb{B}) \arrow[r, "\mathrm{N}(v)"'] & \mathrm{N}(\mathbb{C}) \end{tikzcd}$$

where the two squares are pullback squares, so by proposition 1.9.34, $\tilde{v}$ is a weak homotopy equivalence of simplicial sets if u is a weak homotopy equivalence of categories; but by the 2-out-of-3 property, $\tilde{u}$ is a weak homotopy equivalence of simplicial sets if and only if $\bar{u}$ is a weak homotopy equivalence of categories, so we are done. ■

Definition 1.11.25. The **category of simplices** of a simplicial set X is the category $\mathbf{\Delta}(X)$ defined below:

- The objects are simplices of X.
- For $x \in X_n$ and $x' \in X_{n'}$, the morphisms $x \to x'$ are the morphisms $\varphi : [n] \to [n']$ in $\mathbf{\Delta}$ such that $X(\varphi)(x') = x$.
- Composition and identities are the obvious ones.

We write $\pi_{\Delta} : \mathbf{\Delta}(X) \to \mathbf{\Delta}$ for the evident projection functor that sends an n-simplex of X to the object $[n]$ in $\mathbf{\Delta}$.

¶ 1.11.26. For brevity, if $\mathbb{A}$ is a small category, then we write $\mathbf{\Delta}(\mathbb{A})$ instead of $\mathbf{\Delta}(\mathrm{N}(\mathbb{A}))$. This is consistent with the notation of § 4.10. We will also use the left and right projection functors of definition 4.10.9.

Remark 1.11.27. Of course, $\mathbf{\Delta}(X)$ is (naturally isomorphic to) the comma category $(\Delta^{\bullet} \downarrow X)$.

Definition 1.11.28. The **Quillen subdivision** of a simplicial set X is the simplicial set $\mathrm{Sd}_{\mathrm{Q}}(X) = \mathrm{N}(\mathbf{\Delta}(X))$.

Lemma 1.11.29. *The functor* $\mathrm{Sd}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$ *admits a right adjoint, namely the functor* $\mathrm{Ex}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$ *defined by the following formula:*

$$\mathrm{Ex}_{\mathrm{Q}}(Y)_n = \mathbf{sSet}\big(\mathrm{Sd}_{\mathrm{Q}}(\Delta^n), Y\big)$$

Proof. Let $F : \mathbf{\Delta} \to \mathbf{sSet}$ be the diagram defined by $F([n]) = \mathrm{Sd}_{\mathrm{Q}}(\Delta^n)$ and let $P : \mathbf{\Delta}(X) \to \mathbf{sSet}$ be the diagram defined by $P(x) = \mathrm{N}\big(\mathbf{\Delta}(X)_{/x}\big)$. Note that if x is an n-simplex of X, then $\pi_{\Delta} : \mathbf{\Delta}(X) \to \mathbf{\Delta}$ induces an isomorphism $\mathbf{\Delta}(X)_{/x} \to \mathbf{\Delta}_{/[n]}$; but there is a natural isomorphism $\mathbf{\Delta}_{/[n]} \cong \mathbf{\Delta}(\Delta^n)$, so $P \cong F\pi_{\Delta}$. On the other hand, lemma 1.11.7 says that $\varinjlim_{\mathbf{\Delta}(X)} P$ can be identified with $\mathrm{N}(\mathbf{\Delta}(X)) = \mathrm{Sd}_{\mathrm{Q}}(X)$, so using the formula of theorem A.5.15, we deduce that $\mathrm{Sd}_{\mathrm{Q}}(X) \cong X \star_{\mathbf{\Delta}} F$ (naturally in X). The claim is then an instance of proposition A.6.15. ■

Lemma 1.11.30. *The functor* $\mathbf{\Delta}(-) : \mathbf{sSet} \to \mathbf{Cat}$ *admits a right adjoint, namely the functor* $\mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-)) : \mathbf{Cat} \to \mathbf{sSet}$.

Proof. By proposition 1.2.1, we have an adjunction

$$\tau_1 \dashv \mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$$

and by lemma 1.11.29, we also have

$$\mathrm{Sd}_{\mathrm{Q}} \dashv \mathrm{Ex}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$$

so by composition, we have the following adjunction:

$$\tau_1 \mathrm{Sd}_{\mathrm{Q}} \dashv \mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-)) : \mathbf{Cat} \to \mathbf{sSet}$$

We also know that $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is fully faithful, so by proposition A.1.3, the counit $\tau_1 \mathrm{N} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ is a natural isomorphism; in particular, $\tau_1 \mathrm{Sd}_{\mathrm{Q}} \cong \mathbf{\Delta}(-)$. Thus we have an adjunction of the required form. ■

Lemma 1.11.31. *Let X be a simplicial set.*

(i) *There is a natural isomorphism*

$$\mathrm{Sd}_{\mathrm{Q}}(X) \cong \mathrm{B}(X, \mathbf{\Delta}, \Delta 1)$$

where on the RHS we regard X as a weight $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$.

(ii) *There is a weak homotopy equivalence* $\lambda_X : \mathrm{Sd}_{\mathrm{Q}}(X) \to X$, *and it is natural in X.*

(iii) *If $X = \mathrm{N}(\mathbb{C})$ for some category $\mathbb{C}$, then $\lambda_X = \mathrm{N}\big(\pi_{\mathrm{R}}\big)$ as morphisms* $\mathrm{Sd}_{\mathrm{Q}}(\mathrm{N}(\mathbb{C})) \to \mathrm{N}(\mathbb{C})$. *In particular,* $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathrm{N}(\mathbb{C})) \to \mathbb{C}$ *is a weak homotopy equivalence of categories.*

Proof. (i). This is straightforward.

(ii). We follow the proof of Lemme 2.1.15 in [Cisinski, 2004].

Let $P, Q : \mathbf{\Delta}(X) \to \mathbf{sSet}$ be the diagrams defined by $P(x) = \mathrm{N}\big(\mathbf{\Delta}(X)_{/x}\big)$ and $Q(x) = \Delta^{\pi_\Delta(x)}$. Note that if x is an n-simplex of X, then $\pi_\Delta : \mathbf{\Delta}(X) \to \mathbf{\Delta}$ induces an isomorphism $\mathbf{\Delta}(X)_{/x} \to \mathbf{\Delta}_{/[n]}$; but there is a natural isomorphism $\mathbf{\Delta}_{/[n]} \cong \mathbf{\Delta}(\Delta^n)$, so the right projection $\pi_\mathrm{R} : \mathbf{\Delta}(\mathrm{N}(-)) \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ induces a natural transformation $\theta : P \Rightarrow Q$. Moreover, by remark 1.11.6, each $P(x)$ and $Q(x)$ is contractible, so $\theta : P \Rightarrow Q$ is a natural weak homotopy equivalence.

Now, by proposition 1.8.38, we have the following commutative diagram:

$$\begin{tikzcd}
\varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}(X)} P \arrow[r] \arrow[d, "\varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}(X)} \theta"'] & \varinjlim_{\mathbf{\Delta}(X)} P \arrow[d, "\varinjlim_{\mathbf{\Delta}(X)} \theta"] \\
\varinjlim^{\mathrm{BK}}_{\mathbf{\Delta}(X)} Q \arrow[r] & \varinjlim_{\mathbf{\Delta}(X)} Q
\end{tikzcd}$$

Lemma 1.11.7 says that $\varinjlim_{\mathbf{\Delta}(X)} P$ can be identified with $\mathrm{N}(\mathbf{\Delta}(X)) \cong \mathrm{Sd}_\mathrm{Q}(X)$ and that the morphism $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P) \to \mathrm{Sd}_\mathrm{Q}(X)$ is a weak homotopy equivalence. On the other hand, theorem A.5.15 implies that $\varinjlim_{\mathbf{\Delta}(X)} Q$ can be identified with X, so $\theta : P \Rightarrow Q$ defines a natural morphism $\lambda_X : \mathrm{Sd}_\mathrm{Q}(X) \to X$.

We claim that $\lambda_X : \mathrm{Sd}_\mathrm{Q}(X) \to X$ is the desired natural weak homotopy equivalence. Indeed, corollary 1.9.20 that the left vertical arrow in the diagram is a weak homotopy equivalence, so to prove the claim, it suffices to show that $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), Q) \to X$ is a weak homotopy equivalence. It is not hard to see that

$$\mathrm{B}_n(\Delta 1, \mathbf{\Delta}(X), Q) \cong \coprod_{(k_0, \ldots, k_n)} X_{k_n} \times \Delta^{k_n}_{k_{n-1}} \times \cdots \times \Delta^{k_1}_{k_0} \times \Delta^{k_0} \cong \mathrm{B}_n(X, \mathbf{\Delta}, \mathbf{\Delta})$$

naturally in n, where $(k_0, \ldots, k_n)$ varies over n-tuples of natural numbers. The morphism $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), Q) \to X$ can then be identified with the realisation of the morphism $\mathrm{B}\big(\Delta 1, \mathbf{\Delta}(X), Q_\bullet\big) \to \operatorname{disc} X_\bullet$ in $\mathbf{ssSet}$ defined by

$$\big(x, \varphi_n, \ldots, \varphi_1, \varphi_0\big) \mapsto X\big(\varphi_n \circ \cdots \circ \varphi_0\big)(x)$$

which is a degreewise weak homotopy equivalence, by proposition 1.9.16; hence, $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), Q) \to X$ is a weak homotopy equivalence, by theorem 1.6.10.

(iii). Let $\mathbb{C}$ be a category and let $X = \mathrm{N}(\mathbb{C})$. Since a functor is uniquely determined by its action on objects and morphisms, it suffices to show that λ_X :

$\mathrm{N}(\mathbf{\Delta}(X)) \to X$ agrees with $\mathrm{N}(\pi_{\mathrm{R}}) : \mathrm{N}(\mathbf{\Delta}(\mathrm{N}(\mathbb{C}))) \to \mathrm{N}(\mathbb{C})$ on vertices and edges. For convenience, we make the following identifications,

$$\begin{aligned}
(\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P))_0 &\cong \coprod_k X_k \times \mathrm{ob}\, \mathbf{\Delta}_{/[k]} \\
(\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P))_1 &\cong \coprod_{(k_0,k_1)} X_{k_1} \times \Delta^{k_1}_{k_0} \times \mathrm{mor}\, \mathbf{\Delta}_{/[k_0]} \\
(\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), Q))_0 &\cong \coprod_k X_k \times \mathrm{ob}\, [k] \\
(\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), Q))_1 &\cong \coprod_{(k_0,k_1)} X_{k_1} \times \Delta^{k_1}_{k_0} \times \mathrm{mor}\, [k_0]
\end{aligned}$$

so that a vertex of $(\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P))$ is a pair (x, φ) where x is a k-simplex of X and φ is a morphism in $\mathbf{\Delta}$ with codomain $[k]$, etc.

Let x be a vertex of $\mathrm{N}(\mathbf{\Delta}(X))$, i.e. an n-simplex of X. It is the image of an evident vertex of $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P)$, namely $(x, \mathrm{id}_{[n]})$. An n-simplex of X is a functor $[n] \to \mathbb{C}$, and by definition, $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), \theta)$ sends $(x, \mathrm{id}_{[n]})$ to (x, n). The image of (x, n) under the morphism $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P) \to X$ is $x(n)$, so λ_X indeed agrees with $\mathrm{N}(\pi_{\mathrm{R}})$ on vertices.

Now let $f : x_0 \to x_1$ be an edge of $\mathrm{N}(\mathbf{\Delta}(X))$, i.e. a morphism $\alpha : [n_0] \to [n_1]$ in $\mathbf{\Delta}$ such that $X(\alpha)(x_1) = x_0$. It is the image of the edge $\left(x_1, \alpha, \mathrm{id}_{\mathrm{id}_{[n_0]}}\right)$ in $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P)$ and by definition, $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), \theta)$ sends it to $(x_1, \alpha, \mathrm{id}_{n_0})$. It can be verified that the image of $(x_1, \alpha, \mathrm{id}_{n_0})$ under $\mathrm{B}(\Delta 1, \mathbf{\Delta}(X), P) \to X$ is $X(\beta)(x_1)$, where $\beta : [1] \to [n_1]$ is the morphism in $\mathbf{\Delta}$ defined by $\beta(0) = \alpha(n_0)$ and $\beta(1) = n_1$, and this is precisely the image of $f : x_0 \to x_1$ under π_{R}. Thus λ_X also agrees with $\mathrm{N}(\pi_{\mathrm{L}})$ on edges. ■

Lemma 1.11.32. *For any simplicial set X, there is an anodyne extension $i_X : X \to \mathrm{Ex}_{\mathrm{Q}}(X)$, and it is natural in X.*

Proof. Let $\rho^n = \lambda_{\Delta^n} : \mathrm{Sd}_{\mathrm{Q}}(\Delta^n) \to \Delta^n$, where $\lambda : \mathrm{Sd}_{\mathrm{Q}} \Rightarrow \mathrm{id}_{\mathbf{sSet}}$ is the natural weak homotopy equivalence of lemma 1.11.31. It is not hard to check that Δ^n is naturally isomorphic to $\mathrm{N}([n])$, so we can identify $\rho^n : \mathrm{Sd}_{\mathrm{Q}}(\Delta^n) \to \Delta^n$ with $\mathrm{N}(\pi_{\mathrm{R}}) : \mathrm{N}(\mathbf{\Delta}(\mathrm{N}([n]))) \to \mathrm{N}([n])$. It is then straightforward to verify that $\rho^n : \mathrm{Sd}_{\mathrm{Q}}(\Delta^n) \to \Delta^n$ is an epimorphism. Thus, noting that each $\mathrm{Sd}_{\mathrm{Q}}(\Delta^n)$ is contractible (by corollary 1.3.11), we may apply proposition 1.6.12 to obtain the required natural anodyne extension $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}_{\mathrm{Q}}$. ■

Theorem 1.11.33.

(i) *The functors* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *and* $\boldsymbol{\Delta}(-) : \mathbf{sSet} \to \mathbf{Cat}$ *constitute a homotopically mutually inverse pair of homotopical functors.*

(ii) *We have the following Quillen equivalence:*

$$\mathrm{Sd}_{\mathrm{Q}} \dashv \mathrm{Ex}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$$

(iii) *We have an adjunction of the form below,*

$$\boldsymbol{\Delta}(-) \dashv \mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-)) : \mathbf{Cat} \to \mathbf{sSet}$$

and these constitute an adjoint homotopical equivalence of homotopical categories.

Proof. (i). By definition, $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ preserves and reflects weak homotopy equivalences, and lemma 1.11.31 says there is a natural weak homotopy equivalence $\lambda : \mathrm{Sd}_{\mathrm{Q}} \Rightarrow \mathrm{id}_{\mathbf{sSet}}$, so (using the 2-out-of-3 property) $\boldsymbol{\Delta}(-) : \mathbf{sSet} \to \mathbf{Cat}$ also preserves and reflects weak homotopy equivalences. Moreover, the same lemma implies that $\pi_{\mathrm{R}} : \boldsymbol{\Delta}(\mathrm{N}(-)) \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ is a natural weak homotopy equivalence, so we indeed have a homotopically mutually inverse pair of homotopical functors.

(ii). First, we must show that the indicated adjunction is a Quillen adjunction, and by proposition 4.3.2, it suffices to show that $\mathrm{Sd}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$ is a left Quillen functor. We already know that it preserves weak homotopy equivalences, so we need only verify that it preserves monomorphisms; but it is clear that $\boldsymbol{\Delta} : \mathbf{sSet} \to \mathbf{Cat}$ and $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ both preserve monomorphisms, so the same must be true of $\mathrm{Sd}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$.

Now, consider the derived adjunction:

$$\mathbf{L}\mathrm{Sd}_{\mathrm{Q}} \dashv \mathbf{R}\mathrm{Ex}_{\mathrm{Q}} : \mathrm{Ho}\,\mathbf{sSet} \to \mathrm{Ho}\,\mathbf{sSet}$$

Since every simplicial set is cofibrant, we may take $\mathbf{L}\mathrm{Sd}_{\mathrm{Q}} = \mathrm{Ho}\,\mathrm{Sd}_{\mathrm{Q}}$; and since $\mathrm{Sd}_{\mathrm{Q}} \simeq \mathrm{id}_{\mathbf{sSet}}$, we have $\mathrm{Ho}\,\mathrm{Sd}_{\mathrm{Q}} \cong \mathrm{id}_{\mathrm{Ho}\,\mathbf{sSet}}$. Thus, we must also have $\mathbf{R}\mathrm{Ex}_{\mathrm{Q}} \cong \mathrm{id}_{\mathrm{Ho}\,\mathbf{sSet}}$, and (recalling lemma 1.5.2) we may apply theorem 4.3.13 to deduce that we have a Quillen equivalence.

(iii). Lemma 1.11.32 (and the 2-out-of-3 property) implies that the functor $\mathrm{Ex}_{\mathrm{Q}} : \mathbf{sSet} \to \mathbf{sSet}$ preserves weak homotopy equivalences, and $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$

preserves weak homotopy equivalences by definition, so the same is true of the composite $\mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-)) : \mathbf{Cat} \to \mathbf{sSet}$. Thus, we have an induced adjoint equivalence of categories:

$$\mathrm{Ho}\,\mathbf{\Delta}(-) \dashv \mathrm{Ho}\,\mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-)) : \mathrm{Ho}\,\mathbf{Cat} \to \mathrm{Ho}\,\mathbf{sSet}$$

Since **Cat** and **sSet** are both saturated homotopical categories, it follows that the unit $\mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(\mathbf{\Delta}(-)))$ and the counit $\mathbf{\Delta}\big(\mathrm{Ex}_{\mathrm{Q}}(\mathrm{N}(-))\big) \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ are natural weak homotopy equivalences, as required. ■

We can say a little bit more about the (weak) homotopy type of the fundamental category of a (reflexive) graph (i.e. a 1-skeletal simplicial set).

Definition 1.11.34. Let n be a positive integer.

- A **principal edge** of the standard simplex Δ^n is an edge corresponding to a map $[1] \to [n]$ that sends 0 to i and 1 to $i + 1$.
- The **spine** of the standard simplex Δ^n is the smallest simplicial subset of Δ^n containing its principal edges.

Remark 1.11.35. A simplex of $\mathrm{N}(\mathbb{C})$ is degenerate if and only if (at least) one of its principal edges is degenerate. However, a non-degenerate simplex of $\mathrm{N}(\mathbb{C})$ may still have degenerate edges!

Proposition 1.11.36. *Let G be a 1-skeletal simplicial set. For each positive integer k, let $X^{(k)}$ be the smallest simplicial subset of $\mathrm{N}\big(\tau_1 G\big)$ containing all k-simplices whose principal edges are in the image of the unit $\eta_G : G \to \mathrm{N}\big(\tau_1 G\big)$, i.e. the k-simplices corresponding to diagrams in $\tau_1 G$ of the form below,*

$$x_0 \longrightarrow \cdots \longrightarrow x_k$$

where the arrows are either identity morphisms or non-degenerate edges of G.

(i) *For each positive integer k, $X^{(k)} \subseteq X^{(k+1)}$, and the inclusion $X^{(k)} \hookrightarrow X^{(k+1)}$ is an anodyne extension.*

(ii) *We have $\mathrm{N}\big(\tau_1 G\big) = \bigcup_{k \geq 1} X^{(k)}$.*

(iii) *The unit $\eta_G : G \to \mathrm{N}\big(\tau_1 G\big)$ is an anodyne extension.*

Proof. (i). The definition of $X^{(k+1)}$ ensures that $X^{(k)} \subseteq X^{(k+1)}$. Let α be a non-degenerate $(k+1)$-simplex of $X^{(k+1)}$. Then α corresponds to a diagram in $\tau_1 G$ of the form below,

$$x_0 \xrightarrow{g_1} x_1 \longrightarrow \cdots \longrightarrow x_k \xrightarrow{g_{k+1}} x_{k+1}$$

where each g_i is a non-degenerate edge of G. Clearly, a face $d_i(\alpha)$ is in $X^{(k)}$ if and only if $i = 0$ or $i = k+1$. Let V^{k+1} be the smallest simplicial subset of Δ^{k+1} containing the 0-th and $(k+1)$-th faces. It is not hard to verify that the inclusion $V^{k+1} \hookrightarrow \Delta^{k+1}$ is an anodyne extension and that the evident commutative diagram

$$(*) \qquad \begin{tikzcd} V^{k+1} \arrow[r] \arrow[d, hook] & X^{(k)} \arrow[d, hook] \\ \Delta^{k+1} \arrow[r] & X^{(k+1)} \end{tikzcd}$$

is a pullback square in $\mathbf{sSet}$. Moreover, since G is 1-skeletal, $\tau_1 G$ is *freely* generated by the non-degenerate edges of G, so the canonical pushout comparison morphism $\Delta^{k+1} \cup^{V^{k+1}} X^{(k)} \to X^{(k+1)}$ is a monomorphism.

Now, let I_{k+1} be the set of all non-degenerate $(k+1)$-simplices of $X^{(k+1)}$. By amalgamating diagrams of the form $(*)$, we obtain a commutative diagram

$$(**) \qquad \begin{tikzcd} I_{k+1} \odot V^{k+1} \arrow[r] \arrow[d, hook] & X^{(k)} \arrow[d, hook] \\ I_{k+1} \odot \Delta^{k+1} \arrow[r] & X^{(k+1)} \end{tikzcd}$$

and as before, $(**)$ is a pullback square in $\mathbf{sSet}$. Noting that every degenerate $(k+1)$-simplex of $X^{(k+1)}$ is already in $X^{(k)}$, we deduce that $I_{k+1} \odot \Delta^{k+1} \to X^{(k+1)}$ and $X^{(k)} \hookrightarrow X^{(k+1)}$ are jointly epimorphic; but the canonical pushout comparison morphism is again a monomorphism, so $(**)$ is also a pushout square. In particular, $X^{(k)} \hookrightarrow X^{(k+1)}$ is an anodyne extension.

(ii). Let α be an n-simplex of $\mathrm{N}(\tau_1 G)$. By factoring the edges of α in terms of the generators, we can find a positive integer m and an m-simplex β of $X^{(m)}$ such that α occurs as a subsimplex of β. In particular, α is an n-simplex of $X^{(m)}$. Thus, $\mathrm{N}(\tau_1 G) = \bigcup_{k \geq 1} X^{(k)}$.

(iii). It is clear that the unit $\eta_G : G \to \mathrm{N}(\tau_1 G)$ is a monomorphism and that its image is precisely $X^{(1)}$. It thus suffices to verify that $X^{(1)} \hookrightarrow \mathrm{N}(\tau_1 G)$ is an anodyne extension; but the class of anodyne extensions is closed under transfinite composition, so the claim is a consequence of (i) and (ii). ■

Lemma 1.11.37. *Let G be a 1-skeletal simplicial set and let G' and G'' be simplicial subsets of G such that $G = G' \cup G''$. Then the induced commutative diagram in* **Cat**

$$\begin{tikzcd} \tau_1(G' \cap G'') \arrow[r] \arrow[d] & \tau_1 G'' \arrow[d] \\ \tau_1 G' \arrow[r] & \tau_1 G \end{tikzcd}$$

is a pushout diagram where all the arrows are monomorphisms, and the induced morphism

$$\mathrm{N}\big(\tau_1 G'\big) \cup \mathrm{N}\big(\tau_1 G''\big) \to \mathrm{N}\big(\tau_1 G\big)$$

is an anodyne extension.

Proof. It is clear that the evident commutative diagram

$$\begin{tikzcd} G' \cap G'' \arrow[r, hook] \arrow[d, hook] & G'' \arrow[d, hook] \\ G' \arrow[r, hook] & G \end{tikzcd}$$

is a pushout diagram in **sSet**, and since $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ is a left adjoint (by definition), the corresponding diagram in **Cat** is also pushout diagram. Moreover, one may directly verify that τ_1 sends monomorphisms between 1-skeletal simplicial sets in **sSet** to monomorphisms in **Cat**. It then follows (using the fact that $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is a right adjoint) that the induced morphism $\mathrm{N}\big(\tau_1 G'\big) \cup \mathrm{N}\big(\tau_1 G''\big) \to \mathrm{N}\big(\tau_1 G\big)$ is indeed a monomorphism in **sSet**; thus, by proposition 1.5.10, it suffices to show that it is a weak homotopy equivalence. But the following diagram commutes,

$$\begin{tikzcd} G' \cup G'' \arrow[r, "\eta_{G'} \cup \eta_{G''}"] \arrow[d, equal] & \mathrm{N}\big(\tau_1 G'\big) \cup \mathrm{N}\big(\tau_1 G''\big) \arrow[d] \\ G \arrow[r, "\eta_G"'] & \mathrm{N}\big(\tau_1 G\big) \end{tikzcd}$$

and by proposition 1.11.36 (plus the fact that the class of anodyne extensions is closed under pushout and composition), the horizontal arrows are weak homotopy equivalences, so the claim is a consequence of the 2-out-of-3 property. ■

— II —

Simplicial categories

2.1 Basics

Prerequisites. §§0.2, 1.1, 1.2, A.2, B.2.

In this section, we use the explicit universe convention.

Definition 2.1.1. A **simplicial category** $\mathcal{C}_\bullet$ consists of the following data:

- For each natural number n, a category $\mathcal{C}_n$.
- For each natural number n and $0 \le i \le n$, a functor $d_i^n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ and a functor $s_i^n : \mathcal{C}_n \to \mathcal{C}_{n+1}$.

These functors are moreover required to satisfy the simplicial identities. The **underlying category** of $\mathcal{C}_\bullet$ is the category $\mathcal{C}_0$.

Remark 2.1.2. In short, a simplicial category is a simplicial object in the meta-category of all categories. Thus, we may refer to the functors d_i^n and s_i^n as **face operators** and **degeneracy operators**, just as in the general case.

Definition 2.1.3. Given two simplicial categories $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$, a **simplicial functor** $F_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ consists of a functor $F_n : \mathcal{C}_n \to \mathcal{D}_n$ for each natural number n, such that the functors F_n are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n F_n = F_{n-1} d_i^n \qquad\qquad s_i^n F_n = F_{n+1} s_i^n$$

Definition 2.1.4. Given two simplicial functors $F_\bullet, F'_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$, a **simplicial natural transformation** $\varphi_\bullet : F_\bullet \Rightarrow F'_\bullet$ consists of a natural transformation

$\varphi_n : F_n \Rightarrow F'_n$ for each natural number n, such that the natural transformations φ_n are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n \varphi_n = \varphi_{n-1} d_i^n \qquad\qquad s_i^n \varphi_n = \varphi_{n+1} s_i^n$$

Definition 2.1.5. Let $\mathbf{U}$ be a universe. A **U-small** (resp. **locally U-small**) **simplicial category** is a simplicial category $\mathcal{C}_\bullet$ such that each $\mathcal{C}_n$ is $\mathbf{U}$-small (resp. locally $\mathbf{U}$-small).

Example 2.1.6. If $\mathcal{C}$ is a $\mathbf{U}$-small category, then we have a $\mathbf{U}$-small constant simplicial category $\mathcal{C}_\bullet$, where $\mathcal{C}_n = \mathcal{C}$ for all n, with the trivial face and degeneracy operators.

Definition 2.1.7. The **bisimplicial nerve** of a simplicial category $\mathcal{C}_\bullet$ is the bisimplicial set $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ defined by the following formula:

$$\left(\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)_n\right)_m = \mathrm{N}(\mathcal{C}_m)_n$$

In other words, the m-simplices of the n-th level of $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ are the composable strings of morphisms in $\mathcal{C}_m$ of length n.

Example 2.1.8. Let $\mathcal{C}$ be an ordinary category, and consider the simplicial category $\mathcal{C}_\bullet$ defined by $\mathcal{C}_n = [\mathbf{I}[n], \mathcal{C}]$, where $\mathbf{I}[n]$ denotes the groupoid obtained by freely inverting all the arrows in $[n]$. The bisimplicial nerve $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ is then (isomorphic to) the **classifying diagram** of $\mathcal{C}$, in the sense of Rezk [2001].

Proposition 2.1.9. *Let* $\mathbf{U}$ *be a universe, let* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *be the category of* $\mathbf{U}$*-small simplicial categories, and let* $\mathbf{ssSet}$ *be the category of bisimplicial sets.*

(i) $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *is a locally finitely presentable* $\mathbf{U}$*-category.*

(ii) $\mathrm{N}^{\mathrm{ss}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{ssSet}$ *is a fully faithful* $\aleph_0$*-accessible functor.*

(iii) N^{ss} *has a left adjoint.*

Proof. (i). This is an instance of proposition 0.2.44.

(ii). That $\mathrm{N}^{\mathrm{ss}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{ssSet}$ is a fully faithful $\aleph_0$-accessible functor essentially follows from the fact that $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is so: see proposition 1.2.1 and the accessible adjoint functor theorem (0.2.50).

(iii). It is also clear that N^{ss} preserves limits for $\mathbf{U}$-small diagrams, so we may apply the accessible adjoint functor theorem to construct a left adjoint for N^{ss}. ■

Definition 2.1.10. A **simplicially enriched category** $\underline{C}$ consists of the following data:

- A set of objects, $\operatorname{ob} C$.
- A simplicial set of morphisms, $\operatorname{mor} \underline{C}$.
- A pair of simplicial maps $\operatorname{dom}, \operatorname{codom} : \operatorname{mor} \underline{C} \to \operatorname{disc} \operatorname{ob} C$.
- For each element C of $\operatorname{ob} C$, a vertex id_C in $\operatorname{mor} \underline{C}$ such that $\operatorname{dom} \mathrm{id}_C = C$ and $\operatorname{codom} \mathrm{id}_C = C$.
- A simplicial map $\underline{C}^{[2]} \to \operatorname{mor} \underline{C}$, written as $(\beta, \alpha) \mapsto \beta \circ \alpha$, where $\underline{C}^{[2]}$ is the simplicial set defined by the following pullback diagram:

$$\begin{tikzcd}
\underline{C}^{[2]} \arrow[r] \arrow[d] & \operatorname{mor} \underline{C} \arrow[d, "\operatorname{codom}"] \\
\operatorname{mor} \underline{C} \arrow[r, "\operatorname{dom}"'] & \operatorname{disc} \operatorname{ob} C
\end{tikzcd}$$

These are moreover required to satisfy the following condition:

- For each natural number n, the given identities and binary operation induce a category with $\operatorname{ob} C$ for its object-set and $(\operatorname{mor} \underline{C})_n$ for its morphism-set.

As usual, we write $\underline{C}(C, C')$ for the simplicial subset of $\operatorname{mor} \underline{C}$ consisting of those simplices α such that $\operatorname{dom} \alpha = C$ and $\operatorname{codom} \alpha = C'$.

The **underlying category** of a simplicial category $\underline{C}$ is the category C obtained by taking $C(C', C) = \underline{C}(C', C)_0$, with the evident identity morphisms and induced composition. By **object** or **morphism** in $\underline{C}$, we shall always mean an object or morphism in the underlying category C.

Remark 2.1.11. It is clear from the definition that a simplicially enriched category $\underline{C}$ induces a simplicial category $C_\bullet$, but not every simplicial category arises in this fashion: simplicially enriched categories correspond to the simplicial categories $C_\bullet$ where $\operatorname{ob} C_\bullet$ is a constant simplicial set.

Definition 2.1.12. Given two simplicially enriched categories $\underline{C} \to \underline{D}$, a **simplicially enriched functor** $\underline{F} : \underline{C} \to \underline{D}$ consists of a map $\operatorname{ob} F : \operatorname{ob} C \to \operatorname{ob} D$ and a simplicial map $\operatorname{mor} \underline{F} : \operatorname{mor} \underline{C} \to \operatorname{mor} \underline{D}$ that respect the structure of simplicially enriched categories in the obvious sense.

Remark 2.1.13. There is a natural bijection between simplicially enriched functors $\underline{C} \to \underline{D}$ and simplicial functors $C_\bullet \to D_\bullet$, where $C_\bullet$ and $D_\bullet$ are the simplicial categories associated with $\underline{C}$ and $\underline{D}$.

Of course, just as in the simplicial case, a simplicially enriched functor $\underline{F} : \underline{C} \to \underline{D}$ has a underlying functor $F : C \to D$ between the underlying categories.

Definition 2.1.14. Given two simplicially enriched functors $\underline{F}, \underline{F}' : \underline{C} \to \underline{D}$, a **simplicially enriched natural transformation** $\varphi : \underline{F} \Rightarrow \underline{F}'$ consists of a morphism $\varphi_C : FC \to F'C$ in D for each object C in C, such that the following diagram commutes for all pairs (C, C'):

$$\begin{tikzcd} & \underline{D}(FC, FC') \arrow[dr, "\underline{D}(FC,\varphi_{C'})"] & \\ \underline{C}(C, C') \arrow[ur, "\operatorname{mor} \underline{F}"] \arrow[dr, "\operatorname{mor} \underline{F}'"'] & & \underline{D}(FC, F'C') \\ & \underline{D}(F'C, F'C') \arrow[ur, "\underline{D}(\varphi_C,FC')"'] & \end{tikzcd}$$

Remark 2.1.15. It is not hard to see that any simplicially enriched natural transformation has an underlying natural transformation; but unlike simplicially enriched functors, being a simplicially enriched natural transformation merely a property, rather than an extra structure.

Less obviously, the bijection between simplicially enriched functors and simplicial functors also extends to a bijection between simplicially enriched natural transformations and simplicial natural transformations. In particular, to check whether a natural transformation is simplicially enriched, it is enough to check whether it is levelwise natural.

Definition 2.1.16. Let $\mathbf{U}$ be a universe. A **$\mathbf{U}$-small simplicially enriched category** is a simplicially enriched category $\underline{C}$ such that $\operatorname{ob} C$ is a $\mathbf{U}$-set and $\operatorname{mor} \underline{C}$ is a simplicial $\mathbf{U}$-set. A **locally $\mathbf{U}$-small simplicially enriched category** is a simplicially enriched category $\underline{C}$ such that $\operatorname{ob} C$ is a $\mathbf{U}$-class and, for each pair (C', C) of elements of $\operatorname{ob} C$, the simplicial set $\underline{C}(C', C)$ is a simplicial $\mathbf{U}$-set.

Remark 2.1.17. Let $\mathbf{U}$ be a universe and let $\mathbf{sSet}$ be the category of simplicial $\mathbf{U}$-sets. Then a locally $\mathbf{U}$-small simplicially enriched category is essentially the same thing as a locally $\mathbf{U}$-small $\mathbf{sSet}$-enriched category, where we regard $\mathbf{sSet}$ as a symmetric monoidal closed category via its cartesian closed structure; and under this identification, simplicially enriched functors (resp. natural transformations) are the same thing as **sSet**-enriched functors (resp. natural transformations).

Proposition 2.1.18. *Let* $\mathbf{U}$ *be a universe and let* **sSet** *be the category of simplicial* $\mathbf{U}$*-sets. Then* **sSet** *admits a simplicial enrichment, with*

$$\underline{\mathbf{sSet}}(X, Y) = [X, Y]$$

where $[X, Y]$ *denotes the exponential object.*

Proof. This is a special case of proposition B.2.5. ■

Definition 2.1.19. A **discrete simplicially enriched category** is a simplicially enriched category $\underline{\mathcal{C}}$ such that $\operatorname{mor} \underline{\mathcal{C}}$ is a constant simplicial set.

Proposition 2.1.20. *Let* $\mathbf{U}$ *be a universe. If* $\mathcal{C}$ *is a locally* $\mathbf{U}$*-small category, then there exists a locally* $\mathbf{U}$*-small discrete simplicially enriched category* $\underline{\mathcal{C}}$ *whose underlying category is* $\mathcal{C}$ *such that, for all simplicially enriched categories* $\underline{\mathcal{D}}$*, the map sending a simplicially enriched functor* $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ *to its underlying ordinary functor* $\mathcal{C} \to \mathcal{D}$ *is a bijection.*

Proof. Obvious. ◆

Definition 2.1.21. Let $\underline{\mathcal{C}}$ be a simplicially enriched category and let C be an object in $\mathcal{C}$. The **simplicially enriched slice category** $\underline{\mathcal{C}}_{/C}$ is defined as follows:

- The objects are morphisms $f : X \to C$ in $\mathcal{C}$.
- The simplicial set of morphisms from $f : X \to C$ to $g : Y \to C$ is defined by the following pullback diagram in **sSet**,

$$\begin{tikzcd} \underline{\mathcal{C}}_{/C}(f, g) \arrow[r] \arrow[d] & \underline{\mathcal{C}}(X, Y) \arrow[d, "g_*"] \\ \Delta^0 \arrow[r] & \underline{\mathcal{C}}(X, C) \end{tikzcd}$$

  where $\Delta^0 \to \underline{\mathcal{C}}(X, C)$ is the morphism corresponding to f (considered as a vertex of $\underline{\mathcal{C}}(X, C)$).
- Composition and identities are inherited from $\underline{\mathcal{C}}$.

Remark 2.1.22. It is straightforward to check that the above indeed defines a simplicially enriched category. The morphism $\underline{\mathcal{C}}_{/C}(f, g) \to \underline{\mathcal{C}}(X, Y)$ is monic, so we may regard $\underline{\mathcal{C}}_{/C}(f, g)$ as a simplicial subset of $\underline{\mathcal{C}}(X, Y)$; however, note that it is not a "full" simplicial subset in general: for $n > 0$, the n-simplices that are in $\underline{\mathcal{C}}_{/C}(f, g)$ must become degenerate after applying $g_* : \underline{\mathcal{C}}(X, Y) \to \underline{\mathcal{C}}(X, C)$.

Proposition 2.1.23. *Let* $\mathbf{U}$ *be a universe.*

(i) *If* $\underline{\mathcal{D}}$ *and* $\underline{\mathcal{E}}$ *are* $\mathbf{U}$*-small simplicially enriched categories, then there exist a* $\mathbf{U}$*-small simplicially enriched category* $\underline{\mathcal{D}} \times \underline{\mathcal{E}}$ *and simplicially enriched functors* $p_1 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \to \underline{\mathcal{D}}$ *and* $p_2 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ *such that* (p_1, p_2) *induce a bijection between simplicially enriched functors* $\langle \underline{F}, \underline{G} \rangle : \underline{\mathcal{C}} \to \underline{\mathcal{D}} \times \underline{\mathcal{E}}$ *and pairs* $(\underline{F}, \underline{G})$ *of simplicially enriched functors, where* $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ *and* $\underline{G} : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$*, where* $\underline{\mathcal{C}}$ *varies over all simplicially enriched categories.*

(ii) *If* $\underline{\mathcal{D}}$ *is a* $\mathbf{U}$*-small simplicially enriched category and* $\underline{\mathcal{E}}$ *is a locally* $\mathbf{U}$*-small simplicially enriched category, then there exist a locally* $\mathbf{U}$*-small simplicially enriched category* $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$ *and a simplicially enriched functor* $\mathrm{ev} : [\underline{\mathcal{D}}, \underline{\mathcal{E}}] \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ *such that* ev *induces a bijection between simplicially enriched functors* $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ *and simplicially enriched functors* $\underline{\mathcal{C}} \to [\underline{\mathcal{D}}, \underline{\mathcal{E}}]$*, where* $\underline{\mathcal{C}}$ *varies over all simplicially enriched categories.*

(iii) *If* $\underline{\mathcal{D}}$ *and* $\underline{\mathcal{E}}$ *are both* $\mathbf{U}$*-small simplicially enriched categories, then* $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$ *is also* $\mathbf{U}$*-small.*

Proof. This is a special case of theorem B.3.7. ∎

Remark 2.1.24. Let $\mathcal{C}$ be an ordinary category and let $\underline{\mathcal{D}}$ be a simplicially enriched category. Then all functors $\mathcal{C} \to \mathcal{D}$ are automatically simplicially enriched (by proposition 2.1.20), and as in remark A.6.5, we have a isomorphism

$$[\mathcal{C}, \underline{\mathcal{D}}](\underline{F}, \underline{F}') \cong \int_{C:\mathcal{C}} \underline{\mathcal{D}}(FC, F'C)$$

and this is natural in both $\underline{F}$ and $\underline{F}'$. More generally, see corollary B.3.22.

Proposition 2.1.25. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{SCat}$ *be the category of* $\mathbf{U}$*-small simplicially enriched categories, and let* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *be the category of* $\mathbf{U}$*-small simplicial categories.*

(i) $\mathbf{SCat}$*, regarded as a full subcategory of* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$*, is closed under limits and colimits for all* $\mathbf{U}$*-small diagrams.*

(ii) **SCat** *is a cartesian closed category.*

(iii) *The inclusion* **SCat** $\hookrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *has a left adjoint, and* **SCat** *is a locally finitely presentable* $\mathbf{U}$*-category.*

Proof. (i). The functor $[\mathbf{\Delta}^{\mathrm{op}}, \mathrm{ob}] : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{sSet}$ has a left adjoint and a right adjoint, so it follows that a limit or colimit for diagrams of simplicially enriched categories, computed as a simplicial category, will have object-space a discrete simplicial set and thus be isomorphic to a simplicially enriched category.

(ii). This is implied by proposition 2.1.23.

(iii). It is not hard to directly construct a left adjoint for the inclusion **SCat** $\hookrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$, and once this is done, we may apply the classification theorem for locally presentable categories (0.2.40) to deduce (from proposition 2.1.9) that **SCat** is also locally finitely presentable. Alternatively, one may instead first show that **SCat** is locally finitely presentable and then use the accessible adjoint functor theorem (0.2.50) to construct a left adjoint for the inclusion. ◊

Proposition 2.1.26. *Let* $\mathcal{C}$ *be a category and let* $\mathbb{S}^{[\bullet]} = \left(S^{[\bullet]}, \varepsilon^{[\bullet]}, \delta^{[\bullet]}\right)$ *be a cosimplicial object in the category of comonads on* $\mathcal{C}$*. If* $\mathbb{S}^{[0]} = (\mathrm{id}, \mathrm{id}, \mathrm{id})$*, then* $\mathcal{C}$ *is the underlying ordinary category of a simplicially enriched category* $\underline{\mathcal{C}}$ *where the hom-spaces are given by the formula below,*

$$\underline{\mathcal{C}}(A, B) \cong \mathcal{C}\left(S^{[\bullet]}A, B\right)$$

with composition in level n induced by the comultiplication $\delta^{[n]} : S^{[n]} \Rightarrow S^{[n]}S^{[n]}$.

Proof. Let $\mathcal{C}_n$ be the Kleisli category associated with the comonad $\mathbb{S}^{[n]}$. Clearly, these fit together to form a simplicial category $\mathcal{C}_\bullet$ such that $\mathrm{ob}\, \mathcal{C}_\bullet$ is a constant simplicial set; so by remark 2.1.11, we have the required simplicially enriched category $\underline{\mathcal{C}}$. ■

Lemma 2.1.27 (Weak Yoneda lemma)**.** *Let* $\mathbf{U}$ *be a universe, let* $\underline{\mathbf{sSet}}$ *be the simplicially enriched category of simplicial* $\mathbf{U}$*-sets, and let* $\underline{\mathcal{C}}$ *be a locally* $\mathbf{U}$*-small simplicially enriched category. For each object* A *in* $\mathcal{C}$ *and each simplicially enriched functor* $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathbf{sSet}}$*, the map* $\varphi \mapsto \varphi_A\left(\mathrm{id}_A\right)$ *is a bijection between the set of* $\mathcal{V}$*-enriched natural transformations* $\varphi : \underline{\mathcal{C}}(A, -) \Rightarrow \underline{F}$ *and the set of vertices of* FA*.*

Proof. This is a special case of lemma B.2.14. ■

2.2 Simplicially enriched limits and colimits

Prerequisites. §§2.1, A.5, A.6, B.2, B.3, B.4.

In this section, we use the explicit universe convention.

Definition 2.2.1. Let $\underline{C}$ be a simplicially enriched category, let X be a simplicial set, and let C be an object in C.

- A **tensor product** of X and C in $\underline{C}$ is pair $(X \odot C, \lambda)$ where $X \odot C$ is an object in C and λ is a morphism $X \to \underline{C}(C, X \odot C)$ such that the simplicially enriched natural transformation

$$\underline{C}(X \odot C, -) \Rightarrow [X, \underline{C}(C, -)]$$

induced by the corresponding vertex of $[X, \underline{C}(C, X \odot C)]$ is a simplicially enriched natural isomorphism. We may also refer to $(X \odot C, \lambda)$ as a **simplicial copower** of A by X.

- A **cotensor product** of X and C in C is a pair $(X \pitchfork C, \lambda)$ where $X \pitchfork C$ is an object in $\underline{C}$ and λ is a morphism $X \to \underline{C}(X \pitchfork C, C)$ such that the simplicially enriched natural transformation

$$\underline{C}(-, X \pitchfork C) \Rightarrow [X, \underline{C}(-, C)]$$

induced by the corresponding vertex of $[X, \underline{C}(X \pitchfork C, C)]$ is a simplicially enriched natural isomorphism. We may also refer to $(X \pitchfork C, \lambda)$ as a **simplicial power** of A by X.

Remark 2.2.2. If $\underline{C}$ is a locally **U**-small simplicially enriched category, then the above definition coincides with the definition of tensor/cotensor product in a **sSet**-enriched category, where **sSet** is the category of simplicial **U**-sets.

Definition 2.2.3. Let $\underline{C}$ be a locally **U**-small simplicially enriched category and let $F : \mathbb{D} \to C$ be a diagram in C.

- A **conical colimit** for F in $\underline{C}$ is an object A and a cocone $\lambda : F \Rightarrow \Delta A$ such that, for all objects B in C, the hom-functor $\underline{C}(-, B) : C^{\mathrm{op}} \to \mathbf{sSet}$ sends λ to a limiting cone in **sSet**.

- A **conical limit** for F in $\underline{C}$ is an object B and a cone $\lambda : \Delta B \Rightarrow F$ such that, for all objects A in C, the hom-functor $\underline{C}(A, -) : C \to \mathbf{sSet}$ sends λ to a limiting cone in **sSet**.

Remark 2.2.4. Every conical colimit (resp. limit) for F in $\underline{C}$ is a colimit (resp. limit) for F in the underlying category C, but the converse is not true in general.

Remark 2.2.5. When $\mathbb{D}$ is an ordinary category $\mathbb{D}$, ordinary cocones (resp. cones) on diagrams $F : \mathbb{D} \to C$ are automatically simplicially enriched, and thus conical colimits (resp. limits) for F are the same thing as $\Delta 1$-weighted colimits (resp. limits) for $\underline{F}$, where $\Delta 1$ denotes the constant functor with value 1 in **sSet**.

Proposition 2.2.6. *Let $\underline{C}$ be a locally* **U***-small simplicially enriched category and let $F : \mathbb{D} \to C$ be a diagram in C. If $\underline{C}$ has cotensor products with the standard simplices, then the following are equivalent for any cocone $\lambda : F \Rightarrow \Delta A$:*

(i) *λ is a conical colimit for F in the simplicially enriched category $\underline{C}$.*

(ii) *λ is a colimit for F in the underlying category C.*

Dually, if $\underline{C}$ has tensor products with the standard simplices, then the following are equivalent for any cone $\lambda : \Delta B \Rightarrow F$:

(i′) *λ is a conical limit for F in the simplicially enriched category $\underline{C}$.*

(ii′) *λ is a limit for F in the underlying category C.*

Proof. (i) $\Rightarrow$ (ii). Immediate.

(ii) $\Rightarrow$ (i). It suffices to show that, for each natural number n, the canonical comparison map

$$\underline{C}\Big(\varinjlim_{\mathbb{D}} F, T\Big)_n \to \varprojlim_{\mathbb{D}} \underline{C}(F, T)_n$$

is a bijection; but by the Yoneda lemma,

$$\underline{C}(S, T)_n \cong \mathbf{sSet}(\Delta^n, \underline{C}(S, T))$$

and the definition of $\Delta^n \pitchfork (-)$ implies there is a natural bijection of the form below,

$$\mathbf{sSet}(\Delta^n, \underline{C}(S, T)) \cong C(S, \Delta^n \pitchfork T)$$

therefore the functor $\underline{C}(-, T)_n : C^{\mathrm{op}} \to \mathbf{Set}$ is representable. ■

Definition 2.2.7. Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \subseteq \mathbf{U}^+$.

- A **U-cocomplete simplicially enriched category** is a locally $\mathbf{U}^+$-small simplicially enriched category $\underline{C}$ such that, for all $\mathbf{U}$-small simplicially enriched diagrams $F : \underline{\mathbb{D}} \to \underline{C}$ and all $\mathbf{U}$-small weights $\underline{W} : \underline{\mathbb{D}}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$, $\underline{C}$ has a $\underline{W}$-weighted colimit for $\underline{F}$.

- A **U-complete simplicially enriched category** is a locally $\mathbf{U}^+$-small simplicially enriched category $\underline{C}$ such that, for all $\mathbf{U}$-small simplicially enriched diagrams $\underline{F} : \underline{\mathbb{D}} \to \underline{C}$ and all $\mathbf{U}$-small weights $\underline{W} : \underline{\mathbb{D}} \to \underline{\mathbf{sSet}}$, $\underline{C}$ has a W-weighted limit for F.

Proposition 2.2.8. *Let $\underline{C}$ be a locally $\mathbf{U}$-small simplicially enriched category.*

- *$\underline{C}$ is $\mathbf{U}$-cocomplete if and only if $\underline{C}$ is simplicially tensored and has conical colimits for all $\mathbf{U}$-small diagrams.*

- *$\underline{C}$ is $\mathbf{U}$-complete if and only if $\underline{C}$ is simplicially cotensored and conical limits for all $\mathbf{U}$-small diagrams.*

- *$\underline{C}$ is both $\mathbf{U}$-cocomplete and $\mathbf{U}$-complete if and only if $\underline{C}$ is both simplicially tensored and cotensored and the* underlying *category C is $\mathbf{U}$-cocomplete and $\mathbf{U}$-complete.*

Proof. See [???]. □

2.3 Simplicial and cosimplicial objects

Prerequisites. §§1.1, 2.1, 2.2, A.6.

¶ 2.3.1. Recall that a **simplicial object** in a category is a diagram of shape $\mathbf{\Delta}^{\mathrm{op}}$, and dually, a **cosimplicial object** is a diagram of shape $\mathbf{\Delta}$. Let us write $\mathbf{s}\mathcal{M}$ for the category of simplicial objects in $\mathcal{M}$, and $\mathbf{c}\mathcal{M}$ for the category of cosimplicial objects in $\mathcal{M}$.

Proposition 2.3.2. *Let $\mathcal{M}$ be a locally small category. Let $\underline{\mathrm{Hom}} : (\mathbf{s}\mathcal{M})^{\mathrm{op}} \times \mathbf{s}\mathcal{M} \to \mathbf{sSet}$ be the functor defined by*

$$\underline{\mathrm{Hom}}(A, B) \cong \operatorname{Tot} \mathcal{M}\big(A_\bullet, B_\bullet\big)$$

where we regard $\mathcal{M}\big(A_\bullet, B_\bullet\big)$ as a cosimplicial simplicial set. Then $\mathbf{s}\mathcal{M}$ is a locally small simplicially enriched category with hom-spaces given by $\underline{\mathrm{Hom}}$.

Proof. By lemma 1.6.22, we have the following end formula:

$$\underline{\mathrm{Hom}}(A, B)_n \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m, \mathcal{M}\left(A_m, B_m\right)\right)$$

Concretely, an element of f of $\underline{\mathrm{Hom}}(A, B)_n$ is a $\coprod_{[m]:\mathbf{\Delta}} \mathbf{\Delta}([m], [n])$-indexed family of morphisms $f_\varphi : A_m \to B_m$ in $\mathcal{M}$, such that for any two morphisms $\varphi : [m] \to [n]$, $\psi : [l] \to [m]$ in $\mathbf{\Delta}$, the diagram in $\mathcal{M}$ shown below commutes:

$$\begin{tikzcd} A_m \arrow[r, "f_\varphi"] \arrow[d, "\psi^*"'] & B_m \arrow[d, "\psi^*"] \\ A_l \arrow[r, "f_{\varphi\circ\psi}"'] & B_l \end{tikzcd}$$

Decomposing an element g of $\underline{\mathrm{Hom}}(B, C)_n$ the same way, we obtain the following commutative diagram in $\mathcal{M}$,

$$\begin{tikzcd} A_m \arrow[r, "f_\varphi"] \arrow[d, "\psi^*"'] & B_m \arrow[r, "g_\varphi"] \arrow[d, "\psi^*"'] & C_m \arrow[d, "\psi^*"] \\ A_l \arrow[r, "f_{\varphi\circ\psi}"'] & B_l \arrow[r, "g_{\varphi\circ\psi}"'] & C_l \end{tikzcd}$$

and thus we have an element of $\underline{\mathrm{Hom}}(A, C)_n$. This is certainly natural in n, and this is clearly the required associative composition with identity.

It remains to be shown that there is a natural bijection of the form below:

$$\mathbf{sSet}\left(\Delta^0, \underline{\mathrm{Hom}}(A, B)\right) \cong \mathbf{s}\mathcal{M}(A, B)$$

Given a morphism $f_\bullet : A_\bullet \to B_\bullet$, we define an element $f[n]$ of $\underline{\mathrm{Hom}}(A, B)_n$ for each object $[n]$ in $\mathbf{\Delta}^{\mathrm{op}}$ as follows: given $\varphi : [m] \to [n]$, we set $(f[n])_\varphi = f_m$ and naturality of f_m makes the diagram in $\mathcal{M}$ shown below commute for every morphism $\psi : [l] \to [m]$ in $\mathbf{\Delta}$:

$$\begin{tikzcd} A_m \arrow[r, "(f[n])_\varphi"] \arrow[d, "\psi^*"'] & B_m \arrow[d, "\psi^*"] \\ A_l \arrow[r, "(f[n])_{\varphi\circ\psi}"'] & B_l \end{tikzcd}$$

Thus, we have a morphism $\Delta^0 \to \underline{\mathrm{Hom}}(A, B)$. Conversely, given a family of elements $f[-]$ such that $\theta^*(f[n]) = f\left[n'\right]$ for all $\theta : \left[n'\right] \to [n]$, we discover

$$(f[n])_\varphi = (f[n])_{\varphi\circ\mathrm{id}_{[m]}} = (\varphi^*(f[n]))_{\mathrm{id}_{[m]}} = (f[m])_{\mathrm{id}_{[m]}}$$

for all morphisms $\varphi : [m] \to [n]$ in $\mathbf{\Delta}$, so we get a morphism $f_\bullet : A_\bullet \to B_\bullet$ by setting $f_m = (f[m])_{\mathrm{id}_{[m]}}$. This establishes the required natural bijection. ■

Definition 2.3.3. A **constant simplicial object** in a category $\mathcal{M}$ is a simplicial object in $\mathcal{M}$ whose face and degeneracy operators are isomorphisms.

Proposition 2.3.4. *Let $\mathcal{M}$ be a locally small category. The following are equivalent for a simplicial object $B_\bullet$ in $\mathcal{M}$:*

(i) *$B_\bullet$ is a constant simplicial object in $\mathcal{M}$.*

(ii) *For all objects A in $\mathcal{M}$, the simplicial set $\mathcal{M}(A, B_\bullet)$ is discrete.*

(iii) *For all simplicial objects $A_\bullet$ in $\mathcal{M}$, the simplicial set $\underline{\mathbf{s}\mathcal{M}}(A, B)$ is discrete.*

Proof. (i) ⇔ (ii). Use the fact that the Yoneda embedding $\mathcal{M} \to [\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful.

(ii) ⇒ (iii). Apply lemma 1.6.24.

(iii) ⇒ (ii). Let A be any object in $\mathcal{M}$. If we regard A as a constant simplicial object in the obvious way, then lemma 1.6.23 says there is a natural isomorphism

$$\underline{\mathbf{s}\mathcal{M}}(A, B) \cong \mathcal{M}(A, B_\bullet)$$

so $\mathcal{M}(A, B_\bullet)$ is indeed discrete. ■

Proposition 2.3.5. *Let $\mathcal{M}$ be a locally small category and let X be a finite simplicial set.*

- *If $\mathcal{M}$ has finite colimits, then for any cosimplicial object $A^\bullet$ in $\mathcal{M}$, there exists an object $X \star A$ in $\mathcal{M}$ equipped with bijections*

  $$\mathcal{M}(X \star A, B) \cong \mathbf{sSet}(X, \mathcal{M}(A^\bullet, B))$$

  that are natural in B.

- *If $\mathcal{M}$ has finite limits, then for any simplicial object $B_\bullet$ in $\mathcal{M}$, there exists an object $\{X, B\}$ in $\mathcal{M}$ equipped with bijections*

  $$\mathcal{M}(A, \{X, B\}) \cong \mathbf{sSet}(X, \mathcal{M}(A, B_\bullet))$$

  that are natural in A.

Proof. The two claims are formally dual; we will prove the first version.

Applying the Yoneda lemma, we see that $\Delta^n \star A$ must be (isomorphic to) A^n. It is not hard to see that, if $X : \mathcal{J} \to \mathbf{sSet}$ is a diagram such that $Xj \star A$ exists for all j in $\mathcal{J}$, then $\left(\varinjlim_{j:\mathcal{J}} Xj\right) \star A$ must be (isomorphic to) $\varinjlim_{j:\mathcal{J}} (Xj \star A)$ when the latter exists; thus, the class of simplicial sets X for which $X \star A$ exists must be closed under finite colimits (because $\mathcal{M}$ has colimits for finite diagrams). We may then use proposition 1.1.18 to deduce that $X \star A$ exists if X is a finite simplicial set. ■

Remark 2.3.6. The same is true for a general simplicial set X when $\mathcal{M}$ has limits and colimits for all small diagrams: see theorem A.6.14.

Proposition 2.3.7. *Let $\mathcal{M}$ be a locally small category and let X be a finite simplicial set.*

- *If $\mathcal{M}$ has finite colimits, then for any cosimplicial object $A^\bullet$ in $\mathcal{M}$, the tensor product $(X \odot A)^\bullet$ exists in $\underline{\mathbf{c}\mathcal{M}}$.*
- *If $\mathcal{M}$ has finite limits, then for any simplicial object $B_\bullet$ in $\mathcal{M}$, the cotensor product $(X \pitchfork B)_\bullet$ exists in $\underline{\mathbf{s}\mathcal{M}}$.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that $\Delta^n \times X$ is a finite simplicial set for all $n \geq 0$ when X is a finite simplicial set, so the objects $(\Delta^n \times X) \star A$ exist in $\mathcal{M}$ (by proposition 2.3.5). We then define $(X \odot A)^\bullet$ by taking $(X \odot A)^n = (\Delta^n \times X) \star A$. Let $B^\bullet$ be any cosimplicial object in $\mathcal{M}$. Using the calculus of ends, we have the following natural bijections:

$$\begin{aligned}
\underline{\mathbf{c}\mathcal{M}}(X \odot A, B)_n &\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathcal{M}\big((\Delta^m \times X) \star A, B^m\big)\big) \\
&\qquad \text{by lemma 1.6.22} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m, \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^m_l \times X_l, \mathcal{M}\big(A^l, B^m\big)\big)\right) \\
&\qquad \text{by definition and remark A.6.5} \\
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(\Delta^m_l \times X_l, \mathcal{M}\big(A^l, B^m\big)\big)\big) \\
&\qquad \text{by proposition A.6.11}
\end{aligned}$$

$$\begin{aligned}
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_l, \mathbf{Set}\left(\Delta^n_m \times X_l, \mathcal{M}\left(A^l, B^m\right)\right)\right) \\
&\qquad \text{by exponential adjunction (twice)} \\
&\cong \int_{[l]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_l, \mathbf{Set}\left(\Delta^n_m \times X_l, \mathcal{M}\left(A^l, B^m\right)\right)\right) \\
&\qquad \text{by the interchange law (theorem A.6.17)} \\
&\cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_l \times X_l, \mathcal{M}\left(A^l, B^l\right)\right) \\
&\qquad \text{by the Yoneda lemma for ends (proposition A.6.18)}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
[X, \underline{\mathbf{c}\mathcal{M}}(A, B)]_n &\cong \mathbf{sSet}(\Delta^n \times X, \underline{\mathbf{c}\mathcal{M}}(A, B)) \\
&\qquad \text{by remark A.2.23} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m \times X_m, \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_l, \mathcal{M}\left(A^l, B^l\right)\right)\right) \\
&\qquad \text{by lemma 1.6.22 and remark A.6.5} \\
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m \times X_m, \mathbf{Set}\left(\Delta^m_l, \mathcal{M}\left(A^l, B^l\right)\right)\right) \\
&\qquad \text{by proposition A.6.11} \\
&\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_l, \mathbf{Set}\left(\Delta^n_m \times X_m, \mathcal{M}\left(A^l, B^l\right)\right)\right) \\
&\qquad \text{by exponential adjunction (twice)} \\
&\cong \int_{[l]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^m_l, \mathbf{Set}\left(\Delta^n_m \times X_m, \mathcal{M}\left(A^l, B^l\right)\right)\right) \\
&\qquad \text{by the interchange law (theorem A.6.17)} \\
&\cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_l \times X_l, \mathcal{M}\left(A^l, B^l\right)\right) \\
&\qquad \text{by the Yoneda lemma for ends (proposition A.6.18)}
\end{aligned}$$

Thus, we have isomorphisms

$$\underline{\mathbf{c}\mathcal{M}}(X \odot A, B) \cong [X, \underline{\mathbf{c}\mathcal{M}}(A, B)]$$

that are natural in $B^\bullet$. Moreover,

$$[\underline{\mathbf{c}\mathcal{M}}(X \odot A, B), \underline{\mathbf{c}\mathcal{M}}(X \odot A, C)] \cong [\underline{\mathbf{c}\mathcal{M}}(X \odot A, B) \times X, \underline{\mathbf{c}\mathcal{M}}(A, C)]$$

and so a similar calculation may be used to verify *simplicial* naturality in $B^\bullet$. ■

Proposition 2.3.8. *Let $\mathcal{M}$ be a locally small category and let X be a finite simplicial set (resp. any simplicial set).*

- *If $\mathcal{M}$ has finite copowers (resp. small copowers), then for any simplicial object $A_\bullet$ in $\mathcal{M}$, the simplicial object $(X \odot A)_\bullet$ defined by*

$$(X \odot A)_n = X_n \odot A_n$$

is (the object part of) a tensor product of X and $A_\bullet$ in $\underline{\mathbf{s}\mathcal{M}}$.

- *If $\mathcal{M}$ has finite powers (resp. small powers), then for any cosimplicial object $B^\bullet$ in $\mathcal{M}$, the cosimplicial object $(X \pitchfork B)^\bullet$ defined by*

$$(X \pitchfork B)^n = X^n \pitchfork B^n$$

is (the object part of) a tensor product of X and $B^\bullet$ in $\underline{\mathbf{c}\mathcal{M}}$.

Proof. The two claims are formally dual; we will prove the first version.

Let $B_\bullet$ be any simplicial object in $\mathcal{M}$. By the calculus of ends, we have the following natural bijections:

$$\begin{aligned}
\underline{\mathbf{s}\mathcal{M}}(X \odot A, B)_n &\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathcal{M}\big(X_m \odot A_m, B_m\big)\big) \\
&\qquad \text{by lemma 1.6.22} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m, \mathbf{Set}\big(X_m, \mathcal{M}\big(A_m, B_m\big)\big)\big) \\
&\qquad \text{by definition} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^n_m \times X_m, \mathcal{M}\big(A_m, B_m\big)\big) \\
&\qquad \text{by exponential adjunction}
\end{aligned}$$

On the other hand:

$$\begin{aligned}
[X, \underline{\mathbf{s}\mathcal{M}}(A, B)]_n &\cong \mathbf{sSet}(\Delta^n \times X, \underline{\mathbf{s}\mathcal{M}}(A, B)) \\
&\qquad \text{by remark A.2.23} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m \times X_m, \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\big(\Delta^l_m, \mathcal{M}\big(A_l, B_l\big)\big)\right) \\
&\qquad \text{by lemma 1.6.22 and remark A.6.5}
\end{aligned}$$

$$\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m \times X_m, \mathbf{Set}\left(\Delta^l_m, \mathcal{M}\left(A_l, B_l\right)\right)\right)$$

by proposition A.6.11

$$\cong \int_{[m]:\mathbf{\Delta}} \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^l_m, \mathbf{Set}\left(\Delta^n_m \times X_m, \mathcal{M}\left(A_l, B_l\right)\right)\right)$$

by exponential adjunction (twice)

$$\cong \int_{[l]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^l_m, \mathbf{Set}\left(\Delta^n_m \times X_m, \mathcal{M}\left(A_l, B_l\right)\right)\right)$$

by the interchange law (theorem A.6.17)

$$\cong \int_{[l]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_l \times X_l, \mathcal{M}\left(A_l, B_l\right)\right)$$

by the Yoneda lemma for ends (proposition A.6.18)

Thus, we have isomorphisms

$$\underline{\mathbf{s}\mathcal{M}}(X \odot A, B) \cong [X, \underline{\mathbf{s}\mathcal{M}}(A, B)]$$

that are natural in $B_\bullet$. Moreover,

$$[\underline{\mathbf{s}\mathcal{M}}(X \odot A, B), \underline{\mathbf{s}\mathcal{M}}(X \odot A, C)] \cong [\underline{\mathbf{s}\mathcal{M}}(X \odot A, B) \times X, \underline{\mathbf{s}\mathcal{M}}(A, C)]$$

so a similar calculation may be used to verify *simplicial* naturality in $B_\bullet$. ■

Definition 2.3.9. Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.

- A **realisation** of a simplicial object $A_\bullet$ in $\underline{\mathcal{M}}$ is an object $\left|A_\bullet\right|$ in $\underline{\mathcal{M}}$ with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}\left(\left|A_\bullet\right|, -\right) \cong [\mathbf{\Delta}, \underline{\mathbf{sSet}}]\left(\Delta^\bullet, \underline{\mathcal{M}}\left(A_\bullet, -\right)\right)$$

- A **totalisation** of a cosimplicial object $B^\bullet$ in $\underline{\mathcal{M}}$ is an object $|B^\bullet|$ in $\underline{\mathcal{M}}$ with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}(-, \operatorname{Tot} B^\bullet) \cong [\mathbf{\Delta}, \underline{\mathbf{sSet}}](\Delta^\bullet, \underline{\mathcal{M}}(-, B^\bullet))$$

Remark 2.3.10. In other words, $\left|A_\bullet\right|$ is the simplicially enriched weighted colimit $\Delta^\bullet \star_{\mathbf{\Delta}^{\mathrm{op}}} A_\bullet$, and $\operatorname{Tot} B^\bullet$ is the simplicially enriched weighted limit $\{\Delta^\bullet, B^\bullet\}^{\mathbf{\Delta}^{\mathrm{op}}}$.

Remark 2.3.11. By remark 2.1.24 and theorems B.3.18 and B.3.19, the above definitions agree with the ones given in §1.6. In particular, we have simplicially enriched natural isomorphisms

$$\underline{\mathcal{M}}(|A_\bullet|, -) \cong \underline{\mathrm{Tot}}\,\underline{\mathcal{M}}(A_\bullet, -)$$
$$\underline{\mathcal{M}}(-, \mathrm{Tot}\, B^\bullet) \cong \underline{\mathrm{Tot}}\,\underline{\mathcal{M}}(-, B_\bullet)$$

for a simplicial object $A_\bullet$ and a cosimplicial object $B^\bullet$ in $\mathcal{M}$, respectively.

Proposition 2.3.12. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *Let X be a simplicial set and let $A_\bullet$ be a simplicial object in $\underline{\mathcal{M}}$. If $\mathcal{M}$ is cocomplete and $X \boxdot A_\bullet$ is the simplicial object in $\mathcal{M}$ defined below,*

  $$(X \boxdot A_\bullet)_n = X_n \odot A_n$$

  then there is an isomorphism

  $$|X \boxdot A_\bullet| \cong X \odot |A_\bullet|$$

  and it is natural in both X and $A_\bullet$.

- *Let X be a simplicial set and let $B^\bullet$ be a cosimplicial object in $\underline{\mathcal{M}}$. If $\mathcal{M}$ is complete and $X \boxed{\pitchfork} G^\bullet$ is the cosimplicial object in $\mathcal{M}$ defined below,*

  $$(X \boxed{\pitchfork} B^\bullet)^n = X_n \pitchfork B^n$$

  then there is an isomorphism

  $$\mathrm{Tot}(X \boxed{\pitchfork} B^\bullet) \cong X \pitchfork \mathrm{Tot}\, B^\bullet$$

  and it is natural in both X and $B^\bullet$.

Proof. The two claims are formally dual; we will prove the first version.

Using the calculus of ends, we have the following natural bijections:

$$\mathcal{M}(X \odot |A_\bullet|, B) \cong \mathbf{sSet}(X, \underline{\mathcal{M}}(|A_\bullet|, B))$$

by definition

$$\cong \mathbf{sSet}\left(X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, \underline{\mathcal{M}}(A_n, B)]\right)$$

by theorem A.6.14

$$
\begin{aligned}
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}\left(X, \left[\Delta^n, \underline{\mathcal{M}}(A_n, B)\right]\right) \\
&\qquad \text{by proposition A.6.11} \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}\left(X \times \Delta^n, \underline{\mathcal{M}}(A_n, B)\right) \\
&\qquad \text{by exponential adjunction} \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m \times \Delta^n_m, \underline{\mathcal{M}}(A_n, B)_m\right) \\
&\qquad \text{by remark A.6.5} \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \mathbf{Set}\left(\Delta^n_m, \underline{\mathcal{M}}(A_n, B)_m\right)\right) \\
&\qquad \text{by exponential adjunction} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\left(\Delta^n_m, \underline{\mathcal{M}}(A_n, B)_m\right)\right) \\
&\qquad \text{by the interchange law (theorem A.6.17)} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \underline{\mathcal{M}}(A_m, B)_m\right) \\
&\qquad \text{by the Yoneda lemma for ends (proposition A.6.18)} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathcal{M}\left(X_m \odot A_m, B\right)_m \\
&\qquad \text{by definition} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{sSet}\left(\Delta^m, \underline{\mathcal{M}}(X_m \odot A_m, B)\right) \\
&\qquad \text{by the ordinary Yoneda lemma} \\
&\cong \mathcal{M}\left(\left|X \boxdot A_\bullet\right|, B\right)
\end{aligned}
$$

Applying the Yoneda lemma once more, we deduce that $\left|X \boxdot A_\bullet\right|$ is naturally isomorphic to $X \odot \left|A_\bullet\right|$. ■

Corollary 2.3.13. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *Let $f_\bullet, f'_\bullet : A_\bullet \to B_\bullet$ be a parallel pair of morphisms in $\mathbf{s}\mathcal{M}$. If $\underline{\mathcal{M}}$ is cocomplete as a simplicially enriched category and there exists a morph-*

ism $H : \Delta^1 \boxdot A_\bullet \to B_\bullet$ *making the following diagram commute,*

$$\begin{tikzcd} \Delta^0 \boxdot A_\bullet \arrow[d, "\delta^1 \boxdot \mathrm{id}_{A_\bullet}"'] & A_\bullet \arrow[l, "\cong"'] \arrow[d, "f_\bullet"] \\ \Delta^1 \boxdot A_\bullet \arrow[r, "H"] & B_\bullet \\ \Delta^0 \boxdot A_\bullet \arrow[u, "\delta^0 \boxdot \mathrm{id}_{A_\bullet}"] & A_\bullet \arrow[l, "\cong"] \arrow[u, "f'_\bullet"'] \end{tikzcd}$$

then there is an edge $\alpha : |f| \Rightarrow |f'|$ *in* $\underline{\mathcal{M}}\big(|A_\bullet|, |B_\bullet|\big)$.

- *Let* $f^\bullet, f'^\bullet : A^\bullet \to B^\bullet$ *be a parallel pair of morphisms in* $\mathbf{c}\mathcal{M}$. *If* $\underline{\mathcal{M}}$ *is complete as a simplicially enriched category and there exists a morphism* $H : A^\bullet \to \Delta^1 \boxed{\pitchfork} B^\bullet$ *making the following diagram commute,*

$$\begin{tikzcd} B^\bullet \arrow[r, "\cong"] & \Delta^0 \boxed{\pitchfork} B^\bullet \\ A^\bullet \arrow[u, "f^\bullet"] \arrow[d, "f'^\bullet"'] \arrow[r, "H"] & \Delta^1 \boxed{\pitchfork} B^\bullet \arrow[u, "\delta^1 \boxed{\pitchfork} \mathrm{id}_{B^\bullet}"'] \arrow[d, "\delta^0 \boxed{\pitchfork} \mathrm{id}_{B^\bullet}"] \\ B^\bullet \arrow[r, "\cong"'] & \Delta^0 \boxed{\pitchfork} B^\bullet \end{tikzcd}$$

then there is an edge $\alpha : \mathrm{Tot}\, f \Rightarrow \mathrm{Tot}\, f'$ *in* $\underline{\mathcal{M}}(\mathrm{Tot}\, A^\bullet, \mathrm{Tot}\, B^\bullet)$.

Proof. The Yoneda lemma implies there are natural bijections

$$\mathcal{M}\big(\Delta^1 \odot A, B\big) \cong \underline{\mathcal{M}}(A, B)_1 \cong \mathcal{M}\big(A, \Delta^1 \pitchfork B\big)$$

so the required edge is obtained by applying realisation (resp. totalisation) to the displayed diagrams. ■

Proposition 2.3.14. *Let* $\underline{\mathcal{M}}$ *be a locally small simplicially enriched category.*

- *If* $\underline{\mathcal{M}}$ *is cocomplete and cotensored, then we have the following adjunction of ordinary categories:*

$$|-| \dashv \Delta^\bullet \pitchfork (-) : \mathcal{M} \to \mathbf{s}\mathcal{M}$$

- *If* $\underline{\mathcal{M}}$ *is complete and tensored, then we have the following adjunction of ordinary categories:*

$$\Delta^\bullet \odot (-) \dashv \mathrm{Tot} : \mathbf{c}\mathcal{M} \to \mathcal{M}$$

TODO: Replace this with the enriched version.

Proof. By definition, we have the following natural bijections:

$$\mathcal{M}\big(|A_\bullet|, B\big) \cong [\mathbf{\Delta}, \mathbf{sSet}]\big(\Delta^\bullet, \underline{\mathcal{M}}\big(A_\bullet, B\big)\big) \cong [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]\big(A_\bullet, \Delta^\bullet \pitchfork B\big)$$

$$\mathcal{M}(A, \operatorname{Tot} B^\bullet) \cong [\mathbf{\Delta}, \mathbf{sSet}](\Delta^\bullet, \underline{\mathcal{M}}(A, B^\bullet)) \cong [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}](\Delta^\bullet \odot A, B^\bullet) \qquad \blacksquare$$

Definition 2.3.15. Let (W, ε, δ) be a comonad on a category $\mathcal{M}$. The **standard resolution** of an object A in $\mathcal{M}$ (with respect to this comonad) is the simplicial object $\mathbf{S}(A)_\bullet$ defined by the following formulae,

$$\begin{aligned} \mathbf{S}(A)_n &= W^{n+1} A \\ d^n_i &= W^{n-i} \varepsilon_{W^i A} \\ s^n_i &= W^{n-i} \delta_{W^i A} \end{aligned}$$

together with the **standard augmentation**, which is defined to be the unique morphism $\big(\tilde{\varepsilon}_A\big)_\bullet : \mathbf{S}(A)_\bullet \to A$ in $\mathbf{s}\mathcal{M}$ given in degree 0 by the counit $\varepsilon_A : WA \to A$.

Remark 2.3.16. One does have to verify that the above really does define a simplicial object and a morphism thereof, but this is straightforward.

Definition 2.3.17. Let $A_\bullet$ be a simplicial object in a category $\mathcal{M}$.

- A **forward contracting homotopy** for $A_\bullet$ consists of an object A_{-1} in $\mathcal{M}$ and morphisms $r : A_0 \to A_{-1}$, $s : A_{-1} \to A_0$, and $h^n : A_n \to A_{n+1}$ in $\mathcal{M}$ satisfying these identities:

$$\begin{aligned} r \circ d^1_1 &= r \circ d^1_0 \\ r \circ s &= \mathrm{id} \\ d^1_0 \circ h^0 &= s \circ r \\ d^1_1 \circ h^0 &= \mathrm{id} \\ d^{n+1}_i \circ h^n &= h^{n-1} \circ d^n_i && \text{if } 0 \le i \le n \\ d^{n+1}_{n+1} \circ h^n &= \mathrm{id} \\ h^{n+1} \circ s^n_i &= s^{n+1}_i \circ h^n && \text{if } 0 \le i \le n \\ h^{n+1} \circ h^n &= s^{n+1}_{n+1} \circ h^n \end{aligned}$$

- A **backward contracting homotopy** for $A_\bullet$ consists of an object A_{-1} in $\mathcal{M}$ and morphisms $r : A_0 \to A_{-1}$, $s : A_{-1} \to A_0$, and $h^n : A_n \to A_{n+1}$ in $\mathcal{M}$ satisfying these identities:

$$r \circ d^1_1 = r \circ d^1_0$$

$$\begin{aligned}
r \circ s &= \mathrm{id} \\
d_0^1 \circ h^0 &= \mathrm{id} \\
d_1^1 \circ h^0 &= s \circ r \\
d_0^{n+1} \circ h^n &= \mathrm{id} \\
d_{i+1}^{n+1} \circ h^n &= h^{n-1} \circ d_i^n && \text{if } 0 \leq i \leq n \\
h^{n+1} \circ h^n &= s_0^{n+1} \circ h^n \\
h^{n+1} \circ s_i^n &= s_{i+1}^{n+1} \circ h^n && \text{if } 0 \leq i \leq n
\end{aligned}$$

Remark 2.3.18. The above definition agrees with definition 1.3.19 in the case $\mathcal{M} = \mathbf{Set}$.

Proposition 2.3.19. *Let $A_\bullet$ be a simplicial object in a locally small category $\mathcal{M}$.*

- *Given a forward contracting homotopy for $A_\bullet$, say $r : A_0 \to A_{-1}$, $s : A_{-1} \to A_0$, and $h^n : A_n \to A_{n+1}$, there are unique morphisms $\tilde{r}_\bullet : A_\bullet \to A_{-1}$ and $\tilde{s}_\bullet : A_{-1} \to A_\bullet$ in $\mathbf{s}\mathcal{M}$ defined in degree 0 by r and s respectively, and we have $\tilde{r}_\bullet \circ \tilde{s}_\bullet = \mathrm{id}_{A_{-1}}$ and an edge $\mathrm{id}_{A_\bullet} \Rightarrow \tilde{s}_\bullet \circ \tilde{r}_\bullet$ in $\underline{\mathbf{s}\mathcal{M}}(A, A)$.*
- *Given a forward contracting homotopy for $A_\bullet$, say $r : A_0 \to A_{-1}$, $s : A_{-1} \to A_0$, and $h^n : A_n \to A_{n+1}$, there are unique morphisms $\tilde{r}_\bullet : A_\bullet \to A_{-1}$ and $\tilde{s}_\bullet : A_{-1} \to A_\bullet$ in $\mathbf{s}\mathcal{M}$ defined in degree 0 by r and s respectively, and we have $\tilde{r}_\bullet \circ \tilde{s}_\bullet = \mathrm{id}_{A_{-1}}$ and an edge $\mathrm{id}_{A_\bullet} \Rightarrow \tilde{s}_\bullet \circ \tilde{r}_\bullet$ in $\underline{\mathbf{s}\mathcal{M}}(A, A)$.*

Proof. The two claims are formally dual; we will prove the first version.

By adjointness, there is a unique morphism $\tilde{s}_\bullet : A_{-1} \to A_\bullet$ in $\mathbf{s}\mathcal{M}$ such that $\tilde{s}_0 = s$. It is clear that there is at most one morphism $\tilde{r}_\bullet : A_\bullet \to A_{-1}$ in $\mathbf{s}\mathcal{M}$. such that $\tilde{r}_0 = r$, and since $r \circ d_1^1 = r \circ d_0^1$, the simplicial identities imply there is indeed such a morphism in $\mathbf{s}\mathcal{M}$. Similarly, to verify the equation $\tilde{r}_\bullet \circ \tilde{s}_\bullet = \mathrm{id}_{A_{-1}}$, it suffices to verify the claim in degree 0; but this is just the hypothesis that $r \circ s = \mathrm{id}_{A_{-1}}$.

Now, let T be any object in $\mathcal{M}$, and consider the simplicial set $\mathcal{M}(T, A_\bullet)$. Then we have a natural forward contracting homotopy for each $\mathcal{M}(T, A_\bullet)$; so by proposition 1.3.20, for each morphism $f_\bullet : T \to A_\bullet$ in $\mathbf{s}\mathcal{M}$, there is a natural edge $f_\bullet \Rightarrow \tilde{s}_\bullet \circ \tilde{r}_\bullet \circ f_\bullet$ in $\mathcal{M}(T, A_\bullet)$. But $\underline{\mathrm{Tot}} : [\mathbf{\Delta}, \underline{\mathbf{sSet}}] \to \underline{\mathbf{sSet}}$ is a simplicially enriched functor (by proposition B.3.16), so this implies there is an edge $\mathrm{id}_{A_\bullet} \Rightarrow \tilde{s}_\bullet \circ \tilde{r}_\bullet$ in $\underline{\mathbf{s}\mathcal{M}}(A, A)$, as required. ∎

Proposition 2.3.20. *Let $\mathcal{M}$ and $\mathcal{N}$ be locally small categories, let*

$$F \dashv U : \mathcal{M} \to \mathcal{N}$$

be an adjunction with unit $\eta : \mathrm{id}_{\mathcal{N}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{M}}$, and let A be an object in $\mathcal{M}$. Taking $U\mathbf{S}(A)_{-1} = UA$, $r = U\varepsilon_A$, $s = \eta_{UA}$, and $h^n = \eta_{U(FU)^{n+1}A}$, we have a forward contracting homotopy for $U\mathbf{S}(A)_\bullet$.

Proof. This is a straightforward exercise in using the triangle identities. ◆

2.4 Simplicial model categories

Prerequisites. §§1.5, 2.1, 4.1, 4.3, 4.7, 4.8.

Definition 2.4.1. Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category. A **simplicial model structure** on $\underline{\mathcal{M}}$ is a model structure on the underlying model category $\mathcal{M}$ that satisfies the following axiom:[1]

SM7. If $i : Z \to W$ is a cofibration in $\mathcal{M}$ and $p : X \to Y$ is a fibration in $\mathcal{M}$, and the square in the diagram below is a pullback square in **sSet**,

$$\begin{tikzcd}
\underline{\mathcal{M}}(W,X) \arrow[dr, dashed, "i^* \square p_*"] \arrow[drr, bend left, "\underline{\mathcal{M}}(W,p)"] \arrow[ddr, bend right, "\underline{\mathcal{M}}(i,X)"'] & & \\
& \underline{\mathcal{M}}(Z,X) \times_{\underline{\mathcal{M}}(Z,Y)} \underline{\mathcal{M}}(W,Y) \arrow[r] \arrow[d] & \underline{\mathcal{M}}(W,Y) \arrow[d, "\underline{\mathcal{M}}(i,Y)"] \\
& \underline{\mathcal{M}}(Z,X) \arrow[r, "\underline{\mathcal{M}}(Z,p)"'] & \underline{\mathcal{M}}(Z,Y)
\end{tikzcd}$$

then the unique morphism $i^* \square p_*$ making the diagram commute is a Kan fibration; moreover, if either $i : Z \to W$ or $p : X \to Y$ is a weak equivalence, then $i^* \square p_*$ is a trivial Kan fibration.

A **simplicial model category** is a locally small simplicially enriched category $\underline{\mathcal{M}}$ that is equipped with a simplicial model structure and satisfies the additional axioms below:

SM0. For each finite simplicial set K and each object X in $\underline{\mathcal{M}}$, the tensor product $K \odot X$ and the cotensor product $K \pitchfork X$ exist in $\underline{\mathcal{M}}$.

[1] This presentation is due to Quillen [1967].

CM1. $\mathcal{M}$ has finite limits and finite colimits.

A **simplicial derivable category** is a locally small simplicially enriched category $\underline{\mathcal{M}}$ that is equipped with a simplicial model structure and satisfies the additional axioms below:

- If W is a cofibrant object in $\mathcal{M}$, then the functor $\underline{\mathcal{M}}(W, -) : \mathcal{M} \to \mathbf{sSet}$ preserves fibrant objects, fibrations, and trivial fibrations; and if X is a fibrant object in $\mathcal{M}$, then the functor $\underline{\mathcal{M}}(-) : \mathcal{M}^{\mathrm{op}} \to \mathbf{sSet}$ sends cofibrant objects (resp. cofibrations, trivial cofibrations) in $\mathcal{M}$ to Kan complexes (resp. Kan fibrations, trivial Kan fibrations) in $\mathbf{sSet}$.
- The underlying ordinary category $\mathcal{M}$ equipped with the given model structure is a derivable category.

Remark 2.4.2. Proposition 2.2.6 implies that limits and colimits in a simplicial model category are automatically conical (i.e. limits and colimits in the simplicially enriched sense).

Remark 2.4.3. Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category equipped whose underlying ordinary category is equipped with a model structure. Then $\underline{\mathcal{M}}$ is a simplicial model category if and only if $\underline{\mathcal{M}}^{\mathrm{op}}$ is a simplicial model category.

Proposition 2.4.4. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category whose underlying ordinary category is equipped with a model structure. If $\underline{\mathcal{M}}$ satisfies axioms SM0 and CM1, then the following are equivalent:*

(i) *Axiom SM7 is satisfied.*

(ii) *For all fibrations (resp. trivial fibrations) $p : X \to Y$ in $\mathcal{M}$, if $i : Z \to W$ is a boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ and the square in the diagram below is a pullback square in $\mathcal{M}$,*

$$\begin{tikzcd}
W \pitchfork X \arrow[dr, dashed, "i \square p"] \arrow[drr, bend left, "\mathrm{id}_W \pitchfork p"] \arrow[ddr, bend right, "i \pitchfork \mathrm{id}_X"'] & & \\
& (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) \arrow[r] \arrow[d] & W \pitchfork Y \arrow[d, "i \pitchfork \mathrm{id}_Y"] \\
& Z \pitchfork X \arrow[r, "\mathrm{id}_Z \pitchfork p"'] & Z \pitchfork Y
\end{tikzcd}$$

then the unique morphism $i \boxslash p$ making the diagram commute is a fibration (resp. trivial fibration); and for all fibrations $p : X \to Y$ in $\mathcal{M}$, if $i : Z \to W$ is a horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, then the morphism $i \boxslash p$ defined as above is a trivial fibration.

(iii) *For all cofibrations (resp. trivial cofibrations) $j : X \to Y$ in $\mathcal{M}$, if $i : Z \to W$ is a boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ and the square in the diagram below is a pushout square in $\mathcal{M}$,*

$$\begin{tikzcd}
Z \odot X \arrow[r, "\mathrm{id}_Z \odot j"] \arrow[d, "i \odot \mathrm{id}_X"'] & Z \odot Y \arrow[d] \arrow[ddr, bend left, "i \odot \mathrm{id}_Y"] & \\
W \odot X \arrow[r] \arrow[drr, bend right, "\mathrm{id}_W \odot j"'] & (W \odot X) \cup^{Z \odot X} (Z \odot Y) \arrow[dr, dashed, "i \boxbslash j"] & \\
& & W \odot Y
\end{tikzcd}$$

then the unique morphism $i \boxbslash j$ making the diagram commute is a cofibration (resp. trivial cofibration); and for all cofibrations $j : X \to Y$ in $\mathcal{M}$, if $i : Z \to W$ is a horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, then the morphism $i \boxbslash j$ defined as above is a trivial cofibration.

Proof. This is (essentially) a special case of proposition 5.5.1. ■

Corollary 2.4.5. *Let $\underline{\mathcal{M}}$ be simplicial derivable category that has tensors for finite (resp. all) simplicial sets and colimits for finite (resp. small) diagrams.*

(i) *If $i : Z \to W$ is a monomorphism of finite (resp. arbitrary) simplicial sets and Y is a a cofibrant object in $\mathcal{M}$, then the morphism $i \odot \mathrm{id}_Y : Z \odot Y \to W \odot Y$ is a cofibration.*

(ii) *If $i : Z \to W$ is an anodyne extension of finite (resp. arbitrary) simplicial sets and Y is a a cofibrant object in $\mathcal{M}$, then the morphism $i \odot \mathrm{id}_Y : Z \odot Y \to W \odot Y$ is a trivial cofibration.*

(iii) *If W is any finite (resp. arbitrary) simplicial set and Y is a cofibrant object in $\mathcal{M}$, then $W \odot Y$ is also a cofibrant object in $\mathcal{M}$.*

Proof. (i) and (ii). Proposition 2.4.4 implies the claims in the special cases where $i : Z \to W$ is a boundary inclusion or horn inclusion, and by proposition 1.4.12 (resp. corollary 0.5.13) and proposition A.3.17, this is enough to deduce the claim for the general case.

(iii). Take $Z = \varnothing$. ∎

Corollary 2.4.6. *Let $\underline{\mathcal{M}}$ be simplicial derivable category that has cotensors for finite (resp. all) simplicial sets and limits for finite (resp. small) diagrams.*

(i) *If $i : Z \to W$ is a monomorphism of finite (resp. arbitrary) simplicial sets and X is a a fibrant object in $\mathcal{M}$, then the morphism $i \pitchfork \mathrm{id}_X : W \pitchfork X \to Z \pitchfork X$ is a fibration.*

(ii) *If $i : Z \to W$ is an anodyne extension of finite (resp. arbitrary) simplicial sets and X is a fibrant object in $\mathcal{M}$, then the morphism $i \pitchfork \mathrm{id}_X : W \pitchfork X \to Z \pitchfork X$ is a trivial cofibration.*

(iii) *If W is any finite (resp. arbitrary) simplicial set and X is a fibrant object in $\mathcal{M}$, then $W \pitchfork Y$ is also a fibrant object in $\mathcal{M}$.*

Proof. These claims are formally dual to the ones in corollary 2.4.5. ∎

Proposition 2.4.7. *Let $\underline{\mathcal{M}}$ be a locally small simplically enriched category with an initial object 0 and a terminal object 1 (in the simplicially enriched sense) and suppose $\underline{\mathcal{M}}$ is equipped with a simplicial model structure.*

- *If A is a cofibrant object in $\mathcal{M}$, then the functor $\underline{\mathcal{M}}(A, -) : \mathcal{M} \to \mathbf{sSet}$ preserves weighted limits, fibrant objects, fibrations, and trivial fibrations.*
- *If B is a fibrant object in $\mathcal{M}$, then the functor $\underline{\mathcal{M}}(-, B) : \mathcal{M}^{\mathrm{op}} \to \mathbf{sSet}$ preserves weighted limits, fibrant objects, fibrations, and trivial fibrations.*

In particular, every simplicial model category is a simplicial derivable category.

Proof. The two claims are formally dual; we will prove the first version.

Essentially by definition, the functor $\underline{\mathcal{M}}(A, -) : \mathcal{M} \to \mathbf{sSet}$ preserves any weighted limits that exist in $\underline{\mathcal{M}}$. Lemma 4.1.16 says the unique morphism $0 \to A$ is a cofibration if and only if A is a cofibrant object in $\mathcal{M}$, so we may then apply axiom SM7 to deduce that $\underline{\mathcal{M}}(A, -)$ preserves fibrant objects, fibrations, and trivial fibrations.

To conclude, we need only apply remark 2.4.2 and proposition 4.1.17. ∎

Lemma 2.4.8. *Let $\underline{\mathcal{M}}$ be a simplicial derivable category, let $\mathcal{M}_{\mathrm{c}}$ be the full subcategory of cofibrant objects in $\mathcal{M}$, and let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects in $\mathcal{M}$.*

- *If A is a cofibrant object in $\mathcal{M}$, then $\underline{\mathcal{M}}(A, -) : \mathcal{M}_{\mathrm{f}} \to$ **sSet** is a homotopical functor.*

- *If B is a fibrant object in $\mathcal{M}$, then $\underline{\mathcal{M}}(-, B) : {\mathcal{M}_{\mathrm{c}}}^{\mathrm{op}} \to$ **sSet** is a homotopical functor.*

*In particular, $\underline{\mathcal{M}}(-, -) : {\mathcal{M}_{\mathrm{c}}}^{\mathrm{op}} \times \mathcal{M}_{\mathrm{f}} \to$ **sSet** is a homotopical functor.*

Proof. By definition, $\underline{\mathcal{M}}(A, -)$ sends trivial fibrations in $\mathcal{M}$ to trivial Kan fibrations when A is cofibrant, and $\underline{\mathcal{M}}(-, B)$ sends trivial cofibrations in $\mathcal{M}$ to trivial Kan fibrations when B is fibrant, we may apply lemma 4.1.33. ■

Theorem 2.4.9. *Let $\underline{\mathcal{M}}$ be a simplicial derivable category, let $(\mathcal{M}_{\mathrm{c}}, Q, p)$ be a left Quillen deformation retract of $\mathcal{M}$, and let $(\mathcal{M}_{\mathrm{f}}, R, i)$ be a right Quillen deformation retract of $\mathcal{M}$.*

(i) *$({\mathcal{M}_{\mathrm{c}}}^{\mathrm{op}} \times \mathcal{M}_{\mathrm{f}}, Q \times R, (p, i))$ is a right deformation retract for the functor $\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to$ **sSet**.*

(ii) *$\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to$ **sSet** has a total right derived functor; furthermore, if $(\mathcal{M}_{\mathrm{c}}, Q, p)$ and $(\mathcal{M}_{\mathrm{f}}, R, i)$ are functorial deformation retracts, then $\underline{\mathcal{M}}(-, -)$ also has a homotopical right approximation.*

Proof. (i). This is lemma 2.4.8.

(ii). Apply theorems 3.3.17 and 3.4.11. ■

Definition 2.4.10. Let $\underline{\mathcal{M}}$ be a simplicial derivable category. A **derived hom-space functor** for $\underline{\mathcal{M}}$ is a total right derived functor for the functor $\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to$ **sSet**. We write $\mathbf{R}\mathrm{Hom}_{\mathcal{M}} : (\mathrm{Ho}\,\mathcal{M})^{\mathrm{op}} \times \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathbf{sSet}$ for (the functor part of) a derived hom-space functor for $\underline{\mathcal{M}}$.

Proposition 2.4.11. *Let $\underline{\mathcal{M}}$ be a simplicial model category.*

- *If A is a cofibrant object in $\mathcal{M}$, then the cosimplicial object $\Delta^{\bullet} \odot A$ is (the object part of) a left frame on A.*

- *If B is a fibrant object in $\mathcal{M}$, then the simplicial object $\Delta^{\bullet} \pitchfork B$ is (the object part of) a right frame on B.*

Proof. See Remark 5.2.10 in [Hovey, 1999]. □

Corollary 2.4.12. *Let $\underline{\mathcal{M}}$ be a simplicial model category. If A is a cofibrant object in $\mathcal{M}$ and B is a fibrant object in $\mathcal{M}$, then:*

- *The hom-space $\underline{\mathcal{M}}(A, B)$ is (the object part of) a left homotopy function complex from A to B.*
- *The hom-space $\underline{\mathcal{M}}(A, B)$ is (the object part of) a right homotopy function complex from A to B.*

Proof. The two claims are formally dual; we will prove the first version.

By proposition 2.4.11, the cosimplicial object $\tilde{A}^\bullet = \Delta^\bullet \odot A$ is (the object part of) a left frame on A; but there is a natural isomorphism between the left hom-complex $\mathcal{Hom}_{\mathcal{M}}\big(\tilde{A}, B\big)$ and the hom-space $\underline{\mathcal{M}}(A, B)$, and B is fibrant by hypothesis, so we are done. ■

Remark 2.4.13. In particular, the derived hom-spaces of the simplicial model category $\underline{\mathcal{M}}$ agree with the derived hom-spaces of the underlying model category $\mathcal{M}$.

Proposition 2.4.14. *Let $\underline{\mathcal{M}}$ be a simplicial model category.*

- *If A is a cofibrant object in $\mathcal{M}$, then $\left(\Delta^1 \odot A, \delta^1 \odot \mathrm{id}_A, \delta^0 \odot \mathrm{id}_A, \sigma^0 \odot \mathrm{id}_A\right)$ is a cylinder object for $\Delta^0 \odot A$ (and hence, isomorphic to a cylinder object for A).*
- *If B is a fibrant object in $\mathcal{M}$, then $\left(\Delta^1 \pitchfork B, \delta^1 \pitchfork \mathrm{id}_B, \delta^0 \pitchfork \mathrm{id}_B, \sigma^0 \pitchfork \mathrm{id}_B\right)$ is a path object for $\Delta^0 \pitchfork B$ (and hence, isomorphic to a path object for B).*

Proof. Apply propositions 2.4.11 and 4.7.21; but see also Lemma 3.5 in [GJ], or Lemma 9.5.4 in [Hirschhorn, 2003]. ■

Corollary 2.4.15. *Let $\underline{\mathcal{M}}$ be a simplicial model category. If A is a cofibrant object in $\mathcal{M}$ and B is a fibrant object in $\mathcal{M}$, then the canonical map*

$$\operatorname{Ho}\mathcal{M}(A, B) \to \pi_0 \underline{\mathcal{M}}(A, B)$$

is a natural bijection.

Proof. Proposition 2.4.14 says that $\left(\Delta^1 \odot A, \delta^1 \odot \mathrm{id}_A, \delta^0 \odot \mathrm{id}_A, \sigma^0 \odot \mathrm{id}_A\right)$ is a cylinder object for $\Delta^0 \odot A$, so if B is fibrant, we may apply lemma 4.2.14 and theorem 4.4.1 to deduce that the connected components of $\underline{\mathcal{M}}(A, B)$ are in natural bijection with the homotopy classes of morphisms $A \to B$. ■

Lemma 2.4.16. *Let $f_0, f_1 : A \to B$ be a parallel pair of morphisms in a simplicial model category $\underline{\mathcal{M}}$.*

- *If A is a cofibrant object in $\mathcal{M}$ and f_0 and f_1 are in the same connected component of $\underline{\mathcal{M}}(A, B)$, then f_0 is a weak equivalence in $\mathcal{M}$ if and only if f_1 is a weak equivalence in $\mathcal{M}$.*

- *If B is a fibrant object in $\mathcal{M}$ and f_0 and f_1 are in the same connected component of $\underline{\mathcal{M}}(A, B)$, then f_0 is a weak equivalence in $\mathcal{M}$ if and only if f_1 is a weak equivalence in $\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

By induction, we may assume that there is an edge $\alpha : f_0 \Rightarrow f_1$ in $\underline{\mathcal{M}}(A, B)$. Let $h : \Delta^1 \odot A \to B$ be the corresponding morphism in $\mathcal{M}$. We then have the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd}
\Delta^0 \odot A \arrow[d, "\delta^1 \odot \mathrm{id}_A"'] & A \arrow[l, "\cong"'] \arrow[d, "f_0"] \\
\Delta^1 \boxdot A \arrow[r, "h"] & B \\
\Delta^0 \boxdot A \arrow[u, "\delta^0 \odot \mathrm{id}_A"] & A \arrow[l, "\cong"] \arrow[u, "f_1"']
\end{tikzcd}$$

Since A is cofibrant, corollary 2.4.5 implies that the morphisms $\delta^0 \odot \mathrm{id}_A, \delta^1 \odot \mathrm{id}_A : \Delta^0 \odot A \to \Delta^1 \odot A$ are weak equivalences in $\mathcal{M}$. Thus, by axiom CM2, f_0 is a weak equivalence in $\mathcal{M}$ if and only if f_1 is a weak equivalence in $\mathcal{M}$. ■

Proposition 2.4.17. *Let $\underline{\mathcal{M}}$ be a simplicial model category and let $\mathbb{A}$ be a small category.*

- *If the projective model structure on $[\mathbb{A}, \mathcal{M}]$ exists, then $[\mathbb{A}, \underline{\mathcal{M}}]$ (with the projective model structure) is a simplicial model category.*

- *If the injective model structure on $[\mathbb{A}, \mathcal{M}]$ exists, then $[\mathbb{A}, \underline{\mathcal{M}}]$ (with the injective model structure) is a simplicial model category.*

Proof. The two claims are formally dual; we will prove the first version.

It is straightforward to check that $[\mathbb{A}, \underline{\mathcal{M}}]$ is indeed a locally small simplicially enriched category with finite weighted limits and colimits (which may be computed componentwise). It remains to be shown that the projective model structure on $[\mathbb{A}, \underline{\mathcal{M}}]$ satisfies axiom SM7. But fibrations, weak equivalences, and weighted limits in $[\mathbb{A}, \underline{\mathcal{M}}]$ are defined componentwise, so proposition 2.4.4 implies that the property is indeed inherited from $\underline{\mathcal{M}}$. ■

The following lemma is useful in the construction of simplicial model structures.

Lemma 2.4.18. *Let $\underline{\mathcal{M}}$ be a simplicially enriched category, let $\underline{\mathcal{N}}$ be a simplicial model category, and let $\underline{U} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$ be a simplicially enriched functor. Given a commutative diagram in $\mathcal{M}$ of the form below,*

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & \hat{A} \arrow[d, "\hat{f}"] \\ B \arrow[r, "i_B"'] & \hat{B} \end{tikzcd}$$

the morphism $Uf : UA \to UB$ is a weak equivalence in $\mathcal{N}$ if the following conditions are satisfied:

- *The cotensor products $\Delta^1 \pitchfork \hat{B}$ and $\partial\Delta^1 \pitchfork \hat{B}$ exist in $\underline{\mathcal{M}}$ and are preserved by $\underline{U} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$.*
- *$U i_A : UA \to U\hat{A}$ and $U i_B : UB \to U\hat{B}$ are weak equivalences in $\mathcal{N}$.*
- *There is a morphism $g : B \to \hat{A}$ in $\mathcal{M}$ such that $g \circ f = i_A$, and $U\hat{B}$ is fibrant in $\mathcal{N}$.*
- *$f : A \to B$ has the left lifting property with respect to all morphisms $p : C \to D$ in $\mathcal{M}$ such that $Up : UC \to UD$ is a fibration in $\mathcal{N}$.*

Proof. The following is a generalisation of the proof of Theorem 1 in [Quillen, 1967, Ch. II, §4].

Let $p : \Delta^1 \pitchfork \hat{B} \to \partial\Delta^1 \pitchfork \hat{B}$ be the morphism in $\mathcal{M}$ induced by the boundary inclusion $\partial\Delta^1 \hookrightarrow \Delta^1$, let $a : A \to \Delta^1 \pitchfork \hat{B}$ be the composite

$$A \xrightarrow{f} B \xrightarrow{i_B} \hat{B} \xrightarrow{\cong} \Delta^0 \pitchfork \hat{B} \to \Delta^1 \pitchfork \hat{B}$$

where $\Delta^0 \pitchfork \hat{B} \to \Delta^1 \pitchfork \hat{B}$ is induced by the unique morphism $\Delta^1 \to \Delta^0$, and let $b : B \to \partial\Delta^1 \pitchfork \hat{B}$ be the unique morphism in $\mathcal{M}$ making the diagram below commute:

$$\begin{tikzcd}
\hat{B} \arrow[r, "\cong"] & \Delta^0 \pitchfork \hat{B} \\
B \arrow[u, "i_B"] \arrow[r, dashed, "b"] \arrow[d, "\hat{f} \circ g"'] & \partial\Delta^1 \pitchfork \hat{B} \arrow[u, "\delta^1 \pitchfork \mathrm{id}_{\hat{B}}"'] \arrow[d, "\delta^0 \pitchfork \mathrm{id}_{\hat{B}}"] \\
\hat{B} \arrow[r, "\cong"'] & \Delta^0 \pitchfork \hat{B}
\end{tikzcd}$$

We then have the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & \Delta^1 \pitchfork \hat{B} \arrow[d, "p"] \\
B \arrow[r, "b"'] & \partial\Delta^1 \pitchfork \hat{B}
\end{tikzcd}$$

By corollary 2.4.6 (and the hypothesis on $\underline{U} : \underline{\mathcal{M}} \to \underline{\mathcal{N}}$), $U p : U\left(\Delta^1 \pitchfork \hat{B}\right) \to U\left(\partial\Delta^1 \pitchfork \hat{B}\right)$ is a fibration in $\mathcal{N}$, so (by the hypothesis on $f : A \to B$) there is a morphism $h : B \to \Delta^1 \pitchfork \hat{B}$ such that $h \circ f = a$ and $p \circ h = b$.

In particular, there is an edge $U i_B \Rightarrow U\left(\hat{f} \circ g\right)$ in $\underline{\mathcal{N}}\left(U A, U \hat{B}\right)$, so by lemma 2.4.16, $U\left(\hat{f} \circ g\right)$ is a weak equivalence. But the diagram below commutes,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "i_A"'] & B \arrow[d, "g"] \arrow[dr, "\hat{f} \circ g"] & \\
& \hat{A} \arrow[r, "\hat{f}"'] & \hat{B}
\end{tikzcd}$$

so using theorem 4.4.1 and the 2-out-of-6 property of weak equivalences in $\mathcal{N}$, we may deduce that $U f : U A \to U B$ is a weak equivalence in $\mathcal{N}$. ■

2.5 Homotopical aspects

Prerequisites. §§1.2, 1.3, 1.4, 1.5, 1.7, 2.1, A.4.

Definition 2.5.1. Let $\mathcal{V}$ be a category with finite products and let $F : \mathbf{sSet} \to \mathcal{V}$ be a functor that preserves finite products. The **F-localisation** of a locally small simplicially enriched category $\underline{\mathcal{C}}$ is the following $\mathcal{V}$-enriched category $F[\underline{\mathcal{C}}]$:

- The objects in $F[\underline{\mathcal{C}}]$ are the objects in $\underline{\mathcal{C}}$.

- For each pair (X, Y) of objects in $\underline{C}$, the hom-object $F[\underline{C}](X, Y)$ is the object $F(\underline{C}(X, Y))$.

- Identities and composition in $F[\underline{C}]$ are inherited from $\underline{C}$ via F.

Remark 2.5.2. It is clear that F-localisation is 2-functorial and moreover preserves finite products of simplicially enriched categories; unlike localisation of relative categories, F-localisation may or may not have a universal property. Nonetheless, there is always a localising functor $C \to F[\underline{C}]$ between the underlying categories.

Definition 2.5.3. Let $\underline{C}$ be a locally small simplicially enriched category. A parallel pair of morphisms $g_0, g_1 : A \to B$ in $\underline{C}$ are **F-homotopic** if their images under the localising functor $C \to F[\underline{C}]$ are equal, in which case we write $g_0 \overset{F}{\sim} g_1$.

Example 2.5.4. The notion of intrinsic homotopy in **sSet** is obtained as the special case where F is the connected components functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$.[2]

Definition 2.5.5. Let $\underline{C}$ be a locally small simplicially enriched category. A **weak F-homotopy equivalence** in $\underline{C}$ is a morphism in $\underline{C}$ whose image in $F[\underline{C}]$ is an isomorphism. An **F-homotopy equivalence** in $\underline{C}$ is a pair (f, g), where $f : A \to B$ and $g : B \to A$ are morphisms in $\underline{C}$ such that $g \circ f \overset{F}{\sim} \mathrm{id}_A$ and $f \circ g \overset{F}{\sim} \mathrm{id}_B$. Two morphisms $f : A \to B$ and $g : B \to A$ are **mutual F-homotopy inverses** when (f, g) constitute an F-homotopy equivalence.

Remark 2.5.6. By lemma A.4.14, the class of weak F-homotopy equivalences in $\underline{C}$ automatically has the 2-out-of-6 property in C.

Lemma 2.5.7. *Let $\underline{C}$ be a locally small simplicially enriched category, let $\mathcal{V}$ be a cartesian closed category, and let $F : \mathbf{sSet} \to \mathcal{V}$ be a functor that preserves finite products.*

- *If $\underline{C}$ is tensored over* **sSet**, *$f : X \to Y$ is a weak F-homotopy equivalence in $\underline{\mathbf{sSet}}$, and $g : A \to B$ is a weak F-homotopy equivalence in $\underline{C}$, then the morphism $f \odot g : X \odot A \to Y \odot B$ is a weak F-homotopy equivalence in $\underline{C}$.*

[2] Recall proposition 1.2.4 and remark 1.3.7.

- *If $\underline{C}$ is cotensored over* **sSet**, *$f : X \to Y$ is a weak F-homotopy equivalence in $\underline{\mathbf{sSet}}$, and $g : A \to B$ is a weak F-homotopy equivalence in $\underline{C}$, then the morphism $f \pitchfork g : Y \pitchfork A \to X \pitchfork B$ is a weak F-homotopy equivalence in $\underline{C}$.*

Proof. Since $\odot$ (resp. $\pitchfork$) is a simplicially enriched functor $\underline{\mathbf{sSet}} \times \underline{C} \to \underline{C}$ (resp. $\underline{\mathbf{sSet}}^{\mathrm{op}} \times \underline{C} \to \underline{C}$), it induces a $\mathcal{V}$-enriched functor $F[\underline{\mathbf{sSet}}] \times F[\underline{C}] \to F[\underline{C}]$ (resp. $F[\underline{\mathbf{sSet}}]^{\mathrm{op}} \times F[\underline{C}] \to F[\underline{C}]$) and so *a fortiori* must preserve weak F-homotopy equivalences. ■

Definition 2.5.8. A **simplicial homotopy** $\alpha : f_0 \Rightarrow f_1$ in a simplicially enriched category $\underline{C}$ is an edge α in $\operatorname{mor} \underline{C}$ such that $d^0(\alpha) = f_1$ and $d^1(\alpha) = f_0$. For each morphism $f : X \to Y$ in C, we define $\mathrm{id}_f : f \Rightarrow f$ to be the simplicial homotopy $s_0(f)$.

Remark 2.5.9. Because $\operatorname{ob} \underline{C}$ is a discrete set, we must have $\operatorname{dom} f_0 = \operatorname{dom} f_1$ and $\operatorname{codom} f_0 = \operatorname{codom} f_1$.

Definition 2.5.10. A parallel pair $g_0, g_1 : A \to B$ of morphisms in a simplicially enriched category $\underline{C}$ are **simplicially homotopic** if they are in the same connected component of $\underline{C}(A, B)$, in which case we write $g_0 \sim g_1$.

Lemma 2.5.11. *Let $\underline{C}$ be a locally small simplicially enriched category, and let $\alpha : f_0 \Rightarrow f_1$ be an intrinsic homotopy of morphisms in* **sSet**.

- *If $\underline{C}$ is tensored over* **sSet**, *then for any morphism $g : A \to B$ in $\underline{C}$, $\alpha \odot \mathrm{id}_g : f_0 \odot g \Rightarrow f_1 \odot g$ is a simplicial homotopy of morphisms in $\underline{C}$.*

- *If $\underline{C}$ is cotensored over* **sSet**, *then for any morphism $g : A \to B$ in $\underline{C}$, $\alpha \pitchfork \mathrm{id}_g : f_0 \pitchfork g \Rightarrow f_1 \pitchfork g$ is a simplicial homotopy of morphisms in $\underline{C}$.*

Proof. This is an immediate consequence of the fact that $\odot$ (resp. $\pitchfork$) is a simplicially enriched functor $\underline{\mathbf{sSet}} \times \underline{C} \to \underline{C}$ (resp. $\underline{\mathbf{sSet}}^{\mathrm{op}} \times \underline{C} \to \underline{C}$). ■

Recall the weak homotopy type functor $\boldsymbol{\pi} : \mathbf{sSet} \to \mathbf{H}$, as defined in proposition 1.5.24.

Definition 2.5.12. Let $\underline{C}$ be a locally small simplicially enriched category.

- A **simplicial homotopy equivalence** in $\underline{C}$ is a π_0-homotopy equivalence.

- The **simplicial homotopy category** of $\underline{\mathcal{C}}$ is the locally small category $\pi_0[\underline{\mathcal{C}}]$.
- The **enriched simplicial homotopy category** of $\underline{\mathcal{C}}$ is the $\mathbf{H}$-enriched category $\boldsymbol{\pi}[\underline{\mathcal{C}}]$.

Remark 2.5.13. It is sometimes convenient to consider other localisations; for example, if $\pi_1 : \mathbf{sSet} \to \mathbf{Grpd}$ is the fundamental groupoid functor,[3] then the 2-category $\pi_1[\underline{\mathcal{C}}]$ has the following properties:

(i) The underlying category of $\pi_1[\underline{\mathcal{C}}]$ is naturally isomorphic to the underlying category of $\underline{\mathcal{C}}$ itself.

(ii) Given a parallel pair $f_0, f_1 : A \to B$ in $\mathcal{C}$, there exists a 2-cell $f_0 \Rightarrow f_1$ if and only if f_0 and f_1 are $\boldsymbol{\pi}$-homotopic in $\underline{\mathcal{C}}$.

(iii) A morphism is a simplicial homotopy equivalence in $\underline{\mathcal{C}}$ if and only if it is an equivalence in the 2-category $\pi_1[\underline{\mathcal{C}}]$.

However, if $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ is the fundamental category functor,[4] then the 2-category $\tau_1[\underline{\mathcal{C}}]$ in general only enjoys the first of the above properties.

Proposition 2.5.14. *Let $\underline{\mathcal{C}}$ be a locally small simplicially enriched category.*

(i) *A morphism in $\underline{\mathcal{C}}$ is a weak π_0-homotopy equivalence if and only if it is a weak $\boldsymbol{\pi}$-homotopy equivalence.*

(ii) *The localising functor $\mathcal{C} \to \pi_0[\underline{\mathcal{C}}]$ induces a bijection between simplicially enriched functors $\underline{\mathcal{C}} \to \mathcal{D}$ and ordinary functors $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an ordinary category (regarded as a simplicially enriched category via proposition 2.1.20).*

(iii) *If $\underline{\mathcal{C}}$ is moreover tensored or cotensored over* $\mathbf{sSet}$*, then $\pi_0[\underline{\mathcal{C}}]$ is the localisation of $\mathcal{C}$ at the weak π_0-equivalences.*

Proof. (i). The underlying category of the $\mathbf{H}$-enriched category $\boldsymbol{\pi}[\underline{\mathcal{C}}]$ is naturally isomorphic to the category $\pi_0[\underline{\mathcal{C}}]$, since $\mathbf{H}(1, \boldsymbol{\pi}X) \cong \pi_0 X$, and the property of being an isomorphism in a $\mathbf{H}$-enriched category depends only on the underlying category.

[3] Recall proposition 1.2.7.
[4] Recall proposition 1.2.1.

(ii). By proposition 1.2.4, a morphism from a simplicial set X to a discrete set Y must factor through $\pi_0 X$ in a unique way, so a simplicially enriched functor $\underline{\mathcal{C}} \to \mathcal{D}$ must factor through $\pi_0[\underline{\mathcal{C}}]$.

(iii). Simplicially tensored categories and simplicially cotensored categories are formally dual; we will prove the claim for case where $\underline{\mathcal{C}}$ is tensored over **sSet**.

First, consider a simplicial homotopy $\alpha : f_0 \Rightarrow f_1$ of morphisms $A \to B$ in $\underline{\mathcal{C}}$. Transposing across the tensor–hom adjunction yields $H : \Delta^1 \odot A \to B$ making the diagram below commute:

$$\begin{tikzcd}
\Delta^0 \odot A \arrow[d, "\delta^1 \odot \mathrm{id}_A"'] & A \arrow[l, "\eta_A"'] \arrow[d, "f_0"] \\
\Delta^1 \odot A \arrow[r, "H"] & B \\
\Delta^0 \odot A \arrow[u, "\delta^0 \odot \mathrm{id}_A"] & A \arrow[l, "\eta_A"] \arrow[u, "f_1"']
\end{tikzcd}$$

Using lemma 2.5.7, it is not hard to see that $\delta^0 \odot \mathrm{id}_A$ and $\delta^1 \odot \mathrm{id}_A$ are π_0-homotopy equivalences in $\underline{\mathcal{C}}$ with common π_0-homotopy inverse $\sigma^0 \odot \mathrm{id}_A$, so any functor that sends weak π_0-homotopy equivalences to isomorphisms must also identify f_0 and f_1, and hence, must factor through $\pi_0[\underline{\mathcal{C}}]$. ■

Proposition 2.5.15. *Let $\underline{\mathcal{C}}$ be a simplicially enriched category.*

(i) *The localising functor $\mathcal{C} \to \pi_0[\underline{\mathcal{C}}]$ is full and surjective on objects.*

(ii) *A morphism in $\mathcal{C}$ is a weak π_0-homotopy equivalence if and only if it has a π_0-homotopy inverse.*

(iii) *Two objects in $\mathcal{C}$ are isomorphic in $\pi_0[\mathcal{C}]$ if and only if there is a simplicial homotopy equivalence between them in $\underline{\mathcal{C}}$.*

Proof. Claim (i) is just the observation that the canonical map $X_0 \to \pi_0 X$ is surjective, and the rest follows straightforwardly. ◆

Definition 2.5.16. A **Dwyer–Kan equivalence of simplicially enriched categories** is a simplicially enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that the induced **H**-enriched functor $\boldsymbol{\pi}[\underline{F}] : \boldsymbol{\pi}[\underline{\mathcal{C}}] \to \boldsymbol{\pi}[\underline{\mathcal{D}}]$ is fully faithful and essentially surjective on objects.

Remark 2.5.17. Strictly speaking, the above definition only applies to locally small simplicially enriched categories; but it is clear how to extend the definition to handle general simplicially enriched categories.

Proposition 2.5.18. *Let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a simplicially enriched functor. The following are equivalent:*

(i) *$\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a Dwyer–Kan equivalence.*

(ii) *For each pair (A, B) of objects in $\mathcal{C}$, the hom-space morphism*

$$\underline{F} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$$

is a weak homotopy equivalence of simplicial sets, and the induced functor $\pi_0[\underline{F}] : \pi_0[\underline{\mathcal{C}}] \to \pi_0[\underline{\mathcal{D}}]$ is essentially surjective on objects.

(iii) *For each pair (A, B) of objects in $\mathcal{C}$, the hom-space morphism*

$$\underline{F} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$$

is a weak homotopy equivalence of simplicial sets, and for each object D in $\mathcal{D}$, there exist an object C in $\mathcal{C}$ and a simplicial homotopy equivalence in $\underline{\mathcal{D}}$ between FC and D.

Proof. This is a straightforward corollary of proposition 2.5.15. ■

Definition 2.5.19. The **bisimplicial nerve** of a simplicially enriched category $\underline{\mathcal{C}}$ is the bisimplicial nerve of the corresponding simplicial category $\mathcal{C}_\bullet$.

Lemma 2.5.20. *Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be small simplicially enriched categories and let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a simplicially enriched functor. The following are equivalent:*

(i) *$\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a bijective-on-objects Dwyer–Kan equivalence.*

(ii) *The morphism $\mathrm{N}^{\mathrm{ss}}(\underline{F})_\bullet : \mathrm{N}^{\mathrm{ss}}(\underline{\mathcal{C}})_\bullet \to \mathrm{N}^{\mathrm{ss}}(\underline{\mathcal{D}})_\bullet$ is a degreewise weak homotopy equivalence.*

Proof. It is clear that $\mathrm{N}^{\mathrm{ss}}(\underline{F})_0 : \mathrm{N}^{\mathrm{ss}}(\underline{\mathcal{C}})_0 \to \mathrm{N}^{\mathrm{ss}}(\underline{\mathcal{D}})_0$ is an isomorphism if and only if $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is bijective on objects. For each positive integer n, we have an isomorphism

$$\mathrm{N}^{\mathrm{ss}}(\underline{\mathcal{C}})_n \cong \coprod_{(c_0,\ldots,c_n)} \underline{\mathcal{C}}\left(c_{n-1}, c_n\right) \times \cdots \times \underline{\mathcal{C}}\left(c_0, c_1\right)$$

and it is natural as $\underline{C}$ varies along bijective-on-objects simplicially enriched functors, so by proposition 1.5.14, each $\mathrm{N}^{\mathrm{ss}}(\underline{F})_n : \mathrm{N}^{\mathrm{ss}}(\underline{C})_n \to \mathrm{N}^{\mathrm{ss}}(\underline{D})_n$ is a weak homotopy equivalence if and only if $\underline{F} : \underline{C} \to \underline{D}$ is a bijective-on-objects Dwyer–Kan equivalence. ■

Definition 2.5.21. A **fibrant simplicially enriched category** is a simplicially enriched category $\underline{C}$ such that the hom-spaces $\underline{C}(A, B)$ are Kan complexes for all pairs (A, B) of objects in $\underline{C}$.

For the sake of brevity, we also make the following definition:

Definition 2.5.22. A **Kan-enriched category** is a fibrant locally small simplicially enriched category.

Theorem 2.5.23. *Let $\underline{C}$ be a locally small simplicially enriched category.*

(i) $\mathrm{Ex}^\infty[\underline{C}]$ *is a Kan-enriched category.*

(ii) *The localisation functor $\mathcal{C} \to \mathrm{Ex}^\infty[\underline{C}]$ is a natural isomorphism of (ordinary) categories.*

(iii) *The localisation functor admits a simplicial enrichment $\underline{C} \to \mathrm{Ex}^\infty[\underline{C}]$ that is a natural Dwyer–Kan equivalence of simplicially enriched categories.*

Proof. The claims are immediate consequences of theorem 1.7.14: the functor $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ preserves finite limits, sends simplicial sets to Kan complexes, and is equipped with a natural weak homotopy equivalence $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}^\infty$ that is bijective on vertices. ■

Remark 2.5.24. In other words, every simplicial enrichment of a category can be replaced with a Dwyer–Kan equivalent *fibrant* simplicial enrichment. However, this procedure tends to destroy the good properties of the original simplicial enrichment: for instance, remark 1.7.15 implies that the fibrant replacement may fail to have simplicially enriched infinite products even when the original does.

Definition 2.5.25. A **simplicially enriched natural weak homotopy equivalence** is a simplicially enriched natural transformation of simplicially enriched functors $\underline{C} \to \underline{\mathbf{sSet}}$ whose components are weak homotopy equivalences.

Definition 2.5.26.

- Let $\underline{C}$ be a locally small simplicially enriched category. A **homotopical representation** (resp. **fibrant representation**) of a simplicially enriched functor $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$ is pair (A, x) where A is an object in C and x is a vertex of FA such that the components of the corresponding simplicially enriched natural transformation $\underline{C}(A, -) \Rightarrow F$ (as described by the weak Yoneda lemma) are weak homotopy equivalences (resp. trivial Kan fibrations).

- A **homotopically representable** (resp. **fibrantly representable**) **simplicially enriched functor** is one that admits a homotopical representation (resp. fibrant representation).

Lemma 2.5.27. *Let $\underline{C}$ be a locally small simplicially enriched category and let $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$ be a simplicially enriched functor. The following are equivalent:*

(i) $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$ *is a homotopically representable simplicially enriched functor.*

(ii) $\boldsymbol{\pi}[\underline{F}] : \boldsymbol{\pi}[\underline{C}] \to \mathbf{H}$ *is a representable* $\mathbf{H}$*-enriched functor.*

Proof. (i) ⇒ (ii). Immediate.

(ii) ⇒ (i). The weak Yoneda lemma (B.2.14) implies that the map $\varphi \mapsto \boldsymbol{\pi}[\varphi]$ is a surjection from the ensemble of simplicially enriched natural transformations $\underline{C}(A, -) \Rightarrow \underline{F}$ onto the ensemble of $\mathbf{H}$-enriched natural transformations $\boldsymbol{\pi}\underline{C}(A, -) \Rightarrow \boldsymbol{\pi}[\underline{F}]$; and it is not hard to see this restricts to a surjection from the ensemble of simplicially enriched natural weak homotopy equivalences onto the ensemble of $\mathbf{H}$-enriched natural isomorphisms. Thus, if $\boldsymbol{\pi}[\underline{F}]$ is representable, then $\underline{F}$ is homotopically representable. ∎

Lemma 2.5.28. *Let $\underline{C}$ be a locally small simplicially enriched category and let $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$ be a simplicially enriched functor. Given any two representations of $\underline{F}$, say (A, x) and (B, y):*

(i) *There is a morphism $f : A \to B$ in C such that $F(f)(x) \sim y$ in FB, and it is unique up to simplicial homotopy in $\underline{C}$.*

(ii) *In particular, every such morphism $f : A \to B$ is (half of) a simplicial homotopy equivalence in $\underline{C}$.*

(iii) *If the component* $\underline{C}(A,B) \to FB$ *of the simplicially enriched natural transformation corresponding to* x *is a trivial Kan fibration, then the largest simplicial subset of* $\underline{C}(A,B)$ *whose vertices are the morphisms* $f : A \to B$ *in* C *such that* $F(f)(x) = y$ *is a contractible Kan complex.*

Proof. (i). Let $\varphi : \underline{C}(A,-) \Rightarrow \underline{F}$ and $\psi : \underline{C}(B,-) \Rightarrow \underline{F}$ be the simplicially enriched natural weak homotopy equivalences such that $\varphi_A(\mathrm{id}_A) = x$ and $\psi_B(\mathrm{id}_B) = y$; such exist and are unique by the weak Yoneda lemma (2.1.27). By proposition 1.5.18, $\pi_0\varphi_B : \pi_0\underline{C}(A,B) \to \pi_0 FB$ is a bijection, so there is a morphism $f : A \to B$ in C such that $\varphi_B(f) \sim y$ in FB, and it is unique up to simplicial homotopy in $\underline{C}$. But the following diagram commutes,

$$\begin{tikzcd}
\underline{C}(A,A) \arrow[r, "\varphi_A"] \arrow[d, "\underline{C}(A,f)"'] & FA \arrow[d, "Ff"] \\
\underline{C}(A,B) \arrow[r, "\varphi_B"'] & FB
\end{tikzcd}$$

so $F(f)(x) = \varphi_B(f) \sim y$ as required.

(ii). Let $f : A \to B$ and $g : B \to A$ be morphisms in $\underline{C}$ such that $F(f)(x) \sim y$ and $F(g)(y) \sim x$. Then $F(g \circ f)(x) \sim x$ and $F(f \circ g)(y) \sim y$, so we must have $g \circ f \sim \mathrm{id}_A$ and $f \circ g \sim \mathrm{id}_B$, i.e. (f, g) is a simplicial homotopy equivalence.

(iii). The indicated simplicial subset $X \subseteq \underline{C}(A,B)$ fits into a pullback diagram of the following form,

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \underline{C}(A,B) \arrow[d, "\varphi_B"] \\
\Delta^0 \arrow[r, "\ulcorner y \urcorner"'] & FB
\end{tikzcd}$$

where $\ulcorner y \urcorner : \Delta^0 \to FB$ is the morphism corresponding to the vertex y; so if $\varphi_B : \underline{C}(A,B) \to FB$ is a trivial Kan fibration, then (by proposition A.3.17) $X \to \Delta^0$ is also a trivial Kan fibration, and therefore X is a contractible Kan complex (by proposition 1.5.8). ■

Definition 2.5.29. A **Dwyer–Kan contractible category** is a simplicially enriched category $\underline{C}$ such that the unique simplicially enriched functor $\underline{C} \to \mathbb{1}$ is a Dwyer–Kan equivalence.

Lemma 2.5.30. *Let $\underline{C}$ be a simplicially enriched category. The following are equivalent:*

(i) *$\underline{C}$ is Dwyer–Kan contractible.*

(ii) *For every pair (A, B) of objects in C, the hom-space $\underline{C}(A, B)$ is weakly contractible.*

(iii) *There is an object A in C such that, for every object B in C, the hom-space $\underline{C}(A, B)$ is weakly contractible and A is simplicially homotopy equivalent to B.*

Proof. Obvious. ◆

Proposition 2.5.31. *Let $\underline{C}$ be a locally small simplicially enriched category, let $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$ be a simplicially enriched functor, and let $\underline{D}$ be the simplicially enriched full subcategory of the slice category $[\underline{C}, \underline{\mathbf{sSet}}]_{/\underline{F}}$ spanned by the fibrant representations of $\underline{F}$. If $\underline{F}$ is fibrantly representable, then $\underline{D}$ is fibrant and Dwyer–Kan contractible.*

Proof. Let (A, x) be a fibrant representation of $\underline{F} : \underline{C} \to \underline{\mathbf{sSet}}$. By lemma 2.5.28 and the strong Yoneda lemma (proposition B.3.9), for any fibrant representation (B, y) of $\underline{F}$, the hom-space $\underline{D}((B, y), (A, x))$ is a contractible Kan complex. Thus, $\underline{D}$ is a fibrant simplicially enriched category, and by lemma 2.5.30, it is Dwyer–Kan contractible. ■

Definition 2.5.32. A **Dwyer–Kan adjunction of simplicially enriched categories** consists of the following data:

- A simplicially enriched functor $\underline{F} : \underline{C} \to \underline{D}$, called the **left adjoint**.
- A simplicially enriched functor $\underline{G} : \underline{D} \to \underline{C}$, called the **right adjoint**.
- A simplicially enriched natural transformation $\eta : \mathrm{id}_{\underline{C}} \Rightarrow \underline{GF}$, called the **unit**.
- A simplicially enriched natural transformation $\varepsilon : \underline{FG} \Rightarrow \mathrm{id}_{\underline{D}}$, called the **counit**.

These are moreover required to satisfy the following condition: for all objects C in C and D in D,

$$\boldsymbol{\pi}\big(\underline{D}(FC, \varepsilon_D) \circ \underline{F}_{C,GD}\big) : \boldsymbol{\pi}\underline{C}(C, GD) \to \boldsymbol{\pi}\underline{D}(FC, D)$$

$$\boldsymbol{\pi}\big(\underline{\mathcal{C}}(\eta_C, GD) \circ \underline{G}_{FC,D}\big) : \boldsymbol{\pi}\underline{\mathcal{D}}(FC, D) \to \boldsymbol{\pi}\underline{\mathcal{C}}(C, GD)$$

are mutually inverse.

Remark 2.5.33. Note that the data $(\underline{F}, \underline{G}, \eta, \varepsilon)$ constitute a Dwyer–Kan adjunction if and only if $(\boldsymbol{\pi}[\underline{F}], \boldsymbol{\pi}[\underline{G}], \boldsymbol{\pi}[\eta], \boldsymbol{\pi}[\varepsilon])$ constitute a $\mathbf{H}$-enriched adjunction (at least when the simplicially enriched categories are locally small). In particular, any simplicially enriched adjunction is also a Dwyer–Kan adjunction, but not vice versa.

Unfortunately, we do not have an analogue of corollary B.2.27; instead, we make the following definitions.

Definition 2.5.34. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be locally small simplicially enriched categories.

- A **Dwyer–Kan left pre-adjoint functor** $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a simplicially enriched functor with the following property: for each object D in $\mathcal{D}$, the simplicially enriched functor

  $$\underline{\mathcal{D}}(\underline{F}-, D) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$$

  is homotopically representable in $\underline{\mathcal{C}}$.

- A **Dwyer–Kan right pre-adjoint functor** $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ is a simplicially enriched functor with the following property: for each object C in $\mathcal{C}$, the simplicially enriched functor

  $$\underline{\mathcal{C}}(C, \underline{G}-) : \underline{\mathcal{D}} \to \underline{\mathbf{sSet}}$$

  is homotopically representable in $\underline{\mathcal{D}}$.

Example 2.5.35. The inclusion $\underline{\mathbf{Kan}} \hookrightarrow \underline{\mathbf{sSet}}$ is a Dwyer–Kan right pre-adjoint functor: indeed, by corollary 1.4.16 and theorem 1.7.14,

$$\underline{\mathbf{sSet}}\big(i_X^\infty, -\big) : \underline{\mathbf{sSet}}(\mathrm{Ex}^\infty(X), -) \Rightarrow \underline{\mathbf{sSet}}(X, -)$$

is a simplicially enriched natural weak homotopy equivalence of simplicially enriched functors $\underline{\mathbf{Kan}} \to \underline{\mathbf{sSet}}$, and $\mathrm{Ex}^\infty(X)$ is a Kan complex, as required.

Remark 2.5.36. Of course, any Dwyer–Kan equivalence of simplicially enriched categories is both a Dwyer–Kan left pre-adjoint functor and a Dwyer–Kan right pre-adjoint functor.

Lemma 2.5.37. *Let $\underline{C}$ and $\underline{D}$ be locally small simplicially enriched categories. The following are equivalent for a simplicially enriched functor $\underline{F} : \underline{C} \to \underline{D}$:*

(i) *$\underline{F} : \underline{C} \to \underline{D}$ is a Dwyer–Kan left pre-adjoint functor.*

(ii) *$\boldsymbol{\pi}[\underline{F}] : \boldsymbol{\pi}[\underline{C}] \to \boldsymbol{\pi}[\underline{D}]$ admits a $\mathbf{H}$-enriched right adjoint.*

Dually, the following are equivalent for a simplicially enriched functor $\underline{G} : \underline{D} \to \underline{C}$:

(i′) *$\underline{G} : \underline{D} \to \underline{C}$ is a Dwyer–Kan right pre-adjoint functor.*

(ii′) *$\boldsymbol{\pi}[\underline{G}] : \boldsymbol{\pi}[\underline{D}] \to \boldsymbol{\pi}[\underline{C}]$ admits a $\mathbf{H}$-enriched left adjoint.*

Proof. Apply lemma 2.5.27 to corollary B.2.27. ■

2.6 Homotopy limits and colimits

Prerequisites. §§1.9, 1.7, 1.11, 2.1, 2.5.

In this section, we define homotopy-theoretic limits and colimits in Kan-enriched categories. In principle, one could extend these definitions to (locally small) simplicially enriched categories by applying theorem 2.5.23, but we will avoid this because one often requires a more serious “correction” than just fibrant replacement of the hom-spaces.

Proposition 2.6.1. *Let $\mathcal{J}$ be a small category.*

- *The functor* $\mathrm{B}(-, \mathcal{J}, -) : [\mathcal{J}^{\mathrm{op}}, \mathbf{sSet}] \times [\mathcal{J}, \mathbf{sSet}] \to \mathbf{sSet}$ *admits a simplicial enrichment.*
- *The functor* $\mathrm{C}(-, \mathcal{J}, -) : [\mathcal{J}, \mathbf{sSet}]^{\mathrm{op}} \times [\mathcal{J}, \mathbf{sSet}] \to \mathbf{sSet}$ *admits a simplicial enrichment.*

Proof. Recall that, by proposition 1.8.36, there are isomorphisms of the form

$$\mathrm{B}(X \times G, \mathcal{J}, F) \cong X \times \mathrm{B}(G, \mathcal{J}, F) \cong \mathrm{B}(G, \mathcal{J}, X \times F)$$

that are natural in X (in the ordinary sense), and using lemma 1.8.26, it is straightforward to verify that these induce strengths for $\mathrm{B}(-, \mathcal{J}, F)$ and $\mathrm{B}(G, \mathcal{J}, -)$ (respectively). Thus, by theorem B.4.17, $\mathrm{B}(-, \mathcal{J}, F)$ and $\mathrm{B}(G, \mathcal{J}, -)$ admit simplicial enrichments, and using proposition B.2.18, it is not hard to verify that $\mathrm{B}(-, \mathcal{J}, -)$ itself admits a simplicial enrichment.

Proposition 1.9.7 then says,

$$\mathrm{C}(G, \mathcal{J}, F) \cong \int_{j:\mathcal{J}} \left[\mathrm{B}\left(G, \mathcal{J}^{\mathrm{op}}, \operatorname{disc} \mathcal{h}_j\right), F j\right]$$

so by proposition B.3.16, $\mathrm{C}(-, \mathcal{J}, -)$ also admits a simplicial enrichment. ∎

Proposition 2.6.2. *The functor* $\mathrm{Ex}^\infty : \mathbf{sSet} \to \mathbf{sSet}$ *admits a unique simplicial enrichment making the canonical embedding* $i^\infty : \mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{Ex}^\infty$ *a simplicially enriched natural transformation.*

Proof. By theorem 1.7.14, Ex^∞ preserves finite products, so we may apply Lemmas 2.1.4 and 2.1.6 in [Johnstone, 2002, Part B]. □

Corollary 2.6.3. *There exist a simplicially enriched functor* $\underline{R} : \underline{\mathbf{sSet}} \to \underline{\mathbf{sSet}}$ *and a simplicially enriched natural transformation* $i : \mathrm{id}_{\underline{\mathbf{sSet}}} \Rightarrow \underline{R}$ *satisfying the following condition:*

- *For all simplicial sets* X*,* RX *is a Kan complex and* $i_X : X \to RX$ *is an anodyne extension.* ∎

Definition 2.6.4. Let $\underline{C}$ be a locally small simplicially enriched category.

- A **Bousfield–Kan limit** in $\underline{C}$ for a small diagram $F : \mathcal{J} \to C$ is a representation of the simplicially enriched functor

  $$\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} \underline{C}(-, F) : \underline{C}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$$

  i.e. a pair $\left(\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} F, \lambda\right)$ where $\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} F$ is an object in C and λ is a vertex of $\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} \underline{C}\left(\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} F, F\right)$ such that the induced simplicially enriched natural transformation

  $$\underline{C}\left(-, \varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} F\right) \Rightarrow \varprojlim{}^{\mathrm{BK}}_{\mathcal{J}} \underline{C}(-, F)$$

  is a simplicially enriched natural isomorphism.

- A **Bousfield–Kan colimit** in $\underline{C}$ for a small diagram $F : \mathcal{J} \to C$ is a representation of the simplicially enriched functor

  $$\varprojlim{}^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{C}(F, -) : \underline{C} \to \underline{\mathbf{sSet}}$$

i.e. a pair $\left(\varinjlim_{\mathcal{J}}^{\mathrm{BK}} F, \lambda\right)$ where $\varinjlim_{\mathcal{J}}^{\mathrm{BK}} F$ is an object in C and λ is a vertex of $\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}\left(F, \varinjlim_{\mathcal{J}}^{\mathrm{BK}} F\right)$ such that the induced simplicially enriched natural transformation

$$\underline{C}\left(\varinjlim_{\mathcal{J}}^{\mathrm{BK}} F, -\right) \Rightarrow \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}(F, -)$$

is a simplicially enriched natural isomorphism.

Remark 2.6.5. By remark 1.9.3 and propositions 2.6.1 and B.3.16, if Bousfield–Kan limits for all diagrams of the shape $\mathcal{J}$ exist in $\underline{C}$, then there is a simplicially enriched functor

$$\varprojlim_{\mathcal{J}}^{\mathrm{BK}} : [\mathcal{J}, \underline{C}] \to \underline{C}$$

and a family of isomorphisms in **sSet** of the form

$$\underline{C}\left(A, \varprojlim_{\mathcal{J}}^{\mathrm{BK}} F\right) \cong \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}(A, F)$$

constituting a simplicially enriched natural isomorphism in A *and* F. Dually for Bousfield–Kan colimits.

The following lemma describes the Bousfield–Kan analogue of the product–equaliser formula for limits.

Lemma 2.6.6. *Let $\underline{C}$ be a locally small simplicially enriched category and let $F : \mathcal{J} \to C$ be a small diagram.*

(i) *If $\underline{C}$ has simplicially enriched products for all finite families and all families of size $\leq |\mathrm{mor}\, \mathcal{J}|$, then the simplicially enriched cobar complex* $\mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)$ *exists in $\underline{C}$, i.e. the cosimplicial simplicially enriched functor*

$$\mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F)) : \underline{C}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$$

admits a representation by a cosimplicial object in $\underline{C}$.

(ii) *If the simplicially enriched cobar complex* $\mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)$ *exists in $\underline{C}$, then we have*

$$\varprojlim_{\mathcal{J}}^{\mathrm{BK}} F \cong \mathrm{Tot}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)$$

where the LHS exists if and only if the RHS exists.

Dually:

(i′) *If $\underline{C}$ has simplicially enriched coproducts for all finite families and all families of size $\leq |\operatorname{mor} \mathcal{J}|$, then the simplicially enriched bar complex $\mathrm{B}_\bullet(\Delta 1, \mathcal{J}, F)$ exists in $\underline{C}$, i.e. the cosimplicial simplicially enriched functor*

$$\mathrm{C}^\bullet(\Delta 1, \mathcal{J}^{\mathrm{op}}, \underline{C}(F, -)) : \underline{C} \to \underline{\mathbf{sSet}}$$

admits a representation by a simplicial object in $\underline{C}$.

(ii′) *If the simplicially enriched bar complex $\mathrm{B}_\bullet(\Delta 1, \mathcal{J}, F)$ exists in $\underline{C}$, then we have*

$$\varinjlim_{\mathcal{J}}{}^{\mathrm{BK}} F \cong \left|\mathrm{B}_\bullet(\Delta 1, \mathcal{J}, F)\right|$$

where the LHS exists if and only if the RHS exists.

Proof. (i). By remark 1.8.18,

$$\mathrm{C}^n(\Delta 1, \mathcal{J}, \underline{C}(-, F)) \cong \prod_{(j_0, \ldots, j_n)} \left(\mathcal{J}\left(j_n, j_{n-1}\right) \times \cdots \times \mathcal{J}\left(j_1, j_0\right)\right) \pitchfork F j_0$$

so the simplicially enriched cobar complex $\mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)$ can be constructed using just simplicially enriched products.

(ii). By (definition and) proposition 1.9.7, we have the following simplicially enriched natural isomorphism;

$$\underline{C}\left(-, \varprojlim_{\mathcal{J}}{}^{\mathrm{BK}} F\right) \cong \mathrm{C}(\Delta 1, \mathcal{J}, \underline{C}(-, F))$$

but recalling the definition of cobar constructions,

$$\mathrm{C}(\Delta 1, \mathcal{J}, \underline{C}(-, F)) \cong \underline{\mathrm{Tot}}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F))$$

and hence, by remark 2.3.11,

$$\underline{C}\left(-, \varprojlim_{\mathcal{J}}{}^{\mathrm{BK}} F\right) \cong \underline{C}(-, \operatorname{Tot} \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F))$$

as required. ■

Unfortunately, the notion of Bousfield–Kan limit/colimit is not stable under Dwyer–Kan equivalence. To resolve this, we need a homotopy-invariant notion:

Definition 2.6.7. Let $\underline{C}$ be a Kan-enriched category.

- A **homotopy limit** in $\underline{C}$ for a small diagram $F : \mathcal{J} \to C$ is a homotopical representation of the simplicially enriched functor

$$\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}(-, F) : [\mathcal{J}, \underline{C}] \to \underline{C}$$

i.e. a pair $\left(\operatorname{ho}\varprojlim_{\mathcal{J}} F, \lambda\right)$ where $\operatorname{ho}\varprojlim_{\mathcal{J}} F$ is an object in C and λ is a vertex of $\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}\left(\operatorname{ho}\varprojlim_{\mathcal{J}} F, F\right)$ such that the induced simplicially enriched natural transformation

$$\underline{C}\left(-, \operatorname{ho}\varprojlim_{\mathcal{J}} F\right) \Rightarrow \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}(-, F)$$

is a simplicially enriched natural weak homotopy equivalence.

- A **homotopy colimit** in $\underline{C}$ for a small diagram $F : \mathcal{J} \to C$ is a homotopical representation of the simplicially enriched functor

$$\varprojlim_{\mathcal{J}^{\mathrm{op}}}^{\mathrm{BK}} \underline{C}(F, -) : \underline{C} \to \underline{\mathbf{sSet}}$$

i.e. a pair $\left(\operatorname{ho}\varinjlim_{\mathcal{J}} F, \lambda\right)$ where $\operatorname{ho}\varinjlim_{\mathcal{J}} F$ is an object in C and λ is a vertex of $\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}\left(F, \operatorname{ho}\varinjlim_{\mathcal{J}} F\right)$ such that the induced simplicially enriched natural transformation

$$\underline{C}\left(\operatorname{ho}\varinjlim_{\mathcal{J}} F, -\right) \Rightarrow \varprojlim_{\mathcal{J}^{\mathrm{op}}}^{\mathrm{BK}} \underline{C}(F, -)$$

is a simplicially enriched natural weak homotopy equivalence.

Definition 2.6.8. Let $\underline{C}$ be a Kan-enriched category and let $\mathcal{J}$ be a small category.

- A **homotopy limit functor** for diagrams of shape $\mathcal{J}$ in $\underline{C}$ is a simplicially enriched functor $\operatorname{ho}\varprojlim_{\mathcal{J}} : [\mathcal{J}, \underline{C}] \to \underline{C}$ equipped with a simplicially enriched natural weak homotopy equivalence

$$\underline{C}\left(-, \operatorname{ho}\varprojlim_{\mathcal{J}} -\right) \Rightarrow \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{C}(-, -)$$

of simplicially enriched functors $\underline{C}^{\mathrm{op}} \times [\mathcal{J}, \underline{C}] \to \mathbf{Kan}$.

- A **homotopy colimit functor** for diagrams of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$ is a simplicially enriched functor $\operatorname{holim}_{\longrightarrow \mathcal{J}} : [\mathcal{J}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ equipped with a simplicially enriched natural weak homotopy equivalence

$$\underline{\mathcal{C}}\left(\operatorname{holim}_{\longrightarrow \mathcal{J}} -, -\right) \Rightarrow \lim^{\mathrm{BK}}_{\longleftarrow \mathcal{J}^{\mathrm{op}}} \underline{\mathcal{C}}(-, -)$$

of simplicially enriched functors $[\mathcal{J}, \underline{\mathcal{C}}]^{\mathrm{op}} \times \underline{\mathcal{C}} \to \mathbf{Kan}$.

Remark 2.6.9. By lemma 2.5.28, homotopy limits/colimits are unique up to simplicial homotopy equivalence. Unfortunately, this is not enough to guarantee functoriality in the sense above.

Lemma 2.6.10. *Let $\underline{\mathcal{C}}$ be a Kan-enriched category and let $\mathcal{J}$ be a small category.*

- *If $\underline{\mathcal{C}}$ has Bousfield–Kan limits for all diagrams of shape $\mathcal{J}$, then there is a homotopy limit functor for diagrams of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$.*
- *If $\underline{\mathcal{C}}$ has Bousfield–Kan colimits for all diagrams of shape $\mathcal{J}$, then there is a homotopy colimit functor for diagrams of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$.*

Proof. The two claims are formally dual; we will prove the first version.

By remark 2.6.5, there exist a simplicially enriched functor that sends a diagram $F : \mathcal{J} \to \mathcal{C}$ to the Bousfield–Kan limit $\lim^{\mathrm{BK}}_{\longleftarrow \mathcal{J}} F$ in $\underline{\mathcal{C}}$ and a simplicially enriched natural *isomorphism*

$$\underline{\mathcal{C}}\left(-, \lim^{\mathrm{BK}}_{\longleftarrow \mathcal{J}} -\right) \cong \lim^{\mathrm{BK}}_{\longleftarrow \mathcal{J}} \underline{\mathcal{C}}(-, -)$$

which is a simplicially enriched natural weak homotopy equivalence *a fortiori*. ■

Theorem 2.6.11. *Let $\mathcal{J}$ be a small category.*

(i) $\underline{\mathbf{Kan}}$ *has Bousfield–Kan limits for all diagrams of shape $\mathcal{J}$; in particular, there is a homotopy limit functor for diagrams of shape $\mathcal{J}$ in* $\underline{\mathbf{Kan}}$.

(ii) *There is a homotopy colimit functor for diagrams of shape $\mathcal{J}$ in* $\underline{\mathbf{Kan}}$.

Proof. (i). This is an immediate consequence of proposition 1.9.18 (plus proposition 4.3.4) and lemma 2.6.10.

(ii). By similar arguments, there exist a simplicially enriched functor $\varinjlim^{\mathrm{BK}}{}_{\mathcal{J}} : [\mathcal{J}, \underline{\mathbf{sSet}}] \to \underline{\mathbf{sSet}}$ and a simplicially enriched natural *isomorphism*

$$\underline{\mathbf{sSet}}\left(\varinjlim^{\mathrm{BK}}_{\mathcal{J}} -, -\right) \Rightarrow \varprojlim^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{\mathbf{sSet}}(-, -)$$

but $\varinjlim^{\mathrm{BK}}{}_{\mathcal{J}} F$ may fail to be a Kan complex even when F is a diagram $\mathcal{J} \to \mathbf{Kan}$. To fix this, consider $\underline{R} : \underline{\mathbf{sSet}} \to \underline{\mathbf{sSet}}$ and $i : \mathrm{id}_{\underline{\mathbf{sSet}}} \Rightarrow \underline{R}$ as in corollary 2.6.3. Then we have a simplicially enriched functor $\underline{R} \varinjlim^{\mathrm{BK}}{}_{\mathcal{J}} : [\mathcal{J}, \underline{\mathbf{Kan}}] \to \underline{\mathbf{Kan}}$, and by corollary 1.4.16, $i : \mathrm{id}_{\underline{\mathbf{sSet}}} \Rightarrow \underline{R}$ induces a simplicially enriched natural weak homotopy equivalence

$$\underline{\mathbf{sSet}}\left(\underline{R} \varinjlim^{\mathrm{BK}}_{\mathcal{J}} -, -\right) \Rightarrow \underline{\mathbf{sSet}}\left(\varinjlim^{\mathrm{BK}}_{\mathcal{J}} -, -\right)$$

and so we have a simplicially enriched natural weak homotopy equivalence

$$\underline{\mathbf{sSet}}\left(\underline{R} \varinjlim^{\mathrm{BK}}_{\mathcal{J}} -, -\right) \Rightarrow \varprojlim^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{\mathbf{sSet}}(-, -)$$

as required. ■

Definition 2.6.12. Let $\underline{\mathcal{C}}$ be a Kan-enriched category.

- A **homotopy descent object** for a cosimplicial object $B^{\bullet}$ in $\underline{\mathcal{C}}$ is a homotopy limit for $B^{\bullet}$ (considered as a diagram $\mathbf{\Delta} \to \mathcal{C}$).
- A **homotopy codescent object** for a simplicial object $A_{\bullet}$ in $\underline{\mathcal{C}}$ is a homotopy colimit for $A_{\bullet}$ (considered as a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$).

The following lemma describes the homotopy analogue of the product–equaliser formula for limits. Unfortunately, the non-functoriality of homotopy limits means that the result is not as strong as lemma 2.6.6.

Lemma 2.6.13. *Let $\underline{\mathcal{C}}$ be a Kan-enriched category and let $F : \mathcal{J} \to \mathcal{C}$ be a small diagram.*

- *If the homotopy cobar complex $\mathrm{C}^{\bullet}(\Delta 1, \mathcal{J}, F)$ exists in $\underline{\mathcal{C}}$, i.e. the cosimplicial simplicially enriched functor*

  $$\mathrm{C}^{\bullet}(\Delta 1, \mathcal{J}, \underline{\mathcal{C}}(-, F)) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$$

  admits a homotopical representation by a cosimplicial object in $\underline{\mathcal{C}}$, then we have

  $$\operatorname{ho}\varprojlim_{\mathcal{J}} F \simeq \operatorname{ho}\varprojlim_{\mathbf{\Delta}} \mathrm{C}^{\bullet}(\Delta 1, \mathcal{J}, F)$$

  provided the homotopy limit on the LHS and the homotopy descent object on the RHS both exist.

- *If the homotopy bar complex* $\mathrm{B}_\bullet(\Delta 1, \mathcal{J}, F)$ *exists in* $\underline{C}$*, i.e. the cosimplicial simplicially enriched functor*

$$\mathrm{C}^\bullet(\Delta 1, \mathcal{J}^{\mathrm{op}}, \underline{C}(F, -)) : \underline{C} \to \underline{\mathbf{sSet}}$$

admits a homotopical representation by a simplicial object in $\underline{C}$*, then we have*

$$\operatorname*{holim}_{\longrightarrow \mathcal{J}} F \simeq \operatorname*{holim}_{\longrightarrow \mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(\Delta 1, \mathcal{J}, F)$$

provided the homotopy colimit on the LHS and the homotopy codescent object on the RHS both exist.

Proof. The two claims are formally dual; we will prove the first version.

For convenience, we will work in $\mathbf{H}$ instead of $\mathbf{sSet}$. By (definition and) proposition 1.9.7, we have the following natural isomorphism;

$$\boldsymbol{\pi}\underline{C}\Big(-, \operatorname*{holim}_{\longleftarrow \mathcal{J}} F\Big) \cong \boldsymbol{\pi}\mathrm{C}(\Delta 1, \mathcal{J}, \underline{C}(-, F))$$

but recalling the definition of cobar constructions,

$$\mathrm{C}(\Delta 1, \mathcal{J}, \underline{C}(-, F)) \cong \underline{\mathrm{Tot}}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F))$$

and by lemma 1.8.42 and corollary 1.9.33:

$$\boldsymbol{\pi}(\underline{\mathrm{Tot}}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F))) \cong \boldsymbol{\pi}\Big(\operatorname*{lim}_{\longleftarrow \mathbf{\Delta}}{}^{\mathrm{BK}}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F))\Big)$$

On the other hand,

$$\boldsymbol{\pi}\underline{C}\Big(-, \operatorname*{holim}_{\longleftarrow \mathbf{\Delta}} \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)\Big) \cong \boldsymbol{\pi}\Big(\operatorname*{lim}_{\longleftarrow \mathbf{\Delta}}{}^{\mathrm{BK}}\, \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, \underline{C}(-, F))\Big)$$

so we have the following natural isomorphism:

$$\boldsymbol{\pi}\underline{C}\Big(-, \operatorname*{holim}_{\longleftarrow \mathcal{J}} F\Big) \cong \boldsymbol{\pi}\underline{C}\Big(-, \operatorname*{holim}_{\longleftarrow \mathbf{\Delta}} \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)\Big)$$

Thus, applying the (ordinary) Yoneda lemma, we deduce that

$$\operatorname*{holim}_{\longleftarrow \mathcal{J}} F \cong \operatorname*{holim}_{\longleftarrow \mathbf{\Delta}} \mathrm{C}^\bullet(\Delta 1, \mathcal{J}, F)$$

in $\pi_0[\underline{C}]$, which implies the claim. ■

Definition 2.6.14. Let $\underline{C}$ and $\underline{D}$ be Kan-enriched categories, let $\underline{F} : \underline{C} \to \underline{D}$ be a simplicially enriched functor, and let $C : \mathcal{J} \to C$ be a small diagram.

- We say $\underline{F}$ **preserves homotopy limits for** C if, for every homotopy limit for C, say (L, λ), the pair $\left(FL, \underline{F}_* \lambda\right)$ is a homotopy limit for FC, where $\underline{F}_* \lambda$ is the image in $\varprojlim^{\mathrm{BK}}_{\mathcal{J}} \underline{\mathcal{D}}(FL, FC)$ of vertex λ under the morphism
$$\varprojlim^{\mathrm{BK}}_{\mathcal{J}} \underline{\mathcal{C}}(L, C) \to \varprojlim^{\mathrm{BK}}_{\mathcal{J}} \underline{\mathcal{D}}(FL, FC)$$
induced by $\underline{F} : \underline{\mathcal{C}}(-,-) \Rightarrow \underline{\mathcal{D}}(\underline{F}-, \underline{F}-)$.

- We say $\underline{F}$ **preserves homotopy colimits for** C if, for every homotopy colimit for C, say (L, λ), the pair $\left(FL, \underline{F}_* \lambda\right)$ is a homotopy colimit for FC, where $\underline{F}_* \lambda$ is the image in $\varprojlim^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{\mathcal{D}}(FC, FL)$ of vertex λ under the morphism
$$\varprojlim^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{\mathcal{C}}(C, L) \to \varprojlim^{\mathrm{BK}}_{\mathcal{J}^{\mathrm{op}}} \underline{\mathcal{D}}(FC, FL)$$
induced by $\underline{F} : \underline{\mathcal{C}}(-,-) \Rightarrow \underline{\mathcal{D}}(\underline{F}-, \underline{F}-)$.

Proposition 2.6.15. *Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be Kan-enriched categories.*

- *If $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ is a Dwyer–Kan right pre-adjoint, then it preserves homotopy limits for all small diagrams.*

- *If $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a Dwyer–Kan left pre-adjoint, then it preserves homotopy colimits for all small diagrams.*

Proof. The two claims are formally dual; we will prove the first version.

Let $D : \mathcal{J} \to \mathcal{D}$ be a small diagram and let (L, λ) be a homotopy limit for D in $\underline{\mathcal{D}}$. We wish to prove that $\left(GL, \underline{G}_* \lambda\right)$ is a homotopy limit for GD in $\underline{\mathcal{C}}$. For each object A in $\mathcal{C}$, let $\left(FA, \eta_A\right)$ be a homotopical representation for $\underline{\mathcal{C}}(A, \underline{G}-) : \underline{\mathcal{D}} \to \underline{\mathbf{Kan}}$ and consider the following diagram in $\mathbf{H}$,

$$\begin{tikzcd}
\boldsymbol{\pi}\underline{\mathcal{D}}(FA, L) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \boldsymbol{\pi}\left(\varprojlim^{\mathrm{BK}}_{\mathcal{J}} \underline{\mathcal{D}}(FA, D)\right) \arrow[d, "\cong"] \\
\boldsymbol{\pi}\underline{\mathcal{C}}(A, GL) \arrow[r] & \boldsymbol{\pi}\left(\varprojlim^{\mathrm{BK}}_{\mathcal{J}} \underline{\mathcal{C}}(A, GD)\right)
\end{tikzcd}$$

where the vertical arrows are induced by the simplicially enriched natural transformation
$$\underline{\mathcal{D}}(FA, -) \Rightarrow \underline{\mathcal{C}}(A, \underline{G}-)$$

corresponding to the vertex η_A in $\underline{\mathcal{C}}(A, GFA)$, the top horizontal arrow is induced by the component of the simplicially enriched natural transformation

$$\underline{\mathcal{D}}(-, L) \Rightarrow \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{D}}(-, D)$$

corresponding to the vertex λ in $\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{D}}(L, D)$, and the bottom horizontal arrow is induced by the component of the simplicially enriched natural transformation

$$\underline{\mathcal{C}}(-, GL) \Rightarrow \varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{C}}(-, GD)$$

corresponding to the vertex $\underline{G}_* \lambda$ in $\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{C}}(GL, GD)$. To prove the claim, we must show that the $\underline{G}_* \lambda$ corresponds to a simplicially enriched natural weak homotopy equivalence; and (using the 2-out-of-3 property) it is enough to verify that the diagram above commutes (in $\mathbf{H}$).

Now, lemma 2.5.37 says that $\boldsymbol{\pi}[\underline{G}] : \boldsymbol{\pi}[\underline{\mathcal{D}}] \to \boldsymbol{\pi}[\underline{\mathcal{C}}]$ admits a $\mathbf{H}$-enriched left adjoint, say $\underline{F} : \boldsymbol{\pi}[\underline{\mathcal{C}}] \to \boldsymbol{\pi}[\underline{\mathcal{D}}]$; and moreover, we may choose $\underline{F}$ so that it agrees with our earlier choices of (FA, η_A). By the weak Yoneda lemma (B.2.14) for $\mathbf{H}$-enriched functors, it then suffices to show that the composite

$$\pi_0 \underline{\mathcal{C}}(GL, GL) \to \pi_0 \underline{\mathcal{D}}(FGL, L) \to \pi_0 \Big(\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{D}}(FGL, D) \Big) \to \pi_0 \Big(\varprojlim_{\mathcal{J}}^{\mathrm{BK}} \underline{\mathcal{C}}(GL, GD) \Big)$$

sends (the connected component of) the vertex id_{GL} to (the connected component of) the vertex $\underline{G}_* \lambda$; but this is a straightforward exercise in using naturality and the (right) triangle identity. ■

Definition 2.6.16. Let $\underline{\mathcal{C}}$ be a Kan-enriched category, let X be a simplicial set, and let C be an object $\mathcal{C}$.

- A **homotopy power** of C by X in $\underline{\mathcal{C}}$ is a homotopical representation of the simplicially enriched functor

$$\underline{\mathbf{sSet}}(X, \underline{\mathcal{C}}(-, C)) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$$

  i.e. a pair $(X \pitchfork C, \lambda)$ where $X \pitchfork C$ is an object in $\mathcal{C}$ and λ is a morphism $X \to \underline{\mathcal{C}}(X \pitchfork C, C)$ such that the simplicially enriched natural transformation

$$\underline{\mathcal{C}}(X \odot C, -) \Rightarrow \underline{\mathbf{sSet}}(X, \underline{\mathcal{C}}(-, C))$$

  induced by the corresponding vertex of $[X, \underline{\mathcal{C}}(X \pitchfork C, C)]$ is a simplicially enriched natural weak homotopy equivalence.

- A **homotopy copower** of C by X in $\underline{C}$ is a homotopical representation of the simplicially enriched functor

$$\underline{\mathbf{sSet}}(X, \underline{C}(C, -)) : \underline{C} \to \underline{\mathbf{sSet}}$$

i.e. a pair $(X \odot C, \lambda)$ where $X \odot C$ is an object in C and λ is a morphism $X \to \underline{C}(C, X \odot C)$ such that the simplicially enriched natural transformation

$$\underline{C}(X \odot C, -) \Rightarrow [X, \underline{C}(C, -)]$$

induced by the corresponding vertex of $[X, \underline{C}(C, X \odot C)]$ is a simplicially enriched natural weak homotopy equivalence.

Lemma 2.6.17. *Let $\mathcal{J}$ be a small category. Then there is a simplicially enriched natural isomorphism*

$$\varprojlim_{\mathcal{J}}{}^{\mathrm{BK}} \Delta(-) \cong [\mathrm{N}(\mathcal{J}), -]$$

of simplicially enriched functors $\underline{\mathbf{sSet}} \to \underline{\mathbf{sSet}}$, where $\Delta : \underline{\mathbf{sSet}} \to [\mathcal{J}, \underline{\mathbf{sSet}}]$ sends simplicial sets X to constant diagrams $\Delta X : \mathcal{J} \to \mathbf{sSet}$.

Proof. Lemma 1.9.5 says that there is a natural isomorphism of the underlying ordinary functors, and it is straightforward to verify that it is a simplicially enriched natural isomorphism. ∎

Proposition 2.6.18. *Let $\underline{C}$ be a Kan-enriched category, let $\mathcal{J}$ be a small category, and let C be an object in C.*

- *Any homotopy limit in $\underline{C}$ for the constant diagram $\Delta C : \mathcal{J} \to C$ is a homotopy power of C by $\mathrm{N}(\mathcal{J})$.*

- *Any homotopy colimit in $\underline{C}$ for the constant diagram $\Delta C : \mathcal{J} \to C$ is a homotopy copower of C by $\mathrm{N}(\mathcal{J})$.*

Proof. This is an immediate consequence of the definitions and lemma 2.6.17. ∎

Theorem 2.6.19. *Let $\underline{C}$ be a Kan-enriched category.*

- *If $\underline{C}$ has homotopy limits for all small constant diagrams, then the enriched simplicial homotopy category $\boldsymbol{\pi}[\underline{C}]$ is a cotensored $\mathbf{H}$-enriched category.*

- *If $\underline{C}$ has homotopy colimits for all small constant diagrams, then the enriched simplicial homotopy category $\boldsymbol{\pi}[\underline{C}]$ is a tensored $\mathbf{H}$-enriched category.*

Proof. The two claims are formally dual; we will prove the first version.

Let C be an object in $\mathcal{C}$ and let X be a simplicial set. We wish to show that the $\mathbf{H}$-enriched functor

$$[\boldsymbol{\pi}X, \boldsymbol{\pi}\underline{C}(-, C)] : \boldsymbol{\pi}[\underline{C}]^{\mathrm{op}} \to \mathbf{H}$$

is representable in $\boldsymbol{\pi}[\underline{C}]$. By (lemma 2.5.27 and) proposition 2.6.18, the homotopy limit of the constant diagram $\Delta C : \boldsymbol{\Delta}(X) \to \mathcal{C}$ yields a representation of the $\mathbf{H}$-enriched functor

$$\boldsymbol{\pi}\big(\underline{\mathbf{sSet}}\big(\mathrm{Sd}_{\mathrm{Q}}(X), \underline{C}(-, C)\big)\big) : \boldsymbol{\pi}[\underline{C}]^{\mathrm{op}} \to \mathbf{H}$$

but by lemma 1.11.31, there is a weak homotopy equivalence $\lambda_X : \mathrm{Sd}_{\mathrm{Q}}(X) \to X$, and by proposition 1.4.24 and corollary 1.5.25, there is a $\mathbf{H}$-enriched natural isomorphism

$$[\boldsymbol{\pi}X, \boldsymbol{\pi}\underline{C}(-, C)] \cong \boldsymbol{\pi}(\underline{\mathbf{sSet}}(X, \underline{C}(-, C)))$$

so we may conclude that the homotopy power of C by $\mathrm{Sd}_{\mathrm{Q}}(X)$ in $\underline{C}$ defines a cotensor product of X and C in $\boldsymbol{\pi}[\underline{C}]$. ∎

2.7 The Dwyer–Kan model structure

Prerequisites. §§0.2, 1.5, 1.6, 2.1, 2.3, 2.4, 2.5, 2.6, 4.1, 4.3, 5.2.

In this section, we construct a model structure on the category of small simplicially enriched categories in which the weak equivalences are the Dwyer–Kan equivalences. As a first step, following Dwyer and Kan [1980a], we construct a "local" model structure on the category of small simplicially enriched categories with a fixed set of objects. We then follow Bergner [2007] in constructing the "global" model structure on the category of all small simplicially enriched categories.

Definition 2.7.1. Let O be an ensemble.

- A **category over** O is a category $\mathcal{C}$ where $\mathrm{ob}\,\mathcal{C} = O$.

- A **functor over** O is a functor $\mathcal{C} \to \mathcal{D}$ where the map $\operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$ is $\mathrm{id} : O \to O$.

- A **simplicially enriched** (resp. **Kan-enriched**) **category over** O is a simplicially enriched (resp. Kan-enriched) category $\underline{\mathcal{C}}$ where $\operatorname{ob} \mathcal{C} = O$.

- A **simplicially enriched functor over** O is a simplicially enriched functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ where the map $\operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$ is $\mathrm{id} : O \to O$.

When O is a set, we write $\mathbf{Cat}_O$ for the category of small categories over O and $\mathbf{SCat}_O$ for the category of small simplicially enriched categories over O.

Remark 2.7.2. A simplicially enriched category over O is the essentially same thing as a simplicial category over O; so by proposition 2.3.2, 2.3.7, and 2.3.8, $\mathbf{SCat}_O$ admits a simplicial enrichment that is cotensored and tensored. There is an evident forgetful functor $\mathbf{SCat}_O \to \mathbf{sSet}^{O \times O}$ sending a simplicially enriched category $\underline{\mathcal{C}}$ over O to the $(O \times O)$-indexed family of simplicial sets $\underline{\mathcal{C}}(-, -)$, and it is not hard to see that this functor admits a compatible simplicial enrichment.

Definition 2.7.3. Let O be a set.

- A **reflexive graph over** O is a $(O \times O)$-indexed set E (i.e. an object in $\mathbf{Set}^{O \times O}$) together with a distinguished element of $E(a, a)$ for each element a of O.

- A **morphism of reflexive graphs over** O is a morphism of the underlying $(O \times O)$-indexed sets that preserves the distinguished elements.

- A **simplicially enriched reflexive graph over** O is an $(O \times O)$-indexed simplicial set E (i.e. an object in $\mathbf{sSet}^{O \times O}$) together with a distinguished vertex of $E(a, a)$ for each element a of O.

- A **morphism of simplicially enriched reflexive graphs over** O is a morphism of the underlying $(O \times O)$-indexed simplicial sets that preserves the distinguished vertices.

We write $\mathbf{Grph}_O$ for the category of reflexive graphs over O and $\mathbf{sGrph}_O$ for the category of simplicially enriched reflexive graphs over O.

Remark 2.7.4. A simplicially enriched reflexive graph over O is the essentially same thing as a simplicial reflexive graph over O; so by proposition 2.3.2, 2.3.7, and 2.3.8, $\mathbf{sGrph}_O$ admits a simplicial enrichment that is cotensored and tensored. There are evident forgetful functors $\mathbf{SCat}_O \to \mathbf{sGrph}_O$ and $\mathbf{sGrph}_O \to \mathbf{sSet}^{O \times O}$ and each one admits a compatible simplicial enrichment.

Proposition 2.7.5. *Let O be a set.*

(i) *$\mathbf{sGrph}_O$ is a locally finitely presentable category.*

(ii) *The forgetful functor $\mathbf{sGrph}_O \to \mathbf{sSet}^{O \times O}$ is $\aleph_0$-accessible and monadic.*

(iii) *The simplicially enriched forgetful functor $\underline{\mathbf{sGrph}_O} \to \underline{\mathbf{sSet}}^{O \times O}$ creates cotensor products.*

Proof. (i). We have remarked that $\mathbf{sGrph}_O$ is equivalent to the category of simplicial objects in $\mathbf{Grph}_O$, and it is not hard to see that the latter is a locally finitely presentable category. We may then apply proposition 0.2.44.

(ii). It is clear that the forgetful functor $U : \mathbf{sGrph}_O \to \mathbf{sSet}^{O \times O}$ preserves colimits for small filtered diagrams and limits for small diagrams, so we may use the accessible adjoint functor theorem (0.2.50) to construct a left adjoint. We must then verify that $U : \mathbf{sGrph}_O \to \mathbf{sSet}^{O \times O}$ creates coequalisers for U-split parallel pairs; once that is done, we may apply the well-known theorem of Beck to deduce that $U : \mathbf{sGrph}_O \to \mathbf{sGrph}^{O \times O}$ is monadic.[5]

(iii). Again, by regarding simplicially enriched reflexive graphs over O as simplicial objects in $\mathbf{Grph}_O$, we may use the formula for cotensor products given in the proof of proposition 2.3.7 to deduce that $\underline{U} : \underline{\mathbf{sGrph}_O} \to \underline{\mathbf{sSet}}^{O \times O}$ creates cotensor products. ■

Theorem 2.7.6. *Let O be a set. The following data constitute a cofibrantly generated simplicial model structure on $\mathbf{sGrph}_O$:*

- *The weak equivalences are the componentwise weak homotopy equivalences.*
- *The cofibrations are the componentwise monomorphisms.*
- *The fibrations are the componentwise Kan fibrations.*

[5] See e.g. Theorem 1 in [CWM, Ch. VI, §7].

This model structure is called the **componentwise model structure***, and with respect to this model structure, we have a Quillen adjunction*

$$F \dashv U : \mathbf{sGrph}_O \to \mathbf{sSet}^{O\times O}$$

where $U : \mathbf{sGrph}_O \to \mathbf{sSet}^{O\times O}$ *is the evident forgetful functor.*

Proof. It is not hard to see that there is an isomorphism of categories

$$\mathbf{sGrph}_O \cong \left(\prod_{a\in O} {}^{\Delta^0/}\mathbf{sSet} \right) \times \prod_{\substack{(a,b)\in O\times O \\ a\neq b}} \mathbf{sSet}$$

and the componentwise model structure so induced indeed has the required weak equivalences, cofibrations, and fibrations. Thus, the free–forgetful adjunction is indeed a Quillen adjunction. Moreover, by remark 2.7.4, the forgetful functor $U : \mathbf{sGrph}_O \to \mathbf{sSet}^{O\times O}$ admits a simplicial enrichment that preserves cotensor products, so using (proposition 1.4.15 and) proposition 2.4.4, we see that the componentwise model structure satisfies axiom SM7.

We still have to show that the componentwise model structure on $\mathbf{sGrph}_O$ is cofibrantly generated. Let $\mathcal{J}$ and $\mathcal{J}'$ be the following subsets of $\operatorname{mor} \mathbf{sSet}^{O\times O}$,

$$\mathcal{J} = \left\{ \partial\Delta^n \odot \hbar_{(a,b)} \hookrightarrow \Delta^n \odot \hbar_{(a,b)} \,\middle|\, n \geq 0, (a,b) \in O \times O \right\}$$
$$\mathcal{J}' = \left\{ \Lambda^n_k \odot \hbar_{(a,b)} \hookrightarrow \Delta^n \odot \hbar_{(a,b)} \,\middle|\, n \geq 1, 0 \leq k \leq n, (a,b) \in O \times O \right\}$$

where $\hbar_{(a,b)}$ is the $(O \times O)$-indexed set that is 1 at (a,b) and $\varnothing$ otherwise, and let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the image of $\mathcal{J}$ (resp. $\mathcal{J}'$) under $F : \mathbf{sSet}^{O\times O} \to \mathbf{sGrph}_O$. By adjointness (and the Yoneda lemma), an $\mathcal{I}$-fibration (resp. $\mathcal{I}'$-fibration) is precisely a componentwise Kan fibration (resp. componentwise trivial Kan fibration). Thus, $\mathcal{I}$ and $\mathcal{I}'$ cofibrantly generate the componentwise model structure on $\mathbf{sGrph}_O$. ■

Proposition 2.7.7. *Let O be a set.*

(i) $\mathbf{SCat}_O$ *is a locally finitely presentable category.*

(ii) *The forgetful functor* $\mathbf{SCat}_O \to \mathbf{sGrph}_O$ *is* $\aleph_0$*-accesible and monadic.*

(iii) *The simplicially enriched forgetful functor* $\underline{\mathbf{SCat}_O} \to \underline{\mathbf{sGrph}_O}$ *creates cotensor products.*

(iv) *The forgetful functor* $\mathbf{SCat}_O \to \mathbf{sSet}^{O \times O}$ *is* $\aleph_0$*-accesible and monadic.*

(v) *The simplicially enriched forgetful functor* $\underline{\mathbf{SCat}}_O \to \underline{\mathbf{sSet}}^{O \times O}$ *creates cotensor products.*

Proof. (i). We have remarked that $\mathbf{SCat}_O$ is equivalent to the category of simplicial objects in $\mathbf{Cat}_O$, and it is not hard to see that the latter is a locally finitely presentable category. We may then apply proposition 0.2.44.

(ii). It is clear that the forgetful functor $U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$ preserves colimits for small filtered diagrams and limits for small diagrams, so we may use the accessible adjoint functor theorem (0.2.50) to construct a left adjoint. We must then verify that $U : \mathbf{SCat}_O \to \mathbf{sSet}^{O \times O}$ creates coequalisers for U-split parallel pairs; once that is done, we may apply the well-known theorem of Beck to deduce that $U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$ is monadic.[6]

(iii). Again, by regarding small simplicially enriched categories over O as simplicial objects in $\mathbf{Cat}_O$, we may use the formula for cotensor products given in the proof of proposition 2.3.7 to deduce that $\underline{U} : \underline{\mathbf{SCat}}_O \to \underline{\mathbf{sGrph}}_O$ creates cotensor products.

(iv) and (v). Similar arguments work. ■

Definition 2.7.8.

- A **local fibration of simplicially enriched categories** is a simplicially enriched functor $\underline{P} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ with the following property: for all pairs (A, B) of objects in $\mathcal{C}$, the morphism $\underline{P}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(PA, PB)$ is a Kan fibration.

- Let O be an ensemble. A **fibration of simplicially enriched categories over** O is a simplicially enriched functor over O that is also a local fibration of simplicially enriched categories.

Theorem 2.7.9 (Dwyer and Kan)**.** *Let O be a set. The following data constitute a cofibrantly generated simplicial model structure on* $\underline{\mathbf{SCat}}_O$*:*

- *The weak equivalences are the Dwyer–Kan equivalences.*

[6] See e.g. Theorem 1 in [CWM, Ch. VI, §7].

- *The fibrations are the fibrations of simplicially enriched categories over $\mathcal{O}$.*

- *The cofibrations are the morphisms that have the left lifting property with respect to the fibrations.*

This model structure is called the **Dwyer–Kan model structure***, and the fibrant objects are the Kan-enriched categories over $\mathcal{O}$. With respect to this model structure, we have a Quillen adjunction*

$$F \dashv U : \mathbf{SCat}_{\mathcal{O}} \to \mathbf{sSet}^{\mathcal{O}\times\mathcal{O}}$$

where $U : \mathbf{SCat}_{\mathcal{O}} \to \mathbf{sSet}^{\mathcal{O}\times\mathcal{O}}$ is the evident forgetful functor.

Proof. First, we will use Kan's lifting theorem (5.2.5) to verify that the data indeed constitute a cofibrantly generated model structure on $\mathbf{SCat}_{\mathcal{O}}$ compatible with the indicated free–forgetful adjunction.

By proposition 2.7.7, the forgetful functor admits a simplicial enrichment that preserves cotensor products. Let $\mathcal{J}$ and $\mathcal{J}'$ be the following subsets of $\operatorname{mor} \mathbf{sSet}^{\mathcal{O}\times\mathcal{O}}$,

$$\begin{aligned} \mathcal{J} &= \left\{ \partial\Delta^n \odot \hbar_{(a,b)} \hookrightarrow \Delta^n \odot \hbar_{(a,b)} \,\middle|\, n \geq 0, (a,b) \in \mathcal{O}\times\mathcal{O} \right\} \\ \mathcal{J}' &= \left\{ \Lambda^n_k \odot \hbar_{(a,b)} \hookrightarrow \Delta^n \odot \hbar_{(a,b)} \,\middle|\, n \geq 1, 0 \leq k \leq n, (a,b) \in \mathcal{O}\times\mathcal{O} \right\} \end{aligned}$$

where $\hbar_{(a,b)}$ is the $(\mathcal{O}\times\mathcal{O})$-indexed set that is 1 at (a,b) and $\emptyset$ otherwise, and let $\mathcal{I}$ (resp. $\mathcal{I}'$) be the image of $\mathcal{J}$ (resp. $\mathcal{J}'$) under $F : \mathbf{sSet}^{\mathcal{O}\times\mathcal{O}} \to \mathbf{SCat}_{\mathcal{O}}$. Since $\mathbf{SCat}_{\mathcal{O}}$ is locally finitely presentable, by remark 0.5.9 and Quillen's small object argument (theorem 0.5.12), there exist functorial weak factorisation systems on $\mathbf{SCat}_{\mathcal{O}}$ cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$; and by adjointness (and the Yoneda lemma), an $\mathcal{I}$-fibration (resp. $\mathcal{I}'$-fibration) is precisely a fibration (resp. trivial fibration) of simplicially enriched categories over $\mathcal{O}$. It remains to be shown that $\mathcal{I}'$-cofibrations are Dwyer–Kan equivalences; but this is a straightforward application of lemma 2.4.18 to theorem 2.5.23.

To complete the proof, we must verify that the above model structure satisfies axiom SM7. But $\underline{U} : \underline{\mathbf{SCat}}_{\mathcal{O}} \to \underline{\mathbf{sSet}}^{\mathcal{O}\times\mathcal{O}}$ preserves cotensor products, so by proposition 2.4.4, this is an immediate consequence of the fact that $\underline{\mathbf{sSet}}^{\mathcal{O}\times\mathcal{O}}$ satisfies axiom SM7. ■

Corollary 2.7.10. *Let O be a set. There is a Quillen adjunction*

$$F \dashv U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$$

where $U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$ is the evident forgetful functor.

Proof. Clearly, $U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$ preserves fibrations and trivial fibrations, so by proposition 4.3.2, we have a Quillen adjunction. ■

Proposition 2.7.11. *Let O be a set and let $\mathcal{W}$ be the full subcategory of $[\mathbb{2}, \mathbf{SCat}_O]$ spanned by the Dwyer–Kan equivalences. Then $\mathcal{W}$ is closed under colimits for small filtered diagrams in $[\mathbb{2}, \mathbf{SCat}_O]$.*

Proof. The forgetful functor $U : \mathbf{SCat}_O \to \mathbf{sSet}^{O \times O}$ preserves weak equivalences and colimits for small filtered diagrams and also reflects weak equivalences, so this is a corollary of proposition 1.5.12. ■

Definition 2.7.12. Let O be a set and let $F : \mathbf{sSet}^{O \times O} \to \mathbf{SCat}_O$ be the free simplicially enriched category over O functor. A **standard cofibration** in $\mathbf{SCat}_O$ is a monomorphism $\underline{f} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ for which there exist a chain of monomorphisms in $\mathbf{SCat}_O$

$$\underline{\mathcal{C}} = \underline{\mathcal{D}}^{(-1)} \xrightarrow{\underline{i}^{(0)}} \underline{\mathcal{D}}^{(0)} \xrightarrow{\underline{i}^{(1)}} \underline{\mathcal{D}}^{(1)} \xrightarrow{\underline{i}^{(2)}} \underline{\mathcal{D}}^{(2)} \longrightarrow \cdots$$

such that the following conditions are satisfied:

- There is a colimiting cocone from the above chain to $\underline{\mathcal{D}}$ where the component $\underline{\mathcal{D}}^{(-1)} \to \underline{\mathcal{D}}$ is $\underline{f} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.

- For each natural number n, there is a pushout diagram of the form below,

$$\begin{tikzcd} F\left(\partial\Delta^n \odot I_n\right) \arrow[r, hook] \arrow[d] & F\left(\Delta^n \odot I_n\right) \arrow[d] \\ \underline{\mathcal{D}}^{(n-1)} \arrow[r, "\underline{i}^{(n)}"'] & \underline{\mathcal{D}}^{(n)} \end{tikzcd}$$

  where I_n is an $(O \times O)$-indexed subset of $\underline{\mathcal{D}}(-,-)_n$ *not* meeting the image of $\underline{f} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, $F\left(\partial\Delta^n \odot I_n\right) \hookrightarrow F\left(\Delta^n \odot I_n\right)$ is induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, and $F\left(\Delta^n \odot I_n\right) \to \underline{\mathcal{D}}^{(n)}$ is the tautological simplicially enriched functor induced by the inclusion $I_n \hookrightarrow \underline{\mathcal{D}}(-,-)_n$.

The following lemma implies that every instance of the word 'monomorphism' in the above definition can be replaced by 'morphism'.

Lemma 2.7.13. *Let O be a set, let n be a natural number, and let $f : X \to Y$ be a monomorphism in $\mathbf{sSet}^{O \times O}$. Given a pushout diagram in $\mathbf{SCat}_O$ of the form below,*

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d] & FY \arrow[d] \\ \underline{C} \arrow[r, "\underline{i}"'] & \underline{D} \end{tikzcd}$$

the simplicially enriched functor $\underline{i} : \underline{C} \to \underline{D}$ is a monomorphism in $\mathbf{SCat}_O$.

Proof. It is enough to show that each $\left(\underline{i}_{a,b}\right)_n : \underline{C}(a,b)_n \to \underline{D}(a,b)_n$ is injective, and since colimits in $\mathbf{SCat}_O$ can be computed degreewise, it suffices to prove the analogous claim for categories over O. But every $(O \times O)$-indexed injective map is (isomorphic to) a coproduct insertions, and the free category over O functor $\mathbf{Set}^{O \times O} \to \mathbf{Cat}_O$ preserves coproducts, so it suffices to show that coproduct insertions in $\mathbf{Cat}_O$ are monic. This is clear if we think in terms of generators and relations. ■

Proposition 2.7.14. *Let O be a set.*

(i) *Every standard cofibration in $\mathbf{SCat}_O$ is a cofibration in the Dwyer–Kan model structure.*

(ii) *Every morphism in $\mathbf{SCat}_O$ can be factored as a standard cofibration followed by a trivial fibration.*

(iii) *Every cofibration in $\mathbf{SCat}_O$ is a retract of a standard cofibration.*

Proof. (i). Let $\mathcal{I}$ be the following subset of $\operatorname{mor} \mathbf{SCat}_O$:

$$\mathcal{I} = \left\{ F\left(\partial\Delta^n \odot \mathbb{k}_{(a,b)}\right) \hookrightarrow F\left(\Delta^n \odot \mathbb{k}_{(a,b)}\right) \,\middle|\, n \geq 0, (a,b) \in O \times O \right\}$$

It is clear that standard cofibrations in $\mathbf{SCat}_O$ are relative $\mathcal{I}$-cell complexes. We previously saw that the $\mathcal{I}$-injective morphisms are precisely the trivial fibrations in the Dwyer–Kan model structure on $\mathbf{SCat}_O$, so relative $\mathcal{I}$-cell complexes are cofibrations.

(ii). A variation on Quillen's small object argument (theorem 0.5.12) applied to $\mathcal{I}$ can be used here.

(iii). Let $\underline{f} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a cofibration. There is a factorisation of the form $\underline{f} = \underline{p} \circ \underline{i}$ where $\underline{p}$ is a trivial fibration and $\underline{i}$ is a standard cofibration; but $\underline{f}$ has the left lifting property with respect to $\underline{p}$, so $\underline{f}$ is a retract of $\underline{i}$. ■

Corollary 2.7.15. *Let O be a set. Every cofibration in* $\mathbf{SCat}_O$ *in the Dwyer–Kan model structure is a monomorphism.* ■

¶ 2.7.16. Let us write Δ_O for the reflexive graph over O defined by the following formula:

$$\Delta_O(a, b) = \begin{cases} 1 & \text{if } a = b \\ \varnothing & \text{if } a \neq b \end{cases}$$

Clearly, Δ_O is an initial object in $\mathbf{Grph}_O$. It admits the structure of a category over O in a unique way and is also an initial object in $\mathbf{Cat}_O$. As there is no danger of confusion, we will also write Δ_O for the initial objects in $\mathbf{sGrph}_O$ and $\mathbf{SCat}_O$.

Lemma 2.7.17. *Let $\underline{\mathcal{C}}$ be a small simplicially enriched category and let $O = \operatorname{ob} \mathcal{C}$. Then the unique morphism $\Delta_O \to \underline{\mathcal{C}}$ in* $\mathbf{SCat}_O$ *is a standard cofibration if there exist $(O \times O)$-indexed subsets $J_n \subseteq \underline{\mathcal{C}}(-,-)_n$ satisfying the following conditions:*

- *Regarding $\underline{\mathcal{C}}$ as a simplicial category $\mathcal{C}_\bullet$, the tautological functor $F(J_n) \to \mathcal{C}_n$ induced by the inclusion $J_n \hookrightarrow \underline{\mathcal{C}}(-,-)_n$ is an isomorphism (in* $\mathbf{Cat}_O$*).*
- *For $0 \le k \le n$, the degeneracy operator $s_k : \underline{\mathcal{C}}(-,-)_n \to \underline{\mathcal{C}}(-,-)_{n+1}$ sends elements of J_n into J_{n+1}.*

In particular, for any object X in $\mathbf{sGrph}_O$*, the unique morphism $\Delta_O \to F(X)$ is a standard cofibration.*

Proof. Let I_n be the intersection of J_n and the set of non-degenerate n-simplices of $\underline{\mathcal{C}}(-,-)$ and let $\underline{\mathcal{C}}^{(n)}$ be the simplicially enriched subcategory of $\underline{\mathcal{C}}$ generated by (all) the n-simplices. Let $\underline{\mathcal{C}}^{(-1)} = \Delta_O$. There is an evident commutative diagram in $\mathbf{SCat}_O$ of the form below,

$$\begin{array}{ccc} F(\partial\Delta^n \odot I_n) & \hookrightarrow & F(\Delta^n \odot I_n) \\ \downarrow & & \downarrow \\ \underline{\mathcal{C}}^{(n-1)} & \hookrightarrow & \underline{\mathcal{C}}^{(n)} \end{array}$$

and it is not hard to see that it is a pushout diagram. Clearly, $\bigcup_{n \geq 0} \underline{\mathcal{C}}^{(n)} = \underline{\mathcal{C}}$, so the unique morphism $\underline{\mathcal{C}}^{(-1)} \to \underline{\mathcal{C}}$ is indeed a standard cofibration, as claimed. Finally, if $\underline{\mathcal{C}} = F(X)$, then we can take $J_n = X_n \setminus \Delta_O$, so the unique morphism $\Delta_O \to F(X)$ is a standard cofibration. ■

Recalling lemma 1.6.8, we see that the realisation functor $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ preserves finite products. In particular, the following definition makes sense:

TODO: What is the relationship between this and realisation in the general sense?

Definition 2.7.18. Let O be a set and let $\underline{\mathcal{C}}_\bullet$ be a simplicial object in $\mathbf{SCat}_O$. The **realisation** of $\underline{\mathcal{C}}_\bullet$ is the simplicially enriched category $\underline{\mathcal{D}}$ defined by the following formula:

$$\underline{\mathcal{D}}(a, b) = \left|\underline{\mathcal{C}}_\bullet(a, b)\right|$$

In other words, $\underline{\mathcal{D}}$ is $\left|\left[\underline{\mathcal{C}}_\bullet\right]\right|$ where we regard $\underline{\mathcal{C}}_\bullet$ as a category enriched over $\mathbf{ssSet}$.

Proposition 2.7.19. *Let O be a set and let $\underline{f}_\bullet : \underline{\mathcal{C}}_\bullet \to \underline{\mathcal{D}}_\bullet$ be a morphism of simplicial objects in $\mathbf{SCat}_O$. If each $\underline{f}_n : \underline{\mathcal{C}}_n \to \underline{\mathcal{D}}_n$ is a Dwyer–Kan equivalence, then $\left|\left[\underline{f}_\bullet\right]\right| : \left|\left[\underline{\mathcal{C}}_\bullet\right]\right| \to \left|\left[\underline{\mathcal{D}}_\bullet\right]\right|$ is also a Dwyer–Kan equivalence.*

Proof. This is a straightforward corollary of theorem 1.6.10. ■

Definition 2.7.20. Let $\mathcal{C}$ be a small category and let $O = \operatorname{ob} \mathcal{C}$. The **standard resolution** of $\mathcal{C}$ is the standard resolution of $\mathcal{C}$ (as an object in $\mathbf{Cat}_O$) with respect to the comonad induced by the free–forgetful adjunction between $\mathbf{Cat}_O$ and $\mathbf{Grph}_O$.

Remark 2.7.21. The fact that the standard resolution of $\mathcal{C}$ is stable under universe enlargement is an instance of the stability of accessible adjunctions.

Remark 2.7.22. Although the standard resolution $\mathbf{S}(\mathbb{C})_\bullet$ of a category $\mathbb{C}$ is most naturally defined as a simplicial category, the fact that $\operatorname{ob} \mathbf{S}(\mathbb{C})_\bullet$ is a constant simplicial set enables us to view it as a simplicially enriched category $\underline{\mathbf{S}}(\mathbb{C})$, per remark 2.1.11.

Proposition 2.7.23. *For any small category $\mathbb{C}$, the standard augmentation $\varepsilon_{\mathbb{C}} : \mathbf{S}(\mathbb{C}) \to \mathbb{C}$ is a Dwyer–Kan equivalence of simplicially enriched categories.*

Proof. Recalling proposition 1.3.20 (and proposition 1.5.3), this is a special case of proposition 2.3.20. ■

Corollary 2.7.24. *The functor $\pi_0[\underline{\mathbf{S}}(\mathbb{C})] \to \mathbb{C}$ induced by the standard augmentation is an isomorphism of categories.* ■

Definition 2.7.25. Let $\underline{C}$ be a small simplicially enriched category and let $O = \operatorname{ob} C$.

- The **standard resolution** of $\underline{C}$ is the standard resolution of $\underline{C}$ (as an object in $\mathbf{SCat}_O$) with respect to the comonad induced by the free–forgetful adjunction between $\mathbf{SCat}_O$ and $\mathbf{sGrph}_O$.
- The **degreewise standard resolution** of $\underline{C}$ is $\underline{\mathbf{S}}(C_\bullet)$ together with the evident augmentation, where $C_\bullet$ is $\underline{C}$ considered as a simplicial category and $\underline{\mathbf{S}}(C_n)$ is the standard resolution of C_n (as an object in $\mathbf{Cat}_O$, with respect to the comonad induced by the free–forgetful adjunctino between $\mathbf{Cat}_O$ and $\mathbf{Grph}_O$) considered as a simplicially enriched category.

Remark 2.7.26. The standard resolution and the degreewise standard resolution are related as follows (up to natural bijection):

$$\mathbf{S}(\underline{C})_n(a, b)_m = \underline{\mathbf{S}}(C_m)(a, b)_n$$

In particular, lemma 1.6.8 implies that their realisations are naturally isomorphic.

Proposition 2.7.27. *Let $\underline{C}$ be a small simplicially enriched category and let $O = \operatorname{ob} C$.*

(i) *The degreewise standard augmentation $\underline{\mathbf{S}}(C_\bullet) \to C_\bullet$ is a degreewise Dwyer–Kan equivalence.*

(ii) *The realisation of the standard augmentation is a Dwyer–Kan equivalence $\left|[\mathbf{S}(\underline{C})_\bullet]\right| \to \underline{C}$.*

(iii) *For each natural number n, the unique morphism $\Delta_O \to \mathbf{S}(\underline{C})_n$ in $\mathbf{SCat}_O$ is a standard cofibration.*

(iv) *For each natural number n, the unique morphism $\Delta_O \to \underline{\mathbf{S}}(C_n)$ in $\mathbf{SCat}_O$ is a standard cofibration.*

(v) *The unique morphism $\Delta_O \to \left|[\mathbf{S}(\underline{C})_\bullet]\right|$ in $\mathbf{SCat}_O$ is a standard cofibration.*

Proof. (i). See proposition 2.7.23.

(ii). Thus, by proposition 2.7.19 and remark 2.7.26, the induced morphisms

$$\left|\mathbf{S}(\underline{C})_\bullet(a, b)\right| \to \underline{C}(a, b)$$

are weak homotopy equivalences.

(iii)–(v). This is a straightforward application of lemma 2.7.17. ∎

Proposition 2.7.28. *Let O be a set. For any Dwyer–Kan equivalence $\underline{f} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ in $\mathbf{SCat}_O$, the morphism $\mathbf{S}(\underline{f})_\bullet : \mathbf{S}(\underline{\mathcal{C}})_\bullet \to \mathbf{S}(\underline{\mathcal{D}})_\bullet$ of standard resolutions is a degreewise Dwyer–Kan equivalence.*

Proof. Corollary 2.7.10 says that $F : \mathbf{sGrph}_O \to \mathbf{SCat}_O$ is a left Quillen functor, and since all objects in $\mathbf{sGrph}_O$ are cofibrant, Ken Brown's lemma (4.3.6) implies that $F : \mathbf{sGrph}_O \to \mathbf{SCat}_O$ preserves weak equivalences. On the other hand, $U : \mathbf{SCat}_O \to \mathbf{sGrph}_O$ preserves weak equivalences by definition. Thus, the morphism $\mathbf{S}(\underline{f})_\bullet : \mathbf{S}(\underline{\mathcal{C}})_\bullet \to \mathbf{S}(\underline{\mathcal{D}})_\bullet$ is a degreewise Dwyer–Kan equivalence. ∎

Lemma 2.7.29. *Let O be a set.*

(i) *The forgetful functor $U : \mathbf{Cat}_O \to \mathbf{Grph}_O$ preserves and reflects monomorphisms.*

(ii) *The coproduct functor $(-) +^O (-) : \mathbf{Cat}_O \times \mathbf{Cat}_O \to \mathbf{Cat}_O$ preserves monomorphisms.*

(iii) *Let $\mathbf{Cat}_{O,\mathrm{m}}$ (resp. $\mathbf{Grph}_{O,\mathrm{m}}$) be the subcategory of $\mathbf{Cat}_O$ (resp. $\mathbf{Grph}_O$) consisting of the monomorphisms. There exists a functor*

$$(-) +^O (-) : \mathbf{Grph}_{O,\mathrm{m}} \times \mathbf{Grph}_{O,\mathrm{m}} \to \mathbf{Grph}_{O,\mathrm{m}}$$

making the following diagram commute up to isomorphism:

$$\begin{tikzcd} \mathbf{Cat}_{O,\mathrm{m}} \times \mathbf{Cat}_{O,\mathrm{m}} \arrow[r, "(-)+^O(-)"] \arrow[d, "U\times U"'] & \mathbf{Cat}_{O,\mathrm{m}} \arrow[d, "U"] \\ \mathbf{Grph}_{O,\mathrm{m}} \times \mathbf{Grph}_{O,\mathrm{m}} \arrow[r, dashed, "(-)+^O(-)"'] & \mathbf{Grph}_{O,\mathrm{m}} \end{tikzcd}$$

Moreover, this functor is equipped with natural monomorphisms $X \to X +^O Y$ and $Y \to X +^O Y$, and these are compatible with the coproduct insertions in $\mathbf{Cat}_O$.

Proof. (i). By (the proof of) proposition 2.7.7, $U : \mathbf{Cat}_O \to \mathbf{Grph}_O$ is monadic, and any monadic functor preserves and reflects monomorphisms.

(ii). Let $\mathcal{C}$ and $\mathcal{D}$ be objects in $\mathbf{Cat}_O$. It is not hard to see that morphisms in the coproduct $\mathcal{C} +^O \mathcal{D}$ admit a *unique* factorisation of the form

$$o_0 \longrightarrow \cdots \longrightarrow o_k$$

where k is a natural number (possibly zero!), each arrow is a *non-identity* morphism in $\mathcal{C}$ or $\mathcal{D}$, and no two adjacent arrows are in the same category. (The coproduct insertions $\mathcal{C} \to \mathcal{C} +^O \mathcal{D}$ and $\mathcal{D} \to \mathcal{C} +^O \mathcal{D}$ are monic, so by abuse of notation, we identify $\mathcal{C}$ and $\mathcal{D}$ with their images in $\mathcal{C} +^O \mathcal{D}$.) It is clear what the action of $(-) +^O (-)$ on morphisms is, and by considering the factorisations discussed above, it is easy to see that $(-) +^O (-)$ preserves monomorphisms.

(iii). Let X and Y be objects in $\mathbf{Grph}_O$. In view of the above description of $\mathcal{C} +^O \mathcal{D}$, let us define $X +^O Y$ to be the reflexive graph whose edges are (finite) paths

$$o_0 \longrightarrow \cdots \longrightarrow o_k$$

where k is a natural number (possibly zero!), each arrow is a *non-distinguished* edge of X or Y, and no two adjacent arrows are in the same graph. It is then clear how to extend $(-) +^O (-)$ to a functor $\mathbf{Grph}_{O,\mathrm{m}} \times \mathbf{Grph}_{O,\mathrm{m}} \to \mathbf{Grph}_{O,\mathrm{m}}$ of the required form, and there are evident natural monomorphisms $X \to X +^O Y$ and $Y \to X +^O Y$ compatible with the coproduct insertions in $\mathbf{Cat}_O$. ■

Lemma 2.7.30. *Let O be a set and let $\mathcal{C}$ and $\mathcal{D}$ be objects in $\mathbf{Cat}_O$. If $\underline{g} : \underline{\mathbf{S}}(\mathcal{D}) \to \mathrm{disc}[\mathcal{D}]$ is the standard augmentation (regarded as a morphism in $\mathbf{SCat}_O$), then the coproduct morphism $\mathrm{id}_{\mathrm{disc}[\mathcal{C}]} +^O \underline{g} : \mathrm{disc}[\mathcal{C}] +^O \underline{\mathbf{S}}(\mathcal{D}) \to \mathrm{disc}[\mathcal{C}] +^O \mathrm{disc}[\mathcal{D}]$ is a Dwyer–Kan equivalence in $\mathbf{SCat}_O$.*

Proof. Let $U : \mathbf{Cat}_O \to \mathbf{Grph}_O$ be the forgetful functor and let $r = U g_0 : U\mathbf{S}(\mathcal{D})_0 \to U\mathcal{D}$. By proposition 2.3.20, there exist morphisms $s : U\mathcal{D} \to U\mathbf{S}(\mathcal{D})_0$ and $h^n : U\mathbf{S}(\mathcal{D})_n \to U\mathbf{S}(\mathcal{D})_{n+1}$ constituting a forward contracting homotopy for $U\mathbf{S}(\mathcal{D})_\bullet$. To prove the claim, (by propositions 1.3.20 and 1.5.3) it suffices to show that $\mathcal{C} +^O \mathbf{S}(\mathcal{D})_\bullet$ admits a forward contracting homotopy corresponding to the morphism $U\left(\mathrm{id}_\mathcal{C} +^O g_\bullet\right) : U\left(\mathcal{C} +^O \mathbf{S}(\mathcal{D})_\bullet\right) \to U\left(\mathcal{C} +^O \mathcal{D}\right)$.

By definition, the morphisms $s : U\mathcal{D} \to U\mathbf{S}(\mathcal{D})_0$ and $h^n : U\mathbf{S}(\mathcal{D})_n \to U\mathbf{S}(\mathcal{D})_{n+1}$ are (split) monomorphisms in $\mathbf{Grph}_O$, so by lemma 2.7.29, we may apply $\mathrm{id}_{UC} +^{O} (-)$ to them. If we identify $UC +^{O} U\mathbf{S}(\mathcal{D})_{\bullet}$ with $U\big(C +^{O} \mathbf{S}(\mathcal{D})_{\bullet}\big)$, then we get the following identities for free:

$$
\begin{aligned}
U\big(\mathrm{id}_C +^{O} g_0\big) \circ U\big(\mathrm{id}_C +^{O} d^1_1\big) &= U\big(\mathrm{id}_C +^{O} g_0\big) \circ U\big(\mathrm{id}_C +^{O} d^1_0\big) \\
\big(\mathrm{id}_{UC} +^{O} h^{n+1}\big) \circ U\big(\mathrm{id}_C +^{O} s^n_i\big) &= U\big(\mathrm{id}_C +^{O} s^{n+1}_i\big) \circ \big(\mathrm{id}_{UC} +^{O} h^n\big) \quad \text{if } 0 \le i \le n \\
\big(\mathrm{id}_{UC} +^{O} h^{n+1}\big) \circ \big(\mathrm{id}_{UC} +^{O} h^n\big) &= U\big(\mathrm{id}_C +^{O} s^{n+1}_{n+1}\big) \circ \big(\mathrm{id}_{UC} +^{O} h^n\big)
\end{aligned}
$$

To complete the proof, we must verify the equations shown below:

$$
\begin{aligned}
U\big(\mathrm{id}_C +^{O} g_0\big) \circ \big(\mathrm{id}_{UC} +^{O} s\big) &= \mathrm{id} \\
U\big(\mathrm{id}_C +^{O} d^1_0\big) \circ \big(\mathrm{id}_{UC} +^{O} h^0\big) &= \big(\mathrm{id}_{UC} +^{O} s\big) \circ U\big(\mathrm{id}_C +^{O} g_0\big) \\
U\big(\mathrm{id}_C +^{O} d^1_1\big) \circ \big(\mathrm{id}_{UC} +^{O} h^0\big) &= \mathrm{id} \\
U\big(\mathrm{id}_C +^{O} d^{n+1}_i\big) \circ \big(\mathrm{id}_{UC} +^{O} h^n\big) &= \big(\mathrm{id}_{UC} +^{O} h^{n-1}\big) \circ U\big(\mathrm{id}_C +^{O} d^n_i\big) \qquad \text{if } 0 \le i \le n \\
U\big(\mathrm{id}_C +^{O} d^{n+1}_{n+1}\big) \circ \big(\mathrm{id}_{UC} +^{O} h^n\big) &= \mathrm{id}
\end{aligned}
$$

The verification is straightforward and is omitted. ■

Lemma 2.7.31. *Let O be a set. The coproduct functor*

$$(-) +^{O} (-) : \mathbf{SCat}_O \times \mathbf{SCat}_O \to \mathbf{SCat}_O$$

preserves weak equivalences.

Proof. Since the class of Dwyer–Kan equivalences is closed under composition, by symmetry, it suffices to verify that $\underline{C} +^{O} (-) : \mathbf{SCat}_O \to \mathbf{SCat}_O$ preserves Dwyer–Kan equivalences for an arbitrary object $\underline{C}$ in $\mathbf{SCat}_O$.

First, suppose that $\underline{C} = F(\operatorname{disc} X)$ for some $(O \times O)$-indexed set X. Then, for any object $\underline{D}$ in $\mathbf{SCat}_O$, we have the following formula,

$$\big(F(\operatorname{disc} X) +^{O} \underline{D}\big)(a, b) = \coprod_{k \ge 0} Y^{(k)}(a, b)$$

where $Y^{(0)}(a, b) = \underline{D}(a, b)$ and in general:

$$Y^{(k+1)}(a, b) = \coprod_{(a', b') \in O \times O} Y^{(k)}(a, a') \times \operatorname{disc} X(a', b') \times \underline{D}(b', b)$$

In other words, every n-simplex of $\big(F(\operatorname{disc} X) +^{O} \underline{D}\big)(a, b)$ admits a *unique* factorisation of the form

$$a \xrightarrow{f_1} \bullet \longrightarrow \cdots \longrightarrow \bullet \xrightarrow{f_{2k+1}} b$$

where f_j is in $\underline{\mathcal{D}}$ when j is odd and in X if j is even. It is then clear that $F(\operatorname{disc} X) +^{O} (-)$ preserves Dwyer–Kan equivalences.

Now, suppose $\underline{\mathcal{C}}$ is degreewise free, i.e. regarded as a simplicial category $\mathcal{C}_\bullet$, for each natural number n, there is an $(X \times X)$-indexed set X_n such that $\mathcal{C}_n = F(X_n)$. Let $\underline{g} : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ be a Dwyer–Kan equivalence in $\mathbf{SCat}_O$. The above argument shows that, for each natural number n,

$$\mathrm{id}_{F(\operatorname{disc} X_n)} +^{O} \underline{g} : F(\operatorname{disc} X_n) +^{O} \underline{\mathcal{D}} \to F(\operatorname{disc} X_n) +^{O} \underline{\mathcal{E}}$$

But $\underline{\mathcal{C}}$ is (isomorphic to) the realisation of the simplicial object $\operatorname{disc}[\mathcal{C}_\bullet]$, so by proposition 2.7.19, the simplicially enriched functor

$$\mathrm{id}_{\underline{\mathcal{C}}} +^{O} \underline{g} : \underline{\mathcal{C}} +^{O} \underline{\mathcal{D}} \to \underline{\mathcal{C}} +^{O} \underline{\mathcal{E}}$$

is a Dwyer–Kan equivalence.

Next, let $\underline{\mathcal{C}}$ be any object in $\mathbf{SCat}_O$. Let $\mathcal{D}$ be any object in $\mathbf{SCat}_O$, and consider the degreewise standard augmentation $\underline{g}_\bullet : \underline{\mathbf{S}}(\mathcal{D}_\bullet) \to \operatorname{disc}[\mathcal{D}_\bullet]$. By lemma 2.7.30,

$$\mathrm{id}_{\operatorname{disc}[\mathcal{C}_\bullet]} +^{O} \underline{g}_\bullet : \operatorname{disc}[\mathcal{C}_\bullet] +^{O} \underline{\mathbf{S}}(\mathcal{D}_\bullet) \to \operatorname{disc}[\mathcal{C}_\bullet] +^{O} \operatorname{disc}[\mathcal{D}_\bullet]$$

is a degreewise Dwyer–Kan equivalence, so applying proposition 2.7.19 again, we deduce that

$$\mathrm{id}_{\underline{\mathcal{C}}} +^{O} \left|[\underline{g}_\bullet]\right| : \underline{\mathcal{C}} +^{O} \left|[\mathbf{S}(\underline{\mathcal{D}})_\bullet]\right| \to \underline{\mathcal{C}} +^{O} \underline{\mathcal{D}}$$

is a Dwyer–Kan equivalence.

Finally, let $\underline{g} : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ be any Dwyer–Kan equivalence in $\mathbf{SCat}_O$. Consider the standard resolution of $\underline{g}$ and the *degreewise* standard resolution of $\underline{\mathcal{C}}$. We have the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathbf{S}}(\mathcal{C}_n) +^{O} \mathbf{S}(\underline{\mathcal{D}})_n \arrow[r] \arrow[d, "\mathrm{id}_{\underline{\mathbf{S}}(\mathcal{C}_n)} +^{O} \mathbf{S}(\underline{g})_n"'] & \operatorname{disc}[\mathcal{C}_n] +^{O} \mathbf{S}(\underline{\mathcal{D}})_n \arrow[d, "\mathrm{id}_{\operatorname{disc}[\mathcal{C}_n]} +^{O} \mathbf{S}(\underline{g})_n"] \\
\underline{\mathbf{S}}(\mathcal{C}_n) +^{O} \mathbf{S}(\underline{\mathcal{E}})_n \arrow[r] & \operatorname{disc}[\mathcal{C}_n] +^{O} \mathbf{S}(\underline{\mathcal{E}})_n
\end{tikzcd}$$

The simplicially enriched categories $\mathbf{S}(\underline{\mathcal{D}})_n$ and $\mathbf{S}(\underline{\mathcal{E}})_n$ are degreewise free, so by proposition 2.7.27 and our earlier argument, the horizontal arrows in the diagram are Dwyer–Kan equivalences. On the other hand, by proposition 2.7.28,

$\mathbf{S}(\underline{g})_n : \mathbf{S}(\underline{\mathcal{D}})_n \to \mathbf{S}(\underline{\mathcal{E}})_n$ is a Dwyer–Kan equivalence; and $\underline{\mathbf{S}}(\mathcal{C}_n)$ is degreewise free, so by the same argument again, the left vertical arrow in the diagram is a Dwyer–Kan equivalence. Thus, using the 2-out-of-3 property, we deduce that the right vertical arrow is a Dwyer–Kan equivalence, so we have a degreewise Dwyer–Kan equivalence

$$\mathrm{id}_{\mathrm{disc}[\mathcal{C}_\bullet]} +^O \mathbf{S}(\underline{g})_\bullet : \mathrm{disc}[\mathcal{C}_\bullet] +^O \mathbf{S}(\underline{\mathcal{D}})_\bullet \to \mathrm{disc}[\mathcal{C}_\bullet] +^O \mathbf{S}(\underline{\mathcal{E}})_\bullet$$

and applying proposition 2.7.19, we deduce that

$$\mathrm{id}_{\underline{\mathcal{C}}} +^O \big|[\mathbf{S}(\underline{g})_\bullet]\big| : \underline{\mathcal{C}} +^O \big|[\mathbf{S}(\underline{\mathcal{D}})_\bullet]\big| \to \underline{\mathcal{C}} +^O \big|[\mathbf{S}(\underline{\mathcal{E}})_\bullet]\big|$$

is also a Dwyer–Kan equivalence. But the following diagram commutes,

$$\begin{tikzcd}
\underline{\mathcal{C}} +^O \big|[\mathbf{S}(\underline{\mathcal{D}})_\bullet]\big| \arrow[r] \arrow[d, "\mathrm{id}_{\underline{\mathcal{C}}} +^O |[\mathbf{S}(\underline{g})_\bullet]|"'] & \underline{\mathcal{C}} +^O \underline{\mathcal{D}} \arrow[d, "\mathrm{id}_{\underline{\mathcal{C}}} +^O \underline{g}"] \\
\underline{\mathcal{C}} +^O \big|[\mathbf{S}(\underline{\mathcal{E}})_\bullet]\big| \arrow[r] & \underline{\mathcal{C}} +^O \underline{\mathcal{E}}
\end{tikzcd}$$

and we know that the horizontal arrows are Dwyer–Kan equivalences, so (using the 2-out-of-3 property) we conclude that

$$\mathrm{id}_{\underline{\mathcal{C}}} +^O \underline{g} : \underline{\mathcal{C}} +^O \underline{\mathcal{D}} \to \underline{\mathcal{C}} +^O \underline{\mathcal{E}}$$

is a Dwyer–Kan equivalence, as required. ■

Proposition 2.7.32. *Let O and I be sets. The coproduct functor*

$$\sum_I^O (-) : (\mathbf{SCat}_O)^I \to \mathbf{SCat}_O$$

preserves weak equivalences.

Proof. It is well known that coproducts can be constructed using filtered colimits and finite coproducts, so this is a corollary of proposition 2.7.11 and lemma 2.7.31. ■

Proposition 2.7.33. *Let O be a set. The Dwyer–Kan model structure on* $\mathbf{SCat}_O$ *is proper.*[7]

Proof. See Proposition 7.3 in [Dwyer and Kan, 1980a]. □

[7] See definition 5.1.6.

2.8 Simplicial localisation

Prerequisites. §§1.2, 1.5, 1.6, 1.11, 2.1, 2.5, 2.7, 3.1, 4.5, 4.6, A.4.

When one passes from a relative category to its homotopy category by freely inverting the weak equivalences, one loses much of the homotopical information. Dwyer and Kan [1980a,b,c] instead proposed a more sophisticated notion of localisation that produces a simplicial category retaining all the homotopical information, at least in the case of a simplicial model category.

Definition 2.8.1. The **standard resolution** of a small relative category $\mathcal{C}$ is the simplicial relative category $\mathbf{S}(\mathcal{C})_\bullet$ where $\operatorname{und} \mathbf{S}(\mathcal{C})_\bullet = \mathbf{S}(\operatorname{und} \mathcal{C})_\bullet$ and $\operatorname{weq} \mathbf{S}(\mathcal{C})_\bullet = \mathbf{S}(\operatorname{weq} \mathcal{C})_\bullet$. The **standard simplicial localisation** of $\mathcal{C}$ is the simplicial category $\mathbf{Lo}(\mathcal{C})_\bullet$ obtained by applying Ho to $\mathbf{S}(\mathcal{C})_\bullet$ degreewise, and the **simplicial localising functor** is the induced simplicial functor $\mathbf{S}(\mathcal{C})_\bullet \to \mathbf{Lo}(\mathcal{C})_\bullet$.

Remark 2.8.2. As in remark 2.7.22, the face and degeneracy operators of the simplicial set $\operatorname{ob} \mathbf{Lo}(\mathcal{C})_\bullet$ are trivial, so we may regard it as a simplicially enriched category $\underline{\mathbf{Lo}}(\mathcal{C})$.

Proposition 2.8.3. *Let $\mathcal{C}$ be a small relative category. The standard augmentation for $\mathcal{C}$ induces an isomorphism $\pi_0[\underline{\mathbf{Lo}}(\mathcal{C})] \to \operatorname{Ho} \mathcal{C}$.*

Proof. Let $\mathcal{D}$ be an ordinary category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor that sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. Then, composing with the standard augmentation $(\varepsilon_{\mathcal{C}})_\bullet : \mathbf{S}(\mathcal{C})_\bullet \to \mathcal{C}$ yields a simplicial functor $\mathbf{S}(\mathcal{C})_\bullet \to \mathcal{D}$ that sends weak equivalences in each $\mathbf{S}(\mathcal{C})_n$ to isomorphisms in $\mathcal{D}$, so the degreewise universal property of $\mathbf{Lo}(\mathcal{C})_\bullet$ yields a unique simplicial functor $\mathbf{Lo}(\mathcal{C})_\bullet \to \mathcal{D}$ making the diagram below commute (strictly),

$$\begin{tikzcd}
\mathbf{S}(\mathcal{C})_\bullet \arrow[r, "(\varepsilon_{\mathcal{C}})_\bullet"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\mathbf{Lo}(\mathcal{C})_\bullet \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

where $\mathbf{S}(\mathcal{C})_\bullet \to \mathbf{Lo}(\mathcal{C})_\bullet$ is the simplicial localising functor. $\mathcal{D}$ is an ordinary category, so proposition 2.5.14 says the corresponding simplicially enriched functor $\underline{\mathbf{Lo}}(\mathcal{C}) \to \mathcal{D}$ factors through the π_0-localising functor $\underline{\mathbf{Lo}}(\mathcal{C}) \to \pi_0[\underline{\mathbf{Lo}}(\mathcal{C})]$ in a unique way. Thus, $\pi_0[\underline{\mathbf{Lo}}(\mathcal{C})]$ has the universal property of $\operatorname{Ho} \mathcal{C}$, and the required isomorphism $\pi_0[\underline{\mathbf{Lo}}(\mathcal{C})] \to \operatorname{Ho} \mathcal{C}$ is induced by the ordinary localising functor $\mathcal{C} \to \operatorname{Ho} \mathcal{C}$. ■

Proposition 2.8.4. *Let $\mathcal{C}$ be a small relative category. The following are equivalent for a morphism $f : X \to Y$ in $\mathcal{C}$:*

(i) *The morphism $f : X \to Y$ is a weak equivalence in $\mathcal{C}$.*

(ii) *The morphism in $\mathbf{Lo}(\mathcal{C})$ corresponding to $f : X \to Y$ is an isomorphism.*

(iii) *The morphism in $\mathbf{Lo}(\mathcal{C})_0$ corresponding to $f : X \to Y$ is an isomorphism.*

Proof. (i) ⇒ (ii). For each natural number n, the morphism in $\mathbf{S}(\mathcal{C})_n$ corresponding to $f : X \to Y$ is a weak equivalence (by definition), so its image in $\mathbf{Lo}(\mathcal{C})_n$ is an isomorphism. Thus, the morphism corresponding to f in the simplicially enriched category $\underline{\mathbf{Lo}}(\mathcal{C})$ is an isomorphism.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Since $\operatorname{und} \mathbf{S}(\mathcal{C})_0$ and $\operatorname{weq} \mathbf{S}(\mathcal{C})_0$ are free categories, the morphisms in $\mathbf{Lo}(\mathcal{C})_0$ can be represented by reduced composable strings generated by morphisms in $\operatorname{und} \mathcal{C}$ and the formal inverses of morphisms in $\operatorname{weq} \mathcal{C}$. Thus, a morphism in $\mathcal{C}$ corresponds to an isomorphism in $\mathbf{Lo}(\mathcal{C})_0$ if and only if it is a weak equivalence in $\mathcal{C}$. ■

To justify the definition of the standard simplicial localisation, we must first study the case of the fundamental category of a graph.

Lemma 2.8.5. *Let O be a set and let $\mathcal{C}$ and $\mathcal{D}$ be objects in $\mathbf{Cat}_O$. Regarding $\mathrm{N}(\mathcal{C})$ and $\mathrm{N}(\mathcal{D})$ as simplicial subsets of $\mathrm{N}\left(\mathcal{C} +^O \mathcal{D}\right)$, the inclusion*

$$\mathrm{N}(\mathcal{C}) \cup \mathrm{N}(\mathcal{D}) \hookrightarrow \mathrm{N}\left(\mathcal{C} +^O \mathcal{D}\right)$$

is a weak homotopy equivalence.

Proof. Let $F : \mathbf{sGrph}_O \to \mathbf{SCat}_O$ be the free simplicially enriched category over O functor. If $\mathcal{C} = FX$ and $\mathcal{D} = FY$ for some X and Y in $\mathbf{sGrph}_O$, then the claim reduces to lemma 1.11.37. In general, consider the standard resolutions of $\mathcal{C}$ and $\mathcal{D}$. Since the standard resolution is degreewise free, for each natural number n, the inclusion

$$\mathrm{N}\left(\mathbf{S}(\mathcal{C})_n\right) \cup \mathrm{N}\left(\mathbf{S}(\mathcal{D})_n\right) \hookrightarrow \mathrm{N}\left(\mathbf{S}(\mathcal{C})_n +^O \mathbf{S}(\mathcal{D})\right)$$

is a weak homotopy equivalence; thus, by theorem 1.6.10,

$$\left|\mathrm{N}\left(\mathbf{S}(\mathcal{C})_\bullet\right)\right| \cup \left|\mathrm{N}\left(\mathbf{S}(\mathcal{D})_\bullet\right)\right| \hookrightarrow \left|\mathrm{N}\left(\mathbf{S}(\mathcal{C})_\bullet +^O \mathbf{S}(\mathcal{D})_\bullet\right)\right|$$

is a weak homotopy equivalence. We have the following commutative diagram in **sSet**,

$$\begin{tikzcd} \left|\mathrm{N}\left(\mathbf{S}(\mathcal{C})_\bullet\right)\right| \cup \left|\mathrm{N}\left(\mathbf{S}(\mathcal{D})_\bullet\right)\right| \arrow[r, hook] \arrow[d] & \left|\mathrm{N}\left(\mathbf{S}(\mathcal{C})_\bullet +^O \mathbf{S}(\mathcal{D})_\bullet\right)\right| \arrow[d] \\ \mathrm{N}(\mathcal{C}) \cup \mathrm{N}(\mathcal{D}) \arrow[r, hook] & \mathrm{N}\left(\mathcal{C} +^O \mathcal{D}\right) \end{tikzcd}$$

where the vertical arrows are induced by the respective standard augmentations; but corollary 2.3.13 and proposition 2.3.20 (plus proposition 1.5.3) together with lemmas 1.5.17 and 2.7.31 (plus lemmas 1.6.8 and 2.5.20) imply that the vertical arrows in the diagram are weak homotopy equivalences, so we may use the 2-out-of-3 property to deduce that the bottom arrow is also a weak homotopy equivalence, as desired. ■

Lemma 2.8.6. *Let O be a set, let $\left(\mathcal{C}_i \,\middle|\, i \in I\right)$ be a small family of categories over O, and let $\mathcal{C} = \sum_{i\in I}^{O} \mathcal{C}_i$ be their coproduct in* $\mathbf{SCat}_O$*. Regarding each $\mathrm{N}\left(\mathcal{C}_i\right)$ as a simplicial subset of $\mathrm{N}(\mathcal{C})$, the inclusion*

$$\bigcup_{i\in I} \mathrm{N}\left(\mathcal{C}_i\right) \hookrightarrow \mathrm{N}(\mathcal{C})$$

is a weak homotopy equivalence.

Proof. Let $\mathcal{J}$ be the poset of finite subsets of I. By lemma 2.8.5 (and induction), for each finite $J \subseteq I$, setting $\mathcal{C}_J = \sum_{j\in I}^{O} \mathcal{C}_j$, the inclusion

$$\bigcup_{j\in J} \mathrm{N}\left(\mathcal{C}_j\right) \hookrightarrow \mathrm{N}\left(\mathcal{C}_J\right)$$

is a weak homotopy equivalence; but $\mathcal{J}$ is directed, so by proposition 2.7.11,

$$\varinjlim_{J\in\mathcal{J}} \bigcup_{j\in J} \mathrm{N}\left(\mathcal{C}_j\right) \hookrightarrow \varinjlim_{J\in\mathcal{J}} \mathrm{N}\left(\mathcal{C}_J\right)$$

is also a weak homotopy equivalence, as required. ■

Lemma 2.8.7. *Let G be a 1-skeletal simplicial set. The unit $\eta_G : G \to \mathrm{N}\left(\pi_1 G\right)$ is an anodyne extension.*

Proof. It is not hard to verify that the unit $\eta_G : G \to \mathrm{N}\left(\pi_1 G\right)$ is a monomorphism, so by proposition 1.5.10, it suffices to show that $\eta_G : G \to \mathrm{N}\left(\pi_1 G\right)$ is a weak homotopy equivalence.

Let O be the set of vertices of G and consider $\pi_1 G$ as a category over O. It is easy to check (using the contractibility of Δ^1 and $\mathrm{N}(\pi_1 \Delta^1)$, plus proposition 1.5.14) that $\eta_G : G \to \mathrm{N}(\pi_1 G)$ is a weak homotopy equivalence when G has a unique non-degenerate edge. In general, we note that the functor $\pi_1 : \mathbf{Grph}_O \to \mathbf{Cat}_O$ preserves coproducts (because it is a left adjoint), so we may apply lemma 2.8.6 to deduce the claim for all 1-skeletal simplicial sets G. ■

Proposition 2.8.8. *Let G be a 1-skeletal simplicial set. There is a natural commutative diagram in* **sSet** *of the form below,*

$$\begin{tikzcd} G \arrow[r] \arrow[d, equal] & \mathrm{N}(\tau_1 G) \arrow[d] \\ G \arrow[r] & \mathrm{N}(\pi_1 G) \end{tikzcd}$$

where the horizontal arrows are the components of the respective adjunctions and the vertical arrow is induced by the unit of the evident adjunction

$$\mathbf{I} \dashv U : \mathbf{Grpd} \to \mathbf{Cat}$$

and moreover, every arrow in the diagram is an anodyne extension.

Proof. It is not hard to verify that $\mathrm{N}(\tau_1 G) \to \mathrm{N}(\pi_1 G)$ is a monomorphism, and the commutativity of the diagram is a consequence of the fact that adjunctions can be composed. Thus, recalling the 2-out-of-3 property and proposition 1.5.10, the claim is a corollary of proposition 1.11.36 and lemma 2.8.7. ■

Remark 2.8.9. In view of proposition 6.2.30, we should regard the above result as saying that $\pi_1 G$ is the groupoid completion of $\tau_1 G$ even when regarded as an $(\infty, 1)$-category.

Lemma 2.8.10. *Let C be a small category. If we regard C as a* maximal *relative category, then the arrows in the diagram below are weak homotopy equivalences,*

$$\mathrm{N}(C) \longleftarrow \left| \mathrm{N}^{\mathrm{ss}}(\underline{\mathbf{S}}(C))_\bullet \right| \longrightarrow \left| \mathrm{N}^{\mathrm{ss}}(\underline{\mathbf{Lo}}(C))_\bullet \right|$$

where the leftward-pointing arrow is induced by the standard augmentation and the rightward-pointing arrow is induced by the simplicial localisation functor.

Proof. Proposition 2.7.23 says that the standard augmentation is a Dwyer–Kan equivalence, so by theorem 1.6.10 and lemma 2.5.20, the leftward-pointing arrow is a weak homotopy equivalence. Similarly, by lemma 1.6.8, it suffices to verify that the morphism

$$\mathrm{N}\big(\mathbf{S}(C)_\bullet\big) \to \mathrm{N}\big(\mathbf{Lo}(C)_\bullet\big)$$

is a degreewise weak homotopy equivalence; but each $\mathbf{S}(C)_n$ is freely generated by a reflexive graph, so this a corollary of proposition 2.8.8. ∎

Recalling that the Dwyer–Kan model structure on $\mathbf{SCat}_O$ is left proper (proposition 2.7.33), we may apply proposition 5.1.23 to complete the justification of the definition of the standard simplicial localisation with the following observation:

Lemma 2.8.11. *Let C be a small relative category and let $O = \operatorname{ob} C$. There is a natural pushout diagram in $\mathbf{SCat}_O$ of the form below,*

$$\begin{tikzcd} \underline{\mathbf{S}}(\operatorname{weq} C) \arrow[r, hook] \arrow[d, hook] & \underline{\mathbf{Lo}}(\max \operatorname{weq} C) \arrow[d, hook] \\ \underline{\mathbf{S}}(\operatorname{und} C) \arrow[r, hook] & \underline{\mathbf{Lo}}(C) \end{tikzcd}$$

where the morphism $\underline{\mathbf{S}}(\operatorname{weq} C) \hookrightarrow \underline{\mathbf{S}}(\operatorname{und} C)$ induced by the inclusion is a standard cofibration, the horizontal arrows are induced by the simplicial localisation functors, and every arrow is a monomorphism in $\mathbf{SCat}_O$.

Proof. Regarding objects in $\mathbf{SCat}_O$ as simplicial objects in $\mathbf{Cat}_O$, the claim that we have a pushout diagram can be verified degreewise; but the diagram in degree n is just

$$\begin{tikzcd} \mathbf{S}(\operatorname{weq} C)_n \arrow[r, hook] \arrow[d, hook] & \mathbf{IS}(\operatorname{weq} C)_n \arrow[d, hook] \\ \mathbf{S}(\operatorname{und} C)_n \arrow[r, hook] & \operatorname{Ho} \mathbf{S}(C)_n \end{tikzcd}$$

and it is not hard to see that this is indeed a pushout diagram. Similarly, the property of being a monomorphism can be checked degreewise. A variation on the proof of lemma 2.7.17 shows that $\underline{\mathbf{S}}(\operatorname{weq} C) \hookrightarrow \underline{\mathbf{S}}(\operatorname{und} C)$ is a standard cofibration. ∎

Definition 2.8.12. Let X and Y be objects in a relative category C.

- A **hammock** in $\mathcal{C}$ from X to Y of width k and length n is a commutative diagram in $\mathcal{C}$ of the form below,

$$
\begin{tikzcd}
X \arrow[r, dash] \arrow[d, equal] & Z_{0,1} \arrow[r, dash] \arrow[d] & Z_{0,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & Z_{0,n-2} \arrow[r, dash] \arrow[d] & Z_{0,n-1} \arrow[r, dash] \arrow[d] & Y \arrow[d, equal] \\
X \arrow[r, dash] \arrow[d, equal] & Z_{1,1} \arrow[r, dash] \arrow[d] & Z_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & Z_{1,n-2} \arrow[r, dash] \arrow[d] & Z_{1,n-1} \arrow[r, dash] \arrow[d] & Y \arrow[d, equal] \\
\vdots \arrow[d, equal] & \vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d, equal] \\
X \arrow[r, dash] \arrow[d, equal] & Z_{k-1,1} \arrow[r, dash] \arrow[d] & Z_{k-1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & Z_{k-1,n-2} \arrow[r, dash] \arrow[d] & Z_{k-1,n-1} \arrow[r, dash] \arrow[d] & Y \arrow[d, equal] \\
X \arrow[r, dash] & Z_{k,1} \arrow[r, dash] & Z_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & Z_{k,n-2} \arrow[r, dash] & Z_{k,n-1} \arrow[r, dash] & Y
\end{tikzcd}
$$

  such that the following conditions are satisfied:

  - In each column, all horizontal arrows point in the same direction.
  - All leftward-pointing arrows are weak equivalences.
  - All vertical arrows are weak equivalences.

  We allow both k and n to be zero; if $n = 0$ then we must have $X = Y$.

- A **reduced hammock** in $\mathcal{C}$ is a hammock with these additional properties:

  - In each column, not every horizontal arrow is an identity morphism.
  - Arrows in adjacent columns point in opposite directions.

Remark 2.8.13. In other words, a hammock in $\mathcal{C}$ from X to Y is a composable sequence of morphisms in the category $\mathcal{C}^T(X, Y)$ for some zigzag type T. It is clear that we can transform any hammock into a reduced hammock by iteratively omitting any column of identity morphisms and composing any adjacent columns where possible.

Definition 2.8.14. Let $\mathcal{C}$ be a small relative category. The **hammock localisation** of $\mathcal{C}$ is the simplicial category $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})$ defined below:

- The objects are those in $\mathcal{C}$.
- For each pair (X, Y) of objects, the hom-space $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y)$ is the simplicial set whose k-simplices are the reduced hammocks of width k (and any length), with face (resp. degeneracy) operators defined by omitting (resp. repeating) a row of objects and reducing the resulting hammock if necessary.
- Composition is defined by concatenation of hammocks (reducing as necessary), and identities are hammocks of length 0.

Proposition 2.8.15. *Let $\mathbf{Z}$ be the category of zigzag types, let $\mathbf{Z}_{\rightarrow}$ be the subcategory of monomorphisms, and let $\mathbf{Z}^{\leftarrow}$ be the subcategory of epimorphisms. Then $\mathbf{Z}$ is a Reedy category with direct subcategory $\mathbf{Z}_{\rightarrow}$ and inverse subcategory $\mathbf{Z}^{\leftarrow}$, and moreover it has cofibrant constants.*

Proof. Let $\deg : \operatorname{ob} \mathbf{Z} \to \mathbb{N}$ be defined by $\deg(n, U, V) = n$. It is clear that this function makes $\mathbf{Z}_{\rightarrow}$ a direct category and $\mathbf{Z}^{\leftarrow}$ an inverse category. Proposition A.4.30 then gives the required factorisation, and lemma A.4.35 shows that it is a Reedy category with cofibrant constants. ■

Corollary 2.8.16. *The functor $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} : [\mathbf{Z}^{\mathrm{op}}, \mathbf{sSet}] \to \mathbf{sSet}$ preserves weak equivalences between Reedy-cofibrant diagrams.*

Proof. Apply (proposition 4.3.2 and) Ken Brown's lemma (4.3.6) and proposition 4.6.24 to proposition 2.8.15. ■

Proposition 2.8.17. *Let $\mathcal{C}$ be a small relative category, let X and Y be objects in $\mathcal{C}$, and let $\mathrm{N}(\mathcal{C}^{*}(X, Y)) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{sSet}$ be the diagram described in remark A.4.42.*

(i) *The diagram $\mathrm{N}(\mathcal{C}^{*}(X, Y)) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{sSet}$ is Reedy-cofibrant.*

(ii) *More generally, if $F : \mathbf{Z} \to \mathbf{Z}$ is any functor that preserves (epimorphisms and) pushouts of epimorphisms along epimorphisms,*[8] *then the diagram $\mathrm{N}\big(\mathcal{C}^{F*}(X, Y)\big) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{sSet}$ is Reedy-cofibrant.*

(iii) *The colimit $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y))$ is naturally isomorphic to the hom-space $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y)$.*

[8] Any functor that preserves pushouts of epimorphisms along epimorphisms must also preserve epimorphisms.

Proof. (i) and (ii). First, let us show that $\mathcal{C}^*(X,Y)$ sends epimorphisms in $\mathbf{Z}$ to monomorphisms in $\mathbf{Cat}$. Indeed, if $\sigma : S \to T$ is an epimorphism in $\mathbf{Z}$, then the corresponding functor $\sigma^* : \mathcal{C}^T(X,Y) \to \mathcal{C}^S(X,Y)$ must be one defined by inserting identity morphisms, so it is indeed a monomorphism. Since $F : \mathbf{Z} \to \mathbf{Z}$ preserves epimorphisms, $\mathcal{C}^{F*}(X,Y)$ also sends epimorphisms in $\mathbf{Z}$ to monomorphisms in $\mathbf{Cat}$.

Secondly, we observe that $\mathcal{C}^*(X,Y)$ sends pushouts of epimorphisms in $\mathbf{Z}$ to pullbacks of monomorphisms in $\mathbf{Cat}$: this is clear using the methods of the proof of lemma A.4.36. As before, since $F : \mathbf{Z} \to \mathbf{Z}$ preserves pushouts of epimorphisms along epimorphisms, $\mathcal{C}^{F*}(X,Y)$ also sends pushouts of epimorphisms in $\mathbf{Z}$ to pullbacks of monomorphisms in $\mathbf{Cat}$.

Thus, it follows that the latching object of the diagram $\mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big)$ at a zigzag type T is simply the simplicial subset of $\mathrm{N}\big(\mathcal{C}^{FT}(X,Y)\big)$ corresponding to the full subcategory of $\mathcal{C}^{FT}(X,Y)$ spanned by the joint image of all functors $(F\sigma)^* : \mathcal{C}^{FT'}(X,Y) \to \mathcal{C}^{FT}(X,Y)$ where $\sigma : T \to T'$ is an epimorphism in $\mathbf{Z}$. In particular, the latching morphism of $\mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big)$ at T is a cofibration, so $\mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big)$ is indeed a Reedy-cofibrant diagram.

(ii). It is clear that reduction defines a morphism $\mathrm{N}\big(\mathcal{C}^T(X,Y)\big) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X,Y)$ and (using lemma A.4.33) it is not hard to see that this is a colimiting cocone. ■

Remark 2.8.18. The hom-space $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X,Y)$ can be constructed as a colimit as shown below,

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X,Y) \cong \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^*(X,Y))$$

where $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is the nerve functor and $\mathcal{C}^*(X,Y) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{Cat}$ is the functor described in remark A.4.42.

Remark 2.8.19. Unlike the standard simplicial localisation, the hammock localisation of a relative category $\mathcal{C}$ is equipped with a natural functor $\mathcal{C} \to \mathbf{Lo}^{\mathrm{H}}(\mathcal{C})$ that is bijective on objects and faithful (but not necessarily full).

Proposition 2.8.20. *Let $\mathcal{C}$ be a small relative category. If $f : X \to Y$ is a weak equivalence in $\mathcal{C}$, then:*

- *For each object S in $\mathcal{C}$, the induced morphism*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,f) : \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,X) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,Y)$$

  is a weak homotopy equivalence.

- *For each object T in $\mathcal{C}$, the induced morphism*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(f,T) : \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(Y,T) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X,T)$$

  is a weak homotopy equivalence.

Proof. The two claims are formally dual; we will prove the first version.

Recalling remark A.4.41, we see that for any zigzag in $\mathcal{C}$ of the form below,

$$S \longrightarrow \cdots \longrightarrow X$$

we have the following natural hammock:

$$\begin{tikzcd} S \arrow[r, dash] \arrow[d, equal] & \cdots \arrow[r, dash] & X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, "f"] & X \arrow[d, equal] \\ S \arrow[r, dash] & \cdots \arrow[r, dash] & X \arrow[r, "f"'] & Y & X \arrow[l, "f"] \end{tikzcd}$$

Similarly, for any zigzag in $\mathcal{C}$ of the form below,

$$S \longrightarrow \cdots \longrightarrow Y$$

we have the following natural hammock:

$$\begin{tikzcd} S \arrow[r, dash] \arrow[d, equal] & \cdots \arrow[r, dash] & Y \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d, equal] \\ S \arrow[r, dash] & \cdots \arrow[r, dash] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

Thus, the morphism $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,f) : \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,X) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(S,Y)$ is half of an intrinsic homotopy equivalence, hence is a weak homotopy equivalence (by proposition 1.5.3). ■

Proposition 2.8.21. *Let $\mathcal{C}$ be a small relative category. There is a natural functor*

$$\operatorname{Ho}\mathcal{C} \to \pi_0\big[\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})\big]$$

and it is an isomorphism.

Proof. We have natural functors $\mathcal{C} \to \mathbf{Lo}^{\mathrm{H}}(\mathcal{C})$ and $\mathbf{Lo}^{\mathrm{H}}(\mathcal{C}) \to \pi_0\big[\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})\big]$; and by proposition 2.8.20, the composite functor $\mathcal{C} \to \pi_0\big[\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})\big]$ sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\pi_0\big[\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})\big]$, so it must factor through the localising functor $\mathcal{C} \to \operatorname{Ho}\mathcal{C}$. This yields the required natural functor $\operatorname{Ho}\mathcal{C} \to \pi_0\big[\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})\big]$, and theorem A.4.44 implies that it is an isomorphism of categories. ■

Definition 2.8.22. A small relative category C admits a **homotopical three-arrow calculus** when it satisfies the following condition:

- Let X and Y be objects in C, let k and l be natural numbers, let $\mathcal{H}_0(X, Y)$ be the category of zigzags in C from X to Y of the following type,

$$\bullet \longleftarrow \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \overset{l \text{ arrows}}{\dashrightarrow} \bullet \longleftarrow \bullet$$

  where the dotted arrows stand for a composable chain of rightward-pointing arrows, let $\mathcal{H}_1(X, Y)$ be the category of zigzags in C from X to Y of the following type,

$$\bullet \longleftarrow \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \longleftarrow \bullet \overset{l \text{ arrows}}{\dashrightarrow} \bullet \longleftarrow \bullet$$

  and let $s : \mathcal{H}_0(X, Y) \to \mathcal{H}_1(X, Y)$ be the functor defined by inserting an identity morphism. Then $s : \mathcal{H}_0 \to \mathcal{H}_1$ is a weak homotopy equivalence of categories.

Lemma 2.8.23. *Let C be a small relative category, let $\mathcal{W} = \operatorname{weq} C$, let X and Y be objects in C, let S and T be zigzag types, and let $T * S$ be their concatenation:*

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

- *If the rightmost arrow of S points rightwards and the leftmost arrow of T points rightwards, then*

$$C^{T*S}(X, Y) \cong \mathbf{G}\big(C^T(-, Y), \mathcal{W}, C^S(X, -)\big)$$

  naturally in X, Y, T, and S.

- *If the rightmost arrow of S points leftwards and the leftmost arrow of T points leftwards, then*

$$C^{T*S}(X, Y) \cong \mathbf{G}\big(C^S(X, -), \mathcal{W}, C^T(-, Y)\big)$$

  naturally in X, Y, T, and S.

- *If the rightmost arrow of S points rightwards and the leftmost arrow of T points leftwards, then*

$$C^{T*S}(X, Y) \cong \mathbf{G}\big(\Delta \mathbb{1}, \mathcal{W}, C^T(-, Y) \times C^S(X, -)\big)$$

  naturally in X, Y, T, and S.

- *If the rightmost arrow of S points leftwards and the leftmost arrow of T points rightwards, then*

$$\mathcal{C}^{T*S}(X,Y) \cong \mathbf{G}\big(\mathcal{C}^{T}(-,Y) \times \mathcal{C}^{S}(X,-), \mathcal{W}, \Delta\mathbb{1}\big)$$

naturally in X, Y, T, and S.

Proof. This is a straightforward exercise. ◊

Lemma 2.8.24. *Let $\mathcal{C}$ be a small relative category, let X and Y be objects in $\mathcal{C}$, let S and T be* possibly degenerate *zigzag types, and let k and l be natural numbers, let $\mathcal{H}_0^+$ be the category of zigzags in $\mathcal{C}$ from X to Y of the following type,*

$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l] \arrow[r, dotted, "k \text{ arrows}"] & \bullet \arrow[r, dotted, "l \text{ arrows}"] & \bullet & \bullet \arrow[l] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

where the dotted arrows stand for a composable chain of rightward-pointing arrows, let $\mathcal{H}_1^+$ be the category of zigzags in $\mathcal{C}$ from X to Y of the following type,

$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l] \arrow[r, dotted, "l \text{ arrows}"] & \bullet & \bullet \arrow[l] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

and let $s : \mathcal{H}_0^+ \to \mathcal{H}_1^+$ be the functor defined by inserting an identity morphism. If $\mathcal{C}$ admits a homotopical three-arrow calculus, then $s : \mathcal{H}_0^+ \to \mathcal{H}_1^+$ is a weak homotopy equivalence of categories.

Proof. Apply the 2-out-of-3 property of weak homotopy equivalences of categories (lemma 1.11.3) and corollary 1.11.18 to lemma 2.8.23. ■

Lemma 2.8.25. *Let $\mathcal{C}$ be a small relative category, let X and Y be objects in $\mathcal{C}$, let S and T be* possibly degenerate *zigzag types, let $\mathcal{H}_0$ be the category of zigzags in $\mathcal{C}$ from X to Y of the following type,*

$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

let $\mathcal{H}_1$ be the category of zigzags in $\mathcal{C}$ from X to Y of the following type,

$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l] & \bullet \arrow[l] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

let $s_0, s_1 : \mathcal{H}_0 \to \mathcal{H}_1$ be the two functors defined by inserting an identity morphism, and let $d : \mathcal{H}_1 \to \mathcal{H}_0$ be the functor defined by composing the middle two arrows.

(i) $d \circ s_0 = d \circ s_1 = \mathrm{id}_{\mathcal{H}_1}$.

(ii) $s_0, s_1 : \mathcal{H}_0 \to \mathcal{H}_1$ *and* $d : \mathcal{H}_1 \to \mathcal{H}_0$ *are all weak homotopy equivalences of categories.*

Proof. (i). Obvious.

(ii). There is a natural transformation $\mathrm{id}_{\mathcal{H}_1} \Rightarrow d \circ s$ whose component at an object in $\mathcal{H}_1$, say

$$\begin{tikzcd} X \arrow[r, squiggly, no head] & \bullet & \bullet \arrow[l, "v"'] & \bullet \arrow[l, "u"'] \arrow[r, squiggly, no head] & Y \end{tikzcd}$$

is given by the commutative diagram in C shown below:

$$\begin{tikzcd} X \arrow[r, squiggly, no head] \arrow[d, equal] & \bullet \arrow[d, equal] & \bullet \arrow[l, "v"'] \arrow[d, "v"] & \bullet \arrow[l, "u"'] \arrow[r, squiggly, no head] \arrow[d, equal] & Y \arrow[d, equal] \\ X \arrow[r, squiggly, no head] & \bullet \arrow[r, equal] & \bullet & \bullet \arrow[l, "v \circ u"] \arrow[r, squiggly, no head] & Y \end{tikzcd}$$

Thus, by lemma 1.3.10 and proposition 1.5.3, $s_0, s_1 : \mathcal{H}_0 \to \mathcal{H}_1$ and $d : \mathcal{H}_1 \to \mathcal{H}_0$ are all indeed weak homotopy equivalences of categories. ■

Lemma 2.8.26. *Let C be a small relative category, let X and Y be objects in C, let S be a zigzag type of the form below,*

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r, squiggly, no head] & \bullet & \bullet \arrow[l] \end{tikzcd}$$

let k be the number of forward-pointing arrows in S, let T be the following zigzag type,

$$\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l] \end{tikzcd}$$

let $\sigma : S \to T$ be the evident morphism of zigzag types that sends all interior backward-pointing arrows in S to identity morphisms, and let $C^S(X, Y)$ (resp. $C^T(X, Y)$) be the category of zigzags in C of type S (resp. T) from X to Y. If C admits a homotopical three-arrow calculus, then the functor $s : C^T(X, Y) \to C^S(X, Y)$ induced by $\sigma : S \to T$ is a weak homotopy equivalence of categories.

Proof. Recalling that the class of weak homotopy equivalences of categories is closed under composition (by lemma 1.11.3), we can reduce the claim to lemmas 2.8.24 and lemma 2.8.25 by factoring $\sigma : S \to T$ as a sequence of morphisms of zigzag types of the appropriate nature and considering the corresponding factorisation of $s : C^T(X, Y) \to C^S(X, Y)$. ■

Theorem 2.8.27 (Fundamental theorem of homotopical three-arrow calculi)**.** *Let C be a small relative category, let X and Y be objects in C, and let $\mathcal{H}(X, Y)$ be the category of zigzags in C from X to Y of the following type:*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

If C admits a homotopical three-arrow calculus, then the reduction morphism

$$\mathrm{N}(\mathcal{H}(X, Y)) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X, Y)$$

is a weak homotopy equivalence.

Proof. We follow the proof of Proposition 6.2 in [Dwyer and Kan, 1980b].

Let $F : \mathbf{Z} \to \mathbf{Z}$ be the evident functor that sends each zigzag type T to the zigzag type

$$\bullet \longleftarrow \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet \longleftarrow \bullet$$

and let $G : \mathbf{Z} \to \mathbf{Z}$ be the evident functor that sends each zigzag type S with k rightward-pointing arrows to the following zigzag type:

$$\bullet \longleftarrow \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \longleftarrow \bullet$$

Let $T_{\rightarrow}$ (resp. $T_{\leftarrow}$) be the zigzag type consisting of a single rightward-pointing (resp. leftward-pointing) arrow. First, let us show that the colimiting cocone component

$$\mathbf{Z}\big(GT_{\rightarrow}, T\big) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathbf{Z}(G, T)$$

is a bijection for all zigzag types T. Indeed, by lemma A.4.33, we see that $\mathbf{Z}(GS, T)$ is empty unless T is a zigzag type of the form below,

$$\bullet \overset{n \text{ arrows}}{\dashleftarrow} \bullet \overset{m \text{ arrows}}{\dashrightarrow} \bullet \overset{l \text{ arrows}}{\dashleftarrow} \bullet$$

where l, m, n are natural numbers and $l + m + n \geq 1$; moreover:

- If $m \geq 1$, then there is a unique morphism $GT_{\rightarrow} \to T$.
- If $m = 0$, then there are $l+n+1$ morphisms $GT_{\rightarrow} \to T$, all of which factor through the unique epimorphism $GT_{\rightarrow} \to GT_{\leftarrow}$ (necessarily uniquely).

Similarly, for any zigzag type S' consisting of only rightward-pointing arrows, there is a unique morphism $T_{\rightarrow} \to S'$, and its image under G is the unique morphism $GT_{\rightarrow} \to GS'$, and it is clear that any zigzag type S with at least one rightward-pointing arrow admits a unique morphism $\sigma : S \to S'$ such that S' consists of only rightward-pointing arrows and $G\sigma = \mathrm{id}_{GS'}$. On the other hand, there are unique morphisms $T_{\rightarrow} * T_{\leftarrow} \to T_{\rightarrow}$ and $T_{\rightarrow} * T_{\leftarrow} \to T_{\leftarrow}$, and for any zigzag type S' consisting of only leftward-pointing arrows, there is a unique morphism $T_{\leftarrow} \to S'$, so we conclude that each connected component of the comma category $(G \downarrow T)$ contains a unique object corresponding to each morphism $GT_{\rightarrow} \to T$. But $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathbf{Z}(G, T)$ can be identified with the set of connected components of the comma category, so this proves the claim.

Now, taking $H = \mathbf{Z}(G-, -)$ and applying the Yoneda lemma, we obtain natural isomorphisms

$$\mathrm{N}\big(\mathcal{C}^{G*}(X, Y)\big) \cong H \star_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y))$$

and by proposition A.6.15,

$$\varinjlim_{\mathbf{Z}^{\mathrm{op}}} H \star_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y)) \cong \varinjlim_{\mathbf{Z}^{\mathrm{op}}} H \star_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y))$$

but we have shown that the canonical $\mathbf{Z}\big(GT_{\rightarrow}, -\big) \Rightarrow \varinjlim_{\mathbf{Z}^{\mathrm{op}}} H$ is a natural isomorphism, so applying the Yoneda lemma again, we deduce that the colimiting cocone component

$$j : \mathrm{N}\big(\mathcal{C}^{GT_{\rightarrow}}(X, Y)\big) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{G*}(X, Y)\big)$$

is an isomorphism as well.

Next, consider the evident natural epimorphisms

$$\varphi : F \Rightarrow \mathrm{id}_{\mathbf{Z}} \qquad \psi : F \Rightarrow G$$

the induced natural transformations

$$\varphi^{*} : \mathcal{C}^{*}(X, Y) \Rightarrow \mathcal{C}^{F*}(X, Y) \qquad \psi^{*} : \mathcal{C}^{G*}(X, Y) \Rightarrow \mathcal{C}^{F*}(X, Y)$$

and the induced morphisms

$$\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\varphi^{*}) : \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y)) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{F*}(X, Y)\big)$$
$$\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\psi^{*}) : \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{G*}(X, Y)\big) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{F*}(X, Y)\big)$$

of simplicial sets. By remark 1.11.10 and lemma 1.11.12, $F : \mathbf{Z} \to \mathbf{Z}$ is a coinitial functor, and thus by theorem A.5.32, the canonical comparison morphism

$$\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{F*}(X, Y)\big) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^{*}(X, Y))$$

is an isomorphism of simplicial sets; but the diagram below commutes for every zigzag type T,

$$\begin{tikzcd}
\mathrm{N}\big(\mathcal{C}^T(X,Y)\big) \arrow[r] \arrow[d, "\mathrm{N}(\varphi^*_T)"'] & \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^*(X,Y)) \arrow[d, "\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\varphi^*)"] \\
\mathrm{N}\big(\mathcal{C}^{FT}(X,Y)\big) \arrow[r] \arrow[d, equal] & \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big) \arrow[d, "\cong"] \\
\mathrm{N}\big(\mathcal{C}^{FT}(X,Y)\big) \arrow[r] & \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^*(X,Y))
\end{tikzcd}$$

where the horizontal arrows are the respective colimiting cocone components, so the morphism $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\varphi^*)$ must also be an isomorphism. With that in mind, we see that the following diagram in **sSet** commutes,

$$\begin{tikzcd}
\mathrm{N}\big(\mathcal{C}^{GT_\to}(X,Y)\big) \arrow[r, "j"] \arrow[d, "i"'] & \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{G*}(X,Y)\big) \arrow[d, "\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\psi^*)"] \\
\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^*(X,Y)) \arrow[r, "\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\varphi^*)"'] & \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big)
\end{tikzcd}$$

where $i : \mathrm{N}\big(\mathcal{C}^{GT_\to}(X,Y)\big) \to \varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\mathcal{C}^*(X,Y))$ is the component of the colimiting cocone. Hence, by the 2-out-of-3 property of weak homotopy equivalences (lemma 1.5.2), i is a weak homotopy equivalence if and only if $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\psi^*)$ is a weak homotopy equivalence.

To complete the proof, we make the following observations:

- By lemma 2.8.26, the components

$$\mathrm{N}\big(\psi^*_T\big) : \mathrm{N}\big(\mathcal{C}^{GT}(X,Y)\big) \to \mathrm{N}\big(\mathcal{C}^{FT}(X,Y)\big)$$

  are weak homotopy equivalences.

- By proposition 2.8.17, $\mathrm{N}(\mathcal{C}^*(X,Y)) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{sSet}$ is a Reedy-cofibrant diagram.

- It is not hard to see that $F : \mathbf{Z} \to \mathbf{Z}$ preserves (epimorphisms and) pushouts of epimorphisms along epimorphisms, so $\mathrm{N}\big(\mathcal{C}^{F*}(X,Y)\big) : \mathbf{Z}^{\mathrm{op}} \to \mathbf{sSet}$ is also a Reedy-cofibrant diagram.

Thus, by corollary 2.8.16, $\varinjlim_{\mathbf{Z}^{\mathrm{op}}} \mathrm{N}(\psi^*)$ is a weak homotopy equivalence; but the morphism

$$\mathrm{N}(\mathcal{H}(X,Y)) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X,Y)$$

in question can be identified with i, so we are done. ∎

Corollary 2.8.28. *Let C be a small relative category and let T be the following zigzag type:*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Assuming C admits a homotopical three-arrow calculus:

- *For any weak equivalence $w : X \to X'$ and any object Y in C, the functor*

  $$w_* : C^T(X,Y) \to C^T(X',Y)$$

  defined by sending each zigzag

  $$X \xleftarrow{v} \bullet \xrightarrow{f} \bullet \xleftarrow{u} Y$$

  to the zigzag

  $$X' \xleftarrow{w \circ v} \bullet \xrightarrow{f} \bullet \xleftarrow{u} Y$$

  is a weak homotopy equivalence of categories.

- *For any object X and any weak equivalence $w : Y' \to Y$ in C, the functor*

  $$w^* : C^T(X,Y) \to C^T(X,Y')$$

  defined by sending each zigzag

  $$X \xleftarrow{v} \bullet \xrightarrow{f} \bullet \xleftarrow{u} Y$$

  to the zigzag

  $$X \xleftarrow{v} \bullet \xrightarrow{f} \bullet \xleftarrow{u \circ w} Y'$$

  is a weak homotopy equivalence of categories.

Proof. The two claims are formally dual; we will prove the first version.

There is an evident morphism $w_* : \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X,Y) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X',Y)$ defined by concatenation and reduction making the following diagram in **sSet** commute,

$$\begin{tikzcd}
\mathrm{N}\big(C^T(X,Y)\big) \arrow[r] \arrow[d, "\mathrm{N}(w_*)"'] & \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X,Y) \arrow[d, "w_*"] \\
\mathrm{N}\big(C^T(X',Y)\big) \arrow[r] & \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X',Y)
\end{tikzcd}$$

where the horizontal arrows are defined by reduction, which are weak homotopy equivalences by the fundamental theorem of homotopical three-arrow calculi (2.8.27). Thus, the 2-out-of-3 property implies that

$$w_* : C^T(X,Y) \to C^T(X',Y)$$

is a weak homotopy equivalence of categories if and only if

$$w_* : \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X,Y) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X',Y)$$

is a weak homotopy equivalence of simplicial sets, and one may use the method of the proof of proposition 2.8.20 to show that the latter is indeed a weak homotopy equivalence. ■

Proposition 2.8.29. *Let C be a small relative category, let $\mathcal{W} = \operatorname{weq} C$, and let T be the following zigzag type:*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

(i) *There is a pullback diagram in* **Cat** *of the form below,*

$$\begin{tikzcd}
\operatorname{weq}[T,C]_{\mathrm{h}} \arrow[r] \arrow[d] & \operatorname{weq}[\mathbb{2},C] \arrow[d, "\langle d_0, d_1\rangle"] \\
[\mathbb{2},\mathcal{W}] \times [\mathbb{2},\mathcal{W}] \arrow[r, "d_1 \times d_0"'] & \mathcal{W} \times \mathcal{W}
\end{tikzcd}$$

where d_1 (resp. d_0) denotes the appropriate domain (resp. codomain) projection. Moreover, the horizontal arrows in the diagram are weak homotopy equivalences of categories.

(ii) *For each pair (X, Y) of objects in $\mathcal{C}$, we have the following pullback diagram in* **Cat***,*

$$\begin{tikzcd} \mathcal{C}^T(X,Y) \arrow[r] \arrow[d] & \operatorname{weq}[\mathbb{T},\mathcal{C}]_{\mathrm{h}} \arrow[d, "p"] \\ \mathbb{1} \arrow[r, "\ulcorner(X,Y)\urcorner"'] & \mathcal{W} \times \mathcal{W} \end{tikzcd}$$

where $p : \operatorname{weq}[\mathbb{T},\mathcal{C}]_{\mathrm{h}} \to \mathcal{W} \times \mathcal{W}$ is the functor defined by sending zigzags in $\mathcal{M}$ of the form

$$X \longleftarrow \bullet \longrightarrow \bullet \longleftarrow Y$$

to the pair (X, Y), and $\ulcorner(X,Y)\urcorner : \mathbb{1} \to \mathcal{W} \times \mathcal{W}$ is the functor corresponding to the object (X, Y) in $\mathcal{W} \times \mathcal{W}$.

(iii) *If $\mathcal{C}$ admits a homotopical three-arrow calculus, then we have a homotopy derived pullback diagram in* **Cat** *of the form below,*

$$\begin{tikzcd} \mathcal{C}^T(X,Y) \arrow[r] \arrow[d] & \operatorname{weq}[\mathbb{2},\mathcal{C}] \arrow[d, "\langle d_1, d_0 \rangle"] \\ \mathcal{W}_{/X} \times {}^{Y/}\mathcal{W} \arrow[r] & \mathcal{W} \times \mathcal{W} \end{tikzcd}$$

where the bottom horizontal arrow is defined by the evident projections.

Proof. (i). It is clear that we have a pullback diagram of the required form, and by applying lemma 1.3.10 and proposition 1.5.3, it is not hard to show that the horizontal arrows are indeed weak homotopy equivalences of categories.

(ii). This is a paraphrase of the definition of $\mathcal{C}^T(X, Y)$.

(iii). Consider the following commutative diagram in **Cat**,

$$\begin{tikzcd} \mathcal{C}^T(X,Y) \arrow[r] \arrow[d] & \operatorname{weq}[\mathbb{T},\mathcal{C}]_{\mathrm{h}} \arrow[r] \arrow[d] & \operatorname{weq}[\mathbb{2},\mathcal{C}] \arrow[d] \\ \mathcal{W}_{/X} \times {}^{Y/}\mathcal{W} \arrow[r] \arrow[d] & {[\mathbb{2},\mathcal{W}] \times [\mathbb{2},\mathcal{W}]} \arrow[r, "d_1 \times d_0"'] \arrow[d, "d_0 \times d_1"] & \mathcal{W} \times \mathcal{W} \\ \mathbb{1} \arrow[r, "\ulcorner(X,Y)\urcorner"'] & \mathcal{W} \times \mathcal{W} & \end{tikzcd}$$

where every square is a pullback diagram. We wish to prove that the horizontal rectangle is a homotopy derived pullback diagram, and since its right half is a homotopy derived pullback diagram by lemma 5.1.18, it suffices by lemma 5.1.19 to verify that its left half is a homotopy derived pullback diagram; but it is not hard to check that the vertical arrows in the diagram below are weak homotopy equivalences of categories,

$$\begin{tikzcd} \mathcal{W}_{/X} \times {}^{Y/}\mathcal{W} \arrow[r] \arrow[d] & {[\mathbb{2}, \mathcal{W}] \times [\mathbb{2}, \mathcal{W}]} \arrow[d, "d_0 \times d_1"] \\ \mathbb{1} \arrow[r, "\ulcorner (X,Y) \urcorner"'] & \mathcal{W} \times \mathcal{W} \end{tikzcd}$$

so it is enough in turn to show that the vertical rectangle is a homotopy derived pullback diagram.

Let $\mathbb{H} : \mathcal{W} \times \mathcal{W}^{\mathrm{op}} \to \mathbf{Cat}$ be the diagram defined by $\mathbb{H}^{Y'}_{X'} = \mathcal{C}^T(X', Y')$. Recalling lemma B.5.32, we have the following commutative diagram,

$$\begin{tikzcd} \mathbf{G}(\Delta\mathbb{1}, \mathcal{W}, \mathbf{G}(\mathbb{H}, \mathcal{W}, \Delta\mathbb{1})) \arrow[r, "\cong"] \arrow[d] & \operatorname{weq}[T, \mathcal{C}]_{\mathrm{h}} \arrow[d, "p"] \\ \mathbf{G}(\Delta\mathbb{1}, \mathcal{W}, \Delta\mathcal{W}) \arrow[r, "\cong"'] & \mathcal{W} \times \mathcal{W} \end{tikzcd}$$

where the left vertical arrow is induced by the canonical projection of (the inner) Grothendieck construction. Applying corollary 1.11.18 to corollary 2.8.28, we see that the diagrams

$$\mathbf{G}(\mathbb{H}, \mathcal{W}, \Delta\mathbb{1}) : \mathcal{W} \to \mathbf{Cat} \qquad\qquad \mathbb{H}_X : \mathcal{W}^{\mathrm{op}} \to \mathbf{Cat}$$

both have the property that every reindexing functor is a weak homotopy equivalence of categories, so by lemma 1.11.22, the diagrams

$$\begin{tikzcd} \mathbf{G}\big(\mathbb{H}_X, \mathcal{W}, \Delta\mathbb{1}\big) \arrow[r] \arrow[d] & \operatorname{weq}[T, \mathcal{C}]_{\mathrm{h}} \arrow[d] \\ \mathbb{1} \arrow[r, "\ulcorner X \urcorner"'] & \mathcal{W} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}^T(X, Y) \arrow[r] \arrow[d] & \mathbf{G}\big(\mathbb{H}_X, \mathcal{W}, \Delta\mathbb{1}\big) \arrow[d] \\ \mathbb{1} \arrow[r, "\ulcorner Y \urcorner"'] & \mathcal{W} \end{tikzcd}$$

are homotopy derived pullback diagrams. In addition, by proposition 1.5.15 and lemma 5.1.17, the evident diagram

$$\begin{tikzcd} \mathcal{W} \times \mathcal{W} \arrow[r] \arrow[d] & \mathcal{W} \arrow[d] \\ \mathcal{W} \arrow[r] & \mathbb{1} \end{tikzcd}$$

is a homotopy derived pullback diagram; thus, in the diagram shown below,

$$\begin{tikzcd}
\mathcal{C}^T(X,Y) \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{H}_X, \mathcal{W}, \Delta\mathbb{1}) \arrow[r] \arrow[d] & \mathrm{weq}\,[T,\mathcal{C}]_{\mathrm{h}} \arrow[d] \\
\mathbb{1} \arrow[r, "\ulcorner Y \urcorner"'] & \mathcal{W} \arrow[r, "\ulcorner X \urcorner \times \mathrm{id}_{\mathcal{W}}"'] \arrow[d] & \mathcal{W} \times \mathcal{W} \arrow[d] \\
 & \mathbb{1} \arrow[r, "\ulcorner X \urcorner"'] & \mathcal{W}
\end{tikzcd}$$

every square is a homotopy derived pullback diagram. In particular,

$$\begin{tikzcd}
\mathcal{C}^T(X,Y) \arrow[r] \arrow[d] & \mathrm{weq}\,[T,\mathcal{C}]_{\mathrm{h}} \arrow[d] \\
\mathbb{1} \arrow[r, "\ulcorner (X,Y) \urcorner"'] & \mathcal{W} \times \mathcal{W}
\end{tikzcd}$$

is a homotopy derived pullback diagram, as required. ■

Theorem 2.8.30. *Let $\mathcal{C}$ be a small relative category and let $\underline{\mathcal{D}}$ be the following simplicially enriched category:*

- *The objects are those in $\mathcal{C}$.*
- *For each pair (X,Y) of objects, the hom-space $\underline{\mathcal{D}}(X,Y)$ is given by*

$$\underline{\mathcal{D}}(X,Y)_n = \underline{\mathbf{Lo}}^{\mathrm{H}}\big(\mathbf{S}(\mathcal{C})_n\big)(X,Y)_n$$

  where $\mathbf{S}(\mathcal{C})_\bullet$ is the standard resolution of $\mathcal{C}$.

- *Composition and identities are inherited from $\underline{\mathbf{Lo}}^{\mathrm{H}}\big(\mathbf{S}(\mathcal{C})_\bullet\big)$.*

Let $\underline{\mathbf{Lo}}(\mathcal{C})$ be the standard simplicial localisation of $\mathcal{C}$ and let $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})$ be the hammock localisation of $\mathcal{C}$. Then:

(i) *The simplicially enriched functor $\underline{\mathcal{D}} \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})$ induced by the standard augmentation $\underline{\mathbf{S}}(\mathcal{C}) \to \mathcal{C}$ is a Dwyer–Kan equivalence.*

(ii) *The simplicially enriched functor $\underline{\mathcal{D}} \to \underline{\mathbf{Lo}}(\mathcal{C})$ induced by the localising functors $\mathbf{S}(\mathcal{C})_n \to \mathbf{Lo}(\mathcal{C})_n$ is a Dwyer–Kan equivalence.*

Proof. See Proposition 2.2 in [Dwyer and Kan, 1980b]. □

The following is due to Dugger [2006].

Lemma 2.8.31 (Dugger)**.** *Let $\mathcal{C}$ be a small category, let $\mathcal{U}$ and $\mathcal{V}$ be subclasses of* $\operatorname{mor} \mathcal{C}$*, let X and Y be objects in $\mathcal{C}$, and let $\mathfrak{H}$ be the double category defined below:*

- *The objects in $\mathfrak{H}$ are the zigzags in $\mathcal{C}$ of the form below,*

$$\begin{tikzcd} X & \tilde{X} \arrow[l, "v"'] \arrow[r] & \hat{Y} & Y \arrow[l, "u"'] \end{tikzcd}$$

  where $u : Y \to \hat{Y}$ is in $\mathcal{U}$ and $v : \tilde{X} \to X$ is in $\mathcal{V}$.

- *The horizontal morphisms in $\mathfrak{H}$ are commutative diagrams of the following form:*

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X}_0 \arrow[l] \arrow[r] \arrow[d] & \hat{Y}_0 \arrow[d] & Y \arrow[l] \arrow[d, equal] \\ X & \tilde{X}_1 \arrow[l] \arrow[r] & \hat{Y}_1 & Y \arrow[l] \end{tikzcd}$$

- *The vertical morphisms in $\mathfrak{H}$ are commutative diagrams of the following form:*

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X}_0 \arrow[l] \arrow[r] & \hat{Y}_0 \arrow[d] & Y \arrow[l] \arrow[d, equal] \\ X & \tilde{X}_1 \arrow[l] \arrow[r] \arrow[u] & \hat{Y}_1 & Y \arrow[l] \end{tikzcd}$$

- *The 2-cells in $\mathfrak{H}$ are commutative diagrams of the evident form.*

- *All identities and compositions are inherited from $\mathcal{C}$.*

Let $H_{\bullet,\bullet}$ be the bisimplicial set where $H_{n,m}$ is the set of $n \times m$ composable arrays of 2-cells in $\mathfrak{H}$.

(i) *The canonical morphism $H_{0,\bullet} \to \left| H_{\bullet,\bullet} \right|$ is a weak homotopy equivalence.*

(ii) *The canonical morphism $H_{0,\bullet} \to \left| H_{\bullet,\bullet} \right|$ is a weak homotopy equivalence.*

Proof. (i). First, observe that each $H_{n,\bullet}$ is (isomorphic to) the nerve of a category. As always, we have $d_{0,\bullet} \circ s_{0,\bullet} = \mathrm{id}_{H_{n,\bullet}}$; we claim that $s_{0,\bullet} \circ d_{0,\bullet}$ and $\mathrm{id}_{H_{n+1,\bullet}}$ are intrinsically homotopic. Indeed, consider a vertical morphism in $\mathfrak{H}$, say:

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X}_0 \arrow[l, "v_0"'] \arrow[r, "f_0"] & \hat{Y}_0 \arrow[d, "k"] & Y \arrow[l, "u_0"'] \arrow[d, equal] \\
X & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "f_1"'] \arrow[u, "h"] & \hat{Y}_1 & Y \arrow[l, "u_1"]
\end{tikzcd}$$

We can decompose the above as a cospan of horizontal morphisms in $\mathfrak{H}$,

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X}_0 \arrow[l, "v_0"'] \arrow[r, "f_0"] \arrow[d, equal] & \hat{Y}_0 \arrow[d, "k"] & Y \arrow[l, "u_0"'] \arrow[d, equal] \\
X \arrow[d, equal] & \tilde{X}_0 \arrow[l, "v_0"'] \arrow[r] & \hat{Y}_0 \arrow[d, equal] & Y \arrow[l, "u_1"'] \arrow[d, equal] \\
X & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "f_1"'] \arrow[u, "h"] & \hat{Y}_1 & Y \arrow[l, "u_1"]
\end{tikzcd}$$

thus obtaining the following diagram of 2-cells in $\mathfrak{H}$,

$$\begin{tikzcd}
\left[\overset{v_0}{\longleftarrow}\overset{f_0}{\longrightarrow}\overset{u_0}{\longleftarrow}\right] \arrow[r, "{(\mathrm{id},k)}"] \arrow[d, "{(h,k)}"'] & \left[\overset{v_0}{\longleftarrow}\longrightarrow\overset{u_1}{\longleftarrow}\right] \arrow[d, "{(h,\mathrm{id})}"] & \left[\overset{v_1}{\longleftarrow}\overset{f_1}{\longrightarrow}\overset{u_1}{\longleftarrow}\right] \arrow[l, "{(h,\mathrm{id})}"'] \arrow[d, "{(\mathrm{id},\mathrm{id})}"] \\
\left[\overset{v_1}{\longleftarrow}\overset{f_1}{\longrightarrow}\overset{u_1}{\longleftarrow}\right] \arrow[r, equal] & \left[\overset{v_1}{\longleftarrow}\overset{f_1}{\longrightarrow}\overset{u_1}{\longleftarrow}\right] \arrow[r, equal] & \left[\overset{v_1}{\longleftarrow}\overset{f_1}{\longrightarrow}\overset{u_1}{\longleftarrow}\right]
\end{tikzcd}$$

where the horizontal (resp. vertical) arrows depict horizontal (resp. vertical) morphisms in $\mathfrak{H}$. Thus, for any composable chain of vertical morphisms in $\mathfrak{H}$ of length $n + 1$, we have a natural commutative diagram in $\mathfrak{H}$ of the form below:

$$\begin{tikzcd}
F_0 \arrow[r] \arrow[d] & \bullet \arrow[d] & F_1 \arrow[l] \arrow[d, "\mathrm{id}"] \\
F_1 \arrow[r, equal] \arrow[d] & F_1 \arrow[r, equal] \arrow[d] & F_1 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
F_n \arrow[r, equal] \arrow[d] & F_n \arrow[r, equal] \arrow[d] & F_n \arrow[d] \\
F_{n+1} \arrow[r, equal] & F_{n+1} \arrow[r, equal] & F_{n+1}
\end{tikzcd}$$

Since the rightmost column is the result of applying $s_{0,\bullet} \circ d_{0,\bullet}$ to the leftmost column, this proves the claim. Hence, by proposition 1.5.3, $d_{0,\bullet} : H_{n+1,\bullet} \to H_{n,\bullet}$ is a weak homotopy equivalence. We then apply corollary 1.6.11 and lemma 4.8.2 to deduce that the canonical morphism $H_{0,\bullet} \to \left| H_{\bullet,\bullet} \right|$ is a weak homotopy equivalence.

(ii). A similar argument works. ■

Definition 2.8.32. A **Dwyer–Kan equivalence of relative categories** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ such that the induced simplicially enriched functor $\underline{\mathbf{Lo}}^{\mathrm{H}}(F) : \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C}) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{D})$ is a Dwyer–Kan equivalence of simplically enriched categories.

Definition 2.8.33. Let $\mathcal{C}_\bullet$ be a simplicial category. The **flattening** of $\mathcal{C}_\bullet$ is the relative category $\mathcal{C}^\flat$ defined below:

- The objects are pairs (n, X), where n is a natural number and X is an object in $\mathcal{C}_n$.
- A morphism $(n, X) \to (m, Y)$ is a pair (φ, f), where $\varphi : [m] \to [n]$ is a morphism in $\mathbf{\Delta}$ and $f : \varphi^* X \to Y$ is a morphism in $\mathcal{C}_m$; a weak equivalence is any morphism of the form $(\varphi, \mathrm{id}) : (n, X) \to (m, \varphi^* X)$.
- Given morphisms $(\varphi, f) : (n, X) \to (m, Y)$ and $(\psi, g) : (m, Y) \to (l, Z)$, their composite is $(\varphi \circ \psi, g \circ \psi^* f) : (n, X) \to (l, Z)$.

In other words, $\mathcal{C}^\flat$ is the Grothendieck construction applied to $\mathcal{C}_\bullet$ considered as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$.

Theorem 2.8.34. *Let* **RelCat** *be the category of small relative categories and let* **SCat** *be the category of small simplicially enriched categories.*

(i) *There is a zigzag of natural Dwyer–Kan equivalences between* $\mathrm{id}_{\mathbf{SCat}}$ *and* $\underline{\mathbf{Lo}}^{\mathrm{H}}\big((-)^\flat\big)$.

(ii) *There is a zigzag of natural Dwyer–Kan equivalences between* $\mathrm{id}_{\mathbf{RelCat}}$ *and* $\mathbf{Lo}^{\mathrm{H}}(-)^\flat$.

(iii) *If we regard* **RelCat** *as a homotopical category where the weak equivalences are the Dwyer–Kan equivalences of relative categories, then the functors*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(-) : \mathbf{RelCat} \to \mathbf{SCat} \qquad (-)^\flat : \mathbf{SCat} \to \mathbf{RelCat}$$

are a mutually quasi-inverse pair of homotopical equivalences.

Proof. See paragraph 2.5 and Proposition 3.1 in [Barwick and Kan, 2012]. □

Theorem 2.8.35 (Relative Yoneda embedding)**.** *Let $\mathcal{C}$ be a small relative category and let $\hbar_\bullet : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}$ be the relative functor defined by the formula below:*

$$\hbar_Y(X) = \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y)$$

(i) *For each pair (X, Y) of objects in $\mathcal{C}$, the induced hom-space morphism*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y) \to \underline{\mathbf{Lo}}^{\mathrm{H}}\big([\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}\big)(X, Y)$$

is a weak homotopy equivalence of simplicial sets.

(ii) *Let $\mathcal{D}$ be the full relative subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}$ spanned by the relative functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ that are naturally weakly equivalent to one in the image of $\hbar_\bullet$. Then the functor $\hbar_\bullet : \mathcal{C} \to \mathcal{D}$ is a Dwyer–Kan equivalence of relative categories.*

Proof. See paragraph 4.3 in [Barwick and Kan, 2011]. □

2.9 Homotopy-coherent diagrams

Prerequisites. §§1.1, 1.2, 2.1, 2.3, 2.5, 2.7, 6.1.

Definition 2.9.1. Let $\mathcal{J}$ be an ordinary category. A **homotopy-coherent diagram** of shape $\mathcal{J}$ in a simplicially enriched category $\underline{\mathcal{C}}$ is a simplicially enriched functor $\underline{\mathbf{S}}(\mathcal{J}) \to \underline{\mathcal{C}}$.

Remark 2.9.2. It is worth thinking about the data that comprise a homotopy-coherent diagram of shape $\mathcal{J}$: in degree 0, one must specify a morphism $F(f)$ in $\mathcal{C}$ for every non-trivial morphism f in $\mathcal{J}$ (but this assignment need not be functorial!); in degree 1, for every composable string of non-trivial morphisms of positive length, such as $f_3 \circ f_2 \circ f_1$, one has a simplicial homotopy from the "free" composition to the "true" composition, e.g.

$$\mu_{f_3, f_2, f_1} : F\big(f_3\big) \circ F\big(f_2\big) \circ F\big(f_1\big) \Rightarrow F\big(f_3 \circ f_2 \circ f_1\big)$$

and so on in higher degrees. The phrase 'homotopy-coherent' alludes to the relations imposed by the higher simplices: for instance, for each composable triple (f_3, f_2, f_1) as above, one has a pair of 2-cells in $\operatorname{mor} \underline{\mathcal{C}}$ as in the diagram below:

$$\begin{tikzcd}
& F(f_3 \circ f_2) \circ F(f_1) \arrow[dr, "\mu_{f_3 \circ f_2, f_1}"] \arrow[d, phantom, "\Downarrow"] & \\
F(f_3) \circ F(f_2) \circ F(f_1) \arrow[ur, "\mu_{f_3, f_2} \circ \mathrm{id}_{f_1}"] \arrow[rr, "\mu_{f_3, f_2, f_1}"'] \arrow[dr, "\mathrm{id}_{f_3} \circ \mu_{f_2, f_1}"'] & & F(f_3 \circ f_2 \circ f_1) \\
& F(f_3) \circ F(f_2 \circ f_1) \arrow[ur, "\mu_{f_3, f_2 \circ f_1}"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}$$

In particular, if $\underline{\mathcal{C}}$ is obtained from a 2-category $\mathfrak{C}$ by applying the nerve functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ to its hom-categories, a homotopy-coherent diagram of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$ is the same thing as a normalised lax 2-functor $\mathcal{J} \to \mathfrak{C}$.

Definition 2.9.3. The **homotopy-coherent nerve** of a simplicially enriched category $\underline{\mathcal{C}}$ is the simplicial set defined by the formula below,

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})_n = \{\text{simplicially enriched functors } \underline{\mathbf{S}}([n]) \to \underline{\mathcal{C}}\}$$

with face and degeneracy maps induced by the coface and codegeneracy maps in $\mathbf{\Delta}$.

Proposition 2.9.4. *Let* $\mathbf{SCat}$ *be the category of small simplicially enriched categories.*

(i) $\mathrm{N}^{\mathrm{hc}} : \mathbf{SCat} \to \mathbf{sSet}$ *has a left adjoint, which is the unique (up to unique isomorphism) colimit-preserving functor* $\underline{\mathbf{C}} : \mathbf{sSet} \to \mathbf{SCat}$ *such that* $\underline{\mathbf{C}}(\Delta^n) = \underline{\mathbf{S}}([n])$.

(ii) $\mathrm{N}^{\mathrm{hc}} : \mathbf{SCat} \to \mathbf{sSet}$ *and* $\underline{\mathbf{C}} : \mathbf{sSet} \to \mathbf{SCat}$ *are both accessible functors.*

(iii) *If* $\mathbb{C}$ *is a small category regarded as a simplicially enriched category, then* $\mathrm{N}^{\mathrm{hc}}(\mathbb{C})$ *is naturally isomorphic to* $\mathrm{N}(\mathbb{C})$.

Proof. (i). Apply theorem 1.1.13.

(ii). This is an instance of the accessible adjoint functor theorem (0.2.50).

(iii). This follows from proposition 2.5.14 and corollary 2.7.24. ∎

Definition 2.9.5. Given a simplicial set X, the **associated simplicially enriched category** is the simplicially enriched category $\underline{\mathbf{C}}(X)$ constructed above.

Remark 2.9.6. The stability of accessible adjunctions under universe enlargement implies that the simplicially enriched category $\underline{\mathbf{C}}(X)$ associated with a simplicial set X does not depend on the choice of universe.

Remark 2.9.7. One way of getting a good grip on the hom-spaces of $\underline{\mathbf{C}}(X)$ for a general simplicial set X is to use the formalism of necklaces introduced by Dugger and Spivak [2011b].

Theorem 2.9.8 (Riehl)**.**

(i) *For any simplicial set X and any pair (a, b) of vertices of X, the hom-space $\underline{\mathbf{C}}(X)(a, b)$ is 3-coskeletal.*

(ii) *For any category $\mathbb{C}$ and any pair (A, B) of objects in $\mathbb{C}$, the hom-space $\underline{\mathbf{C}}(\mathrm{N}(\mathbb{C}))(A, B)$ is 2-coskeletal.*

(iii) *For any category $\mathbb{C}$, its associated simplicially enriched category $\underline{\mathbf{C}}(\mathrm{N}(\mathbb{C}))$ is naturally isomorphic to the standard resolution $\underline{\mathbf{S}}(\mathbb{C})$.*

Proof. See Theorems 4.1, 6.4, and 6.7 in [Riehl, 2011c]. □

Corollary 2.9.9. *For any simplicially enriched category $\underline{\mathcal{C}}$ and any ordinary category $\mathcal{J}$, there is a bijection*

$$\left\{\text{simplicial maps } \mathrm{N}(\mathcal{J}) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})\right\} \cong \{\textit{homotopy-coherent diagrams of shape } \mathcal{J} \textit{ in } \underline{\mathcal{C}}\}$$

and it is natural in $\mathcal{J}$ and in $\underline{\mathcal{C}}$. ■

Remark 2.9.10. The above result can also be proven directly, and the uniqueness of representations for functors up to unique isomorphism then implies that $\underline{\mathbf{C}}(\mathrm{N}(\mathcal{J}))$ must be isomorphic to $\underline{\mathbf{S}}(\mathcal{J})$.

Definition 2.9.11. Let F and G be homotopy-coherent diagrams of shape $\mathcal{J}$ in a simplicially enriched category $\underline{\mathcal{C}}$. A **homotopy-coherent natural transformation** $F \Rightarrow G$ is a homotopy-coherent diagram of shape $\mathcal{J} \times [1]$ such that the restriction along $\underline{\mathbf{S}}(\mathrm{id}_{\mathcal{J}} \times \delta^1)$ is F and the restriction along $\underline{\mathbf{S}}(\mathrm{id}_{\mathcal{J}} \times \delta^0)$ is G.

Unfortunately, it is in general not possible to compose homotopy-coherent natural transformations, and even when it is possible, the composite is usually only well-defined up to higher homotopy. Instead, in good situations, what we get is a quasicategory:

Theorem 2.9.12. *Let $\mathcal{J}$ be a small category and let $\underline{\mathcal{C}}$ be a small simplicially enriched category. Consider the following simplicial set:*

$$[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}} = \left[\mathrm{N}(\mathcal{J}), \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})\right]$$

(i) *There is a natural identification of the vertices of $[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ as homotopy-coherent diagrams of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$, and similarly, there is a natural identification of the edges as homotopy-coherent natural transformations.*

(ii) *If $\underline{\mathcal{C}}$ is fibrant, then the homotopy-coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})$ is a small quasicategory.*

(iii) *Under the same hypothesis, $[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is a small quasicategory.*

Proof. (i). Apply corollary 2.9.9 to the explicit description of exponential objects in the category of simplicial **U**-sets.

(ii). See Theorem 2.1 in [Cordier and Porter, 1986].

(iii). Use corollary 6.2.15. □

Let us say that a locally small simplicially enriched category $\underline{\mathcal{C}}$ admits **rectification for homotopy-coherent diagrams** if, for all small categories $\mathcal{J}$, we have a commutative diagram of functors of the form below,

$$\begin{tikzcd} {[\mathcal{J}, \mathcal{C}]} \arrow[r] \arrow[d] & \tau_1 [\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}} \\ \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] \arrow[ur, dashed] & \end{tikzcd}$$

where $[\mathcal{J}, \mathcal{C}] \to \tau_1 [\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is the functor

$$[\mathcal{J}, \mathcal{C}] \cong \tau_1 \mathrm{N}([\mathcal{J}, \mathcal{C}]) \cong \tau_1 [\mathrm{N}(\mathcal{J}), \mathrm{N}(\mathcal{C})] \to \tau_1 [\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$$

induced by the canonical morphism $\mathrm{N}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})$, $[\mathcal{J}, \mathcal{C}] \to \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$ is the localising functor, and $\pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] \to \tau_1 [\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is fully faithful and essentially surjective on objects. (Note that this functor is unique *if* it exists, because the localising functor $[\mathcal{J}, \mathcal{C}] \to \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$ is full and bijective on objects.)

Theorem 2.9.13 (Cordier–Porter)**.** *Let $\underline{C}$ be a locally small simplicially enriched category. Consider the following conditions:*

(i) *$\underline{C}$ is fibrant and complete as a simplicially enriched category.*

(ii) *$\underline{C}$ is fibrant and cocomplete as a simplicially enriched category.*

(iii) *$\underline{C}$ is the simplicially enriched category of Kan complexes.*

If $\underline{C}$ satisfies any one of the above conditions, then $\underline{C}$ admits rectification for homotopy-coherent diagrams.

Proof. (i). See Theorem 4.7 in [Cordier and Porter, 1986].

(ii). This follows from claim (i) by duality.

(iii). See the remark following Corollary 2.3 in [Cordier and Porter, 1997]. □

2.10 The Bergner model structure

Prerequisites. §§1.5, 2.1, 2.5, 4.1, 5.2.

Definition 2.10.1. A **Dwyer–Kan isofibration of simplicially enriched categories** is a simplicially enriched functor $\underline{P} : \underline{C} \to \underline{D}$ with the following properties:

- $\underline{P} : \underline{C} \to \underline{D}$ is a local fibration of simplicially enriched categories.
- If $g : PC \to D$ is a weak simplicial homotopy equivalence in $\underline{D}$, then there is a weak simplicial homotopy equivalence $f : C \to \tilde{D}$ in $\underline{C}$ such that $P\tilde{D} = D$ and $Pf = g$.

Theorem 2.10.2 (Bergner)**.** *The following data constitute a cofibrantly generated model structure on* **SCat***:*

- *The weak equivalences are the Dwyer–Kan equivalences.*
- *The fibrations are the Dwyer–Kan isofibrations.*
- *The cofibrations are the morphisms that have the left lifting property with respect to the Dwyer–Kan isofibrations.*

This model structure is called the **Bergner model structure***, and the fibrant objects are the Kan-enriched categories.*

Proof. See Theorem 1.1 in [Bergner, 2007]. □

Proposition 2.10.3. *The Bergner model structure on* **SCat** *is right proper.*[9]

Proof. See Proposition 3.5 in [Bergner, 2007]. □

[9] See definition 5.1.6.

— III —

Homotopical categories

3.1 Basics

Prerequisites. § A.4.

Definition 3.1.1. A relative category $\mathcal{C}$ is a **category with weak equivalences** if it is semi-saturated and $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property, and it is a **homotopical category** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property. A **homotopical functor** is a relative functor between homotopical categories.

Remark 3.1.2. If $\mathcal{C}$ is a relative category such that $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property, then every isomorphism in $\mathcal{C}$ is automatically a weak equivalence. Indeed, suppose $f : X \to Y$ and $g : Y \to X$ are mutual inverses in $\mathcal{C}$; then the fact that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ are in $\operatorname{weq} \mathcal{C}$ implies that f and g must also be in $\operatorname{weq} \mathcal{C}$. Recalling lemma A.4.14, it follows that every homotopical category is a category with weak equivalences.

¶ 3.1.3. To simplify notation, we will usually not distinguish between $\operatorname{und} \mathcal{C}$ and $\mathcal{C}$. For example, when $\mathcal{C}$ and $\mathcal{D}$ are relative categories, then by 'ordinary functor $\mathcal{C} \to \mathcal{D}$' we mean a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$.

Example 3.1.4. Any saturated relative category is automatically a homotopical category, by corollary A.4.15. In particular, any minimal saturated relative category is a homotopical category. On the other hand, any maximal relative category is obviously a homotopical category.

Remark 3.1.5. A relative category $\mathcal{C}$ is a category with weak equivalences or a homotopical category if and only if the opposite relative category $\mathcal{C}^{\mathrm{op}}$ is.

Lemma 3.1.6. *Let A be an object in a homotopical category (resp. category with weak equivalences) $\mathcal{C}$. Then the slice category $\mathcal{C}_{/A}$ is also a homotopical category (resp. category with weak equivalences) if we declare a morphism in $\mathcal{C}_{/A}$ to be a weak equivalence if and only if it is a weak equivalence in $\mathcal{C}$.*

Proof. Use lemma A.4.14 on the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$. ■

Lemma 3.1.7. *Any relative subcategory $\mathcal{D}$ of a homotopical category (resp. category with weak equivalences) $\mathcal{C}$ is also a homotopical category (resp. category with weak equivalences).*

Proof. Use lemma A.4.14 on the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$. ■

Lemma 3.1.8. *Let $\mathcal{C}$ be a relative category, let $\mathcal{D}$ be a saturated homotopical category, and let $F : \mathcal{C} \to \mathcal{D}$ be a relative functor. If a morphism in $\mathcal{C}$ is a weak equivalence if and only if its image under F is a weak equivalence in $\mathcal{D}$, then $\mathcal{C}$ is also a saturated homotopical category.*

Proof. Consider the induced functor $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that f is an isomorphism in $\operatorname{Ho} \mathcal{C}$. Since $\operatorname{Ho} F$ is a functor, Ff must be an isomorphism in $\operatorname{Ho} \mathcal{D}$; but $\mathcal{D}$ is saturated, so Ff is a weak equivalence in $\mathcal{D}$. We may therefore deduce that f is a weak equivalence in $\mathcal{C}$. ■

Corollary 3.1.9. *Any relative subcategory of a saturated homotopical category is a saturated homotopical category.* ■

Lemma 3.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be two relative categories. If $\mathcal{D}$ is a homotopical category (resp. category with weak equivalences), then the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is also a homotopical category (resp. category with weak equivalences).*

Proof. This is a straightforward check. ◆

Lemma 3.1.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be two relative categories. If $\mathcal{D}$ is a saturated homotopical category, then the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is also a saturated homotopical category.*

Proof. For each object C in $\mathcal{C}$, we have a homotopical functor $C^* : [\mathcal{C}, \mathcal{D}]_{\mathrm{h}} \to \mathcal{D}$ that evaluates an object F in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ at C. Thus, we obtain a functor $\operatorname{Ho} C^* : \operatorname{Ho} [\mathcal{C}, \mathcal{D}]_{\mathrm{h}} \to \operatorname{Ho} \mathcal{D}$.

Consider a morphism $\varphi : F \Rightarrow F'$ in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ such that φ is an isomorphism in $\operatorname{Ho}[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Since $\operatorname{Ho}\mathcal{C}^*$ is a functor, $(\operatorname{Ho}\mathcal{C}^*)(\varphi)$ must be an isomorphism in $\operatorname{Ho}\mathcal{C}$; but $\mathcal{C}$ is a saturated homotopical category, so that implies the component φ_C is a weak equivalence in $\mathcal{C}$. We therefore conclude that φ is a weak equivalence in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. ■

Definition 3.1.12. Two objects in a relative category are **weakly equivalent** if they can be connected by a *zigzag* of weak equivalences; we define $X \overset{\mathrm{w}}{\simeq} Y$ to mean that X and Y are weakly equivalent.

Remark 3.1.13. If X and Y are weakly equivalent in a relative category $\mathcal{C}$, then they are isomorphic in $\operatorname{Ho}\mathcal{C}$. The converse is certainly true if $\mathcal{C}$ is saturated, but is false if $\mathcal{C}$ is not semi-saturated.

Definition 3.1.14. A **homotopically replete subcategory** of a relative category $\mathcal{C}$ is a relative subcategory $\mathcal{D}$ with the following property:

- If D is an object in $\mathcal{D}$ and $f : C \to D$ is a weak equivalence in $\mathcal{C}$, then both C and f are in $\mathcal{D}$.
- If D is an object in $\mathcal{D}$ and $g : D \to C$ is a weak equivalence in $\mathcal{C}$, then both C and g are in $\mathcal{D}$.

Remark 3.1.15. Any full relative subcategory $\mathcal{D}$ of a relative category $\mathcal{C}$ is homotopically replete if and only if it has the following property:

- If D is an object in $\mathcal{D}$ and C an object in $\mathcal{C}$ that is weakly equivalent to D, then C is in $\mathcal{D}$.

Definition 3.1.16. A parallel pair of morphisms in a relative category $\mathcal{C}$ are **weakly homotopic** if they are equal in $\operatorname{Ho}\mathcal{C}$; we define $f \overset{\mathrm{w}}{\sim} g$ to mean that f and g are weakly homotopic.

Definition 3.1.17. An **equivalence** in a relative category $\mathcal{C}$ is a pair (f, g), where $f : X \to Y$ and $g : Y \to X$ are morphisms in $\mathcal{C}$ such that $g \circ f \overset{\mathrm{w}}{\sim} \mathrm{id}_X$ and $f \circ g \overset{\mathrm{w}}{\sim} \mathrm{id}_Y$. Two morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ are **mutual quasi-inverses** when (f, g) constitute an equivalence in $\mathcal{C}$.

Remark 3.1.18. It follows from the definitions that quasi-inverses are unique up to weak homotopy.

Lemma 3.1.19. *If the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ for a relative category $\mathcal{C}$ is full, then the following are equivalent for all morphisms $f : X \to Y$ in $\mathcal{C}$:*

- *f is a morphism in $\mathcal{C}$ and has a quasi-inverse.*
- *γf is an isomorphism in $\mathcal{C}$.*

Proof. Obvious. ♦

Remark 3.1.20. Clearly, any isomorphism in any relative category has a quasi-inverse; but this implies that in a relative category that is *not* semi-saturated, a morphism that has a quasi-inverse need *not* be a weak equivalence. On other hand, if f is a morphism in a *saturated* homotopical category and f has a quasi-inverse, then f must be a weak equivalence.

Definition 3.1.21. A relative category $\mathcal{C}$ has the **Whitehead property** when the following are equivalent:

- f is a weak equivalence in $\mathcal{C}$.
- f is a morphism in $\mathcal{C}$ and has a quasi-inverse.

Theorem 3.1.22. *Let $\mathcal{C}$ be a relative category. The following are equivalent:*

(i) *$\mathcal{C}$ has the Whitehead property.*

(ii) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ is full, and $\mathcal{C}$ is a saturated homotopical category.*

Proof. (i) ⇒ (ii). By theorem A.4.44, every morphism $\gamma X_0 \to \gamma X_n$ in $\operatorname{Ho} \mathcal{C}$ is of the form

$$\left(\gamma f_n\right)^{-1} \circ \cdots \circ \gamma h_2 \circ \left(\gamma f_1\right)^{-1} \circ \gamma h_1$$

for some morphisms $h_1 : X_0 \to Y_1$, $f_1 : X_1 \to Y_1$, $h_2 : X_1 \to Y_2$, etc. in $\mathcal{C}$, where $f_1, \ldots, f_n$ are weak equivalences. By the Whitehead property, each $f_i : X_i \to Y_i$ has a quasi-inverse in $\mathcal{C}$, say $g_i : Y_i \to X_i$. Since $\gamma g_i = \left(\gamma f_i\right)^{-1}$, it follows that

$$\left(\gamma f_n\right)^{-1} \circ \cdots \circ h_2 \circ \left(\gamma f_1\right)^{-1} \circ \gamma h_1 = \gamma\left(g_n \circ \cdots \circ h_2 \circ g_1 \circ h_1\right)$$

and therefore $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ is indeed full.

In particular, every morphism $f : X \to Y$ in $\mathcal{C}$ such that $\gamma f : \gamma X \to \gamma Y$ is an isomorphism in $\operatorname{Ho} \mathcal{C}$ must have a quasi-inverse, and hence must be a weak equivalence, in view of the Whitehead property. We therefore conclude that $\mathcal{C}$ is a saturated homotopical category.

(ii) ⇒ (i). The converse follows from the definitions and lemma 3.1.19. ■

Remark 3.1.23. The Whitehead property is in general not inherited by slice categories or by functor categories. For example, if $q \circ f = p$ and g is a quasi-inverse for f, it is only guaranteed that $q \overset{\mathrm{w}}{\sim} p \circ g$.

Definition 3.1.24. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two ordinary functors between relative categories. A **natural weak equivalence** $\alpha : F \Rightarrow G$ is a natural transformation such that $\alpha_C : FC \to GC$ is a weak equivalence in $\mathcal{D}$ for all objects C in $\mathcal{C}$, and we say F and G are **naturally weakly equivalent** if they can be connected by a *zigzag* of natural weak equivalences.

Remark 3.1.25. This is precisely the notion of weak equivalence in the relative functor category $[\min \operatorname{und} \mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Although the definition above applies to all functors, if $H : \mathcal{D} \to \mathcal{E}$ is an ordinary functor, then the natural transformation $H\alpha : HF \Rightarrow HG$ is only guaranteed to be a natural weak equivalence if we assume H is a relative functor.

Definition 3.1.26. A **relative equivalence** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ for which there exists a relative functor $G : \mathcal{D} \to \mathcal{C}$ such that GF is naturally weakly equivalent to $\mathrm{id}_{\mathcal{C}}$ and FG is naturally weakly equivalent to $\mathrm{id}_{\mathcal{D}}$. Such a G is said to be a **relative inverse** of F. When $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories, we may say **homotopical equivalence** and **homotopical inverse** instead of 'relative equivalence' and 'relative inverse'.

Proposition 3.1.27. *If $F : \mathcal{C} \to \mathcal{D}$ is a relative equivalence of relative categories with relative inverse $G : \mathcal{D} \to \mathcal{C}$, then $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ is an equivalence of categories, with quasi-inverse $\operatorname{Ho} G : \operatorname{Ho} \mathcal{D} \to \operatorname{Ho} \mathcal{C}$.* ■

Definition 3.1.28. An **adjoint relative equivalence** is an adjunction of the form below,

$$F \dashv G : \mathcal{D} \to \mathcal{C}$$

where $\mathcal{C}$ and $\mathcal{D}$ are relative categories, F and G are relative functors, and both the adjunction unit and counit are natural weak equivalences. When $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories, we may say **adjoint homotopical equivalence** instead of 'adjoint relative equivalence'.

Proposition 3.1.29. *An adjoint relative equivalence of relative categories descends to an adjoint equivalence of homotopy categories.*

Proof. Use the 2-functoriality of Ho : $\mathfrak{RelCat} \to \mathfrak{Cat}$ (corollary A.4.20). ■

Definition 3.1.30. A **homotopically contractible category** is a homotopical category C such that the unique (homotopical) functor $C \to \mathbb{1}$ is a homotopical equivalence, where $\mathbb{1}$ is the trivial category with only one object.

Proposition 3.1.31. *Let C be a homotopical category. The following are equivalent:*

(i) *C is homotopically contractible.*

(ii) *C is inhabited, and for every object A in C, the constant functor ΔA is naturally weakly equivalent to id_C.*

(iii) *There exists an object A in C such that ΔA and id_C are naturally weakly equivalent.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ♦

3.2 Homotopical Kan extensions

Prerequisites. §§3.1, A.4.

Definition 3.2.1. Let C be a homotopical category. A **homotopically initial object** in C is an object A for which there exists a zigzag of natural transformations of the form

$$\Delta A \rightsquigarrow F \xrightarrow{\alpha} G \rightsquigarrow \mathrm{id}_C$$

where $\Delta A : C \to C$ is the constant functor with value A, $\alpha_A : FA \to GA$ is a weak equivalence in C, and the squiggles denote (possibly trivial) zigzags of natural weak equivalences. Dually, a **homotopically terminal object** in C is a homotopically initial object in C^{op}.

Proposition 3.2.2. *Let C be a homotopical category. If A is a homotopically initial (resp. homotopically terminal) object in C, then:*

(i) *Any object in C weakly equivalent to A is also a homotopically initial (resp. homotopically terminal) object in C.*

(ii) *A is an initial (resp. terminal) object in* $\operatorname{Ho} \mathcal{C}$.

(iii) *If $\mathcal{C}$ is a minimal homotopical category, then A is an initial (resp. terminal) object in $\mathcal{C}$ as well.*

Conversely, any initial (resp. terminal) object in $\mathcal{C}$ is also homotopically initial (resp. homotopically terminal).

Proof. Obvious. (This is Proposition 38.3 in [DHKS].) ◆

Proposition 3.2.3. *If A is a homotopically initial object in a homotopical category $\mathcal{C}$, then for any object Z in $\mathcal{C}$, the zigzag category $\mathcal{C}^{(\mathbf{T})}(A, Z)$ is connected.*

Proof. By theorem A.4.44, there is a bijection between the connected components of $\mathcal{C}^{(\mathbf{T})}(A, Z)$ and the morphisms $A \to Z$ in $\operatorname{Ho} \mathcal{C}$; but we know A is an initial object in $\operatorname{Ho} \mathcal{C}$, so $\mathcal{C}^{(\mathbf{T})}(A, Z)$ has exactly one connected component. ■

Lemma 3.2.4. *Let $H : \mathcal{C} \to \mathcal{D}$ be a relative functor and let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor. If If* $\operatorname{weq} \mathcal{D}$ *has the 2-out-of-3 property and F is naturally weakly equivalent to H, then F is also a relative functor.*

Proof. Apply the 2-out-of-3 property inductively. ◆

Lemma 3.2.5. *If A and A' be homotopically initial objects in a homotopical category $\mathcal{C}$, then $A \overset{\mathrm{w}}{\simeq} A'$, and moreover every morphism $A \to A'$ in $\mathcal{C}$ is a weak equivalence.*

Proof. This is paragraph 38.5 in [DHKS].

Suppose, as in the definition, that we have endofunctors F, F', G, G' on $\mathcal{C}$ and natural transformations $\alpha : F \Rightarrow G$, $\alpha' : F' \Rightarrow G'$, such that $F \overset{\mathrm{w}}{\simeq} \Delta A$, $F' \overset{\mathrm{w}}{\simeq} \Delta A'$, $G \overset{\mathrm{w}}{\simeq} \mathrm{id}_{\mathcal{C}}$, and $G' \overset{\mathrm{w}}{\simeq} \mathrm{id}_{\mathcal{C}}$, and the morphisms $\alpha_A : FA \to GA$ and $\alpha'_{A'} : FA' \to GA'$ are both weak equivalences. Note that the previous lemma implies G and G' are both homotopical functors, while a similar argument shows that F and F' sends *all* morphisms to weak equivalences.

Let $f : A \to A'$ be a morphism in $\mathcal{C}$. By applying the 2-out-of-3 property repeatedly in the following diagram,

$$\begin{tikzcd} A \arrow[r, squiggly, dash] \arrow[d, equal] & FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[r, squiggly, dash] \arrow[d, "Gf"] & A \arrow[d, "f"] \\ A \arrow[r, squiggly, dash] & FA' \arrow[r, "\alpha_{A'}"'] & GA' \arrow[r, squiggly, dash] & A' \end{tikzcd}$$

we see that f is a weak equivalence if and only if $\alpha_{A'} : FA' \to GA'$ is a weak equivalence. Since $\alpha'_{A'} : F'A' \to G'A'$ is a weak equivalence, and $GA' \overset{\mathrm{w}}{\simeq} A'$, it follows that $\alpha'_{GA'} : FGA' \to G'GA'$ is a weak equivalence, and since G is homotopical, so $G\alpha'_{GA'} : GFGA' \to GG'GA'$ is also a weak equivalence. Similarly, $\alpha_A : FA \to GA$ is a weak equivalence, and $A \overset{\mathrm{w}}{\simeq} FA' \overset{\mathrm{w}}{\simeq} G'FA'$, so $\alpha_{G'FA'} : FG'FA' \to GG'FA'$ is a weak equivalence as well.

Now, by applying the 2-out-of-6 property to the diagram below,

$$\begin{tikzcd}
FF'FA' \arrow[r, "\alpha_{F'FA'}"] \arrow[d, "F\alpha'_{FA'}"'] & GF'FA' \arrow[r, "GF'\alpha_{A'}"] \arrow[d, "G\alpha'_{FA'}"'] & GF'GA' \arrow[d, "G\alpha'_{GA'}"] \\
FG'FA' \arrow[r, "\alpha_{G'FA'}"'] & GG'FA' \arrow[r, "GG'\alpha_{A'}"'] & GG'GA'
\end{tikzcd}$$

we may deduce that $GG'\alpha_{A'} : GG'FA' \to GG'GA'$ is a weak equivalence, and hence that $\alpha_{A'} : FA' \to GA'$ is a weak equivalence, as required. Moreover, $A \overset{\mathrm{w}}{\simeq} FA'$ and $GA' \overset{\mathrm{w}}{\simeq} A'$, so it follows that $A \overset{\mathrm{w}}{\simeq} A'$. ■

¶ 3.2.6. We will say that an object in a homotopical category $\mathcal{C}$ characterised by a homotopical universal property is **homotopically unique** if the full subcategory spanned by such objects inside the homotopical category of objects in $\mathcal{C}$ *equipped with the relevant additional structure* is homotopically contractible.

Proposition 3.2.7. *Let $\mathcal{C}$ be a homotopically contractible category.*

(i) *Every morphism in $\mathcal{C}$ is a weak equivalence.*

(ii) *The unique functor* $\operatorname{Ho}\mathcal{C} \to \mathbb{1}$ *is an equivalence of categories.*

(iii) *If $\mathcal{C}$ is a minimal homotopical category, then $\mathcal{C} \to \mathbb{1}$ is also an equivalence of categories.*

(iv) *The opposite homotopical category $\mathcal{C}^{\mathrm{op}}$ and the homotopical functor category $[\mathcal{D}, \mathcal{C}]_{\mathrm{h}}$ (for any homotopical category $\mathcal{D}$) are also homotopically contractible.*

(v) *Every object in $\mathcal{C}$ is both homotopically initial and homotopically terminal.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ♦

Proposition 3.2.8. *Let $\mathcal{C}$ be a homotopical category. If $\mathcal{D}$ is the full homotopical subcategory of $\mathcal{C}$ spanned by the homotopically initial (or homotopically terminal) objects, then $\mathcal{D}$ is homotopically contractible.*

Proof. This follows from lemma 3.2.5. ■

REMARK 3.2.9. Even if $\mathcal{C}$ is a saturated homotopical category, an object that is initial in $\operatorname{Ho}\mathcal{C}$ need not be homotopically initial in $\mathcal{C}$. Indeed, let $\mathcal{C}$ be the maximal homotopical category generated by a graph of the following form:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots$$

No object in $\mathcal{C}$ is homotopically initial, because the length of the shortest zigzag connecting two objects cannot be bounded above; yet every object in $\operatorname{Ho}\mathcal{C}$ is initial. The same argument shows that $\mathcal{C}$ is not homotopically contractible, but $\operatorname{Ho}\mathcal{C}$ is certainly contractible.

Definition 3.2.10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two ordinary functors between homotopical categories. A **homotopical left Kan extension** (resp. **homotopical right Kan extension**) of G along F is a homotopically initial (resp. homotopically terminal) object of the homotopical category $(G \downarrow F^*)_{\mathrm{h}}$ (resp. $(F^* \downarrow G)_{\mathrm{h}}$) described below:

- The objects are pairs (H, α) where H is a homotopical functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).
- The weak equivalences are the natural weak equivalences.

REMARK 3.2.11. Note that any homotopical Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $G : \mathcal{C} \to \mathcal{E}$ has, by definition, an underlying *homotopical* functor $H : \mathcal{D} \to \mathcal{E}$.

Corollary 3.2.12. *Homotopical Kan extensions are homotopically unique, any two homotopical left (resp. right) Kan extensions of G along F are naturally weakly equivalent.* ■

Definition 3.2.13. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two ordinary functors between homotopical categories, and let $L : \mathcal{E} \to \mathcal{F}$ be a homotopical functor. We say L **preserves** a homotopical left (resp. right) Kan extension (H, α) of G along F if $(LH, L\alpha)$ is a homotopical left (resp. right) Kan extension of LF along G. If a homotopical Kan extension is preserved by *all* homotopical functors, then it is said to be **absolute**.

3.3 Quillen–Verdier derived functors

Prerequisites. §§ 3.1, A.4, A.1, A.5

The fact that $\mathrm{Ho} : \mathfrak{RelCat} \to \mathfrak{Cat}$ is a 2-functor means that relative functors $F : \mathcal{C} \to \mathcal{D}$ descend to functors $\mathrm{Ho}\, F : \mathrm{Ho}\, \mathcal{C} \to \mathrm{Ho}\, \mathcal{D}$ in a very well-behaved way. However, what can we say about ordinary (i.e. not necessarily relative) functors $\mathcal{C} \to \mathcal{D}$?

In this section, we follow [DHKS, §§ 40–43]; however, we will use a weaker definition of 'deformation retract' and a stronger definition of 'total derived functor'.

Definition 3.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories, and let $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}\, \mathcal{C}$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}\, \mathcal{D}$ be the localising functors.

- A **total left derived functor** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is an absolute right (!) Kan extension of $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathrm{Ho}\, \mathcal{D}$ along $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}\, \mathcal{C}$.

- A **total right derived functor** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is an absolute left (!) Kan extension of $\gamma_{\mathcal{C}} G : \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$ along $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}\, \mathcal{D}$.

Remark 3.3.2. The above definition is essentially due to Verdier [1963], but the formulation using Kan extensions is due to Quillen [1967, Ch. I, § 4]. We deviate from convention by demanding that the Kan extensions be *absolute*; this is in order to make theorem 3.3.5 true.

Remark 3.3.3. As with everything defined by a universal property, total derived functors are unique up to unique isomorphism *if* they exist.

Definition 3.3.4. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of ordinary categories. A **derived adjunction** for $F \dashv G$ consists of

- a left derived functor $(\mathbf{L}F, \alpha)$ for F,

- a right derived functor $(\mathbf{R}G, \beta)$ for G, and

- an adjunction $\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}\, \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$ with unit $\bar{\eta} : \mathrm{id}_{\mathrm{Ho}\, \mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ and counit $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\mathrm{Ho}\, \mathcal{D}}$,

such that (α, β) constitute a conjugate pair of natural transformations. We refer to $\bar{\eta}$ as the **derived unit** and $\bar{\varepsilon}$ as the **derived counit**.

The following appears in [Maltsiniotis, 2007].

Theorem 3.3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an ordinary adjunction. If $(\mathbf{L}F, \alpha)$ is a total left derived functor for F and $(\mathbf{R}G, \beta)$ is a total right derived functor for G, then there exist unique natural transformations $\bar{\eta} : \mathrm{id}_{\mathrm{Ho}\,\mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ and $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\mathrm{Ho}\,\mathcal{D}}$ making $\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{C}$ a derived adjunction for $F \dashv G$ with derived unit $\bar{\eta}$ and derived counit $\bar{\varepsilon}$.*

Proof. Let η and ε be the unit and counit of the adjunction $F \dashv G$. First, we prove that $\bar{\eta}$ and $\bar{\varepsilon}$ are unique if they exist. Indeed, if they exist, then (α, β) is a conjugate pair of natural transformations, so we must have the equations shown below:

$$\beta F \bullet \gamma_{\mathcal{C}} \eta = (\mathbf{R}G)\alpha \bullet \bar{\eta}\gamma_{\mathcal{C}} \qquad\qquad \bar{\varepsilon}\gamma_{\mathcal{D}} \bullet (\mathbf{L}F)\beta = \gamma_{\mathcal{D}}\varepsilon \bullet \alpha G$$

However, $((\mathbf{R}G)(\mathbf{L}F), (\mathbf{R}G)\alpha)$ is a left Kan extension of $(\mathbf{R}G)\gamma_{\mathcal{D}} F$ along $\gamma_{\mathcal{C}}$ and $((\mathbf{L}F)(\mathbf{R}G), (\mathbf{L}F)\beta)$ is a right Kan extension of $(\mathbf{L}F)\gamma_{\mathcal{C}} G$ along $\gamma_{\mathcal{D}}$, so $\bar{\eta}$ and $\bar{\varepsilon}$ are uniquely determined as natural transformations by these equations.

Next, we prove that the natural transformations $\bar{\eta}$ and $\bar{\varepsilon}$ defined above satisfy the left and right triangle identities. Using naturality and the defining equations for $\bar{\eta}$ and $\bar{\varepsilon}$, we obtain the following:

$$\begin{aligned}
\alpha \bullet (\bar{\varepsilon}(\mathbf{L}F) \bullet (\mathbf{L}F)\bar{\eta})\gamma_{\mathcal{C}} &= \alpha \bullet \bar{\varepsilon}(\mathbf{L}F)\gamma_{\mathcal{C}} \bullet (\mathbf{L}F)\bar{\eta}\gamma_{\mathcal{C}} \\
&= \bar{\varepsilon}\gamma_{\mathcal{D}} F \bullet (\mathbf{L}F)(\mathbf{R}G)\alpha \bullet (\mathbf{L}F)\bar{\eta}\gamma_{\mathcal{C}} \\
&= \bar{\varepsilon}\gamma_{\mathcal{D}} F \bullet (\mathbf{L}F)\beta F \bullet (\mathbf{L}F)\gamma_{\mathcal{C}}\eta \\
&= \gamma_{\mathcal{D}}\varepsilon F \bullet \alpha G F \bullet (\mathbf{L}F)\gamma_{\mathcal{C}}\eta \\
&= \gamma_{\mathcal{D}}\varepsilon F \bullet \gamma_{\mathcal{D}} F\eta \bullet \alpha \\
&= \gamma_{\mathcal{D}}(\varepsilon F \bullet F\eta) \bullet \alpha
\end{aligned}$$

Since $(\mathbf{L}F, \alpha)$ is a right Kan extension of F along $\gamma_{\mathcal{C}}$, this implies that $\bar{\eta}$ and $\bar{\varepsilon}$ satisfy the left triangle identity if η and ε do. A formally dual calculation shows that the same is true for the right triangle identity. Thus, we have the required derived adjunction. ■

Definition 3.3.6. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **left deformation retract** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a triple $(\mathcal{C}^\circ, Q, p)$ where

- $\mathcal{C}^\circ$ is a full subcategory of $\mathcal{C}$ with the induced relative subcategory structure,
- Q is a pair of maps $\operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{C}$ and $\operatorname{mor}\mathcal{C} \to \operatorname{mor}\mathcal{C}$ (but not necessarily functorial), and
- p assigns to each object X in $\mathcal{C}$ a weak equivalence $p_X : QX \to X$,

and these data are required to satisfy the following axioms:

DR1. For all objects X in $\mathcal{C}$, the object QX is in $\mathcal{C}^\circ$.

DR2. For all morphisms $f : X \to Y$ in $\mathcal{C}$, we have $p_Y \circ Qf = f \circ p_X$, i.e. the diagram in $\mathcal{C}$ shown below commutes,

$$\begin{tikzcd} QX \arrow[r, "p_X"] \arrow[d, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

and if f is a weak equivalence in $\mathcal{C}$, then so is Qf.

DR3. The inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ induces a faithful functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$.

DR4. The restriction $F|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \to \mathcal{D}$ is a relative functor.

An ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is **left deformable** if there exists a left deformation retract for F. A **left deformation retract** of a relative category $\mathcal{C}$ is a left deformation retract for $\mathrm{id}_{\mathcal{C}}$.

Dually, a **right deformation retract** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a triple $(\mathcal{D}^\circ, R, i)$ where

- $\mathcal{D}^\circ$ is a full subcategory of $\mathcal{D}$ with the induced relative subcategory structure,
- R is a pair of maps $\operatorname{ob}\mathcal{D} \to \operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{D} \to \operatorname{mor}\mathcal{D}$ (but not necessarily functorial), and
- i assigns to each object A in $\mathcal{D}$ a weak equivalence $i_A : A \to RA$,

and these data are required to satisfy the following axioms:

DR1. For all objects A in $\mathcal{D}$, the object RA is in $\mathcal{D}^\circ$.

DR2. For all morphisms $g : A \to B$ in $\mathcal{D}$, we have $Rg \circ i_A = i_B \circ g$, i.e. the diagram in $\mathcal{D}$ shown below commutes,

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "g"'] & RA \arrow[d, "Rg"] \\ B \arrow[r, "i_B"'] & RB \end{tikzcd}$$

and if g is a weak equivalence in $\mathcal{D}$, then so is Rg.

DR3. The inclusion $\mathcal{D}^\circ \hookrightarrow \mathcal{D}$ induces a faithful functor $\operatorname{Ho}\mathcal{D}^\circ \to \operatorname{Ho}\mathcal{D}$.

DR4. The restriction $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor.

An ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is **right deformable** if there exists a right deformation retract for G. A **right deformation retract** of a relative category $\mathcal{D}$ is a right deformation retract for $\mathrm{id}_{\mathcal{D}}$.

Remark 3.3.7. Every relative functor is both left deformable and right deformable, with trivial left and right deformation retracts.

Proposition 3.3.8.

- *Let $\mathcal{C}$ be a relative category and let $(\mathcal{C}^\circ, Q, p)$ be a left deformation retract of $\mathcal{C}$. Then the functor $U : \operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ induced by the inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ is fully faithful and essentially surjective on objects, and there exist a unique functor $\bar{Q} : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}^\circ$ and a unique natural isomorphism $\operatorname{Ho} p : U\bar{Q} \Rightarrow \mathrm{id}_{\operatorname{Ho}\mathcal{C}}$ whose components are (the images of) the weak equivalences $p_X : QX \to X$.*

- *Let $\mathcal{D}$ be a relative category and let $(\mathcal{D}^\circ, R, i)$ be a right deformation retract of $\mathcal{D}$. Then the functor $U : \operatorname{Ho}\mathcal{D}^\circ \to \operatorname{Ho}\mathcal{D}$ by the inclusion $\mathcal{D}^\circ \hookrightarrow \mathcal{D}$ is fully faithful and essentially surjective on objects, and there exist a unique functor $\bar{R} : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{D}^\circ$ and a unique natural isomorphism $\operatorname{Ho} i : \mathrm{id}_{\operatorname{Ho}\mathcal{D}} \Rightarrow U\bar{R}$ whose components are (the images of) the weak equivalences $i_A : A \to RA$.*

Proof. The two claims are formally dual; we will prove the first version.

Since each $p_X : QX \to X$ is a weak equivalence in $\mathcal{C}$ and each QX is in $\mathcal{C}^\circ$, we see that the functor $U : \operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ is essentially surjective on objects. Axiom DR3 says that it is also faithful, so it remains to be shown that U is full.

Let $\tilde{X}$ and $\tilde{Y}$ be objects in $\mathcal{C}^\circ$ and consider a morphism $f : \tilde{X} \to \tilde{Y}$ in $\operatorname{Ho} \mathcal{C}$. By theorem A.4.44 and axiom DR2, there is a morphism of zigzags in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} X \arrow[d, equal] & Q\tilde{X} \arrow[l, "p_{\tilde{X}}"'] \arrow[d, "p_{\tilde{X}}"'] \arrow[r, dash] & \cdots \arrow[r, dash] & Q\tilde{Y} \arrow[r, "p_{\tilde{Y}}"] \arrow[d, "p_{\tilde{Y}}"] & Y \arrow[d, equal] \\ X \arrow[r, equal] & X \arrow[r, dash] & \cdots \arrow[r, dash] & Y \arrow[r, equal] & Y \end{tikzcd}$$

where the top row is a zigzag in $\mathcal{C}^\circ$ and the bottom row is a zigzag in $\mathcal{C}$ representing $f : \tilde{X} \to \tilde{Y}$ in $\operatorname{Ho} \mathcal{C}$. Thus, the top row also represents $f : \tilde{X} \to \tilde{Y}$ in $\operatorname{Ho} \mathcal{C}$, and in particular, $f : \tilde{X} \to \tilde{Y}$ is in the image of $U : \operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}$.

Clearly, the assignments $X \mapsto QX$ and $f \mapsto Qf$ induce a well-defined functor $\operatorname{Ho} Q : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ whose image is contained in the image of the fully faithful functor $U : \operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}$. Thus, there is a unique functor $\bar{Q} : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{C}^\circ$ such that $\operatorname{Ho} Q = U\bar{Q}$. It is easy to see that there is a unique natural isomorphism $\operatorname{Ho} i : U\bar{Q} \Rightarrow \mathrm{id}_{\operatorname{Ho} \mathcal{C}}$ of the required form. ■

Corollary 3.3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *Let $(\mathcal{C}^\circ, Q, p)$ be a left deformation retract for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$. For any full subcategory $\mathcal{C}' \subseteq \mathcal{C}$, if $\mathcal{C}'$ contains every QX and the restriction $F : \mathcal{C}' \to \mathcal{D}$ is a relative functor, then $(\mathcal{C}', Q, p)$ is also a left deformation retract for $F : \mathcal{C} \to \mathcal{D}$.*

- *Let $(\mathcal{D}^\circ, R, i)$ be a right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$. For any full subcategory $\mathcal{D}' \subseteq \mathcal{D}$, if $\mathcal{D}'$ contains every RA and the restriction $G : \mathcal{D}' \to \mathcal{C}$ is a relative functor, then $(\mathcal{D}', Q, p)$ is also a right deformation retract for $G : \mathcal{D} \to \mathcal{C}$.*

Proof. The two claims are formally dual; we will prove the first version.

Clearly, $(\mathcal{C}', Q, p)$ satisfies axioms DR1 and DR2, and by hypothesis, it also satisfies axiom DR4. We must now verify axiom DR3. We have the following commutative diagram,

$$\begin{tikzcd} \operatorname{Ho} \mathcal{C}^\circ \arrow[r] \arrow[dr] & \operatorname{Ho} \mathcal{C}^\circ_F \arrow[d] \\ & \operatorname{Ho} \mathcal{C} \end{tikzcd}$$

and since $\operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}$ is fully faithful and essentially surjective on objects (by proposition 3.3.8), $\operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}^\circ_F$ is faithful. In particular, the evident restriction of $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract of $\mathcal{C}_F^\circ$, so the same proposition implies $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}_F^\circ$ is fully faithful and essentially surjective on objects. The 2-out-of-3 property of such functors then implies that $\operatorname{Ho}\mathcal{C}_F^\circ \to \operatorname{Ho}\mathcal{C}$ is faithful, as required. ■

Proposition 3.3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. Given a left deformation retract $(\mathcal{C}^\circ, Q, p)$ for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$:*

(i) *If Q is functorial, then the composite $FQ : \mathcal{C} \to \mathcal{D}$ is a relative functor.*

(ii) *If $\mathcal{C}_F^\circ$ is the full subcategory of $\mathcal{C}$ spanned by the objects X such that the morphism $Fp_X : FQX \to FX$ is weak equivalence in $\mathcal{D}$, then $\mathcal{C}^\circ \subseteq \mathcal{C}_F^\circ$.*

(iii) *If moreover* $\operatorname{weq}\mathcal{D}$ *has the 2-out-of-3 property in $\mathcal{D}$, then $\left(\mathcal{C}_F^\circ, Q, p\right)$ is also a left deformation retract for F.*

Dually, given a right deformation retract $(\mathcal{D}^\circ, R, i)$ be a right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$:

(i′) *If Q is functorial, then the composite $GR : \mathcal{D} \to \mathcal{C}$ is a relative functor.*

(ii′) *If $\mathcal{D}_G^\circ$ is the full subcategory of $\mathcal{D}$ spanned by the objects A such that the morphism $Gi_A : GA \to GRA$ is weak equivalence in $\mathcal{C}$, then $\mathcal{D}^\circ \subseteq \mathcal{D}_G^\circ$.*

(iii′) *If moreover* $\operatorname{weq}\mathcal{C}$ *has the 2-out-of-3 property in $\mathcal{C}$, then $\left(\mathcal{D}_G^\circ, R, i\right)$ is also a right deformation retract for F.*

Proof. (i). Immediate from the definitions.

(ii). Let $\tilde{X}$ be an object in $\mathcal{C}^\circ$. By definition, $Q\tilde{X}$ is also an object in $\mathcal{C}^\circ$, and $F|_{\mathcal{C}^\circ}$ is a relative functor, so $Fp_{\tilde{X}} : FQ\tilde{X} \to F\tilde{X}$ is a weak equivalence in $\mathcal{C}$.

(iii). Let X and Y be objects in $\mathcal{C}_F^\circ$ and let $f : X \to Y$ be a weak equivalence in $\mathcal{C}$. Consider the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
FQX \arrow[r, "Fp_X"] \arrow[d, "FQf"'] & FX \arrow[d, "Ff"] \\
FQY \arrow[r, "Fp_Y"'] & FY
\end{tikzcd}$$

FQf is a weak equivalence in $\mathcal{D}$ by claim (i), and both Fp_X and Fp_Y are weak equivalences by the definition of $\mathcal{C}_F^\circ$, so using the 2-out-of-3 property of $\operatorname{weq}\mathcal{D}$, we may deduce that $F f$ is a weak equivalence in $\mathcal{D}$ too. Thus, $F|_{\mathcal{C}_F^\circ}$ is a relative functor, and by corollary 3.3.9, $\left(\mathcal{C}_F^\circ, Q, p\right)$ is a left deformation retract for F. ■

Lemma 3.3.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories, let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor, and let $\left\{\mathcal{C}_j^\circ \,\middle|\, j \in J\right\}$ be a directed family of full subcategories of $\mathcal{C}$ and let $\mathcal{C}^\circ = \bigcup_{j \in J} \mathcal{C}_j^\circ$. If each restriction $F : \mathcal{C}_j^\circ \to \mathcal{D}$ is a relative functor, then the restriction $F : \mathcal{C}^\circ \to \mathcal{D}$ is also a relative functor.*

Proof. Obvious. ◆

When we have the 2-out-of-6 property, we can say a bit more about enlarging deformation retracts.

Lemma 3.3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *Let $(\mathcal{C}^\circ, Q, p)$ and $(\mathcal{C}'^\circ, Q', p')$ be left deformation retracts for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ and let $\mathcal{C}''^\circ$ be the full subcategory of $\mathcal{C}$ generated by the union of $\mathcal{C}^\circ$ and $\mathcal{C}'^\circ$. If $\mathcal{D}$ is a homotopical category, then the restriction $F : \mathcal{C}''^\circ \to \mathcal{D}$ is a relative functor.*
- *Let $(\mathcal{D}^\circ, R, i)$ and $(\mathcal{D}'^\circ, R', i')$ be right deformation retracts for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ and let $\mathcal{D}''^\circ$ be the full subcategory of $\mathcal{D}$ generated by the union of $\mathcal{D}^\circ$ and $\mathcal{D}'^\circ$. If $\mathcal{C}$ is a homotopical category, then the restriction $G : \mathcal{D}''^\circ \to \mathcal{D}$ is a relative functor.*

Proof. The two claims are formally dual; we will prove the first version. We follow the proof of Proposition 40.4 in [DHKS].

Let $f : X \to Y$ be a weak equivalence in $\mathcal{C}''^\circ$. If X and Y are both in $\mathcal{C}^\circ$ (resp. $\mathcal{C}'^\circ$), then $F f : F X \to F Y$ is a weak equivalence in $\mathcal{D}$, because $(\mathcal{C}^\circ, Q, p)$ (resp. $(\mathcal{C}'^\circ, Q', p')$) is a left deformation retract for $F : \mathcal{C} \to \mathcal{D}$; so instead suppose X (resp. Y) is in $\mathcal{C}^\circ$ (resp. $\mathcal{C}'^\circ$). Consider the following diagram in $\mathcal{C}$:

$$\begin{tikzcd}
FQQ'X \arrow[r, "Fp_{Q'X}"] \arrow[d, "FQp'_X"', "\simeq"] & FQ'X \arrow[r, "FQ'f", "\simeq"'] \arrow[d, "Fp'_X"'] & FQ'Y \arrow[d, "Fp'_Y", "\simeq"'] \\
FQX \arrow[r, "\simeq", "Fp_X"'] & FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

The left square commutes because $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract and the right square commutes because $(\mathcal{C}'^\circ, Q', p')$ is a left deformation retract. Since the marked arrows are weak equivalences in $\mathcal{D}$, the 2-out-of-6 property implies that $F f : F X \to F Y$ is also a weak equivalence in $\mathcal{D}$, as required. ■

Proposition 3.3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *Let $(\mathcal{C}^\circ, Q, p)$ be a left deformation retract for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$. Assuming $\mathcal{D}$ is a homotopical category, there is a full subcategory $\mathcal{C}^\circ_F \subseteq \mathcal{C}$ with the following property: for any full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ that contains every QX, $(\mathcal{C}', Q, p)$ is a left deformation retract for $F : \mathcal{C} \to \mathcal{D}$ if and only if $\mathcal{C}' \subseteq \mathcal{C}^\circ_F$.*

- *Let $(\mathcal{D}^\circ, R, i)$ be a right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$. Assuming $\mathcal{C}$ is a homotopical category, there is a full subcategory $\mathcal{D}^\circ_G \subseteq \mathcal{D}$ with the following property: for any full subcategory $\mathcal{D}' \subseteq \mathcal{D}$ that contains every RA, $(\mathcal{D}', R, i)$ is a right deformation retract for $G : \mathcal{D} \to \mathcal{C}$ if and only if $\mathcal{D}' \subseteq \mathcal{D}^\circ_F$.*

Proof. Recalling corollary 3.3.9, this is a straightforward consequence of lemmas 3.3.11 and 3.3.12. ■

Proposition 3.3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories, and let $\gamma_\mathcal{C} : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ and $\gamma_\mathcal{D} : \mathcal{D} \to \operatorname{Ho} \mathcal{D}$ be the respective localising functors.*

- *If $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$, then there exist a right Kan extension $(\mathbf{L}F, \alpha)$ of $\gamma_\mathcal{D} F$ along $\gamma_\mathcal{C}$ such that $(\mathbf{L}F)\gamma_\mathcal{C} = \gamma_\mathcal{D} F Q$ and $\alpha = \gamma_\mathcal{D} F p$. (In particular, $\gamma_\mathcal{D} F Q$ is functorial even if Q is not.)*

- *If $(\mathcal{D}^\circ, R, i)$ is a right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$, then there exist a left Kan extension $(\mathbf{R}G, \beta)$ of $\gamma_\mathcal{C} G$ along $\gamma_\mathcal{D}$ such that $(\mathbf{R}G)\gamma_\mathcal{D} = \gamma_\mathcal{C} G R$ and $\beta = \gamma_\mathcal{C} G i$. (In particular, $\gamma_\mathcal{C} G R$ is functorial even if R is not.)*

Proof. The two claims are formally dual; we will prove the first version.

To simplify notation, we may assume without loss of generality that $\mathcal{D}$ is a minimal saturated relative category and that $\gamma_\mathcal{D} = \mathrm{id}_\mathcal{D}$. Henceforth, we write γ instead of $\gamma_\mathcal{C}$. First, observe that γQ is functorial (even if Q is not) because each $\gamma p_X : \gamma Q X \to \gamma X$ is an isomorphism, so (using axioms DR1 and DR3) there is a unique functor $\tilde{Q} : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{C}^\circ$ such that $\tilde{Q}\gamma = \gamma Q$. Let $\gamma^\circ : \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}^\circ$ be the localising functor for $\mathcal{C}^\circ$. Since $F|_{\mathcal{C}^\circ}$ is a relative functor (by axiom DR4), we must have $F|_{\mathcal{C}^\circ} = \tilde{F}\gamma^\circ$ for a unique functor $\tilde{F} : \operatorname{Ho} \mathcal{C}^\circ \to \mathcal{D}$. We may then define $\mathbf{L}F$ to be the functor $\tilde{F}\tilde{Q}$. We define $\alpha : (\mathbf{L}F)\gamma \Rightarrow F$ by taking $\alpha_X = F p_X$; by axiom DR2, this is indeed a natural transformation.

It remains to be shown that $(\mathbf{L}F, \alpha)$ is a right Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$. Let $H : \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ be a functor and let $\varphi : H\gamma \Rightarrow F$ be any natural transformation. By restricting along the inclusion $\mathcal{C}^\circ \to \mathcal{C}$, we obtain a natural transformation $\varphi|_{\mathcal{C}^\circ} : H|_{\operatorname{Ho}\mathcal{C}^\circ}\gamma^\circ \Rightarrow F|_{\mathcal{C}^\circ}$, so there is a unique natural transformation $\tilde{\varphi} : H|_{\operatorname{Ho}\mathcal{C}^\circ} \Rightarrow \tilde{F}$ such that $\tilde{\varphi}\gamma^\circ = \varphi|_{\mathcal{C}^\circ}$ (by the 2-dimensional universal property of $\operatorname{Ho}\mathcal{C}$). Since γp is a natural isomorphism, there is then a unique natural transformation $\bar{\varphi} : H \Rightarrow \mathbf{L}F$ such that $\bar{\varphi}_{\gamma X} \circ H\gamma p_X = \tilde{\varphi}_{\gamma^\circ QX}$ for all objects X in $\mathcal{C}$. We then have $\alpha \bullet \bar{\varphi}\gamma = \varphi$, and $\bar{\varphi}$ is the unique such natural transformation because the canonical functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ is essentially surjective on objects. ∎

Definition 3.3.15. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ be relative categories. Given a composable pair of ordinary functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, a **lax left deformation retract** for (G, F) consists of

- a left deformation retract $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ for F, and
- a left deformation retract $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ for G,

such that $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is also a left deformation retract for GF as well. A **strong left deformation retract** for (G, F) is a lax left deformation retract as above such that F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$. We say a composable pair of functors is **laxly left deformable** (resp. **strongly left deformable**) if it admits a lax left deformation (resp. strong left deformation).

Dually, given a composable pair of ordinary functors $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$, an **oplax right deformation retract** for (F, G) consists of

- a right deformation retract $\left(\mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ}\right)$ for F, and
- a right deformation retract $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ for G,

such that $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ is a right deformation retract for GF as well. A **strong right deformation retract** for (F, G) is an oplax right deformation retract as above such that G sends objects in $\mathcal{D}^\circ$ to objects in $\mathcal{C}^\circ$. We say a composable pair of functors is **oplaxly right deformable** (resp. **strongly left deformable**) if it admits an oplax right deformation (resp. strong right deformation).

Lemma 3.3.16.

- *Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ be a left deformation retract for $F : \mathcal{C} \to \mathcal{D}$ and let $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ be a left deformation retract for $G : \mathcal{D} \to \mathcal{E}$. If F maps* *objects in* $\mathcal{C}^\circ$ *to objects in* $\mathcal{D}^\circ$*, then* $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ *is a left deformation retract for* $GF : \mathcal{C} \to \mathcal{E}$*.*

Dually:

- *Let* $\left(\mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ}\right)$ *be a right deformation retract for* $F : \mathcal{C} \to \mathcal{B}$ *and let* $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ *be a right deformation retract for* $G : \mathcal{D} \to \mathcal{C}$*. If* G *maps objects in* $\mathcal{D}^\circ$ *to objects in* $\mathcal{C}^\circ$*, then* $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ *is a right deformation retract for* $FG : \mathcal{D} \to \mathcal{B}$*.*

Proof. Our hypotheses imply that the restriction $GF|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \to \mathcal{E}$ is a relative functor, so $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ satisfies the conditions required to be a left deformation retract for $GF : \mathcal{C} \to \mathcal{E}$. ■

Theorem 3.3.17. *Let* $\mathcal{C}$*,* $\mathcal{D}$*, and* $\mathcal{E}$ *be relative categories, and let* $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$*,* $\gamma_{\mathcal{D}} : \mathcal{D} \to \operatorname{Ho}\mathcal{D}$*, and* $\gamma_{\mathcal{E}} : \mathcal{E} \to \operatorname{Ho}\mathcal{E}$ *be the respective localising functors.*

(i) *Let* $F : \mathcal{C} \to \mathcal{D}$ *be an ordinary functor. If* $(\mathcal{C}^\circ, Q, p)$ *is any left deformation retract for* F*, then* F *has a total left derived functor* $(\mathbf{L}F, \alpha)$ *such that* $(\mathbf{L}F)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}} F Q$ *and* $\alpha = \gamma_{\mathcal{D}} F p$*.*

(ii) *Let* $F, F' : \mathcal{C} \to \mathcal{D}$ *be a parallel pair of ordinary functors. If* $(\mathbf{L}F, \alpha)$ *and* $(\mathbf{L}F', \alpha')$ *are total left derived functors for* F *and* F' *(respectively), then for any natural transformation* $\varphi : F \Rightarrow F'$*, there exists a unique natural transformation* $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$ *such that* $\alpha' \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}}\varphi \bullet \alpha$*.*

(iii) *Moreover, if* $(\mathcal{C}^\circ, Q, p)$ *is a left deformation retract for both* F *and* F'*, then* $(\mathbf{L}\varphi)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}}\varphi Q$*.*

(iv) *Let* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$ *be ordinary functors between relative categories. If* $\left(\mathbf{L}F, \alpha^F\right)$*,* $\left(\mathbf{L}G, \alpha^G\right)$*, and* $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ *are total left derived functors for* F*,* G*, and* GF *(respectively), then there is a unique natural transformation* $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ *such that* $\alpha^{GF} \bullet \boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} = \alpha^G F \bullet (\mathbf{L}G)\alpha^F$*.*

(v) *If* (G, F) *is moreover a strongly left deformable composable pair, then the canonical comparison* $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ *is an isomorphism.*

Dually:

(i′) *Let $G : \mathcal{D} \to \mathcal{C}$ be an ordinary functor. If $(\mathcal{D}^\circ, R, i)$ is any right deformation retract for G, then G has a total right derived functor $(\mathbf{R}G, \beta)$ such that $(\mathbf{R}G)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}} G R$ and $\beta = \gamma_{\mathcal{C}} G i$.*

(ii′) *Let $G, G' : \mathcal{D} \to \mathcal{C}$ be a parallel pair of ordinary functors. If $(\mathbf{R}G, \beta)$ and $(\mathbf{R}G', \beta')$ are total right derived functors for G and G' (respectively), then for any natural transformation $\psi : G' \Rightarrow G$, there exists a unique natural transformation $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$ such that $(\mathbf{R}\psi)\gamma_{\mathcal{D}} \bullet \beta' = \beta \bullet \gamma_{\mathcal{C}}\psi$.*

(iii′) *Moreover, if $(\mathcal{D}^\circ, R, i)$ is a right deformation retract for both G and G', then $(\mathbf{R}\psi)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}}\psi R$.*

(iv′) *Let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$ be ordinary functors between relative categories. If $\left(\mathbf{R}F, \beta^F\right)$, $\left(\mathbf{R}G, \beta^G\right)$, and $\left(\mathbf{R}(FG), \beta^{FG}\right)$ are total right derived functors for F, G, and FG (respectively), then there is a unique natural transformation $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ such that $\boldsymbol{\delta}_{F,G}\gamma_{\mathcal{D}} \bullet \beta^{FG} = (\mathbf{R}F)\beta^G \bullet \beta^F G$.*

(v′) *If (F, G) is moreover a strongly right deformable composable pair, then the canonical comparison $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ is an isomorphism.*

Proof. (i). By proposition 3.3.14, the functor $\gamma_{\mathcal{D}} F : \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ has a right Kan extension along $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$, say $(\mathbf{L}F, \alpha)$, characterised by the announced equations. We must verify that $(\mathbf{L}F, \alpha)$ is an absolute right Kan extension, i.e. that $(H(\mathbf{L}F), H\alpha)$ is a right Kan extension for any functor $H : \operatorname{Ho}\mathcal{D} \to \mathcal{E}$ whatsoever.

It is clear that $(\mathcal{C}^\circ, Q, p)$ is also a left deformation retract for $H\gamma_{\mathcal{D}}F : \mathcal{C} \to \mathcal{E}$, so the cited proposition yields a right Kan extension (L', α') of $H\gamma_{\mathcal{D}}F$ along $\gamma_{\mathcal{C}}$. There is then a unique natural transformation $\varphi : H(\mathbf{L}F) \Rightarrow L'$ such that $\alpha' \bullet \varphi\gamma_{\mathcal{C}} = H\alpha$, i.e. the following diagram commutes for all objects X in $\mathcal{C}$:

$$\begin{tikzcd}
H(\mathbf{L}F)\gamma_{\mathcal{C}}X \arrow[r, "\varphi_{\gamma_{\mathcal{C}}X}"] \arrow[d, "H\alpha_X"'] & L'\gamma_{\mathcal{C}}X \arrow[d, "\alpha'_X"] \\
H\gamma_{\mathcal{D}}FX \arrow[r, equal] & H\gamma_{\mathcal{D}}FX
\end{tikzcd}$$

However, if $\tilde{X}$ is in $\mathcal{C}^\circ$, then $\alpha_{\tilde{X}}$ and $\alpha'_{\tilde{X}}$ are isomorphisms, and so $\varphi_{\gamma_{\mathcal{C}}X}$ must be an isomorphism as well. Since the canonical functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ is essentially surjective on objects, $\varphi : H(\mathbf{L}F) \Rightarrow L'$ must be a natural isomorphism. In particular, $(H(\mathbf{L}F), H\alpha)$ is indeed a right Kan extension.

(ii). Noting that $\gamma_{\mathcal{D}}\varphi \bullet \alpha$ is a natural transformation $(\mathbf{L}F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}F'$, the universal property of $(\mathbf{L}F', \alpha')$ yields a unique natural transformation $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$ such that $\gamma_{\mathcal{D}}\varphi \bullet \alpha = \alpha' \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}}$, as required.

(iii). We must have

$$\gamma_{\mathcal{D}}Fp \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}}\varphi \bullet \gamma_{\mathcal{D}}F'p = \gamma_{\mathcal{D}}Fp \bullet \gamma_{\mathcal{D}}\varphi Q$$

as required.

(iv). Since $\alpha^G F \bullet (\mathbf{L}G)\alpha^F$ is a natural transformation $(\mathbf{L}G)(\mathbf{L}F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}GF$, the universal property of $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ yields the required natural transformation $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$.

(v). Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ and $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ constitute a strong left deformation retract for (G, F), and let $\left(\mathbf{L}F, \alpha^F\right)$, $\left(\mathbf{L}G, \alpha^G\right)$, $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ be the total left derived functors for F and G, respectively, as constructed in claim (i). Then,

$$\begin{aligned}
\alpha^{GF} \bullet \boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} &= \alpha^G F \bullet (\mathbf{L}G)\alpha^F \\
&= \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}F \bullet \gamma_{\mathcal{E}}GQ^{\mathcal{D}^\circ}Fp^{\mathcal{C}^\circ} \\
&= \gamma_{\mathcal{E}}GFp^{\mathcal{C}^\circ} \bullet \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}
\end{aligned}$$

so we must have $\boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}$; but $\gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}$ is a natural isomorphism because F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$ and G preserves weak equivalences in $\mathcal{D}^\circ$, so we deduce that $\boldsymbol{\mu}_{G,F}$ is also a natural isomorphism (using the fact that $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is bijective on objects). ■

Corollary 3.3.18. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *If $F : \mathcal{C} \to \mathcal{D}$ is a relative functor, then* $(\operatorname{Ho} F, \mathrm{id})$ *is a total left derived functor for F.*
- *If $G : \mathcal{D} \to \mathcal{C}$ is a relative functor, then* $(\operatorname{Ho} G, \mathrm{id})$ *is a total right derived functor for G.*

Proof. The two claims are formally dual; we will prove the first version.

By remark 3.3.7, the trivial right deformation retract is a right deformation retract for $F : \mathcal{C} \to \mathcal{D}$. Thus, $\operatorname{Ho} F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ together with $\mathrm{id} : (\operatorname{Ho} F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}F$ constitute a total left derived functor for F. ■

Proposition 3.3.19. *Let $\mathcal{C}$ be a relative category.*

- *If $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract of $\mathcal{C}$ and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ such that* $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \operatorname{weq} \mathcal{C}$*, then the functor* $\mathcal{C}\left[\mathcal{W}^{-1}\right] \to \operatorname{Ho} \mathcal{C}$ *induced by the inclusion* $\mathcal{W} \hookrightarrow \operatorname{weq} \mathcal{C}$ *has a fully faithful left adjoint.*

- *If $(\mathcal{C}^\circ, R, i)$ is a right deformation retract of $\mathcal{C}$ and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ such that* $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \operatorname{weq} \mathcal{C}$*, then the functor* $\mathcal{C}\left[\mathcal{W}^{-1}\right] \to \operatorname{Ho} \mathcal{C}$ *induced by the inclusion* $\mathcal{W} \hookrightarrow \operatorname{weq} \mathcal{C}$ *has a fully faithful right adjoint.*

Proof. The two claims are formally dual; we will prove the first version.

Consider the localising functor $\gamma_{\mathcal{W}} : \mathcal{C} \to \mathcal{C}\left[\mathcal{W}^{-1}\right]$. Since $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W}$, $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for $\gamma_{\mathcal{W}}$, so (by theorem 3.3.17) there exists an absolute right Kan extension (F, α) of $\gamma_{\mathcal{W}} : \mathcal{C} \to \mathcal{C}\left[\mathcal{W}^{-1}\right]$ along the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$. Since γ factors through $\gamma_{\mathcal{W}}$, say $\gamma = G\gamma_{\mathcal{W}}$, the 2-dimensional universal property of $\mathcal{C}\left[\mathcal{W}^{-1}\right]$ yields a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}[\mathcal{W}^{-1}]}$ such that $\varepsilon\gamma_{\mathcal{W}} = \alpha$; similar arguments show that (F, ε) is an absolute right Kan extension of $\mathrm{id} : \mathcal{C}\left[\mathcal{W}^{-1}\right] \to \mathcal{C}\left[\mathcal{W}^{-1}\right]$ along $G : \mathcal{C}\left[\mathcal{W}^{-1}\right] \to \operatorname{Ho} \mathcal{C}$, so F is a left adjoint for G with counit ε, by proposition A.5.21.

It remains to be shown that $F : \operatorname{Ho} \mathcal{C} \to \mathcal{C}\left[\mathcal{W}^{-1}\right]$ is fully faithful. Consider the natural transformation $G\varepsilon : GFG \Rightarrow G$. The total derived functor theorem says $\varepsilon\gamma_{\mathcal{W}} : FG\gamma_{\mathcal{W}} \Rightarrow \gamma_{\mathcal{W}}$ is given by $\gamma_{\mathcal{W}} p$, so $G\varepsilon\gamma_{\mathcal{W}}$ is given by $G\gamma_{\mathcal{W}} p$, which is a natural isomorphism. Since $\gamma_{\mathcal{W}}$ is bijective on objects, we deduce that $G\varepsilon$ itself is a natural isomorphism. Thus, $\eta G : G \Rightarrow GFG$ is a natural isomorphism (by the right triangle identity), and since G is bijective on objects, we may use proposition A.1.3 to see that F is fully faithful. ■

Definition 3.3.20. The **2-category of small left deformation retracts** is defined as follows:

- The objects are pairs $\left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ where $\mathcal{C}$ is a small relative category and $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract of $\mathcal{C}$.

- A 1-morphism $F : \left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right) \to \left(\mathcal{D}, \mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ is an ordinary functor $F : \mathcal{C} \to \mathcal{D}$, such that $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract for F, and F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$.

- The 2-morphisms are ordinary natural transformations.

- All compositions and identities are inherited from 2-category of small categories.

We write $\mathfrak{LDef}$ for this 2-category, and we write LDefFun for its hom-sets.

The **2-category of small right deformation retracts** is defined dually:

- The objects are pairs $\left(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ where $\mathcal{D}$ is a small relative category and $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ is a right deformation retract of $\mathcal{D}$.
- A 1-morphism $G : \left(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right) \to \left(\mathcal{C}, \mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ}\right)$ is an ordinary functor $G : \mathcal{D} \to \mathcal{C}$, such that $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ is a right deformation retract for G, and G sends objects in $\mathcal{D}^\circ$ to objects in $\mathcal{C}^\circ$.
- The 2-morphisms are ordinary natural transformations.
- All compositions and identities are inherited from 2-category of small categories.

We write $\mathfrak{RDef}$ for this 2-category, and we write RDefFun for its hom-sets.

Remark 3.3.21. The duality principle for deformation retracts can be formalised as follows: there is a 2-functor $\mathfrak{LDef}^{\mathrm{co}} \to \mathfrak{RDef}$ that sends $\left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ to its opposite $\left(\mathcal{C}^{\mathrm{op}}, (\mathcal{C}^\circ)^{\mathrm{op}}, \left(Q^{\mathcal{C}^\circ}\right)^{\mathrm{op}}, \left(p^{\mathcal{C}^\circ}\right)^{\mathrm{op}}\right)$, and it has an evident strict inverse $\mathfrak{RDef}^{\mathrm{co}} \to \mathfrak{LDef}$. Note that these two 2-functors reverse the direction of 2-morphisms but preserve the direction of 1-morphisms!

Corollary 3.3.22. *There are two pseudofunctors,* $\mathbf{L}$ *and* $\mathbf{R}$*, where:*

- $\mathbf{L}$ *is a pseudofunctor* $\mathfrak{LDef} \to \mathfrak{Cat}$ *that sends an object* $\left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ *to the homotopy category* $\operatorname{Ho}\mathcal{C}$*, a 1-morphism* $F : \left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right) \to \left(\mathcal{D}, \mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ *to its total left derived functor* $\mathbf{L}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$*, and a 2-morphism* $\varphi : F \Rightarrow F'$ *to the derived natural transformation* $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$*, and* $\mathbf{L}$ *preserves identity 1-morphisms strictly.*
- $\mathbf{R}$ *is a pseudofunctor* $\mathfrak{RDef} \to \mathfrak{Cat}$ *that sends an object* $\left(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ *to the homotopy category* $\operatorname{Ho}\mathcal{C}$*, a 1-morphism* $G : \left(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right) \to \left(\mathcal{C}, \mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ}\right)$ *to its total right derived functor* $\mathbf{R}G : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$*, and a 2-morphism* $\psi : G' \Rightarrow G$ *to the derived natural transformation* $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$*, and* $\mathbf{R}$ *preserves identity 1-morphisms strictly.*

- $\mathbf{L}$ *and* $\mathbf{R}$ *are compatible with the duality principle, in the sense that the following diagrams commute (strictly):*

$$\begin{tikzcd} \mathfrak{LDef}^{\mathrm{co}} \arrow[r, "\mathbf{L}^{\mathrm{co}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathfrak{Cat}^{\mathrm{co}} \arrow[d, "(-)^{\mathrm{op}}"] \\ \mathfrak{RDef} \arrow[r, "\mathbf{R}"'] & \mathfrak{Cat} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{RDef}^{\mathrm{co}} \arrow[r, "\mathbf{R}^{\mathrm{co}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathfrak{Cat}^{\mathrm{co}} \arrow[d, "(-)^{\mathrm{op}}"] \\ \mathfrak{LDef} \arrow[r, "\mathbf{L}"'] & \mathfrak{Cat} \end{tikzcd}$$

Proof. The main claims follow from theorem 3.3.17; the only thing left to check is that the collection of 2-isomorphisms $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ satisfy the coherence laws for pseudofunctors; that is, we should show that the following diagrams commute:

$$\begin{tikzcd} (\mathbf{L}H)(\mathbf{L}G)(\mathbf{L}F) \arrow[r, "(\mathbf{L}H)\boldsymbol{\mu}_{G,F}"] \arrow[d, "\boldsymbol{\mu}_{H,G}(\mathbf{L}F)"'] & (\mathbf{L}H)\mathbf{L}(GF) \arrow[d, "\boldsymbol{\mu}_{H,GF}"] \\ \mathbf{L}(HG)(\mathbf{L}F) \arrow[r, "\boldsymbol{\mu}_{HG,F}"'] & \mathbf{L}(HGF) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{R}(FGH) \arrow[r, "\boldsymbol{\delta}_{F,GH}"] \arrow[d, "\boldsymbol{\delta}_{FG,H}"'] & (\mathbf{R}F)\mathbf{R}(GH) \arrow[d, "(\mathbf{R}F)\boldsymbol{\delta}_{G,H}"] \\ \mathbf{R}(FG)(\mathbf{R}H) \arrow[r, "\boldsymbol{\delta}_{F,G}(\mathbf{R}H)"'] & (\mathbf{R}F)(\mathbf{R}G)(\mathbf{R}H) \end{tikzcd}$$

However, using the explicit formulae for $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ in the proof of the theorem, it is easy to see that these diagrams do indeed commute. ∎

Definition 3.3.23. A **deformable adjunction** between two relative categories is an ordinary adjunction where the left adjoint is left deformable and the right adjoint is right deformable.

Theorem 3.3.24. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of ordinary categories, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

(i) *If $F \dashv G : \mathcal{D} \to \mathcal{C}$ is a deformable adjunction, then it admits a derived adjunction.*

(ii) *Let $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be another adjunction, with unit η' and counit ε', and let $H : \mathcal{C}' \to \mathcal{C}$ and $K : \mathcal{D}' \to \mathcal{D}$ be relative functors. If*

   - *$(\mathcal{C}^{\circ}, Q, p)$ is a left deformation retract for F,*

- *$(\mathcal{C}'^{\circ}, Q', p')$ is a left deformation retract for F',*
- *H sends objects in $\mathcal{C}'^{\circ}$ to objects in $\mathcal{C}^{\circ}$,*
- *$(\mathcal{D}^{\circ}, R, i)$ is a right deformation retract for G,*
- *$(\mathcal{D}'^{\circ}, R', i')$ is a right deformation retract for G', and*
- *K sends objects in $\mathcal{D}'^{\circ}$ to objects in $\mathcal{D}^{\circ}$,*

then for any conjugate pair of natural transformations,

$$\varphi : FH \Rightarrow KF' \qquad\qquad \psi : HG' \Rightarrow GK$$

the derived natural transformations

$$\mathbf{L}\varphi : (\mathbf{L}F)(\mathrm{Ho}\, H) \Rightarrow (\mathrm{Ho}\, K)(\mathbf{L}F') \quad \mathbf{R}\psi : (\mathrm{Ho}\, K)(\mathbf{R}G') \Rightarrow (\mathbf{R}G)(\mathrm{Ho}\, K)$$

also constitute a conjugate pair.

(iii) *Let $F' \dashv G' : \mathcal{D}' \to \mathcal{D}$ be another adjunction, with unit η' and counit ε'. If (F', F) is strongly left deformable and (G, G') is strongly right deformable, then the three derived adjunctions*

$$\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}\, \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$$
$$\mathbf{L}F' \dashv \mathbf{R}G' : \mathrm{Ho}\, \mathcal{D}' \to \mathrm{Ho}\, \mathcal{D}$$
$$\mathbf{L}(F'F) \dashv \mathbf{R}(GG') : \mathrm{Ho}\, \mathcal{D}' \to \mathrm{Ho}\, \mathcal{C}$$

make $\left(\boldsymbol{\mu}_{F',F}, \boldsymbol{\delta}_{G,G'}\right)$ a conjugate pair of natural transformations, i.e.

$$\left(\boldsymbol{\delta}_{G,G'}\mathbf{L}(F'F)\right) \bullet \bar{\eta}'' = \left((\mathbf{R}G)(\mathbf{R}G')\boldsymbol{\mu}_{F',F}\right) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}$$
$$\bar{\varepsilon}'' \bullet \left(\boldsymbol{\mu}_{F',F}\mathbf{R}(GG')\right) = \bar{\varepsilon}' \bullet (\mathbf{L}F')\bar{\varepsilon}(\mathbf{R}G') \bullet \left((\mathbf{L}F)(\mathbf{L}F')\boldsymbol{\delta}_{G,G'}\right)$$

where $\bar{\eta}''$ and $\bar{\varepsilon}''$ are the unit and counit for $\mathbf{L}(F'F) \dashv \mathbf{R}(GG')$.

Proof. (i). We appeal to theorems 3.3.5 and 3.3.17.

(ii). Recall the following characterisations of $\mathbf{L}\varphi$ and $\mathbf{R}\psi$:

$$\gamma_{\mathcal{D}}KF'p' \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}'} = \gamma_{\mathcal{D}}\varphi \bullet \gamma_{\mathcal{D}}FpH$$
$$(\mathbf{R}\psi)\gamma_{\mathcal{D}'} \bullet \gamma_{\mathcal{C}}HG'i' = \gamma_{\mathcal{C}}GiK \bullet \gamma_{\mathcal{C}}\psi$$

We wish to show that these equations hold:

$$\bar{\varepsilon}(\mathrm{Ho}\, K) \bullet (\mathbf{L}F)(\mathbf{R}\psi) = (\mathrm{Ho}\, K)\bar{\varepsilon}' \bullet (\mathbf{L}\varphi)(\mathbf{R}G') \tag{1}$$

(2) $$(\mathbf{R}G)(\mathbf{L}\varphi) \bullet \bar{\eta}(\operatorname{Ho} H) = (\mathbf{R}\psi)(\mathbf{L}F') \bullet (\operatorname{Ho} H)\bar{\eta}'$$

By proposition A.1.5, it suffices to show that equation (1) is satisfied, and since the canonical functor $\operatorname{Ho}\mathcal{D}'^{\circ} \to \operatorname{Ho}\mathcal{D}'$ is essentially surjective on objects, equation (1) holds if and only if the following equation holds for all $\hat{A}$ in $\mathcal{D}'^{\circ}$:

(3) $$\bar{\varepsilon}_{\gamma_D K\hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'}\hat{A}} = (\operatorname{Ho} K)\bar{\varepsilon}'_{\gamma_{D'}\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'}G'R'\hat{A}}$$

We observe that $G'i'_{\hat{A}} : G'\hat{A} \to G'R'\hat{A}$ is a weak equivalence in $\mathcal{C}'$ (because $(\mathcal{D}'^{\circ}, R', i')$ is a right deformation retract for G'), so $\gamma_C HG'i'_{\hat{A}}$ is invertible, and we must have

$$(\mathbf{R}\psi)_{\gamma_{D'}\hat{A}} = \gamma_C Gi_{K\hat{A}} \circ \gamma_C \psi_{\hat{A}} \circ \left(\gamma_C HG'i'_{\hat{A}}\right)^{-1}$$

and hence,

$$(\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'}\hat{A}} = \gamma_D FQGi_{K\hat{A}} \circ \gamma_D FQ\psi_{\hat{A}} \circ \left(\gamma_D FQHG'i'_{\hat{A}}\right)^{-1}$$

therefore:

$$\begin{aligned}\bar{\varepsilon}_{\gamma_D K\hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'}\hat{A}} &= \gamma_D \varepsilon_{K\hat{A}} \circ \gamma_D F p_{G\hat{A}} \circ \gamma_D FQ\psi_{\hat{A}} \circ \left(\gamma_D FQHG'i'_{\hat{A}}\right)^{-1} \\ &= \gamma_D \varepsilon_{K\hat{A}} \circ \gamma_D F\psi_{\hat{A}} \circ \gamma_D F p_{HG'\hat{A}} \circ \left(\gamma_D FQHG'i'_{\hat{A}}\right)^{-1}\end{aligned}$$

On the other hand,

$$\bar{\varepsilon}'_{\gamma_{D'}\hat{A}} = \gamma_{D'}\varepsilon'_{\hat{A}} \circ \gamma_{D'} F' p'_{G'\hat{A}} \circ \left(\gamma_{D'} F'Q'G'i'_{\hat{A}}\right)^{-1}$$

and so,

$$\begin{aligned}&(\operatorname{Ho} K)\bar{\varepsilon}'_{\gamma_{D'}\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'}G'R'\hat{A}} \\ &\qquad = \gamma_D K\varepsilon'_{\hat{A}} \circ \gamma_D KF'p'_{G'\hat{A}} \circ \left(\gamma_D KF'Q'G'i'_{\hat{A}}\right)^{-1} \circ (\mathbf{L}\varphi)_{\gamma_{C'}G'R'\hat{A}} \\ &\qquad = \gamma_D K\varepsilon'_{\hat{A}} \circ \gamma_D KF'p'_{G'\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'}G'\hat{A}} \circ \left(\gamma_D FQHG'i'_{\hat{A}}\right)^{-1} \\ &\qquad = \gamma_D K\varepsilon'_{\hat{A}} \circ \gamma_D \varphi_{G'\hat{A}} \circ \gamma_D F p_{HG'\hat{A}} \circ \left(\gamma_D FQHG'i'_{\hat{A}}\right)^{-1}\end{aligned}$$

but $\varepsilon_{K\hat{A}} \circ F\psi_{\hat{A}} = K\varepsilon'_{\hat{A}} \circ \varphi_{G'\hat{A}}$ by hypothesis, so equation (3) indeed holds.

(iii). Suppose

- $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for F,
- $(\mathcal{C}'^\circ, Q', p')$ is a left deformation retract for F',
- F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{C}'^\circ$,
- $(\mathcal{D}^\circ, R, i)$ is a right deformation retract for G,
- $(\mathcal{D}'^\circ, R', i')$ is a right deformation retract for G', and
- G' sends objects in $\mathcal{D}'^\circ$ to objects in $\mathcal{D}^\circ$,

and recall that the comparison isomorphisms are characterised by the following equations:

$$\boldsymbol{\mu}_{F',F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}'}F'p'FQ \qquad\qquad \boldsymbol{\delta}_{G,G'}\gamma_{\mathcal{D}'} = \gamma_{\mathcal{C}}GiG'R'$$

Thus, $\big(\big((\mathbf{R}G)(\mathbf{R}G') \circ \boldsymbol{\mu}_{F',F}\big) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}\big)\gamma_{\mathcal{C}}$ expands to

$$\begin{aligned}
&\gamma_{\mathcal{C}}GRG'R'F'p'FQ \\
&\qquad\qquad \bullet \gamma_{\mathcal{C}}(GRG'i'F'Q'FQ \bullet GR\eta'Q'FQ) \bullet \big(\gamma_{\mathcal{C}}GRp'FQ\big)^{-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \bullet \gamma_{\mathcal{C}}(GiFQ \bullet \eta Q) \bullet \big(\gamma_{\mathcal{C}}p\big)^{-1}
\end{aligned}$$

and a straightforward calculation then shows

$$\begin{aligned}
&\big(\big(\boldsymbol{\delta}_{G,G'}^{-1} \circ \boldsymbol{\mu}_{F',F}\big) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}\big)\gamma_{\mathcal{C}} \\
&\qquad\qquad = \gamma_{\mathcal{C}}GiG'R'F'FQ \bullet \gamma_{\mathcal{C}}(GG'i'F'FQ \bullet G\eta FQ \bullet \eta Q) \bullet \big(\gamma_{\mathcal{C}}p\big)^{-1}
\end{aligned}$$

but the RHS is precisely the definition of $\big(\big(\boldsymbol{\delta}_{G,G'}\mathbf{L}(F'F)\big) \bullet \bar{\eta}''\big)\gamma_{\mathcal{C}}$. The dual calculation proves the other equation. ■

Corollary 3.3.25. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be relative categories, let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be two adjunctions of ordinary categories, and let $H : \mathcal{C}' \to \mathcal{C}$ and $K : \mathcal{D}' \to \mathcal{D}$ be homotopical functors. Suppose we have a conjugate pair of natural transformations as in the diagrams below:*

(L)
$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "H"] \arrow[d, "F'"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C} \arrow[d, "F"] \\
\mathcal{D}' \arrow[r, "K"'] & \mathcal{D}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D}' \arrow[r, "K"] \arrow[d, "G'"'] \arrow[dr, phantom, "\psi\nearrow"] & \mathcal{D} \arrow[d, "G"] \\
\mathcal{C}' \arrow[r, "H"'] & \mathcal{C}
\end{tikzcd}$$
(R)

Assume the following hypotheses:

- (C°, Q, p) *is a left deformation retract for* F.
- (C'°, Q', p') *is a left deformation retract for* F'.
- H *sends objects in* C'° *to objects in* C°.
- $(\mathcal{D}^\circ, R, i)$ *is a right deformation retract for* G.
- $(\mathcal{D}'^\circ, R', i')$ *is a right deformation retract for* G'.
- K *sends objects in* $\mathcal{D}'^\circ$ *to objects in* $\mathcal{D}^\circ$.

Then, considering the derived natural transformations $\mathbf{L}\varphi$ *and* $\mathbf{R}\varphi$*:*

$$\text{(L}'\text{)} \quad \begin{tikzcd} \operatorname{Ho} C' \arrow[r, "\operatorname{Ho} H"] \arrow[d, "\mathbf{L}F'"'] \arrow[dr, phantom, "\Swarrow_{\mathbf{L}\varphi}"] & \operatorname{Ho} C \arrow[d, "\mathbf{L}F"] \\ \operatorname{Ho} \mathcal{D}' \arrow[r, "\operatorname{Ho} K"'] & \operatorname{Ho} \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Ho} \mathcal{D}' \arrow[r, "\operatorname{Ho} K"] \arrow[d, "\mathbf{R}G'"'] \arrow[dr, phantom, "\mathbf{R}\psi \Nearrow"] & \operatorname{Ho} \mathcal{D} \arrow[d, "\mathbf{R}G"] \\ \operatorname{Ho} C' \arrow[r, "\operatorname{Ho} H"'] & \operatorname{Ho} C \end{tikzcd} \quad \text{(R}'\text{)}$$

- *If diagram* (R) *satisfies the left Beck–Chevalley condition, then so does* (R′).
- *If diagram* (L) *satisfies the right Beck–Chevalley condition, then so does* (L′).

Proof. The theorem says that $\mathbf{L}\varphi$ and $\mathbf{R}\psi$ constitute a conjugate pair of natural transformations, and by theorem 3.3.17 it is clear that $\mathbf{L}\varphi$ (resp. $\mathbf{R}\psi$) is a natural isomorphism if φ (resp. ψ) is a natural isomorphism. ∎

Proposition 3.3.26. *Let* $\mathcal{B}, C, \mathcal{D}, \mathcal{E}$ *be relative categories.*

- *Let* $F : C \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$ *be functors and suppose* (G, F) *is laxly left deformable. If the canonical comparison* $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ *is a natural isomorphism and* $\mathcal{E}$ *is a saturated homotopical category, then* (G, F) *is a left deformable composable pair.*

Dually:

- *Let* $F : C \to \mathcal{B}$ *and* $G : \mathcal{D} \to C$ *be functors and suppose* (F, G) *is oplaxly right deformable. If the canonical comparison* $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ *is a natural isomorphism and* C *is a saturated homotopical category, then* (F, G) *is a left deformable composable pair.*

Proof. Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ and $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ constitute a lax left deformation retract for (G, F). By theorem 3.3.17, we may assume without loss of generality that $(\mathbf{L}F)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}} F Q$, $(\mathbf{L}G)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}} G Q^{\mathcal{D}^\circ}$, and $\boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{C}} G p^{\mathcal{D}^\circ} F Q^{\mathcal{C}^\circ}$. Our hypothesis says $\boldsymbol{\mu}_{G,F}$ is a natural isomorphism and $\mathcal{E}$ is a saturated homotopical category, so the morphisms $G p^{\mathcal{D}^\circ}_{F Q^{\mathcal{C}^\circ} X} : G Q^{\mathcal{D}^\circ} F Q^{\mathcal{C}^\circ} X \Rightarrow G F Q^{\mathcal{C}^\circ} X$ are weak equivalences, for all objects X in $\mathcal{C}$.

Now, let $\tilde{X}$ be an object in $\mathcal{C}^\circ$. The following diagram commutes,

$$\begin{tikzcd}
G Q^{\mathcal{D}^\circ} F Q^{\mathcal{C}^\circ} \tilde{X} \arrow[r, "G p^{\mathcal{D}^\circ}_{F Q^{\mathcal{C}^\circ} \tilde{X}}"] \arrow[d, "G Q^{\mathcal{D}^\circ} F p^{\mathcal{C}^\circ}_{\tilde{X}}"'] & G F Q^{\mathcal{C}^\circ} \tilde{X} \arrow[d, "G F p^{\mathcal{C}^\circ}_{\tilde{X}}"] \\
G Q^{\mathcal{D}^\circ} F \tilde{X} \arrow[r, "G p^{\mathcal{D}^\circ}_{F \tilde{X}}"'] & G F \tilde{X}
\end{tikzcd}$$

and since $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract for both F and $G F$, it follows that the downward-pointing arrows in the above diagrams are weak equivalences in $\mathcal{E}$; so using the 2-out-of-3 property of weq $\mathcal{E}$ and the fact that $G p^{\mathcal{D}^\circ}_{F Q^{\mathcal{C}^\circ} \tilde{X}}$ is a weak equivalence, we deduce that $G p^{\mathcal{D}^\circ}_{F \tilde{X}}$ is a weak equivalence in $\mathcal{E}$. Thus, recalling proposition 3.3.10, we obtain a left deformation retract $\left(\mathcal{D}^\circ_G, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ for G such that F sends every object in $\mathcal{C}^\circ$ to an object in $\mathcal{D}^\circ_G$, and so (G, F) is indeed strongly left deformable. ∎

Corollary 3.3.27. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be relative categories, and let*

$$F_! \dashv F^* : \mathcal{D} \to \mathcal{C} \qquad\qquad G_! \dashv G^* : \mathcal{E} \to \mathcal{D}$$

be adjunctions of ordinary categories. If $\mathcal{C}$ and $\mathcal{E}$ are saturated homotopical categories, then the following are equivalent:

(i) *$\left(G_!, F_!\right)$ is strongly left deformable and (F^*, G^*) is strongly right deformable.*

(ii) *$\left(G_!, F_!\right)$ is laxly left deformable and (F^*, G^*) is strongly right deformable.*

(iii) *$\left(G_!, F_!\right)$ is strongly left deformable and (F^*, G^*) is oplaxly right deformable.*

Proof. Theorem 3.3.24 says $\left(\boldsymbol{\mu}_{G_!, F_!}, \boldsymbol{\delta}_{F^*, G^*}\right)$ is a conjugate pair of natural transformations, and the pasting lemma (A.1.11) implies $\boldsymbol{\mu}_{G_!, F_!}$ is a natural isomorphism if and only if $\boldsymbol{\delta}_{F^*, G^*}$ is a natural isomorphism, so the equivalence of the three statements follows from the proposition above. ∎

Proposition 3.3.28. *Let C and $\mathcal{D}$ be two relative categories, let $F \dashv G : \mathcal{D} \to C$ be an adjunction of ordinary categories with unit η and counit ε, let (C°, Q, p) be a left deformation retract for F, and let $(\mathcal{D}^\circ, R, i)$ be a right deformation retract for G. Consider the following statements:*

(i) *For all objects $\tilde{X}$ in C° and all objects $\hat{B}$ in $\mathcal{D}^\circ$, if $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$, then its right adjoint transpose $\tilde{X} \to G\hat{B}$ is a weak equivalence in C.*

(ii) *For all objects X in C, The morphism $Gi_{FQX} \circ \eta_{QX} : QX \to GRFQX$ is a weak equivalence in C.*

(iii) *The derived unit $\bar{\eta} : \mathrm{id}_{\mathrm{Ho}\, C} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ is a natural isomorphism.*

(i′) *For all objects $\tilde{X}$ in C° and all objects $\hat{B}$ in $\mathcal{D}^\circ$, if $\tilde{X} \to G\hat{B}$ is a weak equivalence in C, then its left adjoint transpose $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$.*

(ii′) *For all objects B in $\mathcal{D}$, the morphism $\varepsilon_{RB} \circ Fp_{GRB} : FQGRB \Rightarrow RB$ is a weak equivalence in $\mathcal{D}$.*

(iii′) *The derived counit $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\mathrm{Ho}\, \mathcal{D}}$ is a natural isomorphism.*

We have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*; if* $\operatorname{weq} C$ *has the 2-out-of-3 property, then* (ii) $\Rightarrow$ (i)*; and if C is a saturated homotopical category, then* (iii) $\Rightarrow$ (ii). *Dually,* (i′) $\Rightarrow$ (ii′) $\Rightarrow$ (iii′)*; if* $\operatorname{weq} \mathcal{D}$ *has the 2-out-of-3 property, then* (ii′) $\Rightarrow$ (i′)*; and if $\mathcal{D}$ is a saturated homotopical category, then* (iii′) $\Rightarrow$ (ii′).

Proof. (i) $\Rightarrow$ (ii). We have a weak equivalence $i_{FQX} : FQX \to RFQX$, and QX is an object in C°, so by the hypothesis, its right adjoint transpose $Gi_{FQX} \circ \eta_{QX}$ is also a weak equivalence.

(ii) $\Rightarrow$ (iii). The derived unit is given by $\bar{\eta}\gamma_C = \gamma_C(GiFQ \bullet \eta Q) \circ \left(\gamma_C p\right)^{-1}$, which is certainly a natural isomorphism if $Gi_{FQX} \circ \eta_{QX}$ is a weak equivalence for all X.

(ii) ⇒ (i). Assume weq $\mathcal{C}$ has the 2-out-of-3 property. Given $\tilde{X}$ in $\mathcal{C}^\circ$, the diagram below commutes,

$$\begin{tikzcd}
Q\tilde{X} \arrow[r, "\eta_{Q\tilde{X}}"] \arrow[d, "p_{\tilde{X}}"'] & GFQ\tilde{X} \arrow[r, "Gi_{FQ\tilde{X}}"] & GRFQ\tilde{X} \arrow[d, "GRFp_{\tilde{X}}"] \\
\tilde{X} \arrow[r, "\eta_{\tilde{X}}"'] & GF\tilde{X} \arrow[r, "Gi_{F\tilde{X}}"'] & GRF\tilde{X}
\end{tikzcd}$$

but the top row and the two vertical arrows are weak equivalences in $\mathcal{C}$, so the bottom row must be a weak equivalence as well, by the 2-out-of-3 property.

Let $g : F\tilde{X} \to \hat{B}$ be a weak equivalence in $\mathcal{D}$, and let $f = Gg \circ \eta_{\tilde{X}}$ be its right adjoint transpose in $\mathcal{C}$. We know $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor, so $GRg : GRF\tilde{X} \to GR\hat{B}$ is a weak equivalence in $\mathcal{C}$; but

$$Gi_{\hat{B}} \circ f = Gi_{\hat{B}} \circ Gg \circ \eta_{\tilde{X}} = GRg \circ \left(Gi_{F\tilde{X}} \circ \eta_{\tilde{X}}\right)$$

and we know $Gi_{\hat{B}} : G\hat{B} \to GR\hat{B}$ is a weak equivalence in $\mathcal{C}$, so by the 2-out-of-3 property again, f must be a weak equivalence in $\mathcal{C}$.

(iii) ⇒ (ii). Now assume $\mathcal{C}$ is a saturated homotopical category. If $\bar{\eta}$ is a natural isomorphism, then each $\gamma_{\mathcal{C}}(GiFQ \bullet \eta Q)$ must also be a natural isomorphism, and so each $Gi_{FQX} \circ \eta_{QX}$ is a weak equivalence, by the saturation hypothesis. ■

Corollary 3.3.29. *With notation as above, suppose the* **Quillen equivalence condition** *is satisfied:*

- *For all objects $\tilde{X}$ in $\mathcal{C}^\circ$ and all objects $\hat{B}$ in $\mathcal{D}^\circ$, a morphism $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$ if and only if its right adjoint transpose $\tilde{X} \to G\hat{B}$ is a weak equivalence in $\mathcal{C}$.*

Then the derived adjunction is an adjoint equivalence of categories. ■

3.4 DHKS derived functors

Prerequisites. §§3.1, 3.2, 3.3.

Notice that in theorem 3.3.17, we constructed derived functors by restricting to a relatively equivalent full subcategory on which the functor respects weak equivalences. This suggests that, by strengthening the definition of 'deformation retract', we may be able to construct derived functors without first passing to the homotopy category.

In this section we follow [DHKS, Ch. VII].

Definition 3.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **functorial left deformation retract** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a triple $(\mathcal{C}^\circ, Q, p)$ where

- $\mathcal{C}^\circ$ is a full subcategory of $\mathcal{C}$ with the induced relative subcategory structure,
- $Q : \mathcal{C} \to \mathcal{C}$ is a relative functor, and
- $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction $F|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \to \mathcal{D}$ is a relative functor.
- For all objects X in $\mathcal{C}$, the object QX is in $\mathcal{C}^\circ$.

An ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is **functorially left deformable** if there exists a functorial left deformation retract for F.

Dually, a **functorial right deformation retract** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a triple $(\mathcal{D}^\circ, R, i)$ where

- $\mathcal{D}^\circ$ is a full subcategory of $\mathcal{D}$ with the induced relative subcategory structure,
- $R : \mathcal{D} \to \mathcal{D}$ is a relative functor, and
- $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor.
- For all objects A in $\mathcal{D}$, the object RA is in $\mathcal{D}^\circ$.

An ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is **functorially right deformable** if there exists a functorial right deformation retract for G.

Remark 3.4.2. Every relative functor is both functorially left deformable and functorially right deformable, with trivial functorial left and right deformation retracts.

Remark. The definition above is the one found in [DHKS, §40] under the name 'deformation retract'; they do not consider the non-functorial version.

Lemma 3.4.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$, then $(\mathcal{C}^\circ, Q, p)$ is also a left deformation retract for F.*
- *If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$, then $(\mathcal{D}^\circ, R, i)$ is also a right deformation retract for G.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that axioms DR1, DR2, and DR4 are satisfied, so we need only check axiom DR3. For this, we simply observe that the inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ and the relative functor $Q : \mathcal{C} \to \mathcal{C}^\circ$ (together with the natural weak equivalence $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$) constitute a relative equivalence of relative categories; thus, proposition 3.1.27 implies the canonical functor $\operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}$ is fully faithful, as required. ■

Remark 3.4.4. Conversely, by replacing a relative category with its homotopy category, we may obtain functorial left (resp. right) deformation retracts from ordinary left (resp. right) deformation retracts.

Proposition 3.4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *Let $Q : \mathcal{C} \to \mathcal{C}$ be a relative functor, let $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ be a natural weak equivalence, and let $\mathcal{C}^\circ$ be the full subcategory of $\mathcal{C}$ spanned by the image of Q. If $\operatorname{weq} \mathcal{D}$ has the 2-out-of-3 property in $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor such that FQ is a relative functor and $FqQ : FQQ \Rightarrow FQ$ is a natural weak equivalence, then $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for F.*

Dually:

- *Let $R : \mathcal{D} \to \mathcal{D}$ be a relative functor, let $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ be a natural weak equivalence, and let $\mathcal{D}^\circ$ be the full subcategory of $\mathcal{D}$ spanned by the image of R. If $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property in $\mathcal{C}$ and $G : \mathcal{D} \to \mathcal{C}$ is a functor such that GR is a relative functor and $GiR : GR \Rightarrow GRR$ is a natural weak equivalence, then $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for G.*

Proof. Let $f : QX \to QY$ be a weak equivalence in $\mathcal{C}^\circ$. By naturality, the following diagram commutes:

$$\begin{tikzcd} FQQX \arrow[r, "FQf"] \arrow[d, "Fp_{QX}"'] & FQQY \arrow[d, "Fp_{QY}"] \\ FQX \arrow[r, "Ff"'] & FQY \end{tikzcd}$$

We know FQf, Fp_{QX}, and Fp_{QY} are weak equivalences in $\mathcal{D}$, so using the 2-out-of-3 property of $\operatorname{weq}\mathcal{D}$, we deduce that Ff is also a weak equivalence in $\mathcal{D}$. Thus $F|_{\mathcal{C}^\circ}$ is a relative functor, as required. ∎

Definition 3.4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories. A **homotopical left approximation** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopical right (!) Kan extension of F along $\mathrm{id}_{\mathcal{C}}$. Dually, a **homotopical right approximation** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a homotopical left (!) Kan extension of G along $\mathrm{id}_{\mathcal{D}}$.

Remark 3.4.7. More explicitly, a homotopical left approximation for $F : \mathcal{C} \to \mathcal{D}$ is a homotopically terminal object in the homotopical category $\left([\mathcal{C}, \mathcal{D}]_{\mathrm{h}} \downarrow F\right)_{\mathrm{h}}$ described below:

- The objects are pairs (K, α) where K is a homotopical functor $\mathcal{C} \to \mathcal{D}$ and α is a natural transformation of type $K \Rightarrow F$.
- The morphisms $(K', \alpha') \to (K, \alpha)$ are those natural transformations $\psi : K' \Rightarrow K$ such that $\alpha \bullet \psi = \alpha'$.
- The weak equivalences are the natural weak equivalences.

Dually, a homotopical right approximation for $G : \mathcal{D} \to \mathcal{C}$ is a homotopically initial object in the homotopical category $\left(F \downarrow [\mathcal{D}, \mathcal{C}]_{\mathrm{h}}\right)_{\mathrm{h}}$. By corollary 3.2.12, homotopical approximations are homotopically unique.

We have the following special case:

Proposition 3.4.8. *Let Q be a homotopical endofunctor on a homotopical category $\mathcal{C}$ and let $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ be a natural transformation. The following are equivalent:*

(i) *(Q, p) is a homotopical left approximation for* $\mathrm{id}_{\mathcal{C}}$.

(ii) $(\mathcal{C}, Q, p)$ *is a functorial left deformation retract for* $\mathrm{id}_{\mathcal{C}}$.

Dually, let R be a homotopical endofunctor on a homotopical category $\mathcal{D}$, and let $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ be a natural transformation. The following are equivalent:

(i′) (R, i) *is a homotopical right approximation for* $\mathrm{id}_{\mathcal{C}}$.

(ii′) $(\mathcal{D}, R, i)$ *is a functorial right deformation retract for* $\mathrm{id}_{\mathcal{D}}$.

Proof. (i) ⇒ (ii). If (Q, p) is a homotopical left approximation for $\mathrm{id}_{\mathcal{C}}$, then there must exist a commutative diagram of the form below,

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r] \arrow[d, "\mathrm{id}"'] & Q_1 \arrow[d, "p_1"'] & Q_2 \arrow[l] \arrow[r] \arrow[d, "p_2"'] & \cdots & Q \arrow[l] \arrow[d, "p"] \\
\mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \cdots \arrow[r, equal] & \mathrm{id}_{\mathcal{C}}
\end{tikzcd}$$

where all the arrows in the top row are natural weak equivalences. Using 2-out-of-3 property, we deduce (by induction) that $p_1, p_2, \ldots, p$ are also natural weak equivalences; thus $(\mathcal{C}, Q, p)$ is indeed a functorial left deformation retract for $\mathrm{id}_{\mathcal{C}}$.

(ii) ⇒ (i). If $(\mathcal{C}, Q, p)$ is a functorial left deformation retract for $\mathrm{id}_{\mathcal{C}}$, then $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ is a natural weak equivalence; but $\left(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}\right)$ is a terminal object in $\left([\mathcal{C}, \mathcal{C}]_{\mathrm{h}} \downarrow \mathrm{id}_{\mathcal{C}}\right)_{\mathrm{h}}$, so by proposition 3.2.2, (Q, p) must be a homotopically terminal object. ■

Lemma 3.4.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories.*

- *Let (F', p') and (F'', p'') be any two homotopical left approximations for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$. For any object X in $\mathcal{C}$, $p'_X : F'X \to FX$ is a weak equivalence in $\mathcal{D}$ if and only if $p''_X : F''X \to FX$ is a weak equivalence in $\mathcal{D}$.*

- *Let (G', i') and (G'', i'') be any two homotopical left approximations for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$. For any object Y in $\mathcal{D}$, $i'_Y : GY \to G'Y$ is a weak equivalence in $\mathcal{C}$ if and only if $i''_Y : GY \to G''_Y$ is a weak equivalence in $\mathcal{C}$.*

Proof. The two claims are formally dual; we will prove the first version.

By lemma 3.2.5, there is a (finite) commutative diagram of the form below,

$$\begin{tikzcd} F' \arrow[r, no head] & \cdots \arrow[r, no head] & F'' \arrow[d, "p''"] \\ F \arrow[u, leftarrow, "p'"'] \arrow[r, equal] & \cdots \arrow[r, equal] & F \end{tikzcd}$$

where the top row is a zigzag of natural weak equivalences. Thus, using 2-out-of-3 property, we deduce (by induction) that $p'_X : F'X \to FX$ is a weak equivalence in $\mathcal{D}$ if and only if $p''_X : F''X \to FX$ is a weak equivalence in $\mathcal{D}$. ∎

Definition 3.4.10. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be ordinary functors between homotopical categories, and let $\varphi : F \Rightarrow F'$ be a natural transformation. We define the homotopical category $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$ as follows:

- The objects are tuples $(H, H', \alpha, \alpha', \theta)$ where H and H' are homotopical functors $\mathcal{C} \to \mathcal{D}$, α and α' are natural transformations of type $H \Rightarrow F$ and $H' \Rightarrow F'$ (respectively), and $\theta : H \Rightarrow H'$ is a natural transformation such that $\varphi \bullet \alpha = \alpha' \bullet \theta$.

- The morphisms $(H, H', \alpha, \alpha', \theta) \to (K, K', \beta, \beta', \chi)$ are pairs (ζ, ζ') of natural transformations, where $\zeta : H \Rightarrow K$ and $\zeta' : H' \Rightarrow K'$, such that $\chi \bullet \zeta = \zeta' \bullet \theta$, $\beta \bullet \zeta = \alpha$, and $\beta' \bullet \zeta' = \alpha'$.

- The weak equivalences are those (ζ, ζ') where both ζ and ζ' are natural weak equivalences.

A **homotopical left approximation** for φ is a homotopically terminal object $(\mathbb{L}F, \mathbb{L}F', \delta, \delta', \mathbb{L}\varphi)$ in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$ *such that* $(\mathbb{L}F, \delta)$ is a homotopical left approximation for F and $(\mathbb{L}F', \delta')$ is a homotopical left approximation for F'.

Dually, let $G, G' : \mathcal{D} \to \mathcal{C}$ be ordinary functors between homotopical categories, and let $\psi : G' \Rightarrow G$ be a natural transformation. We define the homotopical category $\left(\psi \downarrow \left[\min \mathbb{2}, [\mathcal{D}, \mathcal{C}]_{\mathrm{h}}\right]_{\mathrm{h}}\right)_{\mathrm{h}}$ as follows:

- The objects are tuples $(H, H', \alpha, \alpha', \theta)$ where H and H' are homotopical functors $\mathcal{D} \to \mathcal{C}$, α and α' are natural transformations of type $G \Rightarrow H$ and $G' \Rightarrow H'$ (respectively), and $\theta : H' \Rightarrow H$ is a natural transformation such that $\alpha \bullet \psi = \theta \bullet \alpha'$.

- The morphisms $(K, K', \beta, \beta', \chi) \to (H, H', \alpha, \alpha', \theta)$ are pairs (ζ, ζ') of natural transformations, where $\zeta : K \Rightarrow H$ and $\zeta' : K' \Rightarrow H'$, such that $\zeta \bullet \chi = \theta \bullet \zeta'$, $\zeta \bullet \beta = \alpha$, and $\zeta' \bullet \beta' = \alpha'$.

- The weak equivalences are those (ζ, ζ') where both ζ and ζ' are natural weak equivalences.

A **homotopical right approximation** for ψ is a homotopically initial object $(\mathbb{R}G, \mathbb{R}G', \delta, \delta', \mathbb{R}\psi)$ in $\left(\psi \downarrow \left[\min \mathbb{2}, [\mathcal{D}, \mathcal{C}]_{\mathrm{h}}\right]_{\mathrm{h}}\right)_{\mathrm{h}}$ *such that* $(\mathbb{R}G, \delta)$ is a homotopical right approximation for G and $(\mathbb{R}G', \delta')$ is a homotopical right approximation for G'.

Theorem 3.4.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories.*

(i) *Let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor. If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for F, then (FQ, Fp) is a homotopical absolute right Kan extension of F along* $\mathrm{id}_{\mathcal{C}}$.

(ii) *Let $F, F' : \mathcal{C} \to \mathcal{D}$ be a parallel pair of ordinary functors. If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for both F and F', then for any natural transformation $\varphi : F \Rightarrow F'$, $(FQ, F'Q, Fp, F'p, \varphi Q)$ is a homotopical left approximation for φ.*

(iii) *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be ordinary functors between homotopical categories. If (G, F) is strongly left deformable, then, for any homotopical left approximation $\big((\mathbb{L}F), \delta^F\big)$ for F and any homotopical left approximation $\big((\mathbb{L}G), \delta^G\big)$ for G, $\big((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\big)$ is a homotopical left approximation for GF.*

Dually:

(i′) *Let $G : \mathcal{D} \to \mathcal{C}$ be an ordinary functor. If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for F, then (GR, Gi) is a homotopical absolute left Kan extension of G along* $\mathrm{id}_{\mathcal{D}}$.

(ii′) *Let $G, G' : \mathcal{D} \to \mathcal{C}$ be a parallel pair of ordinary functors. If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for both G and G', then for any natural transformation $\psi : G' \Rightarrow G$, $(GR, G'R, Gi, G'i, \psi R)$ is a homotopical right approximation for ψ.*

(iii′) *Let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$ be ordinary functors between homotopical categories. If (F, G) is strongly right deformable, then, for any homotopical right approximation $\left((\mathbb{R}F), \delta^F\right)$ for F and any homotopical right approximation $\left((\mathbb{R}G), \delta^G\right)$ for G, $\left((\mathbb{R}F)(\mathbb{R}G), \delta^F \circ \delta^G\right)$ is a homotopical right approximation for FG.*

Proof. (i). Let $H : \mathcal{D} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{E}$ be any two homotopical functors, and let $\alpha : K \Rightarrow HF$ be any natural transformation. Then, we have the following commutative diagram of natural transformations,

$$\begin{tikzcd}
K \arrow[dr, "\alpha"'] & KQ \arrow[l, "Kp"'] \arrow[r, "\alpha Q"] \arrow[d] & HFQ \arrow[dl, "HFp"] \\
& HF &
\end{tikzcd}$$

and, for any other homotopical functor $K' : \mathcal{C} \to \mathcal{E}$ and natural transformation $\psi : K' \Rightarrow K$, for $\alpha' = \alpha \bullet \psi$, the diagram

$$\begin{tikzcd}
K' \arrow[d, "\psi"'] & K'Q \arrow[l, "K'p"'] \arrow[r, "\alpha' Q"] \arrow[d, "\psi Q"] & HFQ \arrow[d, equal] \\
K \arrow[dr, "\alpha"'] & KQ \arrow[l, "Kp"'] \arrow[r, "\alpha Q"] \arrow[d] & HFQ \arrow[dl, "HFp"] \\
& HF &
\end{tikzcd}$$

also commutes; thus, (HFQ, HFp) is indeed a homotopically terminal object in $\left([\mathcal{C}, \mathcal{E}]_{\mathrm{h}} \downarrow HF\right)_{\mathrm{h}}$.

(ii). Suppose $(H, H', \alpha, \alpha', \theta)$ is an object in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$. The diagram below commutes,

$$\begin{tikzcd}
& HQ \arrow[rr, "\theta Q"] \arrow[dl, "Hp"'] \arrow[dd, "\alpha Q"' near end] & & H'Q \arrow[dl, "H'p"'] \arrow[dd, "\alpha' Q"] \\
H \arrow[rr, "\theta" near end] \arrow[dd, "\alpha"'] & & H' \arrow[dd, "\alpha'" near start] & \\
& FQ \arrow[rr, "\varphi Q"' near start] \arrow[dl, "Fp"] & & F'Q \arrow[dl, "F'p"] \\
F \arrow[rr, "\varphi"'] & & F' &
\end{tikzcd}$$

and $(Hp, H'p)$ is a weak equivalence, so $(FQ, F'Q, Fp, F'p, \varphi Q)$ is indeed a homotopically terminal object in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$.

(iii). Let $\left(\mathcal{C}^{\circ}, Q^{\mathcal{C}^{\circ}}, p^{\mathcal{C}^{\circ}}\right)$ and $\left(\mathcal{D}^{\circ}, Q^{\mathcal{D}^{\circ}}, p^{\mathcal{D}^{\circ}}\right)$ be functorial left deformation retracts for F and G respectively, and suppose F maps objects in $\mathcal{C}^{\circ}$ to objects in $\mathcal{D}^{\circ}$. To begin, observe that $Gp^{\mathcal{D}^{\circ}} FQ^{\mathcal{C}^{\circ}} : GQ^{\mathcal{D}^{\circ}} FQ^{\mathcal{C}^{\circ}} \Rightarrow GFQ^{\mathcal{C}^{\circ}}$ is a natural weak equivalence; and, as established above, both $\delta^F Q^{\mathcal{C}^{\circ}} : (\mathbb{L}F)Q^{\mathcal{C}^{\circ}} \Rightarrow FQ^{\mathcal{C}^{\circ}}$ and $\delta^G Q^{\mathcal{D}^{\circ}} : (\mathbb{L}G)Q^{\mathcal{D}^{\circ}} \Rightarrow GQ^{\mathcal{D}^{\circ}}$ are natural weak equivalences, so their horizontal composite $\left(\delta^G Q^{\mathcal{C}^{\circ}}\right) \circ \left(\delta^F Q^{\mathcal{D}^{\circ}}\right)$ is also a natural weak equivalence. We also know that $\left(\mathcal{C}^{\circ}, Q^{\mathcal{C}^{\circ}}, p^{\mathcal{C}^{\circ}}\right)$ is a functorial left deformation retract for GF, so $\left(GFQ^{\mathcal{C}^{\circ}}, GFp^{\mathcal{C}^{\circ}}\right)$ is a homotopical left approximation for GF. Now, noting that the following diagram commutes,

$$\begin{tikzcd}
(\mathbb{L}G)Q^{\mathcal{D}^{\circ}}(\mathbb{L}F)Q^{\mathcal{C}^{\circ}} \arrow[r, "(\delta^G Q^{\mathcal{D}^{\circ}})\circ(\delta^F Q^{\mathcal{C}^{\circ}})"] \arrow[d, "((\mathbb{L}G)p^{\mathcal{D}^{\circ}})\circ((\mathbb{L}F)p^{\mathcal{C}^{\circ}})"'] & GQ^{\mathcal{D}^{\circ}} FQ^{\mathcal{C}^{\circ}} \arrow[r, "Gp^{\mathcal{D}^{\circ}} FQ^{\mathcal{C}^{\circ}}"] \arrow[d, "(Gp^{\mathcal{D}^{\circ}})\circ(Fp^{\mathcal{C}^{\circ}})"'] & GFQ^{\mathcal{C}^{\circ}} \arrow[d, "GFp^{\mathcal{C}^{\circ}}"] \\
(\mathbb{L}G)(\mathbb{L}F) \arrow[r, "\delta^G \circ \delta^F"'] & GF \arrow[r, equal] & GF
\end{tikzcd}$$

we conclude that $\left((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\right)$ and $\left(GFQ^{\mathcal{C}^{\circ}}, GFp^{\mathcal{C}^{\circ}}\right)$ are weakly equivalent in $\left([\mathcal{C}, \mathcal{E}]_{\mathrm{h}} \downarrow GF\right)_{\mathrm{h}}$, and so $\left((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\right)$ is also a homotopical left approximation for GF, by proposition 3.2.2. ■

Remark 3.4.12. Unlike the situation we had with total derived functors, the assignment $F \mapsto FQ$ (resp. $G \mapsto GR$) is not a lax (resp. oplax) 2-functor, because we do not have a natural transformation $\mathrm{id}_{\mathcal{C}} \Rightarrow Q$ (resp. $R \Rightarrow \mathrm{id}_{\mathcal{D}}$).

Corollary 3.4.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and let $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \operatorname{Ho} \mathcal{D}$ be the respective localising functors.*

- *If $F : \mathcal{C} \to \mathcal{D}$ is a left deformable functor and $(\mathbb{L}F, \delta)$ is any homotopical left approximation for F, then $\left(\operatorname{Ho}(\mathbb{L}F), \gamma_{\mathcal{D}}\delta\right)$ is a total left derived functor for F.*

- *If $G : \mathcal{D} \to \mathcal{C}$ is a right deformable functor and $(\mathbb{R}G, \delta)$ is any homotopical right approximation for G, then $\left(\operatorname{Ho}(\mathbb{R}G), \gamma_{\mathcal{C}}\delta\right)$ is a total right derived functor for G.*

Proof. Combine theorems 3.3.17 and 3.4.11. ■

3.5 Two-arrow calculi

Prerequisites. §§3.1, A.4.

Definition 3.5.1. Let C be a relative category.

- We say C admits a **calculus of spans** if, for any morphism $f : X \to Y$ and any weak equivalence $v : \tilde{Y} \to Y$ in C, there exists a pullback square in C of the form below,

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "f'"] \arrow[d, dashed, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  where $v' : \tilde{X} \to X$ is also a weak equivalence in C.

- We say C admits a **calculus of cospans** if, for any weak equivalence $u : Y \to \hat{Y}$ and any morphism $g : Y \to Z$ in C, there exists a pushout square in C of the form below,

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[d, dashed, "u'"] \\ \hat{Y} \arrow[r, dashed, "g'"'] & \hat{Z} \end{tikzcd}$$

  where $u' : Z \to \hat{Z}$ is also a weak equivalence in C.

We follow Jardine [2009] in using the following terminology:

Definition 3.5.2. Let C be a relative category.

- A **cocycle** $(f, v) : X \nrightarrow Y$ in C is a span of the form below,

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} Y$$

  where $v : \tilde{X} \to X$ is a weak equivalence in C and $f : \tilde{X} \to Y$ is any morphism. The **cocycle category** $C^{\sim\to}(X, Y)$ is the category whose objects are cocycles $X \nrightarrow Y$ in C and whose morphisms are commutative diagrams of the following form,

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] \arrow[d] & Y \arrow[d, equal] \\ X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] & Y \end{tikzcd}$$

with composition and identities inherited from $\mathcal{C}$.

- A **cycle** $(u, f) : X \nrightarrow Y$ in $\mathcal{C}$ is a cospan of the form below,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

where $u : Y \to \hat{Y}$ is a weak equivalence in $\mathcal{C}$ and $f : X \to \hat{Y}$ is any morphism. The **cycle category** $\mathcal{C}^{\to\sim}(X, Y)$ is the category whose objects are cycles $X \nrightarrow Y$ in $\mathcal{C}$ and whose morphisms are commutative diagrams of the following form,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & \hat{Y} \arrow[d] & Y \arrow[l, "u"'] \arrow[d, equal] \\ X \arrow[r, "f'"'] & \hat{Y}' & Y \arrow[l, "u'"] \end{tikzcd}$$

with composition and identities inherited from $\mathcal{C}$.

Remark 3.5.3. In many cases of interest, $\mathcal{C}$ will be a relative category where $\operatorname{weq} \mathcal{C}$ does *not* have the 2-out-of-3 property; as such, we cannot assume that the underlying morphism of a morphism of cocycles or cycles is a weak equivalence.

¶ 3.5.4. Let $\mathcal{C}$ be a relative category that admits a calculus of spans. Given a pair of cocycles in $\mathcal{C}$, say (f, v) and (g, v') as below,

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} Y \xleftarrow{v'} \tilde{Y} \xrightarrow{g} Z$$

a **composition** for the pair is a commutative diagram of the following form,

$$\begin{tikzcd} & & W \arrow[dl, "v''"'] \arrow[dr, "f''"] & & \\ & \tilde{X} \arrow[dl, "v"'] \arrow[dr, "f"] & & \tilde{Y} \arrow[dl, "v'"'] \arrow[dr, "f'"] & \\ X & & Y & & Z \end{tikzcd}$$

where the diamond is a pullback square with $v'' : W \to \tilde{X}$ a weak equivalence in $\mathcal{C}$, and the **composite** is the cocycle $(f' \circ f'', v \circ v'')$. It is clear that compositions exist and are unique up to unique isomorphism (in the appropriate sense). Moreover, composition is associative and unital up to coherent natural isomorphism, so we get a **bicategory of cocycles** in $\mathcal{C}$, which we denote by $\mathcal{C}^{\sim\to}$, and we have an obvious pseudofunctor $\mathcal{C} \to \mathcal{C}^{\sim\to}$ that sends a morphism $f : X \to Y$ in $\mathcal{C}$ to the cocycle (f, id_X).

Dually, if $\mathcal{C}$ is a relative category that admits a calculus of cospans, then we get a **bicategory of cycles** in $\mathcal{C}$, which we denote by $\mathcal{C}^{\to\sim}$, and we have an obvious pseudofunctor $\mathcal{C} \to \mathcal{C}^{\to\sim}$ that sends a morphism $f : X \to Y$ in $\mathcal{C}$ to the cycle (id_Y, f).

Remark 3.5.5. If $\mathcal{C}$ is a small relative category, then the category of cocycles or cycles between any two objects is a small category; but if $\mathcal{C}$ is merely locally small, then the category of cocycles or cycles may not even be essentially small.

Theorem 3.5.6 (Fundamental theorem of calculi of spans and cospans)**.** *Let $\mathcal{C}$ be a small relative category and let $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ be the connected components functor.*[1]

- *If $\mathcal{C}$ admits a calculus of spans and $\pi_0[\mathcal{C}^{\sim\to}]$ is the category obtained by applying π_0 to the hom-categories of the bicategory of cocycles, then the pseudofunctor $\mathcal{C} \to \mathcal{C}^{\sim\to}$ induces an isomorphism* $\operatorname{Ho}\mathcal{C} \to \pi_0[\mathcal{C}^{\sim\to}]$.
- *If $\mathcal{C}$ admits a calculus of cospans and $\pi_0[\mathcal{C}^{\to\sim}]$ is the category obtained by applying π_0 to the hom-categories of the bicategory of cycles, then the pseudofunctor $\mathcal{C} \to \mathcal{C}^{\to\sim}$ induces an isomorphism* $\operatorname{Ho}\mathcal{C} \to \pi_0[\mathcal{C}^{\to\sim}]$.

Proof. The two claims are formally dual; we will prove the first version.

Let $v : \tilde{X} \to X$ be a weak equivalence in $\mathcal{C}$. We must first show that the cocycle $(v, \mathrm{id}_{\tilde{X}}) : \tilde{X} \nrightarrow X$ becomes an isomorphism in $\pi_0[\mathcal{C}^{\sim\to}]$. Consider the cocycle $(\mathrm{id}_{\tilde{X}}, v) : X \nrightarrow \tilde{X}$. The following diagram commutes,

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "v"] \arrow[d, "v"] & X \arrow[d, equal] \\
X & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & X
\end{tikzcd}$$

so $(v, \mathrm{id}_{\tilde{X}}) \circ (\mathrm{id}_{\tilde{X}}, v) = (\mathrm{id}_X, \mathrm{id}_X)$ in $\pi_0[\mathcal{C}^{\sim\to}]$. On the other hand, given a pullback square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
K \arrow[r, "p_1"] \arrow[d, "p_0"'] & \tilde{X} \arrow[d, "v"] \\
\tilde{X} \arrow[r, "v"'] & X
\end{tikzcd}$$

[1] Recall proposition A.2.15.

where $p_0 : K \to \tilde{X}$ is a weak equivalence, the universal property of K yields a unique morphism $\Delta : X \to K$ making the diagram below commute:

$$\begin{tikzcd} X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "\Delta"'] & X \arrow[d, equal] \\ X & K \arrow[l, "p_0"] \arrow[r, "p_1"'] & X \end{tikzcd}$$

Thus, $\left(\mathrm{id}_{\tilde{X}}, v\right) \circ \left(v, \mathrm{id}_{\tilde{X}}\right) = \left(\mathrm{id}_{\tilde{X}}, \mathrm{id}_{\tilde{X}}\right)$ in $\pi_0[C^{\sim\to}]$. It now follows that every morphism $X \to Y$ in $\pi_0[C^{\sim\to}]$ is of the form $\left(f, \mathrm{id}_{\tilde{X}}\right) \circ \left(v, \mathrm{id}_{\tilde{X}}\right)^{-1}$ for some weak equivalence $v : \tilde{X} \to X$ in C and some morphism $f : \tilde{X} \to Y$; hence, the induced functor $\mathrm{Ho}\, C \to \pi_0[C^{\sim\to}]$ is a bijection on objects and full.

It remains to be shown that the functor $\mathrm{Ho}\, C \to \pi_0[C^{\sim\to}]$ is faithful. Suppose we have the following commutative diagram in C,

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, equal] \\ X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] & X \end{tikzcd}$$

where $v : \tilde{X} \to X$ and $v' : \tilde{X}' \to X$ are weak equivalences in C. The 2-out-of-3 property of isomorphisms in $\mathrm{Ho}\, C$ ensures $h : \tilde{X} \to \tilde{X}'$ is an isomorphism in $\mathrm{Ho}\, C$, so:

$$f \circ v^{-1} = (f' \circ h) \circ (h \circ v')^{-1} = f' \circ v'^{-1}$$

We may therefore define a functor $\pi_0[C^{\sim\to}] \to \mathrm{Ho}\, C$ that sends the connected component of a cocycle $(f, v) : X \nrightarrow Y$ in C to the morphism $f \circ v^{-1}$ in $\mathrm{Ho}\, C$; and using the fact that localising functor $C \to \mathrm{Ho}\, C$ is an epimorphism in **Cat**, we see that this functor is a left inverse for the functor $\mathrm{Ho}\, C \to \pi_0[C^{\sim\to}]$ constructed in the previous paragraph. Thus $\mathrm{Ho}\, C \to \pi_0[C^{\sim\to}]$ is indeed an isomorphism. ■

Proposition 3.5.7. *Let C be a relative category in which* weq C *has the 2-out-of-3 property, and let X and Y be objects in C.*

- *If C admits a calculus of spans, then the cocycle category $C^{\sim\to}(X, Y)$ (considered as a maximal relative category) also admits a calculus of spans.*
- *If C admits a calculus of cospans, then the cycle category $C^{\to\sim}(X, Y)$ (considered as a maximal relative category) also admits a calculus of cospans.*

Proof. The two claims are formally dual; we will prove the first version.

Since weq C has the 2-out-of-3 property in C, the underlying morphisms of morphisms of cocycles must be weak equivalences in C. It follows that pullbacks in $C^{\sim\to}(X,Y)$ exist and can be constructed componentwise in C. ■

Corollary 3.5.8. *Let C be a relative category in which* weq C *has the 2-out-of-3 property.*

- *Let (f,v) and (f',v') be two cocycles $X \twoheadrightarrow Y$ in C. If C admits a calculus of spans, then (f,v) and (f',v') are in the same connected component of $C^{\sim\to}(X,Y)$ if and only if there exists a commutative diagram in C of the following form,*

$$\begin{tikzcd}
X \arrow[d, equal] & \bullet \arrow[l, "v"'] \arrow[r, "f"] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "w_3"'] \arrow[r, "f_3"] \arrow[u, "w_1"'] \arrow[d, "w_2"] & Y \arrow[d, equal] \\
X & \bullet \arrow[l, "v'"] \arrow[r, "f'"'] & Y
\end{tikzcd}$$

  where w_1, w_2, w_3 are weak equivalences in C.

- *Let (u,g) and (u',g') be two cycles $X \twoheadrightarrow Y$ in C. If C admits a calculus of cospans, then (u,g) and (u',g') are in the same connected component of $C^{\to\sim}(X,Y)$ if and only if there exists a commutative diagram in C of the following form,*

$$\begin{tikzcd}
X \arrow[d, equal] \arrow[r, "g"] & \bullet \arrow[d, "w_1"] & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[d, equal] \arrow[r, "g_3"] & \bullet & Y \arrow[l, "w_3"'] \arrow[d, equal] \\
X \arrow[r, "g'"'] & \bullet \arrow[u, "w_2"'] & Y \arrow[l, "u'"]
\end{tikzcd}$$

  where w_1, w_2, w_3 are weak equivalences in C.

Proof. Combine the fundamental theorem of calculi of spans and cospans (3.5.6) with the previous proposition. ■

The following definition is due to Gabriel and Zisman [GZ].

Definition 3.5.9. Let C be a relative category. We say C admits a **calculus of right fractions** if the following axioms are satisfied:

- (Right Ore condition). Given any morphism $f : X \to Y$ in C and any weak equivalence $v : X \to \tilde{X}$, there exists a commutative diagram of the form below,
$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "f'"] \arrow[d, dashed, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
where $v' : \tilde{X} \to X$ is also a weak equivalence in C.

- (Right cancellability). Given any parallel pair $f_0, f_1 : X \to Y$ in C, if $t : Y \to T$ is a weak equivalence in C such that $t \circ f_0 = t \circ f_1$, then there exists a weak equivalence $s : S \to X$ such that $f_0 \circ s = f_1 \circ s$.

Dually, we say C admits a **calculus of left fractions** if the following axioms are satisfied:

- (Left Ore condition). Given any weak equivalence $u : Y \to \hat{Y}$ and any morphism $g : Y \to Z$ in C, there exists a commutative diagram of the form below,
$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[d, dashed, "u'"] \\ \hat{Y} \arrow[r, dashed, "g'"'] & \hat{Z} \end{tikzcd}$$
where $u' : Z \to \hat{Z}$ is also a weak equivalence in C.

- (Left cancellability). Given any parallel pair $g_0, g_1 : Y \to Z$ in C, if $s : S \to Y$ is a weak equivalence in C such that $g_0 \circ s = g_1 \circ s$, then there exists a weak equivalence $t : Z \to T$ such that $t \circ g_0 = t \circ g_1$.

Remark 3.5.10. Although we cannot compose cocycles (resp. cycles) using pullbacks (resp. pushouts) and form a bicategory of cocycles (resp. cycles) in a relative category C with a calculus of right fractions (resp. calculus of left fractions), the axioms are still enough to give a well-defined category $\pi_0[C^{\sim\to}]$ (resp. $\pi_0[C^{\to\sim}]$).

Lemma 3.5.11. *Let Y be an object in a relative category C.*

- *Let $(\mathcal{C}_{/Y})_{\mathrm{w}}$ be the full subcategory of the slice category $\mathcal{C}_{/Y}$ spanned by the objects $v : \tilde{Y} \to Y$ where v is a weak equivalence in $\mathcal{C}$. If $\mathcal{C}$ admits a calculus of right fractions, then $(\mathcal{C}_{/Y})^{\mathrm{op}}_{\mathrm{w}}$ is a filtered category.*[2]

- *Let $({}^{Y/}\mathcal{C})_{\mathrm{w}}$ be the full subcategory of the slice category ${}^{Y/}\mathcal{C}$ spanned by the objects $u : Y \to \hat{Y}$ where u is a weak equivalence in $\mathcal{C}$. If $\mathcal{C}$ admits a calculus of left fractions, then $({}^{Y/}\mathcal{C})_{\mathrm{w}}$ is a filtered category.*

Proof. The two claims are formally dual; we will prove the first version.

To begin, we observe that $\mathrm{id} : Y \to Y$ is an object in $(\mathcal{C}_{/Y})_{\mathrm{w}}$, so $(\mathcal{C}_{/Y})_{\mathrm{w}}$ is indeed an inhabited category. Now suppose we have two objects in $(\mathcal{C}_{/Y})_{\mathrm{w}}$, say $v' : \tilde{Y} \to Y$ and $v : \tilde{Y}' \to Y$. Then the right Ore condition ensures there is a commutative diagram in $\mathcal{C}$ of the form below,

$$
\begin{tikzcd}
& \tilde{Y}'' \arrow[dl] \arrow[dr] \arrow[dd, "v''"] & \\
\tilde{Y} \arrow[dr, "v"'] & & \tilde{Y}' \arrow[dl, "v'"] \\
& Y &
\end{tikzcd}
$$

where $v'' : \tilde{Y}'' \to Y$ is a weak equivalence in $\mathcal{C}$. Finally, suppose we have a parallel pair of morphisms in $(\mathcal{C}_{/Y})_{\mathrm{w}}$, say $f_0, f_1 : \tilde{Y} \to \tilde{Y}'$ such that $v' \circ f_0 = v' \circ f_1 = v$. The right cancellability condition then yields a weak equivalence $s : S \to \tilde{Y}$ such that $f_0 \circ s = f_1 \circ s$. This completes the proof that $(\mathcal{C}_{/Y})_{\mathrm{w}}$ is a cofiltered category. ∎

Theorem 3.5.12 (Fundamental theorem of calculi of fractions)**.** *Let $\mathcal{C}$ be a relative category.*

- *Let Y and Z be objects in $\mathcal{C}$. If $\mathcal{C}$ admits a calculus of right fractions, then the hom-ensemble maps*

$$
\begin{aligned}
\mathcal{C}(\tilde{Y}, Z) &\to \operatorname{Ho}\mathcal{C}(Y, Z) \\
f &\mapsto f \circ v^{-1}
\end{aligned}
$$

defined by each weak equivalence $v : \tilde{Y} \to Y$ in $\mathcal{C}$ constitute a colimiting cocone over the evident filtered diagram of shape $(\mathcal{C}_{/Y})^{\mathrm{op}}_{\mathrm{w}}$.

[2] See definition 0.2.1.

- *Let X and Y be objects in C. If C admits a calculus of left fractions, then the hom-ensemble maps*

$$\begin{aligned} C\big(X, \hat{Y}\big) &\to \operatorname{Ho} C(X, Y) \\ g &\mapsto u^{-1} \circ g \end{aligned}$$

defined by each weak equivalence $u : Y \to \hat{Y}$ in C constitute a colimiting cocone over the evident filtered diagram of shape $\left({}^{Y}/C\right)_{\mathrm{w}}$.

Proof. See Proposition 2.4 in [GZ, Ch. I]. ☐

Proposition 3.5.13. *Let C be a relative category. Let (f, v) and (f', v') be two cocycles $Y \nrightarrow Z$ in C. If C admits a calculus of right fractions, then the following are equivalent:*

(i) *The cocycles (f, v) and (f', v') are in the same connected component of the cocycle category $C^{\sim\to}(Y, Z)$.*

(ii) *We have $f \circ v^{-1} = f' \circ v'^{-1}$ in $\operatorname{Ho} C$.*

(iii) *There exists a commutative diagram in C of the form below,*

$$\begin{tikzcd} Y \arrow[d, equal] & \bullet \arrow[l, "v"'] \arrow[r, "f"] & Z \arrow[d, equal] \\ Y \arrow[d, equal] & \bullet \arrow[l, "w_3"'] \arrow[r] \arrow[u] \arrow[d] & Z \arrow[d, equal] \\ Y & \bullet \arrow[l, "v'"] \arrow[r, "f'"'] & Z \end{tikzcd}$$

where w_3 is a weak equivalence in C.

Dually, let (u, g) and (u', g') be two cocycles $X \nrightarrow Y$ in C. If C admits a calculus of left fractions, then the following are equivalent:

(i′) *The cycles (u, g) and (u', g') are in the same connected component of the cycle category $C^{\to\sim}(X, Y)$.*

(ii′) *We have $u^{-1} \circ g = u'^{-1} \circ g'$ in $\operatorname{Ho} C$.*

(iii′) *There exists a commutative diagram in* $\mathcal{C}$ *of the form below,*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & \bullet \arrow[d] & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[r] \arrow[d, equal] & \bullet & Y \arrow[l, "w_3"'] \arrow[d, equal] \\
X \arrow[r, "g'"'] & \bullet \arrow[u] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where w_3 *is a weak equivalence in* $\mathcal{C}$.

Proof. (i) ⇒ (ii). It is clear that any two cocycles in the same connected component of $\mathcal{C}^{\sim\rightarrow}(Y, Z)$ must represent the same morphism $Y \to Z$ in $\operatorname{Ho}\mathcal{C}$.

(ii) ⇒ (iii). Suppose (f, v) and (f', v') represent the same morphism in $\operatorname{Ho}\mathcal{C}$. Using the explicit description of filtered colimits of ensembles, we deduce that there is a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
& \tilde{Y}'' \arrow[dl, "h_1"'] \arrow[dr, "h_2"] \arrow[dd, "v''"] & \\
\tilde{Y} \arrow[dr, "v"'] & & \tilde{Y}' \arrow[dl, "v'"] \\
& Y &
\end{tikzcd}$$

where v'' is a weak equivalence in $\mathcal{C}$ and $f \circ h_1 = f' \circ h_2$. Thus, the following diagram commutes, as required:

$$\begin{tikzcd}
Y \arrow[d, equal] & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f"] & Z \arrow[d, equal] \\
Y \arrow[d, equal] & \tilde{Y}'' \arrow[l, "v''"'] \arrow[r] \arrow[u, "h_1"'] \arrow[d, "h_2"] & Z \arrow[d, equal] \\
Y & \tilde{Y}' \arrow[l, "v'"] \arrow[r, "f'"'] & Z
\end{tikzcd}$$

(iii) ⇒ (i). Immediate. ■

Proposition 3.5.14. *Let* $\mathcal{C}$ *be a homotopical category. If* $\mathcal{C}$ *admits*

- *a calculus of spans, or*
- *a calculus of cospans, or*
- *a calculus of right fractions, or*
- *a calculus of left fractions*

then $\mathcal{C}$ is a saturated homotopical category.

Proof. The four cases are similar; we will assume that $\mathcal{C}$ admits a calculus of spans.

Suppose $f : X \to Y$ is a morphism that is invertible in $\operatorname{Ho}\mathcal{C}$. Then there exists a cocycle $(g, v) : Y \nrightarrow X$ in $\mathcal{C}$ such that $g \circ v^{-1}$ is a two-sided inverse for f in $\operatorname{Ho}\mathcal{C}$. Construct a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} \tilde{X} \arrow[r, "f'"] \arrow[d, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $v' : \tilde{X} \to X$ is a weak equivalence in $\mathcal{C}$. The fundamental theorem of calculi of spans (3.5.6) implies that $(f \circ g, v) = \left(\mathrm{id}_Y, \mathrm{id}_Y\right)$ and $(g \circ f', v') = \left(\mathrm{id}_X, \mathrm{id}_X\right)$ in $\pi_0[\mathcal{C}^{\sim\to}]$, so by corollary 3.5.8, we must have commutative diagrams of the form below:

$$\begin{tikzcd} Y \arrow[d, equal] & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f\circ g"] & Y \arrow[d, equal] \\ Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[d, equal] \\ Y & Y \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v'"'] \arrow[r, "g\circ f'"] & X \arrow[d, equal] \\ X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \arrow[d, equal] \\ X & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

Thus, by repeatedly using the 2-out-of-3 property of $\operatorname{weq}\mathcal{C}$ in $\mathcal{C}$, we see that $f \circ g$ and $g \circ f'$ are weak equivalences in $\mathcal{C}$, and by using the 2-out-of-6 property, we deduce that f (as well as g and f') is indeed a weak equivalence in $\mathcal{C}$. ∎

One advantage of calculi of fractions over calculi of spans and cospans is the following:

Proposition 3.5.15. *Let C be a relative category and let $\gamma : C \to \operatorname{Ho} C$ be the localising functor.*

- *If C admits a calculus of right fractions, then $\gamma : C \to \operatorname{Ho} C$ preserves limits for any finite diagram in C.*
- *If C admits a calculus of left fractions, then $\gamma : C \to \operatorname{Ho} C$ preserves colimits for any finite diagram in C.*

Proof. Apply theorems 0.2.13 and 3.5.12. ■

Definition 3.5.16. Let C be a relative category.

- A **colocal object** (or **right-closed object**) in C is an object X in C such that the hom-ensemble map

$$C(X, v) : C\left(X, \tilde{Y}\right) \to C(X, Y)$$

  is a bijection for all weak equivalences $v : \tilde{Y} \to Y$ in C.

- A **local object** (or **left-closed object**) in C is an object Y in C such that the hom-ensemble map

$$C(u, Y) : C\left(\hat{X}, Y\right) \to C(X, Y)$$

  is a bijection for all weak equivalences $u : X \to \hat{X}$ in C.

Proposition 3.5.17. *Let C be a relative category. If C admits a calculus of right fractions, then the following are equivalent for an object X in C:*

(i) *X is a colocal object in C.*

(ii) *For all weak equivalences $v : \tilde{Y} \to Y$ in C, the hom-ensemble map*

$$C(X, v) : C\left(X, \tilde{Y}\right) \to C(X, Y)$$

is a surjection.

(iii) *The map $C(X, Y) \to \operatorname{Ho} C(\gamma X, \gamma Y)$ induced by the localising functor $\gamma : C \to \operatorname{Ho} C$ is a bijection.*

Dually, if C admits a calculus of left fractions, then the following are equivalent for an object Y in C:

(i′) *Y is a local object in $\mathcal{C}$.*

(ii′) *For all weak equivalences $u : X \to \hat{X}$ in $\mathcal{C}$, the hom-ensemble map*

$$\mathcal{C}(u, Y) : \mathcal{C}\big(\hat{X}, Y\big) \to \mathcal{C}(X, Y)$$

is a surjection.

(iii′) *The map $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ induced by the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is a bijection.*

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). The fundamental theorem of calculi of fractions (3.5.12) says that there is a natural bijection

$$\varinjlim_{v:(\mathcal{C}_{/X})_{\mathrm{w}}^{\mathrm{op}}} \mathcal{C}(\operatorname{dom} v, Y) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

where v varies over the weak equivalences in $\mathcal{C}$ with codomain X (considered as a full subcategory of the slice category $\mathcal{C}_{/X}$). Note that each weak equivalence $v : \tilde{X} \to X$ is a split epimorphism, so $\operatorname{Ho}\mathcal{C}(X, Y)$ is a filtered colimit for a diagram of injective maps. In particular, the map $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ is injective. On the other hand, if $i : X \to \tilde{X}$ is a section of a weak equivalence $v : \tilde{X} \to X$, then $\gamma(v)^{-1} = \gamma(i)$. Thus, the map $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ is also surjective.

(iii) ⇒ (i). Let $v : \tilde{Y} \to Y$ be any weak equivalence in $\mathcal{C}$. The hom-ensemble bijection in the hypothesis is natural, so we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C}\big(X, \tilde{Y}\big) \arrow[r, "\gamma"] \arrow[d, "\mathcal{C}(X,v)"'] & \operatorname{Ho}\mathcal{C}\big(\gamma X, \gamma\tilde{Y}\big) \arrow[d, "\operatorname{Ho}\mathcal{C}(\gamma(v))"] \\
\mathcal{C}(X, Y) \arrow[r, "\gamma"'] & \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)
\end{tikzcd}$$

Since $\gamma(v) : \gamma\tilde{Y} \to \gamma Y$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$, the map $\mathcal{C}(X, v)$ must be a bijection. Thus, X is a colocal object in $\mathcal{C}$. ■

¶ 3.5.18. Given a functor $F : \mathcal{C} \to \mathcal{D}$, an F**-isomorphism** is a morphism in $\mathcal{C}$ that F sends to an isomorphism in $\mathcal{D}$. Note that $\mathcal{C}$, together with the class of F-isomorphisms, is then a saturated homotopical category by lemma 3.1.8.

Proposition 3.5.19. *Let $\mathcal{C}$ be a relative category. Consider the following statements:*

(i) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a left adjoint.*

(ii) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a fully faithful left adjoint.*

(iii) *For each object X in $\mathcal{C}$, there exists a colocal object $\tilde{X}$ and a γ-isomorphism $p : \tilde{X} \to X$.*

We always have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *and if $\mathcal{C}$ admits a calculus of right factions, then* (iii) $\Rightarrow$ (i) *as well.*

Dually:

(i′) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a right adjoint.*

(ii′) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a fully faithful right adjoint.*

(iii′) *For each object Y in $\mathcal{C}$, there exists a local object $\hat{Y}$ and a γ-isomorphism $i : Y \to \hat{Y}$.*

We always have the implications (i′) $\Rightarrow$ (ii′) $\Rightarrow$ (iii′), *and if $\mathcal{C}$ admits a calculus of left fractions, then* (iii′) $\Rightarrow$ (i′) *as well.*

Proof. (i) $\Rightarrow$ (ii). This is proposition A.4.21.

(ii) $\Rightarrow$ (iii). Let $L : \operatorname{Ho}\mathcal{C} \to \mathcal{C}$ be a left adjoint for $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$. We then have the following natural bijection:

$$\mathcal{C}(L\gamma X, Y) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

Since $\gamma v : \gamma\tilde{Y} \to \gamma Y$ is an isomorphism for any weak equivalence $v : \tilde{Y} \to Y$ in $\mathcal{C}$, it follows that $L\gamma X$ is a colocal object in $\mathcal{C}$.

Now, consider the adjunction counit component $\varepsilon_X : L\gamma X \to X$. Proposition A.1.3 says the adjunction unit $\eta : \mathrm{id}_{\operatorname{Ho}\mathcal{C}} \Rightarrow \gamma L$ is a natural isomorphism, so the right triangle identity implies $\gamma\varepsilon_X : \gamma L\gamma X \to \gamma X$ is an isomorphism, i.e. ε_X is a γ-isomorphism, as required.

(iii) $\Rightarrow$ (i). Suppose $\mathcal{C}$ admits a calculus of right fractions. Proposition 3.5.17 says the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ induces a natural map

$$\mathcal{C}\big(\tilde{X}, Y\big) \to \operatorname{Ho}\mathcal{C}\big(\gamma\tilde{X}, \gamma Y\big)$$

that is a bijection whenever $\tilde{X}$ is a colocal object, so if $p : \tilde{X} \to X$ is a γ-isomorphism, we obtain a bijection

$$\mathcal{C}\big(\tilde{X}, Y\big) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

that is natural in Y. Since γ is bijective on objects, this implies γ has a left adjoint. ■

Theorem 3.5.20 (Reflective localisations)**.** *Let $U : \mathcal{D} \to \mathcal{C}$ be a fully faithful functor. If U has a left adjoint, say $F : \mathcal{C} \to \mathcal{D}$, then:*

(i) *Let $\mathcal{U}$ be the smallest subcategory of $\mathcal{C}$ that contains all identity morphisms and the components of the adjunction unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$. Then $(\mathcal{C}, \mathcal{U})$ admits a calculus of left fractions.*

(ii) *Any localisation of $\mathcal{C}$ at $\mathcal{U}$ is also a localisation of $\mathcal{C}$ at F-isomorphisms.*

(iii) *The canonical functor $\bar{F} : \mathcal{C}\big[\mathcal{U}^{-1}\big] \to \mathcal{D}$ induced by $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective on objects.*

Dually, if U has a right adjoint, say $H : \mathcal{C} \to \mathcal{D}$, then:

(i′) *Let $\mathcal{V}$ be the smallest subcategory of $\mathcal{C}$ that contains all identity morphisms and the components of the adjunction counit $\varepsilon : UH \Rightarrow \mathrm{id}_{\mathcal{C}}$. Then $(\mathcal{C}, \mathcal{V})$ admits a calculus of right fractions.*

(ii′) *Any localisation of $\mathcal{C}$ at $\mathcal{V}$ is also a localisation of $\mathcal{C}$ at H-isomorphisms.*

(iii′) *The canonical functor $\bar{H} : \mathcal{C}\big[\mathcal{V}^{-1}\big] \to \mathcal{D}$ induced by $H : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective on objects.*

Proof. (i). The naturality of η ensures that $(\mathcal{C}, \mathcal{U})$ satisfies the left Ore condition. Suppose $f_0, f_1 : UFX \to Y$ are morphisms in $\mathcal{C}$ such that $f_0 \circ \eta_X = f_1 \circ \eta_X$. By proposition A.1.3, the adjunction counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, so the triangle identities imply that $\eta UF = FU\eta$. But $\eta_Y \circ f_0 = UFf_0 \circ \eta_{UFX}$ and $\eta_Y \circ f_1 = UFf_1 \circ \eta_{UFX}$, so we may deduce that $\eta_Y \circ f_0 = \eta_Y \circ f_1$. Thus $(\mathcal{C}, \mathcal{U})$ is left cancellable.

(ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By naturality of η, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\ Y \arrow[r, "\eta_Y"'] & UFY \end{tikzcd}$$

Thus, any functor that sends the components of η to isomorphisms must also make F-isomorphisms invertible. On the other hand, $F\eta$ is a natural isomorphism because ε is, so any functor that makes F-isomorphisms invertible must also send the components of η to isomorphisms.

(iii). Since $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, the functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects, and so $\bar{F} : \mathcal{C}[\mathcal{V}^{-1}] \to \mathcal{D}$ must also be essentially surjective on objects.

It remains to be shown that $\bar{F}$ is a fully faithful functor. Let Y be an object in $\mathcal{C}$, and let $f : X \to X'$ be an F-isomorphism. Since $F \dashv U$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{D}(FX', FY) \arrow[r, "\cong"] \arrow[d, "\mathcal{D}(Ff, FY)"'] & \mathcal{C}(X', UFY) \arrow[d, "\mathcal{C}(f, UFY)"] \\ \mathcal{D}(FX, FY) \arrow[r, "\cong"'] & \mathcal{C}(X, UFY) \end{tikzcd}$$

We then see that UFY is a local object in $\mathcal{C}$ (with respect to F-isomorphisms). Since $\eta_Y : Y \to UFY$ is an F-isomorphism, we may then apply proposition 3.5.19 to deduce that the localising functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{V}^{-1}]$ has a fully faithful right adjoint that sends each object γY to UFY. Thus $\bar{F}$ is indeed fully faithful. ■

3.6 Three-arrow calculi

Prerequisites. §§3.1, A.4.

In this section, we follow [DHKS, §36] and [Thomas, 2011].

Definition 3.6.1. Let $\mathcal{C}$ be a relative category, let $\mathcal{W} = \operatorname{weq} \mathcal{C}$ be the subcategory of weak equivalences in $\mathcal{C}$, and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W}$. We say $\mathcal{C}$ **admits a three-arrow calculus** for $\mathcal{C}$ with respect to $(\mathcal{U}, \mathcal{V})$ if the following conditions are satisfied:

A1. For each weak equivalence w in $\mathcal{C}$, there exist u in $\mathcal{U}$ and v in $\mathcal{V}$ such that $w = v \circ u$.

A2. Given a diagram of the form $\hat{Y} \xleftarrow{u} Y \xrightarrow{g} Z$ in $\mathcal{C}$ with u in $\mathcal{U}$, there exists a diagram of the form $\hat{Y} \xrightarrow{g'} \hat{Z} \xleftarrow{u'} Z$ such that

- $g' \circ u = u' \circ g$,
- u' is in $\mathcal{U}$, and
- given any diagram of the form $\hat{Y} \xrightarrow{y} T \xleftarrow{z} Z$ such that $y \circ u = z \circ g$, there exists a (not necessarily unique) morphism $T \to \hat{Z}$ making the diagram below commute:

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "g'"] \arrow[ddr, bend left, "y"] & \\
Z \arrow[r, "u'"'] \arrow[drr, bend right, "z"'] & \hat{Z} \arrow[dr, dashed] & \\
 & & T
\end{tikzcd}$$

A3. Given a diagram of the form $X \xrightarrow{f} Y \xleftarrow{v} \tilde{Y}$ in $\mathcal{C}$ with v in $\mathcal{V}$, there exists a diagram of the form $X \xleftarrow{v'} \tilde{X} \xrightarrow{f'} \tilde{Y}$ such that

- $f \circ v' = v \circ g'$,
- v' is in $\mathcal{V}$, and
- given any diagram of the form $X \xleftarrow{x} S \xrightarrow{y} Y$ such that $f \circ x = v \circ y$, there exists a (not necessarily unique) morphism $S \to \tilde{X}$ making the diagram below commute:

$$\begin{tikzcd}
S \arrow[dr, dashed] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\
 & \tilde{X} \arrow[r, "v'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
 & \tilde{Y} \arrow[r, "v"'] & Y
\end{tikzcd}$$

A **uni-fractionable category** is a relative category $\mathcal{C}$ together with a pair of subcategories $(\mathcal{U}, \mathcal{V})$ such that $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property in $\mathcal{C}$ and $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$.

Remark 3.6.2. Note that axiom A1 implies that $\operatorname{ob}\mathcal{U} = \operatorname{ob}\mathcal{V} = \operatorname{ob}\mathcal{C}$; in particular, every identity morphism in $\mathcal{C}$ is also in $\mathcal{U}$ and $\mathcal{V}$.

Remark 3.6.3. Consider diagrams of the following forms,

$$
\begin{tikzcd}
 & Y \arrow[dl, "u"'] \arrow[dr, "g"] & \\
\hat{Y} \arrow[dr, "g'"'] & & Z \arrow[dl, "u'"] \\
 & \hat{Z} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \tilde{X} \arrow[dl, "v'"'] \arrow[dr, "f'"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "v"] \\
 & \tilde{Y} &
\end{tikzcd}
$$

where u, u' are in $\mathcal{U}$ and v, v' are in $\mathcal{V}$. Under the assumption that $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{C}$, the morphism g is in $\mathcal{W}$ if and only if g' is in $\mathcal{W}$, and the morphism f is in $\mathcal{W}$ if and only if f' is in $\mathcal{W}$.

Definition 3.6.4. Let $\mathcal{C}$ be a relative category, let $\mathcal{W} = \operatorname{weq}\mathcal{C}$ be the subcategory of weak equivalences in $\mathcal{C}$, and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W}$. A **functorial three-arrow calculus** for $\mathcal{C}$ with respect to $(\mathcal{U}, \mathcal{V})$ consists of the following data:

FA1. A functorial factorisation system on $\mathcal{W}$ with left class contained in $\operatorname{mor}\mathcal{U}$ and right class contained in $\operatorname{mor}\mathcal{V}$.

FA2. A functor from the full subcategory of $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$ spanned by those diagrams of the form $\hat{Y} \overset{u}{\leftarrow} Y \overset{g}{\rightarrow} Z$, where u is in $\mathcal{U}$, to the category $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$, such that each diagram $\hat{Y} \overset{u}{\leftarrow} Y \overset{g}{\rightarrow} Z$ is sent to a diagram of the form $\hat{Y} \overset{g'}{\rightarrow} \hat{Z} \overset{u'}{\leftarrow} Z$, where $g' \circ u = u' \circ g$, u' is in $\mathcal{U}$, and u' is an isomorphism if u is.

FA3. A functor from the full subcategory of $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$ spanned by those diagrams of the form $X \overset{f}{\rightarrow} Y \overset{v}{\leftarrow} \tilde{Y}$, where v is in $\mathcal{V}$, to the category $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$, such that each diagram $X \overset{f}{\rightarrow} Y \overset{v}{\leftarrow} \tilde{Y}$ is sent to a diagram of the form $X \overset{v'}{\leftarrow} \tilde{X} \overset{f'}{\rightarrow} \tilde{Y}$, where $f \circ v' = v \circ g'$, v' is in $\mathcal{V}$, and v' is an isomorphism if v is.

If such data exist, then we say $\mathcal{C}$ **admits a functorial three-arrow calculus** with respect to $(\mathcal{U}, \mathcal{V})$.

Remark 3.6.5. If $\operatorname{mor}\mathcal{U}$ is closed under pushout in $\mathcal{C}$, then we may take pushouts to construct datum FA2; similarly, if $\operatorname{mor}\mathcal{V}$ is closed under pullback in $\mathcal{C}$, then we may take pullbacks to construct datum FA3.

Remark 3.6.6. A relative category C admits a (functorial) three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$ if and only if the opposite relative category C^{op} admits a (functorial) three-arrow calculus with respect to $(\mathcal{V}, \mathcal{U})$.

Proposition 3.6.7. *Let C be a relative category and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W} = \operatorname{weq} C$ (itself considered as a subcategory of C). If C admits a functorial three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, then C admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$.*

Proof. Obviously, having datum FA1 implies axiom A1 is satisfied. Now suppose we have a commutative square of the form below in C,

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "y"] \\
Z \arrow[r, "z"'] & T
\end{tikzcd}$$

where u is in $\mathcal{U}$. The datum FA2 then gives us the following commutative diagram,

$$\begin{tikzcd}
Y \arrow[rr, "u"] \arrow[dr] \arrow[dd, "g"'] & & \hat{Y} \arrow[dr, "y"] \arrow[dd, "g'"] & \\
& T \arrow[rr, equal] \arrow[dd, equal] & & T \arrow[dd, "w"] \\
Z \arrow[rr, "u'"] \arrow[dr, "z"'] & & \hat{Z} \arrow[dr, dashed] & \\
& T \arrow[rr, "w"'] & & \hat{T}
\end{tikzcd}$$

and $w : T \to \hat{T}$ is an isomorphism, thus, there exists a morphism $\hat{Z} \to T$ making the diagram below commute:

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "g'"] \arrow[ddr, bend left, "y"] & \\
Z \arrow[r, "u'"] \arrow[drr, bend right, "z"'] & \hat{Z} \arrow[dr, dashed] & \\
& & T
\end{tikzcd}$$

This shows that axiom A2 is satisfied, and the dual argument proves axiom A3. ■

Proposition 3.6.8. *Let $\mathcal{A}$ and $\mathcal{C}$ be relative categories. If $\mathcal{C}$ admits a functorial three-arrow calculus, and either*

- $\operatorname{weq}\mathcal{C}$ *has the 2-out-of-3 property in $\mathcal{C}$, or*
- $\mathcal{A}$ *is a minimal relative category,*

then the relative functor category $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$ admits a functorial three-arrow calculus constructed componentwise from $\mathcal{C}$.

Proof. Let $(\mathcal{U}, \mathcal{V})$ be a functorial three-arrow calculus for $\mathcal{C}$. It is clear that, when $\mathcal{A}$ is a minimal relative category, all the data constituting a three-arrow calculus for $\mathcal{C}$ may be lifted componentwise to define a three-arrow calculus for $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$.

In general, we must check that $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$ is closed under the various componentwise constructions. However, if $f : A \to B$ is a weak equivalence in $\mathcal{A}$ and $\theta : X \Rightarrow Y$ is a natural weak equivalence of relative functors $X, Y : \mathcal{A} \to \mathcal{M}$, and $\psi \bullet \varphi$ is the componentwise $(\mathcal{U}, \mathcal{V})$-factorisation of θ, then the diagram below commutes,

$$\begin{tikzcd} XA \arrow[r, "\varphi_A"] \arrow[d, "Xf"'] & ZA \arrow[r, "\psi_A"] \arrow[d, "Zf"'] & YA \arrow[d, "Yf"] \\ XB \arrow[r, "\varphi_B"'] & ZB \arrow[r, "\psi_B"'] & YB \end{tikzcd}$$

and so by the 2-out-of-3 property of $\operatorname{weq}\mathcal{C}$, Zf is also a weak equivalence in $\mathcal{C}$, thus $Z : \mathcal{A} \to \mathcal{M}$ is a relative functor. Similarly, one uses the 2-out-of-3 property of $\operatorname{weq}\mathcal{C}$ to ensure that the componentwise constructions satisfy the conditions to be data FA2 and FA3 for a functorial three-arrow calculus. ■

Theorem 3.6.9 (Fundamental theorem of three-arrow calculi)**.** *Let $\mathcal{C}$ be a relative category such that $\operatorname{weq}\mathcal{C}$ has the 2-out-of-3 property in $\mathcal{C}$. If $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, then:*

(i) *Every morphism in $\operatorname{Ho}\mathcal{C}$ can be represented by a zigzag in $\mathcal{C}$ of the form below,*

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

where u is in $\mathcal{U}$ and v is in $\mathcal{V}$.

(ii) *Two such zigzags represent the same morphism in* Ho $\mathcal{C}$ *if and only if there exists a commutative diagram in* $\mathcal{C}$ *of the form*

$$
\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[r, "f_3"] \arrow[u, "v_1"] \arrow[d, "w_1"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] \arrow[u, "u_1"] & \hat{Y}' \arrow[u, "u_2"'] & Y \arrow[l, "u'"]
\end{tikzcd}
$$

where u_1, u_2, u_3, u_4 *are in* $\mathcal{U}$, v_1, v_2, v_3, v_4 *are in* $\mathcal{V}$, *and* w_1, w_2 *are weak equivalences in* $\mathcal{C}$.

Proof. For the functorial case, see paragraph 36.3 in [DHKS]; for the general case, see Lemma 4.9 and Theorem 5.13 in [Thomas, 2011]. □

Proposition 3.6.10. *If* $\mathcal{C}$ *is a homotopical category that admits a three-arrow calculus, then* $\mathcal{C}$ *is a saturated homotopical category.*

Proof. Suppose $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ whose image in Ho $\mathcal{C}$ is an isomorphism, with inverse represented by the following zigzag,

$$Y \xleftarrow{v} \tilde{Y} \xrightarrow{g} \hat{X} \xleftarrow{u} X$$

where u is in $\mathcal{U}$ and v is in $\mathcal{V}$. Then, by axioms A2 and A3, there exist v' in $\mathcal{V}$, f' in $\mathcal{C}$, u'' in $\mathcal{U}$, and f'' in $\mathcal{C}$ such that the diagrams below commute,

$$
\begin{tikzcd}
\tilde{X} \arrow[r, "f'"] \arrow[d, "v'"'] & \tilde{Y} \arrow[d, "v"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "u''"] \\
\hat{X} \arrow[r, "f''"'] & \hat{Y}
\end{tikzcd}
$$

and by theorem 3.6.9, we have commutative diagrams in $\mathcal{C}$ of the following form,

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u] \arrow[d] & X \arrow[l] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] & \bullet & X \arrow[l] \arrow[d, equal] \\
X & \tilde{X} \arrow[l, "v'"'] \arrow[r, "g \circ f'"'] \arrow[u] & \hat{X} \arrow[u] & X \arrow[l, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u] \arrow[d] & Y \arrow[l] \arrow[d, equal] \\
Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] & \bullet & Y \arrow[l] \arrow[d, equal] \\
Y & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f'' \circ g"'] \arrow[u] & \hat{Y} \arrow[u] & Y \arrow[l, "u''"']
\end{tikzcd}
$$

where all leftward- and upward-pointing arrows are weak equivalences in $\mathcal{C}$. We may then deduce that *every* arrow appearing in the above diagrams are in $\operatorname{weq} \mathcal{C}$ by iteratively applying the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$. In particular, $g \circ f'$ and $f'' \circ g$ are weak equivalences in $\mathcal{C}$, so the 2-out-of-6 property of $\operatorname{weq} \mathcal{C}$ implies that f', f'', g are all in $\operatorname{weq} \mathcal{C}$. We then conclude that f is in $\operatorname{weq} \mathcal{C}$, by using the 2-out-of-3 property again. ■

3.7 Categories of fibrant objects

Prerequisites. §§3.1, 3.5, A.4.

One particularly common kind of relative category with a calculus of spans is obtained by taking the full subcategory of fibrant objects in a model category. We can study these categories axiomatically following Brown [1973]:

Definition 3.7.1. A **category of fibrant objects** is a locally small category $\mathcal{E}$ with finite products and equipped with a pair $(\mathcal{W}, \mathcal{F})$ of subclasses of $\operatorname{mor} \mathcal{E}$ satisfying these axioms:

(A) $(\mathcal{E}, \mathcal{W})$ is a category with weak equivalences, i.e. every isomorphism is in $\mathcal{W}$ and $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{E}$.

(B) Every isomorphism is in $\mathcal{F}$, and $\mathcal{F}$ is closed under composition.

(C) Pullbacks along morphisms in $\mathcal{F}$ exist, and the pullback of a morphism that is in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$) is also a morphism that is in $\mathcal{F}$ (resp. $\mathcal{W} \cap \mathcal{F}$).

(D) For each object X in $\mathcal{E}$, there is a commutative diagram of the form below,

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "\Delta"'] & \mathrm{Path}(X) \arrow[d, "p"] \\ & X \times X \end{tikzcd}$$

where $\Delta : X \to X \times X$ is the diagonal morphism, $i : X \to \mathrm{Path}(X)$ is in $\mathcal{W}$, and $\mathrm{Path}(X) \to X \times X$ is in $\mathcal{F}$.

(E) For any object X in $\mathcal{E}$, the unique morphism $X \to 1$ is in $\mathcal{F}$.

In a category of fibrant objects as above,

- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **fibration** is a morphism in $\mathcal{F}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$; and

Definition 3.7.2. Let X be an object in a category of fibrant objects $\mathcal{E}$. A **path object** for X is a quadruple $\big(\mathrm{Path}(X), i, p_0, p_1\big)$, where $\mathrm{Path}(X)$ is an object in $\mathcal{E}$, $i : X \to \mathrm{Path}(X)$ is a weak equivalence, and p_0 and p_1 are retractions of i such that the morphism $\langle p_0, p_1 \rangle : \mathrm{Path}(X) \to X \times X$ is a fibration.

Remark 3.7.3. Axiom D is precisely the statement that path objects exist in a category of fibrant objects.

Lemma 3.7.4. *Let X be an object in a category of fibrant objects $\mathcal{E}$ and let $\big(\mathrm{Path}(X), i, p_0, p_1\big)$ be a path object for X. Then $p_0, p_1 : \mathrm{Path}(X) \to X$ are trivial fibrations.*

Proof. Axioms C and E imply the two projections $X \times X \to X$ are fibrations, so axiom B implies $p_0, p_1 : \mathrm{Path}(X) \to X$ must be fibrations. By definition, we have $p_0 \circ i = p_1 \circ i = \mathrm{id}_X$, so axiom A implies p_0 and p_1 are both weak equivalences, as required. ■

Lemma 3.7.5 (Factorisation lemma)**.** *Let $f : X \to Y$ be a morphism in a category of fibrant objects $\mathcal{E}$. Then there exist a fibration $v : E_f \to Y$ and a weak equivalence $u : X \to E_f$ in $\mathcal{E}$ such that $f = v \circ u$ and u is a section of a trivial fibration.*

Proof. Let $(\mathrm{Path}(Y), i, p_0, p_1)$ be a path object for Y in $\mathcal{E}$. Form a pullback diagram in $\mathcal{E}$ of the form below:

$$\begin{tikzcd} E_f \arrow[r, "g"] \arrow[d, "q"'] & \mathrm{Path}(Y) \arrow[d, "p_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Note that $p_0 : \mathrm{Path}(Y) \to Y$ is a trivial fibration (by lemma 3.7.4), so this makes sense, and $q : E_f \to X$ is also a trivial fibration (by axiom C). Let $u : X \to E_f$ be the unique morphism such that $q \circ u = \mathrm{id}_X$ and $g \circ u = i \circ f$, and let $v = p_1 \circ g$. Then u is a section of a trivial fibration, hence is a weak equivalence by axiom A, and v is a fibration because we have the following commutative diagram,

$$\begin{tikzcd} E_f \arrow[r, "g"] \arrow[d, "{\langle q, p_1 \circ g \rangle}"] \arrow[dd, bend right=60, "v"'] & \mathrm{Path}(Y) \arrow[d, "{\langle p_0, p_1 \rangle}"] \\ X \times Y \arrow[r, "f \times \mathrm{id}_Y"'] \arrow[d] & Y \times Y \\ Y & \end{tikzcd}$$

where the square in the diagram is a pullback square and $X \times Y \to Y$ is the product projection (thus, a fibration by axioms C and E). ■

Proposition 3.7.6. *Let A be an object in a category of fibrant objects $\mathcal{E}$.*

(i) *Let $(\mathcal{E}_{/A})_{\mathrm{f}}$ be the full subcategory of the slice category $\mathcal{E}_{/A}$ spanned by the fibrations over A. Then $(\mathcal{E}_{/A})_{\mathrm{f}}$ is a category of fibrant objects where a morphism in $(\mathcal{E}_{/A})_{\mathrm{f}}$ is a weak equivalence (resp. fibration) if and only if the underlying morphism in $\mathcal{E}$ is a weak equivalence (resp. fibration).*

(ii) *The slice category ${}^{A/}\mathcal{E}$ is a category of fibrant objects where a morphism in ${}^{A/}\mathcal{E}$ is a weak equivalence (resp. fibration) if and only if the underlying morphism in $\mathcal{E}$ is a weak equivalence (resp. fibration).*

Proof. (i). Since $\mathcal{E}$ has pullbacks along fibrations, $(\mathcal{E}_{/A})_{\mathrm{f}}$ has finite products (and the inclusion $(\mathcal{E}_{/A})_{\mathrm{f}} \hookrightarrow \mathcal{E}_{/A}$ preserves them). It is clear that $(\mathcal{E}_{/A})_{\mathrm{f}}$ with the given weak equivalences and fibrations satisfies axioms A, B, C, and E; and for axiom D, we may apply the factorisation lemma (3.7.5).

(ii). It is not hard to see that ${}^{A/}\mathcal{E}$ has finite products and pullbacks along fibrations (and the forgetful functor ${}^{A/}\mathcal{E} \to \mathcal{E}$ preserves them). Thus ${}^{A/}\mathcal{E}$ with the given weak equivalences and fibrations satisfies axioms A, B, C, and E; and for axiom D, it is clear that path objects in ${}^{A/}\mathcal{E}$ can be constructed as in $\mathcal{E}$. ■

Proposition 3.7.7. *Let $\mathcal{E}$ be a category of fibrant objects and let $\mathbb{D}$ be a small category. If $\mathcal{E}$ admits functorial (weak equivalence, fibration)-factorisations, then the functor category $[\mathbb{D}, \mathcal{E}]$ is a category of fibrant objects where a natural transformation is a weak equivalence (resp. fibration) if and only if its components are weak equivalences (resp. fibrations) in $\mathcal{E}$.*

Proof. Since $\mathcal{E}$ has finite products, $[\mathbb{D}, \mathcal{E}]$ also has finite products, and they are computed componentwise. It is clear that $[\mathbb{D}, \mathcal{E}]$ with the given weak equivalences and fibrations satisfies axioms A, B, C, and E; and for axiom D, we may use the functorial (weak equivalence, fibration)-factorisation system on $\mathcal{E}$. ■

Lemma 3.7.8. *Let $\mathcal{E}$ be a category of fibrant objects and let $\mathcal{D}$ be a category of weak equivalences. If $F : \mathcal{E} \to \mathcal{D}$ is a functor that sends trivial fibrations in $\mathcal{E}$ to weak equivalences in $\mathcal{D}$, then F also sends weak equivalences in $\mathcal{E}$ to weak equivalences in $\mathcal{D}$.*

Proof. Let f be a weak equivalence in $\mathcal{E}$. By the factorisation lemma (3.7.5), there exist morphisms u and v in $\mathcal{E}$ such that $f = v \circ u$, u is a section of a trivial fibration, and v is a trivial fibration. Since weak equivalences have the 2-out-of-3 property in $\mathcal{D}$, Fu is a weak equivalence in $\mathcal{D}$; hence, Ff is a weak equivalence in $\mathcal{D}$, as required. ■

Corollary 3.7.9. *Let $\mathcal{E}$ be a category of fibrant objects and let $\mathcal{W}$ be a subclass of* $\operatorname{mor} \mathcal{E}$ *that has the 2-out-of-3 property.*

(i) *If every trivial fibration in $\mathcal{E}$ is in $\mathcal{W}$, then every weak equivalence in $\mathcal{E}$ is also in $\mathcal{W}$.*

(ii) *In particular, the class of weak equivalences in $\mathcal{E}$ is the smallest class of morphisms in $\mathcal{E}$ that has the 2-out-of-3 property and contains the trivial fibrations.* ■

Definition 3.7.10. The **homotopy category** of a category of fibrant objects $\mathcal{E}$ is the category $\operatorname{Ho} \mathcal{E}$ obtained by freely inverting the weak equivalences in $\mathcal{E}$, as in definition A.4.9.

Proposition 3.7.11. *Let $\mathcal{E}$ be a category of fibrant objects and let $\mathcal{V}$ be the class of trivial fibrations in $\mathcal{E}$.*

(i) *$\mathcal{E}$, considered as a relative category with* trivial fibrations *as weak equivalences, admits a calculus of spans.*

(ii) *The canonical comparison functor $\mathcal{E}[\mathcal{V}^{-1}] \to \operatorname{Ho}\mathcal{E}$ is an isomorphism of categories.*

(iii) *Every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{E}$ can be represented by a cocycle in $\mathcal{E}$ of the form below,*

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} Y$$

where $p : \tilde{X} \to X$ is a trivial fibration.

Proof. (i). This is just axiom C for a category of fibrant objects.

(ii). Lemma 3.7.8 implies every weak equivalence in $\mathcal{E}$ becomes invertible in $\mathcal{E}[\mathcal{V}^{-1}]$, so it has the same universal property as $\operatorname{Ho}\mathcal{E}$; thus the canonical comparison functor $\mathcal{E}[\mathcal{V}^{-1}] \to \operatorname{Ho}\mathcal{E}$ must be an isomorphism of categories.

(iii). This is a consequence of the fundamental theorem of calculi of spans (3.5.6). ■

Proposition 3.7.12. *Let $f : X \to Y$ be a morphism in a category of fibrant objects $\mathcal{E}$.*

(i) *There is a functor $f^* : (\mathcal{E}_{/Y})_{\mathrm{f}} \to (\mathcal{E}_{/X})_{\mathrm{f}}$ sending an fibration over Y to its pullback along $f : X \to Y$.*

(ii) *The pullback functor $f^* : (\mathcal{E}_{/Y})_{\mathrm{f}} \to (\mathcal{E}_{/X})_{\mathrm{f}}$ preserves weak equivalences and fibrations.*

Proof. (i). This is just axiom C for a category of fibrant objects.

(ii). Recalling the pullback pasting lemma, this is a consequence of axiom C and lemma 3.7.8. ■

Lemma 3.7.13. *Let $\mathcal{E}$ be a category of fibrant objects and let $s : X \to Y$ be a section of a trivial fibration in $\mathcal{E}$. Given a pullback diagram in $\mathcal{E}$ of the form below,*

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "s'"'] & X \arrow[d, "s"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

where $g : Y' \to Y$ is a fibration in $\mathcal{E}$, the morphism $s' : X' \to Y'$ is a weak equivalence in $\mathcal{E}$.

Proof. Let $p : Y \to X$ be a trivial fibration in $\mathcal{E}$ such that $p \circ s = \mathrm{id}_X$. Form the following pullback diagram in $\mathcal{E}$:

$$\begin{tikzcd} Z \arrow[r, "q"] \arrow[d, "r"'] & Y \arrow[d, "p"] \\ Y' \arrow[r, "p \circ g"'] & X \end{tikzcd}$$

By axiom C, $r : Z \to Y'$ is a trivial fibration and $q : Z \to Y$ is a fibration. There is a unique morphism $t : Y' \to Z$ such that $r \circ t = \mathrm{id}_{Y'}$ and $q \circ t = g$; note that t is then a weak equivalence by axiom A. Now, consider the commutative diagram in $\mathcal{E}$ shown below:

$$\begin{tikzcd} Y' \arrow[r, "p \circ g"] \arrow[d, "t"] \arrow[dd, bend right=60, "\mathrm{id}"'] & X \arrow[d, "s"'] \arrow[dd, bend left=60, "\mathrm{id}"] \\ Z \arrow[r, "q"] \arrow[d, "r"] & Y \arrow[d, "p"'] \\ Y' \arrow[r, "p \circ g"'] & X \end{tikzcd}$$

The lower square and the outer rectangle are pullback diagrams in $\mathcal{E}$, so the pullback pasting lemma implies the upper square is also a pullback diagram in $\mathcal{E}$. Similarly, the following diagram in $\mathcal{E}$ commutes,

$$\begin{tikzcd} X' \arrow[r, "s'"'] \arrow[rr, bend left, "f"] \arrow[d, "s'"'] & Y' \arrow[r, "p \circ g"'] \arrow[d, "t"'] & X \arrow[d, "s"] \\ Y' \arrow[r, "t"] \arrow[rr, bend right, "g"'] & Z \arrow[r, "q"] & Y \end{tikzcd}$$

and both the right square and the outer rectangle are pullback diagrams in $\mathcal{E}$, so the left square is a pullback diagram in $\mathcal{E}$ as well. But proposition 3.7.12 says that $s^* : (\mathcal{E}_{/Y})_{\mathrm{f}} \to (\mathcal{E}_{/X})_{\mathrm{f}}$ preserves weak equivalences, so $s' : X' \to Y'$ is indeed a weak equivalence, as required. ■

Proposition 3.7.14. *In a category of fibrant objects, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

Proof. Axiom C says that the pullback of a trivial fibration is a trivial fibration, so the factorisation lemma (3.7.5) implies it is enough to prove that the pullback of a section of a trivial fibration is a weak equivalence; but that is precisely the statement of lemma 3.7.13. ■

Definition 3.7.15. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a category of fibrant objects $\mathcal{E}$ and let $(\mathrm{Path}(Y), i, p_0, p_1)$ be a path object for Y.

- A **right homotopy** from f_0 to f_1 with respect to $(\mathrm{Path}(Y), i, p_0, p_1)$ is a morphism $H : X \to \mathrm{Path}(Y)$ such that $p_0 \circ H = f_0$ and $p_1 \circ H = f_1$.
- We say f_0 and f_1 are **right homotopic** if there exists a right homotopy from f_0 to f_1 with respect to some path object for Y.

Remark 3.7.16. If f_0 and f_1 are right homotopic, then $f_0 = f_1$ in $\operatorname{Ho}\mathcal{E}$ (because $p_0, p_1 : \mathrm{Path}(Y) \to Y$ are isomorphisms in $\operatorname{Ho}\mathcal{E}$ with a common section, namely $i : Y \to \mathrm{Path}(Y)$). The converse is not true in general; but see also theorem 3.7.34.

Lemma 3.7.17. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a category of fibrant objects $\mathcal{E}$.*

(i) *Given any path object $(\mathrm{Path}(Y), i, p_0, p_1)$ for Y, $i \circ f_0 : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to itself.*

(ii) *If $H : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to f_1 with respect to a path object $(\mathrm{Path}(Y), i, p_0, p_1)$ for Y, then the same H is a right homotopy from f_1 to f_0 for the path object $(\mathrm{Path}(Y), i, p_1, p_0)$.*

Proof. Obvious. ♦

Lemma 3.7.18. *Let Y be an object in a category of fibrant objects $\mathcal{E}$. Given two path objects for Y, say $\left(\mathrm{Path}(Y)', i', p_0', p_1'\right)$ and $\left(\mathrm{Path}(Y)'', i'', p_0'', p_1''\right)$, then there exists a third path object $\left(\mathrm{Path}(Y), i, p_0, p_1\right)$ such that the diagram below commutes,*

$$\begin{tikzcd}
& & Y \arrow[d, "i"'] \arrow[ddl, bend right, "i'"'] \arrow[ddr, bend left, "i''"] & & \\
& & \mathrm{Path}(Y) \arrow[dl, "q'"] \arrow[dr, "q''"'] \arrow[ddll, bend right, "p_0"'] \arrow[ddrr, bend left, "p_1"] & & \\
& \mathrm{Path}(Y)' \arrow[dl, "p_0'"'] \arrow[dr, "p_1'"] & & \mathrm{Path}(Y)'' \arrow[dl, "p_0''"'] \arrow[dr, "p_1''"] & \\
Y & & Y & & Y
\end{tikzcd}$$

and the diamond is a pullback diagram.

Proof. Axiom C ensures that we can construct a diagram of the required form in $\mathcal{E}$. Moreover, axiom A and lemma 3.7.4 imply that $p_0, p_1 : \mathrm{Path}(Y) \to Y$ are trivial fibrations, so $i : Y \to \mathrm{Path}(Y)$ is a weak equivalence. Finally, we note that $\langle p_0, p_1 \rangle : \mathrm{Path}(Y) \to Y \times Y$ can be factorised as follows,

$$\mathrm{Path}(Y) \xrightarrow{\langle q', p_1 \rangle} \mathrm{Path}(Y)' \times Y \xrightarrow{p_0' \times \mathrm{id}_Y} Y \times Y$$

but $p_0' \times \mathrm{id}_Y : \mathrm{Path}(Y)' \times Y \to Y \times Y$ is a (trivial) fibration, and in the diagram below,

$$\begin{tikzcd}
\mathrm{Path}(Y) \arrow[r, "\langle q', p_1 \rangle"] \arrow[d, "q''"'] & \mathrm{Path}(Y)' \times Y \arrow[r, "\pi_1"] \arrow[d, "p_1' \times \mathrm{id}_Y"] & \mathrm{Path}(Y') \arrow[d, "p_1'"] \\
\mathrm{Path}(Y)'' \arrow[r, "\langle p_0'', p_1'' \rangle"'] & Y \times Y \arrow[r, "\pi_1"'] & Y
\end{tikzcd}$$

the outer rectangle and the right square are pullback diagrams, so the left square is also a pullback diagram, and therefore $\langle q', p_1 \rangle : \mathrm{Path}(Y) \to \langle q', p_1 \rangle$ is also a fibration; thus, $\langle p_0, p_1 \rangle : \mathrm{Path}(Y) \to Y \times Y$ is indeed a fibration. ■

Corollary 3.7.19. *Let $f_0, f_1, f_2 : X \to Y$ be three parallel morphisms in a category of fibrant objects $\mathcal{E}$. If f_0 and f_1 are right homotopic, and f_1 and f_2 are right homotopic, then f_0 and f_2 are also right homotopic.* ■

Corollary 3.7.20. *Let X be an object in a category of fibrant objects $\mathcal{E}$. Any two path objects for X are weakly equivalent as objects in the category ${}^{\Delta_X}/\left(\mathcal{E}_{/X \times X}\right)_{\mathrm{f}}$, where $\Delta_X : X \to X \times X$ is the diagonal embedding.* ■

Lemma 3.7.21. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a category of fibrant objects $\mathcal{E}$.*

(i) *If f_0 and f_1 are right homotopic and $g : W \to X$ is any morphism in $\mathcal{E}$, then $f_0 \circ g$ and $f_1 \circ g$ are also right homotopic.*

(ii) *If f_0 and f_1 are right homotopic and $g : Y \to Z$ is any morphism in $\mathcal{E}$, then for any path object $\left(\mathrm{Path}(Z), i, p_0, p_1\right)$ for Z, there exist a trivial fibration $q : \tilde{X} \to X$ and a right homotopy from $g \circ f_0 \circ q$ to $g \circ f_1 \circ q$ with respect to $\left(\mathrm{Path}(Z), i, p_0, p_1\right)$.*

Proof. (i). Obvious.

(ii). See Proposition 1 in [Brown, 1973]. □

Definition 3.7.22. A parallel pair of morphisms in a category of fibrant objects $\mathcal{E}$, say $f_0, f_1 : X \to Y$, are **homotopic** if there is a trivial fibration $q : \tilde{X} \to X$ in $\mathcal{E}$ such that $f_0 \circ q$ and $f_1 \circ q$ are right homotopic.

Remark 3.7.23. Since trivial fibrations are weak equivalences, remark 3.7.16 implies that homotopic pairs of morphisms in $\mathcal{E}$ become equal in $\mathrm{Ho}\,\mathcal{E}$. Moreover, the converse is true: see theorem 3.7.34.

Proposition 3.7.24. *Let $\mathcal{E}$ be a category of fibrant objects. The relation of homotopy is a congruence on $\mathcal{E}$.*

Proof. First, let us show that the relation of homotopy is an equivalence relation on $\mathrm{mor}\,\mathcal{E}$. It is reflexive and symmetric because the relation of right homotopy is reflexive and symmetric (lemma 3.7.17). It is also transitive: indeed, given trivial fibrations $q_2 : \tilde{X}_2 \to X$ and $q_0 : \tilde{X}_0 \to X$ such that $f_0 \circ q_2$ and $f_1 \circ q_2$ are right homotopic and $f_1 \circ q_0$ and $f_2 \circ q_0$ are right homotopic, by taking $\tilde{X}_1 = \tilde{X}_2 \times_X \tilde{X}_0$ and applying axioms A, B, and C, we can find a trivial fibration $q_1 : \tilde{X}_1 \to X$ such that $f_0 \circ q_1$ and $f_1 \circ q_1$ are right homotopic and $f_1 \circ q_1$ and $f_2 \circ q_1$ are right homotopic; thus, $f_0 \circ q_1$ and $f_2 \circ q_1$ are right homotopic, by corollary 3.7.19.

It remains to be shown that the relation of homotopy is compatible with composition, but this follows by a straightforward application of axioms A, B, and C to lemma 3.7.21. ■

Definition 3.7.25. The **primitive homotopy category** of a category of fibrant objects $\mathcal{E}$ is the category $\mathrm{Ho}_\pi\,\mathcal{E}$ obtained by identifying homotopic morphisms.

Remark 3.7.26. The name 'primitive homotopy category' alludes to two facts: first, that homotopic morphisms in $\mathcal{E}$ become identified in $\operatorname{Ho}_\pi \mathcal{E}$; and second, that weak equivalences in $\mathcal{E}$ do not necessarily become invertible in $\operatorname{Ho}_\pi \mathcal{E}$.

Lemma 3.7.27. *Let $\mathcal{E}$ be a category of fibrant objects. If $p : W \to X$ is a trivial fibration in $\mathcal{E}$, then it is an epimorphism in* $\operatorname{Ho}_\pi \mathcal{E}$.

Proof. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{E}$. Suppose $f_0 \circ p = f_1 \circ p$ in $\operatorname{Ho}_\pi \mathcal{E}$, i.e. there is a trivial fibration $q : \tilde{W} \to W$ such that $f_0 \circ p \circ q$ and $f_1 \circ p \circ q$ are right homotopic. Then $p \circ q : \tilde{W} \to X$ is a trivial fibration (by axioms A and B) and so we have $f_0 = f_1$ in $\operatorname{Ho}_\pi \mathcal{E}$, as required. ■

To relate $\operatorname{Ho}_\pi \mathcal{E}$ and $\operatorname{Ho} \mathcal{E}$, we will need a homotopy-theoretic generalisation of pullbacks.

Lemma 3.7.28. *Let $\mathcal{E}$ be a category of fibrant objects. Given a commutative diagram in $\mathcal{E}$ of the form below,*

$$\begin{tikzcd} X_0 \arrow[r, "p_0"] \arrow[d, "f"'] & Y_0 \arrow[d, "g"] & T_0 \arrow[l, "y_0"'] \arrow[d, "h"] \\ X_1 \arrow[r, "p_1"'] & Y_1 & T_1 \arrow[l, "y_1"] \end{tikzcd}$$

if $p_0 : X_0 \to Y_0$ and $p_1 : X_1 \to Y_1$ are fibrations, $f : X_0 \to X_1$ and $g : Y_0 \to Y_1$ are trivial fibrations, and $h : T_0 \to T_1$ is a weak equivalence, then the induced morphism

$$T_0 \times_{Y_0} X_0 \to T_1 \times_{Y_1} X_1$$

is a weak equivalence.

Proof. First, construct a pullback square in $\mathcal{E}$ of the following form:

$$\begin{tikzcd} T' \arrow[r, "y'"] \arrow[d, "h'"'] & Y_0 \arrow[d, "g"] \\ T_1 \arrow[r, "y_1"'] & Y_1 \end{tikzcd}$$

Axiom C ensures the existence of such a pullback square and that $h' : T' \to T_1$ is a trivial fibration. There is then a unique morphism $t' : T_0 \to T'$ making the diagram below commute,

$$\begin{tikzcd} T_1 \arrow[d, equal] & T_0 \arrow[l, "h"'] \arrow[r, "y_0"] \arrow[d, "t'"] & Y_0 \arrow[d, equal] \\ T_1 & T' \arrow[l, "h'"] \arrow[r, "y'"'] & Y_0 \end{tikzcd}$$

and by axiom A, $t' : T_0 \to T'$ is a weak equivalence in $\mathcal{E}$. Next, consider the following diagram in $\mathcal{E}$:

$$\begin{tikzcd} T_0 \times_{Y_0} X_0 \arrow[r] \arrow[d] & T' \times_{Y_0} X_0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ T_0 \arrow[r, "t'"] \arrow[rr, bend right, "y_0"'] & T' \arrow[r, "y'"] & Y_0 \end{tikzcd}$$

The right square and the outer rectangle are pullback diagrams, so the left square is also a pullback diagram. Moreover, since $T' \times_{Y_0} X_0 \to T'$ is a fibration, by proposition 3.7.14, the morphism $T_0 \times_{Y_0} X_0 \to T' \times_{Y_0} X_0$ is a weak equivalence in $\mathcal{E}$. Finally, consider the following diagram in $\mathcal{E}$:

$$\begin{tikzcd} T_0 \times_{Y_0} X_0 \arrow[r] \arrow[d] & T' \times_{Y_0} X_0 \arrow[r] \arrow[d] & T_1 \times_{Y_1} X_1 \arrow[d] \\ T_0 \arrow[r, "t'"] \arrow[rr, bend right, "h"'] & T' \arrow[r, "h'"] & T_1 \end{tikzcd}$$

It is straightforward to verify that $T' \times_{Y_0} X_0 \to T_1 \times_{Y_1} X_1$ is the pullback of a trivial fibration, so using axioms A and C again, we may deduce that the morphism $T_0 \times_{Y_0} X_0 \to T_1 \times_{Y_1} X_1$ is a weak equivalence. ■

Definition 3.7.29. Let $\mathcal{E}$ be a category of fibrant objects. A **homotopy pullback** of a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in $\mathcal{E}$ consists of the following data:

- An object in $\mathcal{E}$, $X \overset{\mathrm{h}}{\times}_Z Y$.
- A pair of morphisms in $\mathcal{E}$, $q_0 : X \overset{\mathrm{h}}{\times}_Z Y \to X$ and $q_1 : X \overset{\mathrm{h}}{\times}_Z Y \to Y$, called **projections**.
- A path object $\left(\mathrm{Path}(Z), i, p_0, p_1\right)$ for Z.

- A morphism $u : X \overset{\mathrm{h}}{\times}_Z Y \to \operatorname{Path}(Z)$ fitting into a pullback diagram in $\mathcal{E}$ of the following form:

$$\begin{tikzcd} X \overset{\mathrm{h}}{\times}_Z Y \arrow[r, "u"] \arrow[d, "\langle q_0, q_1 \rangle"'] & \operatorname{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ X \times Y \arrow[r, "f \times g"'] & Z \times Z \end{tikzcd}$$

We will often abuse notation and refer to $X \overset{\mathrm{h}}{\times}_Z Y$ as the homotopy pullback.

Remark 3.7.30. Given $f : X \to Z$ and $g : Y \to Z$, we may form a category whose objects are homotopy pullbacks of f and g and whose morphisms are tuples of morphisms in $\mathcal{E}$ making all the relevant diagrams commute, and corollary 3.7.20 and lemma 3.7.28 imply that any two objects in this category are connected by a span of weak equivalences.

Proposition 3.7.31. *Let $\mathcal{E}$ be a category of fibrant objects and let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{E}$. Given a path object $\left(\operatorname{Path}(Z), i, p_0, p_1\right)$ for Z, consider a pullback diagram in $\mathcal{E}$ of the following form:*

$$\begin{tikzcd} X \overset{\mathrm{h}}{\times}_Z Y \arrow[r, "u"] \arrow[d, "\langle q_0, q_1 \rangle"'] & \operatorname{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ X \times Y \arrow[r, "f \times g"'] & Z \times Z \end{tikzcd}$$

(i) *The morphism $\langle q_0, q_1 \rangle : X \overset{\mathrm{h}}{\times}_Z Y \to X \times Y$ is a fibration in $\mathcal{E}$, as are the projections $q_0 : X \overset{\mathrm{h}}{\times}_Z Y \to X$ and $q_1 : X \overset{\mathrm{h}}{\times}_Z Y \to Y$.*

(ii) *The dotted arrows in the diagram shown below form a limiting cone over the diagram of solid arrows:*

$$\begin{tikzcd} X \overset{\mathrm{h}}{\times}_Z Y \arrow[rr, dashed, "q_1"] \arrow[dd, dashed, "q_0"'] \arrow[dr, dashed, "u"] \arrow[drr, dashed] \arrow[ddr, dashed] & & Y \arrow[d, "g"] \\ & \operatorname{Path}(Z) \arrow[r, "p_1"'] \arrow[d, "p_0"] & Z \\ X \arrow[r, "f"'] & Z & \end{tikzcd}$$

In particular, the following diagram commutes in $\mathrm{Ho}_\pi\,\mathcal{E}$*:*

$$\begin{tikzcd} X \overset{\mathrm{h}}{\times}_Z Y \arrow[r, "q_1"] \arrow[d, "q_0"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

(iii) *If* $f : X \to Z$ *is a weak equivalence in* $\mathcal{E}$*, then* $q_1 : X \overset{\mathrm{h}}{\times}_Z Y \to Y$ *is a trivial fibration in* $\mathcal{E}$*. Symmetrically, if* $g : Y \to Z$ *is a weak equivalence in* $\mathcal{E}$*, then* $q_0 : X \overset{\mathrm{h}}{\times}_Z Y \to X$ *is a trivial fibration in* $\mathcal{E}$*.*

(iv) *If either* $f : X \to Z$ *or* $g : Y \to Z$ *is a fibration in* $\mathcal{E}$*, then the comparison morphism* $X \times_Z Y \to X \overset{\mathrm{h}}{\times}_Z Y$ *induced by* $i : Z \to \mathrm{Path}(Z)$ *is a weak equivalence.*

Proof. (i). By definition, $\langle p_0, p_1 \rangle : \mathrm{Path}(Z) \to Z \times Z$ is a fibration in $\mathcal{E}$, so by axiom C, so is $\langle q_0, q_1 \rangle : X \overset{\mathrm{h}}{\times}_Z Y \to X \times Y$. Axioms C and E imply that the projections $X \times Y \to X$ and $X \times Y \to Z$ are fibrations, so by axiom B, the projections $X \overset{\mathrm{h}}{\times}_Z Y \to X$ and $X \overset{\mathrm{h}}{\times}_Z Y \to Y$ are also fibrations.

(ii). This is a straightforward check.

(iii). We will prove the claim in the case where $f : X \to Z$ is a weak equivalence in $\mathcal{E}$. Consider the following diagram in $\mathcal{E}$,

$$\begin{tikzcd} X \overset{\mathrm{h}}{\times}_Z Y \arrow[r] \arrow[rr, bend left, "u"] \arrow[d, "\langle q_0, q_1 \rangle"'] & M_g \arrow[r] \arrow[d] & \mathrm{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ X \times Y \arrow[r, "f \times \mathrm{id}_Y"'] & Z \times Y \arrow[r, "\mathrm{id}_Z \times g"'] & Z \times Z \end{tikzcd}$$

where the outer rectangle and right square are pullback diagrams. The pullback pasting lemma says the left square is also a pullback diagram, so by proposition 3.7.14, the morphism $X \overset{\mathrm{h}}{\times}_Z Y \to M_g$ is a weak equivalence in $\mathcal{E}$. On the other hand, lemma 3.7.4 and axiom C imply that the composite $M_g \to Z \times Y \to Y$ is a trivial fibration. It is clear that the composite $Z \overset{\mathrm{h}}{\times}_Z Y \to M_g \to Z \times Y \to Y$ is equal to $q_1 : X \overset{\mathrm{h}}{\times}_Z Y \to Y$, so by axiom A, it is a weak equivalence.

(iv). Assume $f : X \to Z$ is a fibration in $\mathcal{E}$, and using the factorisation lemma (3.7.5), factor $g : Y \to Z$ as a weak equivalence $j : Y \to \hat{Y}$ followed by a fibration $\hat{g} : \hat{Y} \to Z$. Then by axioms B and C, $f \times \hat{g} : X \times \hat{Y} \to Z \times Z$ is also a fibration in $\mathcal{E}$. Thus, there is a diagram in $\mathcal{E}$ of the form below,

$$\begin{tikzcd} X \times_Z \hat{Y} \arrow[r] \arrow[d] & Z \arrow[d, "i"] \\ X \overset{\mathrm{h}}{\times}_Z \hat{Y} \arrow[r, "\hat{u}"] \arrow[d] & \mathrm{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ X \times \hat{Y} \arrow[r, "f \times \hat{g}"'] & Z \times Z \end{tikzcd}$$

where the outer rectangle and both squares are pullback diagrams. Since $i : Z \to \mathrm{Path}(Z)$ is a weak equivalence and $\hat{u} : X \overset{\mathrm{h}}{\times}_Z \hat{Y} \to \mathrm{Path}(Z)$ is a fibration (by axiom C), we may use proposition 3.7.14 to deduce that $X \times_Z \hat{Y} \to X \times_Z \hat{Y}$ is a weak equivalence. Similarly, the morphisms $\mathrm{id}_X \times_Z j : X \times_Z Y \to X \times_Z \hat{Y}$ and $\mathrm{id}_X \overset{\mathrm{h}}{\times}_Z j : X \overset{\mathrm{h}}{\times}_Z Y \to X \overset{\mathrm{h}}{\times}_Z \hat{Y}$ induced by $j : Y \to \hat{Y}$ are weak equivalences, so by considering the following commutative diagram in $\mathcal{E}$,

$$\begin{tikzcd} X \times_Z Y \arrow[r, "\mathrm{id}_Z \times_Z j"] \arrow[d] & X \times_Z \hat{Y} \arrow[d] \\ X \overset{\mathrm{h}}{\times}_Z Y \arrow[r, "\mathrm{id}_Z \overset{\mathrm{h}}{\times}_Z j"'] & X \overset{\mathrm{h}}{\times}_Z \hat{Y} \end{tikzcd}$$

we deduce (using axiom A) that the comparison morphism $X \times_Z Y \to X \overset{\mathrm{h}}{\times}_Z Y$ is indeed a weak equivalence in $\mathcal{E}$. ■

Corollary 3.7.32. *Let $f : X \to Y$ be a weak equivalence in a category of fibrant objects $\mathcal{E}$, let $\left(\mathrm{Path}(Y), i, p_0, p_1\right)$ be a path object for Y, and let K be defined by the following pullback diagram in $\mathcal{E}$:*

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "\langle k_0, k_1 \rangle"'] & \mathrm{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

(i) *There is a unique morphism $r : X \to K$ such that $k_0 \circ r = k_1 \circ r = \mathrm{id}_Y$ and $u \circ r = i \circ f$; moreover, $r : X \to K$ is a weak equivalence.*

(ii) (K, r, k_0, k_1) *is a path object for* X.

Proof. (i). The existence and uniqueness of $r : X \to K$ is clear. Proposition 3.7.31 says that $k_0, k_1 : K \to X$ are trivial fibrations in $\mathcal{E}$, so by axiom A, $r : X \to K$ is a weak equivalence in $\mathcal{E}$.

(ii). It remains to be shown that $\langle k_0, k_1 \rangle : K \to X \times X$ is a fibration in $\mathcal{E}$; but that is an immediate consequence of axiom C. ■

Corollary 3.7.33. *Let* $f_0, f_1 : X \to Y$ *be a parallel pair of morphisms in a category of fibrant objects* $\mathcal{E}$. *If* $g : Y \to Z$ *is a weak equivalence in* $\mathcal{E}$ *such that* $t \circ f_0$ *and* $t \circ f_1$ *are right homotopic, then* f_0 *and* f_1 *are also right homotopic.*

Proof. Suppose $(\mathrm{Path}(Z), i, p_0, p_1)$ is a path object for Y and $H : X \to \mathrm{Path}(Z)$ is a right homotopy from $g \circ f_0$ to $g \circ f_1$. Let $K = Y \overset{\mathrm{h}}{\times}_Z Y$ be defined by the following pullback diagram in $\mathcal{E}$:

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "\langle k_0, k_1 \rangle"'] & \mathrm{Path}(Z) \arrow[d, "\langle p_0, p_1 \rangle"] \\ Y \times Y \arrow[r, "g \times g"'] & Z \times Z \end{tikzcd}$$

By construction, there is a unique morphism $F : X \to K$ such that $k_0 \circ F = f_0$, $k_1 \circ F = f_1$, and $u \circ F = H$; and there is a unique morphism $r : Y \to K$ such that $k_0 \circ r = \mathrm{id}_Y$, $k_1 \circ r = \mathrm{id}_Y$, and $u \circ r = i \circ g$. Moreover, corollary 3.7.32 says that (K, r, k_0, k_1) is a path object for Y. Thus, $F : X \to K$ is a right homotopy from f_0 to f_1, as required. ■

Theorem 3.7.34 (K. S. Brown)**.** *Let* $\mathcal{E}$ *be a category of fibrant objects.*

(i) *For any weak equivalence* $v : \tilde{Y} \to Y$ *in* $\mathcal{E}$ *and any morphism* $f : X \to Y$ *in* $\mathcal{E}$, *there exists a commutative diagram in* $\mathrm{Ho}_\pi\,\mathcal{E}$ *of the form below,*

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "f'"] \arrow[d, dashed, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $v' : \tilde{X} \to X$ *is a trivial fibration in* $\mathcal{E}$.

(ii) *If $t : Y \to T$ is a weak equivalence in $\mathcal{E}$, then $t : Y \to T$ is a monomorphism in* $\operatorname{Ho}_\pi \mathcal{E}$.

(iii) *The localisation functor* $\operatorname{Ho}_\pi \mathcal{E} \to \operatorname{Ho} \mathcal{E}$ *is faithful, i.e. for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{E}$, we have $f_0 = f_1$ in* $\operatorname{Ho} \mathcal{E}$ *if and only if there exists a trivial fibration $q : \tilde{X} \to X$ such that $f_0 \circ q$ and $f_1 \circ q$ are right homotopic.*

Proof. (i). See proposition 3.7.31.

(ii). Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{E}$ and suppose $t : Y \to T$ is a weak equivalence in $\mathcal{E}$ such that $t \circ f_0 = t \circ f_1$ in $\operatorname{Ho}_\pi \mathcal{E}$, i.e. there is a trivial fibration $q : \tilde{X} \to X$ such that $t \circ f_0 \circ q$ and $t \circ f_1 \circ q$ are right homotopic. Then, by corollary 3.7.33, $f_0 \circ q$ and $f_1 \circ q$ are right homotopic. Thus, $f_0 = f_1$ in $\operatorname{Ho}_\pi \mathcal{E}$, as required.

(iii). It now follows that $\operatorname{Ho}_\pi \mathcal{E}$ with the class of *trivial fibrations* constitute a relative category that admits a calculus of right fractions. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{E}$. Suppose $f_0 = f_1$ in $\operatorname{Ho} \mathcal{E}$. By proposition 3.5.13, there must be a commutative diagram in $\operatorname{Ho}_\pi \mathcal{E}$ of the form below,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f_0"] & Y \arrow[d, equal] \\
X \arrow[d, equal] & \tilde{X}' \arrow[l, "q'"'] \arrow[u] \arrow[r] \arrow[d] & Y \arrow[u, equal] \arrow[d, equal] \\
X \arrow[u, equal] \arrow[r, equal] & X \arrow[r, "f_1"'] & Y
\end{tikzcd}$$

where $q' : \tilde{X}' \to X$ is a trivial fibration in $\mathcal{E}$. In other words, there is a trivial fibration $q' : \tilde{X}' \to X$ in $\mathcal{E}$ such that $f_0 \circ q' = f_1 \circ q'$ in $\operatorname{Ho}_\pi \mathcal{E}$. But lemma 3.7.27 says $q' : \tilde{X}' \to X$ is an epimorphism in $\operatorname{Ho}_\pi \mathcal{E}$, so we may deduce that $f_0 = f_1$ in $\operatorname{Ho}_\pi \mathcal{E}$, as required. ■

Lemma 3.7.35. *Let $f : X \to Y$ be a weak equivalence in a category of fibrant objects $\mathcal{E}$.*

(i) *The morphism $f : X \to Y$ is both a monomorphism and an epimorphism in* $\operatorname{Ho}_\pi \mathcal{E}$.

(ii) *If $f : X \to Y$ has a retraction in $\mathcal{E}$, then $f : X \to Y$ is an isomorphism in* $\operatorname{Ho}_\pi \mathcal{E}$.

(iii) *If $f : X \to Y$ has a section in $\mathcal{E}$, then $f : X \to Y$ is an isomorphism in* $\operatorname{Ho}_\pi \mathcal{E}$.

Proof. (i). Since $f : X \to Y$ becomes invertible in $\operatorname{Ho} \mathcal{E}$, it must be both a monomorphism and an epimorphism in $\operatorname{Ho} \mathcal{E}$; but faithful functors reflect monomorphisms and epimorphisms, so theorem 3.7.34 implies that $f : X \to Y$ is also both a monomorphism and an epimorphism in $\operatorname{Ho}_\pi \mathcal{E}$.

(ii). If $f : X \to Y$ is a split monomorphism in $\mathcal{E}$, then the same is true in $\operatorname{Ho}_\pi \mathcal{E}$; but in any category, a split monomorphism that is also an epimorphism must be an isomorphism.

(iii). Similarly, this follows from the dual fact: in any category, a split epimorphism that is also a monomorphism must be an isomorphism. ■

Proposition 3.7.36. *Let $\mathcal{E}$ be a category of fibrant objects. For any functor $F : \mathcal{E} \to \mathcal{C}$, the following are equivalent:*

(i) *$F : \mathcal{E} \to \mathcal{C}$ factors through the functor $\mathcal{E} \to \operatorname{Ho}_\pi \mathcal{E}$ be the functor that sends morphisms to their homotopy classes.*

(ii) *For any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{E}$, if there is a weak equivalence $g : W \to X$ in $\mathcal{E}$ such that $F f_0 \circ F g = F f_1 \circ F g$, then $F f_0 = F f_1$.*

(iii) *For any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{E}$, if there is a trivial fibration $p : Y \to Z$ in $\mathcal{E}$ such that $F p \circ F f_0 = F p \circ F f_1$, then $F f_0 = F f_1$; and if there is a trivial fibration $g : W \to X$ in $\mathcal{E}$ such that $F f_0 \circ F g = F f_1 \circ F g$, then $F f_0 = F f_1$.*

Proof. (i) ⇒ (ii), (i) ⇒ (iii). These are consequences of lemma 3.7.35.

(ii) ⇒ (i). First, let us show that $F : \mathcal{E} \to \mathcal{C}$ identifies right homotopic morphisms. Let $(\operatorname{Path}(Y), i, p_0, p_1)$ be any path object for Y in $\mathcal{E}$. By definition, $i : Y \to \operatorname{Path}(Y)$ is a weak equivalence and a common section for $p_0, p_1 : \operatorname{Path}(Y) \to Y$. In particular, $F p_0 \circ F i = F p_1 \circ F i$, so the hypothesis implies $F p_0 = F p_1$. Thus, if $f_0, f_1 : X \to Y$ are right homotopic in $\mathcal{E}$, then $F f_0 = F f_1$.

It remains to be shown that $F : \mathcal{E} \to \mathcal{C}$ identifies homotopic morphisms. Suppose $f_0, f_1 : X \to Y$ is a parallel pair of morphisms in $\mathcal{E}$ and $q : \tilde{X} \to X$ is a trivial fibration such that $f_0 \circ q$ and $f_1 \circ q$ are right homotopic. Then $F f_0 \circ F q =$

$F f_1 \circ F q$, by the above paragraph. But trivial fibrations are weak equivalences, so the hypothesis implies $F f_0 = F f_1$, as required.

(iii) ⇒ (i). As above, we first show that $F : \mathcal{E} \to \mathcal{C}$ identifies right homotopic morphisms. Let $(\mathrm{Path}(Y), i, p_0, p_1)$ be any path object for Y in $\mathcal{E}$. Lemma 3.7.4 says that $p_0, p_1 : \mathrm{Path}(Y) \to Y$ are trivial fibrations in $\mathcal{E}$. Since $p_0 \circ i = p_1 \circ i = \mathrm{id}_Y$, we have $F p_0 \circ F(i \circ p_0) = F p_0$ and $F p_1 \circ F(i \circ p_1) = F p_1$; thus, the hypothesis implies $F(i \circ p_0) = F(i \circ p_1) = \mathrm{id}_{F\mathrm{Path}(Y)}$. We may then deduce that $F p_0 = F p_1$, and it follows that right homotopic morphisms in $\mathcal{E}$ become equal in $\mathcal{C}$.

It remains to be shown that $F : \mathcal{E} \to \mathcal{C}$ identifies homotopic morphisms; but the argument used above works under these hypotheses. ■

Remark 3.7.37. The above proposition implies that the primitive homotopy category $\mathrm{Ho}_\pi\, \mathcal{E}$ we defined is *isomorphic* to the category $\pi\mathcal{E}$ defined in [Brown, 1973, §2].

— IV —

Model categories

In [1967], Quillen introduced the notion of a 'closed model category' (but we shall say simply 'model category') for homotopy theory, so as to formalise the similarities between the homotopy theory of spaces and homological algebra. The idea was that, to do homotopy theory, one only really needs to know which morphisms are cofibrations, which are weak equivalences, and which are fibrations.

4.1 Basics

Prerequisites. §§3.1, 3.5, 3.6, A.3, A.4.

Definition 4.1.1. A **model structure** on a category $\mathcal{M}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of subensembles of mor $\mathcal{M}$ satisfying the following axioms:[1]

CM2. $\mathcal{W}$ has the 2-out-of-3 property.

CM3. $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{F}$ are closed under retracts.

CM4. Given a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ W \arrow[r] & Y \end{tikzcd}$$

where i is in $\mathcal{C}$ and p is in $\mathcal{F}$, if at least one of i or p is also in $\mathcal{W}$, then there exists a morphism $W \to X$ making both of the evident triangles commute.

[1] This presentation is due to Quillen [1969].

CM5. Any morphism f in $\mathcal{M}$ may be factored in two ways:

- $f = p \circ i$ where i is in $\mathcal{C} \cap \mathcal{W}$ and p is in $\mathcal{F}$, and
- $f = q \circ j$, where j is in $\mathcal{C}$ and q is in $\mathcal{W} \cap \mathcal{F}$.

Given a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{M}$,

- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **cofibration** is a morphism in $\mathcal{C}$,
- a **fibration** is a morphism in $\mathcal{F}$,
- a **trivial cofibration** (or **acyclic cofibration**) is a morphism in $\mathcal{C} \cap \mathcal{W}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$;
- a **cofibrant object** is an object W that is projective with respect to the class of trivial fibrations, i.e. for every trivial fibration $p : X \to Y$ and every morphism $w : W \to Y$, there exists a morphism $W \to X$ making the diagram below commute:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ur, dashed] \arrow[r, "w"'] & Y \end{tikzcd}$$

- a **fibrant object** is an object X that is injective with respect to the class of trivial cofibrations, i.e. for every trivial cofibration $i : Z \to W$ and every morphism $z : Z \to X$, there exists a morphism $W \to X$ making the diagram below commute:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "i"'] & X \\ W \arrow[ur, dashed] & \end{tikzcd}$$

- a **cofibrant–fibrant object** is an object that is both cofibrant and fibrant.

A **model category** is a locally small category $\mathcal{M}$ that is equipped with a model structure and satisfies the additional axiom below:

CM1. $\mathcal{M}$ has finite limits and finite colimits.

A **derivable category** is a locally small category $\mathcal{M}$ that is equipped with a model structure and satisfies the additional axioms below:

DC0. For each object X in $\mathcal{M}$, there exist

- a trivial cofibration $X \to \hat{X}$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$, and
- a trivial fibration $\tilde{X} \to X$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$.

DC1. $\mathcal{M}$ has pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$, and pullbacks along morphisms in $\mathcal{W} \cap \mathcal{F}$; i.e. given diagrams in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & Z' \\ W & \end{tikzcd} \qquad\qquad \begin{tikzcd} & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

if $i : Z \to W$ is in $\mathcal{C} \cap \mathcal{W}$, then the diagram on the left can be completed to a pushout square; and if $p : X \to Y$ is in $\mathcal{W} \cap \mathcal{F}$, then the diagram on the right can be completed to a pullback square.

Remark 4.1.2. Our definition of 'cofibrant object' (resp. 'fibrant object') is necessarily non-standard, because we do not always have initial objects (resp. terminal objects). Nonetheless, in a model category, our definitions agree with the standard ones: see lemma 4.1.16.

Definition 4.1.3. A **DHK model category** is a model category satisfying the following variants of CM1 and CM5:

CM1*. $\mathcal{M}$ is complete and cocomplete.

CM5*. The $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations can be chosen *functorially* in the sense of definition A.3.28.

Remark 4.1.4. Hovey [1999] and Hirschhorn [2003] attribute the stronger definition of 'model category' to Dwyer, Hirschhorn, and Kan [DHK], hence the name 'DHK model category'; of course, this is the definition used in the cited works, as well as in [DHKS]. Note also that the definition in [Hovey, 1999] includes the functorial factorisations as a *structure* instead of a property. On the other hand, [DS] and [GJ] use Quillen's 1969 definition essentially verbatim.

Example 4.1.5. Let $\mathcal{M}$ be any category. The **discrete model structure** on $\mathcal{M}$ is defined by the following data:

- The weak equivalences are the isomorphisms.
- Every morphism is both a cofibration and a fibration.

It is straightforward to directly verify that the axioms are satisfied in this case. Notice that if $\mathcal{M}$ is complete and cocomplete, then the discrete model structure even makes $\mathcal{M}$ into a DHK model category.

Example 4.1.6. The **mono–epi model structure** on **Set** is defined by the following data:

- Every morphism is a weak equivalence.
- The cofibrations are the injective maps.
- The fibrations are the surjective maps.

The key observation is that **Set** admits a mono–epi weak factorisation system;[2] in fact, we can even choose the mono–epi factorisations functorially: for example, given a map $f : X \to Y$, we may take the cograph factorisation $X \to X \amalg Y \to Y$, where $X \to X \amalg Y$ is the coproduct insertion and $X \amalg Y \to Y$ is the map $(\!(f, \mathrm{id}_Y)\!)$.

Remark 4.1.7. Let $\mathcal{M}$ be a category. Then, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{M}$ if and only if $(\mathcal{F}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ is a model structure on $\mathcal{M}^{\mathrm{op}}$.

Lemma 4.1.8. *Let $\mathcal{M}$ be a category equipped with a model structure.*

- *If $i : Z \to W$ is a cofibration in $\mathcal{M}$ and Z is a cofibrant object, then W is also a cofibrant object.*
- *If $p : X \to Y$ is a fibration in $\mathcal{M}$ and Y is a fibrant object, then X is also a fibrant object.*

Proof. The two claims are formally dual; we will prove the first version.

[2] — not to be confused with the epi–mono *orthogonal* factorisation system!

Let $p : X \to Y$ be a trivial fibration in $\mathcal{M}$ and let $w : W \to Y$ be any morphism in $\mathcal{M}$. Since Z is cofibrant, there exists a morphism $z : Z \to X$ such that the diagram below commutes,

$$\begin{tikzcd} Z \arrow[r, dashed, "z"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and since $i : Z \to W$ is a cofibration, axiom CM4 gives a morphism $s : W \to X$ such that $p \circ s = w$. Thus W is also cofibrant. ■

Lemma 4.1.9. *In a category equipped with a model structure:*

- *Every trivial fibration with cofibrant codomain is a split epimorphism.*
- *Every trivial cofibration with fibrant domain is a split monomorphism.*

Proof. The two claims are formally dual; we will prove the first version.

Let $p : X \to Y$ be a trivial fibration, and suppose Y is cofibrant. Consider the following diagram in $\mathcal{M}$:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

By definition, there exists a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$. This shows that $p : X \to Y$ is a split epimorphism. ■

Lemma 4.1.10. *Let $\mathcal{M}$ be a category equipped with a model structure. The following are equivalent for a morphism f in $\mathcal{M}$:*

(i) *f is a weak equivalence in $\mathcal{M}$.*

(ii) *For any factorisation $f = p \circ j$ in $\mathcal{M}$ where p is a fibration and j is a trivial cofibration, p must be a trivial fibration.*

(iii) *There exist a trivial cofibration j and a trivial fibration q such that $f = q \circ j$.*

Proof. (i) ⇒ (ii). Use axiom CM2.

(ii) ⇒ (iii). Use axiom CM5.

(iii) ⇒ (i). Use axiom CM2 again. ■

Lemma 4.1.11. *Let $\mathcal{M}$ be a category with a pair of weak factorisation systems $(\mathcal{C}', \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}')$. Assume $\mathcal{W}$ is a subensemble of* $\operatorname{mor} \mathcal{C}$ *satisfying the following condition:*

$$\mathcal{W} \subseteq \{ q \circ j \mid j \in \mathcal{C}', q \in \mathcal{F}' \}$$

(i) $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$.

(ii) *If $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$, then $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{C} \cap \mathcal{W} = \mathcal{C}'$.*

Dually:

(i′) $\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{F}'$.

(ii′) *If $\mathcal{F}' \subseteq \mathcal{W} \cap \mathcal{F}$, then $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{W} \cap \mathcal{F} = \mathcal{F}'$.*

In particular, assuming $\mathcal{C}' \cup \mathcal{F}' \subseteq \mathcal{W}$, we have $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$ if and only if $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$.

Proof. (i). Suppose $i : X \to Z$ is in $\mathcal{C} \cap \mathcal{W}$; then there must be $j : X \to Y$ in $\mathcal{C}'$ and $q : Y \to Z$ in $\mathcal{F}'$ such that $i = q \circ j$, and so we have the commutative diagram shown below:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "q"] \\ Z \arrow[r, "\mathrm{id}"'] & Z \end{tikzcd}$$

Since $i \boxslash q$, i must be a retract of j; hence, by proposition A.3.17, i is in $\mathcal{C}'$, and therefore $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$.

(ii). If we know $\mathcal{C}' \subseteq \mathcal{C}$, then $\mathcal{F}' \subseteq \mathcal{F}$ by proposition A.3.3, and $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$, so from claim (i) it follows that $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$. ■

Theorem 4.1.12. *Let $\mathcal{M}$ be a category and let $\mathcal{C}, \mathcal{W}, \mathcal{F}$ be subclasses of* $\operatorname{mor} \mathcal{M}$. *Assuming $\mathcal{M}$ has* either *pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$* or *pullbacks along morphisms in $\mathcal{W} \cap \mathcal{F}$, the following are equivalent:*

(i) *$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure for $\mathcal{M}$.*

(ii) *$\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$, and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems for $\mathcal{M}$.*

Proof. (i) ⇒ (ii). Axiom CM5 says that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$- and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations exist, and axiom CM4 says we have the following inclusions:

$$\mathcal{C} \subseteq {}^{\boxslash}(\mathcal{W} \cap \mathcal{F}) \qquad \mathcal{W} \cap \mathcal{F} \subseteq \mathcal{C}^{\boxslash}$$
$$\mathcal{F} \subseteq (\mathcal{C} \cap \mathcal{W})^{\boxslash} \qquad \mathcal{C} \cap \mathcal{W} \subseteq {}^{\boxslash}\mathcal{F}$$

Axiom CM3 implies each one of $\mathcal{C}, \mathcal{F}, \mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F}$ is closed under retracts, so we may apply proposition A.3.19 to deduce that both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are indeed weak factorisation systems.

(ii) ⇒ (i). We may deduce from proposition A.3.17 that $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts, and it remains to be shown that $\mathcal{W}$ is closed under retracts. The two cases are formally dual; we will assume that $\mathcal{M}$ has pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$.

Let $w : X \to Z$ be a morphism in $\mathcal{W}$, and consider a commutative diagram of the form below:

$$\begin{tikzcd}
X' \arrow[r, "s_X"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "w'"'] & X \arrow[r, "r_X"'] \arrow[d, "w"'] & X' \arrow[d, "w'"] \\
Z' \arrow[r, "s_Z"] \arrow[rr, bend right, "\mathrm{id}"'] & Z \arrow[r, "r_Z"] & Z'
\end{tikzcd}$$

Choose a $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ factorisation for w', say $w' = p' \circ j'$, with $j' : X' \to Y'$ in $\mathcal{C} \cap \mathcal{W}$ and $p' : Y' \to Z'$ in $\mathcal{F}$. Construct the following commutative diagram,

$$\begin{tikzcd}
X' \arrow[r, "s_X"] \arrow[d, "j'"'] & X \arrow[r, "r_X"] \arrow[d, "u"'] & X' \arrow[d, "j'"] \\
Y' \arrow[r, "s_Y"] \arrow[d, "p'"'] & Y \arrow[r, "r_Y"] \arrow[d, "v"'] & Y' \arrow[d, "p'"] \\
Z' \arrow[r, "s_Z"'] & Z \arrow[r, "r_Z"'] & Z'
\end{tikzcd}$$

where the top left square is a pushout square, $v \circ u = w$, and $r_Y \circ s_Y = \mathrm{id}_Y$. Since $\mathcal{C} \cap \mathcal{W}$ is closed under pushouts, u is also in $\mathcal{C} \cap \mathcal{W}$, and by the 2-out-of-3 property, v is in $\mathcal{W}$. Thus, p' is in $\mathcal{F}$ and is a retract of v:

$$\begin{tikzcd}
Y' \arrow[r, "s_Y"'] \arrow[d, "p'"'] \arrow[rr, bend left, "\mathrm{id}"] & Y \arrow[r, "r_Y"'] \arrow[d, "v"'] & Y' \arrow[d, "p'"] \\
Z' \arrow[r, "s_Z"] \arrow[rr, bend right, "\mathrm{id}"'] & Z \arrow[r, "r_Z"] & Z'
\end{tikzcd}$$

Using the 2-out-of-3 property again, choose a $(\mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F})$-factorisation of v, say $v = q \circ j$. Since $j \boxslash p'$, there exists a morphism r such that $r \circ j = r_Y$ and $p' \circ r = r_Z \circ q$. Putting $s = j \circ s_Y$, we obtain $r \circ s = r_Y \circ s_Y = \mathrm{id}_Y$; thus p' is a retract of q and must therefore be in $\mathcal{F} \cap \mathcal{W}$. Hence, $w' = p' \circ j'$ is in $\mathcal{W}$. ■

Corollary 4.1.13. *Let $\mathcal{M}$ be a derivable category.*

- *Pushouts of trivial cofibrations along any morphism in $\mathcal{M}$ exist, and any such is a trivial cofibration.*
- *Pullbacks of trivial fibrations along any morphism in $\mathcal{M}$ exist, and any such is a trivial fibration.*

Proof. Apply proposition A.3.17. ■

Remark 4.1.14. May and Ponto [2012, Ch. 14] define 'model category' to mean a complete and cocomplete locally small category $\mathcal{M}$ equipped with a triple of classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfying condition (ii) of the above proposition; if the two weak factorisation systems can be extended to a pair of functorial factorisation systems, then this is a DHK model category.

Lemma 4.1.15. *Let $\mathcal{M}$ be a category equipped with a model structure.*

- *The class of cofibrant objects in $\mathcal{M}$ is closed under retracts.*
- *The class of fibrant objects in $\mathcal{M}$ is closed under retracts.*

Proof. The two claims are formally dual; we will prove the first version.

Let X be a cofibrant object and let $s : X' \to X$ and $r : X \to X'$ be morphisms in $\mathcal{M}$ such that $r \circ s = \mathrm{id}_{X'}$. We must show that, for any cofibration $i : Z \to W$ in $\mathcal{M}$ and any morphism $z' : Z \to X'$ in $\mathcal{M}$, there is a morphism $h' : W \to X'$ in $\mathcal{M}$ such that $h' \circ i = z'$. Let $z = s \circ z'$. Since X is cofibrant, there is a morphism $h : W \to X$ such that $h \circ i = z$; so if $h' = r \circ h$, then $h' \circ i = r \circ s \circ z' = z'$, as required. ■

Lemma 4.1.16. *Let $\mathcal{M}$ be a category equipped with a model structure. If $\mathcal{M}$ has an initial object 0, then the following are equivalent for any object W in $\mathcal{M}$:*

(i) *W is a cofibrant object in $\mathcal{M}$.*

(ii) *The unique morphism $0 \to W$ has the left lifting property with respect to all trivial fibrations in $\mathcal{M}$.*

(iii) *The unique morphism $0 \to W$ is a cofibration.*

Dually, if $\mathcal{M}$ has a terminal object 1, then the following are equivalent for any object X in $\mathcal{M}$:

(i′) *X is a fibrant object in $\mathcal{M}$.*

(ii′) *The unique morphism $X \to 1$ has the right lifting property with respect to all trivial fibrations in $\mathcal{M}$.*

(iii′) *The unique morphism $X \to 1$ is a fibration.*

Proof. (i) ⇔ (ii). Obvious.

(ii) ⇔ (iii). By theorem 4.1.12, any morphism that has the left lifting property with respect to all trivial fibrations must be a cofibration. ■

Proposition 4.1.17. *Let $\mathcal{M}$ be a category equipped with a model structure. If $\mathcal{M}$ satisfies axiom DC1 and has both an initial object and a terminal object, then $\mathcal{M}$ is a derivable category. In particular, any model category is a derivable category.*

Proof. Use axiom CM5 to factorise the unique morphisms $0 \to X$ and $X \to 1$, and then apply lemma 4.1.16 to deduce that axiom DC0 is satisfied. ■

Lemma 4.1.18. *Let $\mathcal{M}$ be a category equipped with a model structure and let A be an object in $\mathcal{M}$.*

(i) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$) admits a* **slice model structure***, where a morphism in $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$) is a weak equivalence, cofibration, or fibration if it is so in $\mathcal{M}$.*

(ii) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$), equipped with the slice model structure, is a derivable category if $\mathcal{M}$ is a derivable category.*

(iii) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$), equipped with the slice model structure, is a model category if $\mathcal{M}$ is a model category.*

Proof. The two halves of each claim are formally dual; we will prove the versions for $\mathcal{M}_{/A}$.

(i). Use lemmas 3.1.6 and A.3.21.

(ii). $\mathcal{M}_{/A}$ always has a terminal object, so axiom CM5 and lemma 4.1.16 imply one half of axiom DC0 in $\mathcal{M}_{/A}$; for the other half, we may use axiom DC0 in $\mathcal{M}$ directly.

It is well known that the projection functor $\mathcal{M}_{/A} \to \mathcal{M}$ preserves and reflects pullbacks and pushouts, so pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in $\mathcal{M}_{/A}$ if pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in $\mathcal{M}$. Thus $\mathcal{M}_{/A}$ satisfies axiom DC1 if $\mathcal{M}$ does.

(iii). The argument above also shows that $\mathcal{M}_{/A}$ has finite limits and colimits if $\mathcal{M}$ does. ■

Lemma 4.1.19. *Let $\left(\mathcal{M}_i \,\middle|\, i \in I\right)$ be a sequence of categories equipped with model structures.*

(i) *The product category $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$ admits a* **product model structure***, where a morphism in $\mathcal{M}$ is a weak equivalence, cofibration, or fibration if each component is so.*

(ii) *$\mathcal{M}$, equipped with the product model structure, is a derivable category if each $\mathcal{M}_i$ is a derivable category.*

(iii) *$\mathcal{M}$, equipped with the product model structure, is a model category if each $\mathcal{M}_i$ is a model category.*

Proof. Everything can be checked componentwise. ◆

Lemma 4.1.20. *Let $\mathcal{M}$ be a category equipped with a model structure.*

- *Suppose we have a commutative diagram in $\mathcal{M}$ of the form below,*

$$\begin{tikzcd} & Z \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ W' \arrow[rr, "g"'] & & W \end{tikzcd}$$

where $i : Z \to W$ and $i' : Z \to W'$ are cofibrations and $g : W' \to W$ is a weak equivalence. If a fibration $p : X \to Y$ has the right lifting property with respect to $i' : Z \to W'$, then $p : X \to Y$ also has the right lifting property with respect to $i : Z \to W$.

- *Suppose we have a commutative diagram in $\mathcal{M}$ of the form below,*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

where $p : X \to Y$ and $p' : X' \to Y$ are fibrations and $f : X \to X'$ is a weak equivalence. If a cofibration $i : Z \to W$ has the left lifting property with respect to $p' : X' \to Y$, then $i : Z \to W$ also has the left lifting property with respect to $p : X \to Y$.

Proof. The two claims are formally dual; we will prove the first version.

Consider the following lifting problem in $\mathcal{M}$:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

Suppose $p : X \to Y$ is a fibration that has the right lifting property with respect to $i' : Z \to W'$. Then there must exist a morphism $h' : W' \to X$ such that the diagram shown below commutes:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "i'"'] & X \arrow[d, "p"] \\ W' \arrow[r, "w \circ g"'] \arrow[ur, dashed, "h'"] & Y \end{tikzcd}$$

Using lemma 4.1.10, choose a trivial cofibration $j : W' \to W''$ and a trivial fibration $q : W'' \to W$ such that $g = q \circ j$. Since $p : X \to Y$ is a fibration, there is a morphism $h'' : W'' \to X$ making the following diagram commute:

$$\begin{tikzcd} W' \arrow[r, "h'"] \arrow[d, "j"'] & X \arrow[d, "p"] \\ W'' \arrow[r, "w \circ q"'] \arrow[ur, dashed, "h''"] & Y \end{tikzcd}$$

On the other hand, $q : W'' \to W$ is a trivial fibration, so there is a morphism $s : W \to W''$ such that the diagram shown below commutes:

$$\begin{tikzcd} Z \arrow[r, "j \circ i'"] \arrow[d, "i"'] & W'' \arrow[d, "q"] \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s"] & W \end{tikzcd}$$

Let $h = h'' \circ s$. Then,

$$h \circ i = h'' \circ s \circ i = h'' \circ j \circ i' = h' \circ i' = z$$

and since $q \circ s = \mathrm{id}$,

$$p \circ h = p \circ h'' \circ s = w \circ q \circ s = w$$

so $h : W \to X$ is indeed a solution to the lifting problem. ■

Definition 4.1.21. Let X be an object in a category $\mathcal{M}$ equipped with a model structure.

- A **cofibrant replacement** for X is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$ and p is a weak equivalence $\tilde{X} \to X$.
- A **fibrant replacement** for X is a pair $(\hat{X}, i)$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$ and i is a weak equivalence $X \to \hat{X}$.
- A **fibrant cofibrant replacement** for X is a cofibrant replacement $(\tilde{X}, p)$ where $p : \tilde{X} \to X$ is a trivial fibration.
- A **cofibrant fibrant replacement** for X is a fibrant replacement $(\hat{X}, i)$ where $i : X \to \hat{X}$ is a trivial cofibration.

Definition 4.1.22. Let $\mathcal{M}$ be a category equipped with a model structure.

- A **cofibrant replacement functor** for $\mathcal{M}$ is a pair (Q, p), where Q is an endofunctor on $\mathcal{M}$ and p is a natural transformation $Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that, for every object X in $\mathcal{M}$, (QX, p_X) is a cofibrant replacement for X.
- A **fibrant replacement functor** for $\mathcal{M}$ is a pair (R, i), where R is an endofunctor on $\mathcal{M}$ and i is a natural transformation $\mathrm{id}_{\mathcal{M}} \Rightarrow R$ such that, for every object X in $\mathcal{M}$, (RX, i_X) is a fibrant replacement for X.

- A **fibrant cofibrant replacement functor** for $\mathcal{M}$ is a pair (Q, p), where Q is an endofunctor on $\mathcal{M}$ and p is a natural transformation $Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that, for every object X in $\mathcal{M}$, $\left(QX, p_X\right)$ is a fibrant cofibrant replacement for X.

- A **cofibrant fibrant replacement functor** for $\mathcal{M}$ is a pair (R, i), where R is an endofunctor on $\mathcal{M}$ and i is a natural transformation $\mathrm{id}_{\mathcal{M}} \Rightarrow R$ such that, for every object X in $\mathcal{M}$, $\left(RX, i_X\right)$ is a cofibrant fibrant replacement for X.

Remark 4.1.23. Note that a fibrant cofibrant replacement for X is precisely a cofibrant replacement for X that is fibrant as an object in $\mathcal{M}_{/X}$, and a cofibrant fibrant replacement for X is precisely a fibrant replacement for X that is cofibrant as an object in ${}^{X/}\mathcal{M}$.

Moreover, if X is fibrant and $\left(\tilde{X}, p\right)$ is a fibrant cofibrant replacement for X, then $\tilde{X}$ is both fibrant and cofibrant in $\mathcal{M}$, and if X is cofibrant and $\left(\hat{X}, i\right)$ is a cofibrant fibrant replacement for X, then $\hat{X}$ is both cofibrant and fibrant in $\mathcal{M}$.

Proposition 4.1.24.

(i) *Any object in a derivable category has both a fibrant cofibrant replacement and a cofibrant fibrant replacement.*

(ii) *Any DHK model category has both a fibrant cofibrant replacement functor and a cofibrant fibrant replacement functor.*

Proof. (i). This is axiom DC0.

(ii). Use axiom CM5* to factorise the unique natural transformations $\Delta 0 \Rightarrow \mathrm{id}_{\mathcal{M}}$ and $\mathrm{id}_{\mathcal{M}} \Rightarrow \Delta 1$, and then apply lemma 4.1.16. ■

It should go without saying that any two cofibrant or fibrant replacements for a fixed object are weakly equivalent; however, more is true:

Lemma 4.1.25. *Let X be an object in a derivable category $\mathcal{M}$.*

- *Any two cofibrant replacements for X are weakly equivalent as objects in the slice model category $\mathcal{M}_{/X}$.*

- *Any two fibrant replacements for X are weakly equivalent as objects in the slice model category ${}^{X/}\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $(\tilde{X}, p)$ be a fibrant cofibrant replacement for X; such exist, by proposition 4.1.24. Let $(\tilde{X}', p')$ be any cofibrant replacement for X. Then, $p : \tilde{X} \to X$ is a trivial fibration, so there exists a morphism $f : \tilde{X}' \to \tilde{X}$ such that $p \circ f = p'$. The 2-out-of-3 property of weak equivalences implies any such $f : \tilde{X}' \to \tilde{X}$ is a weak equivalence, so we may deduce that every cofibrant replacement for X is weakly equivalent to $(\tilde{X}, p)$ as objects in $\mathcal{M}_{/X}$. ■

In the presence of functorial cofibrant and fibrant replacements, we can say something stronger still:

Proposition 4.1.26. *Let X be an object in a derivable category $\mathcal{M}$.*

- *If $\mathcal{M}$ has a cofibrant replacement functor, then the full subcategory of the slice category $\mathcal{M}_{/X}$ spanned by the cofibrant replacements for X is homotopically contractible.*
- *If $\mathcal{M}$ has a fibrant replacement functor, then the full subcategory of the slice category ${}^{X/}\mathcal{M}$ spanned by the fibrant replacements for X is homotopically contractible.*

Proof. The two claims are formally dual; we will prove the first version.

Let (Q, p) be a cofibrant replacement functor for $\mathcal{M}$. Then, for each cofibrant replacement $(\tilde{X}, q)$ for X, we have the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} \tilde{X} \arrow[dr, "q"'] & Q\tilde{X} \arrow[l, "p_{\tilde{X}}"'] \arrow[r, "Qq"] \arrow[d] & QX \arrow[dl, "p_X"] \\ & X & \end{tikzcd}$$

Thus, the constant functor at (QX, p_X) is naturally weakly equivalent to the identity functor of the category of cofibrant replacements for X, and we may then apply proposition 3.1.31 to deduce that it is homotopically contractible. ■

Remark 4.1.27. In other words, cofibrant replacements (resp. fibrant replacements) are homotopically unique in a model category with functorial cofibrant replacements (resp. functorial fibrant replacements).

Proposition 4.1.28. *Let $\mathcal{M}$ be a category with a model structure, let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be the model structure on $\mathcal{M}$, and let $\mathcal{N}$ be a full subcategory of $\mathcal{M}$.*

(i) *If $\mathcal{N}$ is homotopically replete in $\mathcal{M}$, then the data*

$$\left(\mathcal{C} \cap \operatorname{mor} \mathcal{N}, \mathcal{W} \cap \operatorname{mor} \mathcal{N}, \mathcal{F} \cap \operatorname{mor} \mathcal{N}\right)$$

constitute a model structure on $\mathcal{N}$.

(ii) *If W is a cofibrant object in $\mathcal{M}$ and is in $\mathcal{N}$, then W is a cofibrant object in $\mathcal{N}$, i.e. projective with respect to $\mathcal{W} \cap \mathcal{F} \cap \operatorname{mor} \mathcal{N}$; dually, if X is a fibrant object in $\mathcal{M}$ and is in $\mathcal{N}$, then X is a fibrant object in $\mathcal{N}$, i.e. injective with respect to $\mathcal{C} \cap \mathcal{W} \cap \operatorname{mor} \mathcal{N}$.*

(iii) *If $\mathcal{M}$ satisfies axiom DC0, then $\mathcal{N}$ also satisfies axiom DC0, and every cofibrant (resp. fibrant) object in $\mathcal{N}$ is also a cofibrant (resp. fibrant) object in $\mathcal{M}$.*

(iv) *If $\mathcal{M}$ is a derivable category, then so is $\mathcal{N}$ when equipped with the above model structure;*

Proof. (i). Lemma 3.1.7 implies that axiom CM2 is satisfied. Since $\mathcal{N}$ is a full subcategory of $\mathcal{M}$, the data $\left(\mathcal{C} \cap \operatorname{mor} \mathcal{N}, \mathcal{W} \cap \operatorname{mor} \mathcal{N}, \mathcal{F} \cap \operatorname{mor} \mathcal{N}\right)$ satisfy axioms CM3 and CM4 because $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ do. Finally, for axiom CM5, we appeal to the hypothesis that $\mathcal{N}$ is homotopically replete.

(ii). This follows from the assumption that $\mathcal{N}$ is a full subcategory of $\mathcal{M}$.

(iii). Given the above, $\mathcal{N}$ satisfies axiom DC0 if $\mathcal{M}$ does. Now, suppose W is a cofibrant object in $\mathcal{N}$. Then, by axiom DC0, there is a cofibrant object $\tilde{W}$ in $\mathcal{M}$ and a trivial fibration $\tilde{W} \to W$ in $\mathcal{M}$; but $\mathcal{N}$ is a homotopically replete subcategory, so $\tilde{W} \to W$ is also a trivial fibration in $\mathcal{N}$, so by lemma 4.1.9, W is a retract of $\tilde{W}$ and is therefore a cofibrant object in $\mathcal{M}$, by lemma 4.1.15. Dually, if X is a fibrant object in $\mathcal{N}$, then it is a fibrant object in $\mathcal{M}$ as well.

(iv). It remains to be shown that pushouts along trivial cofibrations and pullbacks along trivial fibrations exist in $\mathcal{N}$. For this, simply apply corollary 4.1.13 to the hypothesis that $\mathcal{N}$ is homotopically replete and full. ■

Definition 4.1.29. The **Quillen homotopy category** (or, more simply, **homotopy category**) of a derivable category $\mathcal{M}$ is the category $\operatorname{Ho} \mathcal{M}$ obtained by freely inverting the weak equivalences in $\mathcal{M}$, as in definition A.4.9.

Definition 4.1.30. A **saturated derivable category** is a derivable category that is saturated as a category with weak equivalences.

Theorem 4.1.31. *Let $\mathcal{M}$ be a derivable category and let $\gamma : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ be the localising functor.*

(i) *Let $\mathcal{U}$ and $\mathcal{V}$ be the classes of trivial cofibrations and trivial fibrations in $\mathcal{M}$, respectively. Then $\mathcal{M}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, which is functorial if $\mathcal{M}$ satisfies axiom CM5*.*

(ii) *Let X and Y be objects in $\mathcal{M}$, let $v : X \to \tilde{X}$ and $v' : X \to \tilde{X}'$ be trivial fibrations, let $u : Y \to \hat{Y}$ and $u' : Y \to \hat{Y}'$ be trivial cofibrations, and let $f : \tilde{X} \to \hat{Y}$ and $f' : \tilde{X}' \to \hat{Y}'$ be morphisms in $\mathcal{M}$. Then,*

$$\gamma(u)^{-1} \circ \gamma(f) \circ \gamma(v)^{-1} = \gamma(u')^{-1} \circ \gamma(f') \circ \gamma(v')^{-1}$$

if and only if there exists a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[r, "f_3"] \arrow[u, "v_1"] \arrow[d, "w_1"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] \arrow[u, "u_1"] & \hat{Y}' \arrow[u, "u_2"'] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where u_1, u_2, u_3, u_4 are trivial cofibrations, v_1, v_2, v_3, v_4 are trivial fibrations, and w_1, w_2 are weak equivalences. In any such diagram, v_1 is a split epimorphism if $\tilde{X}$ is cofibrant, and u_2 is a split monomorphism if $\hat{Y}'$ is fibrant.

(iii) *$\mathcal{M}$ is a saturated derivable category if and only if the weak equivalences in $\mathcal{M}$ have the 2-out-of-6 property.*

(iv) *If X is a cofibrant object in $\mathcal{M}$ and Y is a fibrant object in $\mathcal{M}$, then the hom-set map $\mathcal{M}(X, Y) \to \operatorname{Ho} \mathcal{M}(\gamma X, \gamma Y)$ is surjective.*

(v) $\operatorname{Ho} \mathcal{M}$ *is a locally small category.*

Proof. (i). Axioms CM2 and CM5 imply axiom A1 is satisfied, and axioms A2 and A3 follow from the above claims; that we get a functorial three-arrow calculus under axiom CM5* is an obvious consequence of the universal property of pushouts and pullbacks.

(ii). This is a special case of the fundamental theorem of three-arrow calculi (3.6.9), plus lemma 4.1.9.

(iii). Apply proposition 3.6.10 and lemma A.4.14.

(iv). Consider a zigzag of the following form in $\mathcal{M}$,

$$X \xleftarrow{v} X' \xrightarrow{f'} Y' \xleftarrow{u} Y$$

where $u : Y \to Y'$ is a trivial cofibration and $v : X' \to X$ is a trivial fibration. Let $\bar{f} = \gamma(u)^{-1} \circ \gamma(f') \circ \gamma(v)^{-1}$ be the corresponding morphism in $\operatorname{Ho} \mathcal{M}$; note that the fundamental theorem of three-arrow calculi says that every morphism $\gamma X \to \gamma Y$ in $\operatorname{Ho} \mathcal{M}$ is of this form. Suppose X is cofibrant and Y is fibrant. Then lemma 4.1.9 says u is a split monomorphism and v is a split epimorphism, so choose $r : Y' \to Y$ and $s : X \to X'$ such that $r \circ u = \mathrm{id}_Y$ and $v \circ s = \mathrm{id}_X$. Since $\gamma(u)$ and $\gamma(v)$ are isomorphisms in $\operatorname{Ho} \mathcal{M}$, we must have $\gamma(u)^{-1} = \gamma(r)$ and $\gamma(v)^{-1} = \gamma(s)$. Hence, taking $f = r \circ f' \circ s$, we have $\bar{f} = \gamma(f)$, as required.

(v). By proposition 4.1.24, every object in $\mathcal{M}$ is weakly equivalent to both a cofibrant object and a fibrant object, so we may deduce that $\operatorname{Ho} \mathcal{M}$ is locally small from claim (iii). ∎

Corollary 4.1.32. *Let $\mathcal{M}$ be a derivable category. For any two objects X and Y in $\mathcal{M}$, every morphism $X \to Y$ in* $\operatorname{Ho} \mathcal{M}$ *can be represented by a zigzag of the following form,*

$$X \xleftarrow{p} \tilde{X} \longrightarrow \hat{Y} \xleftarrow{i} Y$$

where $(\tilde{X}, p)$ is any cofibrant replacement for X and $(\hat{Y}, i)$ is any fibrant replacement for Y. ∎

Lemma 4.1.33. *Let $\mathcal{M}$ be a derivable category and let $\mathcal{C}$ be a relative category where* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-3 property and the special 2-out-of-4 property.*

- *Let $\mathcal{M}_{\mathrm{c}}$ be the full subcategory of cofibrant objects in $\mathcal{M}$. If a functor $F : \mathcal{M}_{\mathrm{c}} \to \mathcal{C}$ sends trivial cofibrations in $\mathcal{M}_{\mathrm{c}}$ to weak equivalences in $\mathcal{C}$, then F preserves all weak equivalences.*
- *Let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects in $\mathcal{M}$. If a functor $G : \mathcal{M}_{\mathrm{f}} \to \mathcal{C}$ sends trivial fibrations in $\mathcal{M}_{\mathrm{f}}$ to weak equivalences in $\mathcal{C}$, then G preserves all weak equivalences.*

Proof. The two claims are formally dual; we will prove the first version.

Axioms CM2 and CM5 imply that every weak equivalence in $\mathcal{M}$ can be factored as a trivial cofibration followed by a trivial fibration, so it is enough to show that F sends trivial fibrations in $\mathcal{M}_{\mathrm{c}}$ to weak equivalences in $\mathcal{C}$. Let $p : X \to Y$ be a trivial fibration in $\mathcal{M}_{\mathrm{c}}$. Y is cofibrant, so lemma 4.1.9 says $p : X \to Y$ has a section $s : Y \to X$.

Let $e = s \circ p$. Since $p : X \to Y$ is a trivial fibration, we may form a pullback square in $\mathcal{M}$ of the following form:

$$\begin{tikzcd}
K \arrow[r, "k_1"] \arrow[d, "k_0"'] & X \arrow[d, "p"] \\
X \arrow[r, "p"'] & Y
\end{tikzcd}$$

There is then a unique morphism $\Delta : X \to K$ such that $k_0 \circ \Delta = k_1 \circ \Delta = \mathrm{id}_X$. Since $k_0 : K \to X$ is a trivial fibration (by corollary 4.1.13), $\Delta : X \to K$ is a weak equivalence in $\mathcal{M}$ and therefore factorises as $q \circ j$ for some trivial cofibration $j : X \to \tilde{K}$ and some trivial fibration $q : \tilde{K} \to K$; note that $\tilde{K}$ is a cofibrant object. There is also a unique morphism $t : X \to K$ such that $k_0 \circ t = \mathrm{id}_X$ and $k_1 \circ t = e$; and X is a cofibrant object, so there exists a morphism $h : X \to \tilde{K}$ such that $q \circ h = t$. Taking $q_0 = k_0 \circ q$ and $q_1 = k_1 \circ q$, we obtain the following commutative diagram in $\mathcal{M}_{\mathrm{c}}$:

$$\begin{tikzcd}
X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "e"] \arrow[d, "h"] & X \arrow[d, equal] \\
X & \tilde{K} \arrow[l, "q_0"'] \arrow[r, "q_1"] & X \\
X \arrow[u, equal] & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] \arrow[u, "j"'] & X \arrow[u, equal]
\end{tikzcd}$$

Consider the image of the above diagram in $\mathcal{C}$. By hypothesis, $F j : F X \to F \tilde{K}$ is a weak equivalence in $\mathcal{C}$, and by repeatedly applying the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$, we may deduce that $F e : F X \to F X$ is a weak equivalence in $\mathcal{C}$ as well. But $\operatorname{weq} \mathcal{C}$ has the special 2-out-of-4 property, and $F e = F s \circ F p$, so we may conclude that $F p : F X \to F Y$ is a weak equivalence in $\mathcal{C}$, as required. ■

Proposition 4.1.34. *Let $\mathcal{M}$ be a derivable category. Let $\mathcal{M}_{\mathrm{c}}$ be the full subcategory of cofibrant objects in $\mathcal{M}$.*

(i) *$\mathcal{M}_{\mathrm{c}}$, considered as a relative category with* trivial cofibrations *as weak equivalences, admits a calculus of cospans.*

(ii) *The localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to trivial cofibrations is isomorphic to the localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to all weak equivalences.*

(iii) *Every morphism $X \to Y$ in $\operatorname{Ho} \mathcal{M}_{\mathrm{c}}$ can be represented by a cycle in $\mathcal{M}_{\mathrm{c}}$ of the form below,*

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y$$

where $\left(\hat{Y}, i\right)$ is any *cofibrant fibrant replacement for Y.*

Dually, let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects in $\mathcal{M}$.

(i′) *$\mathcal{M}_{\mathrm{f}}$, considered as a relative category with* trivial fibrations *as weak equivalences, admits a calculus of spans.*

(ii′) *The localisation of $\mathcal{M}_{\mathrm{f}}$ with respect to trivial fibrations is isomorphic to the localisation of $\mathcal{M}_{\mathrm{f}}$ with respect to all weak equivalences.*

(iii′) *Every morphism $X \to Y$ in $\operatorname{Ho} \mathcal{M}_{\mathrm{f}}$ can be represented by a cocycle in $\mathcal{M}_{\mathrm{f}}$ of the form below,*

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} Y$$

where $\left(\tilde{X}, p\right)$ is any *fibrant cofibrant replacement for X.*

Proof. (i). This is an immediate consequence of corollary 4.1.13.

(ii). Suppose $F : \mathcal{M}_{\mathrm{c}} \to \mathcal{C}$ is a functor that sends trivial cofibrations in $\mathcal{M}_{\mathrm{c}}$ to isomorphisms in $\mathcal{C}$. It is clear that isomorphisms have the special 2-out-of-4 property, so we may apply lemma 4.1.33 to deduce that F sends weak equivalences in $\mathcal{M}_{\mathrm{c}}$ to isomorphisms in $\mathcal{C}$ as well. Hence, any localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to trivial cofibrations must also be a localisation of $\mathcal{M}_c$ with respect to weak equivalences.

(iii). The fundamental theorem of calculi of cospans (3.5.6) says every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{M}_c$ can be represented by a cycle in $\mathcal{M}_c$ of the form below,

$$X \xrightarrow{g} Y' \xleftarrow{u} Y$$

where $u : Y \to Y'$ is a trivial cofibration, and that two such cycles represent the same morphism if and only if they are in the same connected component of the cycle category $\mathcal{M}_c^{\to\sim}(X, Y)$. Let $(\hat{Y}, i)$ be any cofibrant fibrant replacement for Y. Since $u : Y \to Y'$ is a trivial cofibration and $\hat{Y}$ is fibrant, axiom CM4 yields a morphism $h : Y' \to \hat{Y}$ such that $h \circ u = i$. Taking $f = h \circ g$, we have the following commutative diagram in $\mathcal{M}_c$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, equal] & Y' \arrow[d, "h"] & Y \arrow[l, "u"'] \arrow[d, equal] \\ X \arrow[r, "f"'] & \hat{Y} & Y \arrow[l, "i"] \end{tikzcd}$$

Thus, the cycles (u, g) and (i, f) represent the same morphism in $\operatorname{Ho}\mathcal{M}_c$. ■

Proposition 4.1.35. *Let $\mathcal{M}$ be a derivable category.*

- *Let $\mathcal{M}_c$ be the full subcategory of cofibrant objects in $\mathcal{M}$. The canonical functor* $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ *induced by the inclusion $\mathcal{M}_c \hookrightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects.*
- *Let $\mathcal{M}_f$ be the full subcategory of fibrant objects in $\mathcal{M}$. The canonical functor* $\operatorname{Ho}\mathcal{M}_f \to \operatorname{Ho}\mathcal{M}$ *induced by the inclusion $\mathcal{M}_f \hookrightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that proposition 4.1.24 implies the functor $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ is essentially surjective on objects; it remains to be shown that the functor is fully faithful. Consider the full subcategory $\mathcal{M}_{cf}$ spanned by the cofibrant–fibrant objects in $\mathcal{M}$. By restricting the localising functors, we obtain the following commutative diagram,

$$\begin{tikzcd} \mathcal{M}_{cf} \arrow[r] \arrow[d, equal] & \operatorname{Ho}\mathcal{M}_c \arrow[d] \\ \mathcal{M}_{cf} \arrow[r] & \operatorname{Ho}\mathcal{M} \end{tikzcd}$$

where $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}_{\mathrm{c}}$ and $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}$ are essentially surjective on objects. Theorem 4.1.31 implies $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}$ is a full functor, so $\operatorname{Ho}\mathcal{M}_{\mathrm{c}} \to \operatorname{Ho}\mathcal{M}$ must also be full.

Now, consider a parallel pair of morphisms in $\operatorname{Ho}\mathcal{M}_{\mathrm{c}}$. Proposition 4.1.34 says they can be represented by cycles of the following form,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y \qquad\qquad X \xrightarrow{f'} \hat{Y} \xleftarrow{i} Y$$

where $(\hat{Y}, i)$ is any cofibrant fibrant replacement for Y. Suppose the two morphisms are equal in $\operatorname{Ho}\mathcal{M}$. Then, there must be a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "i"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[u, "v_1"] \arrow[r, "f_3"] \arrow[d, "w_1"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & X \arrow[l, "\mathrm{id}"] \arrow[u, "u_1"] \arrow[r, "f'"'] & \hat{Y} \arrow[u, "u_2"'] & Y \arrow[l, "i"]
\end{tikzcd}$$

where u_1, u_2, u_3, u_4 are trivial cofibrations, v_1, v_2, v_3, v_4 are trivial fibrations, and w_1, w_2 are weak equivalences. Since X is cofibrant, there exists a morphism s in $\mathcal{M}$ such that $v_1 \circ s = \mathrm{id}_X$, so (using lemma 4.1.8) we obtain the following commutative diagram in $\mathcal{M}_{\mathrm{c}}$:

$$\begin{tikzcd}
X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "i"'] \arrow[d, equal] \\
X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[u, "\mathrm{id}"] \arrow[r, "f_3 \circ s"] \arrow[d, "w_1 \circ s"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & X \arrow[l, "\mathrm{id}"] \arrow[u, "u_1"] \arrow[r, "f'"'] & \hat{Y} \arrow[u, "u_2"'] & Y \arrow[l, "i"]
\end{tikzcd}$$

Noting that axiom CM2 implies $w_1 \circ s$ is a weak equivalence in $\mathcal{M}_{\mathrm{c}}$, we may then deduce that the two zigzags also represent the same morphism in $\operatorname{Ho}\mathcal{M}_{\mathrm{c}}$. Thus, the functor $\operatorname{Ho}\mathcal{M}_{\mathrm{c}} \to \operatorname{Ho}\mathcal{M}$ is indeed faithful. ■

4.2 Left and right homotopy

Prerequisites. §4.1.

Definition 4.2.1. Let X be an object in a model category $\mathcal{M}$.

- A **cylinder object** for X is a quadruple $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$, where $\mathrm{Cyl}(X)$ is an object in $\mathcal{M}$, $p : \mathrm{Cyl}(X) \to X$ is a weak equivalence, and i_0 and i_1 are sections of p such that the morphism $(\!(i_0, i_1)\!) : X + X \to \mathrm{Cyl}(X)$ is a cofibration.

- A **path object** for X is a quadruple $\big(\mathrm{Path}(X), i, p_0, p_1\big)$, where $\mathrm{Path}(X)$ is an object in $\mathcal{M}$, $i : X \to \mathrm{Path}(X)$ is a weak equivalence, and p_0 and p_1 are retractions of i such that the morphism $\langle p_0, p_1\rangle : \mathrm{Path}(X) \to X \times X$ is a fibration.

Remark 4.2.2. Let $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ be a cylinder object for X. By definition, $p \circ i_0 = p \circ i_1 = \mathrm{id}_X$, and p is a weak equivalence, so by the 2-out-of-3 property, i_0 and i_1 must also be weak equivalences $X \to \mathrm{Cyl}(X)$.

Dually, if $\big(\mathrm{Path}(X), i, p_0, p_1\big)$ is a path object for X, then p_0 and p_1 must be weak equivalences $\mathrm{Path}(X) \to X$.

Proposition 4.2.3. *Let X be an object in a model category $\mathcal{M}$.*

- *There exists a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, where the morphism $p : \mathrm{Cyl}(X) \to X$ is a trivial fibration.*

- *There exists a path object $\big(\mathrm{Path}(X), i, p_0, p_1\big)$ for X, where the morphism $i : X \to \mathrm{Path}(X)$ is a trivial cofibration.*

Proof. Use axioms CM1 and CM5. ∎

Definition 4.2.4. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$, let $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ be a cylinder object for X, and let $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ be a path object for Y.

- A **left homotopy** from f_0 to f_1 with respect to $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ is a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$.

- A **right homotopy** from f_0 to f_1 with respect to $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ is a morphism $H : X \to \mathrm{Path}(Y)$ such that $p_0 \circ H = f_0$ and $p_1 \circ H = f_1$.

- We say f_0 and f_1 are **left homotopic** if there exists a left homotopy from f_0 to f_1 with respect to some cylinder object for X.
- We say f_0 and f_1 are **right homotopic** if there exists a right homotopy from f_0 to f_1 with respect to some path object for Y.

Remark 4.2.5. If f_0 and f_1 are either left homotopic or right homotopic, then they must represent the same morphism in $\operatorname{Ho}\mathcal{M}$. For definiteness, let us write $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ for the localising functor, and suppose $H : \operatorname{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1. Since i_0 and i_1 are both sections of the weak equivalence $p : \operatorname{Cyl}(X) \to X$, we must have $\gamma i_0 = (\gamma p)^{-1} = \gamma i_1$; but $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$, so indeed $\gamma f_0 = \gamma f_1$. This is one of the reasons for calling $\operatorname{Ho}\mathcal{M}$ the *homotopy* category of $\mathcal{M}$.

However, it is not quite true that $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are either left homotopic or right homotopic; this only happens in special cases. In general, being left/right homotopic fails to even be an equivalence relation.

Definition 4.2.6. Let $f : X \to Y$ be a morphism in a model category $\mathcal{M}$.

- A **left homotopy left inverse** for f is a morphism $g : Y \to X$ in $\mathcal{M}$ such that $g \circ f$ and id_X are left homotopic.
- A **right homotopy right inverse** for f is a morphism $h : Y \to X$ in $\mathcal{M}$ such that $f \circ h$ and id_Y are right homotopic.
- A **right homotopy left inverse** for f is a morphism $g : Y \to X$ in $\mathcal{M}$ such that $g \circ f$ and id_X are right homotopic.
- A **left homotopy right inverse** for f is a morphism $h : Y \to X$ in $\mathcal{M}$ such that $f \circ h$ and id_Y are left homotopic.

A **homotopy equivalence** in $\mathcal{M}$ is a pair (f, g) such that g (resp. f) is both a left homotopy left inverse and a right homotopy right inverse for f (resp. g). Two morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{M}$ are **mutual homotopy inverses** when (f, g) constitute a homotopy equivalence in $\mathcal{M}$.

Remark 4.2.7. Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a model category.

- g is a left homotopy left inverse for f if and only if f is a left homotopy right inverse for g.

- g is a right homotopy left inverse for f if and only if f is a right homotopy left inverse for g.

However, note that the dual of 'left homotopy left inverse' is 'right homotopy right inverse', and the dual of 'right homotopy left inverse' is 'left homotopy right inverse'!

Lemma 4.2.8. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category, and suppose f_0 and f_1 are either left or right homotopic. Then, f_0 is a weak equivalence if and only if f_1 is a weak equivalence.*

Proof. Assume f_0 and f_1 are left homotopic; the other case is formally dual. So, there exist a cylinder object $\left(\mathrm{Cyl}(X), i_0, i_1, p\right)$ for X and a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Suppose f_0 is a weak equivalence. By remark 4.2.2, i_0 is a weak equivalence, so the 2-out-of-3 property implies H is also a weak equivalence; but i_1 is a weak equivalence as well, so f_1 must be a weak equivalence too. A symmetrical argument proves that f_0 is a weak equivalence if f_1 is. ■

Lemma 4.2.9. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a model category $\mathcal{M}$.*

(i) *If $g \circ f$ is either left or right homotopic to id_X, and $f \circ g$ is either left or right homotopic to id_Y, then (f, g) is an equivalence in $\mathcal{M}$ (in the sense of definition 3.1.17).*

(ii) *If there exist morphisms $g, h : Y \to X$ such that $g \circ f$ is either left or right homotopic to id_X and $f \circ h$ is either left or right homotopic to id_Y, then (the image of) f is an isomorphism in $\mathrm{Ho}\,\mathcal{M}$.*

Proof. Obvious, given remark 4.2.5. ◆

Lemma 4.2.10. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *Given any cylinder object $\left(\mathrm{Cyl}(X), i_0, i_1, p\right)$ for X, $f_0 \circ p : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to itself.*

(ii) *If $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1 with respect to a cylinder object $\left(\mathrm{Cyl}(X), i_0, i_1, p\right)$ for X, then the same H is a left homotopy from f_1 to f_0 for the cylinder object $\left(\mathrm{Cyl}(X), i_1, i_0, p\right)$.*

Dually:

(i′) *Given any path object* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *for* Y, $i \circ f_0 : X \to \mathrm{Path}(Y)$ *is a right homotopy from* f_0 *to itself.*

(ii′) *If* $H : X \to \mathrm{Path}(Y)$ *is a right homotopy from* f_0 *to* f_1 *with respect to a path object* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *for* Y, *then the same* H *is a right homotopy from* f_1 *to* f_0 *for the path object* $\big(\mathrm{Path}(Y), i, p_1, p_0\big)$.

Proof. Obvious. ◆

Lemma 4.2.11. *Let* $\mathcal{M}$ *be a model category.*

- *If* $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ *is a cylinder object for a cofibrant object in* $\mathcal{M}$, *then the insertions* $i_0, i_1 : X \to \mathrm{Cyl}(X)$ *are trivial cofibrations, and* $\mathrm{Cyl}(X)$ *is a cofibrant object in* $\mathcal{M}$.

- *If* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *is a path object for a fibrant object in* $\mathcal{M}$, *then the projections* $p_0, p_1 : Y \to \mathrm{Path}(Y)$ *are trivial fibrations, and* $\mathrm{Path}(X)$ *is a fibrant object in* $\mathcal{M}$.

Proof. See Lemmas 1.5 and 1.7 in [GJ], or Lemma 7.3.6 in [Hirschhorn, 2003]. □

Lemma 4.2.12. *Let* $\mathcal{M}$ *be model category.*

- *Let* X *be a cofibrant object in* $\mathcal{M}$. *Given two cylinder objects for* X, *say* $\big(\mathrm{Cyl}(X)', i_0', i_1', p'\big)$ *and* $\big(\mathrm{Cyl}(X)'', i_0'', i_1'', p''\big)$, *there exists a third cylinder object* $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ *such that the diagram below commutes,*

$$
\begin{tikzcd}
X \arrow[dr, "i_0'"] \arrow[ddrr, bend right, "i_0"'] & & X \arrow[dl, "i_1'"'] \arrow[dr, "i_0''"] & & X \arrow[dl, "i_1''"'] \arrow[ddll, bend left, "i_1"] \\
& \mathrm{Cyl}(X)' \arrow[dr] \arrow[ddr, "p'"'] & & \mathrm{Cyl}(X)'' \arrow[dl] \arrow[ddl, "p''"] & \\
& & \mathrm{Cyl}(X) \arrow[d, "p"] & & \\
& & X & &
\end{tikzcd}
$$

and the diamond is a pushout diagram.

- *If Y is a fibrant object in $\mathcal{M}$, and we have two path objects for Y, say $\big(\mathrm{Path}(Y)', i', p_0', p_1'\big)$ and $\big(\mathrm{Path}(Y)'', i'', p_0'', p_1''\big)$, then there exists a third path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ such that the diagram below commutes,*

$$
\begin{tikzcd}
 & & Y \arrow[ddl, bend right, "i'"'] \arrow[d, "i"'] \arrow[ddr, bend left, "i''"] & & \\
 & & \mathrm{Path}(Y) \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "p_0"'] \arrow[ddrr, bend left, "p_1"] & & \\
 & \mathrm{Path}(Y)' \arrow[dl, "p_0'"] \arrow[dr, "p_1'"] & & \mathrm{Path}(Y)'' \arrow[dl, "p_0''"'] \arrow[dr, "p_1''"'] & \\
Y & & Y & & Y
\end{tikzcd}
$$

and the diamond is a pullback diagram.

Proof. See Lemmas 1.5 and 1.7 in [GJ, Ch. II], or Lemma 7.4.2 in [Hirschhorn, 2003]. □

Corollary 4.2.13. *Let $f_0, f_1, f_2 : X \to Y$ be three parallel morphisms in a model category $\mathcal{M}$.*

- *Assuming X is cofibrant, if f_0 and f_1 are left homotopic, and f_1 and f_2 are left homotopic, then f_0 and f_2 are also left homotopic.*

- *Assuming Y is fibrant, if f_0 and f_1 are right homotopic, and f_1 and f_2 are right homotopic, then f_0 and f_2 are also right homotopic.* ■

Lemma 4.2.14. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

- *If X is cofibrant, and f_0 and f_1 are left homotopic, given any path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, there is a right homotopy $H : X \to \mathrm{Path}(Y)$ from f_0 to f_1.*

- *If Y is fibrant, and f_0 and f_1 are right homotopic, given any cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, there is a left homotopy $H : \mathrm{Cyl}(X) \to Y$ from f_0 to f_1.*

Proof. See Proposition 1.8 in [GJ, Ch. II], or Proposition 7.4.7 in [Hirschhorn, 2003]. □

Proposition 4.2.15. *Let X and Y be objects in a model category $\mathcal{M}$.*

(i) *If X is cofibrant, then being left homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(ii) *If Y is fibrant, then being right homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(iii) *If X is cofibrant and Y is fibrant, then these two equivalence relations on $\mathcal{M}(X, Y)$ coincide.*

Proof. Use the preceding lemmas. ■

Lemma 4.2.16. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

- *If f_0 and f_1 are right homotopic and $g : W \to X$ is any morphism in $\mathcal{M}$, then $f_0 \circ g$ and $f_1 \circ g$ are also right homotopic.*
- *If f_0 and f_1 are left homotopic and $g : Y \to Z$ is any morphism in $\mathcal{M}$, then $g \circ f_0$ and $g \circ f_1$ are also left homotopic.*

Proof. Obvious. ◆

Corollary 4.2.17. *Let $\mathcal{M}$ be a model category, and let $\mathcal{M}_{\mathrm{cf}}$ be the full subcategory spanned by the cofibrant–fibrant objects. Then the equivalence relation induced by homotopy is a congruence on $\mathcal{M}_{\mathrm{cf}}$; in particular, there exist a locally small category $\mathcal{M}_{\mathrm{h}}$ and a full functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ with these properties:*

- *The objects of $\mathcal{M}_{\mathrm{h}}$ are those of $\mathcal{M}_{\mathrm{cf}}$.*
- *The hom-set $\mathcal{M}_{\mathrm{h}}(X, Y)$ is $\mathcal{M}(X, Y)$ modulo homotopy.*
- *The functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}_{\mathrm{h}}$ sends each morphism in $\mathcal{M}_{\mathrm{cf}}$ to its homotopy class.* ■

The next result is a version of Whitehead's theorem; however, this is a purely formal consequence of the model category axioms and has no real content, unlike the original theorem.

Proposition 4.2.18. *Let X and Y be cofibrant–fibrant objects in a model category $\mathcal{M}$. If $f : X \to Y$ is a weak equivalence, then f has a homotopy inverse in $\mathcal{M}$.*

Proof. See Theorem 1.10 in [GJ, Ch. II], or Theorem 7.5.10 in [Hirschhorn, 2003]. □

Lemma 4.2.19. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

- *If $g : W \to X$ is a morphism with a right homotopy right inverse in $\mathcal{M}$, then $f_0 \circ g$ and $f_1 \circ g$ are right homotopic if and only if f_0 and f_1 are right homotopic.*
- *If $g : Y \to Z$ is a morphism with a left homotopy left inverse in $\mathcal{M}$, then $g \circ f_0$ and $g \circ f_1$ are left homotopic if and only if f_0 and f_1 are left homotopic.*

Proof. This follows immediately from the definitions and lemma 4.2.16. ■

Corollary 4.2.20. *Let W, X, Y, Z be cofibrant–fibrant objects in a model category $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms.*

- *If $g : W \to X$ is a weak equivalence such that $f_0 \circ g$ and $f_1 \circ g$ are homotopic, then f_0 and f_1 are homotopic.*
- *If $g : Y \to Z$ is a weak equivalence such that $g \circ f_0$ and $g \circ f_1$ are homotopic, then f_0 and f_1 are homotopic.*

Proof. Apply proposition 4.2.18 in conjunction with the above lemma. ■

4.3 Quillen functors

Prerequisites. §§ 3.1, 3.3, 3.4, 4.1, A.5.

Definition 4.3.1.

- A **left Quillen functor** is a functor between derivable categories that has a right adjoint and preserves cofibrations and trivial cofibrations.
- A **right Quillen functor** is a functor between derivable categories that has a left adjoint and preserves fibrations and trivial fibrations.
- A **Quillen adjunction** is an adjunction

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

  where F is a left Quillen functor and G is a right Quillen functor.

- A **Quillen equivalence** is a Quillen adjunction as above satisfying this additional condition:
  - Given a cofibrant object A in $\mathcal{N}$ and fibrant object Y in $\mathcal{M}$, a morphism $FA \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if its right adjoint transpose $A \to GY$ is a weak equivalence in $\mathcal{N}$.

Proposition 4.3.2. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be an adjunction between categories with model structures. The following are equivalent:*

(i) *F preserves cofibrations and trivial cofibrations.*

(ii) *G preserves fibrations and trivial fibrations.*

(iii) *F preserves cofibrations and G preserves fibrations.*

(iv) *F preserves trivial cofibrations and G preserves trivial fibrations.*

(v) *(Assuming $\mathcal{M}$ and $\mathcal{N}$ are derivable categories.) $F \dashv G$ is a Quillen adjunction.*

Proof. Use proposition A.3.26. ■

Remark 4.3.3. A functor between categories with model structures that preserves both trivial cofibrations and trivial fibrations must also preserve weak equivalences, since axioms CM2 and CM5 together imply that a morphism is a weak equivalence if and only if it is of the form $p \circ i$ where i is a trivial cofibration and p is a trivial fibration. In particular, a functor that is both left and right Quillen must be homotopical.

Proposition 4.3.4. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

- *F sends cofibrant objects in $\mathcal{N}$ to cofibrant objects in $\mathcal{M}$.*
- *G sends fibrant objects in $\mathcal{M}$ to fibrant objects in $\mathcal{N}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let B be a cofibrant object in $\mathcal{N}$ and let $p : X \to Y$ be a trivial fibration in $\mathcal{M}$. Since $F \dashv G$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{M}(FB, X) \arrow[r, "\cong"] \arrow[d, "\mathcal{M}(FB,p)"'] & \mathcal{N}(B, GX) \arrow[d, "\mathcal{M}(B,Gp)"] \\ \mathcal{M}(FB, Y) \arrow[r, "\cong"'] & \mathcal{N}(B, GY) \end{tikzcd}$$

By hypothesis, $Gp : GX \to GY$ is a trivial fibration in $\mathcal{N}$, so the hom-set map $\mathcal{N}(B, Gp)$ is a surjection. It follows that $\mathcal{M}(FB, p)$ is also a surjection, and thus FB is a cofibrant object in $\mathcal{M}$. ■

Proposition 4.3.5.

(i) *The composite of two Quillen adjunctions is also a Quillen adjunction.*

(ii) *The composite of two Quillen equivalences is also a Quillen equivalence.*

Proof. Obvious. ◆

Lemma 4.3.6 (Ken Brown's lemma)**.** *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a category with weak equivalences.*

- *Let $\mathcal{M}_c$ be the full subcategory of cofibrant objects in $\mathcal{M}$. If $F : \mathcal{M}_c \to \mathcal{C}$ sends trivial cofibrations in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$, then F also sends weak equivalences in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$.*

- *Let $\mathcal{M}_f$ be the full subcategory of fibrant objects in $\mathcal{M}$. If $F : \mathcal{M}_f \to \mathcal{C}$ sends trivial fibrations in $\mathcal{M}_f$ to weak equivalences in $\mathcal{C}$, then F also sends weak equivalences in $\mathcal{M}_f$ to weak equivalences in $\mathcal{C}$.*

Proof. See Lemma 9.9 in [DS], Lemma 7.7.1 in [Hirschhorn, 2003], or Lemma 14.5 in [DHKS]. □

The usual proof of the Ken Brown's lemma uses binary coproducts (or binary products, as the case may be), so it cannot be used in the case where the domain is merely a derivable category. Nonetheless, we have already proved something very similar, namely lemma 4.1.33.

Proposition 4.3.7 (Dugger)**.** *Let $F \dashv G$ be an adjunction between DHK model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(iii) *G preserves fibrations between fibrant objects and all trivial fibrations.*

Proof. See Proposition 8.5.4 in [Hirschhorn, 2003], or Corollary A.2 in [Dugger, 2001b]. □

Proposition 4.3.8. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be an adjunction between derivable categories and assume that both the adjunction unit $\eta : \mathrm{id}_{\mathcal{N}} \Rightarrow GF$ and the adjunction counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{M}}$ are natural weak equivalences. If the functor $G : \mathcal{M} \to \mathcal{N}$ preserves and reflects weak equivalences, then:*

(i) *A morphism $FA \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if its right adjoint transpose $A \to GY$ is a weak equivalence in $\mathcal{N}$.*

(ii) *$F : \mathcal{N} \to \mathcal{M}$ preserves weak equivalences.*

(iii) *If $F : \mathcal{N} \to \mathcal{M}$ preserves cofibrations, then the adjunction is a Quillen adjunction.*

Dually, if the functor $F : \mathcal{N} \to \mathcal{M}$ preserves and reflects weak equivalences, then:

(i′) *A morphism $FA \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if its right adjoint transpose $A \to GY$ is a weak equivalence in $\mathcal{N}$.*

(ii′) *$G : \mathcal{M} \to \mathcal{N}$ preserves weak equivalences.*

(iii′) *If $G : \mathcal{M} \to \mathcal{N}$ preserves fibrations, then the adjunction is a Quillen adjunction.*

Proof. (i). Let Y be an object in $\mathcal{M}$, let A be an object in $\mathcal{N}$, let $f : FA \to Y$ be a morphism in $\mathcal{M}$, and let $g : A \to GY$ be its right adjoint transpose. First, suppose $f : FA \to Y$ is a weak equivalence in $\mathcal{M}$. Then $Gf : GFA \to GY$ and $g = Gf \circ \eta_A : A \to GY$ are weak equivalences in $\mathcal{N}$.

Conversely, suppose $g : A \to GY$ is a weak equivalence in $\mathcal{N}$. Then so are $GFg : GFA \to GFGY$ and $Gf = G\varepsilon_Y \circ GFg : GFA \to GY$. But $G : \mathcal{M} \to \mathcal{N}$ reflects weak equivalences, so $f : FA \to Y$ is a weak equivalence in $\mathcal{M}$.

(ii). Axiom CM2 implies that $GF : \mathcal{N} \to \mathcal{N}$ preserves weak equivalences, and $G : \mathcal{M} \to \mathcal{N}$ reflects weak equivalences by hypothesis, so $F : \mathcal{N} \to \mathcal{M}$ must preserve weak equivalences.

(iii). It now follows that $F : \mathcal{N} \to \mathcal{M}$ preserves trivial cofibrations if it preserves cofibrations. We may then apply proposition 4.3.2 to complete the proof. ■

Definition 4.3.9. Let $\mathcal{M}$ be a derivable category.

- A **left Quillen deformation retract** (resp. **functorial left Quillen deformation retract**) of $\mathcal{M}$ is a left deformation retract of $\mathcal{M}$ of the form $(\mathcal{M}_c, Q, p)$ where $\mathcal{M}_c$ is the full subcategory of cofibrant objects in $\mathcal{M}$.

- A **right Quillen deformation retract** (resp. **functorial right Quillen deformation retract**) of $\mathcal{M}$ is a right deformation retract of $\mathcal{M}$ of the form $(\mathcal{M}_f, R, i)$ where $\mathcal{M}_f$ is the full subcategory of fibrant objects in $\mathcal{M}$.

Lemma 4.3.10. *Let $\mathcal{M}$ be a derivable category.*

- *Left Quillen deformation retracts of $\mathcal{M}$ exist.*

- *Right Quillen deformation retracts of $\mathcal{M}$ exist.*

Proof. The two claims are formally dual; we will prove the first version.

For each object X in $\mathcal{M}$, choose a fibrant cofibrant replacement (QX, p_X); such exist by proposition 4.1.24. Then, for each morphism $f : X \to Y$ in $\mathcal{M}$, there exists a morphism $Qf : QX \to QY$ making the diagram commute,

$$\begin{tikzcd} QX \arrow[r, "p_X"] \arrow[d, dashed, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

because $p_Y : QY \to Y$ is a trivial fibration and QX is cofibrant; note that axiom CM2 implies Qf is a weak equivalence if (and only if!) f is. Thus, axioms DR1–2 are satisfied. For axiom DR3, we refer to proposition 4.1.35. Finally, we simply need to observe that axiom DR4 is trivial. ■

Lemma 4.3.11. *Let $\mathcal{M}$ be a derivable category.*

- *$(\mathcal{M}_c, Q, p)$ is a functorial left Quillen deformation for $\mathcal{M}$ if and only if (Q, p) is a cofibrant replacement functor for $\mathcal{M}$.*

- *$(\mathcal{M}_f, R, i)$ is a functorial left Quillen deformation for $\mathcal{M}$ if and only if (R, i) is a cofibrant replacement functor for $\mathcal{M}$.*

Proof. Obvious. ♦

Theorem 4.3.12. *Let $\mathcal{M}$ be a derivable category, let $\mathcal{C}$ be a relative category, and let $\gamma_{\mathcal{M}} : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ and $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ be the respective localising functors. Suppose* $\operatorname{weq}\mathcal{C}$ *has the 2-out-of-3 property and the special 2-out-of-4 property. If $F : \mathcal{M} \to \mathcal{C}$ is a functor that sends trivial cofibrations in $\mathcal{M}$ to weak equivalences in $\mathcal{C}$, then:*

(i) *Any left Quillen deformation retract of $\mathcal{M}$ is a left deformation retract for F; in particular, a total left derived functor for F exists.*

(ii) *If $\mathcal{M}$ has a cofibrant replacement functor, then F is functorially left deformable and has a homotopical left approximation.*

(iii) *If $(\mathbf{L}F, \alpha)$ is any total left derived functor for F, then the extension counit component $\alpha_X : (\mathbf{L}F)\gamma_{\mathcal{M}}X \to \gamma_{\mathcal{C}}FX$ is an isomorphism for all cofibrant objects X in $\mathcal{M}$.*

Dually, if $F : \mathcal{M} \to \mathcal{C}$ is a functor that sends trivial fibrations in $\mathcal{M}$ to weak equivalences in $\mathcal{C}$, then:

(i′) *Any right Quillen deformation retract of $\mathcal{M}$ is a right deformation retract for F; in particular, a total right derived functor for F exists.*

(ii′) *If $\mathcal{M}$ has a fibrant replacement functor, then F is functorially right deformable and has a homotopical right approximation.*

(iii′) *If $(\mathbf{R}F, \beta)$ is any total right derived functor for F, then the extension counit component $\beta_X : (\mathbf{R}G)\gamma_{\mathcal{M}}X \to \gamma_{\mathcal{C}}FX$ is an isomorphism for all fibrant objects X in $\mathcal{M}$.*

Proof. (i). Let $(\mathcal{M}_c, Q, p)$ be a left Quillen deformation retract of $\mathcal{M}$. Then F sends weak equivalences in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$ by lemma 4.1.33, so $(\mathcal{M}_c, Q, p)$ is indeed a left deformation retract for $\mathcal{C}$. We may then apply theorem 3.3.17 to obtain a total left derived functor.

(ii). By the same argument, if (Q, p) is a cofibrant replacement functor for $\mathcal{M}$, then $(\mathcal{M}_c, Q, p)$ is a functorial left deformation retract for F. We then appeal to theorem 3.4.11.

(iii). The extension counit has the required property because, for all cofibrant objects X in $\mathcal{M}$, the morphism $F p_X : FQX \to FX$ is a weak equivalence in $\mathcal{C}$; but this is precisely the component of the extension counit at X. ■

Theorem 4.3.13. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

(i) *Any left Quillen deformation retract of $\mathcal{N}$ is a left deformation retract for F; dually, any right Quillen deformation retract of $\mathcal{M}$ is a right deformation retract for G.*

(ii) *$F \dashv G$ is a deformable adjunction; in particular, a derived adjunction exists.*

(iii) *If $F \dashv G$ is a Quillen equivalence, then the derived adjunction*

$$\mathbf{L}F \dashv \mathbf{R}G : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$$

is an adjoint equivalence of categories; and if $\mathcal{M}$ and $\mathcal{N}$ are saturated derivable categories, then the converse is true.

Proof. (i). Since weak equivalences in derivable categories are closed under retracts (by axiom CM3), we may use theorem 4.3.12.

(ii). That $F \dashv G$ is a derivable adjunction follows immediately; then apply theorem 3.3.24 for the existence of the derived adjunction.

(iii). This is a special case of proposition 3.3.28. ■

Proposition 4.3.14. *Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be derivable categories.*

- *If $F : \mathcal{N} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{L}$ are left Quillen functors, then the composite $(\mathbf{L}G)(\mathbf{L}F)$ is (the functor part of) a total left derived functor for GF.*

- *If $F : \mathcal{N} \to \mathcal{P}$ and $G : \mathcal{M} \to \mathcal{N}$ are right Quillen functors, then the composite $(\mathbf{R}F)(\mathbf{R}G)$ is (the functor part of) a total right derived functor for FG.*

Assuming $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{L}$ have fibrant and cofibrant replacement functors:

- *If $F : \mathcal{N} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{L}$ are left Quillen functors, then the composite $(\mathbb{L}G)(\mathbb{L}F)$ is (the functor part of) a homotopical left approximation for GF.*

- *If $F : \mathcal{N} \to \mathcal{P}$ and $G : \mathcal{M} \to \mathcal{N}$ are right Quillen functors, then the composite $(\mathbb{R}F)(\mathbb{R}G)$ is (the functor part of) a homotopical right approximation for FG.*

Proof. Use theorems 3.3.17, 3.4.11, and 4.3.12 with proposition 4.3.5. ■

Definition 4.3.15. Let $\mathbb{A}$ be a small category and let $\mathcal{M}$ be a category equipped with a model structure.

- The **injective model structure** on the functor category $[\mathbb{A}, \mathcal{M}]$ is a model structure such that a morphism in $[\mathbb{A}, \mathcal{M}]$ is a cofibration (resp. weak equivalence) if and only if all its components are cofibrations (resp. weak equivalences) in $\mathcal{M}$.

- The **projective model structure** on the functor category $[\mathbb{A}, \mathcal{M}]$ is a model structure such that a morphism in $[\mathbb{A}, \mathcal{M}]$ is a fibration (resp. weak equivalence) if and only if all its components are fibrations (resp. weak equivalences) in $\mathcal{M}$.

Remark 4.3.16. The injective (resp. projective) model structure on $[\mathbb{A}, \mathcal{M}]$ is unique *if it exists*, by theorem 4.1.12.

Proposition 4.3.17. *Let $\mathcal{M}$ be a derivable category, let $\mathbb{A}$ be a small category, and let $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ be the functor that sends an object X in $\mathcal{M}$ to the constant functor $\Delta X : \mathbb{A} \to \mathcal{M}$ with value X.*

- *If $\mathcal{M}$ has colimits for diagrams of shape $\mathbb{A}$, then $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a right Quillen functor with respect to the projective model structure on $[\mathbb{A}, \mathcal{M}]$ when it exists.*

- *If $\mathcal{M}$ has limits for diagrams of shape $\mathbb{A}$, then $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a left Quillen functor with respect to the injective model structure on $[\mathbb{A}, \mathcal{M}]$ when it exists.*

Proof. Δ certainly preserves fibrations (resp. cofibrations) and weak equivalences with respect to the projective (resp. injective) model structure, so by proposition 4.3.2, $\varinjlim_{\mathbb{A}} \dashv \Delta$ (resp. $\Delta \dashv \varprojlim_{\mathbb{A}}$) is a Quillen adjunction.[3] ■

[3] Recall proposition 0.1.12.

Proposition 4.3.18. *Let $\mathcal{M}$ be a category and let I be a set.*

(i) *The functor category $[I, \mathcal{M}]$ admits a model structure that is simultaneously an injective model structure and a projective model structure.*

(ii) *If $\mathcal{M}$ is a derivable category (resp. saturated derivable category, model category), then $[I, \mathcal{M}]$ equipped with the above model structure is a derivable category (resp. saturated derivable category, model category).*

(iii) *If $\mathcal{M}$ is a derivable category and has products and coproducts for families of objects indexed by I, then $\Delta : \mathcal{M} \to [I, \mathcal{M}]$ is both a left Quillen functor and a right Quilen functor.*

(iv) *If $\mathcal{M}$ is a model category, then the canonical exponential comparison functor* $\operatorname{Ho}[I, \mathcal{M}] \to [I, \operatorname{Ho}\mathcal{M}]$ *is an isomorphism of categories.*

Proof. (i). If we declare the cofibrations (resp. weak equivalences, fibrations) in $[I, \mathcal{M}]$ to be precisely the morphisms that are cofibrations (resp. weak equivalences, fibrations) componentwise, then the axioms CM2–5 may be verified componentwise as well.

(ii). Axioms DC0, DC1, and CM1 can be verified componentwise. If $\mathcal{M}$ is saturated, then we can use lemma 3.1.11 to deduce that $[I, \mathcal{M}]$ is also saturated.

(iii). Apply proposition 4.3.17.

(iv). Use theorem 4.4.1 and the fact that the congruence of homotopy is componentwise in $[I, \mathcal{M}]$. ■

Corollary 4.3.19. *Let $\mathcal{M}$ be a saturated derivable category and let I be a set.*

- *If $\mathcal{M}$ has products for families of objects indexed by I, then the product of an I-indexed family of weak equivalences between fibrant objects is also a weak equivalence between fibrant objects.*

- *If $\mathcal{M}$ has coproducts for families of objects indexed by I, then the coproduct of an I-indexed family of weak equivalences between cofibrant objects is also a weak equivalence between cofibrant objects.*

Proof. Apply lemma 4.1.33 to the previous proposition. ■

Proposition 4.3.20. *Let $\mathcal{M}$ be a derivable category and let $\mathbb{A}$ be a small category.*

- *If $\mathcal{M}$ has coproducts for families of size $\leq |\mathrm{mor}\,\mathbb{A}|$, then the evaluation functors $[\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ are right Quillen functors with respect to the injective model structure on $[\mathbb{A}, \mathcal{M}]$ (if it exists).*

- *If $\mathcal{M}$ has products for families of size $\leq |\mathrm{mor}\,\mathbb{A}|$, then the evaluation functors $[\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ are left Quillen functors with respect to the projective model structure on $[\mathbb{A}, \mathcal{M}]$ (if it exists).*

Proof. The two claims are formally dual; we will prove the first version.

Let A be an object in $\mathbb{A}$ and let $A^* : [\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ be the functor $F \mapsto FA$. It is not hard to check that A^* has a left adjoint $A_! : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$, namely the functor $X \mapsto \mathbb{A}(A, -) \odot X$. Since the class of cofibrations and the class of trivial cofibrations are both closed under coproducts, we see that $A_! : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a left Quillen functor with respect to the injective model structure. Thus, by proposition 4.3.2, $A^* : [\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ is a right Quillen functor. ■

Corollary 4.3.21. *Let $\mathcal{M}$ be a derivable category and let $\mathbb{A}$ be a small category. Suppose the injective and projective model structures on $[\mathbb{A}, \mathcal{M}]$ both exist. If $\mathcal{M}$ has both coproducts and products for families of size $\leq |\mathrm{mor}\,\mathbb{A}|$, then:*

- *Every fibration (resp. trivial fibration) in the injective model structure on $[\mathbb{A}, \mathcal{M}]$ is a fibration (resp. trivial fibration) in the projective model structure.*

- *Every cofibration (resp. trivial cofibration) in the projective model structure on $[\mathbb{A}, \mathcal{M}]$ is a cofibration (resp. trivial cofibration) in the injective model structure.*

- *The trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : [\mathbb{A}, \mathcal{M}] \to [\mathbb{A}, \mathcal{M}]$$

  is a Quillen equivalence between the injective and projective model structures. ■

4.4 The homotopy category

Prerequisites. §§4.1, 4.2, 4.3, A.4, A.5, A.6.

Theorem 4.4.1. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the localising functor.*

(i) $\operatorname{Ho}\mathcal{M}$ *is equivalent to the locally small category $\mathcal{M}_{\mathrm{h}}$ defined in corollary 4.2.17, and $\mathcal{M}$ is a saturated homotopical category.*

(ii) *If X and Y are cofibrant–fibrant objects in $\mathcal{M}$, then the hom-set map $\mathcal{M}(X, Y) \to \operatorname{Ho}\mathcal{M}(X, Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

Proof. (i). This is Theorem 1.11 in [GJ, Ch. II], or Proposition 5.8 in [DS].

(ii). Implied by claim (i). □

Corollary 4.4.2. *Let $f : X \to Y$ be a morphism in a model category $\mathcal{M}$. If f has a quasi-inverse in $\mathcal{M}$ (in the sense of definition 3.1.17), then f is a weak equivalence in $\mathcal{M}$.*

Proof. If f has a quasi-inverse in $\mathcal{M}$, then (the image of) f is an isomorphism in $\operatorname{Ho}\mathcal{M}$; but $\mathcal{M}$ is a saturated homotopical category, so f must be a weak equivalence in $\mathcal{M}$. ■

Corollary 4.4.3. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the localising functor.*

(i) *For any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, if X is cofibrant and Y is fibrant, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

(ii) *The full subcategory $\mathcal{M}_{\mathrm{cf}}$ of cofibrant–fibrant objects in $\mathcal{M}$ has the Whitehead property (in the sense of definition 3.1.21).*

Proof. (i). As noted in remark 4.2.5, if $f_0, f_1 : X \to Y$ are homotopic, then we must have $\gamma f_0 = \gamma f_1$. Conversely, suppose $\gamma f_0 = \gamma f_1$ with X cofibrant and Y fibrant. Let (RX, i') be a cofibrant fibrant replacement for X and (QY, p') be a fibrant cofibrant replacement for Y. Then, there exists morphisms $f_0', f_1' : RX \to QY$ such that $f_0 = p' \circ f_0' \circ i'$ and $f_1 = p' \circ f_1' \circ i'$. Since $i' : X \to RX$

and $p' : QY \to Y$ are weak equivalences, we must have $\gamma f_0' = \gamma f_1'$ in $\operatorname{Ho}\mathcal{M}$. The theorem then implies f_0' and f_1' are homotopic; thus f_0 and f_1 are also homotopic, by lemmas 4.2.14 and 4.2.16.

(ii). Apply theorem 3.1.22 in conjunction with lemma 4.2.9 and the above corollary. ∎

Corollary 4.4.4. *Let $f : X \to Y$ be a morphism between two cofibrant objects in a derivable category $\mathcal{M}$. If $\mathcal{M}$ is a saturated homotopical category, then the following are equivalent:*

(i) *The morphism $f : X \to Y$ is a weak equivalence in $\mathcal{M}$.*

(ii) *The hom-set map map $\operatorname{Ho}\mathcal{M}(f, Z) : \operatorname{Ho}\mathcal{M}(Y, Z) \to \operatorname{Ho}\mathcal{M}(X, Z)$ is a bijection for all cofibrant–fibrant objects Z in $\mathcal{M}$.*

(iii) *The hom-set map $\mathcal{M}_{\mathrm{h}}(f, Z) : \mathcal{M}_{\mathrm{h}}(Y, Z) \to \mathcal{M}_{\mathrm{h}}(X, Z)$ is a bijection for all cofibrant–fibrant objects Z in $\mathcal{M}$, where $\mathcal{M}_{\mathrm{h}}(Y, Z)$ (resp. $\mathcal{M}_{\mathrm{h}}(X, Z)$) denotes the set of all morphisms $Y \to Z$ (resp. $X \to Z$) in $\mathcal{M}$* modulo homotopy.

Proof. (i) ⇒ (ii). Every weak equivalence in $\mathcal{M}$ becomes an isomorphism in $\operatorname{Ho}\mathcal{M}$, so in particular $\operatorname{Ho}\mathcal{M}(f, Z) : \operatorname{Ho}\mathcal{M}(Y, Z) \to \operatorname{Ho}\mathcal{M}(X, Z)$ must be a bijection.

(ii) ⇔ (iii). Corollary 4.4.3 implies that the horizontal arrows in the following commutative diagram are bijections,

$$\begin{tikzcd} \mathcal{M}_{\mathrm{h}}(Y, Z) \arrow[r] \arrow[d, "\mathcal{M}'(f,Z)"'] & \operatorname{Ho}\mathcal{M}(Y, Z) \arrow[d, "\operatorname{Ho}\mathcal{M}(f,Z)"] \\ \mathcal{M}_{\mathrm{h}}(X, Z) \arrow[r] & \operatorname{Ho}\mathcal{M}(X, Z) \end{tikzcd}$$

and so $\mathcal{M}_{\mathrm{h}}(f, Z)$ is a bijection if and only if $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection.

(ii) ⇒ (i). Suppose $\left(\hat{X}, i_X\right)$ is a cofibrant fibrant replacement for X and $\left(\hat{Y}, i_Y\right)$ is a cofibrant fibrant replacement for Y. Then, (by axiom CM4) there exists a morphism $\hat{f} : \hat{X} \to \hat{Y}$ making the diagram below commute,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Y} \end{tikzcd}$$

and by the 2-out-of-3 property, f is a weak equivalence if and only if $\hat{f}$ is a weak equivalence. On the other hand, the following diagram also commutes,

$$\begin{tikzcd}
\operatorname{Ho}\mathcal{M}(\hat{Y}, Z) \arrow[r, "\operatorname{Ho}\mathcal{M}(\hat{f},Z)"] \arrow[d, "\operatorname{Ho}\mathcal{M}(i_Y,Z)"'] & \operatorname{Ho}\mathcal{M}(\hat{X}, Z) \arrow[d, "\operatorname{Ho}\mathcal{M}(i_X,Z)"] \\
\operatorname{Ho}\mathcal{M}(Y, Z) \arrow[r, "\operatorname{Ho}\mathcal{M}(f,Z)"'] & \operatorname{Ho}\mathcal{M}(X, Z)
\end{tikzcd}$$

and so $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection if and only if $\operatorname{Ho}\mathcal{M}(\hat{f}, Z)$ is a bijection; but $\hat{X}$ and $\hat{Y}$ are both cofibrant–fibrant objects, so if $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection for all cofibrant–fibrant objects Z, then $\hat{f}$ must be a weak equivalence (because $\mathcal{M}$ is a saturated homotopical category). ■

Proposition 4.4.5. *Let $\mathcal{C}$ be a locally small relative category, let $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ let X and Y be objects in $\mathcal{C}$, and let $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to$ **Set** be defined by the following formula:*

$$H(C', C) = \operatorname{Ho}\mathcal{C}(\gamma C', \gamma Y) \times \operatorname{Ho}\mathcal{C}(\gamma X, \gamma C)$$

Then the evident maps $H(C, C) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ defined by composition exhibit $\operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ *as the coend of H.*

Proof. By proposition A.6.6, the coend $\int^{C:\mathcal{C}} \operatorname{Ho}\mathcal{C}(\gamma C, \gamma Y) \times \operatorname{Ho}\mathcal{C}(\gamma X, \gamma C)$ can be identified with the ensemble of connected components of the following category:

- The objects are pairs (g, C, f), where C is an object in $\mathcal{C}$ and $f : \gamma X \to \gamma C$ and $g : \gamma C \to \gamma Y$ are morphisms in $\operatorname{Ho}\mathcal{C}$.
- The morphisms $(g', C', f') \to (g, C, f)$ are morphisms $h : C' \to C$ in $\mathcal{C}$ such that $\gamma h \circ f' = f$ and $g \circ \gamma h = g'$.
- Identities and composition are inherited from $\mathcal{C}$.

Given any object (g, C, f) in this category, it is not hard to see that there is a zigzag of morphisms connecting (g, C, f) to $(\mathrm{id}_Y, Y, g \circ f)$. Thus, the canonical comparison map $\int^{C:\mathcal{C}} H(C, C) \to \operatorname{Ho}\mathcal{C}(X, Y)$ is a bijection, as claimed. ■

Corollary 4.4.6. *Let $\mathcal{C}$ be a locally small relative category. Then the hom-functor*

$$\mathcal{C}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

admits a pointwise left Kan extension along the localising functor $\gamma^{\mathrm{op}} \times \gamma : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \operatorname{Ho} \mathcal{C}^{\mathrm{op}} \times \operatorname{Ho} \mathcal{C}$, *namely the hom-functor*

$$\operatorname{Ho} \mathcal{C}(-,-) : \operatorname{Ho} \mathcal{C}^{\mathrm{op}} \times \operatorname{Ho} \mathcal{C} \to \mathbf{Set}$$

with unit $\mathcal{C}(-,-) \Rightarrow \operatorname{Ho} \mathcal{C}(\gamma -, \gamma -)$ *induced by* $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$.

Proof. By theorem A.5.15, the value of a pointwise left Kan extension of $\mathcal{C}(-,-)$ along $\gamma^{\mathrm{op}} \times \gamma$ is computed by the following coend,

$$\int^{(C',C):\mathcal{C}^{\mathrm{op}}\times\mathcal{C}} \operatorname{Ho} \mathcal{C}(\gamma C', \gamma -) \times \operatorname{Ho} \mathcal{C}(\gamma -, \gamma C) \times \mathcal{C}(C', C)$$

and by the interchange law (theorem A.6.17) and the Yoneda lemma (proposition A.6.18) for coends, there is a natural bijection between the above and the coend

$$\int^{C:\mathcal{C}} \operatorname{Ho} \mathcal{C}(\gamma C, \gamma -) \times \operatorname{Ho} \mathcal{C}(\gamma -, \gamma C)$$

so the claim is a consequence of proposition 4.4.5. ■

Proposition 4.4.7. *Let* $\mathcal{M}$ *be a derivable category, let* $\mathcal{U}$ *be the class of trivial cofibrations in* $\mathcal{M}$, *and let* $\mathcal{V}$ *be the class of trivial fibrations in* $\mathcal{M}$.

- *Let* $(\mathcal{M}_{\mathrm{c}}, Q, p)$ *be a left Quillen deformation retract of* $\mathcal{M}$ *and let* $\mathcal{M}[\mathcal{U}^{-1}]$ *be the localisation of* $\mathcal{M}$ *with respect to the trivial cofibrations. Then the inclusion* $\mathcal{M}_{\mathrm{c}} \hookrightarrow \mathcal{M}$ *induces a fully faithful functor* $\operatorname{Ho} \mathcal{M}_{\mathrm{c}} \to \mathcal{M}[\mathcal{U}^{-1}]$, *and* (Q, p) *induces a right adjoint for that functor.*

- *Let* $(\mathcal{M}_{\mathrm{f}}, R, i)$ *be a right Quillen deformation retract of* $\mathcal{M}$ *and let* $\mathcal{M}[\mathcal{V}^{-1}]$ *be the localisation of* $\mathcal{M}$ *with respect to the trivial fibrations. Then the inclusion* $\mathcal{M}_{\mathrm{f}} \hookrightarrow \mathcal{M}$ *induces a fully faithful functor* $\operatorname{Ho} \mathcal{M}_{\mathrm{f}} \to \mathcal{M}[\mathcal{V}^{-1}]$, *and* (R, i) *induces a left adjoint for that functor.*

Proof. The two claims are formally dual; we will prove the first version.

By corollary 4.1.13, $(\mathcal{M}, \mathcal{U})$ admits a calculus of cospans, so we may use the fundamental theorem of calculi of cospans (3.5.6) and lemma 4.1.8 to deduce that the canonical functor $\operatorname{Ho} \mathcal{M}_{\mathrm{c}} \to \mathcal{M}[\mathcal{U}^{-1}]$ is indeed fully faithful.

On the other hand, lemma 4.1.33 says the localising functor $\mathcal{M} \to \mathcal{M}[\mathcal{U}^{-1}]$ sends weak equivalences in $\mathcal{M}_{\mathrm{c}}$ to isomorphisms in $\mathcal{M}[\mathcal{U}^{-1}]$, so we may apply proposition 3.3.19 to deduce that the canonical functor $\mathcal{M}[\mathcal{U}^{-1}] \to \operatorname{Ho} \mathcal{M}$ has a fully faithful left adjoint defined by Q. On the other hand, proposition 4.1.35 implies that the canonical functor $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ is fully faithful with a *right* adjoint defined by Q. Thus, we have the following hom-set bijections,

$$\operatorname{Ho}\mathcal{M}_c(X, QY) \cong \operatorname{Ho}\mathcal{M}(X, Y) \cong \mathcal{M}\big[\mathcal{W}^{-1}\big](QX, Y)$$

and these bijections are moreover natural in X. Since X is a cofibrant object, the morphism $p_X : QX \to X$ is invertible in $\mathcal{M}\big[\mathcal{W}^{-1}\big]$; and p defines a natural transformation of functors $\operatorname{Ho}\mathcal{M}_c \to \mathcal{M}\big[\mathcal{W}^{-1}\big]$, so we have obtained from (Q, p) a right adjoint for the functor $\operatorname{Ho}\mathcal{M}_c \to \mathcal{M}\big[\mathcal{W}^{-1}\big]$ induced by the inclusion $\mathcal{M}_c \hookrightarrow \mathcal{M}$, as required. ∎

The following proposition extends a result of Joyal [2010].

Proposition 4.4.8. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two* saturated *derivable categories with the same underlying category and let $\mathcal{M}_f$ and $\mathcal{M}'_f$ be the full subcategories of fibrant objects in $\mathcal{M}$ and $\mathcal{M}'$, respectively. Consider the following statements:*

(i) *Every weak equivalence in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}'$.*

(ii) *$\mathcal{M}'_f$ is a full relative subcategory of $\mathcal{M}_f$.*

(iii) *Every fibrant object in $\mathcal{M}'$ is a fibrant object in $\mathcal{M}$.*

If $\mathcal{M}$ and $\mathcal{M}'$ have the same cofibrations, then (i) ⇒ (ii)*; we always have* (ii) ⇒ (iii)*; and if $\mathcal{M}'$ is a saturated derivable category with the same cofibrations as $\mathcal{M}$, then* (iii) ⇒ (i).

Proof. (i) ⇒ (ii). Every trivial cofibration in $\mathcal{M}$ is a trivial cofibration in $\mathcal{M}'$, so every fibrant object in $\mathcal{M}'$ is a fibrant object in $\mathcal{M}$. Since $\mathcal{M}$ and $\mathcal{M}'$ have the same cofibrations, they also have the same trivial fibrations (by theorem 4.1.12), so lemma 4.1.33 implies every weak equivalence in $\mathcal{M}'_f$ is also a weak equivalence in $\mathcal{M}_f$. But we assumed every weak equivalence in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}'_f$, so this implies that a morphism in $\mathcal{M}'_f$ is a weak equivalence if and only if it is a weak equivalence in $\mathcal{M}_f$, as required.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Since $\mathcal{M}$ and $\mathcal{M}'$ have the same cofibrations, $\mathcal{M}'_f$ is a full relative subcategory of $\mathcal{M}_f$, and proposition 4.4.7 implies that the induced functor $\operatorname{Ho}\mathcal{M}'_f \to \operatorname{Ho}\mathcal{M}_f$ has a left adjoint, say $L : \operatorname{Ho}\mathcal{M}_f \to \operatorname{Ho}\mathcal{M}'_f$.

Let (R, i) and (R', i') be right Quillen deformations for $\mathcal{M}$ and $\mathcal{M}'$, respectively. Axiom CM2 implies a morphism $f : X \to Y$ is a weak equivalence in $\mathcal{M}$ (resp. in $\mathcal{M}'$) if and only if $Rf : RX \to RY$ is a weak equivalence in $\mathcal{M}$ (resp. $R'f : R'X \to R'Y$ is a weak equivalence in $\mathcal{M}'$). The uniqueness of left adjoints implies $LR \cong R'$ as functors $\mathcal{M}[\mathcal{V}^{-1}] \to \operatorname{Ho} \mathcal{M}'_{\mathrm{f}}$, where $\mathcal{V}$ is the class of trivial fibrations in $\mathcal{M}$ (or $\mathcal{M}'$), so if $\mathcal{M}'$ is a saturated derivable category, it follows that every weak equivalence in $\mathcal{M}$ is also a weak equivalence in $\mathcal{M}'$. ■

Theorem 4.4.9 (Determination principle)**.** *The model structure on a derivable category is uniquely determined by any one of the following sets of data:*

(i) *The cofibrations and the weak equivalences.*

(ii) *The cofibrations and the trivial cofibrations.*

(iii) *The cofibrations and the fibrant objects.*

(iv) *The cofibrations and the fibrations.*

(v) *The trivial cofibrations and the trivial fibrations.*

(i′) *The fibrations and the weak equivalences.*

(ii′) *The fibrations and the trivial fibrations.*

(iii′) *The fibrations and the cofibrant objects.*

Proof. (i) and (ii). By theorem 4.1.12, the fibrations are precisely the morphisms with the right lifting property with respect to every trivial cofibration.

(iii). Apply proposition 4.4.8 and reduce to case (i).

(iv). The trivial cofibrations are precisely the morphisms with the left lifting property with respect to all fibrations, and the trivial fibrations are precisely the morphisms with the right lifting property with respect to all cofibrations, so this reduces to case (v).

(v). Axioms CM2 and CM5 imply that every weak equivalence is of the form $p \circ i$ where i is a trivial cofibration and p is a trivial fibration. Thus, the trivial cofibrations and the trivial fibrations together determine the weak equivalences. On the other hand, the trivial cofibrations determine the fibrations, and the trivial fibrations determine the cofibrations, thus the entire model structure is determined. ■

4.5 Reedy diagrams

Prerequisites. §§0.2, 0.5, A.3, A.5, A.6.

Definition 4.5.1.

- A **direct category** is a category $\mathcal{C}$ for which there exists a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ such that, if $f : A \to B$ is a morphism in $\mathcal{C}$, then $\deg A \leq \deg B$ with equality if and only if $f = \mathrm{id}_A = \mathrm{id}_B$.

- An **inverse category** is a category $\mathcal{C}$ for which there exists a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ such that, if $f : A \to B$ is a morphism in $\mathcal{C}$, then $\deg A \geq \deg B$ with equality if and only if $f = \mathrm{id}_A = \mathrm{id}_B$.

Proposition 4.5.2. *Let $\mathcal{C}$ be a category and let $\preccurlyeq$ be the binary relation on* $\operatorname{ob} \mathcal{C}$ *defined below:*

$$A \preccurlyeq B \text{ if and only if there is a morphism } A \to B$$

Then the following are equivalent:

(i) *There exists a function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ *making $\mathcal{C}$ a direct category.*

(ii) *If $f : A \to A$ is an endomorphism in $\mathcal{C}$, then $f = \mathrm{id}_A$; $\preccurlyeq$ is an antisymmetric relation on* $\operatorname{ob} \mathcal{C}$*; and for any object A in $\mathcal{C}$, there is a natural number* $\deg A$ *such that, for any chain in* $\operatorname{ob} \mathcal{C}$ *of the form below,*

$$A_0 \preccurlyeq \cdots \preccurlyeq A_n = A$$

if $A_0, \ldots, A_n$ are distinct, then $n \leq \deg A$. (In particular, $\preccurlyeq$ is a well-founded partial order.)

Dually, let $\preccurlyeq$ be the binary relation on $\operatorname{ob} \mathcal{C}$ *defined below:*

$$A \preccurlyeq B \text{ if and only if there is a morphism } B \to A$$

Then the following are equivalent:

(i′) *There exists a function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ *making $\mathcal{C}$ an inverse category.*

(ii′) *If $f : A \to A$ is an endomorphism in $\mathcal{C}$, then $f = \mathrm{id}_A$; $\preccurlyeq$ is an antisymmetric relation on* $\operatorname{ob} \mathcal{C}$*; and for any object A in $\mathcal{C}$, there is a natural number* $\deg A$ *such that, for any chain in* $\operatorname{ob} \mathcal{C}$ *of the form below,*

$$A_0 \preccurlyeq \cdots \preccurlyeq A_n = A$$

if $A_0, \ldots, A_n$ are distinct, then $n \leq \deg A$. (In particular, $\preccurlyeq$ is a well-founded partial order.)

Proof. This is a straightforward exercise. ◇

Remark 4.5.3. The degree function for a direct or inverse category is not determined by the underlying category: for example, if deg is a degree function for $\mathcal{C}$, then so is $A \mapsto 1 + \deg A$. Nonetheless, the above proposition shows that any direct or inverse category has a canonical degree function.

Definition 4.5.4. A **Reedy category** is a category $\mathcal{C}$ equipped with two subcategories, the **direct subcategory** $\mathcal{C}_{\rightarrow}$ and the **inverse subcategory** $\mathcal{C}^{\leftarrow}$, such that the following conditions are satisfied:

- $\operatorname{ob} \mathcal{C} = \operatorname{ob} \mathcal{C}_{\rightarrow} = \operatorname{ob} \mathcal{C}^{\leftarrow}$.
- There exists a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ simultaneously making $\mathcal{C}_{\rightarrow}$ a direct category and $\mathcal{C}^{\leftarrow}$ an inverse category.
- Every morphism in $\mathcal{C}$ admits a unique factorisation of the form $\sigma \circ \delta$, where δ is in $\mathcal{C}_{\leftarrow}$ and σ is in $\mathcal{C}_{\rightarrow}$.

A **Reedy diagram** in a category $\mathcal{M}$ is a functor $\mathcal{C} \to \mathcal{M}$, where $\mathcal{C}$ is a Reedy category.

Remark 4.5.5. Any direct (resp. inverse) category is a Reedy category in a trivial way: take the whole category as the direct (resp. inverse) subcategory, and take $\operatorname{disc} \operatorname{ob} \mathcal{C}$ as the inverse (resp. direct) subcategory.

Example 4.5.6. The simplex category $\mathbf{\Delta}$ is a Reedy category, where the direct subcategory consists of all injective maps, and the inverse subcategory consists of all surjective maps; note that the unique factorisation condition is implied by theorem 1.1.4.

Remark 4.5.7. The opposite of any Reedy category is automatically a Reedy category, after exchanging the direct and inverse subcategories.

Proposition 4.5.8. *Let $\mathcal{C}$ be a category, let $\mathcal{C}_{\rightarrow}$ and $\mathcal{C}^{\leftarrow}$ be subcategories with $\operatorname{ob} \mathcal{C} = \operatorname{ob} \mathcal{C}_{\rightarrow} = \operatorname{ob} \mathcal{C}^{\leftarrow}$, and let $\preccurlyeq$ be the smallest transitive binary relation on $\operatorname{ob} \mathcal{C}$ such that $A \preccurlyeq B$ if there is either a morphism $A \to B$ in $\mathcal{C}_{\rightarrow}$ or a morphism $B \to A$ in $\mathcal{C}^{\leftarrow}$. The following are equivalent:*

(i) *$\mathcal{C}$ is a Reedy category with direct category $\mathcal{C}_{\rightarrow}$ and inverse category $\mathcal{C}^{\leftarrow}$.*

(ii) *$\mathcal{C}_{\rightarrow}$ is a direct category; $\mathcal{C}^{\leftarrow}$ is an inverse category; $\preccurlyeq$ is an antisymmetric relation on* $\operatorname{ob}\mathcal{C}$*; and for any object A in $\mathcal{C}$, there is a natural number* $\deg A$ *such that, for any chain in* $\operatorname{ob}\mathcal{C}$ *of the form below,*

$$A_0 \preccurlyeq \cdots \preccurlyeq A_n = A$$

if $A_0, \ldots, A_n$ are distinct, then $n \le \deg A$. (In particular, $\preccurlyeq$ is a well-founded partial order.)

Proof. This is a straightforward exercise. ◊

Lemma 4.5.9. *Let $\mathcal{C}$ be a Reedy category, let $\alpha : A \to B$ and $\beta : B \to C$ be morphisms in $\mathcal{C}$, let $\delta : A \to D$ be in $\mathcal{C}^{\leftarrow}$ and let $\sigma : D \to C$ be in $\mathcal{C}_{\rightarrow}$.*

(i) *If $\beta \circ \alpha = \sigma \circ \delta$, then* $\deg D \le \deg B$.

(ii) *If $\beta \circ \alpha$ is in $\mathcal{C}_{\rightarrow}$, then α is also in $\mathcal{C}_{\rightarrow}$.*

(iii) *If $\beta \circ \alpha$ is in $\mathcal{C}^{\leftarrow}$, then β is also in $\mathcal{C}^{\leftarrow}$.*

Proof. See (the proof of) Lemma 2.9 in [Riehl and Verity, 2014]. ■

Definition 4.5.10. Let A be an object in a Reedy category $\mathcal{C}$.

- The **latching category** of $\mathcal{C}$ at A, denoted by $\partial\mathcal{C}_{\rightarrow A}$, is the largest full subcategory of the slice category $\left(\mathcal{C}_{\rightarrow} \downarrow A\right)$ that does *not* contain the object $\mathrm{id}_A : A \to A$.
- The **matching category** of $\mathcal{C}$ at A, denoted by $\partial\mathcal{C}^{\leftarrow A}$, is the largest full subcategory of the slice category $(A \downarrow \mathcal{C}^{\leftarrow})$ that does *not* contain the object $\mathrm{id}_A : A \to A$.

Remark 4.5.11. If $\mathcal{C}$ is a Reedy category whose direct (resp. inverse) subcategory is discrete, then all its latching (resp. matching) categories are empty.

Proposition 4.5.12. *Let $\mathcal{C}$ be a Reedy category with degree function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$. *For any natural number n:*

(i) *The full subcategory $\mathcal{C}_{\le n}$ spanned by the objects A in $\mathcal{C}$ such that* $\deg A \le n$ *is a Reedy category.*

(ii) *Let A be an object in $\mathcal{C}$ with* $\deg A = n + 1$. *Then the inclusion $\partial \mathcal{C}_{\to A} \hookrightarrow (\mathcal{C}_{\le n} \downarrow A)$ is cofinal, and the inclusion $\partial \mathcal{C}^{\leftarrow A} \hookrightarrow (A \downarrow \mathcal{C}_{\le n})$ is coinitial.*

Proof. (i). This is a straightforward exercise.

(ii). See Proposition 15.2.8 in [Hirschhorn, 2003]. □

Definition 4.5.13. A **locally finite Reedy category** is a Reedy category such that every latching category and every matching category is finite.

Remark 4.5.14. The factorisation axiom implies that a locally finite Reedy category is a category whose hom-sets are finite; but not every Reedy category with that property is locally finite.

Example 4.5.15. The simplex category $\mathbf{\Delta}$ is a locally finite Reedy category.

Definition 4.5.16. Let A be an object in a small Reedy category $\mathcal{C}$.

- The **boundary** of the representable functor $h_A : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is the subfunctor $\partial h_A \subseteq h_A$ consisting of all morphisms $A' \to A$ in $\mathcal{C}$ that are *not* in the inverse subcategory $\mathcal{C}^{\leftarrow}$.
- The **boundary** of the representable functor $h^A : \mathcal{C} \to \mathbf{Set}$ is the subfunctor $\partial h^A \subseteq h^A$ consisting of all morphisms $A \to A'$ in $\mathcal{C}$ that are *not* in the direct subcategory $\mathcal{C}^{\to}$.

Remark 4.5.17. Lemma 4.5.9 ensures that ∂h_A and ∂h^A are indeed subfunctors.

Lemma 4.5.18. *Let A be an object in a small Reedy category $\mathcal{C}$.*

- *Let $P : \partial \mathcal{C}_{\to A} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ be the functor that sends an object $A' \to A$ in $\partial \mathcal{C}_{\to A}$ to $h_{A'}$. Then the canonical morphism $\varinjlim_{\partial \mathcal{C}_{\to A}} P \to h_A$ is a monomorphism and has ∂h_A as its image.*
- *Let $P : (\partial \mathcal{C}^{\leftarrow A})^{\mathrm{op}} \to [\mathcal{C}, \mathbf{Set}]$ be the functor that sends an object $A \to A'$ in $\partial \mathcal{C}^{\leftarrow A}$ to $h^{A'}$. Then the canonical morphism $\varinjlim_{\partial \mathcal{C}^{\leftarrow A}} P \to h^A$ is a monomorphism and has ∂h^A as its image.*

Proof. Apply proposition 4.5.12 to Observation 3.18 in [Riehl and Verity, 2014]. ■

Definition 4.5.19. Let C be a small Reedy category.

- A **Reedy-acyclic morphism** in $[C^{\mathrm{op}}, \mathbf{Set}]$ is a morphism that has the right lifting property with respect to every boundary inclusion $\partial \hbar_A \hookrightarrow \hbar_A$.
- A **Reedy-acyclic morphism** in $[C, \mathbf{Set}]$ is a morphism that has the right lifting property with respect to every boundary inclusion $\partial \hbar^A \hookrightarrow \hbar^A$.

REMARK 4.5.20. In the special case of the simplex category $\mathbf{\Delta}$, we have $\partial \hbar_{[n]} = \partial \Delta^n$, as expected. Thus, the Reedy-acyclic morphisms in $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}] = \mathbf{sSet}$ are the trivial Kan fibrations.

Definition 4.5.21. Let C be a small Reedy category, let $\mathcal{M}$ be a locally small category, and let $X : C \to \mathcal{M}$ be a diagram.

- A **latching object** $\mathrm{L}_A(X)$ is a weighted colimit $\partial \hbar_A \star_C X$ in $\mathcal{M}$. The **latching morphism** $\mathrm{L}_A(X) \to XA$ is the morphism in $\mathcal{M}$ induced by the boundary inclusion $\partial \hbar_A \hookrightarrow \hbar_A$.
- A **matching object** $\mathrm{M}_A(X)$ is a weighted limit $\left\{\partial \hbar^A, X\right\}^C$ in $\mathcal{M}$. The **matching morphism** $XA \to \mathrm{M}_A(X)$ is the morphism in $\mathcal{M}$ induced by the boundary inclusion $\partial \hbar^A \hookrightarrow \hbar^A$.

REMARK 4.5.22. Assuming existence, the latching object $\mathrm{L}_A(X)$ is functorial in A (as A varies in the direct subcategory), and the matching object $\mathrm{M}_A(X)$ is functorial in A (as A varies in the inverse subcategory). Of course, it should go without saying that $\mathrm{L}_A(X)$ and $\mathrm{M}_A(X)$ are both functorial in X (as X varies in $[C, \mathcal{M}]$), and moreover, we have the following natural bijections:

$$\mathcal{M}\big(\mathrm{L}_A(X), -\big) \cong \mathrm{M}_A(\mathcal{M}(X, -))$$
$$\mathcal{M}\big(-, \mathrm{M}_A(X)\big) \cong \mathrm{M}_A(\mathcal{M}(-, X))$$

Definition 4.5.23. Let C be a small Reedy category, let $\mathcal{M}$ be a locally small category, and let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $\mathbb{C} \to \mathcal{M}$.

- A **relative latching object** $\mathrm{L}_A(X, Y, \varphi)$ is an object in $\mathcal{M}$ equipped with a pullback diagram in $[\mathcal{M}, \mathbf{Set}]$ of the form below,

$$\begin{tikzcd}
\mathcal{M}\big(\mathrm{L}_A(X,Y,\varphi), -\big) \arrow[r] \arrow[d] & \big\{\partial\hbar_A, \mathcal{M}(Y,-)\big\}^{C^{\mathrm{op}}} \arrow[d, "\{\partial\hbar_A, \mathcal{M}(\varphi,-)\}^{C^{\mathrm{op}}}"] \\
\big\{\hbar_A, \mathcal{M}(X,-)\big\}^{C^{\mathrm{op}}} \arrow[r] & \big\{\partial\hbar_A, \mathcal{M}(X,-)\big\}^{C^{\mathrm{op}}}
\end{tikzcd}$$

  where the bottom horizontal arrow is induced by the boundary inclusion $\partial\hbar_A \hookrightarrow \hbar_A$, the **relative latching morphism** $\mathrm{L}_A(X,Y,\varphi) \to YA$ corresponds to $\varphi_A : XA \to YA$, and the **insertion** $XA \to \mathrm{L}_A(X,Y,\varphi)$ corresponds to $\mathrm{id} : \mathrm{L}_A(X,Y,\varphi) \to \mathrm{L}_A(X,Y,\varphi)$.

- A **relative matching object** $\mathrm{M}_A(X,Y,\varphi)$ is an object in $\mathcal{M}$ equipped with a pullback diagram in $[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$ of the form below,

$$\begin{tikzcd}
\mathcal{M}\big(-, \mathrm{M}_A(X,Y,\varphi)\big) \arrow[r] \arrow[d] & \big\{\partial\hbar^A, \mathcal{M}(-,X)\big\}^{C} \arrow[d, "\{\partial\hbar^A, \mathcal{M}(-,\varphi)\}^{C}"] \\
\big\{\hbar^A, \mathcal{M}(-,Y)\big\}^{C} \arrow[r] & \big\{\partial\hbar^A, \mathcal{M}(-,Y)\big\}^{C}
\end{tikzcd}$$

  where the bottom horizontal arrow is induced by the boundary inclusion $\partial\hbar^A \hookrightarrow \hbar^A$, the **relative matching morphism** $XA \to \mathrm{M}_A(X,Y,\varphi)$ corresponds to $\varphi_A : XA \to YA$, and the **projection** $\mathrm{M}_A(X,Y,\varphi) \to YA$ corresponds to $\mathrm{id} : \mathrm{M}_A(X,Y,\varphi) \to \mathrm{M}_A(X,Y,\varphi)$.

Remark 4.5.24. Recalling lemma 4.5.18:

- If the latching category $\partial C_{\to A}$ is empty, then we may identify the relative latching morphism $\mathrm{L}_A(X,Y,\varphi) \to YA$ with the component $\varphi_A : XA \to YA$.

- If the matching category $\partial C^{\leftarrow A}$ is empty, then we may identify the relative matching morphism $XA \to \mathrm{M}_A(X,Y,\varphi)$ with the component $\varphi_A : XA \to YA$.

Remark 4.5.25.

- If $\mathcal{M}$ has enough colimits, then we have a pushout diagram in $\mathcal{M}$ of the form below,
$$\begin{tikzcd} \mathrm{L}_A(X) \arrow[r] \arrow[d, "\mathrm{L}_A(\varphi)"'] & XA \arrow[d] \\ \mathrm{L}_A(Y) \arrow[r] & \mathrm{L}_A(X,Y,\varphi) \end{tikzcd}$$
where the right vertical arrow is the insertion.

- If $\mathcal{M}$ has enough limits, then we have a pullback diagram in $\mathcal{M}$ of the form below,
$$\begin{tikzcd} \mathrm{M}_A(X,Y,\varphi) \arrow[r] \arrow[d] & \mathrm{M}_A(X) \arrow[d, "\mathrm{M}_A(\varphi)"] \\ YA \arrow[r] & \mathrm{M}_A(Y) \end{tikzcd}$$
where the left vertical arrow is the projection.

Definition 4.5.26. Let C be a small Reedy category and let $\mathcal{M}$ be a locally small category.

- A natural transformation $\varphi : X \Rightarrow Y$ of diagrams $C \to \mathcal{M}$ has the **Reedy left lifting property** with respect to a morphism $g : M \to N$ in $\mathcal{M}$ if the relative matching morphism $\hbar^{YA} \to \mathrm{M}_A\left(\hbar^Y, \hbar^X, \varphi^*\right)$ has the *right* lifting property with respect to $g^* : \hbar^N \to \hbar^M$ in $[\mathcal{M}, \mathbf{Set}]$.

- A natural transformation $\varphi : X \Rightarrow Y$ of diagrams $C \to \mathcal{M}$ has the **Reedy right lifting property** with respect to a morphism $g : M \to N$ in $\mathcal{M}$ if the relative matching morphism $\hbar_{XA} \to \mathrm{M}_A\left(\hbar_X, \hbar_Y, \varphi_*\right)$ has the *right* lifting property with respect to $g_* : \hbar_M \to \hbar_N$ in $[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$.

Lemma 4.5.27. *Let C be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $\overline{(-)} : \mathcal{M} \to \overline{\mathcal{M}}$ be a fully faithful functor, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $C \to \mathcal{M}$, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *Assuming the relative latching object $\mathrm{L}_A\left(\overline{X}, \overline{Y}, \overline{\varphi}\right)$ exists in $\overline{\mathcal{M}}$, $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to $g : M \to N$ if and only if the relative latching morphism $\mathrm{L}_A\left(\overline{X}, \overline{Y}, \overline{\varphi}\right) \to \overline{YA}$ has the left lifting property with respect to $\overline{g} : \overline{M} \to \overline{N}$.*

- *Assuming the relative matching object* $\mathrm{M}_A\left(\overline{X}, \overline{Y}, \overline{\varphi}\right)$ *exists in* $\overline{\mathcal{M}}$, $\varphi : X \Rightarrow Y$ *has the Reedy right lifting property with respect to* $g : M \to N$ *if and only if the relative matching morphism* $\overline{XA} \to \mathrm{M}_A\left(\overline{X}, \overline{Y}, \overline{\varphi}\right)$ *has the right lifting property with respect to* $\overline{g} : \overline{M} \to \overline{N}$.

Proof. This is a straightforward exercise. ◊

Lemma 4.5.28. *Let* C *be a small Reedy category and let*

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

be an adjunction between locally small categories.

- *Given a natural transformation* $\varphi : X \Rightarrow Y$ *of diagrams* $C \to \mathcal{N}$ *and a morphism* $g : M \to M'$ *in* $\mathcal{M}$, $\varphi : X \Rightarrow Y$ *has the Reedy left lifting property with respect to* $Gg : GM \to GM'$ *if and only if* $F\varphi : FX \Rightarrow FY$ *has the Reedy left lifting property with respect to* $g : M \to M'$.

- *Given a natural transformation* $\varphi : X \Rightarrow Y$ *of diagrams* $C \to \mathcal{M}$ *and a morphism* $g : N' \to N$ *in* $\mathcal{N}$, $\varphi : X \Rightarrow Y$ *has the Reedy right lifting property with respect to* $Fg : FN' \to FN$ *if and only if* $G\varphi : GX \Rightarrow GY$ *has the Reedy right lifting property with respect to* $g : N' \to N$.

Proof. This is a straightforward exercise. ◊

Proposition 4.5.29. *Let* C *be a small Reedy category, let* $\mathcal{M}$ *be a locally small category, let* $\varphi : X \Rightarrow Y$ *be a natural transformation of diagrams* $C \to \mathcal{M}$, *and let* $g : M \to N$ *be a morphism in* $\mathcal{M}$. *The following are equivalent:*

(i) $\varphi : X \Rightarrow Y$ *has the Reedy left lifting property with respect to* $g : M \to N$.

(ii) *The morphism in* $[C^{\mathrm{op}}, \mathbf{Set}]$

$$\mathcal{M}(Y, M) \to \mathcal{M}(X, M) \times_{\mathcal{M}(X,N)} \mathcal{M}(Y, N)$$

induced by the evident commutative square is a Reedy-acyclic morphism.

(iii) *The morphism in* $[[C, \mathcal{M}]^{\mathrm{op}}, \mathbf{Set}]$

$$\mathcal{M}((-)A, M) \to \mathrm{M}_A(\mathcal{M}(-, M), \mathcal{M}(-, N), \mathcal{M}(-, g))$$

induced by the relative matching morphisms has the right *lifting property with respect to* $\varphi_* : \mathcal{k}_X \to \mathcal{k}_Y$.

Dually, the following are equivalent:

(i′) $\varphi : X \Rightarrow Y$ *has the Reedy right lifting property with respect to* $g : M \to N$.

(ii′) *The morphism in* $[\mathcal{C}, \mathbf{Set}]$

$$\mathcal{M}(N, X) \to \mathcal{M}(M, X) \times_{\mathcal{M}(M,Y)} \mathcal{M}(N, Y)$$

induced by the evident commutative square is a Reedy-acyclic morphism.

(iii′) *The morphism in* $[[\mathcal{C}, \mathcal{M}], \mathbf{Set}]$

$$\mathcal{M}(N, (-)A) \to \mathrm{M}_A(\mathcal{M}(N, -), \mathcal{M}(M, -), \mathcal{M}(g, -))$$

induced by the relative matching morphisms has the right *lifting property with respect to* $\varphi^* : \text{ሐ}^Y \to \text{ሐ}^X$.

Proof. Let A be any object in $\mathcal{C}$. Consider the following commutative diagram in $\mathbf{Set}$,

$$\begin{tikzcd}
 & & \mathcal{M}(XA, N) \arrow[dd] & & \\
\mathcal{M}(XA, M) \arrow[urr, "\mathcal{M}(XA,g)"] \arrow[dd] & & & & \mathcal{M}(YA, N) \arrow[ull, "\mathcal{M}(\varphi_A,N)"'] \arrow[dd] \\
 & & \mathrm{M}_A(\mathcal{M}(X, N)) & & \\
\mathrm{M}_A(\mathcal{M}(X, M)) \arrow[urr, "\mathrm{M}_A(\mathcal{M}(X,g))"] & & & & \mathrm{M}_A(\mathcal{M}(Y, N)) \arrow[ull, "\mathrm{M}_A(\mathcal{M}(\varphi,N))"'] \\
 & & \mathrm{M}_A(\mathcal{M}(Y, M)) \arrow[ull, "\mathrm{M}_A(\mathcal{M}(\varphi,M))"] \arrow[urr, "\mathrm{M}_A(\mathcal{M}(Y,g))"'] & &
\end{tikzcd}$$

where the vertical arrows are the respective matching morphisms. Let L be the limit of the above diagram. It is not hard to see that the induced diagrams

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathrm{M}_A(\mathcal{M}(Y, M), \mathcal{M}(X, M), \mathcal{M}(\varphi, M)) \arrow[d] \\
\mathcal{M}(YA, N) \arrow[r] & \mathrm{M}_A(\mathcal{M}(Y, N), \mathcal{M}(X, N), \mathcal{M}(\varphi, N))
\end{tikzcd} \tag{1}$$

(2)
$$
\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathcal{M}(XA, N) \times_{\mathcal{M}(XA,N)} \mathcal{M}(YA, N) \arrow[d] \\
\mathrm{M}_A(\mathcal{M}(Y, M)) \arrow[r] & \mathrm{M}_A\big(\mathcal{M}(X, N) \times_{\mathcal{M}(X,N)} \mathcal{M}(Y, N)\big)
\end{tikzcd}
$$

(3)
$$
\begin{tikzcd}
L \arrow[r] \arrow[d] & \mathrm{M}_A(\mathcal{M}(Y, M), \mathcal{M}(Y, N), \mathcal{M}(Y, g)) \arrow[d] \\
\mathcal{M}(XA, M) \arrow[r] & \mathrm{M}_A(\mathcal{M}(X, M), \mathcal{M}(X, N), \mathcal{M}(X, g))
\end{tikzcd}
$$

are pullback diagrams in **Set**. Thus, by the Yoneda lemma, the set L can be identified with the following:

1. The set of all commutative squares of the form

$$
\begin{tikzcd}
\hbar^N \arrow[r] \arrow[d, "g^*"'] & \hbar^{YA} \arrow[d] \\
\hbar^M \arrow[r] & \mathrm{M}_A\big(\hbar^Y, \hbar^X, \varphi^*\big)
\end{tikzcd}
$$

   in $[\mathcal{M}, \mathbf{Set}]$, where the right vertical arrow is the relative matching morphism.

2. The set of all commutative squares of the form

$$
\begin{tikzcd}
\partial \hbar_A \arrow[r] \arrow[d, hook] & \mathcal{M}(Y, M) \arrow[d] \\
\hbar_A \arrow[r] & \mathcal{M}(X, N) \times_{\mathcal{M}(X,N)} \mathcal{M}(Y, N)
\end{tikzcd}
$$

   in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, where the right vertical arrow is induced by the evident commutative square.

3. The set of all commutative squares of the form

$$
\begin{tikzcd}
\hbar_X \arrow[r] \arrow[d, "\varphi_*"'] & \mathcal{M}((-)A, M) \arrow[d] \\
\hbar_Y \arrow[r] & \mathrm{M}_A(\mathcal{M}(-, M), \mathcal{M}(-, N), \mathcal{M}(-, g))
\end{tikzcd}
$$

   in $[[\mathcal{C}, \mathcal{M}]^{\mathrm{op}}, \mathbf{Set}]$, where the right vertical arrow is induced by the relative matching morphisms.

Thus, the surjectivity of the comparison map $\mathcal{M}(YA, M) \to L$ is equivalent to each of conditions (i), (ii), and (iii). ■

Proposition 4.5.30. *Let C be a small Reedy category, let $U : \mathcal{M}' \to \mathcal{M}$ be an orthogonality-reflecting functor between locally small categories, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $C \to \mathcal{M}'$, and let $g : M \to N$ be a morphism in $\mathcal{M}'$.*

- *If $U\varphi : UX \Rightarrow UY$ has the Reedy left lifting property with respect to $Ug : UM \to UN$, then $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to $g : M \to N$.*
- *If $U\varphi : UX \Rightarrow UY$ has the Reedy right lifting property with respect to $Ug : UM \to UN$, then $\varphi : X \Rightarrow Y$ has the Reedy right lifting property with respect to $g : M \to N$.*

Proof. The two claims are formally dual; we will prove the first version.

Consider the following commutative diagram in $[C^{\mathrm{op}}, \mathbf{Set}]$:

$$\begin{tikzcd}
\mathcal{M}'(Y, M) \arrow[r] \arrow[d, "U"'] & \mathcal{M}'(X, M) \times_{\mathcal{M}'(X,N)} \mathcal{M}'(Y, N) \arrow[d, "U"] \\
\mathcal{M}(UY, UM) \arrow[r] & \mathcal{M}(UX, UM) \times_{\mathcal{M}(UX,UN)} \mathcal{M}(UY, UN)
\end{tikzcd}$$

By lemma A.3.6, the above diagram is a pullback square in $[C^{\mathrm{op}}, \mathbf{Set}]$, and by proposition 4.5.29, $U\varphi : UX \Rightarrow UY$ has the Reedy left lifting property with respect to $Ug : UM \to UN$ if and only if the bottom horizontal arrow in the diagram is a Reedy-acyclic morphism in $[C^{\mathrm{op}}, \mathbf{Set}]$. Since the class of Reedy-acyclic morphisms is closed under pullback (by proposition A.3.17), we conclude that $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to $g : M \to N$. ■

Definition 4.5.31. Let C be a small Reedy category and let $\mathcal{M}$ be a locally small category.

- A diagram $Y : C \to \mathcal{M}$ is **Reedy-projective** with respect to a morphism $g : M \to N$ if the matching morphism $\hbar^{YA} \to \mathrm{M}_A(\hbar^Y)$ has the *right* lifting property with respect to $g^* : \hbar^N \to \hbar^M$ in $[\mathcal{M}, \mathbf{Set}]$.
- A diagram $X : C \to \mathcal{M}$ is **Reedy-injective** with respect to a morphism $g : M \to N$ if the matching morphism $\hbar_{XA} \to \mathrm{M}_A(\hbar_Y)$ has the *right* lifting property with respect to $g_* : \hbar_M \to \hbar_N$ in $[\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$.

Lemma 4.5.32. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $\overline{(-)} : \mathcal{M} \to \overline{\mathcal{M}}$ be a fully faithful functor, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *Assuming $\overline{\mathcal{M}}$ has an initial object 0, a diagram $Y : \mathcal{C} \to \mathcal{M}$ is Reedy-projective with respect to $g : M \to N$ if and only if the unique natural transformation $\Delta 0 \Rightarrow \overline{Y}$ has the Reedy left lifting property with respect to $\overline{g} : \overline{M} \to \overline{N}$.*
- *Assuming $\overline{\mathcal{M}}$ has a terminal object 1, a diagram $X : \mathcal{C} \to \mathcal{M}$ is Reedy-injective with respect to $g : M \to N$ if and only if the unique natural transformation $\overline{X} \Rightarrow \Delta 1$ has the Reedy right lifting property with respect to $\overline{g} : \overline{M} \to \overline{N}$.*

Proof. This is a straightforward exercise. □

Lemma 4.5.33. *Let $\mathcal{C}$ be a small Reedy category and let*

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

be an adjunction.

- *Given a diagram $Y : \mathcal{C} \to \mathcal{M}$ and a morphism $g : M \to M'$ in $\mathcal{M}$, Y is Reedy-projective with respect to $Gg : GM \to GM'$ if and only if FY is Reedy-projective with respect to $g : M \to M'$.*
- *Given a diagram $X : \mathcal{C} \to \mathcal{M}$ and a morphism $g : N' \to N$ in $\mathcal{N}$, X is Reedy-injective with respect to $Fg : FN' \to FN$ if and only if GX is Reedy-injective with respect to $g : N' \to N$.*

Proof. This is a straightforward exercise. ◊

Proposition 4.5.34. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category and let $g : M \to N$ be a morphism in $\mathcal{M}$. The following are equivalent for a diagram $Y : \mathcal{C} \to \mathcal{M}$:*

(i) *Y is Reedy-projective with respect to $g : M \to N$.*

(ii) *The morphism in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*

$$\mathcal{M}(Y, g) : \mathcal{M}(Y, M) \to \mathcal{M}(Y, N)$$

has the right *lifting property with respect to every boundary inclusion $\partial \text{よ}_A \hookrightarrow \text{よ}_A$.*

(iii) *The morphism in* $[[\mathcal{C},\mathcal{M}]^{\mathrm{op}},\mathbf{Set}]$

$$\mathcal{M}((-)A,M) \to \mathrm{M}_A(\mathcal{M}(-,M),\mathcal{M}(-,N),\mathcal{M}(-,g))$$

induced by the relative matching morphisms has the right *lifting property with respect to the unique morphism* $0 \to \mathcal{k}_Y$.

Dually, the following are equivalent for a diagram $X : \mathcal{C} \to \mathcal{M}$*:*

(i′) X *is Reedy-injective with respect to* $g : M \to N$.

(ii′) *The morphism in* $[\mathcal{C},\mathbf{Set}]$

$$\mathcal{M}(g,X) : \mathcal{M}(N,X) \to \mathcal{M}(M,X)$$

has the right *lifting property with respect to every boundary inclusion* $\partial\mathcal{k}^A \hookrightarrow \mathcal{k}^A$.

(iii′) *The morphism in* $[[\mathcal{C},\mathcal{M}],\mathbf{Set}]$

$$\mathcal{M}(N,(-)A) \to \mathrm{M}_A(\mathcal{M}(N,-),\mathcal{M}(M,-),\mathcal{M}(g,-))$$

induced by the relative matching morphisms has the right *lifting property with respect to the unique morphism* $0 \to \mathcal{k}^X$.

Proof. The proof is essentially the same as proposition 4.5.29. ■

Proposition 4.5.35. *Let* $\mathcal{C}$ *be a small Reedy category, let* $U : \mathcal{M}' \to \mathcal{M}$ *be an orthogonality-reflecting functor between locally small categories, and let* $g : M \to N$ *be a morphism in* $\mathcal{M}'$.

- *Let* $Y : \mathcal{C} \to \mathcal{M}'$ *be a diagram. If* UY *is Reedy-projective with respect to* $Ug : UM \to UN$, *then* Y *is Reedy-projective with respect to* $g : M \to N$.
- *Let* $X : \mathcal{C} \to \mathcal{M}'$ *be a diagram. If* UX *is Reedy-injective with respect to* $Ug : UM \to UN$, *then* X *is Reedy-injective with respect to* $g : M \to N$.

Proof. The proof is essentially the same as proposition 4.5.30. ■

Definition 4.5.36. Let $\mathcal{C}$ be a small Reedy category.

- A **Reedy cell complex** (resp. **relative Reedy cell complex**) in $[\mathcal{C}^{\mathrm{op}},\mathbf{Set}]$ is an $\mathcal{I}$-cell complex (resp. relative $\mathcal{I}$-cell complex), where $\mathcal{I}$ is the set of all boundary inclusions $\partial\mathcal{k}_A \hookrightarrow \mathcal{k}_A$.

- A **Reedy cell complex** (resp. **relative Reedy cell complex**) in $[\mathcal{C}, \mathbf{Set}]$ is an $\mathcal{I}$-cell complex (resp. relative $\mathcal{I}$-cell complex), where $\mathcal{I}$ is the set of all boundary inclusions $\partial \hbar^A \hookrightarrow \hbar^A$.

Lemma 4.5.37. *Let $\mathcal{C}$ be a small Reedy category with degree function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$*, let $\mathcal{C}_{\leq n}$ be the full subcategory spanned by the objects A in $\mathcal{C}$ such that* $\deg A \leq n$*, and let $j : \mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ be the inclusion.*

- *The restriction functor $j^* : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \left[(\mathcal{C}_{\leq n})^{\mathrm{op}}, \mathbf{Set}\right]$ preserves relative Reedy cell complexes.*

- *The restriction functor $j^* : [\mathcal{C}, \mathbf{Set}] \to \left[\mathcal{C}_{\leq n}, \mathbf{Set}\right]$ preserves relative Reedy cell complexes.*

Proof. The two claims are formally dual; we will prove the first version.

Since $j^* : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \left[(\mathcal{C}_{\leq n})^{\mathrm{op}}, \mathbf{Set}\right]$ preserve colimits, it suffices to verify that it sends boundary inclusions to relative Reedy cell complexes. Let A be an object in $\mathcal{C}$. If $\deg A \leq n$, then $j^* \partial \hbar_A \hookrightarrow j^* \hbar_A$ is (isomorphic to) a boundary inclusion in $\left[(\mathcal{C}_{\leq n})^{\mathrm{op}}, \mathbf{Set}\right]$. Otherwise, $\deg A > n$, and since no morphism $A' \to A$ with $\deg A' \leq n$ is in the inverse subcategory $\mathcal{C}^{\leftarrow}$, we deduce that $j^* \partial \hbar_A \hookrightarrow j^* \hbar_A$ is an isomorphism (so a relative Reedy cell complex *a fortiori*). ■

Proposition 4.5.38. *Let $\mathcal{C}$ be a small Reedy category.*

- *For any object A in $\mathcal{C}$, both $\hbar_A$ and its boundary $\partial \hbar_A$ are Reedy cell complexes in $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]$.*

- *For any object A in $\mathcal{C}$, both $\hbar^A$ and its boundary $\partial \hbar^A$ are Reedy cell complexes in $[\mathcal{C}, \mathbf{sSet}]$.*

Proof. See Observation 6.2 in [Riehl and Verity, 2014]. □

Lemma 4.5.39. *Let $\mathcal{C}$ be a locally finite Reedy category with degree function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$*, let $\mathcal{C}_{\leq n}$ be the full subcategory spanned by the objects A in $\mathcal{C}$ such that* $\deg A \leq n$*, and let $j : \mathcal{C}_{\leq n} \hookrightarrow \mathcal{C}$ be the inclusion.*

- *If* $\deg A = n + 1$*, then $j^* \partial \hbar_A$ is an $\aleph_0$-compact object in $\left[(\mathcal{C}_{\leq n})^{\mathrm{op}}, \mathbf{Set}\right]$.*

- *If* $\deg A = n + 1$*, then $j^* \partial \hbar^A$ is an $\aleph_0$-compact object in $\left[\mathcal{C}_{\leq n}, \mathbf{Set}\right]$.*

Proof. Apply proposition 0.2.46 to lemma 4.5.18. (Recall that the latching category $\partial \mathcal{C}_{\to A}$ is finite by hypothesis.) ■

Proposition 4.5.40. *Let C be a small Reedy category.*

- *A morphism in $[C^{\mathrm{op}}, \mathbf{Set}]$ is a relative Reedy cell complex if and only if its relative latching morphisms are injective maps.*
- *A morphism in $[C, \mathbf{Set}]$ is a relative Reedy cell complex if and only if its relative latching morphisms are injective maps.*

Proof. See Corollary 6.8 in [Riehl and Verity, 2014]. □

Proposition 4.5.41. *Let C be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $C \to \mathcal{M}$, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *Assuming $\mathcal{M}$ is cocomplete, if $\alpha : F \to G$ is a relative Reedy cell complex in $[C^{\mathrm{op}}, \mathbf{Set}]$ and $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to $g : M \to N$, then in the commutative diagram in $\mathcal{M}$ shown below,*

$$\begin{tikzcd}
F \star_C X \arrow[r, "\alpha \star_C \mathrm{id}_X"] \arrow[d, "\mathrm{id}_F \star_C \varphi"'] & G \star_C X \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_G \star_C \varphi"] & \\
F \star_C Y \arrow[r] \arrow[drr, bend right, "\alpha \star_C \mathrm{id}_Y"'] & F \star_C Y \cup^{F \star_C X} G \star_C Y \arrow[dr, dashed] & \\
& & G \star_C Y
\end{tikzcd}$$

where the square is a pushout, the indicated arrow

$$F \star_C Y \cup^{F \star_C X} G \star_C Y \to G \star_C Y$$

has the left lifting property with respect to $g : M \to N$.

- *Assuming $\mathcal{M}$ is complete, if $\alpha : F \to G$ is a relative Reedy cell complex in $[C, \mathbf{Set}]$ and $\varphi : X \Rightarrow Y$ has the Reedy right lifting property with respect to $g : M \to N$, then in the commutative diagram in $\mathcal{M}$ shown below,*

$$\begin{tikzcd}
\{G, X\}^C \arrow[dr, dashed] \arrow[drr, bend left, "\{G,\varphi\}^C"] \arrow[ddr, bend right, "\{\alpha,X\}^C"'] & & \\
& \{F, X\}^C \times_{\{F,Y\}^C} \{G, Y\}^C \arrow[r] \arrow[d] & \{G, Y\}^C \arrow[d, "\{\alpha,Y\}^C"] \\
& \{F, X\}^C \arrow[r, "\{F,\varphi\}^C"'] & \{F, Y\}^C
\end{tikzcd}$$

where the square is a pushout, the indicated arrow

$$\{G, X\}^{\mathcal{C}} \to \{F, X\}^{\mathcal{C}} \times_{\{F,Y\}^{\mathcal{C}}} \{G, Y\}^{\mathcal{C}}$$

has the right lifting property with respect to $g : M \to N$.

Proof. Apply proposition A.3.17 to Lemma 5.7 in [Riehl and Verity, 2014]. ■

Corollary 4.5.42. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *If $\alpha : F \to G$ is a relative Reedy cell complex in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and $Y : \mathcal{C} \to \mathcal{M}$ is Reedy-projective with respect to $g : M \to N$, then the morphism*

  $$\alpha \star_{\mathcal{C}} \mathrm{id}_Y : F \star_{\mathcal{C}} Y \to G \star_{\mathcal{C}} Y$$

  has the left lifting property with respect to $g : M \to N$ (if it exists in $\mathcal{M}$).

- *If $\alpha : F \to G$ is a relative Reedy cell complex in $[\mathcal{C}, \mathbf{Set}]$ and $X : \mathcal{C} \to \mathcal{M}$ is Reedy-injective with respect to $g : M \to N$, then the morphism*

  $$\{\alpha, X\}^{\mathcal{C}} : \{G, X\}^{\mathcal{C}} \to \{F, X\}^{\mathcal{C}}$$

  has the right lifting property with respect to $g : M \to N$ (if it exists in $\mathcal{M}$).

Proof. The two claims are formally dual; we will prove the first version.

By enlarging the universe or shrinking $\mathcal{M}$ if necessary, we may assume $\mathcal{M}$ is a small category. The Yoneda embedding $\mathcal{h}^\bullet : \mathcal{M} \to [\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is then fully faithful and preserves all colimits that exist in $\mathcal{M}$, and $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is a cocomplete locally small category. We then apply lemma 4.5.32 and proposition 4.5.41. ■

Corollary 4.5.43. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $\mathcal{C} \to \mathcal{M}$, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *If F is a Reedy cell complex in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ and $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to $g : M \to N$, then the morphism*

  $$\mathrm{id}_F \star_{\mathcal{C}} \varphi : F \star_{\mathcal{C}} X \to F \star_{\mathcal{C}} Y$$

  has the left lifting property with respect to $g : M \to N$ (if it exists in $\mathcal{M}$).

- *If F is a Reedy cell complex in $[\mathcal{C}, \mathbf{Set}]$ and $\varphi : X \Rightarrow Y$ has the Reedy right lifting property with respect to $g : M \to N$, then the morphism*

$$\{F, \varphi\}^{\mathcal{C}} : \{F, X\}^{\mathcal{C}} \to \{F, Y\}^{\mathcal{C}}$$

has the right lifting property with respect to $g : M \to N$ (if it exists in $\mathcal{M}$).

Proof. The two claims are formally dual; we will prove the first version.

By enlarging the universe or shrinking $\mathcal{M}$ if necessary, we may assume $\mathcal{M}$ is a small category. The Yoneda embedding $\mathcal{h}^\bullet : \mathcal{M} \to [\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is then fully faithful and preserves all colimits that exist in $\mathcal{M}$, and $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is a cocomplete locally small category. We then apply lemma 4.5.27 and proposition 4.5.41. ■

Corollary 4.5.44. *Let $\mathcal{C}$ be a small Reedy category with degree function* $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$, *let A be an object in $\mathcal{C}$ with* $\deg A = n + 1$, *let $\mathcal{M}$ be a locally small category, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $\mathcal{C} \to \mathcal{M}$, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *If the restriction of $\varphi : X \Rightarrow Y$ in $\left[\mathcal{C}_{\le n}, \mathcal{M}\right]$ has the Reedy left lifting property with respect to $g : M \to N$ and the relative latching object $\mathrm{L}_A(X, Y, \varphi)$ exists in $\mathcal{M}$, then the insertion $X A \to \mathrm{L}_A(X, Y, \varphi)$ has the left lifting property with respect to $g : M \to N$.*

- *If the restriction of $\varphi : X \Rightarrow Y$ in $\left[\mathcal{C}_{\le n}, \mathcal{M}\right]$ has the Reedy right lifting property with respect to $g : M \to N$ and the relative matching object $\mathrm{M}_A(X, Y, \varphi)$ exists in $\mathcal{M}$, then the projection $\mathrm{M}_A(X, Y, \varphi) \to Y A$ has the right lifting property with respect to $g : M \to N$.*

Proof. The two claims are formally dual; we will prove the first version.

By enlarging the universe or shrinking $\mathcal{M}$ if necessary, we may assume $\mathcal{M}$ is a small category. The Yoneda embedding $\mathcal{h}^\bullet : \mathcal{M} \to [\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is then fully faithful and preserves all colimits that exist in $\mathcal{M}$, and $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is a cocomplete locally small category. We may then apply lemma 4.5.27 and replace $\mathcal{M}$ with $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$, i.e. we may assume $\mathcal{M}$ is a cocomplete locally small category.

Now, as in remark 4.5.25, we have a pushout diagram in $\mathcal{M}$ of the form below:

$$\begin{tikzcd}
\mathrm{L}_A(X) \arrow[r] \arrow[d, "\mathrm{L}_A(\varphi)"'] & X A \arrow[d] \\
\mathrm{L}_A(Y) \arrow[r] & \mathrm{L}_A(X, Y, \varphi)
\end{tikzcd}$$

Proposition 4.5.38 says that the boundary $\partial \mathcal{k}_A$ is a Reedy cell complex, so by corollary 4.5.43, $\mathrm{L}_A(\varphi) : \mathrm{L}_A(X) \to \mathrm{L}_A(Y)$ has the left lifting property with respect to $g : M \to N$; hence, by proposition A.3.17, the right vertical arrow in the above pushout diagram has the same left lifting property. This proves the claim. ■

Proposition 4.5.45. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $(\mathcal{L}, \mathcal{R})$ be a pair of subclasses of* mor $\mathcal{M}$ *such that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$, let $\psi : Z \Rightarrow W$ be a morphism in $[\mathcal{C}, \mathcal{M}]$ that has the Reedy left lifting property with respect to (every morphism that is in) $\mathcal{R}$, and let $\varphi : X \Rightarrow Y$ be a morphism in $[\mathcal{C}, \mathcal{M}]$ that has the Reedy right lifting property with respect to (every morphism that is in) $\mathcal{L}$.*

- *If the relative latching objects $\mathrm{L}_A(Z, W, \psi)$ exist in $\mathcal{M}$ for all objects A in $\mathcal{C}$, then $\varphi : Z \Rightarrow W$ has the left lifting property with respect to $\varphi : X \Rightarrow Y$.*
- *If the relative matching objects $\mathrm{M}_A(X, Y, \varphi)$ exist in $\mathcal{M}$ for all objects A in $\mathcal{C}$, then $\varphi : X \Rightarrow Y$ has the right lifting property with respect to $\psi : Z \Rightarrow W$.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose we have a commutative square in $[\mathcal{C}, \mathcal{M}]$ of the form below:

$$\begin{tikzcd} Z \arrow[r, "\zeta"] \arrow[d, "\psi"'] & X \arrow[d, "\varphi"] \\ W \arrow[r, "\omega"'] & Y \end{tikzcd}$$

Choose a degree function $\deg : \mathrm{ob}\, \mathcal{C} \to \mathbb{N}$ and let A be an object in $\mathcal{C}$ with $\deg A = n+1$. Suppose we have defined for all objects A' in $\mathcal{C}_{\leq n}$ a morphism $\eta_{A'} : XA' \to WA'$, such that these morphisms constitute a natural transformation of diagrams $\mathcal{C}_{\leq n} \to \mathcal{M}$ such that the following diagram commute in $\left[\mathcal{C}_{\leq n}, \mathcal{M}\right]$:

$$\begin{tikzcd} Z \arrow[r, "\zeta"] \arrow[d, "\psi"'] & X \arrow[d, "\varphi"] \\ W \arrow[r, "\omega"'] \arrow[ur, "\eta"] & Y \end{tikzcd}$$

Note that the relative latching morphism $\mathrm{L}_A(Z, W, \psi) \to WA$ exists in $\mathcal{M}$ and is in $\mathcal{L}$ by hypothesis. Thus (by passing through the Yoneda embedding

$\mathcal{M} \to [\mathcal{M}^{\mathrm{op}}, \mathbf{Set}]$ if necessary), there is an induced commutative diagram

$$\begin{tikzcd} \mathrm{L}_A(Z, W, \psi) \arrow[r] \arrow[d] & X A \arrow[d] \\ W A \arrow[r] & \mathrm{M}_A(X, Y, \varphi) \end{tikzcd}$$

in which the right vertical arrow has the right lifting property with respect to the left vertical arrow (by lemma 4.5.27). We thus obtain a morphism $\eta_A : W A \to X A$ making the evident triangles commute. Using lemma 4.5.18, we deduce that the diagram

$$\begin{tikzcd} X A' \arrow[r, "\eta_{A'}"] \arrow[d, "X\delta"'] & W A' \arrow[d, "W\delta"] \\ X A \arrow[r, "\eta_A"'] & W A \end{tikzcd}$$

commutes for every morphism $\delta : A' \to A$ in $\mathcal{C}_{\to}$, and similarly, the diagram

$$\begin{tikzcd} X A \arrow[r, "\eta_A"] \arrow[d, "X\sigma"'] & W A \arrow[d, "W\sigma"] \\ X A' \arrow[r, "\eta_{A'}"'] & W A' \end{tikzcd}$$

commutes for every morphism $\sigma : A \to A'$ in $\mathcal{C}^{\leftarrow}$; so by using the factorisation axiom, we can extend η to a natural transformation of diagrams $\mathcal{C}_{\leq n+1}$. Hence, by induction, we obtain a solution to our original lifting problem in $[\mathcal{C}, \mathcal{M}]$. ■

Proposition 4.5.46. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ be a locally small category, let $(\mathcal{L}, \mathcal{R})$ be a pair of subclasses of* mor $\mathcal{M}$ *such that $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.*

- *If $X : \mathcal{C} \to \mathcal{M}$ is Reedy-injective with respect to (every morphism that is in) $\mathcal{L}$, $\psi : Z \Rightarrow W$ is a morphism in $[\mathcal{C}, \mathcal{M}]$ that has the Reedy left lifting property with respect to $\mathcal{R}$, and the relative latching objects $\mathrm{L}_A(Z, W, \psi)$ exist in $\mathcal{M}$ for all objects A in $\mathcal{C}$, then X is injective with respect to $\psi : Z \Rightarrow W$.*
- *If $W : \mathcal{C} \to \mathcal{M}$ is Reedy-projective with respect to (every morphism that is in) $\mathcal{R}$, $\varphi : X \Rightarrow Y$ is a morphism in $[\mathcal{C}, \mathcal{M}]$ that has the Reedy right lifting property with respect to $\mathcal{L}$, and the relative matching objects $\mathrm{M}_A(X, Y, \varphi)$ exist in $\mathcal{M}$ for all objects A in $\mathcal{C}$, then W is projective with respect to $\varphi : X \Rightarrow Y$.*

Proof. The proof is essentially the same as that of proposition 4.5.45. ■

Proposition 4.5.47. *Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories. Then $\mathcal{C} \times \mathcal{D}$ is a Reedy category, with direct subcategory $\mathcal{C}_{\rightarrow} \times \mathcal{D}_{\rightarrow}$ and inverse subcategory $\mathcal{C}^{\leftarrow} \times \mathcal{D}^{\leftarrow}$.*

Proof. See Proposition 15.1.6 in [Hirschhorn, 2003]. □

¶ 4.5.48. Given functor $F : \mathcal{C} \to \mathbf{Set}$ and $G : \mathcal{D} \to \mathbf{Set}$, let $F \boxtimes G : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ be the functor defined by $(F \boxtimes G)(C, D) = F(C) \times G(D)$. Note that we may identify $\hbar^{(C,D)}$ with $\hbar^{C} \boxtimes \hbar^{D}$.

Lemma 4.5.49. *Let $\mathcal{C}$ and $\mathcal{D}$ be small Reedy categories.*

- *For any object C in $\mathcal{C}$ and any object D in $\mathcal{D}$, the following diagram is a pushout square in $[\mathcal{C}^{\mathrm{op}} \times \mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$:*

$$\begin{tikzcd}
\partial \hbar_C \boxtimes \partial \hbar_D \arrow[r, hook] \arrow[d, hook] & \partial \hbar_C \boxtimes \hbar_D \arrow[d, hook] \\
\hbar_C \boxtimes \partial \hbar_D \arrow[r, hook] & \partial\left(\hbar_C \boxtimes \hbar_D\right)
\end{tikzcd}$$

- *For any object C in $\mathcal{C}$ and any object D in $\mathcal{D}$, the following diagram is a pushout square in $[\mathcal{C} \times \mathcal{D}, \mathbf{Set}]$:*

$$\begin{tikzcd}
\partial \hbar^C \boxtimes \partial \hbar^D \arrow[r, hook] \arrow[d, hook] & \partial \hbar^C \boxtimes \hbar^D \arrow[d, hook] \\
\hbar^C \boxtimes \partial \hbar^D \arrow[r, hook] & \partial\left(\hbar^C \boxtimes \hbar^D\right)
\end{tikzcd}$$

Proof. This is a straightforward exercise. ◊

Lemma 4.5.50. *Let $\mathcal{C}$ and $\mathcal{D}$ be small Reedy categories, let $\mathcal{M}$ be a locally small category, let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $\mathcal{C} \to [\mathcal{D}, \mathcal{M}]$, and let $g : M \to N$ be a morphism in $\mathcal{M}$.*

- *Let $\mathcal{L}$ be the class of morphisms in $[\mathcal{D}, \mathcal{M}]$ that have the Reedy left lifting property with respect to $g : M \to N$. Assuming the relative latching objects $\mathrm{L}_C(X, Y, \varphi)$ exist in $[\mathcal{D}, \mathcal{M}]$ for every object C in $\mathcal{C}$, the relative latching morphisms $\mathrm{L}_C(X, Y, \varphi) \to YC$ are in $\mathcal{L}$ if and only if $\varphi : X \Rightarrow Y$ (regarded as a morphism in $[\mathcal{C} \times \mathcal{D}, \mathcal{M}]$) has the Reedy left lifting property with respect to $g : M \to N$.*

- *Let $\mathcal{R}$ be the class of morphisms in $[\mathcal{D}, \mathcal{M}]$ that have the Reedy right lifting property with respect to $g : M \to N$. Assuming the relative matching objects $\mathrm{M}_C(X, Y, \varphi)$ exist in $[\mathcal{D}, \mathcal{M}]$ for every object C in $\mathcal{C}$, the relative matching morphisms $XC \to \mathrm{M}_C(X, Y, \varphi)$ are in $\mathcal{R}$ if and only if $\varphi : X \Rightarrow Y$ (regarded as a morphism in $[\mathcal{C} \times \mathcal{D}, \mathcal{M}]$) has the Reedy right lifting property with respect to $g : M \to N$.*

Proof. The two claims are formally dual; we will prove the first version.

By enlarging the universe or shrinking $\mathcal{M}$ if necessary, we may assume $\mathcal{M}$ is a small category. The Yoneda embedding $\hbar^\bullet : \mathcal{M} \to [\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is then fully faithful and preserves all colimits that exist in $\mathcal{M}$, and $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$ is a cocomplete locally small category. We may then apply lemma 4.5.27 and replace $\mathcal{M}$ with $[\mathcal{M}, \mathbf{Set}]^{\mathrm{op}}$, i.e. we may assume $\mathcal{M}$ is a cocomplete locally small category.

Let C be an object in $\mathcal{C}$, let $\lambda_C : \mathrm{L}_C(X, Y, \varphi) \to YC$ be the relative latching morphism, and let D be an object in $\mathcal{D}$. It can be shown that the relative latching object $\mathrm{L}_D\big(\mathrm{L}_C(X, Y, \varphi), YC, \lambda_C\big)$ is a the colimit for the commutative diagram in $\mathcal{M}$ shown below:

$$
\begin{tikzcd}
& X(C)(D) & \\
\mathrm{L}_D(XC) \arrow[ur] \arrow[d, "\mathrm{L}_D(\varphi_C)"'] & & \mathrm{L}_C(X)(D) \arrow[ul] \arrow[d, "\mathrm{L}_C(\varphi)_D"] \\
& \mathrm{L}_D\big(\mathrm{L}_C(X)\big) \arrow[ul] \arrow[ur] \arrow[d, "\mathrm{L}_D(\mathrm{L}_C(\varphi))"'] & \\
\mathrm{L}_D(YC) & & \mathrm{L}_C(Y)(D) \\
& \mathrm{L}_D\big(\mathrm{L}_C(Y)\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

Indeed, by remark 4.5.25 and proposition A.6.15, we have the following pushout square in $\mathcal{M}$,

$$
\begin{tikzcd}
\mathrm{L}_D\big(\mathrm{L}_C(X)\big) \arrow[r] \arrow[d, "\mathrm{L}_D(\mathrm{L}_C(\varphi))"'] & \mathrm{L}_D(XC) \arrow[d] \\
\mathrm{L}_D\big(\mathrm{L}_C(X)\big) \arrow[r] & \mathrm{L}_D\big(\mathrm{L}_C(X, Y, \varphi)\big)
\end{tikzcd}
$$

and since (weighted) colimits in $[\mathcal{D}, \mathcal{M}]$ can be computed componentwise, we also have the pushout square shown below:

$$\begin{tikzcd}
\mathrm{L}_C(X)(D) \arrow[r] \arrow[d, "\mathrm{L}_C(\varphi)_D"'] & X(C)(D) \arrow[d] \\
\mathrm{L}_C(Y)(D) \arrow[r] & \mathrm{L}_C(X, Y, \varphi)(D)
\end{tikzcd}$$

On the other hand, by lemma 4.5.49, the following are pushout squares in $\mathcal{M}$,

$$\begin{tikzcd}
\mathrm{L}_D\big(\mathrm{L}_C(X)\big) \arrow[r] \arrow[d] & \mathrm{L}_D(XC) \arrow[d] \\
\mathrm{L}_C(X)(D) \arrow[r] & \mathrm{L}_{(C,D)}(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{L}_D\big(\mathrm{L}_C(Y)\big) \arrow[r] \arrow[d] & \mathrm{L}_D(YC) \arrow[d] \\
\mathrm{L}_C(Y)(D) \arrow[r] & \mathrm{L}_{(C,D)}(Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{L}_{(C,D)}(X) \arrow[r] \arrow[d, "\mathrm{L}_{(C,D)}(\varphi)"'] & X(C)(D) \arrow[d] \\
\mathrm{L}_{(C,D)}(Y) \arrow[r] & \mathrm{L}_{(C,D)}(X, Y, \varphi)
\end{tikzcd}$$

thus,

$$\mathrm{L}_D\big(\mathrm{L}_C(X, Y, \varphi), YC, \lambda_C\big) \cong \mathrm{L}_{(C,D)}(X, Y, \varphi)$$

and moreover, this isomorphism is compatible with the relative latching morphisms. Thus, the relative latching morphisms $\lambda_C : \mathrm{L}_C(X, Y, \varphi) \to YC$ has the Reedy left lifting property with respect to $g : M \to N$ (for every C) if and only if the relative latching morphisms $\mathrm{L}_{(C,D)}(X, Y, \varphi) \to Y(C)(D)$ have the left lifting property with respect to $g : M \to N$ (for every C and D), as claimed. ■

4.6 Reedy model structures

Prerequisites. §§4.1, 4.3, 4.5.

Definition 4.6.1. Let $\mathcal{C}$ be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure.

- A **Reedy weak equivalence** in $[\mathcal{C}, \mathcal{M}]$ is a natural transformation such that all its components are weak equivalences in $\mathcal{M}$.
- A **Reedy cofibration** in $[\mathcal{C}, \mathcal{M}]$ is a natural transformation that has the Reedy left lifting property with respect to all trivial fibrations in $\mathcal{M}$.

- A **Reedy trivial cofibration** in $[\mathcal{C}, \mathcal{M}]$ is a natural transformation that has the Reedy left lifting property with respect to all fibrations in $\mathcal{M}$.
- A **Reedy fibration** in $[\mathcal{C}, \mathcal{M}]$ is a natural transformation that has the Reedy right lifting property with respect to all trivial cofibrations in $\mathcal{M}$.
- A **Reedy trivial fibration** in $[\mathcal{C}, \mathcal{M}]$ is a natural transformation that has the Reedy right lifting property with respect to all cofibrations in $\mathcal{M}$.
- A **Reedy-cofibrant object** in $[\mathcal{C}, \mathcal{M}]$ is a diagram that is Reedy-projective with respect to all trivial fibrations in $\mathcal{M}$.
- A **Reedy-fibrant object** in $[\mathcal{C}, \mathcal{M}]$ is a diagram that is Reedy-injective with respect to all trivial cofibrations in $\mathcal{M}$.

Remark 4.6.2. Since every trivial cofibration is a cofibration, every Reedy trivial fibration is a Reedy fibration; dually, since every trivial fibration is a fibration, every Reedy trivial cofibration is a Reedy cofibration.

Proposition 4.6.3. *Let $\mathcal{C}$ be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure.*

- *If $Y : \mathcal{C} \to \mathcal{M}$ is a Reedy-cofibrant diagram, then, for each object A in $\mathcal{C}$, the object YA is cofibrant in $\mathcal{M}$.*
- *If $X : \mathcal{C} \to \mathcal{M}$ is a Reedy-fibrant diagram, then, for each object A in $\mathcal{C}$, the object XA is fibrant in $\mathcal{M}$.*

Proof. Recalling proposition 4.5.38, this is a special case of corollary 4.5.42. ∎

Proposition 4.6.4. *Let $\mathcal{C}$ be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure.*

- *If $\varphi : X \Rightarrow Y$ is a Reedy cofibration (resp. Reedy trivial cofibration) in $[\mathcal{C}, \mathcal{M}]$, then, for each object A in $\mathcal{C}$, the morphisms $\mathrm{L}_A(\varphi) : \mathrm{L}_A(X) \to \mathrm{L}_A(Y)$ (if it exists) and $\varphi_A : XA \to YA$ are cofibrations (resp trivial cofibrations).*
- *If $\varphi : X \Rightarrow Y$ is a Reedy fibration (resp. Reedy trivial fibration) in $[\mathcal{C}, \mathcal{M}]$, then, for each object A in $\mathcal{C}$, the morphisms $\mathrm{M}_A(\varphi) : \mathrm{M}_A(X) \to \mathrm{M}_A(Y)$ (if it exists) and $\varphi_A : XA \to YA$ are fibrations (resp trivial fibrations).*

Proof. Recalling proposition 4.5.38, this is a special case of corollary 4.5.43. ■

Proposition 4.6.5. *Let $\mathcal{C}$ be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure. For each object S in $\mathcal{M}$, assuming ${}^{S/}\mathcal{M}$ is equipped with the slice model structure:*

(i) *A natural transformation of diagrams $\mathcal{C} \to {}^{S/}\mathcal{M}$ is a Reedy weak equivalence if and only if the underlying natural transformation of diagrams*

(ii) *A natural transformation of diagrams $\mathcal{C} \to {}^{S/}\mathcal{M}$ is a Reedy cofibration (resp. Reedy trivial cofibration, Reedy fibration, Reedy trivial fibration) if underlying natural transformation of diagrams $\mathcal{C} \to \mathcal{M}$ is.*

(iii) *A diagram $\mathcal{C} \to {}^{S/}\mathcal{M}$ is a Reedy-cofibrant (resp. Reedy-fibrant) object if the underlying diagram $\mathcal{C} \to \mathcal{M}$ is.*

Dually, for each object T in $\mathcal{M}$, assuming $\mathcal{M}_{/T}$ is equipped with the slice model structure:

(i) *A natural transformation of diagrams $\mathcal{C} \to \mathcal{M}_{/T}$ is a Reedy weak equivalence if and only if the underlying natural transformation of diagrams*

(ii) *A natural transformation of diagrams $\mathcal{C} \to \mathcal{M}_{/T}$ is a Reedy cofibration (resp. Reedy trivial cofibration, Reedy fibration, Reedy trivial fibration) if underlying natural transformation of diagrams $\mathcal{C} \to \mathcal{M}$ is.*

(iii) *A diagram $\mathcal{C} \to \mathcal{M}_{/T}$ is a Reedy-cofibrant (resp. Reedy-fibrant) object if the underlying diagram $\mathcal{C} \to \mathcal{M}$ is.*

Proof. (i). This is an immediate consequence of the definition of weak equivalence in ${}^{S/}\mathcal{M}$.

(ii). The four subclaims are similar; we will prove the first.

Let $\varphi : X \Rightarrow Y$ be a natural transformation of diagrams $\mathcal{C} \to {}^{S/}\mathcal{M}$ and let $U : {}^{S/}\mathcal{M} \to \mathcal{M}$ be the projection. By proposition 4.5.30 and lemma A.3.7, if $U\varphi : UX \Rightarrow UY$ has the Reedy left lifting property with respect to all trivial fibrations in $\mathcal{M}$, then $\varphi : X \Rightarrow Y$ has the Reedy left lifting property with respect to all trivial fibrations in ${}^{S/}\mathcal{M}$. In other words, if $U\varphi : UX \Rightarrow UY$ is a Reedy cofibration in $[\mathcal{C}, \mathcal{M}]$, then $\varphi : X \Rightarrow Y$ is a Reedy cofibration in $\left[\mathcal{C}, {}^{S/}\mathcal{M}\right]$.

(iii). The two subclaims are similar; we will prove the first.

Let $Y : \mathcal{C} \to \mathcal{M}$ be a diagram (and let proposition 4.5.30 and lemma A.3.7). By proposition 4.5.35 (and lemma A.3.7 again), if UY is Reedy-projective with respect to all trivial fibrations in $\mathcal{M}$, then Y is Reedy-projective with respect to all trivial fibrations in ${}^{S/}\mathcal{M}$. Thus, if UY is a Reedy-cofibrant object in $[\mathcal{C}, \mathcal{M}]$, then Y is a Reedy-cofibrant object in $\left[\mathcal{C}, {}^{S/}\mathcal{M}\right]$. ■

Definition 4.6.6. Let $\mathcal{C}$ be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure. A **sub-Reedy model structure** on $[\mathcal{C}, \mathcal{M}]$ is a model structure that satisfies the following conditions:

- The weak equivalences are the Reedy weak equivalences.
- Every cofibration (resp. trivial cofibration, fibration, trivial fibration) in $[\mathcal{C}, \mathcal{M}]$ is a Reedy cofibration (resp. Reedy trivial cofibration, Reedy fibration, Reedy trivial fibration).
- Every cofibrant (resp. fibrant) object in $[\mathcal{C}, \mathcal{M}]$ is a Reedy-cofibrant (resp. Reedy-fibrant) object.

Proposition 4.6.7. *Let $\mathcal{C}$ be a small Reedy category, let $\mathcal{M}$ is a locally small category with a model structure, and let $\mathcal{N}$ be a homotopically replete full subcategory of $\mathcal{M}$. Given a sub-Reedy model structure on $[\mathcal{C}, \mathcal{M}]$, if $[\mathcal{C}, \mathcal{M}]$ satisfies axiom DC0, then its restriction to $\left[\mathcal{C}, \mathcal{N}\right]$ is a sub-Reedy model structure (with respect to the model structure on $\mathcal{N}$ restricted from $\mathcal{M}$), and $\left[\mathcal{C}, \mathcal{N}\right]$ also satisfies axiom DC0.*

Proof. By proposition 4.1.28, the model structure on $[\mathcal{C}, \mathcal{M}]$ restricted to $\left[\mathcal{C}, \mathcal{N}\right]$ is a model structure, and if $[\mathcal{C}, \mathcal{M}]$ satisfies axiom DC0, then so does $\left[\mathcal{C}, \mathcal{N}\right]$, and moreover the cofibrant (resp. fibrant) objects in $\left[\mathcal{C}, \mathcal{N}\right]$ are cofibrant (resp. fibrant) objects in $[\mathcal{C}, \mathcal{M}]$.

It remains to be shown that the model structure on $\left[\mathcal{C}, \mathcal{N}\right]$ is sub-Reedy if the model structure on $[\mathcal{C}, \mathcal{M}]$ is. Clearly, the Reedy weak equivalences in $\left[\mathcal{C}, \mathcal{N}\right]$ are the Reedy weak equivalences in $[\mathcal{C}, \mathcal{M}]$ that are in $\left[\mathcal{C}, \mathcal{N}\right]$. On the other hand, every Reedy cofibration (resp. Reedy trivial cofibration, etc.) in $[\mathcal{C}, \mathcal{M}]$ that is in $\left[\mathcal{C}, \mathcal{N}\right]$ is also a Reedy cofibration (resp. Reedy trivial cofibration, etc.) in $\left[\mathcal{C}, \mathcal{N}\right]$, so we indeed have a sub-Reedy model structure on $\left[\mathcal{C}, \mathcal{N}\right]$. ■

Definition 4.6.8. A **Reedy-admissible derivable category** is a derivable category $\mathcal{M}$ that satisfies the following additional axioms:

RD0. For any locally finite Reedy category C and any diagram $X : C \to \mathcal{M}$, there exist

- a Reedy trivial cofibration $X \to \hat{X}$ where $\hat{X}$ is a Reedy-fibrant object in $[C, \mathcal{M}]$, and
- a Reedy trivial fibration $\tilde{X} \to X$ where $\tilde{X}$ is a Reedy-cofibrant object in $[C, \mathcal{M}]$.

RD1. For any locally finite Reedy category C with degree function $\deg : \operatorname{ob} C \to \mathbb{N}$ and any object A in C with $\deg A = n + 1$:

- For every morphism $\varphi : X \Rightarrow Y$ in $[C, \mathcal{M}]$ whose restriction is a Reedy trivial cofibration in $\left[C_{\le n}, \mathcal{M}\right]$, the relative latching object $\mathrm{L}_A(X, Y, \varphi)$ exists in $\mathcal{M}$.
- For every morphism $\varphi : X \Rightarrow Y$ in $[C, \mathcal{M}]$ whose restriction is a Reedy trivial fibration in $\left[C_{\le n}, \mathcal{M}\right]$, the relative matching object $\mathrm{M}_A(X, Y, \varphi)$ exists in $\mathcal{M}$.

RD5. For any locally finite Reedy category C, every morphism in $[C, \mathcal{M}]$ can be factored in two ways:

- a Reedy trivial cofibration followed by a Reedy fibration, and
- a Reedy cofibration followed by a Reedy trivial fibration.

Remark 4.6.9. By remark 4.5.25, every model category automatically satisfies axiom RD1; and by lemma 4.1.16, if a model category satisfies axiom RD5, then it also satisfies axiom RD0.

Lemma 4.6.10. *Let C be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a derivable category that satisfies axiom RD1 (resp. axiom CM1*).*

- *A morphism in $[C, \mathcal{M}]$ is a Reedy trivial cofibration if and only if it is both a Reedy cofibration and a Reedy weak equivalence.*
- *A morphism in $[C, \mathcal{M}]$ is a Reedy trivial fibration if and only if it is both a Reedy fibration and a Reedy weak equivalence.*

Proof. The two claims are formally dual; we will prove the first version.

We have already noted that every Reedy trivial cofibration is a Reedy cofibration, and proposition 4.6.4 says that every Reedy trivial cofibration is a componentwise trivial cofibration, hence is a Reedy weak equivalence *a fortiori*.

Now suppose $\varphi : X \Rightarrow Y$ is a natural transformation of diagrams $\mathcal{C} \to \mathcal{M}$ that is both a Reedy cofibration and a Reedy weak equivalence. Choose a degree function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$. Let A be an object in $\mathcal{C}$ and suppose that the relative latching morphism $\mathrm{L}_{A'}(X, Y, \varphi) \to Y A'$ exists and is a trivial cofibration in $\mathcal{M}$ for any object A' in $\mathcal{C}$ with $\deg A' < \deg A$. Then, by lemma 4.5.27, the restriction of $\varphi : X \Rightarrow Y$ in $\left[\mathcal{C}_{\leq \deg A - 1}, \mathcal{M}\right]$ is a Reedy trivial cofibration, so the relative latching object $\mathrm{L}_A(X, Y, \varphi)$ also exists in $\mathcal{M}$. Moreover, we have a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X A \arrow[d] \arrow[ddr, bend left, "\varphi_A"] & \\
\mathrm{L}_A(X, Y, \varphi) \arrow[dr] & \\
& Y A
\end{tikzcd}$$

and corollary 4.5.44 says that the insertion $X A \to \mathrm{L}_A(X, Y, \varphi)$ is a trivial cofibration in $\mathcal{M}$, so by axiom CM2, $\mathrm{L}_A(X, Y, \varphi) \to Y A$ is a weak equivalence. But $\varphi : X \Rightarrow Y$ is a Reedy cofibration by hypothesis, so the relative latching morphism is also a cofibration in $\mathcal{M}$, so this proves that the relative latching morphism is a trivial cofibration. Thus, by induction, we deduce that $\varphi : X \Rightarrow Y$ is indeed a Reedy trivial cofibration in $[\mathcal{C}, \mathcal{M}]$. ■

Lemma 4.6.11. *Let $\mathcal{C}$ be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a derivable category that satisfies axiom RD1 (resp. axiom CM1*).*

- *The Reedy trivial cofibrations in $[\mathcal{C}, \mathcal{M}]$ have the left lifting property with respect to the Reedy fibrations in $[\mathcal{C}, \mathcal{M}]$.*
- *The Reedy cofibrations in $[\mathcal{C}, \mathcal{M}]$ have the left lifting property with respect to the Reedy trivial fibrations in $[\mathcal{C}, \mathcal{M}]$.*

Proof. This is a special case of proposition 4.5.45. ■

Proposition 4.6.12. *Let $\mathcal{C}$ be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a locally small category with a model structure that satisfies axiom RD1 (resp. axiom CM1*). Then there is at most one sub-Reedy model structure on $[\mathcal{C}, \mathcal{M}]$, namely the one defined by the following data:*

- *The weak equivalences are the Reedy weak equivalences.*
- *The cofibrations are the Reedy cofibrations.*
- *The fibrations are the Reedy fibrations.*

This model structure (if it exists) is called the **Reedy model structure** *on $[\mathcal{C}, \mathcal{M}]$. In addition, in the Reedy model structure:*

(i) *The trivial cofibrations (resp. trivial fibrations) are the Reedy trivial cofibrations (resp. Reedy trivial fibrations).*

(ii) *The cofibrant (resp. fibrant) objects are the Reedy-cofibrant (resp. Reedy-fibrant) objects in $[\mathcal{C}, \mathcal{M}]$.*

Proof. Suppose we have a sub-Reedy model structure on $[\mathcal{C}, \mathcal{M}]$. Lemma 4.6.11 implies that every Reedy cofibration (resp. Reedy fibration) in $[\mathcal{C}, \mathcal{M}]$ has the left lifting property (resp. right lifting property) with respect to every trivial fibration (resp. trivial cofibration) in $[\mathcal{C}, \mathcal{M}]$, hence is a cofibration (resp. fibration) in $[\mathcal{C}, \mathcal{M}]$ (by theorem 4.1.12). To identify the trivial cofibrations and the trivial fibrations, we apply lemma 4.6.10; and to identify the cofibrant objects and the fibrant objects, we apply proposition 4.5.46. ∎

Corollary 4.6.13. *Let $\mathcal{M}$ be a derivable category that satisfies axiom RD1. The following are equivalent:*

(i) *$\mathcal{M}$ is a Reedy-admissible derivable category.*

(ii) *For every locally finite Reedy category, $[\mathcal{C}, \mathcal{M}]$ (with the Reedy model structure) is a derivable category where the cofibrant (resp. fibrant) objects are the Reedy-cofibrant (resp. Reedy-fibrant) objects.* ∎

Corollary 4.6.14. *Let $\mathcal{C}$ be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a Reedy-admissible derivable category (resp. a complete and cocomplete model category).*

- *If $C = C_{\rightarrow}$, then the Reedy model structure on $[C, \mathcal{M}]$ is the injective model structure.*
- *If $C = C^{\leftarrow}$, then the Reedy model structure on $[C, \mathcal{M}]$ is the projective model structure.*

Proof. This follows from corollary 4.6.13 and remark 4.5.24. ∎

Theorem 4.6.15 (Kan)**.** *Let C be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a model category with limits and colimits for finite (resp. small) diagrams. The following data define a model structure on $[C, \mathcal{M}]$:*

- *The weak equivalences are the Reedy weak equivalences.*
- *The cofibrations are the Reedy cofibrations.*
- *The fibrations are the Reedy fibrations.*

This model structure is called the **Reedy model structure** *on $[C, \mathcal{M}]$. Moreover, if $\mathcal{M}$ satisfies axiom CM5*, then so does $[C, \mathcal{M}]$.*

Proof. See Theorem 5.2.5 in [Hovey, 1999], or Theorem 15.3.4 in [Hirschhorn, 2003]. □

Corollary 4.6.16. *If $\mathcal{M}$ is a model category, then $\mathcal{M}$ is a Reedy-admissible derivable category.*

Proof. Combine corollary 4.6.13 and theorem 4.6.15. ∎

Proposition 4.6.17. *Let C be a small Reedy category and let $\mathcal{M}$ be a locally small category with a model structure.*

- *If the injective model structure on $[C, \mathcal{M}]$ exists, then the trivial adjunction*
  $$\mathrm{id} \dashv \mathrm{id} : [C, \mathcal{M}] \to [C, \mathcal{M}]$$
  is a Quillen equivalence between the injective model structure and any sub-Reedy model structure.
- *If the projective model structure on $[C, \mathcal{M}]$ exists, then the trivial adjunction*
  $$\mathrm{id} \dashv \mathrm{id} : [C, \mathcal{M}] \to [C, \mathcal{M}]$$
  is a Quillen equivalence between any sub-Reedy model structure and the projective model structure.

Proof. This is an immediate consequence of proposition 4.6.4. ■

Proposition 4.6.18. *Let $\mathcal{M}$ and $\mathcal{N}$ be derivable categories, let $\mathcal{C}$ be a small Reedy category, and let*

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

be a Quillen adjunction.

- *The induced left adjoint $[\mathcal{C}, F] : [\mathcal{C}, \mathcal{N}] \to [\mathcal{C}, \mathcal{M}]$ preserves Reedy cofibrations, Reedy trivial cofibrations, and Reedy-cofibrant objects.*

- *The induced right adjoint $[\mathcal{C}, G] : [\mathcal{C}, \mathcal{M}] \to [\mathcal{C}, \mathcal{N}]$ preserves Reedy fibrations, Reedy trivial fibrations, and Reedy-fibrant objects.*

In particular, the induced adjunction

$$[\mathcal{C}, F] \dashv [\mathcal{C}, G] : [\mathcal{C}, \mathcal{M}] \to [\mathcal{C}, \mathcal{N}]$$

is a Quillen adjunction with respect to the Reedy model structures on $[\mathcal{C}, \mathcal{M}]$ and $[\mathcal{C}, \mathcal{N}]$ (if they exist).

Proof. Apply lemmas 4.5.28 and 4.5.33. ■

Lemma 4.6.19. *Let $\mathcal{C}$ be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a model category with limits and colimits for all finite (resp. small) diagrams.*

- *A diagram $X : \mathcal{C} \to \mathcal{M}$ is Reedy-cofibrant if and only if every latching morphism $\mathrm{L}_A(X) \to XA$ is a cofibration in $\mathcal{M}$.*

- *A diagram $X : \mathcal{C} \to \mathcal{M}$ is Reedy-fibrant if and only if every matching morphism $XA \to \mathrm{M}_A(X)$ is a fibration in $\mathcal{M}$.*

Proof. Let 0 be an initial object in $\mathcal{M}$ and let 1 be a terminal object in $\mathcal{M}$. It is a standard fact that $\Delta 0$ is an initial object in $[\mathcal{C}, \mathcal{M}]$ and $\Delta 1$ is a terminal object in $[\mathcal{C}, \mathcal{M}]$, so the claims follow from lemma 4.1.16, remark 4.5.25, and the observation that the latching morphism $\mathrm{L}_A(\Delta 0) \to 0$ and the matching morphism $1 \to \mathrm{M}_A(\Delta 1)$ are isomorphisms for all objects A in $\mathcal{C}$. ■

Lemma 4.6.20. *Let $\mathcal{C}$ be a locally finite Reedy category (resp. a small Reedy category) and let $\mathcal{M}$ be a model category with limits and colimits for all finite (resp. small) diagrams.*

- *If $Y : \mathcal{C} \to \mathcal{M}$ is a Reedy cofibrant diagram, then, for every object A in $\mathcal{C}$, the latching object $\mathrm{L}_A(Y)$ are cofibrant objects in $\mathcal{M}$.*
- *If $X : \mathcal{C} \to \mathcal{M}$ is a Reedy fibrant diagram, then, for every object A in $\mathcal{C}$, the matching object $\mathrm{M}_A(X)$ are fibrant objects in $\mathcal{M}$.*

Proof. Recalling proposition 4.5.38, this is a special case of corollary 4.5.42. ■

Theorem 4.6.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally finite Reedy categories (resp. small Reedy categories) and let $\mathcal{M}$ be a model category with limits and colimits for finite (resp. small) diagrams. Then the canonical isomorphisms*

$$[\mathcal{C}, [\mathcal{D}, \mathcal{M}]] \cong [\mathcal{C} \times \mathcal{D}, \mathcal{M}] \cong [\mathcal{D}, [\mathcal{C}, \mathcal{M}]]$$

are compatible with the respective (iterated) Reedy model structures.

Proof. It is clear that the above isomorphisms are compatible with the weak equivalences in each model structure. To see that they are also compatible with the Reedy cofibrations and the Reedy fibrations, apply lemma 4.5.50. ■

Definition 4.6.22. Let $\mathcal{C}$ be a Reedy category.

- $\mathcal{C}$ has **cofibrant constants** if, for every object A in $\mathbb{C}$, the latching category $\partial\mathcal{C}_{\to A}$ has at most one connected component.
- $\mathcal{C}$ has **fibrant constants** if, for every object A in $\mathbb{C}$, the matching category $\partial\mathcal{C}^{\leftarrow A}$ has at most one connected component.

Example 4.6.23. Let $\mathcal{C}$ be a Reedy category.

- If $\mathcal{C} = \mathcal{C}^{\leftarrow}$, then $\mathcal{C}$ has cofibrant constants. (In fact, every latching category is empty.)
- If $\mathcal{C} = \mathcal{C}_{\to}$, then $\mathcal{C}$ has fibrant constants. (In fact, every matching category is empty.)

Proposition 4.6.24. *Let $\mathcal{M}$ be a model category, let $\mathcal{C}$ be a finite (resp. small) Reedy category, and assume $\mathcal{M}$ has limits and colimits for all finite (resp. small) diagrams.*

- *If $\mathcal{C}$ has cofibrant constants, then the functor $\Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ is a left Quillen functor.*

- *If $\mathcal{C}$ has fibrant constants, then the functor $\Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ is a right Quillen functor.*

Proof. The two claims are formally dual; we will prove the second version.

If the matching category $\partial\mathcal{C}^{\leftarrow A}$ is empty, then the matching object of ΔX at A is a terminal object in $\mathcal{M}$, so the relative matching morphism of Δf at A is isomorphic to $f : X \to Y$ in this case.

On the other hand, if the matching category $\partial\mathcal{C}^{\leftarrow A}$ of $\mathcal{C}$ has only one connected component, then the matching morphism $X \to \mathrm{M}_A(\Delta X)$ must be an isomorphism, so the relative matching morphism of Δf at A is an isomorphism, hence a (trivial) fibration in particular.

We now conclude that, for any fibration $f : X \to Y$ in $\mathcal{M}$, every relative matching morphism of $\Delta f : \Delta X \to \Delta Y$ is a fibration. Clearly, the functor $\Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ preserves weak equivalences, so this completes the proof that Δ is a right Quillen functor. ■

Theorem 4.6.25 (Hirschhorn)**.** *Let $\mathcal{C}$ be a small Reedy category.*

(i) *$\mathcal{C}$ has cofibrant constants.*

(ii) *$\Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ is a left Quillen functor for all DHK model categories $\mathcal{M}$.*

(iii) *For every cofibrant object X in any DHK model category $\mathcal{M}$, the constant diagram $\Delta X : \mathcal{C} \to \mathcal{M}$ is Reedy cofibrant.*

Dually, the following are equivalent:

(i′) *$\mathcal{C}$ has fibrant constants.*

(ii′) *$\Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ is a right Quillen functor for all DHK model categories $\mathcal{M}$.*

(iii′) *For every fibrant object X in any DHK model category $\mathcal{M}$, the constant diagram $\Delta X : \mathcal{C} \to \mathcal{M}$ is Reedy fibrant.*

Proof. (i) ⇒ (ii). This is proposition 4.6.24.

(ii) ⇒ (iii). Left Quillen functors preserve cofibrant objects, by proposition 4.3.4.

(iii) ⇒ (i). Take $\mathcal{M}$ to be **Set** equipped with the mono–epi model structure,[4] and consider the constant diagram $\Delta 1$. Since 1 is a cofibrant object in $\mathcal{M}$, $\Delta 1$ must be a Reedy cofibrant object in $[\mathbb{C}, \mathcal{M}]$. It is not hard to see that the latching object $\mathrm{L}_A(\Delta 1)$ is the set of connected components of the latching category $\partial \mathcal{C}_{\to A}$, so by lemma 4.6.19, $\partial \mathcal{C}_{\to A}$ has at most one connected component. ∎

Corollary 4.6.26. *Let $\mathcal{M}$ be a DHK model category and let $\mathcal{C}$ be a small Reedy category.*

- *If $\mathcal{C}$ has fibrant constants, then the adjunction $\varinjlim_{\mathcal{C}} \dashv \Delta : \mathcal{M} \to [\mathcal{C}, \mathcal{M}]$ is deformable.*
- *If $\mathcal{C}$ has cofibrant constants, then the adjunction $\Delta \dashv \varprojlim_{\mathcal{C}} : [\mathcal{C}, \mathcal{M}] \to \mathcal{M}$ is deformable.*

Proof. Apply theorem 4.3.13 to the above result. ∎

For the remainder of this section, we follow [Barwick, 2007a] and discuss the functoriality of the Reedy model structure.

Definition 4.6.27. Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories. A **morphism of Reedy categories** $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that sends every morphism in $\mathcal{C}_{\to}$ to $\mathcal{D}_{\to}$ and every morphism in $\mathcal{C}^{\leftarrow}$ to $\mathcal{D}^{\leftarrow}$, or equivalently, a commutative diagram of functors of the form below:

$$\begin{tikzcd}
\mathcal{C}_{\to} \arrow[r, "F_{\to}"] \arrow[d, hook] & \mathcal{D}_{\to} \arrow[d, hook] \\
\mathcal{C} \arrow[r, "F"] & \mathcal{D} \\
\mathcal{C}^{\leftarrow} \arrow[r, "F^{\leftarrow}"'] \arrow[u, hook] & \mathcal{D}^{\leftarrow} \arrow[u, hook]
\end{tikzcd}$$

Lemma 4.6.28. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of Reedy categories. If D is any object in $\mathcal{D}$, then:*

- *There is a unique Reedy category structure on the comma category $(F \downarrow D)$ making the projection $(F \downarrow D) \to \mathcal{C}$ a morphism of Reedy categories.*

[4] See example 4.1.6.

- *There is a unique Reedy category structure on the comma category $(D \downarrow F)$ making the projection $(D \downarrow F) \to \mathcal{C}$ a morphism of Reedy categories.*

Proof. Obvious. ⧫

While it is true that any functor $F : \mathcal{C} \to \mathcal{D}$ induces a homotopical functor $F^* : [\mathcal{D}, \mathcal{M}] \to [\mathcal{C}, \mathcal{M}]$, even if F is a morphism of Reedy categories, F^* need not be either a left Quillen functor or a right Quillen functor. Instead, we must consider the following:

Definition 4.6.29. Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories.

- A **left fibration of Reedy categories** is a morphism $F : \mathcal{C} \to \mathcal{D}$ such that, for any object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has fibrant constants.
- A **right fibration of Reedy categories** is a morphism $F : \mathcal{C} \to \mathcal{D}$ such that, for any object D in $\mathcal{D}$, the comma category $(D \downarrow F)$ has cofibrant constants.

Remark 4.6.30. A Reedy category $\mathcal{C}$ has fibrant (resp. cofibrant) constants if and only if the unique morphism $\mathcal{C} \to \mathbb{1}$ is a left (resp. right) fibration.

Remark 4.6.31. A morphism $F : \mathcal{C} \to \mathcal{D}$ of Reedy categories is a left (resp. right) fibration if and only if $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a right (resp. left) fibration.

Theorem 4.6.32 (Barwick)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism between small Reedy categories. The following are equivalent:*

(i) *The morphism $F : \mathcal{C} \to \mathcal{D}$ is a left fibration of Reedy categories.*

(ii) *For every object D in $\mathcal{D}$ and every object (C, h) in $(F \downarrow D)$, the matching category $\partial(F \downarrow D)^{\leftarrow(C,h)}$ has at most one connected component.*

(iii) *The functor $F^* : [\mathcal{D}, \mathcal{M}] \to [\mathcal{C}, \mathcal{M}]$ is a right Quillen functor for all DHK model categories $\mathcal{M}$.*

Dually, the following are equivalent:

(i′) *The morphism $F : \mathcal{C} \to \mathcal{D}$ is a right fibration of Reedy categories.*

(ii′) *For every object D in $\mathcal{D}$ and every object (C, h) in $(D \downarrow F)$, the latching category $\partial(D \downarrow F)_{\to(C,h)}$ has at most one connected component.*

(iii′) *The functor* $F^* : [\mathcal{D}, \mathcal{M}] \to [\mathcal{C}, \mathcal{M}]$ *is a left Quillen functor for all DHK model categories* $\mathcal{M}$.

Proof. See Lemma 2.6 and Theorem 2.7 in [Barwick, 2007a], or Lemma 3.20 and Theorem 3.22 in [Barwick, 2010]. ■

4.7 Framings and resolutions

Prerequisites. §§1.1, 1.2, 1.5, 2.3, 4.1, 4.2, 4.3, 4.5, 4.6.

In homological algebra, one studies objects in categories without homotopical structure by embedding them in one that does, in such a way that objects in the original category become weakly equivalent to their presentations. The notion of 'resolution' in the sense of Dwyer and Kan [1980c] was invented for similar reasons: though not every model category has a simplicial enrichment, we can still replace objects with homotopically better-behaved simplicial (or cosimplicial) ones. It is also useful to simultaneously discuss the closely related notion of 'framing' introduced by Dwyer, Hirschhorn, and Kan [DHK].

In this section, we follow [Hirschhorn, 2003, Ch. 16].

¶ 4.7.1. Throughout this section, $\mathbf{c}\mathcal{M}$ and $\mathbf{s}\mathcal{M}$ will always be equipped with the Reedy model structure, which exists (by theorem 4.6.15) when $\mathcal{M}$ is a model category.

Proposition 4.7.2. *Let* $\mathcal{M}$ *be a model category.*

- *A cosimplicial object* $B^\bullet$ *in* $\mathcal{M}$ *is Reedy-cofibrant if and only if, for all monomorphisms* $i : Z \to W$ *between finite simplicial sets, the morphism*

$$i \star \mathrm{id}_B : Z \star B \to W \star B$$

induced by i *is a cofibration in* $\mathcal{M}$.

- *A simplicial object* $A_\bullet$ *in* $\mathcal{M}$ *is Reedy-fibrant if and only if, for all monomorphisms* $i : Z \to W$ *between finite simplicial sets, the morphism*

$$\{i, A\} : \{W, A\} \to \{Z, A\}$$

induced by i *is a fibration in* $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

Proposition 1.2.20 implies that the class of monomorphisms between finite simplicial sets is the smallest class of morphisms between finite simplicial sets that contains the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ and is closed under pushout and composition. Thus, by corollary 4.5.43, $i \star \mathrm{id}_B : Z \star B \to W \star B$ is a cofibration in $\mathcal{M}$ for any monomorphism $i : Z \to W$ between finite simplicial sets and any Reedy-cofibrant cosimplicial object $B^\bullet$. Conversely, any such cosimplicial object must be Reedy-cofibrant. ∎

Lemma 4.7.3. *Let $\mathcal{M}$ be any category.*

(i) *There exist adjunctions of the form below,*

$$(-)^0 \dashv \mathrm{cosk}^0 : \mathcal{M} \to \mathbf{c}\mathcal{M}$$
$$\mathrm{sk}_0 \dashv (-)_0 : \mathbf{s}\mathcal{M} \to \mathcal{M}$$

where $(-)^0 : \mathbf{c}\mathcal{M} \to \mathcal{M}$ is the functor that sends a cosimplicial object $A^\bullet$ to the object A^0, and dually, $(-)_0 : \mathbf{s}\mathcal{M} \to \mathcal{M}$ is the functor that sends a simplicial object $A_\bullet$ to the object A_0.

(ii) *Moreover, if $\mathcal{M}$ has finite powers, then $(-)^0 : \mathbf{c}\mathcal{M} \to \mathcal{M}$ has a left adjoint $\mathrm{sk}^0 : \mathcal{M} \to \mathbf{c}\mathcal{M}$; dually, if $\mathcal{M}$ has finite copowers, then $(-)_0 : \mathcal{M} \to \mathbf{s}\mathcal{M}$ has a right adjoint $\mathrm{cosk}_0 : \mathcal{M} \to \mathbf{s}\mathcal{M}$.*

Proof. It is straightforward to verify that the following formulae work,

$$\mathrm{sk}_0(A)_n = A \qquad\qquad \mathrm{cosk}_0(A)_n = [n] \pitchfork A$$
$$\mathrm{sk}^0(A)^n = [n] \odot A \qquad\qquad \mathrm{cosk}^0(A)^n = A$$

where $[n] \pitchfork A$ is the $(n+1)$-fold power of A, and $[n] \odot A$ is the $(n+1)$-fold copower of A. ∎

Definition 4.7.4. Let A be an object in a derivable category $\mathcal{M}$.

- A **cosimplicial resolution** of A is a Reedy-cofibrant replacement in $\mathbf{c}\mathcal{M}$ for the cosimplicial object $\mathrm{cosk}^0(A)$.
- A **simplicial resolution** of A is a Reedy-fibrant replacement in $\mathbf{s}\mathcal{M}$ for the simplicial object $\mathrm{sk}_0(A)$.

Remark 4.7.5. Proposition 4.7.2 implies that (in the case of a model category) the above definition is equivalent to the original definition of 'resolution' given by Dwyer and Kan [1980c].

Definition 4.7.6.

- A **cosimplicially resolvable category** is a derivable category $\mathcal{M}$ that satisfies the following additional axioms:
  - For every object B in $\mathcal{M}$, there is a cosimplicial resolution $\left(\tilde{B}^\bullet, p^\bullet\right)$ of B where $p^\bullet : \tilde{B}^\bullet \to \operatorname{cosk}^0(A)$ is a degreewise trivial fibration.
  - For every weak equivalence $w : A \to B$ in $\mathcal{M}$, there exist a Reedy cofibration $u^\bullet : \operatorname{cosk}^0(A) \to \tilde{B}^\bullet$ and a degreewise trivial fibration $v^\bullet : \tilde{B}^\bullet \to \operatorname{cosk}^0(B)$ such that $v^\bullet \circ u^\bullet = \operatorname{cosk}^0(w)$.
- A **simplicially resolvable category** is a derivable category $\mathcal{M}$ that satisfies the following additional axioms:
  - For every object A in $\mathcal{M}$, there is a simplicial resolution $\left(\hat{A}_\bullet, i_\bullet\right)$ of A where $i_\bullet : \operatorname{sk}_0(A) \to \hat{A}_\bullet$ is a Reedy trivial cofibration.
  - For every weak equivalence $w : A \to B$ in $\mathcal{M}$, there exist a degreewise trivial cofibration $u_\bullet : \operatorname{sk}_0(A) \to \hat{A}_\bullet$ and a Reedy fibration $v_\bullet : \hat{A}_\bullet \to \operatorname{sk}_0(B)$ such that $v_\bullet \circ u_\bullet = \operatorname{sk}_0(w)$.
- A **resolvable category** is a derivable category that is both a cosimplicially resolvable category and a simplicially resolvable category.

Proposition 4.7.7. *Let $\mathcal{M}$ be a derivable category.*

(i) *If $\mathcal{M}$ is a Reedy-admissible derivable category, then $\mathcal{M}$ is a resolvable category.*

(ii) *If $\mathcal{M}$ is a model category, then $\mathcal{M}$ is a resolvable category.*

(iii) *If $\mathcal{M}$ is a DHK model category, then cosimplicial resolutions and simplicial resolutions can both be chosen functorially.*

Proof. This follows from proposition 4.1.24 and theorem 4.6.15. ■

Proposition 4.7.8. *Let $\mathcal{M}$ be a resolvable category and let $\mathcal{N}$ be a homotopically replete full subcategory of $\mathcal{M}$.*

- *If $\mathcal{M}$ is a cosimplicially resolvable category, then so is $\mathcal{N}$ (with the model structure on $\mathcal{N}$ inherited from $\mathcal{M}$).*
- *If $\mathcal{M}$ is a simplicially resolvable category, then so is $\mathcal{N}$ (with the model structure on $\mathcal{N}$ inherited from $\mathcal{M}$).*
- *If $\mathcal{M}$ is a resolvable category, then so is $\mathcal{N}$ (with the model structure on $\mathcal{N}$ inherited from $\mathcal{M}$).*

Proof. The first two claims are formally dual and the third claim is their conjunction; we will prove the first claim.

By the proof of proposition 4.6.7, Reedy-cofibrant objects (resp. Reedy cofibrations) in $\mathbf{c}\mathcal{M}$ are also Reedy-cofibrant objects (resp. Reedy cofibrations) in $\mathbf{c}\mathcal{N}$, so $\mathcal{N}$ is a cosimplicially resolvable category if $\mathcal{M}$ is. ■

Proposition 4.7.9. *Let C be an object in a DHK model category $\mathcal{M}$.*

- *The full subcategory of the slice category $\mathbf{c}\mathcal{M}_{/\operatorname{cosk}^0(C)}$ spanned by the cosimplicial resolutions of C is homotopically contractible.*[5]
- *The full subcategory of the slice category ${}^{\operatorname{sk}_0(C)/}\mathbf{s}\mathcal{M}$ spanned by the simplicial resolutions of A is homotopically contractible.*

Proof. This follows from proposition 4.1.26 and theorem 4.6.15. ■

Lemma 4.7.10. *Let $\mathcal{M}$ be a model category.*

- $\operatorname{cosk}_0 : \mathcal{M} \to \mathbf{s}\mathcal{M}$ *is a right Quillen functor.*
- $\operatorname{sk}^0 : \mathcal{M} \to \mathbf{c}\mathcal{M}$ *is a left Quillen functor.*

Proof. The two claims are formally dual; we will prove the first version.

By proposition 4.3.2, it is enough to show that $(-)_0 : \mathbf{s}\mathcal{M} \to \mathcal{M}$ preserves cofibrations and trivial cofibrations. However, the latching category at $[0]$ is empty, so if $f : A_\bullet \to B_\bullet$ is a Reedy cofibration, then $f_0 : A_0 \to B_0$ must be a cofibration in $\mathcal{M}$. Since $(-)_0$ preserves weak equivalences, it follows that $(-)_0$ preserves trivial cofibrations. ■

[5] See definition 3.1.30.

Lemma 4.7.11. *Let $\mathcal{M}$ be a model category.*

- *There is a unique natural transformation $\Delta : \mathrm{sk}_0 \Rightarrow \mathrm{cosk}_0$ such that $\varepsilon_A \circ (\Delta_A)_0 \circ \eta_A = \mathrm{id}_A$ for all objects A in $\mathcal{M}$, where $\eta_A : A \to \mathrm{sk}_0(A)_0$ and $\varepsilon_A : \mathrm{cosk}_0(A)_0 \to A$ are the components of the unit and counit of the respective adjunctions.*

- *There is a unique natural transformation $\nabla : \mathrm{sk}^0 \Rightarrow \mathrm{cosk}^0$ such that $\varepsilon_A \circ (\nabla_A)_0 \circ \eta_A = \mathrm{id}_A$ for all objects A in $\mathcal{M}$, where $\eta_A : A \to \mathrm{sk}^0(A)^0$ and $\varepsilon_A : \mathrm{cosk}^0(A)^0 \to A$ are the components of the unit and counit of the respective adjunctions.*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to check that η_A is an isomorphism, so $\varepsilon_A \circ (\Delta_A)_0$ is uniquely determined. The universal property of $\mathrm{cosk}_0(A)$ implies $\Delta_A : \mathrm{sk}_0(A) \to \mathrm{cosk}_0(A)$ is determined by its adjoint transpose $\varepsilon_A \circ (\Delta_A)_0 : \mathrm{sk}_0(A)_0 \to A$, so Δ_A is also uniquely determined. ∎

Definition 4.7.12. Let A be an object in a model category $\mathcal{M}$.

- A **cosimplicial frame** on A is a pair $(\tilde{A}^\bullet, p^\bullet)$, where $\tilde{A}^\bullet$ is a cosimplicial object in $\mathcal{M}$, $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a Reedy weak equivalence with $p^0 : \tilde{A}^0 \to \mathrm{cosk}^0(A)^0$ an isomorphism, and $\tilde{A}^\bullet$ is Reedy-cofibrant if A is cofibrant.

- A **simplicial frame** on A is a pair $(\hat{A}_\bullet, i_\bullet)$, where $\hat{A}_\bullet$ is a simplicial object in $\mathcal{M}$, $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a Reedy weak equivalence with $i_0 : \mathrm{sk}_0(A)_0 \to \hat{A}_0$ an isomorphism, and $\hat{A}_\bullet$ is Reedy-fibrant if A is fibrant.

- A **left frame** on A is a tuple $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$, where $\tilde{A}^\bullet$ is a cosimplicial object in $\mathcal{M}$, $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a Reedy weak equivalence with $p^0 : \tilde{A}^0 \to \mathrm{cosk}^0(A)^0$ an isomorphism, $i^\bullet$ is a Reedy cofibration, and $p^\bullet \circ i^\bullet = \nabla_A$.

- A **right frame** on A is a tuple $(\hat{A}_\bullet, i_\bullet, p_\bullet)$, where $\hat{A}_\bullet$ is a simplicial object in $\mathcal{M}$, $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a Reedy weak equivalence with $i_0 : \mathrm{sk}_0(A)_0 \to \hat{A}_0$ an isomorphism, $p_\bullet$ is a Reedy fibration, and $p_\bullet \circ i_\bullet = \Delta_A$.

Proposition 4.7.13. *Let A be an object in a model category $\mathcal{M}$.*

(i) *If $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ is a left frame on A, then $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A.*

(ii) *If A is cofibrant, then every cosimplicial frame on A is a cosimplicial resolution of A.*

(iii) *If $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial resolution of A, then $\tilde{A}^\bullet$ is (the underlying cosimplicial object of) a cosimplicial frame on $\tilde{A}^0$, and $\left(\tilde{A}^0, p^0\right)$ is (isomorphic to) a cofibrant replacement for A.*

Dually:

(i′) *If $\left(\hat{A}_\bullet, i_\bullet, p_\bullet\right)$ is a right frame on A, then $\left(\hat{A}_\bullet, i_\bullet\right)$ is a simplicial frame on A.*

(ii′) *If A is fibrant, then every simplicial frame on A is a simplicial resolution of A.*

(iii′) *If $\left(\hat{A}_\bullet, i_\bullet\right)$ is a simplicial resolution of A, then $\hat{A}_\bullet$ is (the underlying simplicial object of) a simplicial frame on $\hat{A}_0$, and $\left(\hat{A}_0, i_0\right)$ is (isomorphic to) a fibrant replacement for A.*

Proof. (i). Suppose $\left(\tilde{A}^\bullet, i^\bullet, p^\bullet\right)$ is a left frame on A. Lemma 4.7.10 implies that $\operatorname{cosk}^0(A)$ is Reedy-cofibrant when A is cofibrant, so $\tilde{A}^\bullet$ is Reedy-cofibrant when A is cofibrant. Thus $\left(\tilde{A}^\bullet, p^\bullet\right)$ is indeed a cosimplicial frame on A.

(ii). If A is cofibrant and $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial frame on A, then $\tilde{A}^\bullet$ is Reedy-cofibrant, and hence $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a Reedy-cofibrant replacement for $\operatorname{cosk}^0(A)$.

(iii). Let $q^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0\left(\tilde{A}^0\right)$ be the component of the adjunction unit at $\tilde{A}^\bullet$. Since $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ implies $q^\bullet$ is also a Reedy weak equivalence. Now, $\tilde{A}^\bullet$ is Reedy-cofibrant by definition, it follows that $\left(\tilde{A}^\bullet, q^\bullet\right)$ is a cosimplicial frame on $\tilde{A}^0$.

Finally, we note that proposition 4.7.2 implies that $\tilde{A}^0$ is a cofibrant object in $\mathcal{M}$, and $p^0 : \tilde{A}^0 \to \operatorname{cosk}^0(A)^0$ is a weak equivalence by definition, so $\left(\tilde{A}^0, p^0\right)$ is (isomorphic to) a cofibrant replacement for A. ■

Remark 4.7.14. The notions of 'left frame' and 'right frame' are originally due to Hovey [1999, §5.2], but he calls them 'cosimplicial frame' and 'simplicial frame' and does not give a name to the weaker notion. It is explained in loc. cit. that a left (resp. right) frame on A is a cosimplicial (resp. simplicial) frame that is almost Reedy-cofibrant (resp. Reedy-fibrant), in the sense that all but one of its latching (resp. matching) morphisms are cofibrations (resp. fibrations). One consequence of this is given in proposition 4.7.21.

Definition 4.7.15. Let $\mathcal{M}$ be a model category.

- A **left framing** for $\mathcal{M}$ is a tuple $(Q^\bullet, i^\bullet, p^\bullet)$, where $Q^\bullet : \mathcal{M} \to \mathbf{c}\mathcal{M}$ is a functor, $i^\bullet : \mathrm{sk}^0 \Rightarrow Q^\bullet$ and $p^\bullet : Q^\bullet \Rightarrow \mathrm{cosk}^0$ are natural transformations, and $\left(Q^\bullet A, (i_A)^\bullet, (p_A)^\bullet\right)$ is a left frame for all *cofibrant* objects A in $\mathcal{M}$.

- A **right framing** for $\mathcal{M}$ is a tuple $\left(R_\bullet, i_\bullet, p_\bullet\right)$, where $R_\bullet : \mathcal{M} \to \mathbf{s}\mathcal{M}$ is a functor, $i_\bullet : \mathrm{sk}_0 \Rightarrow R_\bullet$ and $p_\bullet : R_\bullet \Rightarrow \mathrm{cosk}_0$ are natural transformations, and $\left(R_\bullet A, (i_A)_\bullet, (p_A)_\bullet\right)$ is a right frame for all *fibrant* objects A in $\mathcal{M}$.

A **framed model category** is a model category equipped with a left framing and a right framing.

Theorem 4.7.16. *Let $\mathcal{M}$ be a model category.*

(i) *On each object A in $\mathcal{M}$, there exist a left frame $\left(\tilde{A}^\bullet, i^\bullet, p^\bullet\right)$ and a right frame $\left(\hat{A}_\bullet, i_\bullet, p_\bullet\right)$ such that $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a trivial Reedy fibration and $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a trivial Reedy cofibration.*

(ii) *If $\mathcal{M}$ satisfies axiom CM5*, then the left and right frames in (i) can be chosen functorially; in particular, left and right framings for $\mathcal{M}$ exist.*

Proof. See Theorem 5.2.8 in [Hovey, 1999]. ☐

Theorem 4.7.17. *Let A be an object in a DHK model category $\mathcal{M}$.*

- *The nerve of the full subcategory of the slice category $\mathbf{c}\mathcal{M}_{/\mathrm{cosk}^0(A)}$ spanned by the cosimplicial frames on A is weakly contractible.*

- *The nerve of the full subcategory of the slice category ${}^{\mathrm{sk}^0(A)/}\mathbf{s}\mathcal{A}$ spanned by the simplicial frames on A is weakly contractible.*

Proof. See Theorem 16.6.18 in [Hirschhorn, 2003]. ☐

Proposition 4.7.18. *Let $\mathcal{M}$ be a model category.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then the morphism*
$$\Lambda^n_k \star \tilde{A} \to \Delta^n \star \tilde{A}$$
induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial cofibration in $\mathcal{M}$.

- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then the morphism*
$$\{\Delta^n, \hat{B}\} \to \{\Lambda^n_k, \hat{B}\}$$
induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

First, note that proposition 4.7.2 implies that $\Lambda^n_k \star \tilde{A} \to \Delta^n \star \tilde{A}$ is a cofibration in $\mathcal{M}$. Since $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ implies that the morphism $\Delta^n \star \tilde{A} \to \Delta^0 \star \tilde{A}$ is a weak equivalence in $\mathcal{M}$ for all $n \geq 0$. It is clear that $- \star \tilde{A}$ preserves finite colimits of finite simplicial sets, so we may then apply lemma 1.5.20. □

Corollary 4.7.19. *Let $\mathcal{M}$ be a model category and let $i : Z \to W$ be an anodyne extension between finite simplicial sets.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then the morphism*
$$i \star \mathrm{id}_{\tilde{A}} : Z \star \tilde{A} \to W \star \tilde{A}$$
induced by $i : Z \to W$ is a trivial cofibration in $\mathcal{M}$.

- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then the morphism*
$$\{i, \hat{B}\} : \{W, \hat{B}\} \to \{Z, \hat{B}\}$$
induced by $i : Z \to W$ is a trivial fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

By proposition 1.4.12, the class of anodyne extensions between finite simplicial sets is generated by the boundary inclusions under composition, pushouts, and retracts. We already know that $- \star \tilde{A}$ sends horn inclusions to trivial cofibrations in $\mathcal{M}$, and it is clear that $- \star \tilde{A}$ preserves composition, pushouts, and retracts, so theorem 4.1.12 and proposition A.3.17 imply that $i \star \mathrm{id}_{\tilde{A}}$ is a trivial cofibration in $\mathcal{M}$. ■

Cosimplicial frames and left frames (resp. simplicial frames and right frames) should be regarded as higher cylinder objects (resp. higher path objects). We can make this precise in two different ways:

Proposition 4.7.20. *Let $\mathcal{M}$ be a model category.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$ is a cylinder object for $\tilde{A}^0$ (and hence, isomorphic to a cylinder object for A).*
- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then $(\hat{B}_1, d_1, d_0, s_0)$ is a path object for $\hat{B}_0$ (and hence, isomorphic to a path object for B).*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism $(\!(\delta^1, \delta^0)\!) : \tilde{A}^0 + \tilde{A}^0 \to \tilde{A}^1$ is isomorphic to the morphism $\partial\Delta^1 \star \tilde{A} \to \Delta^1 \star \tilde{A}$ induced by $\partial\Delta^1 \hookrightarrow \Delta^1$, and the latter is a cofibration by proposition 4.7.2. On the other hand, the morphism $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ is a retraction for $\delta^1 : \tilde{A}^0 \to \tilde{A}^1$, and proposition 4.7.18 implies the latter is (isomorphic to) a trivial cofibration; thus, by the 2-out-of-3 property of weak equivalences, $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ must be a weak equivalence. ■

Proposition 4.7.21. *Let $\mathcal{M}$ be a model category.*

- *If $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ is a left frame on an object in $\mathcal{M}$, then $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$ is a cylinder object for $\tilde{A}^0$.*
- *If $(\tilde{B}_\bullet, i_\bullet, p_\bullet)$ is a right frame on an object in $\mathcal{M}$, then $(\tilde{B}_1, d_1, d_0, s_0)$ is a path object for $\hat{B}_0$.*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism $(\!(\delta^1, \delta^0)\!) : \tilde{A}^0 + \tilde{A}^0 \to \tilde{A}^1$ is isomorphic to the relative latching morphism at $[1]$ for $i^\bullet : \mathrm{sk}^0(A) \to \tilde{A}^\bullet$, and the latter is a Reedy cofibration, so $(\!(\delta^1, \delta^0)\!)$ is a cofibration in $\mathcal{M}$. On the other hand, we have the following commutative diagram,

$$\begin{tikzcd} \tilde{A}^1 \arrow[r, "p^1"] \arrow[d, "\sigma^0"'] & \operatorname{cosk}^0(A)^1 \arrow[d, "\sigma^0"] \\ \tilde{A}^0 \arrow[r, "p^0"'] & \operatorname{cosk}^0(A)^0 \end{tikzcd}$$

where p^0 and p^1 are weak equivalences in $\mathcal{M}$. Since $\sigma^0 : \operatorname{cosk}^0(A)^1 \to \operatorname{cosk}(A)^0$ is an isomorphism (and so a weak equivalence *a fortiori*), the 2-out-of-3 property of weak equivalences implies $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ is also a weak equivalence. ∎

Proposition 4.7.22. *Let $\mathcal{M}$ be a model category and let X be a finite simplicial set.*

- *If $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on a cofibrant object A in $\mathcal{M}$, then the cosimplicial object $(X \odot \tilde{A})^\bullet$ is (the object part of) a cosimplicial frame on $X \star \tilde{A}$.*
- *If $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on a fibrant object B in $\mathcal{M}$, then the simplicial object $(X \pitchfork \hat{B})_\bullet$ is (the object part of) a simplicial frame on $\{X, \hat{B}\}$.*

Proof. The two claims are formally dual; we will prove the first version.

To show that the cosimplicial object $(X \odot \tilde{A})^\bullet$ is indeed (the object part of) a cosimplicial frame on $X \star \tilde{A}$, it suffices to verify that $(X \odot \tilde{A})^\bullet$ is Reedy-cofibrant and all its codegeneracy operators are weak equivalences: the latter condition ensures that the counit component $(X \odot \tilde{A})^\bullet \to \operatorname{cosk}^0((X \odot \tilde{A})^0)$ is a Reedy weak equivalence, and we know that $(X \odot \tilde{A})^0 \cong X \star \tilde{A}$. By definition, we have the following natural bijections:

$$\begin{aligned} \mathcal{M}(Z \star (X \odot \tilde{A}), B) &\cong \mathbf{sSet}(Z, \mathcal{M}((X \odot \tilde{A})^\bullet, B)) \\ &\cong \mathbf{sSet}(Z, [X, \mathcal{M}(\tilde{A}^\bullet, B)]) \\ &\cong \mathbf{sSet}(Z \times X, \mathcal{M}(\tilde{A}^\bullet, B)) \\ &\cong \mathcal{M}((Z \times X) \star \tilde{A}, B) \end{aligned}$$

Since $i \times \mathrm{id}_X : Z \times X \to W \times X$ is a monomorphism between finite simplicial sets when $i : Z \to W$ is, we may then use proposition 4.7.2 to deduce that $(X \odot \tilde{A})^\bullet$ is indeed Reedy-cofibrant.

It remains to be shown that the codegeneracy operators of $(X \odot \tilde{A})^\bullet$ are weak equivalences. The cosimplicial identities and axiom CM2 implies it is enough to show that each coface operator $\delta_n^i : \left(X \odot \tilde{A}\right)^{n-1} \to \left(X \odot \tilde{A}\right)^n$ is a weak equivalence. Since the unique morphism $\Delta^n \to \Delta^0$ is a (weak) homotopy equivalence, we can use proposition 1.5.15 and the 2-out-of-3 property of weak homotopy equivalences to deduce that, for each $\delta_n^i : \Delta^{n-1} \to \Delta^n$, the induced morphism $\delta_n^i \times \mathrm{id}_X : \Delta^0 \times X \to \Delta^n \times X$ is a weak homotopy equivalence. Proposition 1.5.10 then says that $\delta_n^i \times \mathrm{id}_X$ is an anodyne extension, so by corollary 4.7.19, the induced morphism $\left(\Delta^{n-1} \times X\right) \star \tilde{A} \to (\Delta^n \times X) \star \tilde{A}$ is a trivial cofibration in $\mathcal{M}$. Thus, every coface operator $\left(X \odot \tilde{A}\right)^0 \to \left(X \odot \tilde{A}\right)^n$ is a weak equivalence in $\mathcal{M}$. ■

4.8 Derived hom-spaces

Prerequisites. §§1.5, 1.6, 3.1, 3.3, 4.1, 4.3, 4.5, 4.6, 4.7.

Given a cofibrant object A and a fibrant object B in a model category $\mathcal{M}$, there ought to be a space of morphisms $A \to B$, at least well-defined up to weak equivalence, such that the set of connected components is in natural bijection with the hom-set $\operatorname{Ho}\mathcal{M}(A, B)$, while homotopy classes of paths correspond to homotopy classes of homotopies of morphisms $A \to B$, and so on. For this, we will use the notion of ‘resolution’ introduced in the previous section.

Definition 4.8.1. Let $\mathcal{M}$ be a category with weak equivalences.

- A **weakly constant cosimplicial object** in $\mathcal{M}$ is a cosimplicial object in $\mathcal{M}$ such that every coface and codegeneracy operator is a weak equivalence in $\mathcal{M}$. We write $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ for the full subcategory of $\mathbf{c}\mathcal{M}$ spanned by the weakly constant cosimplicial objects in $\mathcal{M}$.

- A **weakly constant simplicial object** in $\mathcal{M}$ is a simplicial object in $\mathcal{M}$ such that every face and degeneracy operator is a weak equivalence in $\mathcal{M}$. We write $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ for the full subcategory of $\mathbf{s}\mathcal{M}$ spanned by the weakly constant simplicial objects in $\mathcal{M}$.

Lemma 4.8.2. *Let $\mathcal{M}$ be a category with weak equivalences.*

- *A cosimplicial object $A^\bullet$ in $\mathcal{M}$ is weakly constant if and only if $d^0 : A^n \to A^{n+1}$ is a weak equivalence in $\mathcal{M}$ for every natural number n.*

- *A simplicial object $B_\bullet$ in $\mathcal{M}$ is weakly constant if and only if $d_0 : B_{n+1} \to B_n$ is a weak equivalence in $\mathcal{M}$ for every natural number n.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $d^0 : A^n \to A^{n+1}$ is a weak equivalence. Then, by the cosimplicial identities (theorem 1.1.4), $d^0 \circ s^0 = \mathrm{id}_{A^n}$, so by the 2-out-of-3 property of weak equivalences in $\mathcal{M}$, $s^0 : A^{n+1} \to A^n$ is also a weak equivalence in $\mathcal{M}$. But $d^1 \circ s^0 = \mathrm{id}_{A^n}$, so $d^1 : A^n \to A^{n+1}$ is a weak equivalence in $\mathcal{M}$, etc. ■

Lemma 4.8.3. *Let $\mathcal{M}$ be a category with weak equivalences.*

- *Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let $p^\bullet : A^\bullet \to \operatorname{cosk}^0\left(A^0\right)$ be the component of the adjunction unit at $A^\bullet$. Then $A^\bullet$ is a weakly constant cosimplicial object in $\mathcal{M}$ if and only if the morphism $p^\bullet : A^\bullet \to \operatorname{cosk}^0\left(A^0\right)$ is a Reedy weak equivalence.*

- *Let $B_\bullet$ be a simplicial object in $\mathcal{M}$ and let $i_\bullet : \operatorname{sk}_0\left(B_0\right) \to B$ be the component of the adjunction counit at $B_\bullet$. Then $B_\bullet$ is a weakly constant simplicial object in $\mathcal{M}$ if and only if the morphism $i_\bullet : \operatorname{sk}_0\left(B_0\right) \to B_\bullet$ is a Reedy weak equivalence.*

Proof. This is a straightforward exercise in using the 2-out-of-3 property of weak equivalences. ◇

Definition 4.8.4. Let $\mathcal{M}$ be a derivable category.

- A **cosimplicial resolution** in $\mathcal{M}$ is a cosimplicial object $\tilde{A}^\bullet$ in $\mathcal{M}$ for which there exist an object A and a morphism $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ such that $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial resolution on A. We write $\mathbf{c}_{\mathrm{r}}\mathcal{M}$ for the full subcategory of $\mathbf{c}\mathcal{M}$ spanned by the cosimplicial resolutions in $\mathcal{M}$.

- A **simplicial resolution** in $\mathcal{M}$ is a simplicial object $\hat{B}_\bullet$ in $\mathcal{M}$ for which there exist an object B and a morphism $i_\bullet : \operatorname{sk}_0(B) \to \hat{B}_\bullet$ such that $\left(\hat{B}_\bullet, i_\bullet\right)$ is a simplicial resolution on B. We write $\mathbf{s}_{\mathrm{r}}\mathcal{M}$ for the full subcategory of $\mathbf{s}\mathcal{M}$ spanned by the simplicial resolutions in $\mathcal{M}$.

Lemma 4.8.5. *Let $\mathcal{M}$ be a derivable category. Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let $p^\bullet : A^\bullet \to \operatorname{cosk}^0\left(A^0\right)$ be the component of the adjunction unit at $A^\bullet$.*

(i) *$A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ if and only if $(A^\bullet, p^\bullet)$ is a cosimplicial resolution of A^0.*

(ii) *$A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ if and only if $A^\bullet$ is Reedy-cofibrant and weakly constant.*

Dually, let $B_\bullet$ be a simplicial object in $\mathcal{M}$ and let $i_\bullet : \mathrm{sk}_0(B_0) \to B$ be the component of the adjunction counit at $B_\bullet$.

(i′) *$B_\bullet$ is a simplicial resolution in $\mathcal{M}$ if and only if $(B_\bullet, i_\bullet)$ is a simplicial resolution of B_0.*

(ii′) *$B_\bullet$ is a simplicial resolution in $\mathcal{M}$ if and only if $B_\bullet$ is Reedy-fibrant and weakly constant.*

Proof. These are straightforward consequences of the definitions and proposition 4.7.13. ◆

Lemma 4.8.6. *Let $\mathcal{M}$ be a derivable category.*

- *If $\mathcal{M}$ is a cosimplicially resolvable category, then there is a left deformation retract of $\mathbf{c}_\mathrm{w}\mathcal{M}$ of the form $(\mathbf{c}_\mathrm{r}\mathcal{M}, Q^\bullet, p^\bullet)$.*
- *If $\mathcal{M}$ is a simplicially resolvable category, then there is a right deformation retract of $\mathbf{s}_\mathrm{w}\mathcal{M}$ of the form $(\mathbf{s}_\mathrm{r}\mathcal{M}, R_\bullet, i_\bullet)$.*

Proof. This is a straightforward matter of unwinding the definitions. ◆

Proposition 4.8.7. *Let $\mathcal{M}$ be a homotopical category.*

- *The following adjunction is an adjoint homotopical equivalence of homotopical categories:*

$$(-)^0 \dashv \mathrm{cosk}^0 : \mathcal{M} \to \mathbf{c}_\mathrm{w}\mathcal{M}$$

  In particular, we have an adjoint equivalence of homotopy categories:

$$\mathrm{Ho}\,(-)^0 \dashv \mathrm{Ho}\,\mathrm{cosk}^0 : \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathbf{c}_\mathrm{w}\mathcal{M}$$

- *The following adjunction is an adjoint homotopical equivalence of homotopical categories:*

$$\mathrm{sk}_0 \dashv (-)_0 : \mathbf{s}_\mathrm{w}\mathcal{M} \to \mathcal{M}$$

  In particular, we have an adjoint equivalence of homotopy categories:

$$\mathrm{Ho}\,\mathrm{sk}_0 \dashv \mathrm{Ho}\,(-)_0 : \mathrm{Ho}\,\mathbf{s}_\mathrm{w}\mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$$

Proof. The two claims are formally dual; we will prove the first version.

First of all, we note that $\operatorname{cosk}^0(A)$ is a weakly constant cosimplicial object in $\mathcal{M}$ for every object A in $\mathcal{M}$, so the adjunction in lemma 4.7.3 restricts to an adjunction between $\mathcal{M}$ and $\mathbf{c}_{\mathrm{w}}\mathcal{M}$. It is clear that the adjunction counit is a natural isomorphism, and lemma 4.8.3 says that the adjunction unit is a natural weak equivalence, so we indeed have an adjoint homotopical equivalence of homotopical categories. Finally, we use proposition 3.1.29 to deduce the claim about homotopy categories. ■

Definition 4.8.8. Let $\mathcal{M}$ be locally small category.

- Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let B be an object in $\mathcal{M}$. The **left hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\big(\mathcal{H}om_{\mathcal{M}}(A, B)\big)_n = \mathcal{M}(A^n, B)$$
- Let A be an object in $\mathcal{M}$ and let $B_\bullet$ be a simplicial object in $\mathcal{M}$. The **right hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\big(\mathcal{H}om_{\mathcal{M}}(A, B)\big)_m = \mathcal{M}\big(A, B_m\big)$$
- Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let $B_\bullet$ be a simplicial object in $\mathcal{M}$. The **total hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\big(\mathcal{H}om_{\mathcal{M}}(A, B)\big)_k = \mathcal{M}\big(A_k, B_k\big)$$

Remark 4.8.9. Let $\mathcal{M}$ be a locally small category.

- For each pair (A, B) of objects in $\mathcal{M}$, we have the following natural isomorphisms:
$$\begin{aligned} \mathcal{H}om_{\mathcal{M}}\big(\operatorname{cosk}^0(A), B\big) &\cong \operatorname{disc}\mathcal{M}(A, B) \\ \mathcal{H}om_{\mathcal{M}}\big(A, \operatorname{sk}_0(B)\big) &\cong \operatorname{disc}\mathcal{M}(A, B) \end{aligned}$$
- For each cosimplicial object $A^\bullet$ in $\mathcal{M}$ and each object B in $\mathcal{M}$, we have the following natural isomorphism:
$$\mathcal{H}om_{\mathcal{M}}\big(A, \operatorname{sk}_0(B)\big) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$

- For each object A in $\mathcal{M}$ and each simplicial object $B_\bullet$ in $\mathcal{M}$, we have the following natural isomorphism:

$$\mathcal{H}om_{\mathcal{M}}\big(\operatorname{cosk}^0(A), B\big) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$

This justifies our use of the same notation for left, right, and total hom-complexes.

Remark 4.8.10. Let $\mathcal{M}$ be a model category.

- For each finite simplicial set Z, each cosimplicial object $A^\bullet$ in $\mathcal{M}$, and each object B in $\mathcal{M}$, we have the following natural bijections:

$$\mathbf{sSet}\big(Z, \mathcal{H}om_{\mathcal{M}}(A, B)\big) \cong \mathcal{M}\big(Z \star A, B\big) \cong \mathbf{c}\mathcal{M}(A, Z \pitchfork B)$$

- For each finite simplicial set Z, each object A in $\mathcal{M}$, and each simplicial object $B_\bullet$ in $\mathcal{M}$, we have the following natural bijections:

$$\mathbf{sSet}\big(Z, \mathcal{H}om_{\mathcal{M}}(A, B)\big) \cong \mathbf{s}\mathcal{M}(Z \odot A, B) \cong \mathcal{M}(A, \{Z, B\})$$

Lemma 4.8.11. *Let $\mathcal{M}$ be a category with weak equivalences.*

- *Let $A^\bullet$ be a weakly constant cosimplicial object in $\mathcal{M}$ and let B be an object in $\mathcal{M}$. Given a parallel pair $f_0, f_1 : A^0 \to B$ in $\mathcal{M}$ such that the corresponding vertices in the left hom-complex $\mathcal{H}om_{\mathcal{M}}(A, B)$ are in the same connected component, we have $f_0 = f_1$ in $\operatorname{Ho}\mathcal{M}$, and $f_0 : A^0 \to B$ is a weak equivalence in $\mathcal{M}$ if and only if $f_1 : A^0 \to B$ is a weak equivalence in $\mathcal{M}$.*

- *Let A be an object in $\mathcal{M}$ and let B be a weakly constant simplicial object in $\mathcal{M}$. Given a parallel pair $f^0, f^1 : A \to B_0$ in $\mathcal{M}$ such that the corresponding vertices in the right hom-complex $\mathcal{H}om_{\mathcal{M}}(A, B)$ are in the same connected component, we have $f^0 = f^1$ in $\operatorname{Ho}\mathcal{M}$, and $f^0 : A \to B_0$ is a weak equivalence in $\mathcal{M}$ if and only if $f^1 : A \to B_0$ is a weak equivalence in $\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

By induction, we may assume that there is an edge in $\mathcal{H}om_{\mathcal{M}}(A, B)$ from f_0 to f_1, i.e. that there is a morphism $h : A^1 \to B$ making the following diagram in

$\mathcal{M}$ commute:

$$\begin{tikzcd}
A^0 \arrow[r, "f_0"] \arrow[d, "\delta^1"'] & B \arrow[d, equal] \\
A^1 \arrow[r, "h"] & B \\
A^0 \arrow[r, "f_1"'] \arrow[u, "\delta^0"] & B \arrow[u, equal]
\end{tikzcd}$$

Since $A^\bullet$ is weakly constant, the coface operators $\delta^0, \delta^1 : A^0 \to A^1$ are weak equivalences in $\mathcal{M}$; but $\sigma^0 \circ \delta^0 = \sigma^0 \circ \delta^1 = \mathrm{id}_{A^0}$, so we must have $f_0 = f_1$ in $\operatorname{Ho} \mathcal{M}$, as required. Similarly, the 2-out-of-3 property of weak equivalences ensures that $f_0 : A^0 \to B$ is a weak equivalence in $\mathcal{M}$ if and only if $f_1 : A^0 \to B$ is a weak equivalence in $\mathcal{M}$. ■

Lemma 4.8.12. *Let $\mathcal{M}$ be a derivable category.*

- *A cofibrant object with respect to any cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ in $\mathcal{M}$ is a degreewise cofibrant as a cosimplicial object in $\mathcal{M}$.*
- *A fibrant object with respect to any simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ in $\mathcal{M}$ is a degreewise fibrant as a simplicial object in $\mathcal{M}$.*

Proof. This is a special case of proposition 4.6.3 ■

Lemma 4.8.13. *Let $\mathcal{M}$ be a derivable category.*

- *If $i^\bullet : A^\bullet \to B^\bullet$ is a Reedy cofibration (resp. trivial Reedy cofibration) in $\mathbf{c}\mathcal{M}$, $p : C \to D$ is a trivial fibration (resp. fibration) in $\mathcal{M}$, and the square in the diagram below is a pullback square in* **sSet***,*

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}}(B, C) \arrow[dr, dashed, "i^* \boxdot p_*"] \arrow[drr, bend left, "\mathcal{H}om_{\mathcal{M}}(B,p)"] \arrow[ddr, bend right, "\mathcal{H}om_{\mathcal{M}}(i,C)"'] & & \\
& \bullet \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{M}}(B, D) \arrow[d, "\mathcal{H}om_{\mathcal{M}}(i,D)"] \\
& \mathcal{H}om_{\mathcal{M}}(A, C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(A,p)"'] & \mathcal{H}om_{\mathcal{M}}(A, D)
\end{tikzcd}$$

then the unique morphism $i^* \boxdot p_*$ making the diagram commute is a trivial Kan fibration.

- *If $i : A \to B$ is a cofibration (resp. trivial cofibration) in $\mathcal{M}$, $p_\bullet : C_\bullet \to D_\bullet$ is a trivial Reedy fibration (resp. Reedy fibration) in $\mathbf{s}\mathcal{M}$, and the square in the diagram below is a pullback square in* **sSet***,*

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}}(B,C) \arrow[dr, dashed, "i^*\square p_*"] \arrow[drr, bend left, "\mathcal{H}om_{\mathcal{M}}(B,p)"] \arrow[ddr, bend right, "\mathcal{H}om_{\mathcal{M}}(i,C)"'] & & \\
& \bullet \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{M}}(B,D) \arrow[d, "\mathcal{H}om_{\mathcal{M}}(i,D)"] \\
& \mathcal{H}om_{\mathcal{M}}(A,C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(A,p)"'] & \mathcal{H}om_{\mathcal{M}}(A,D)
\end{tikzcd}$$

then the unique morphism $i^* \square p_*$ making the diagram commute is a trivial Kan fibration.

Proof. This is a special case of proposition 4.5.29. ■

Lemma 4.8.14. *Let $\mathcal{M}$ be a derivable category.*

- *If $B^\bullet$ is a Reedy-cofibrant cosimplicial object in $\mathcal{M}$, then the left hom-complex functor $\mathcal{H}om_{\mathcal{M}}(B,-) : \mathcal{M} \to$* **sSet** *sends trivial fibrations in $\mathcal{M}$ to trivial Kan fibrations.*

- *If $C_\bullet$ is a Reedy-fibrant simplicial object in $\mathcal{M}$, then the right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(-,C) : \mathcal{M}^{\mathrm{op}} \to$* **sSet** *sends trivial cofibrations in $\mathcal{M}$ to trivial Kan fibrations.*

Proof. This is a special case of proposition 4.5.34. ■

Corollary 4.8.15. *Let $\mathcal{M}$ be a derivable category. For any cosimplicial resolution $\tilde{A}^\bullet$ in $\mathcal{M}$:*

(i) *The left hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A,-) : \mathcal{M} \to$* **sSet** *sends weak equivalences between fibrant objects in $\mathcal{M}$ to weak homotopy equivalences of simplicial sets.*

(ii) *The total hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A,-) : \mathbf{s}\mathcal{M} \to$* **sSet** *sends Reedy weak equivalences between* degreewise *fibrant simplicial objects in $\mathcal{M}$ to weak homotopy equivalences of simplicial sets.*

Dually, for any simplicial resolution $B_\bullet$ in $\mathcal{M}$:

(i′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(-, B) : \mathcal{M}^{\mathrm{op}} \to$ **sSet** sends weak equivalences between cofibrant objects in $\mathcal{M}$ to weak homotopy equivalences of simplicial sets.*

(ii′) *The total hom-complex functor $\mathcal{H}om_{\mathcal{M}}(-, B) : (\mathbf{c}\mathcal{M})^{\mathrm{op}} \to$ **sSet** sends Reedy weak equivalences between* degreewise *cofibrant cosimplicial objects in $\mathcal{M}$ to weak homotopy equivalences of simplicial sets.*

Proof. (i). Apply lemma 4.1.33 to lemma 4.8.14.

(ii). Consider the functor $\mathcal{M}(A^\bullet, -) : \mathbf{s}\mathcal{M} \to \mathbf{ssSet}$ that sends an object $B_\bullet$ in $\mathbf{s}\mathcal{M}$ to the bisimplicial set defined by the formula below:

$$\left(\mathcal{M}\left(A^\bullet, B_\bullet\right)\right)_m = \mathcal{H}om_{\mathcal{M}}\left(A, B_m\right)$$

Then, by claim (i), $\mathcal{M}(A^\bullet, -)$ sends Reedy weak equivalences between degreewise fibrant simplicial objects in $\mathcal{M}$ to Reedy weak equivalences in **ssSet**. We may then use lemma 1.6.8 and theorem 1.6.10 to deduce that the total hom-complex functor has the required property. ∎

Proposition 4.8.16. *Let $\mathcal{M}$ be a derivable category.*

- *If $\tilde{A}^\bullet$ is a cosimplicial resolution in $\mathcal{M}$, then for each fibrant object B in $\mathcal{M}$, there is a natural bijection*

$$\operatorname{Ho}\mathcal{M}\left(\tilde{A}^0, B\right) \to \pi_0 \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, B\right)$$

  sending the class (in $\operatorname{Ho}\mathcal{M}$*) of a morphism $\tilde{A}^0 \to B$ in $\mathcal{M}$ to the connected component of the corresponding vertex of $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, B\right)$.*

- *If $\hat{B}_\bullet$ is a simplicial resolution in $\mathcal{M}$, then for each cofibrant object A in $\mathcal{M}$, there is a natural bijection*

$$\operatorname{Ho}\mathcal{M}\left(A, \hat{B}_0\right) \to \pi_0 \mathcal{H}om_{\mathcal{M}}\left(A, \hat{B}_0\right)$$

  sending the class (in $\operatorname{Ho}\mathcal{M}$*) of a morphism $A \to \hat{B}_0$ in $\mathcal{M}$ to the connected component of the corresponding vertex of $\mathcal{H}om_{\mathcal{M}}\left(A, \hat{B}\right)$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $\tilde{A}^\bullet$ be a Reedy-cofibrant cosimplicial object in $\mathcal{M}$ and let $\mathcal{V}$ be the class of trivial fibrations in $\mathcal{M}$. By lemma 4.8.14, know $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M} \to \mathbf{sSet}$ preserves trivial fibrations, and lemma 1.3.27 implies that $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ sends trivial Kan fibrations to bijections, so we have an induced functor

$$\pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M}[\mathcal{V}^{-1}] \to \mathbf{Set}$$

and in particular, we have natural maps

$$\mathcal{M}[\mathcal{V}^{-1}](\tilde{A}^0, B) \to \mathbf{Set}(\pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, \tilde{A}^0), \pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B))$$

so by evaluating at the vertex of $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \tilde{A}^0)$ corresponding to $\mathrm{id} : \tilde{A}^0 \to \tilde{A}^0$, we obtain a natural map $\mathcal{M}[\mathcal{V}^{-1}](\tilde{A}^0, B) \to \pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$. However, by propositions 4.1.35 and 4.4.7, the natural map

$$\mathcal{M}[\mathcal{V}^{-1}](\tilde{A}^0, B) \to \operatorname{Ho} \mathcal{M}(\tilde{A}^0, B)$$

induced by the localising functor $\mathcal{M}[\mathcal{V}^{-1}] \to \operatorname{Ho} \mathcal{M}$ is a bijection for each *fibrant* object B in $\mathcal{M}$, so we have a natural map $\operatorname{Ho} \mathcal{M}(\tilde{A}^0, B) \to \pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$ of the required form.

Observe that we have the following commutative diagram in $\mathbf{Set}$,

$$\begin{tikzcd}
\mathcal{M}(\tilde{A}^0, B) \arrow[r] \arrow[d, equal] & \operatorname{Ho} \mathcal{M}(\tilde{A}^0, B) \arrow[d] \\
\mathcal{M}(\tilde{A}^0, B) \arrow[r] & \pi_0 \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)
\end{tikzcd}$$

where the top horizontal arrow is the map that sends a morphism in $\mathcal{M}$ to its class in $\operatorname{Ho} \mathcal{M}$. The bottom horizontal arrow is a surjective map, so the right vertical arrow must also be a surjective map. To complete the proof of the claim, it now suffices to show that it is also an injective map; but this is implied by lemma 4.8.11, so we are done. ∎

Proposition 4.8.17. *Let $\mathcal{M}$ be a derivable category.*

- *If $A^\bullet$ is a* degreewise *cofibrant weakly constant cosimplicial object in $\mathcal{M}$, then the functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}\mathcal{M} \to \mathbf{sSet}$ preserves Reedy weak equivalences between simplicial* resolutions.

- *If $B_\bullet$ is a* degreewise *fibrant weakly constant simplicial object in $\mathcal{M}$, then the functor $\mathcal{H}om_{\mathcal{M}}(-, B) : (\mathbf{c}\mathcal{M})^{\mathrm{op}} \to \mathbf{sSet}$ preserves Reedy weak equivalences between cosimplicial* resolutions.

Proof. The two claims are formally dual; we will prove the first version.

Let $A^\bullet$ be a degreewise cofibrant weakly constant cosimplicial object in $\mathcal{M}$ and let $p^\bullet : A^\bullet \to \operatorname{cosk}^0(A^0)$ be the component of the adjunction unit. By lemma 4.8.5, $p^\bullet$ is a Reedy weak equivalence. Let $f_\bullet : B_\bullet \to C_\bullet$ be a Reedy weak equivalence between simplicial resolutions. We then have the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}}(A, B) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(p,B)"] \arrow[d, "\mathcal{H}om_{\mathcal{M}}(A,f)"'] & \mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), B\right) \arrow[d, "\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right),f\right)"] \\
\mathcal{H}om_{\mathcal{M}}(A, C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(p,C)"'] & \mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), C\right)
\end{tikzcd}$$

Corollary 4.8.15 says that $\mathcal{H}om_{\mathcal{M}}(p, B)$ and $\mathcal{H}om_{\mathcal{M}}(p, C)$ are weak homotopy equivalences; but recalling lemma 4.8.12, we may then use remark 4.8.9 to deduce that $\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), f\right)$ is a weak homotopy equivalence. Finally, we apply the 2-out-of-3 property of weak homotopy equivalences to conclude that $\mathcal{H}om_{\mathcal{M}}(A, f)$ itself is a weak homotopy equivalence. ■

Corollary 4.8.18. *For any derivable category $\mathcal{M}$, the total hom-complex functor*

$$\mathcal{H}om_{\mathcal{M}} : \left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{r}}\mathcal{M} \to \mathbf{sSet}$$

preserves weak equivalences.

Proof. Apply lemma 4.8.12 to proposition 4.8.17. ■

Proposition 4.8.19. *Let $\mathcal{M}$ be a derivable category. If $A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ and $B_\bullet$ is a simplicial resolution in $\mathcal{M}$, then there is a natural diagram of weak homotopy equivalences in* $\mathbf{sSet}$ *of the form below,*

$$\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right) \longrightarrow \mathcal{H}om_{\mathcal{M}}(A, B) \longleftarrow \mathcal{H}om_{\mathcal{M}}\left(A^0, B\right)$$

where $\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right)$ is the left hom-complex, $\mathcal{H}om_{\mathcal{M}}\left(A^0, B\right)$ is the right hom-complex, $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the total hom-complex, the rightward arrow is the morphism induced by the adjunction counit component $i_\bullet : \operatorname{sk}_0\left(B_0\right) \to B_\bullet$, and the leftward arrow is the morphism induced by the adjunction unit component $p^\bullet : A^\bullet \to \operatorname{cosk}^0\left(A^0\right)$.

Proof. The two halves of the claim are formally dual; we will show that there is a natural weak homotopy equivalence $\mathcal{H}om_{\mathcal{M}}(A, B_0) \to \mathcal{H}om_{\mathcal{M}}(A, B)$.

Lemma 4.8.12 says that each B_m is a fibrant object in $\mathcal{M}$, so by lemma 4.8.5, $i_\bullet : \mathrm{sk}_0(B_0) \to B_\bullet$ is a Reedy weak equivalence between degreewise fibrant objects. Thus, $\mathcal{H}om_{\mathcal{M}}(A, \mathrm{sk}_0(B_0)) \to \mathcal{H}om_{\mathcal{M}}(A, B)$ is a weak homotopy equivalence, by corollary 4.8.15. Since the total hom-complex $\mathcal{H}om_{\mathcal{M}}(A, \mathrm{sk}_0(B_0))$ is naturally isomorphic to the left hom-complex $\mathcal{H}om_{\mathcal{M}}(A, B_0)$ (by remark 4.8.9), this is the required natural weak homotopy equivalence. ∎

Definition 4.8.20. Let A and B be objects in a derivable category $\mathcal{M}$.

- A **left homotopy function complex** from A to B consists of the data $(\tilde{A}^\bullet, p^\bullet, \hat{B}, i, \mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B}))$, where $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial resolution of A, $(\hat{B}, i)$ is a fibrant replacement for B, and $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ is the left hom-complex.

- A **right homotopy function complex** from A to B consists of the data $(\tilde{A}, p, \hat{B}_\bullet, i_\bullet, \mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B}))$, where (A, p) is a cofibrant replacement for A, $(\hat{B}_\bullet, i_\bullet)$ is a simplicial resolution of B, and $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ is the right hom-complex.

- A **two-sided homotopy function complex** from A to B consists of the data $(\tilde{A}^\bullet, p^\bullet, \hat{B}_\bullet, i_\bullet, \mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B}))$, where $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial resolution of A, $(\hat{B}_\bullet, i)$ is a simplicial resolution of B, and $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ is the total hom-complex.

We will often abuse notation and say $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ is a (left, right, or two-sided) homotopy function complex from A to B, omitting mention of the other data.

Remark 4.8.21. The weak homotopy type of $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ depends only on the isomorphism class of A and B in $\mathrm{Ho}\,\mathcal{M}$, by corollary 4.8.15.

Proposition 4.8.22. *Let $f : A \to B$ be a morphism in a derivable category $\mathcal{M}$.*

- *Let $(\hat{A}, i_A)$ and $(\hat{B}, i_B)$ be fibrant replacements for A and B, respectively, and let $\hat{f} : \hat{A} \to \hat{B}$ be any morphism in $\mathcal{M}$ making the diagram below commute:*

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & \hat{A} \arrow[d, "\hat{f}"] \\ B \arrow[r, "i_B"'] & \hat{B} \end{tikzcd}$$

Assuming $\mathcal{M}$ is a cosimplicially resolvable category, $f : A \to B$ is an isomorphism in Ho $\mathcal{M}$ *if and only if the induced morphism of left homotopy function complexes*

$$\mathcal{H}om_{\mathcal{M}}(C, \hat{f}) : \mathcal{H}om_{\mathcal{M}}(C, \hat{A}) \to \mathcal{H}om_{\mathcal{M}}(C, \hat{B})$$

is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$.

- *Let $(\tilde{A}, p_A)$ and $(\tilde{B}, p_B)$ be cofibrant replacements for A and B, respectively, and let $\tilde{f} : \tilde{A} \to \tilde{B}$ be any morphism in $\mathcal{M}$ making the diagram below commute:*

$$\begin{tikzcd} \tilde{A} \arrow[r, "p_A"] \arrow[d, "\tilde{f}"'] & A \arrow[d, "f"] \\ \tilde{B} \arrow[r, "p_B"'] & B \end{tikzcd}$$

Assuming $\mathcal{M}$ is a simplicially resolvable category, $f : A \to B$ is an isomorphism in Ho $\mathcal{M}$ *if and only if the induced morphism of right homotopy function complexes*

$$\mathcal{H}om_{\mathcal{M}}(\tilde{f}, C) : \mathcal{H}om_{\mathcal{M}}(\tilde{B}, C) \to \mathcal{H}om_{\mathcal{M}}(\tilde{A}, C)$$

is a weak homotopy equivalence for all simplicial resolutions $C_\bullet$.

Proof. The two claims are formally dual; we will prove the first version.

First, suppose $f : A \to B$ is an isomorphism in Ho $\mathcal{M}$. Then, by proposition 4.1.35, $\hat{f} : \hat{A} \to \hat{B}$ is an isomorphism in Ho $\mathcal{M}_{\mathrm{f}}$, so we may use corollary 4.8.15 (and lemma 1.5.2) to deduce that $\mathcal{H}om_{\mathcal{M}}(C, \hat{f})$ is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$.

Conversely, suppose $\mathcal{H}om_{\mathcal{M}}(C, \hat{f})$ is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$. Proposition 4.8.16 then implies that the hom-set map

$$\mathrm{Ho}\,\mathcal{M}(C^0, \hat{f}) : \mathrm{Ho}\,\mathcal{M}(C^0, \hat{A}) \to \mathrm{Ho}\,\mathcal{M}(C^0, \hat{B})$$

is a bijection for all cosimplicial resolutions $C^\bullet$; but by hypothesis, every object in $\mathcal{M}$ is weakly equivalent to one that occurs as C^0 for some cosimplicial resolution $C^\bullet$, so $\hat{f} : \hat{A} \to \hat{B}$ and $f : A \to B$ are isomorphisms in Ho $\mathcal{M}$. ■

Definition 4.8.23. Let $\mathcal{M}$ be a derivable category.

- Assuming $\mathcal{M}$ is a cosimplicially resolvable category, a **derived left hom-space functor** for an object B in $\mathcal{M}$ is a functor
$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(-, B) : (\mathrm{Ho}\,\mathcal{M})^{\mathrm{op}} \to \mathrm{Ho}\,\mathbf{sSet}$$
equipped with natural isomorphisms
$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\big(A^0, B\big) \cong \mathcal{H}om_{\mathcal{M}}\big(A, \hat{B}\big)$$
in $\mathrm{Ho}\,\mathbf{sSet}$, where $A^\bullet$ varies over the cosimplicial resolutions in $\mathcal{M}$, $\big(\hat{B}, i\big)$ varies over the fibrant replacements for B, and $\mathcal{H}om_{\mathcal{M}}\big(A, \hat{B}\big)$ is the left hom-complex.

- Assuming $\mathcal{M}$ is a simplicially resolvable category, a **derived right hom-space functor** for an object A in $\mathcal{M}$ is a functor
$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathbf{sSet}$$
equipped with natural isomorphisms
$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\big(A, B_0\big) \cong \mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, B\big)$$
in $\mathrm{Ho}\,\mathbf{sSet}$, where $\big(\tilde{A}, p\big)$ varies over the cofibrant replacements for A, $B_\bullet$ varies over the simplicial resolutions in $\mathcal{M}$, and $\mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, B\big)$ is the right hom-complex.

- Assuming $\mathcal{M}$ is a resolvable category, a **derived hom-space functor** for $\mathcal{M}$ is a functor $\mathbf{R}\mathrm{Hom}_{\mathcal{M}} : (\mathrm{Ho}\,\mathcal{M})^{\mathrm{op}} \times \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathbf{sSet}$ equipped with natural isomorphisms
$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\big(A^0, B_0\big) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$
in $\mathrm{Ho}\,\mathbf{sSet}$, where $A^\bullet$ varies over the cosimplicial resolutions in $\mathcal{M}$, $B_\bullet$ varies over the simplicial resolutions in $\mathcal{M}$, and $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the total hom-complex.

We will often refer to the object $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, B)$ as a **derived hom-space**, omitting mention of the other data.

The name 'derived hom-space' is justified by the following theorem.

Theorem 4.8.24. *Let $\mathcal{M}$ be a* resolvable *category, let* $\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}, Q^{\bullet}, p^{\bullet}\right)$ *be a left deformation retract of* $\mathbf{c}_{\mathrm{w}}\mathcal{M}$*, and let* $\left(\mathbf{s}_{\mathrm{r}}\mathcal{M}, R_{\bullet}, i_{\bullet}\right)$ *be a right deformation retract of* $\mathbf{s}_{\mathrm{w}}\mathcal{M}$*.*

(i) $\left(\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{r}}\mathcal{M}, Q^{\bullet} \times R_{\bullet}, \left(p^{\bullet}, i_{\bullet}\right)\right)$ *is a right deformation retract for the total hom-complex functor* $\mathcal{H}om_{\mathcal{M}} : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$.

(ii) $\mathcal{H}om_{\mathcal{M}} : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$ *has a total right derived functor; furthermore, if* $\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}, Q^{\bullet}, p^{\bullet}\right)$ *and* $\left(\mathbf{s}_{\mathrm{r}}\mathcal{M}, R_{\bullet}, i_{\bullet}\right)$ *are functorial deformation retracts, then* $\mathcal{H}om_{\mathcal{M}}$ *also has a homotopical right approximation.*

(iii) *The functor* $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0(-), \operatorname{sk}_0(-)\right) : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \times \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ *is a derived hom-space functor for* $\mathcal{M}$*.*

Proof. (i). It suffices to show that the restriction the total hom-complex functor $\mathcal{H}om_{\mathcal{M}}$ preserves weak equivalences as a functor $\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{r}}\mathcal{M} \to \mathbf{sSet}$; but this is a consequence of lemma 4.8.12 and corollary 4.8.18.

(ii). Apply theorems 3.3.17 and 3.4.11.

(iii). This follows from claims (i) and (ii). ■

Theorem 4.8.25. *Let $\mathcal{M}$ be a* resolvable *category. If B is a fibrant object in $\mathcal{M}$, then:*

(i) *The left hom-complex functor* $\mathcal{H}om_{\mathcal{M}}(-, B) : (\mathbf{c}\mathcal{M})^{\mathrm{op}} \to \mathbf{sSet}$ *sends degreewise trivial cofibrations in* $\mathbf{c}\mathcal{M}$ *and Reedy weak equivalences in* $\mathbf{c}_{\mathrm{r}}\mathcal{M}$ *to weak homotopy equivalences in* $\mathbf{sSet}$.

(ii) *The left hom-complex functor* $\mathcal{H}om_{\mathcal{M}}(-, B) : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \to \mathbf{sSet}$ *admits a total right derived functor.*

(iii) *The functor* $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0(-), B\right) : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \to \operatorname{Ho}\mathbf{sSet}$ *is a derived left hom-space functor.*

Dually, if A is a cofibrant object in $\mathcal{M}$, then:

(i′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}\mathcal{M} \to \mathbf{sSet}$ sends degreewise trivial fibrations in $\mathbf{s}\mathcal{M}$ and Reedy weak equivalences in $\mathbf{s}_{\mathrm{r}}\mathcal{M}$ to weak homotopy equivalences in $\mathbf{sSet}$.*

(ii′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$ admits a total right derived functor.*

(iii′) *The functor $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(A, \mathrm{sk}_0(-)\big) : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ is a derived right hom-space functor.*

Proof. (i). Let $f^{\bullet} : A^{\bullet} \to C^{\bullet}$ be a degreewise trivial cofibration in $\mathbf{c}\mathcal{M}$ (resp. Reedy weak equivalence in $\mathbf{c}_{\mathrm{r}}\mathcal{M}$), and choose a simplicial resolution $\big(\hat{B}_{\bullet}, i_{\bullet}\big)$ of B. We then have a morphism of bisimplicial sets

$$\mathcal{M}\big(f^{\bullet}, \hat{B}_{\bullet}\big) : \mathcal{M}\big(C^{\bullet}, \hat{B}_{\bullet}\big) \to \mathcal{M}\big(A^{\bullet}, \hat{B}_{\bullet}\big)$$

and since each $f^n : A^n \to C^n$ is a trivial cofibration (resp. weak equivalence) in $\mathcal{M}$, lemma 4.8.14 (resp. corollary 4.8.15) says that the components

$$\mathcal{M}\big(f^{n}, \hat{B}_{\bullet}\big) : \mathcal{M}\big(C^{n}, \hat{B}_{\bullet}\big) \to \mathcal{M}\big(A^{n}, \hat{B}_{\bullet}\big)$$

are trivial Kan fibrations, hence weak homotopy equivalences *a fortiori*. Thus, applying lemma 1.6.8 and theorem 1.6.10, we deduce that the morphism

$$\mathcal{H}om_{\mathcal{M}}\big(f, \hat{B}\big) : \mathcal{H}om_{\mathcal{M}}\big(C, \hat{B}\big) \to \mathcal{H}om_{\mathcal{M}}\big(A, \hat{B}\big)$$

is a weak homotopy equivalence. Using corollary 4.8.15, proposition 4.8.19, and the 2-out-of-3 property of weak homotopy equivalences, we then conclude that the morphism $\mathcal{H}om_{\mathcal{M}}(f, B) : \mathcal{H}om_{\mathcal{M}}(C, B) \to \mathcal{H}om_{\mathcal{M}}(A, B)$ is indeed a weak homotopy equivalence.

(ii). By lemma 4.8.6, there is a left deformation retract $\big(\mathbf{c}_{\mathrm{r}}\mathcal{M}, Q^{\bullet}, p^{\bullet}\big)$ of $\mathbf{c}_{\mathrm{w}}\mathcal{M}$; and we have seen that $\mathcal{H}om_{\mathcal{M}}(-, B)$ preserves weak equivalences as a functor $\big(\mathbf{c}_{\mathrm{r}}\mathcal{M}\big)^{\mathrm{op}} \to \mathbf{sSet}$, so we may apply theorem 3.3.17.

(iii). The total derived functor theorem implies that $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(\mathrm{cosk}^0(A), B\big)$ is naturally isomorphic to the weak homotopy type of $\mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, B\big)$ for any cosimplicial resolution $\big(\tilde{A}^{\bullet}, p^{\bullet}\big)$ of $\mathrm{cosk}^0(A)$, so $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(\mathrm{cosk}^0(-), B\big)$ is indeed a derived left hom-space functor. ■

Definition 4.8.26. Let $\mathcal{M}$ be a derivable category. A **cosimplicial resolution model structure** on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a model structure that satisfies the following conditions:

- $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a derivable category with this model structure.
- The weak equivalences in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ are the Reedy weak equivalences.
- Every fibration (resp. trivial fibration) in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a degreewise fibration (resp. degreewise trivial fibration) in $\mathbf{c}\mathcal{M}$.
- Every cofibration (resp. trivial cofibration) in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a Reedy cofibration (resp. Reedy trivial cofibration) in $\mathbf{c}\mathcal{M}$.
- Every cofibrant object in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a Reedy-cofibrant object in $\mathbf{c}\mathcal{M}$.
- If $\tilde{A}^{\bullet}$ is a cofibrant object in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, then the left hom-complex functor

  $$\mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, -\big) : \mathcal{M} \to \mathbf{sSet}$$

  sends fibrations (resp. fibrant objects) in $\mathcal{M}$ to Kan fibrations (resp. Kan complexes).
- If $i^{\bullet} : A^{\bullet} \to B^{\bullet}$ is a cofibration between cofibrant objects in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, $p : C \to D$ is a fibration in $\mathcal{M}$, and the square in the diagram below is a pullback square in $\mathbf{sSet}$,

  $$\begin{tikzcd}
  \mathcal{H}om_{\mathcal{M}}(B,C) \arrow[dr, dashed, "i^* \boxbslash p_*"] \arrow[drr, bend left, "\mathcal{H}om_{\mathcal{M}}(B,p)"] \arrow[ddr, bend right, "\mathcal{H}om_{\mathcal{M}}(i,C)"'] & & \\
  & \bullet \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{M}}(B,D) \arrow[d, "\mathcal{H}om_{\mathcal{M}}(i,D)"] \\
  & \mathcal{H}om_{\mathcal{M}}(A,C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(A,p)"'] & \mathcal{H}om_{\mathcal{M}}(A,D)
  \end{tikzcd}$$

  then the unique morphism $i^* \boxbslash p_*$ making the diagram commute is a Kan fibration.

Dually, a **simplicial resolution model structure** on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a model structure that satisfies the following conditions:

- $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a derivable category with this model structure.

- The weak equivalences in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ are the Reedy weak equivalences.
- Every cofibration (resp. trivial cofibration) in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a degreewise cofibration (resp. degreewise trivial cofibration) in $\mathbf{s}\mathcal{M}$.
- Every fibration (resp. trivial fibration) in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a Reedy fibration (resp. Reedy trivial fibration) in $\mathbf{s}\mathcal{M}$.
- Every fibrant object in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a Reedy-fibrant object in $\mathbf{s}\mathcal{M}$.
- If $\hat{B}_\bullet$ is a fibrant object in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, then the right hom-complex functor
$$\mathcal{H}om_{\mathcal{M}}\big(-, \hat{B}\big) : \mathcal{M}^{\mathrm{op}} \to \mathbf{sSet}$$
sends cofibrations (resp. cofibrant objects) in $\mathcal{M}$ to Kan fibrations (resp. Kan complexes).
- If $i : A \to B$ is a cofibration in $\mathcal{M}$, $p_\bullet : C_\bullet \to D_\bullet$ is a fibration between fibrant objects in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, and the square in the diagram below is a pullback square in $\mathbf{sSet}$,
$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}}(B,C) \arrow[dr, dashed, "i^* \boxslash p_*"] \arrow[drr, bend left, "\mathcal{H}om_{\mathcal{M}}(B,p)"] \arrow[ddr, bend right, "\mathcal{H}om_{\mathcal{M}}(i,C)"'] & & \\
& \bullet \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{M}}(B,D) \arrow[d, "\mathcal{H}om_{\mathcal{M}}(i,D)"] \\
& \mathcal{H}om_{\mathcal{M}}(A,C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(A,p)"'] & \mathcal{H}om_{\mathcal{M}}(A,D)
\end{tikzcd}$$
then the unique morphism $i^* \boxslash p_*$ making the diagram commute is a Kan fibration.

Remark 4.8.27. If $\mathcal{M}$ has initial and terminal objects, then the first condition on left hom-complexes (resp. right hom-complexes) is a special case of the second condition.

Remark 4.8.28. If there is a cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, then $\mathcal{M}$ is a cosimplicially resolvable category; dually, if there is a simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, then $\mathcal{M}$ is a simplicially resolvable category.

Remark 4.8.29. If $\mathcal{M}$ is a model category, then:

- A cofibrant object with respect to any cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a cosimplicial resolution in $\mathcal{M}$.
- A fibrant object with respect to any simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a simplicial resolution in $\mathcal{M}$.

Conversely, we will show (as theorem 4.8.34) that there exist a cosimplicial (resp. simplicial) resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ (resp. $\mathbf{s}_{\mathrm{w}}\mathcal{M}$) where the cofibrant (resp. fibrant) objects are precisely the cosimplicial (resp. simplicial) resolutions in $\mathcal{M}$.

Proposition 4.8.30. *Let $\mathcal{M}$ be a derivable category.*

- *For any cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, the following adjunction is a Quillen equivalence of derivable categories:*

$$(-)^0 \dashv \operatorname{cosk}^0 : \mathcal{M} \to \mathbf{c}_{\mathrm{w}}\mathcal{M}$$

- *For any simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, the following adjunction is a Quillen equivalence of derivable categories:*

$$\operatorname{sk}_0 \dashv (-)_0 : \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathcal{M}$$

Proof. The two claims are formally dual; we will prove the first version.

Proposition 4.6.4 implies that $(-)^0 : \mathbf{c}_{\mathrm{w}}\mathcal{M} \to \mathcal{M}$ is a left Quillen functor, so by proposition 4.3.2, we indeed have a Quillen adjunction. Moreover, for any weakly constant cosimplicial object $A^\bullet$ in $\mathcal{M}$ and any object B in $\mathcal{M}$, a morphism $A^0 \to B$ is a weak equivalence in $\mathcal{M}$ if and only if its right adjoint transpose $A^\bullet \to \operatorname{cosk}^0(A)$ is a weak equivalence in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, so the adjunction is a Quillen equivalence. ■

Proposition 4.8.31. *Let $\mathcal{M}$ be a derivable category and let $\mathcal{N}$ be a homotopically replete full subcategory of $\mathcal{M}$.*

- *Given a cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, its restriction to $\mathbf{c}_{\mathrm{w}}\mathcal{N}$ is cosimplicial resolution model structure (with respect to the model structure on $\mathcal{N}$ inherited from $\mathcal{M}$).*

- *Given a simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, its restriction to $\mathbf{s}_{\mathrm{w}}\mathcal{N}$ is cosimplicial resolution model structure (with respect to the model structure on $\mathcal{N}$ inherited from $\mathcal{M}$).*

Proof. The two claims are formally dual; we will prove the first version.

By proposition 4.1.28, the model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ restricted to $\mathbf{c}_{\mathrm{w}}\mathcal{N}$ is a model structure; and by the proof of proposition 4.6.7, the weak equivalences (resp. cofibrations, trivial cofibrations, cofibrant objects) in $\mathbf{c}_{\mathrm{w}}\mathcal{N}$ are Reedy weak equivalences (resp. Reedy cofibrations, Reedy trivial cofibrations, Reedy-cofibrant objects), as required. Finally, it is clear that the conditions on left hom-complexes are satisfied by $\mathbf{c}_{\mathrm{w}}\mathcal{N}$ if they are satisfied by $\mathbf{c}_{\mathrm{w}}\mathcal{M}$. ∎

Lemma 4.8.32. *Let $\mathcal{M}$ be a model category.*

- *A morphism $A^\bullet \to B^\bullet$ in $\mathcal{M}$ is a Reedy cofibration if and only if the induced morphism*
$$\big(W \star A\big) \cup^{W\star A} \big(Z \star B\big) \to W \star B$$
is a cofibration in $\mathcal{M}$ for all monomorphisms $Z \to W$ between finite simplicial sets.

- *A morphism $A_\bullet \to B_\bullet$ in $\mathcal{M}$ is a Reedy fibration if and only if the induced morphism*
$$\{W, A\} \to \{Z, A\} \times_{\{Z,B\}} \{W, B\}$$
is a fibration in $\mathcal{M}$ for all monomorphisms $Z \to W$ between finite simplicial sets.

Proof. Since monomorphisms in **sSet** are relative Reedy cell complexes (by proposition 1.2.20), this is just a special case of proposition 4.5.41. ∎

Lemma 4.8.33. *Let $\mathcal{M}$ be a model category.*

- *Given a Reedy cofibration $A^\bullet \to B^\bullet$ between cosimplicial resolutions in $\mathcal{M}$, the morphism*
$$\big(\Delta^n \star A\big) \cup^{\Lambda^n_k \star A} \big(\Lambda^n_k \star B\big) \to \Delta^n \star B$$
induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial cofibration in $\mathcal{M}$.

- *Given a Reedy fibration $A_\bullet \to B_\bullet$ between simplicial resolutions in $\mathcal{M}$, the morphism*

$$\{\Delta^n, A\} \to \left\{\Lambda^n_k, A\right\} \times_{\{\Lambda^n_k, B\}} \{\Delta^n, B\}$$

induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

Consider the following commutative square in $\mathcal{M}$:

$$\begin{tikzcd} \Lambda^n_k \star A \arrow[r] \arrow[d] & \Lambda^n_k \star B \arrow[d] \\ \Delta^n \star A \arrow[r] & \Delta^n \star B \end{tikzcd}$$

Since $A^\bullet$ and $B^\bullet$ are cosimplicial resolutions in $\mathcal{M}$, we may apply proposition 4.7.18 to deduce that the vertical arrows are trivial cofibrations in $\mathcal{M}$. Thus, by axiom CM2 and corollary 4.1.13, the morphism $\left(\Delta^n \star A\right) \cup^{\Lambda^n_k \star A} \left(\Lambda^n_k \star B\right) \to \Delta^n \star B$ is a weak equivalence in $\mathcal{M}$. It remains to be shown that the morphism is a cofibration in $\mathcal{M}$; but that is a special case of lemma 4.8.32, so we are done. ■

Theorem 4.8.34. *Let $\mathcal{M}$ be a model category.*

- *The restriction of the Reedy model structure on $\mathbf{c}\mathcal{M}$ is a cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$.*

- *The restriction of the Reedy model structure on $\mathbf{s}\mathcal{M}$ is a simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

By proposition 4.1.28, the restriction of the Reedy model structure on $\mathbf{c}\mathcal{M}$ makes $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ a derivable category where the weak equivalences, cofibrations, trivial cofibrations, and cofibrant objects are the expected ones. The condition on left hom-complexes remains to be verified; but recalling lemma 4.8.33 and remark 4.8.10, this is (essentially) a special case of proposition 5.5.1. ■

Theorem 4.8.35. *Let $\mathcal{M}$ be a derivable category. If $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ has a cosimplicial resolution model structure and $h_\bullet : \mathcal{M} \to \left[\left(\mathbf{c}_{\mathrm{c}}\mathcal{M}\right)^{\mathrm{op}}, \mathbf{sSet}\right]_{\mathrm{h}}$ is the functor defined by*

$$h_B(A) = \mathcal{H}om_{\mathcal{M}}(A, B)$$

where $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the left hom-complex and $\mathbf{c}_{\mathrm{c}}\mathcal{M}$ is the full subcategory of cofibrant objects in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, then:

(i) $\hbar_\bullet$ *sends fibrations (resp. fibrant objects, trivial fibrations) in* $\mathcal{M}$ *to componentwise Kan fibrations (resp. componentwise Kan complexes, componentwise trivial Kan fibrations).*

(ii) $\hbar_\bullet$ *admits a total right derived functor.*

(iii) *For each cofibrant object* $A^\bullet$ *in* $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ *and each object* B *in* $\mathcal{M}$*,* $\mathbf{R}\hbar_B(A)$ *is a derived hom-space* $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A^0, B\right)$*.*

Dually, if $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ *has a simplicial resolution model structure and* $\hbar^\bullet : \mathcal{M}^{\mathrm{op}} \to \left[\mathbf{s}_{\mathrm{f}}\mathcal{M}, \mathbf{sSet}\right]_{\mathrm{h}}$ *is the functor defined by*

$$\hbar^A(B) = \mathcal{H}om_{\mathcal{M}}(A, B)$$

where $\mathcal{H}om_{\mathcal{M}}(A, B)$ *is the right hom-complex and* $\mathbf{s}_{\mathrm{f}}\mathcal{M}$ *is the full subcategory of fibrant objects in* $\mathbf{s}_{\mathrm{w}}\mathcal{M}$*, then:*

(i′) $\hbar^\bullet$ *sends cofibrations (resp. trivial cofibrations) in* $\mathcal{M}$ *to componentwise Kan fibrations (resp. componentwise trivial Kan fibrations).*

(ii′) $\hbar^\bullet$ *admits a total right derived functor.*

(iii′) *For each object* A *in* $\mathcal{M}$ *and each simplicial resolution* $B_\bullet$ *in* $\mathcal{M}$*,* $\mathbf{R}\hbar^A(B)$ *is a derived hom-space* $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A, B_0\right)$*.*

Proof. (i). The preservation of fibrations and fibrant objects is a consequence of the hypothesis that $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ has a cosimplicial resolution model structure, and the preservation of trivial fibrations is lemma 4.8.13; note that corollary 4.8.15 implies that each $\hbar_B : \left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \to \mathbf{sSet}$ indeed preserves weak equivalences.

(ii). Since the weak equivalences in $\left[\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}}, \mathbf{sSet}\right]_{\mathrm{h}}$ are componentwise (by definition), we may apply theorem 4.3.12.

(iii). The total derived functor theorem implies that $\mathbf{R}\hbar_B(A)$ is isomorphic to the weak homotopy type of the left hom-complex $\mathcal{H}om_{\mathcal{M}}\left(A, \hat{B}\right)$, where $\left(\hat{B}, i\right)$ is any fibrant replacement for B, so $\mathbf{R}\hbar_B(A)$ is a derived hom-space $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A^0, B\right)$. ■

Lemma 4.8.36. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be derivable categories and let*

$$F \dashv G : \mathcal{N} \to \mathcal{M}$$

be a Quillen adjunction.

- *The induced functor* $\mathbf{c}F : \mathbf{c}\mathcal{M} \to \mathbf{c}\mathcal{N}$ *sends cosimplicial resolutions in* $\mathcal{M}$ *to cosimplicial resolutions in* $\mathcal{N}$.

- *The induced functor* $\mathbf{s}G : \mathbf{s}\mathcal{N} \to \mathbf{s}\mathcal{M}$ *sends simplicial resolutions in* $\mathcal{N}$ *to simplicial resolutions in* $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

By propositions 4.3.4 and 4.6.18, $\mathbf{c}F$ sends Reedy-cofibrant cosimplicial objects in $\mathcal{M}$ to Reedy-cofibrant cosimplicial objects in $\mathcal{N}$. Lemma 4.1.33 implies that $\mathbf{c}F$ preserves weak constancy for degreewise cofibrant cosimplicial objects; but lemma 4.8.12 says cosimplicial resolutions are degreewise cofibrant, so we are done. ■

Theorem 4.8.37. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be derivable categories, let*

$$F \dashv G : \mathcal{N} \to \mathcal{M}$$

be a Quillen adjunction, and let

$$\mathbf{L}F \dashv \mathbf{R}G : \operatorname{Ho}\mathcal{N} \to \operatorname{Ho}\mathcal{M}$$

be the derived adjunction. If either

- *both* $\mathcal{M}$ *and* $\mathcal{N}$ *are cosimplicially resolvable categories, or*

- *both* $\mathcal{M}$ *and* $\mathcal{N}$ *are simplicially resolvable categories,*

then there are natural isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{N}}((\mathbf{L}F)A, B) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, (\mathbf{R}G)B)$$

in $\operatorname{Ho}\mathbf{sSet}$, *where* A *varies in* $\operatorname{Ho}\mathcal{M}$ *and* B *varies in* $\operatorname{Ho}\mathcal{N}$.

Proof. The two subclaims are formally dual; we will prove the first version.

Let $\tilde{A}$ be a cosimplicial resolution in $\mathcal{M}$ and let B be a fibrant object in $\mathcal{N}$. Since $F \dashv G$ is an adjunction, we have the following natural isomorphism of left hom-complexes;

$$\mathcal{H}om_{\mathcal{N}}\big(F\tilde{A}, B\big) \cong \mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, GB\big)$$

moreover, by proposition 4.3.4 and lemma 4.8.36, both simplicial sets are (part of) left homotopy function complexes. Theorem 4.3.12 and proposition 4.8.30 then imply we have the required natural isomorphism in $\operatorname{Ho}\mathbf{sSet}$. ■

Lemma 4.8.38. *Let $\mathcal{M}$ be a derivable category.*

- *If $i^\bullet : A^\bullet \to B^\bullet$ is a cofibration between cofibrant objects with respect to a cosimplicial resolution model structure on $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, $p : C \to D$ is a fibration in $\mathcal{M}$, and the induced morphism*

$$\mathcal{H}om_{\mathcal{M}}(B, C) \to \mathcal{H}om_{\mathcal{M}}(A, C) \times_{\mathcal{H}om_{\mathcal{M}}(A,D)} \mathcal{H}om_{\mathcal{M}}(B, D)$$

is a weak homotopy equivalence, then $p : C \to D$ has the right lifting property with respect to each $i^n : A^n \to B^n$.

- *If $i : A \to B$ is a cofibration in $\mathcal{M}$, $p_\bullet : C_\bullet \to D_\bullet$ is a fibration between fibrant objects with respect to a simplicial resolution model structure on $\mathbf{s}_{\mathrm{w}}\mathcal{M}$, and the induced morphism*

$$\mathcal{H}om_{\mathcal{M}}(B, C) \to \mathcal{H}om_{\mathcal{M}}(A, C) \times_{\mathcal{H}om_{\mathcal{M}}(A,D)} \mathcal{H}om_{\mathcal{M}}(B, D)$$

is a weak homotopy equivalence, then $i : A \to B$ has the left lifting property with respect to each $p_n : C_n \to D_n$.

Proof. The two claims are formally dual; we will prove the first version.

By definition, the indicated morphism of simplicial sets is a Kan fibration, so the hypothesis implies it is a trivial Kan fibration. Since every simplicial set is cofibrant, the morphism is a (split) epimorphism; thus, for each natural number n, the map

$$\mathcal{M}(B^n, C) \to \mathcal{M}(A^n, C) \times_{\mathcal{M}(A^n,D)} \mathcal{M}(B^n, D)$$

is a surjection. We may then apply lemma A.3.2 to deduce that $p : C \to D$ has the right lifting property with respect to each $i^n : A^n \to B^n$. ∎

4.9 The Dwyer–Kan comparison theorem

Prerequisites. §§1.1, 1.5, 1.6, 1.9, 1.10, 1.11, 2.8, 4.1, 4.6, 4.7, 4.8, A.4.

In this section, we examine the relationship between the derived hom-spaces of a resolvable category $\mathcal{M}$ and the hom-spaces of its hammock localisation $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{M})$.

Remark 4.9.1. If we do not restrict our attention to small resolvable categories, then it will be necessary to work with categories that may not be essentially small as well as various simplicial ensembles constructed from the nerves of such categories. Fortunately, the main result of this section says that many of these (large) simplicial ensembles of interest are actually weakly homotopy equivalent to (small) simplicial sets.

It will be convenient to make the following definition:

Definition 4.9.2. Let $\mathcal{M}$ be a category with a model structure.

- A **marked zigzag type** (T, V', V'') is a zigzag type T, regarded as a relative category, equipped with a pair of subcategories V' and V'' of the subcategory of weak equivalences.

- A **zigzag** in $\mathcal{M}$ of type (T, V', V'') is a zigzag of type T where the morphisms in V' (resp. V'') are mapped to trivial cofibrations (resp. trivial fibrations) in $\mathcal{M}$.

Proposition 4.9.3. *Let C be an object in a derivable category $\mathcal{M}$.*

- *Let $(\mathcal{M}_{/C})_{\mathrm{wf}}$ be the full subcategory of $\mathcal{M}_{/C}$ spanned by the trivial fibrations $B \to C$ and let $(\tilde{A}^\bullet, p^\bullet)$ be a cosimplicial resolution of A. If $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(C)$ is a degreewise trivial fibration in $\mathbf{c}\mathcal{M}$, then the corresponding functor $P : \mathbf{\Delta} \to (\mathcal{M}_{/C})_{\mathrm{wf}}$ is a homotopy coinitial functor.*

- *Let $({}^{C/}\mathcal{M})_{\mathrm{wc}}$ be the full subcategory of ${}^{C/}\mathcal{M}$ spanned by the trivial cofibrations $C \to A$ and let $(\hat{B}_\bullet, i_\bullet)$ be a simplicial resolution of C. If $i_\bullet : \operatorname{sk}_0(C) \to \hat{B}_\bullet$ is a degreewise trivial cofibration in $\mathbf{s}\mathcal{M}$, then the corresponding functor $I : \mathbf{\Delta}^{\mathrm{op}} \to ({}^{C/}\mathcal{M})_{\mathrm{wc}}$ is a homotopy cofinal functor.*

Proof. The two claims are formally dual; we will prove the first version. We follow the proof of Proposition 6.12 in [Dwyer and Kan, 1980c].

Let (B, q) be an object in $(\mathcal{M}_{/C})_{\mathrm{wf}}$, i.e. a trivial fibration $q : B \to C$ in $\mathcal{M}$. We must show that the comma category $(P \downarrow (B, q))$ is aspherical, i.e. that the nerve $\mathrm{N}((P \downarrow (B, q)))$ is weakly contractible. By corollary 1.9.31,

$$\mathrm{N}((P \downarrow (B, q))) \simeq \mathcal{H}om_{\mathcal{M}_{/C}}\big((\tilde{A}, p), (B, q)\big)$$

and we have the following pullback diagram in **sSet**,

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}_{/C}}\big((\tilde{A}, p), (B, q)\big) \arrow[r] \arrow[d] & \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B) \arrow[d, "\mathcal{M}(\tilde{A}, q)"] \\
\Delta^0 \arrow[r] & \mathcal{H}om_{\mathcal{M}}(\tilde{A}, C)
\end{tikzcd}$$

where the bottom horizontal arrow corresponds to the vertex $i^0 : \tilde{A}^0 \to C$ and (by lemma 4.8.14) the right vertical arrow is a trivial Kan fibration, so (by proposition A.3.17) the left vertical arrow is also a trivial Kan fibration. Thus, $(P \downarrow (B, q))$ is indeed an aspherical category. ■

Corollary 4.9.4. *Let C be an object in a derivable category $\mathcal{M}$.*

- *Let $\left(\mathcal{M}_{/C}\right)_{\mathrm{c,wf}}$ be the full subcategory of $\mathcal{M}_{/C}$ spanned by the fibrant cofibrant replacements for C. If $\mathcal{M}$ is a cosimplicially resolvable category, then $\left(\mathcal{M}_{/C}\right)_{\mathrm{c,wf}}$ is an aspherical category.*
- *Let $\left({}^{C/}\mathcal{M}\right)_{\mathrm{f,wc}}$ be the full subcategory of ${}^{C/}\mathcal{M}$ spanned by the cofibrant fibrant replacements for C. If $\mathcal{M}$ is a simplicially resolvable category, then $\left({}^{C/}\mathcal{M}\right)_{\mathrm{f,wc}}$ is an aspherical category.*

Proof. Since $\mathbf{\Delta}$ is aspherical (by remark 1.11.6), and cosimplicial (resp. simplicial) resolutions are degreewise cofibrant (resp. fibrant) by lemma 4.8.12, the claim is a consequence of Quillen's Theorem A (corollary 1.11.15) and proposition 1.11.13 applied to proposition 4.9.3. ■

Corollary 4.9.5. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let $\mathcal{H}$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,*

$$\bullet \xleftarrow{\text{t. fib.}} \bullet \longrightarrow \bullet \xleftarrow{\text{t. cofib.}} \bullet$$

let $\left(\tilde{A}^\bullet, p^\bullet\right)$ be a cosimplicial resolution of A, and let $\left(\hat{B}_\bullet, i_\bullet\right)$ be a simplicial resolution of B. If $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ is a degreewise trivial fibration and $i_\bullet : \operatorname{sk}_0(B) \to \hat{B}_\bullet$ is a degreewise trivial cofibration, then

$$\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right) \simeq \mathrm{N}(\mathcal{H})$$

by a zigzag of weak homotopy equivalences.

Proof. Considering $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}^\bullet, \hat{B}\right)$ as a diagram $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, by (lemma 1.9.28 and) corollary 1.9.30, we have a natural weak homotopy equivalence

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}^\bullet, \hat{B}\right) \to \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$$

where we have used lemma 1.6.8 to identify the realisation $\left|\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}^\bullet, \hat{B}\right)\right|$ with the two-sided hom-complex $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$; similarly, by corollary 1.6.9, there is a natural weak homotopy equivalence

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}\left(\tilde{A}^n, \hat{B}_\bullet\right) \to \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}^n, \hat{B}\right)$$

so applying Ken Brown's lemma (4.3.6) to theorem 1.6.10 and proposition 1.9.19, we have a natural weak homotopy equivalence

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}\left(\tilde{A}^{\bullet}, \hat{B}_{\bullet}\right) \to \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$$

On the other hand, writing $Q : \left(\mathcal{M}_{/A}\right)_{\mathrm{wf}} \to \mathcal{M}$ and $R : \left({}^{B/}\mathcal{M}\right)_{\mathrm{wc}} \to \mathcal{M}$ for the evident projections, proposition 4.9.3 and theorem 1.10.27 imply that there is a weak homotopy equivalence

$$\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}\left(\tilde{A}^{\bullet}, \hat{B}_{\bullet}\right) \to \varinjlim_{\left(\mathcal{M}_{/A}\right)_{\mathrm{wf}}{}^{\mathrm{op}}}{}^{\mathrm{BK}} \varinjlim_{\left({}^{B/}\mathcal{M}\right)_{\mathrm{wc}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}(Q, R)$$

and by proposition 1.9.9:

$$\varinjlim_{\left(\mathcal{M}_{/A}\right)_{\mathrm{wf}}{}^{\mathrm{op}}}{}^{\mathrm{BK}} \varinjlim_{\left({}^{B/}\mathcal{M}\right)_{\mathrm{wc}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}(Q, R) \cong \varinjlim_{\left(\mathcal{M}_{/A}\right)_{\mathrm{wf}}{}^{\mathrm{op}} \times \left({}^{B/}\mathcal{M}\right)_{\mathrm{wc}}}{}^{\mathrm{BK}} \operatorname{disc} \mathcal{M}(Q, R)$$

Furthermore, remark 1.8.5 says the RHS is (isomorphic to) the nerve of the following category $\tilde{\mathcal{H}}$:

- The objects in $\tilde{\mathcal{H}}$ are objects in $\mathcal{H}$, i.e. zigzags in $\mathcal{M}$ of the following type:

$$A \xleftarrow{\text{t. fib.}} \bullet \longrightarrow \bullet \xleftarrow{\text{t. cofib.}} B$$

- The morphisms in $\tilde{\mathcal{H}}$ are commutative diagrams of the form below,

$$\begin{tikzcd} A \arrow[d, equal] & \bullet \arrow[l] \arrow[r] & \bullet \arrow[d] & B \arrow[l] \arrow[d, equal] \\ A & \bullet \arrow[l] \arrow[r] \arrow[u] & \bullet & B \arrow[l] \end{tikzcd}$$

  where the top row is the domain and the bottom row is the codomain.

Thus, by lemma 2.8.31, we have the following zigzag of weak homotopy equivalences of simplicial sets:

$$\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right) \longleftarrow \bullet \longrightarrow \mathrm{N}\left(\tilde{\mathcal{H}}\right) \longleftarrow \bullet \longrightarrow \mathrm{N}(\mathcal{H})$$

In particular, $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right) \simeq \mathrm{N}(\mathcal{H})$. ∎

Lemma 4.9.6. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let S and T be* possibly degenerate *marked zigzag types, let $\mathcal{H}_0$ be the category of zigzags in $\mathcal{M}$ of the following type,*

$$A \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} B$$

let $\mathcal{H}_1$ be the category of zigzags in $\mathcal{M}$ of the following type,

$$A \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} B$$

let $d : \mathcal{H}_1 \to \mathcal{H}_0$ be the functor that composes the two arrows in the middle of the zigzag, and let f be an object in $\mathcal{H}_0$, say:

$$A \rightsquigarrow C \xleftarrow{w} D \rightsquigarrow B$$

- *If $u^\bullet : \operatorname{cosk}^0(D) \to \tilde{C}^\bullet$ is a Reedy cofibration in $\mathbf{c}\mathcal{M}$ and $v^\bullet : \tilde{C}^\bullet \to \operatorname{cosk}^0(C)$ is a degreewise trivial fibration in $\mathbf{c}\mathcal{M}$ such that $v^\bullet \circ u^\bullet = \operatorname{cosk}^0(w)$, then $u^\bullet$ and $v^\bullet$ define a cosimplicial object in the comma category $(f \downarrow d)$, and the corresponding functor $\tilde{F} : \mathbf{\Delta} \to (f \downarrow d)$ is a right aspherical functor.*

- *If $u_\bullet : \operatorname{sk}_0(D) \to \hat{D}_\bullet$ is a degreewise trivial cofibration and $v_\bullet : \hat{D}_\bullet \to \operatorname{sk}_0(C)$ is a Reedy fibration in $\mathbf{s}\mathcal{M}$ in $\mathbf{s}\mathcal{M}$ such that $v_\bullet \circ u_\bullet = \operatorname{sk}_0(w)$, then $u_\bullet$ and $v_\bullet$ define a simplicial object in the comma category $(d \downarrow f)$, and the corresponding functor $\hat{F} : \mathbf{\Delta}^{\text{op}} \to (d \downarrow f)$ is a left aspherical functor.*

Proof. The two claims are formally dual; we will prove the first version. We follow paragraph 8.1 in [Dwyer and Kan, 1980c].

First, note that by proposition 4.6.4, each $u^n : D \to \tilde{C}^n$ is a cofibration in $\mathcal{M}$, and since $u^n \circ u^n = w$, axiom CM2 implies that each $u^n : D \to \tilde{C}^n$ is a trivial cofibration in $\mathcal{M}$. Thus $u^\bullet$ and $v^\bullet$ do indeed define a cosimplicial object $\left(\tilde{f}^\bullet, \mathrm{id}\right)$ in $(f \downarrow d)$.

Let $\tilde{F} : \mathbf{\Delta} \to (f \downarrow d)$ be the functor corresponding to $\left(\tilde{f}^\bullet, \mathrm{id}\right)$, let g be an object in $\mathcal{H}_1$, say

$$A \rightsquigarrow K \xleftarrow{q} M \xleftarrow{j} L \rightsquigarrow B$$

and let $h : f \to d(g)$ be a morphism in $\mathcal{H}_0$, say:

$$\begin{tikzcd}
A \arrow[r, squiggly, dash] \arrow[d, equal] & C \arrow[d, "k"'] & D \arrow[l, "w"'] \arrow[r, squiggly, dash] \arrow[d, "l"] & B \arrow[d, equal] \\
A \arrow[r, squiggly, dash] & K & L \arrow[l, "q \circ j"] \arrow[r, squiggly, dash] & B
\end{tikzcd}$$

We must show that the comma category $\big(\tilde{F} \downarrow (g, h)\big)$ is aspherical. But by corollary 1.9.31,

$$\mathrm{N}\big(\big(\tilde{F} \downarrow (g, h)\big)\big) \simeq \mathcal{H}om_{(f\downarrow d)}\big(\big(\tilde{f}, \mathrm{id}\big), (g, h)\big)$$

and it is not hard to see that there is a pullback diagram in **sSet** of the form below,

$$\begin{tikzcd}
\mathcal{H}om_{(f\downarrow d)}\big(\tilde{f}, g\big) \arrow[r] \arrow[d] & \mathcal{H}om_{D/\mathcal{M}}\big(\big(\tilde{C}, u\big), (M, j \circ l)\big) \arrow[d, "\mathcal{H}om_{D/\mathcal{M}}((\tilde{C}, v), q)"] \\
\Delta^0 \arrow[r] & \mathcal{H}om_{D/\mathcal{M}}\big(\big(\tilde{C}, u\big), (K, l)\big)
\end{tikzcd}$$

where the bottom horizontal arrow corresponds to the vertex $k \circ v^0 : \tilde{C}^0 \to K$; thus (by proposition A.3.17) it suffices to prove that the right vertical arrow is a trivial Kan fibration. But by proposition 4.6.5, the unique morphism $\big(\operatorname{cosk}^0(D), \mathrm{id}\big) \to \big(\tilde{C}, u\big)$ in ${}^{\operatorname{cosk}^0(D)/}\mathbf{c}\mathcal{M}$ is a Reedy cofibration in $\mathbf{c}\big({}^{D/}\mathcal{M}\big)$ (because $u^\bullet : \operatorname{cosk}^0(D) \to \tilde{C}^\bullet$ is a Reedy cofibration in $\mathbf{c}\mathcal{M}$), so by lemma 4.1.16, $\big(\tilde{C}, u\big)$ is a Reedy-cofibrant object in $\mathbf{c}\big({}^{D/}\mathcal{M}\big)$, and therefore

$$\mathcal{H}om_{D/\mathcal{M}}\big(\big(\tilde{C}, v\big), q\big) : \mathcal{H}om_{D/\mathcal{M}}\big(\big(\tilde{C}, v\big), (M, j \circ l)\big) \to \mathcal{H}om_{D/\mathcal{M}}\big(\big(\tilde{C}, v\big), (K, l)\big)$$

is a trivial Kan fibration by lemma 4.8.14. This shows that $\big(\tilde{F} \downarrow (g, h)\big)$ is an aspherical category, and since (g, h) is arbitrary, we conclude that $\tilde{F} : \mathbf{\Delta} \to (f \downarrow d)$ is a right aspherical functor. ■

Proposition 4.9.7. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let S and T be* possibly degenerate *marked zigzag types, let $\mathcal{H}_0$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,*

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

let $\mathcal{H}_1$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

and let $d : \mathcal{H}_1 \to \mathcal{H}_0$ be the functor that composes the two arrows in the middle of the zigzag.

- *If $\mathcal{M}$ is a cosimplicially resolvable category, then $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a left aspherical functor.*

- *If $\mathcal{M}$ is a simplicially resolvable category, then $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a right aspherical functor.*

In particular, if $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a weak homotopy equivalence of categories.

Proof. The two claims are formally dual; we will prove the first version.

To show that $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a left aspherical functor, we must verify that the comma category $(f \downarrow d)$ is aspherical for every object f in $\mathcal{H}_0$. Since $\mathbf{\Delta}$ is an aspherical category (by remark 1.11.6), it suffices to find a weak homotopy equivalence $\mathbf{\Delta} \to (f \downarrow d)$; but Quillen's Theorem A (corollary 1.11.15) says any left or right aspherical functor is a weak homotopy equivalence of categories, so the claim is a consequence of lemma 4.9.6. ■

Remark 4.9.8. In view of Quillen's Theorem A (corollary 1.11.15), the above proposition essentially says that the factorisation of a weak equivalence into a trivial cofibration and a trivial fibration (as in lemma 4.1.10) is homotopically unique: indeed, if we take S and T to be the trivial zigzag types of length 0, then $\mathcal{H}_0$ is a discrete category, and since $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a weak homotopy equivalence of categories, the (strict) fibres of d must therefore be aspherical categories.

Lemma 4.9.9. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let S and T be* possibly degenerate *marked zigzag types, let k be a natural number, let $\mathcal{H}_i$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,*

$(i = 1)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\text{of type } S"] & \bullet \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, squiggly, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 2)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, squiggly, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 3)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\text{of type } S"] & \bullet \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, squiggly, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 4)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, squiggly, "\text{of type } T"] & \bullet \end{tikzcd}$$

and let $\mathcal{H}'_i$ $(0 \leq i \leq 4)$ be the category of zigzags in $\mathcal{M}$ of the following type:

$(i = 1)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet \arrow[r, squiggly, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 2)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 3)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

$(i = 4)$
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

- *Let $s : \mathcal{H}_1 \to \mathcal{H}_2$ (resp. $s : \mathcal{H}_3 \to \mathcal{H}_4$) be the functor defined by inserting an identity morphism. There is a functor $d : \mathcal{H}_2 \to \mathcal{H}_1$ (resp. $d : \mathcal{H}_4 \to \mathcal{H}_3$) for which there is a natural transformation $\mathrm{id}_{\mathcal{H}_2} \Rightarrow s \circ d$ (resp. $\mathrm{id}_{\mathcal{H}_4} \Rightarrow s \circ d$) where the component at an object, say*

  $$\begin{tikzcd} A \arrow[r, squiggly, no head] & \bullet & \bullet \arrow[l, "u"'] \arrow[r] & \cdots \arrow[r] & \bullet & \bullet \arrow[l, "w"'] \arrow[r, squiggly, no head] & B \end{tikzcd}$$

  is given by the following commutative diagram in $\mathcal{M}$,

  $$\begin{tikzcd} A \arrow[r, squiggly, no head] \arrow[d, equal] & \bullet \arrow[d, equal] & \bullet \arrow[l, "u"'] \arrow[r] \arrow[d, "u"'] & \cdots \arrow[r] & \bullet \arrow[d, "u'"] & \bullet \arrow[l, "w"'] \arrow[r, squiggly, no head] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, squiggly, no head] & \bullet \arrow[r, equal] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet & \bullet \arrow[l, "w \circ u'"] \arrow[r, squiggly, no head] & B \end{tikzcd}$$

  where the middle squares are pushout diagrams in $\mathcal{M}$. Moreover, $d \circ s \cong \mathrm{id}_{\mathcal{H}_1}$ (resp. $d \circ s \cong \mathrm{id}_{\mathcal{H}_3}$).

- *Let $s : \mathcal{H}'_1 \to \mathcal{H}'_2$ (resp. $s : \mathcal{H}'_3 \to \mathcal{H}'_4$) be the functor defined by inserting an identity morphism. There is a functor $d : \mathcal{H}'_2 \to \mathcal{H}'_1$ (resp. $d : \mathcal{H}'_4 \to \mathcal{H}'_3$) for which there is a natural transformation $\mathrm{id}_{\mathcal{H}'_2} \Rightarrow s \circ d$ (resp. $\mathrm{id}_{\mathcal{H}'_4} \Rightarrow s \circ d$) where the component at an object, say*

  $$\begin{tikzcd} A \arrow[r, squiggly, no head] & \bullet & \bullet \arrow[l, "w"'] \arrow[r] & \cdots \arrow[r] & \bullet & \bullet \arrow[l, "v"'] \arrow[r, squiggly, no head] & B \end{tikzcd}$$

  is given by the following commutative diagram in $\mathcal{M}$,

  $$\begin{tikzcd} A \arrow[r, squiggly, no head] \arrow[d, equal] & \bullet \arrow[d, equal] & \bullet \arrow[l, "w \circ v'"'] \arrow[r] \arrow[d, "v'"'] & \cdots \arrow[r] & \bullet \arrow[r, equal] \arrow[d, "v"] & \bullet \arrow[r, squiggly, no head] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, squiggly, no head] & \bullet & \bullet \arrow[l, "w"] \arrow[r] & \cdots \arrow[r] & \bullet & \bullet \arrow[l, "v"] \arrow[r, squiggly, no head] & B \end{tikzcd}$$

  where the middle squares are pullback diagrams in $\mathcal{M}$. Moreover, $d \circ s \cong \mathrm{id}_{\mathcal{H}'_1}$ (resp. $d \circ s \cong \mathrm{id}_{\mathcal{H}'_3}$).

Proof. This is a straightforward consequence of corollary 4.1.13. ♦

Corollary 4.9.10. *With notation as in the lemma:*

- *The functors $s : \mathcal{H}_1 \to \mathcal{H}_2$ and $s : \mathcal{H}_3 \to \mathcal{H}_4$ are weak homotopy equivalences of categories.*
- *The functors $s : \mathcal{H}'_1 \to \mathcal{H}'_2$ and $s : \mathcal{H}'_3 \to \mathcal{H}'_4$ are weak homotopy equivalences of categories.*

Proof. Apply lemma 1.3.10 and proposition 1.5.3 to lemma 4.9.9. ■

Proposition 4.9.11. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let k and l be natural numbers, let S and T be* possibly degenerate *marked zigzag types, let $\mathcal{H}_0$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,*

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \overset{l \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

let $\mathcal{H}_1$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{l \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

and let $s : \mathcal{H}_0 \to \mathcal{H}_1$ be the functor defined by inserting an identity morphism. If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then $s : \mathcal{H}_0 \to \mathcal{H}_1$ is a weak homotopy equivalence of categories.

Proof. Let $\mathcal{H}_2$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,

$$\bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{l \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

let $s^2 : \mathcal{H}_0 \to \mathcal{H}_2$ be the functor defined by inserting two identity morphisms, and let $d : \mathcal{H}_2 \to \mathcal{H}_1$ be the functor defined by composing the two arrows in the middle. By factoring s^2 appropriately, we may apply corollary 4.9.10 (twice) to deduce that $s^2 : \mathcal{H}_0 \to \mathcal{H}_2$ is a weak homotopy equivalence of categories. On the other hand, by proposition 4.9.7, if $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then $d : \mathcal{H}_2 \to \mathcal{H}_1$ is a weak homotopy equivalence of categories. It is clear that $d \circ s^2 = s$, so (by lemma 1.11.3) $s : \mathcal{H}_0 \to \mathcal{H}_1$ is also a weak homotopy equivalence of categories. ■

Corollary 4.9.12. *If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then:*

(i) *$\mathcal{M}$ admits a homotopical three-arrow calculus.*

(ii) *Let A and B be objects in $\mathcal{M}$ and let $\mathcal{H}(A, B)$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type:*

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet$$

Then the reduction morphism $\mathrm{N}(\mathcal{H}(A, B)) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{M})(A, B)$ *is a weak homotopy equivalence.*

Proof. (i). This is the special case of proposition 4.9.11 where S and T are the degenerate zigzag type.

(ii). Apply the fundamental theorem of homotopical three-arrow calculi (2.8.27). ■

Lemma 4.9.13. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let k be a natural number, let S and T be* possibly degenerate *marked zigzag types, and consider the categories of zigzags in $\mathcal{M}$ from A to B of the following types:*

($\mathcal{H}_0$)
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

($\tilde{\mathcal{H}}_0$)
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

($\hat{\mathcal{H}}_0$)
$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then:

- *The evident inclusion $\tilde{\mathcal{H}}_0 \hookrightarrow \mathcal{H}_0$ is a weak homotopy equivalence of categories.*
- *The evident inclusion $\hat{\mathcal{H}}_0 \hookrightarrow \mathcal{H}_0$ is a weak homotopy equivalence of categories.*

Proof. The two claims are formally dual; we will prove the first version.

Let $\mathcal{H}_1$ be the category of zigzags in $\mathcal{M}$ from A to B of the following type,

$$\begin{tikzcd} \bullet \arrow[r, squiggly, no head, "\text{of type } S"] & \bullet & \bullet \arrow[l, "\text{w.e.}"'] \arrow[r, dotted, "k \text{ arrows}"] & \bullet & \bullet \arrow[l, "\text{t. fib.}"'] & \bullet \arrow[l, "\text{t. cofib.}"'] \arrow[r, squiggly, no head, "\text{of type } T"] & \bullet \end{tikzcd}$$

let $d : \mathcal{H}_1 \to \mathcal{H}_0$ be the functor defined by composing the evident pair of morphism, and let $s : \tilde{\mathcal{H}}_0 \to \mathcal{H}_1$ be defined by inserting an identity morphism. By corollary 4.9.10, $s : \tilde{\mathcal{H}}_0 \to \mathcal{H}_1$ is a weak homotopy equivalence of categories, and proposition 4.9.7 says that $d : \mathcal{H}_1 \to \mathcal{H}_0$ is a weak homotopy equivalence of categories if $\mathcal{M}$ is either cosimplicially or simplicially resolvable; but it is clear that $d \circ s : \tilde{\mathcal{H}}_0 \to \mathcal{H}_0$ is just the inclusion, so the claim follows (by lemma 1.11.3). ■

Lemma 4.9.14. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let k be a natural number, let S and T be* possibly degenerate *marked zigzag types, and consider the categories of zigzags in $\mathcal{M}$ from A to B of the following types,*

$$(\mathcal{H}_0) \qquad \bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

$$(\tilde{\mathcal{H}}_0) \qquad \bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

$$(\mathcal{H}'_0) \qquad \bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

$$(\hat{\mathcal{H}}'_0) \qquad \bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{t. fib.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{t. cofib.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

where in the middle we have k rightward-pointing arrows. If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then:

- *The evident inclusion $\tilde{\mathcal{H}}_0 \hookrightarrow \mathcal{H}_0$ is a weak homotopy equivalence of categories.*
- *The evident inclusion $\hat{\mathcal{H}}'_0 \hookrightarrow \mathcal{H}'_0$ is a weak homotopy equivalence of categories.*

Proof. The proof is essentially the same as lemma 4.9.13. ■

Proposition 4.9.15. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, , let k be a natural number, let S and T be* possibly degenerate *marked zigzag types, and consider the categories of zigzags in $\mathcal{M}$ from A to B of the following types:*

$$(\mathcal{H}) \qquad \bullet \overset{\text{of type } S}{\rightsquigarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{k \text{ arrows}}{\dashrightarrow} \bullet \xleftarrow{\text{w.e.}} \bullet \overset{\text{of type } T}{\rightsquigarrow} \bullet$$

$$(\mathcal{H}'') \qquad \bullet \xrightarrow[\sim\sim]{\text{of type } S} \bullet \xleftarrow{\text{t. fib.}} \bullet \overset{k\ arrows}{\dashrightarrow} \bullet \xleftarrow{\text{t. cofib.}} \bullet \xrightarrow[\sim\sim]{\text{of type } T} \bullet$$

If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then the evident inclusion $\mathcal{H}'' \hookrightarrow \mathcal{H}$ is a weak homotopy equivalence of categories.

Proof. Recalling that the class of weak homotopy equivalences of categories is closed under composition (by lemma 1.11.3), this is an immediate consequence of lemmas 4.9.13 and 4.9.14. ∎

Corollary 4.9.16. *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, and consider the categories of zigzags in $\mathcal{M}$ from A to B of the following types:*

$$(\mathcal{H}) \qquad \bullet \xleftarrow{\text{w.e.}} \bullet \longrightarrow \bullet \xleftarrow{\text{w.e.}} \bullet$$

$$(\mathcal{H}'') \qquad \bullet \xleftarrow{\text{t. fib.}} \bullet \longrightarrow \bullet \xleftarrow{\text{t. cofib.}} \bullet$$

If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then the evident inclusion $\mathcal{H}'' \hookrightarrow \mathcal{H}$ is a weak homotopy equivalence of categories. ∎

Theorem 4.9.17 (Dwyer and Kan)**.** *Let $\mathcal{M}$ be a derivable category, let A and B be objects in $\mathcal{M}$, let $\tilde{A}^\bullet$ be (the object part of) a cosimplicial resolution of A, and let $\hat{B}_\bullet$ be (the object part of) a simplicial resolution of B. If $\mathcal{M}$ is either cosimplicially or simplicially resolvable, then*

$$\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right) \simeq \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{M})(A, B)$$

by a zigzag of weak homotopy equivalences.

Proof. Combine corollaries 4.9.5, 4.9.12, and 4.9.16. ∎

4.10 Virtual cofibrancy and fibrancy

Prerequisites. §§1.1, 3.1, 3.3, 4.1, 4.6, A.1, A.5.

In this section, we follow [DHKS, §23]. As usual, for each natural number n, let $[n]$ denote the category $\{0 \to \cdots \to n\}$ corresponding to the finite ordinal $\{0, \ldots, n\}$, and let $\mathbf{\Delta}$ be the category whose objects are the $[n]$ and whose morphisms are functors.

Definition 4.10.1. The **category of simplices** of a (small) category $\mathbb{C}$ is the category $\mathbf{\Delta}(\mathbb{C})$ defined below:

- The objects are functors $[n] \to \mathbb{C}$.
- The morphisms $(f : [m] \to \mathbb{C}) \to (g : [n] \to \mathbb{C})$ are functors $[m] \to [n]$ making the evident triangle commute (strictly).
- Composition and identities are the obvious ones.

We write $\pi_\Delta : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}$ for the evident projection functor that sends an object $[n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ to the object $[n]$ in $\mathbf{\Delta}$.

¶ 4.10.2. To elucidate the above definition, it is helpful to introduce some notation for the objects in $\mathbf{\Delta}(\mathbb{C})$. It is not hard to see that a functor $f : [n] \to \mathbb{C}$ is the same thing as a string of n composable morphisms in $\mathbb{C}$, e.g.

$$A_0 \xrightarrow{f_1} A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \xrightarrow{f_n} A_n$$

so let us write $\left[A_0 \overset{f_1}{\to} A_1 \cdots A_{n-1} \overset{f_n}{\to} A_n\right]$ for the corresponding object in $\mathbf{\Delta}(\mathbb{C})$. Since the projection $\pi_\Delta : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}$ is faithful, we may borrow the notation of §1.1 and write e.g. $\delta^1 : \left[A_0\right] \to \left[A_0 \overset{f_1}{\to} A_1\right]$ for the unique morphism whose image under π_Δ is $\delta^1 : [0] \to [1]$.

Observe that, given a commutative triangle in $\mathbb{C}$ of the form below,

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

we obtain the following commutative diagram in $\mathbf{\Delta}(\mathbb{C})$:

$$\begin{tikzcd}
 & & {[B]} \arrow[dl, "\delta^1"'] \arrow[dr, "\delta^0"] \arrow[dd] & & \\
 & \left[A \xrightarrow{f} B\right] \arrow[dr, "\delta^2"] & & \left[B \xrightarrow{g} C\right] \arrow[dl, "\delta^0"] & \\
 & & \left[A \xrightarrow{f} B \xrightarrow{g} C\right] & & \\
{[A]} \arrow[uur, "\delta^1"] \arrow[urr] \arrow[rr, "\delta^1"'] & & \left[A \xrightarrow{h} C\right] \arrow[u, "\delta^1"] & & {[C]} \arrow[uul, "\delta^0"'] \arrow[ull] \arrow[ll, "\delta^0"]
\end{tikzcd}$$

Similar phenomena occur for longer strings of composable morphisms. Thus, one may think of $\mathbf{\Delta}(\mathbb{C})$ as being a kind of barycentric subdivision of $\mathbb{C}$; notice also that the Mac Lane subdivision category $\mathbb{C}^{\S}$ occurs as a subcategory of $\mathbf{\Delta}(\mathbb{C})$.

Remark 4.10.3. There is an obvious natural isomorphism $\mathbf{\Delta}(\mathbb{C}) \cong \mathbf{\Delta}(\mathbb{C}^{\mathrm{op}})$ such that the following diagram of functors commutes,

$$\begin{tikzcd}
\mathbf{\Delta}(\mathbb{C}) \arrow[r, "\cong"] \arrow[d, "\pi_\Delta"'] & \mathbf{\Delta}(\mathbb{C}^{\mathrm{op}}) \arrow[d, "\pi_\Delta"] \\
\mathbf{\Delta} \arrow[r, "(-)^{\mathrm{op}}"'] & \mathbf{\Delta}
\end{tikzcd}$$

but in general there is no isomorphism between $\mathbf{\Delta}(\mathbb{C})$ and $\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}$.

Proposition 4.10.4. *Let X be a simplicial set, and let $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ be the inclusion of the standard simplices.*

(i) *The comma category $(\Delta^\bullet \downarrow X)$ is a Reedy category, where the direct subcategory consists of all face operators and the inverse subcategory consists of all degeneracy operators.*

(ii) *Moreover, $(\Delta^\bullet \downarrow X)$ has fibrant constants.*

Proof. (i). The evident projection $(\Delta^\bullet \downarrow X) \to \mathbf{\Delta}$ is a discrete right fibration, so the Reedy category structure on $\mathbf{\Delta}$ induces one on $(\Delta^\bullet \downarrow X)$.

(ii). See Proposition 15.10.4 in [Hirschhorn, 2003]. $\square$

Corollary 4.10.5. *The category $\mathbf{\Delta}(\mathbb{C})$ of simplices of a (small) category $\mathbb{C}$ admits a Reedy category structure with fibrant constants.*

Proof. It is not hard to see that the category $\mathbf{\Delta}(\mathbb{C})$ as defined above is isomorphic to the comma category $(\Delta^\bullet \downarrow \mathrm{N}(\mathbb{C}))$, where $\mathrm{N}(\mathbb{C})$ is the nerve of $\mathbb{C}$. ∎

Corollary 4.10.6. *If $\mathcal{M}$ is a DHK model category and $\mathbb{C}$ is a small category, then:*

- *The functor $\varinjlim_{\mathbf{\Delta}(\mathbb{C})} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to \mathcal{M}$ sends Reedy weak equivalences between Reedy-cofibrant diagrams to weak equivalences between cofibrant objects.*
- *The functor $\varprojlim_{\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to \mathcal{M}$ sends Reedy weak equivalences between Reedy-fibrant diagrams to weak equivalences between fibrant objects.*

Proof. Apply Ken Brown's lemma (4.3.6) and corollary 4.6.26. ∎

Lemma 4.10.7. *Let $F : \mathbb{C} \to \mathbb{D}$ be a functor between (small) categories.*

(i) $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ *is a left fibration of Reedy categories.*

(ii) $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ *is a right fibration of Reedy categories.*

Proof. (i). Let $[D_0 \cdots D_n]$ be an object in $\mathbf{\Delta}(\mathbb{D})$, let $\big([C_0 \cdots C_m], h\big)$ be an object in the comma category $\big(\mathbf{\Delta}(F) \downarrow [D_0 \cdots D_n]\big)$. We will show that the matching category

$$\partial\Big(\big([C_0 \cdots C_m], h\big) \downarrow \big(\mathbf{\Delta}(F) \downarrow [D_0 \cdots D_n]\big)_{\leftarrow}\Big)$$

has at most one connected component.

First, note that the objects of this matching category are pairs (k, l), where k is in $\mathbf{\Delta}(\mathbb{C})_{\leftarrow}$, $k \neq \mathrm{id}_{[C_0 \cdots C_m]}$, l is in $\mathbf{\Delta}(\mathbb{D})$, and $h = l \circ \mathbf{\Delta}(F)k$. Let (σ, δ) be the codegeneracy–coface factorisation of $\pi_\Delta h$ in $\mathbf{\Delta}$.

- If $\sigma = \mathrm{id}_{[m]}$, then the matching category must be empty.

- If $\sigma \neq \mathrm{id}_{[m]}$, then we may lift (σ, δ) along the respective π_Δ projections to obtain a terminal object in the matching category, so the matching category is connected *a fortiori*.

Thus, by theorem 4.6.32, $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ is a left fibration of Reedy categories.

(ii). A similar argument shows that $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ is a right fibration of Reedy categories. ∎

Corollary 4.10.8. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *The functor $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ is a right Quillen functor.*

(ii) *The functor $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ is a left Quillen functor.*

Proof. Apply theorem 4.6.32. ∎

Definition 4.10.9. Let $\mathbb{C}$ be a (small) category and let $\mathbf{\Delta}(\mathbb{C})$ be its category of simplices.

- The **left projection functor** $\pi_{\mathrm{L}} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathbb{C}$ is the functor defined by evaluating objects $f : [n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ at the initial object in $[n]$.
- The **right projection functor** $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{C}) \to \mathbb{C}$ is the functor defined by evaluating objects $f : [n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ at the terminal object in $[n]$.
- A **strong left equivalence** in $\mathbf{\Delta}(\mathbb{C})$ is a morphism such that the underlying map in $\mathbf{\Delta}$ preserves the initial object.
- A **strong right equivalence** in $\mathbf{\Delta}(\mathbb{C})$ is a morphism such that the underlying map in $\mathbf{\Delta}$ preserves the terminal object.
- The class of **weak left equivalences** in $\mathbf{\Delta}(\mathbb{C})$ is the smallest subcategory that has the 2-out-of-6 property and contains all the strong left equivalences.
- The class of **weak right equivalences** in $\mathbf{\Delta}(\mathbb{C})$ is the smallest subcategory that has the 2-out-of-6 property and contains all the strong right equivalences.

We write $\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}$ for the category of simplices of $\mathbb{C}$ regarded as a relative category with weak equivalences the strong left equivalences, and we write $\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}$ for the category of simplices of $\mathbb{C}$ regarded as a relative category with weak equivalences the strong right equivalences.

Remark 4.10.10. The strong left (resp. right) equivalences in $\mathbf{\Delta}(\mathbb{C})$ are closed under composition, and the left (resp. right) projection to $\mathbb{C}$ sends strong left (resp. right) equivalences to identity morphisms, so if we regard $\mathbf{\Delta}(\mathbb{C})$ as a relative category with weak equivalences the strong left (resp. right) equivalences, then the left (resp. right) projection functor becomes a relative functor.

Unfortunately, the subcategory of strong left (resp. right) equivalences in $\mathbf{\Delta}(\mathbb{C})$ does not generally have the 2-out-of-6 property, or even the 2-out-of-3 property; one may rectify this by instead considering the class of weak left (resp. right) equivalences. An example of a weak left equivalence that is not a strong left equivalence is the morphism $\delta^0 : \left[A \overset{\mathrm{id}}{\to} A\right] \to [A]$: this is a weak left equivalence because $\sigma^0 : [A] \to \left[A \overset{\mathrm{id}}{\to} A\right]$ is a strong left equivalence and $\delta^0 \circ \sigma^0 = \mathrm{id}_{[A]}$, but δ^0 is not a strong left equivalence because the underlying cosimplicial operator in $\mathbf{\Delta}$ sends 0 in [0] to 1 in [1].

Remark 4.10.11. It is not hard to see that $\mathbf{\Delta}(-)$ is a functor $\mathbf{Cat} \to \mathbf{Cat}$ and that π_{L} (resp. π_{R}) defines a natural transformation $\mathbf{\Delta}(-)^{\mathrm{op}} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ (resp. $\mathbf{\Delta}(-) \Rightarrow \mathrm{id}_{\mathbf{Cat}}$).

Lemma 4.10.12. *Let $F : \mathbb{C} \to \mathbb{D}$ be a functor, let $\pi_{\mathrm{L}} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathbb{C}$ be the left projection functor, and let $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{C}) \to \mathbb{C}$ be the right projection functor. Then, for any object D in $\mathbb{D}$:*

- *The canonical comparison functor $\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \to \left(D \downarrow F\pi_{\mathrm{L}}\right)$ is an isomorphism.*
- *The canonical comparison functor $\mathbf{\Delta}((F \downarrow D)) \to \left(F\pi_{\mathrm{R}} \downarrow D\right)$ is an isomorphism.*

Proof. The two claims are formally dual; we will prove the first version.

As always, the comma category $(D \downarrow F)$ fits into a comma square,

$$\begin{tikzcd}
(D \downarrow F) \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbb{C} \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "D"'] & \mathbb{D}
\end{tikzcd}$$

and the following diagram of functors commutes,

$$\begin{tikzcd}
\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \arrow[r, "\mathbf{\Delta}(P)^{\mathrm{op}}"] \arrow[d, "\pi_{\mathrm{L}}"'] & \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \arrow[d, "\pi_{\mathrm{L}}"] \\
(D \downarrow F) \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

so the universal property of $(D \downarrow F)$ gives us a canonical comparison functor $\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \to \left(D \downarrow F\pi_{\mathrm{L}}\right)$, as claimed. It is not hard to check that the second diagram is a pullback square, so the pasting lemma for comma squares implies that the comparison functor is an isomorphism. ■

Proposition 4.10.13. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

- *The functor* $\mathrm{Ran}_{F\pi_{\mathrm{L}}} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends Reedy weak equivalences between Reedy-fibrant diagrams to componentwise weak equivalences between componentwise fibrant diagrams.*

- *The functor* $\mathrm{Lan}_{F\pi_{\mathrm{R}}} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends Reedy weak equivalences between Reedy-cofibrant diagrams to componentwise weak equivalences between componentwise cofibrant diagrams.*

Proof. The two claims are formally dual; we will prove the second version.

Using the formula for $\mathrm{Lan}_{F\pi_{\mathrm{R}}}$ given by theorem A.5.15, we see that, for each object D in $\mathbb{D}$, the functor $\left(\mathrm{Lan}_{F\pi_{\mathrm{R}}} -\right)(D) : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to \mathcal{M}$ is naturally isomorphic to the functor $\varinjlim : \left[\left(F\pi_{\mathrm{R}} \downarrow D\right), \mathcal{M}\right] \to \mathcal{M}$; but by lemma 4.10.12, there is a canonical isomorphism $\left(F\pi_{\mathrm{R}} \downarrow D\right) \cong \mathbf{\Delta}((F \downarrow D))$, so $\left(\mathrm{Lan}_{F\pi_{\mathrm{R}}} -\right)(D)$ is in turn naturally isomorphic to $\varinjlim : [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \to \mathcal{M}$. The claim now follows from corollary 4.10.6. ■

Theorem 4.10.14. *Let $\mathcal{M}$ be a DHK model category and let $\mathbb{C}$ be a small category.*

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\pi_{\mathrm{L}}^{*} \dashv \mathrm{Ran}_{\pi_{\mathrm{L}}} : \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}^{\mathrm{op}}, \mathcal{M}\right]_{\mathrm{h}} \to [\mathbb{C}, \mathcal{M}]$$

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [\mathbb{C}, \mathcal{M}] \to \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$

Proof. See Proposition 23.2 in [DHKS]. □

Definition 4.10.15. Let $\mathcal{M}$ be a DHK model category and let $\mathbb{C}$ be a small category.

- A **virtually cofibrant diagram** $X : \mathbb{C} \to \mathcal{M}$ is one for which there exists a Reedy-cofibrant diagram $\tilde{X} : \mathbf{\Delta}(\mathbb{C}) \to \mathcal{M}$ such that $\tilde{X}$ is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$ and $X \cong \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{X}$.

- A **virtually fibrant diagram** $Y : \mathbb{C} \to \mathcal{M}$ is one for which there exists a Reedy-fibrant diagram $\hat{Y} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathcal{M}$ such that $\hat{Y}$ is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}^{\mathrm{op}}, \mathcal{M}\right]_{\mathrm{h}}$ and $Y \cong \mathrm{Ran}_{\pi_{\mathrm{L}}} \hat{Y}$.

We write $[\mathbb{C}, \mathcal{M}]_{\mathrm{vc}}$ for the full subcategory of $[\mathbb{C}, \mathcal{M}]$ spanned by the virtually cofibrant diagrams, and we write $[\mathbb{C}, \mathcal{M}]_{\mathrm{vf}}$ for the full subcategory of $[\mathbb{C}, \mathcal{M}]$ spanned by the virtually fibrant diagrams.

Theorem 4.10.16. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *The functor* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends virtually cofibrant diagrams to componentwise cofibrant diagrams and preserves componentwise weak equivalences between such diagrams.*

(ii) *If* $\mathrm{Lan}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}]$ *moreover restricts to a functor* $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \to \left[\mathbf{\Delta}(\mathbb{D})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$*, then* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *preserves virtually cofibrant diagrams.*

(iii) *If (Q, p) is a cofibrant replacement functor for* $[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$*, then*

$$\left([\mathbb{C}, \mathcal{M}]_{\mathrm{vc}}, \mathrm{Lan}_{\pi_{\mathrm{R}}} \circ Q \circ \pi_{\mathrm{R}}^*, \varepsilon \bullet \left(\mathrm{Lan}_{\pi_{\mathrm{R}}} \circ p \circ \pi_{\mathrm{R}}^*\right)\right)$$

is a functorial left deformation retract for Lan_F*, where ε is the counit of the adjunction* $\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^*$.

(iv) *The adjunction shown below is deformable:*

$$\mathrm{Lan}_F \dashv F^* : [\mathbb{D}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$$

(v) *Given another functor* $G : \mathbb{D} \to \mathbb{E}$ *between small categories,* $(\mathrm{Lan}_G, \mathrm{Lan}_F)$ *is strongly left deformable.*

Dually:

(i′) *The functor* $\mathrm{Ran}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends virtually fibrant diagrams to componentwise fibrant diagrams and preserves componentwise weak equivalences between such diagrams.*

(ii′) *If* $\mathrm{Ran}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D})^{\mathrm{op}}, \mathcal{M}]$ *moreover restricts to a functor* $\left[\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}_{\mathrm{L}}, \mathcal{M}\right]_{\mathrm{h}} \to \left[\mathbf{\Delta}(\mathbb{D})^{\mathrm{op}}_{\mathrm{L}}, \mathcal{M}\right]_{\mathrm{h}}$*, then* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *preserves virtually cofibrant diagrams.*

(iii′) *If* (R, i) *is a fibrant replacement functor for* $[\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}]$*, then*

$$\left([\mathbb{C}, \mathcal{M}]_{\mathrm{vf}}, \mathrm{Ran}_{\pi_{\mathrm{L}}} \circ R \circ \pi_{\mathrm{L}}^*, \left(\mathrm{Ran}_{\pi_{\mathrm{L}}} \circ i \circ \pi_{\mathrm{L}}^*\right) \bullet \eta\right)$$

is a functorial right deformation retract for Ran_F*, where* η *is the unit of the adjunction* $\pi_{\mathrm{L}}^* \dashv \mathrm{Ran}_{\pi_{\mathrm{L}}}$.

(iv′) *The adjunction shown below is deformable:*

$$F^* \dashv \mathrm{Ran}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$$

(v′) *Given another functor* $G : \mathbb{D} \to \mathbb{E}$ *between small categories,* $(\mathrm{Ran}_G, \mathrm{Ran}_F)$ *is strongly right deformable.*

Proof. (i). Let $\tilde{X}$ be a Reedy-cofibrant diagram $\mathbb{C} \to \mathcal{M}$ that is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$ and let $X = \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{X}$. There is a canonical isomorphism $\mathrm{Lan}_{F\pi_{\mathrm{R}}} \cong \mathrm{Lan}_F \circ \mathrm{Lan}_{\pi_{\mathrm{R}}}$, so proposition 4.10.13 implies $\mathrm{Lan}_F X$ is a componentwise cofibrant diagram $\mathbb{D} \to \mathcal{M}$.

Let $\tilde{Y}$ be another Reedy-cofibrant diagram $\mathbb{C} \to \mathcal{M}$ that is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$, let $Y = \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{Y}$, and let $\varphi : X \Rightarrow Y$ be a componentwise weak equivalence. Proposition 3.3.28 applied to theorem 4.10.14 implies the adjunction unit components $\tilde{X} \to \pi_{\mathrm{R}}^* X$ and $\tilde{Y} \to \pi_{\mathrm{R}}^* Y$ are Reedy weak equivalences. Using axiom CM2 and CM5, factor $\tilde{Y} \to \pi_{\mathrm{R}}^* Y$ as a trivial cofibration $\theta : \tilde{Y} \to \tilde{Z}$ followed by a trivial fibration $\tilde{Z} \to \pi_{\mathrm{R}}^* Y$; then by axiom CM4 there exists a natural transformation $\psi : \tilde{X} \Rightarrow \tilde{Z}$ making the diagram in $[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ shown below commute:

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "\psi"] \arrow[d] & \tilde{Z} \arrow[d] & \tilde{Y} \arrow[l, "\theta"'] \arrow[d] \\
\pi_{\mathrm{R}}^*(X) \arrow[r, "\pi_{\mathrm{R}}^*(\varphi)"'] & \pi_{\mathrm{R}}^*(Y) \arrow[r, equal] & \pi_{\mathrm{R}}^*(Y)
\end{tikzcd}$$

Since $\pi_{\mathrm{R}}^*(\varphi)$ is a Reedy weak equivalence, it follows from axiom CM2 that ψ is also a Reedy weak equivalence. Transposing across the adjunction $\operatorname{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^*$, we obtain a commutative diagram in $[\mathbb{C}, \mathcal{M}]$,

$$\begin{tikzcd}
\operatorname{Lan}_{\pi_{\mathrm{R}}} \tilde{X} \arrow[r, "\operatorname{Lan}_{\pi_{\mathrm{R}}} \psi"] \arrow[d, equal] & \operatorname{Lan}_{\pi_{\mathrm{R}}} \tilde{Z} \arrow[d] & \operatorname{Lan}_{\pi_{\mathrm{R}}} \tilde{Y} \arrow[l, "\operatorname{Lan}_{\pi_{\mathrm{R}}} \theta"'] \arrow[d, equal] \\
X \arrow[r, "\varphi"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

to which we may then apply Lan_F, yielding the following commutative diagram in $[\mathbb{D}, \mathcal{M}]$:

$$\begin{tikzcd}
\operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{X} \arrow[r, "\operatorname{Lan}_{F\pi_{\mathrm{R}}} \psi"] \arrow[d, "\cong"'] & \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z} \arrow[d] & \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{Y} \arrow[l, "\operatorname{Lan}_{F\pi_{\mathrm{R}}} \theta"'] \arrow[d, "\cong"] \\
\operatorname{Lan}_F X \arrow[r, "\operatorname{Lan}_F \varphi"'] & \operatorname{Lan}_F Y \arrow[r, equal] & \operatorname{Lan}_F Y
\end{tikzcd}$$

Now, $\operatorname{Lan}_{F\pi_{\mathrm{R}}} \psi : \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{X} \to \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z}$ and $\operatorname{Lan}_{F\pi_{\mathrm{R}}} \theta : \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{Y} \to \operatorname{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z}$ are componentwise weak equivalences between componentwise cofibrant diagrams, by proposition 4.10.13, so we deduce that $\operatorname{Lan}_F \varphi$ is also a componentwise weak equivalence between componentwise cofibrant diagrams as claimed, using the 2-out-of-3 property of weak equivalences in $\mathcal{M}$.

(ii). The following diagram of functors is commutative,

$$\begin{tikzcd}
\mathbf{\Delta}(\mathbb{C}) \arrow[r, "\mathbf{\Delta}(F)"] \arrow[d, "\pi_{\mathrm{R}}"'] & \mathbf{\Delta}(\mathbb{D}) \arrow[d, "\pi_{\mathrm{R}}"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

so there is a canonical natural isomorphism $\operatorname{Lan}_F \circ \operatorname{Lan}_{\pi_{\mathrm{R}}} \cong \operatorname{Lan}_{\pi_{\mathrm{R}}} \circ \operatorname{Lan}_{\mathbf{\Delta}(F)}$. Corollary 4.10.8 implies $\operatorname{Lan}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}]$ preserves Reedy-cofibrant diagrams, so it follows from the hypothesis that the functor $\operatorname{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ preserves virtually cofibrant diagrams.

(iii). Having proved claim (i), it is now enough to show that the natural transformation $\varepsilon \bullet \left(\mathrm{Lan}_{\pi_{\mathrm{R}}} \circ p \circ \pi_{\mathrm{R}}^*\right) : \mathrm{Lan}_{\pi_{\mathrm{R}}} \circ Q \circ \pi_{\mathrm{R}}^* \Rightarrow \mathrm{id}_{[\mathbb{C},\mathcal{M}]}$ is a natural weak equivalence; but this is also a consequence of proposition 3.3.28 applied to theorem 4.10.14.

(iv). The functor F^* is a homotopical functor, hence trivially right deformable, and claim (iii) implies Lan_F is left deformable.

(v). Since F^* and G^* are both homotopical functors, (F^*, G^*) is strongly right deformable, and we may deduce from claim (i) that $\left(\mathrm{Lan}_G, \mathrm{Lan}_F\right)$ is laxly left deformable. Thus, by lemma 3.1.11, theorem 4.4.1, and corollary 3.3.27, the composable pair $\left(\mathrm{Lan}_G, \mathrm{Lan}_F\right)$ is strongly left deformable. ■

Lemma 4.10.17. *Let $\mathcal{M}$ be a DHK model category, let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, and let D be an object in $\mathbb{D}$.*

- *Given the following comma square,*

$$\begin{tikzcd} \mathbf{\Delta}((F \downarrow D)) \arrow[r] \arrow[d, "\mathbf{\Delta}(P)"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{1} \arrow[d, "D"] \\ \mathbf{\Delta}(\mathbb{C}) \arrow[r, "F\pi_{\mathrm{R}}"'] & \mathbb{D} \end{tikzcd}$$

  the derived left Beck–Chevalley transformation

$$\left(\mathbf{L}\varinjlim_{\mathbf{\Delta}((F\downarrow D))}\right) \circ (\mathrm{Ho}\,\mathbf{\Delta}(P)^*) \Rightarrow (\mathrm{Ho}\,D^*) \circ \left(\mathbf{L}\mathrm{Lan}_{F\pi_{\mathrm{R}}}\right)$$

  is a natural isomorphism.

Dually:

- *Given the following comma square,*

$$\begin{tikzcd} \mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \arrow[r, "\mathbf{\Delta}(P)^{\mathrm{op}}"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \arrow[d, "F\pi_{\mathrm{L}}"] \\ \mathbb{1} \arrow[r, "D"'] & \mathbb{D} \end{tikzcd}$$

  the derived right Beck–Chevalley transformation

$$\left(\mathbf{R}\varprojlim_{\mathbf{\Delta}((D\downarrow F))^{\mathrm{op}}}\right) \circ (\mathrm{Ho}\,\mathbf{\Delta}(P)^*) \Rightarrow (\mathrm{Ho}\,D^*) \circ \left(\mathbf{R}\mathrm{Ran}_{F\pi_{\mathrm{L}}}\right)$$

  is a natural isomorphism.

Proof. Lemma 4.10.7 says $\mathbf{\Delta}(P) : \mathbf{\Delta}((F \downarrow D)) \to \mathbf{\Delta}(\mathbb{C})$ is a right fibration of Reedy categories, so by theorem 4.6.32, $\mathbf{\Delta}(P)^* : [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ preserves Reedy-cofibrant diagrams. Proposition 7.1.19 implies that the left Beck–Chevalley transformation $\varinjlim_{(F\pi_{\mathrm{R}} \downarrow D)}(-\mathbf{\Delta}(P)) \Rightarrow \left(\mathrm{Lan}_{F\pi_{\mathrm{R}}} -\right)(D)$ is a natural isomorphism, hence by corollary 3.3.25, so too is its derived natural transformation. ∎

Proposition 4.10.18. *Let $\mathcal{M}$ be a DHK model category, let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, and let D be an object in $\mathbb{D}$.*

- *Given the following comma square,*

$$\begin{tikzcd}
(F \downarrow D) \arrow[r] \arrow[d, "P"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{1} \arrow[d, "D"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

  the derived left Beck–Chevalley transformation

$$\left(\mathbf{L}\varinjlim_{(F\downarrow D)}\right) \circ (\mathrm{Ho}\, P^*) \Rightarrow (\mathrm{Ho}\, D^*) \circ \left(\mathbf{L}\mathrm{Lan}_F\right)$$

  is a natural isomorphism.

Dually:

- *Given the following comma square,*

$$\begin{tikzcd}
(D \downarrow F) \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbb{C} \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "D"'] & \mathbb{D}
\end{tikzcd}$$

  the derived right Beck–Chevalley transformation

$$\left(\mathbf{R}\varprojlim_{(D\downarrow F)}\right) \circ (\mathrm{Ho}\, P^*) \Rightarrow (\mathrm{Ho}\, D^*) \circ \left(\mathbf{R}\mathrm{Ran}_F\right)$$

  is a natural isomorphism.

Proof. Consider the following diagram, where the 2-cells are the respective left Beck–Chevalley transformations:

$$\begin{tikzcd}
\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[r, "\mathbf{\Delta}(P)^*"] \arrow[d, hook] & \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[d, hook] \\
[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \arrow[r, "\mathbf{\Delta}(P)^*"] \arrow[d, "\mathrm{Lan}_{\pi_{\mathrm{R}}}"'] & [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \arrow[d, "\mathrm{Lan}_{\pi_{\mathrm{R}}}"] \arrow[dl, phantom, "\Leftarrow"] \\
[\mathbb{C}, \mathcal{M}] \arrow[r, "P^*"] \arrow[d, "\mathrm{Lan}_F"'] & [(F \downarrow D), \mathcal{M}] \arrow[d, "\varinjlim"] \arrow[dl, phantom, "\Leftarrow"] \\
[\mathbb{D}, \mathcal{M}] \arrow[r, "D^*"'] & \mathcal{M}
\end{tikzcd}$$

The pasting lemma (A.1.11) implies that left Beck–Chevalley transformations can be pasted together, and the preceding lemma says the derived left Beck–Chevalley transformation

$$\left(\mathbf{L}\varinjlim_{\mathbf{\Delta}((F\downarrow D))}\right) \circ (\mathrm{Ho}\,\mathbf{\Delta}(P)^*) \Rightarrow (\mathrm{Ho}\,D^*) \circ \left(\mathbf{L}\mathrm{Lan}_{F\pi_{\mathrm{R}}}\right)$$

is a natural isomorphism; but theorem 4.10.14 says that the adjunctions

$$\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [\mathbb{C}, \mathcal{M}] \to \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$
$$\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [(F \downarrow D), \mathcal{M}] \to \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$

satisfy the Quillen equivalence condition, so the commutative diagram shown below automatically satisfies the derived left Beck–Chevalley condition,

$$\begin{tikzcd}
[\mathbb{C}, \mathcal{M}] \arrow[r, "P^*"] \arrow[d, "\pi_{\mathrm{R}}^*"'] & [(F \downarrow D), \mathcal{M}] \arrow[d, "\pi_{\mathrm{R}}^*"] \\
\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[r, "\mathbf{\Delta}(P)^*"'] & \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}
\end{tikzcd}$$

and therefore, by cancelling natural isomorphisms, we conclude that the derived left Beck–Chevalley transformation

$$\left(\mathbf{L}\varinjlim_{(F\downarrow D)}\right) \circ (\mathrm{Ho}\,P^*) \Rightarrow (\mathrm{Ho}\,D^*) \circ \left(\mathbf{L}\mathrm{Lan}_F\right)$$

is a natural isomorphism, as claimed. ■

— V —

Topics in model categories

5.1 Proper model categories

Prerequisites. §§3.1, 4.1, 4.3, 4.4, 4.6.

Definition 5.1.1. Let C be a category with weak equivalences.

- A **homotopically quadrable morphism** in C is a morphism $p : E \to Y$ with the following property: for any morphism $f : Y' \to Y$ and any weak equivalence $v : Y'' \to Y'$, there is a commutative diagram in C of the form below,

$$\begin{tikzcd} E'' \arrow[r, "u"] \arrow[d] & E' \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ Y'' \arrow[r, "v"'] & Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

  where both squares are pullbacks and $u : E'' \to E'$ is a weak equivalence in C.

- A **homotopically coquadrable morphism** in C is a morphism $i : Z \to W$ with the following property: for any morphism $f : Z \to Z'$ and any weak equivalence $v : Z' \to Z''$, there is a commutative diagram in C of the form below,

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "i"'] & Z' \arrow[r, "v"] \arrow[d] & Z'' \arrow[d] \\ W \arrow[r] & W' \arrow[r, "u"'] & W'' \end{tikzcd}$$

  where both squares are pushouts and $u : W' \to W''$ is a weak equivalence in C.

Remark. Homotopically quadrable (resp. coquadrable) morphisms are called 'sharp maps' (resp. 'flat maps') in [Rezk, 1998].

Remark 5.1.2. The pullback (resp. pushout) pasting lemma implies the class of homotopically quadrable (resp. coquadrable) morphisms in $\mathcal{C}$ is closed under pullback (resp. pushout).

Lemma 5.1.3. *Let $\mathcal{M}$ be a model category.*

- *Every trivial fibration in $\mathcal{M}$ is homotopically quadrable.*
- *Every trivial cofibration in $\mathcal{M}$ is homotopically coquadrable.*

Proof. The two claims are formally dual; we will prove the first version.

Consider a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} E'' \arrow[r, "u"] \arrow[d, "p''"'] & E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ Y'' \arrow[r, "v"'] & Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

where both squares are pullbacks. If $p : E \to Y$ is a trivial fibration, then by proposition A.3.17, so are $p' : E' \to Y'$ and $p'' : E'' \to Y''$. Axiom CM2 then implies $u : E'' \to E'$ is a weak equivalence if (and only if) $v : Y'' \to Y'$ is a weak equivalence. ■

Lemma 5.1.4. *Let $\mathcal{M}$ be a model category and let $f : X \to Y$ be a morphism in $\mathcal{M}$. Then the base change adjunction*[1]

$$\Sigma_f \dashv f^* : \mathcal{M}_{/Y} \to \mathcal{M}_{/X}$$

is a Quillen adjunction.

Proof. By proposition 4.3.2, it suffices to verify that the dependent sum functor $\Sigma_f : \mathcal{M}_{/X} \to \mathcal{M}_{/Y}$ preserves cofibrations and trivial cofibrations; but this is an immediate consequence of the definition of slice model structures. ■

The following appears as Proposition 2.3 in [Rezk, 2002].

[1] See lemma A.2.17.

Proposition 5.1.5. *Let $\mathcal{M}$ be a model category and let $f : X \to Y$ be a morphism in $\mathcal{M}$. The following are equivalent:*

(i) *The pullback of $f : X \to Y$ along any fibration $p : E \to Y$ in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}$. (In particular, $f : X \to Y$ is a weak equivalence in $\mathcal{M}$.)*

(ii) *The base change adjunction*

$$\Sigma_f \dashv f^* : \mathcal{M}_{/Y} \to \mathcal{M}_{/X}$$

is a Quillen equivalence.

(iii) *The derived base change adjunction*

$$\mathbf{L}\Sigma_f \dashv \mathbf{R}f^* : \operatorname{Ho}\mathcal{M}_{/Y} \to \operatorname{Ho}\mathcal{M}_{/X}$$

is an adjoint equivalence of categories.

Proof. (i) ⇔ (ii). Consider a fibrant object in $\mathcal{M}_{/Y}$, i.e. a fibration $p : E \to Y$ in $\mathcal{M}$. Then we have a pullback square in $\mathcal{M}$:

$$\begin{tikzcd} f^*E \arrow[r, "u"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Let $q : T \to X$ be a cofibrant object in $\mathcal{M}_{/X}$, i.e. any morphism $q : T \to X$ in $\mathcal{M}$ where T is a cofibrant object in $\mathcal{M}$. If $u : f^*E \to E$ is a weak equivalence in $\mathcal{M}$, then (by axiom CM2) a morphism $q \to f^*p$ in $\mathcal{M}_{/X}$ is a weak equivalence if and only if its right adjoint transpose $\Sigma_f q \to p$ is a weak equivalence in $\mathcal{M}_{/Y}$. Thus, recalling lemma 5.1.4, condition (i) implies condition (ii).

Conversely, suppose $(\tilde{E}, v)$ is a cofibrant replacement for f^*E in $\mathcal{M}$. Let $q = f^*p \circ v$. Then $q : \tilde{E} \to X$ is a cofibrant object in $\mathcal{M}_{/X}$ and $v : q \to f^*p$ is a weak equivalence in $\mathcal{M}_{/X}$; so if the base change adjunction is a Quillen equivalence, then the right adjoint transpose $\Sigma_f q \to p$ is a weak equivalence in $\mathcal{M}_{/Y}$. But the underlying morphism of the right adjoint transpose is $u \circ v : \tilde{E} \to E$, and axiom CM2 implies $u \circ v$ is a weak equivalence in $\mathcal{M}$ if and only if $u : f^*E \to E$ is a weak equivalence in $\mathcal{M}$; thus condition (ii) implies condition (i).

(ii) ⇔ (iii). Since model categories are saturated homotopical categories (by theorem 4.4.1), we may apply proposition 3.3.28. ■

Definition 5.1.6. Let $\mathcal{M}$ be a category.

- A **right proper model structure** on $\mathcal{M}$ is a model structure where every fibration is homotopically quadrable.
- A **left proper model structure** on $\mathcal{M}$ is a model structure where every cofibration is homotopically coquadrable.
- A **proper model structure** on $\mathcal{M}$ is a model structure that is both left proper and right proper.

The following appears as Proposition 2.5 in [Rezk, 2002].

Proposition 5.1.7. *Let $\mathcal{M}$ be a model category. The following are equivalent:*

(i) *$\mathcal{M}$ is a right proper model category.*

(ii) *The pullback of any weak equivalence $f : X \to Y$ along any fibration $p : E \to Y$ in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}$.*

(iii) *For any weak equivalence $f : X \to Y$, the derived base change adjunction*

$$\mathbf{L}\Sigma_f \dashv \mathbf{R}f^* : \operatorname{Ho}\mathcal{M}_{/Y} \to \operatorname{Ho}\mathcal{M}_{/X}$$

is an adjoint equivalence of categories.

Proof. (i) ⇔ (ii). The class of fibrations is closed under pullbacks, by proposition A.3.17.

(ii) ⇔ (iii). Apply proposition 5.1.5. ∎

Proposition 5.1.8 (Reedy)**.** *Let $\mathcal{M}$ be a model category.*

- *If all objects in $\mathcal{M}$ are cofibrant, then $\mathcal{M}$ is a left proper model category.*
- *If all objects in $\mathcal{M}$ are fibrant, then $\mathcal{M}$ is a right proper model category.*

Proof. See Proposition 13.1.2 in [Hirschhorn, 2003], or use proposition 3.7.14. □

Proposition 5.1.9. *Let $\mathcal{M}$ be a model category and let $\mathbb{A}$ be a small category.*

(i) *If the injective model structure on $[\mathbb{A}, \mathcal{M}]$ exists and $\mathcal{M}$ is a left proper model category, then $[\mathbb{A}, \mathcal{M}]$ (with the injective model structure) is also a left proper model category.*

(ii) *If the projective model structure on $[\mathbb{A}, \mathcal{M}]$ exists and $\mathcal{M}$ is a left proper model category with products for families of size $\leq |\mathrm{mor}\, \mathbb{A}|$, then $[\mathbb{A}, \mathcal{M}]$ (with the projective model structure) is also a left proper model category.*

Dually:

(i′) *If the projective model structure on $[\mathbb{A}, \mathcal{M}]$ exists and $\mathcal{M}$ is a right proper model category, then $[\mathbb{A}, \mathcal{M}]$ (with the projective model structure) is also a right proper model category.*

(ii′) *If the injective model structure on $[\mathbb{A}, \mathcal{M}]$ exists and $\mathcal{M}$ is a right proper model category with coproducts for families of size $\leq |\mathrm{mor}\, \mathbb{A}|$, then $[\mathbb{A}, \mathcal{M}]$ (with the injective model structure) is also a right proper model category.*

Proof. (i). Since cofibrations, weak equivalences, and pushouts are all defined componentwise in $[\mathbb{A}, \mathcal{M}]$, left properness is indeed inherited from $\mathcal{M}$.

(ii). Corollary 4.3.21 says that the cofibrations in the projective model structure on $[\mathbb{A}, \mathcal{M}]$ are also cofibrations in the injective model structure, so the projective model structure is left proper if the injective model structure is. ■

For the remainder of this section, we study pullbacks (resp. pushouts) in the context of left (resp. right) proper model categories.

Definition 5.1.10. Let $\mathcal{M}$ be a model category.

- A **(right) derived pullback** for a pair of morphisms $f : X \to Z$ and $g : Y \to Z$ in $\mathcal{M}$ consists of the following data:
  - An object in $\mathcal{M}$, $X \overset{\mathbf{R}}{\times}_Z Y$.
  - A pullback diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} X \overset{\mathbf{R}}{\times}_Z Y \arrow[r] \arrow[d] & \hat{Y} \arrow[d, "\hat{g}"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} \end{tikzcd}$$

where $\hat{f} : \hat{X} \to \hat{Z}$ and $\hat{g} : \hat{Y} \to \hat{Z}$ are fibrations between fibrant objects in $\mathcal{M}$.

– A commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Z \arrow[d] & Y \arrow[l, "g"'] \arrow[d] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} & \hat{Y} \arrow[l, "\hat{g}"] \end{tikzcd}$$

where the vertical arrows are weak equivalences in $\mathcal{M}$.

We will often abuse notation and refer to $X \overset{\mathbf{R}}{\times}_Z Y$ as the derived pullback.

- A **(left) derived pushout** for a pair of morphisms $f : Z \to X$ and $g : Z \to Y$ in $\mathcal{M}$ consists of the following data:

  – An object in $\mathcal{M}$, $X \underset{\mathbf{L}}{\cup}^Z Y$.

  – A pushout diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} \tilde{Z} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & \tilde{Y} \arrow[d] \\ \tilde{X} \arrow[r] & X \underset{\mathbf{L}}{\cup}^Z Y \end{tikzcd}$$

where $\tilde{f} : \tilde{Z} \to \tilde{X}$ and $\tilde{g} : \tilde{Z} \to \tilde{Y}$ are cofibrations between cofibrant objects in $\mathcal{M}$.

  – A commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} \tilde{X} \arrow[d] & \tilde{Z} \arrow[l, "\tilde{f}"'] \arrow[r, "\tilde{g}"] \arrow[d] & \tilde{Y} \arrow[d] \\ X & Z \arrow[l, "f"] \arrow[r, "g"'] & Y \end{tikzcd}$$

where the vertical arrows are weak equivalences in $\mathcal{M}$.

We will often abuse notation and refer to $X \underset{\mathbf{L}}{\cup}^Z Y$ as the derived pushout.

Remark 5.1.11. The free cospan $\{\bullet \to \bullet \leftarrow \bullet\}$ is clearly an inverse category; dually, the free span $\{\bullet \leftarrow \bullet \to \bullet\}$ is a direct category.

Lemma 5.1.12. *Let $\mathcal{M}$ be a model category.*

- *The injective model structure on the category of cospans in $\mathcal{M}$ exists and coincides with the Reedy model structure.*

- *The projective model structure on the category of spans in $\mathcal{M}$ exists and coincides with the Reedy model structure.*

Proof. In view of remark 5.1.11, this is a special case of corollary 4.6.14. ■

Corollary 5.1.13. *Let $\mathcal{M}$ be a model category.*

- *The limit functor $\varprojlim : [\{\bullet \to \bullet \leftarrow \bullet\}, \mathcal{M}] \to \mathcal{M}$ admits a right derived functor.*

- *The colimit functor $\varprojlim : [\{\bullet \leftarrow \bullet \to \bullet\}, \mathcal{M}] \to \mathcal{M}$ admits a left derived functor.*

Proof. Apply proposition 4.3.2 and theorem 4.3.12 to proposition 4.3.17. ■

Lemma 5.1.14. *Let $\mathcal{M}$ be a model category.*

- *A cospan in $\mathcal{M}$ is injective-fibrant if and only if its vertices are fibrant objects in $\mathcal{M}$ and its arrows are fibrations in $\mathcal{M}$.*

- *A span in $\mathcal{M}$ is projective-cofibrant if and only if its vertices are cofibrant objects in $\mathcal{M}$ and its arrows are cofibrations in $\mathcal{M}$.*

Proof. Apply lemma 5.1.12 and the explicit description of cofibrations (resp. fibrations) in the Reedy model structure. ■

Remark 5.1.15. Thus, a derived pullback (resp. derived pushout) for a cospan (resp. span) is essentially the same thing as a injective-fibrant (resp. projective-cofibrant) replacement together with a pullback (resp. pushout) for the replacement cospan (resp. span). In particular, by lemma 4.1.25 and Ken Brown's lemma (4.3.6), their underlying objects are unique up to weak equivalence.

Definition 5.1.16. Let $\mathcal{M}$ be a model category.

- A **derived pullback diagram** in $\mathcal{M}$ is a commutative square in $\mathcal{M}$, say

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

  such that the canonical comparison morphism $W \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence for all derived pullbacks $X \overset{\mathbf{R}}{\times}_Z Y$.

- A **derived pushout diagram** in $\mathcal{M}$ is a commutative square in $\mathcal{M}$, say

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & W \end{tikzcd}$$

  such that the canonical comparison morphism $X \underset{\mathbf{L}}{\cup}^Z Y \to W$ is a weak equivalence for all derived pushouts $X \underset{\mathbf{L}}{\cup}^Z Y$.

Lemma 5.1.17. *Let $\mathcal{M}$ be a model category.*

- *Let 1 be a terminal object in $\mathcal{M}$. If the class of weak equivalences in $\mathcal{M}$ is closed under binary products, then for any pair (X, Y) of objects in $\mathcal{M}$, we have the following derived pullback diagram in $\mathcal{M}$,*

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & 1 \end{tikzcd}$$

  where the two non-trivial arrows are the product projections.

- *Let 0 be an initial object in $\mathcal{M}$. If the class of weak equivalences in $\mathcal{M}$ is closed under binary coproducts, then for any pair (X, Y) of objects in $\mathcal{M}$, we have the following derived pushout diagram in $\mathcal{M}$,*

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X + Y \end{tikzcd}$$

  where the two non-trivial arrows are the coproduct insertions.

Proof. The two claims are formally dual; we will prove the first version.

By proposition 4.1.17, there are fibrant replacements $(\hat{X}, i_X)$ and $(\hat{Y}, i_Y)$ for X and Y (respectively), and by remark 5.1.15, the diagram in question is a derived pullback diagram if and only if $i_X \times i_Y : X \times Y \to \hat{X} \times \hat{Y}$ is a weak equivalence in $\mathcal{M}$; but we assumed that the class of weak equivalences in $\mathcal{M}$ is closed under binary products, so we are done. ■

Lemma 5.1.18. *Let $\mathcal{M}$ be a model category.*

- *Consider a commutative square in $\mathcal{M}$:*

$$\begin{tikzcd} W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

  Assuming $f : X \to Z$ is a weak equivalence in $\mathcal{M}$, the square is a derived pullback diagram in $\mathcal{M}$ if and only if $q : W \to Y$ is a weak equivalence in $\mathcal{M}$.

- *Consider a commutative square in $\mathcal{M}$:*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "j"] \\ X \arrow[r, "i"'] & W \end{tikzcd}$$

  Assuming $f : Z \to X$ is a weak equivalence in $\mathcal{M}$, the square is a derived pushout diagram in $\mathcal{M}$ if and only if $j : Y \to W$ is a weak equivalence in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

Using proposition 4.1.17 and axiom CM5, we may choose:

- a fibrant replacement $(\hat{Z}, i_Z)$ for Z,
- a fibration $\hat{f} : \hat{X} \to \hat{Z}$ and a weak equivalence $i_X : X \to \hat{X}$ such that $\hat{f} \circ i_X = i_X \circ f$, and
- a fibration $\hat{g} : \hat{Y} \to \hat{Z}$ and a weak equivalence $i_Y : Y \to \hat{Y}$ such that $\hat{g} \circ i_Y = i_Z \circ f$.

Observe that axiom CM2 implies that $\hat{f} : \hat{X} \to \hat{Z}$ is a trivial fibration in $\mathcal{M}$. We then construct the following pullback square in $\mathcal{M}$,

$$\begin{tikzcd} \hat{W} \arrow[r, "\hat{q}"] \arrow[d, "\hat{p}"'] & \hat{Y} \arrow[d, "\hat{g}"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} \end{tikzcd}$$

and we note that (by corollary 4.1.13) $\hat{q} : \hat{W} \to \hat{Y}$ is also a trivial fibration in $\mathcal{M}$. Thus, the canonical comparison morphism $i_W : W \to \hat{W}$ is a weak equivalence in $\mathcal{M}$ if and only if $q : W \to Y$ is a weak equivalence in $\mathcal{M}$; and recalling remark 5.1.15, the claim follows. ■

Lemma 5.1.19. *Let $\mathcal{M}$ be a model category.*

- *Consider a commutative diagram in $\mathcal{M}$ of the form below:*

$$\begin{tikzcd} W' \arrow[r, "w"] \arrow[d, "p'"'] & W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X' \arrow[r, "x"'] & X \arrow[r, "f"'] & Z \end{tikzcd}$$

  Assuming the right square is a derived pullback diagram, the outer rectangle is a derived pullback diagram if and only if the left square is a derived pullback diagram.

- *Consider a commutative diagram in $\mathcal{M}$ of the form below:*

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[r, "y"] \arrow[d, "i"] & Y' \arrow[d, "i'"] \\ X \arrow[r, "j"'] & W \arrow[r, "w"'] & W' \end{tikzcd}$$

  Assuming the left square is a derived pushout diagram, the outer rectangle is a derived pushout diagram if and only if the right square is a derived pushout diagram.

Proof. The two claims are formally dual; we will prove the first version.

Using proposition 4.1.17 and axiom CM5, we may choose:

- a fibrant replacement $(\hat{Z}, i_Z)$ for Z,

- a fibration $\hat{f} : \hat{X} \to \hat{Z}$ and a weak equivalence $i_X : X \to \hat{X}$ such that $\hat{f} \circ i_X = i_X \circ f$,
- a fibration $\hat{g} : \hat{Y} \to \hat{Z}$ and a weak equivalence $i_Y : Y \to \hat{Y}$ such that $\hat{g} \circ i_Y = i_Z \circ f$, and
- a fibration $\hat{x} : \hat{X}' \to \hat{X}$ and a weak equivalence $i_{X'} : X' \to \hat{X}'$ such that $\hat{x} \circ i_{X'} = i_X \circ x$.

We then construct the following diagram in $\mathcal{M}$,

$$\begin{tikzcd}
\hat{W}' \arrow[r] \arrow[d] & \hat{W} \arrow[r] \arrow[d, "\hat{p}"] & \hat{Y} \arrow[d, "\hat{g}"] \\
\hat{X}' \arrow[r, "\hat{x}"'] & \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z}
\end{tikzcd}$$

where both squares are pullback diagram; note that the ordinary pullback pasting lemma says that the outer rectangle is then also a pullback diagram. We thus have canonical comparison morphisms $i_{W'} : W' \to \hat{W}'$ and $i_W : W \to \hat{W}$, and by hypothesis, $i_W : W \to \hat{W}$ is a weak equivalence. Moreover, by proposition A.3.17, $\hat{p} : \hat{W} \to \hat{Y}$ is a fibration such that $\hat{p} \circ i_Y = i_W \circ p$, so by remark 5.1.15, the following are equivalent:

(i) $i_{W'} : W' \to \hat{W}'$ is a weak equivalence.

(ii) The diagram shown below is a derived pullback square in $\mathcal{M}$:

$$\begin{tikzcd}
W' \arrow[r, "w"] \arrow[d, "p'"'] & W \arrow[d, "p"] \\
X' \arrow[r, "x"'] & X
\end{tikzcd}$$

(iii) The diagram shown below is a derived pullback square in $\mathcal{M}$:

$$\begin{tikzcd}
W' \arrow[r, "q \circ w"] \arrow[d, "p'"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "f \circ x"'] & Z
\end{tikzcd}$$

This completes the proof. ∎

Proposition 5.1.20. *Let $\mathcal{M}$ be a model category.*

- *Consider a commutative cube in $\mathcal{M}$:*

$$\begin{tikzcd}
W' \arrow[rr] \arrow[dd] \arrow[dr] & & Y' \arrow[dd] \arrow[dr] & \\
& W \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
X' \arrow[rr] \arrow[dr] & & Z' \arrow[dr] & \\
& X \arrow[rr] & & Z
\end{tikzcd}$$

Assuming the diagonal arrows are weak equivalences in $\mathcal{M}$, the front face is a derived pullback diagram in $\mathcal{M}$ if and only if the back face is a derived pullback diagram in $\mathcal{M}$.

- *Consider a commutative cube in $\mathcal{M}$:*

$$\begin{tikzcd}
Z' \arrow[rr] \arrow[dd] \arrow[dr] & & Y' \arrow[dd] \arrow[dr] & \\
& Z \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
X' \arrow[rr] \arrow[dr] & & W' \arrow[dr] & \\
& X \arrow[rr] & & W
\end{tikzcd}$$

Assuming the diagonal arrows are weak equivalences in $\mathcal{M}$, the front face is a derived pushout diagram in $\mathcal{M}$ if and only if the back face is a derived pushout diagram in $\mathcal{M}$.

Proof. Apply lemmas 5.1.18 and 5.1.19 (twice for each direction of each claim). ∎

Proposition 5.1.21. *Let $\mathcal{M}$ be a model category.*

- *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms in $\mathcal{M}$. Given any derived pullback $X \overset{\mathbf{R}}{\times}_Z Y$ of f and g, if Z is fibrant and either f or g is a homotopically quadrable fibration, then the canonical comparison morphism $X \times_Z Y \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence in $\mathcal{M}$.*

- *Let $f : Z \to X$ and $g : Z \to Y$ be morphisms in $\mathcal{M}$. Given any derived pushout $X \underset{\mathbf{L}}{\cup}^Z Y$ of f and g, if Z is cofibrant and either f or g is a homotopically coquadrable cofibration, then the canonical comparison morphism $X \underset{\mathbf{L}}{\cup}^Z Y \to X \cup^Z Y$ is a weak equivalence in $\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $f : X \to Z$ is a homotopically quadrable fibration. By axiom CM5, the morphism $g : Y \to Z$ admits a factorisation $g = p \circ v$ where $p : Y' \to Z$ is a fibration and $v : Y \to Y'$ is a weak equivalence. Thus, we have a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & X \times_Z Y' \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r, "v"'] & Y' \arrow[r, "p"'] & Z
\end{tikzcd}$$

where the two squares are pullbacks and the morphism $X \times_Z Y \to X \times_Z Y'$ is a weak equivalence in $\mathcal{M}$. Recalling lemma 5.1.14, we see that the right square is (part of) a derived pullback for $f : X \to Z$ and $g : Y \to Z$, so remark 5.1.15 implies that the canonical comparison morphism $X \times_Z Y \to X \overset{\mathbf{R}}{\times}_Z Y$ is the composite of two weak equivalences, hence is itself a weak equivalence by axiom CM2. ∎

Proposition 5.1.22. *Let $\mathcal{M}$ be a model category.*

- *Let $f : X \to Z$ be a morphism in $\mathcal{M}$. If for any morphism $g : Y \to Z$ and any derived pullback $X \overset{\mathbf{R}}{\times}_Z Y$ for f and g, the canonical comparison morphism $X \times_Z Y \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence in $\mathcal{M}$, then $f : X \to Z$ is a homotopically quadrable morphism in $\mathcal{M}$.*

- *Let $f : Z \to X$ be a morphism in $\mathcal{M}$. If for any morphism $g : Z \to Y$ and any derived pushout $X \underset{\mathbf{L}}{\cup}^Z Y$ for f and g, the canonical comparison morphism $X \underset{\mathbf{L}}{\cup}^Z Y \to X \cup^Z Y$ is a weak equivalence in $\mathcal{M}$, then $f : Z \to X$ is a homotopically coquadrable morphism in $\mathcal{M}$.*

Proof. Consider a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} X \times_Z Y' \arrow[r, "f^* v"] \arrow[d] & X \times_Z Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"'] & Y \arrow[r, "g"'] & Z \end{tikzcd}$$

where the squares are pullbacks and $v : Y' \to Y$ is a weak equivalence. Using axiom CM5, we may construct the following commutative diagram in $\mathcal{M}$,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "w_X"'] & Z \arrow[d, "w_Z"] & Y \arrow[l, "g"'] \arrow[d, "w_Y"] & Y' \arrow[l, "v"'] \arrow[d, "w_{Y'}"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} & \hat{Y} \arrow[l, "\hat{g}"] & \hat{Y}' \arrow[l, "\hat{v}"] \end{tikzcd}$$

where the vertical arrows are weak equivalences in $\mathcal{M}$ and the arrows in the bottom row are fibrations between fibrant objects in $\mathcal{M}$. We then have a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} X \times_Z Y' \arrow[r, "f^* v"] \arrow[d, "w_X \times_{w_Z} w_{Y'}"'] & X \times_Z Y \arrow[d, "w_X \times_{w_Z} w_Y"] \\ \hat{X} \times_{\hat{Z}} \hat{Y}' \arrow[r, "\hat{f}^* \hat{v}"'] & \hat{X} \times_{\hat{Z}} \hat{Y} \end{tikzcd}$$

where all the arrows are canonical comparison morphisms. The vertical arrows are weak equivalences by hypothesis, and $\hat{f}^* \hat{v} : \hat{X} \times_{\hat{Z}} \hat{Y}' \to \hat{X} \times_{\hat{Z}} \hat{Y}$ is a weak equivalence by Ken Brown's lemma (4.3.6) and lemma 5.1.4; hence $f^* v : X \times_Z Y' \to X \times_Z Y$ is a weak equivalence (by axiom CM2), as required. ■

The following appears as Proposition 2.7 in [Rezk, 1998]:

Proposition 5.1.23. *The following are equivalent for a morphism $f : X \to Z$ in a* right proper *model category $\mathcal{M}$:*

(i) *The morphism $f : X \to Z$ is homotopically quadrable in $\mathcal{M}$.*

(ii) *For any morphism $g : Y \to Z$ and any derived pullback $X \overset{\mathbf{R}}{\times}_Z Y$ for f and g, the canonical comparison morphism $X \times_Z Y \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence in $\mathcal{M}$.*

Dually, the following are equivalent for a morphism $f : Z \to X$ is a left proper *model category $\mathcal{M}$:*

(i′) *The morphism $f : Z \to X$ is homotopically coquadrable in $\mathcal{M}$.*

(ii′) *For any morphism $g : Z \to Y$ and any derived pullback $X \underset{\mathbf{L}}{\cup}^Z Y$ for f and g, the canonical comparison morphism $X \underset{\mathbf{L}}{\cup}^Z Y \to X \cup^Z Y$ is a weak equivalence in $\mathcal{M}$.*

Proof. (i) ⇒ (ii). Suppose we have a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "w_X"'] & Z \arrow[d, "w_Z"] & Y \arrow[l, "g"'] \arrow[d, "w_Y"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} & \hat{Y} \arrow[l, "\hat{g}"] \end{tikzcd}$$

where the vertical arrows are weak equivalences in $\mathcal{M}$ and $\hat{f} : \hat{X} \to \hat{Z}$ and $\hat{g} : \hat{Y} \to \hat{Z}$ are fibrations between fibrant objects in $\mathcal{M}$ for which we have a pullback diagram in $\mathcal{M}$:

$$\begin{tikzcd} X \overset{\mathbf{R}}{\times}_Z Y \arrow[r] \arrow[d] & \hat{Y} \arrow[d, "\hat{g}"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} \end{tikzcd}$$

Consider the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} X \times_{\hat{Z}} \hat{Y} \arrow[r] \arrow[d] & X \overset{\mathbf{R}}{\times}_Z Y \arrow[r] \arrow[d] & \hat{Y} \arrow[d, "\hat{g}"] \\ X \arrow[r, "w_X"'] & \hat{X} \arrow[r, "\hat{f}"'] & \hat{Z} \end{tikzcd}$$

By the pullback pasting lemma, the left square is a pullback; hence by right properness, the canonical comparison morphism $X \times_{\hat{Z}} \hat{Y} \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence. Applying the pullback pasting lemma again, we find that the squares in the diagram below are pullbacks:

$$\begin{tikzcd}
X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, "w_Y"] & \\
X \times_{\hat{Z}} \hat{Y} \arrow[r] \arrow[d] & (w_Z)^* \hat{Y} \arrow[r] \arrow[d, "(w_Z)^* \hat{g}"] & \hat{Y} \arrow[d, "\hat{g}"] \\
X \arrow[r, "f"'] & Z \arrow[r, "w_Z"'] & \hat{Z}
\end{tikzcd}$$

By right properness (and axiom CM2), the morphism $Y \to (w_Z)^* \hat{Y}$ is a weak equivalence, so if $f : X \to Z$ is a homotopically quadrable morphism, then the canonical comparison morphism $X \times_Z Y \to X \times_{\hat{Z}} \hat{Y}$ is a weak equivalence, in which case the canonical comparison morphism $X \times_Z Y \to X \overset{\mathbf{R}}{\times}_Z Y$ is a weak equivalence (by axiom CM2 again).

(ii) $\Rightarrow$ (i). See proposition 5.1.22. ■

5.2 Combinatorial model categories

Prerequisites. §§0.2, 0.3, 0.5, 4.1, A.3.

Definition 5.2.1. A **cofibrantly generated model category** is a complete and cocomplete model category $\mathcal{M}$ such that there exist a set $\mathcal{I}$ of cofibrations and a set $\mathcal{I}'$ of trivial cofibrations satisfying these conditions:

- $(\mathcal{I}, \mathcal{M})$ admits the small object argument, and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of all cofibrations in $\mathcal{M}$.
- $(\mathcal{I}', \mathcal{M})$ admits the small object argument, and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ is the class of all trivial cofibrations in $\mathcal{M}$.

Remark 5.2.2. By Quillen's small object argument (theorem 0.5.12), any cofibrantly generated model category satisfies axiom CM5* and thus is a DHK model category.

Theorem 5.2.3 (Kan's recognition principle)**.** *Let $\mathcal{M}$ be a complete and cocomplete locally small category, let $\mathcal{W}$ be a subcategory of $\mathcal{M}$ containing all the objects, and let $\mathcal{I}$ and $\mathcal{I}'$ be subsets of* $\operatorname{mor} \mathcal{M}$*. Assume the following hypotheses:*

- *$\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property in $\mathcal{M}$.*
- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ both admit the small object argument.*
- $\operatorname{inj}^{\mathcal{M}} \mathcal{I} \subseteq \mathcal{W} \cap \operatorname{inj}^{\mathcal{M}} \mathcal{I}'$.
- $\operatorname{cof}_{\mathcal{M}} \mathcal{I}' \subseteq \mathcal{W} \cap \operatorname{cof}_{\mathcal{M}} \mathcal{I}$.

If, in addition, either

- $\operatorname{inj}^{\mathcal{M}} \mathcal{I} = \mathcal{W} \cap \operatorname{inj}^{\mathcal{M}} \mathcal{I}'$, *or*
- $\operatorname{cof}_{\mathcal{M}} \mathcal{I}' = \mathcal{W} \cap \operatorname{cof}_{\mathcal{M}} \mathcal{I}$.

then there exists a unique model structure on $\mathcal{M}$ such that $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of cofibrations, $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ is the class of trivial cofibrations, and $\mathcal{W}$ is the class of weak equivalences.

Proof. See Theorem 11.3.1 in [Hirschhorn, 2003]. □

Corollary 5.2.4. *Let $\mathcal{M}$ be a model category, let $\mathcal{W}$ be the class of weak equivalences in $\mathcal{M}$, let $\mathcal{I}$ be a set of cofibrations in $\mathcal{M}$, and let $\mathcal{I}'$ be a set of trivial cofibrations in $\mathcal{M}$. Assume the following hypotheses:*

- *$\mathcal{M}$ is complete and cocomplete.*
- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ both admit the small object argument.*
- $\operatorname{inj}^{\mathcal{M}} \mathcal{I} \subseteq \mathcal{W}$.
- $\operatorname{cof}_{\mathcal{M}} \mathcal{I}' \subseteq \operatorname{cof}_{\mathcal{M}} \mathcal{I}$.

If, in addition, either

- $\operatorname{inj}^{\mathcal{M}} \mathcal{I} = \mathcal{W} \cap \operatorname{inj}^{\mathcal{M}} \mathcal{I}'$, *or*
- $\operatorname{cof}_{\mathcal{M}} \mathcal{I}' = \mathcal{W} \cap \operatorname{cof}_{\mathcal{M}} \mathcal{I}$.

then there exists a unique model structure on $\mathcal{M}$ such that $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of cofibrations, $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ is the class of trivial cofibrations, and $\mathcal{W}$ is the class of weak equivalences.

Proof. To use Kan's recognition principle (theorem 5.2.3), it suffices to verify that $\operatorname{inj}^{\mathcal{M}} \mathcal{I} \subseteq \operatorname{inj}^{\mathcal{M}} \mathcal{I}'$ and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}' \subseteq \mathcal{W}$. The first inclusion is a consequence of proposition A.3.3 and the assumption that every $\mathcal{I}'$-cofibration is a $\mathcal{I}$-cofibration. To prove the second inclusion, recall that theorem 4.1.12 and proposition A.3.17 imply that the class of trivial cofibrations in $\mathcal{M}$ is closed under pushouts, transfinite composition, and retracts, so every $\mathcal{I}'$-cofibration is a trivial cofibration (hence a weak equivalence *a fortiori*). ■

Theorem 5.2.5 (Kan's lifting theorem)**.** *Let $\mathcal{M}$ be a complete and cocomplete locally small category and let $\mathcal{N}$ be a cofibrantly generated model category. Assume the following hypotheses:*

- *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is an adjunction of categories.*
- *$\mathcal{J}$ (resp. $\mathcal{J}'$) is a generating set of cofibrations (resp. trivial cofibrations) in $\mathcal{N}$.*
- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ admit the small object argument, where $\mathcal{I}$ and $\mathcal{I}'$ are the following sets:*

$$\mathcal{I} = \{ Ff \mid f \in \mathcal{J} \} \qquad\qquad \mathcal{I}' = \{ Ff \mid f \in \mathcal{J}' \}$$

- *G sends relative $\mathcal{I}'$-cell complexes in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.*

Then:

(i) *There is a unique model structure on $\mathcal{M}$ with $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ as the class of cofibrations and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ as the class of trivial cofibrations.*

(ii) *A morphism $g : A \to B$ in $\mathcal{M}$ is a weak equivalence (resp. fibration, trivial fibration) in this model structure if and only if $Gg : GA \to GB$ is a weak equivalence (resp. fibration, trivial fibration) in $\mathcal{N}$.*

(iii) *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is a Quillen adjunction with respect to this model structure.*

Proof. See Theorem 11.3.2 in [Hirschhorn, 2003]. □

Corollary 5.2.6. *Let $\mathcal{M}$ be a complete and cocomplete locally small category, let $\mathcal{N}$ be a cofibrantly generated model category. Assume the following hypotheses:*

- *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is an adjunction of categories.*

- *$\mathcal{J}$ (resp. $\mathcal{J}'$) is a generating set of cofibrations (resp. trivial cofibrations) in $\mathcal{N}$.*

- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ admit the small object argument, where $\mathcal{I}$ and $\mathcal{I}'$ are the following sets:*

$$\mathcal{I} = \{Ff \mid f \in \mathcal{J}\} \qquad\qquad \mathcal{I}' = \{Ff \mid f \in \mathcal{J}'\}$$

- *$G : \mathcal{M} \to \mathcal{N}$ is fully faithful.*

- *G sends relative $\mathcal{I}$-cell complexes in $\mathcal{M}$ to cofibrations in $\mathcal{N}$.*

- *The adjunction unit $\eta : \mathrm{id}_{\mathcal{N}} \Rightarrow GF$ is a natural weak equivalence.*

Then:

(i) *There is a unique model structure on $\mathcal{M}$ with $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ as the class of cofibrations and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ as the class of trivial cofibrations.*

(ii) *A morphism $g : A \to B$ in $\mathcal{M}$ is a weak equivalence (resp. fibration, trivial fibration) in this model structure if and only if $Gg : GA \to GB$ is a weak equivalence (resp. fibration, trivial fibration) in $\mathcal{N}$.*

(iii) *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is a Quillen* equivalence *with respect to this model structure.*

Proof. (i) and (ii). Since $\eta : \mathrm{id}_{\mathcal{N}} \Rightarrow GF$ is a natural weak equivalence, a morphism f in $\mathcal{N}$ is a weak equivalence if and only if GFf is a weak equivalence. Thus, $GF : \mathcal{N} \to \mathcal{N}$ also preserves trivial cofibrations. In particular, if f is in $\mathcal{J}'$, then GFf is a trivial cofibration.

Next, consider a diagram $X : \mathbb{D} \to \mathcal{M}$. We have the following commutative diagram in $\mathcal{N}$,

$$\begin{tikzcd}
GXd \arrow[r] \arrow[d, "\eta_{GXd}"'] & \varinjlim_{\mathbb{D}} GX \arrow[d, "\eta_{\varinjlim_{\mathbb{D}} GX}"] \\
GFGXd \arrow[r] \arrow[d, equal] & GF \varinjlim_{\mathbb{D}} GX \arrow[d, "\cong"] \\
GFGXd \arrow[r] \arrow[d, "G\varepsilon_{Xd}"'] & G \varinjlim_{\mathbb{D}} FGX \arrow[d, "G \varinjlim_{\mathbb{D}} \varepsilon X"] \\
GXd \arrow[r] & G \varinjlim_{\mathbb{D}} X
\end{tikzcd}$$

where the horizontal arrows are either the components of the colimiting cocones or the images thereof. The right triangle identity says that the composite of the left column is id_{GXd}, so the composite of the right column must be the canonical comparison $\varinjlim_{\mathbb{D}} GX \to G \varinjlim_{\mathbb{D}} X$; but proposition A.1.3 says $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{M}}$ is a natural isomorphism, so the canonical comparison morphism must be a weak equivalence.

It follows from the observations above (plus theorem 4.1.12 and proposition A.3.17) that G sends relative $\mathcal{I}'$-cell complexes in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$. We may now apply Kan's lifting theorem (5.2.5) to deduce that $\mathcal{I}$ and $\mathcal{I}'$ cofibrantly generate a model structure on $\mathcal{M}$ with the required properties.

(iii). We already know that $F \dashv G : \mathcal{M} \to \mathcal{N}$ is a Quillen adjunction. To complete the proof, we simply appeal to propositions 4.3.8 and A.1.3. ■

Theorem 5.2.7 (Existence of cofibrantly generated projective model structures)**.** *Let $\mathbb{A}$ be a small category. If $\mathcal{M}$ is a cofibrantly generated model category, then the projective model structure on* $[\mathbb{A}, \mathcal{M}]$ *exists and is cofibrantly generated.*

Proof. See Theorem 11.6.1 in [Hirschhorn, 2003]. □

The following is due to Smith [1998].

Definition 5.2.8. A **combinatorial model category** is a cofibrantly generated model category that is also a locally presentable category.

Remark 5.2.9. Since locally presentable categories are automatically complete and cocomplete,[2] in light of remark 0.5.9, to show that a locally presentable model category $\mathcal{M}$ is a combinatorial model category, it is enough to verify that there exist sets $\mathcal{I}$ and $\mathcal{I}'$ such that $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of all cofibrations in $\mathcal{M}$ and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ is the class of all trivial cofibrations in $\mathcal{M}$.

Theorem 5.2.10 (Smith's recognition principle)**.** *Let $\mathcal{M}$ be a locally presentable category, let $\mathcal{W}$ be a subcategory of $\mathcal{M}$ containing all the objects, and let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{M}$*. Assume the following hypotheses:*

- *$\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property in $\mathcal{M}$.*
- *Every $\mathcal{I}$-injective morphism in $\mathcal{M}$ is in $\mathcal{W}$.*
- *The class $\mathcal{W} \cap \operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is closed under pushouts and transfinite composition.*
- *$\mathcal{W}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{M}]$, is an accessible subcategory of $[\mathbb{2}, \mathcal{M}]$.*

TODO: Replace this with the solution set condition.

Then there exists a unique model structure on $\mathcal{M}$ such that $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of cofibrations and $\mathcal{W}$ is the class of weak equivalences, and this makes $\mathcal{M}$ a combinatorial model category.

Proof. See Theorem 1.7 in [Beke, 2000]. □

Theorem 5.2.11 (Existence of combinatorial injective model structures)**.** *Let $\mathbb{A}$ be a small category. If $\mathcal{M}$ is a combinatorial model category, then the injective model structure on $[\mathbb{A}, \mathcal{M}]$ exists and is cofibrantly generated.*

Proof. This theorem is due to Lurie; see [HTT, Proposition A.2.8.2]. □

Definition 5.2.12. Let κ and λ be regular cardinals. A **strongly (κ, λ)-combinatorial model category** is a combinatorial model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.
- $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.
- Each hom-set in $\mathbf{K}_\kappa(\mathcal{M})$ is λ-small.

[2] See theorem 0.2.40.

- There exist λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

Proposition 5.2.13. *For any combinatorial model category $\mathcal{M}$, there exist regular cardinals κ and λ making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category.*

Proof. Apply proposition 0.2.35, lemma 0.2.38, and remark 0.3.4. ■

Proposition 5.2.14. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category.*

(i) *There exist (trivial cofibration, fibration)- and (cofibration, trivial fibration)-factorisation functors that are κ-accessible and strongly λ-accessible.*

(ii) *Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be the full subcategory of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations (resp. trivial fibrations). Then $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.*

Proof. (i). Since the weak factorisation systems on $\mathcal{M}$ are cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ and $\mathbf{K}_\kappa(\mathcal{M})$ is locally λ-small, we may apply the small object argument of either Quillen (theorem 0.5.12 and corollary 0.5.14) or Garner (proposition 0.5.23 and theorem 0.5.24) to obtain the required functorial weak factorisation systems.

(ii). This is corollary 0.5.27. ■

Theorem 5.2.15. *Let (L', R) and (L, R') be functorial weak factorisation systems on a locally presentable category $\mathcal{M}$ and let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the morphisms in the right class of of the weak factorisation systems induced by (L', R) and (L, R'), respectively. Suppose κ and λ are regular cardinals satisfying the following hypotheses:*

- *$\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.*
- *$\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.*
- *$R, R' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ preserve colimits for small κ-filtered diagrams and are strongly λ-accessible functors.*

Let $\mathcal{C}'$ be the full subcategory of $[\mathbb{2}, \mathcal{M}]$ spanned by the morphisms in the left class of the weak factorisation system induced by (L', R) and let $\mathcal{W}$ be the preimage of $\mathcal{F}'$ under the functor $R : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$. Then:

(i) *The functorial weak factorisation systems* (L', R) *and* (L, R') *restrict to functorial weak factorisation systems on* $\mathbf{K}_\lambda(\mathcal{M})$*.*

(ii) *The inclusions* $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ *and* $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ *are strongly* λ*-accessible functors.*

(iii) $\mathcal{W}$ *is closed under colimits for small* κ*-filtered diagrams in* $[\mathbb{2}, \mathcal{M}]$*, and the inclusion* $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ *is a strongly* λ*-accessible functor.*

(iv) $\mathcal{C}' \subseteq \mathcal{W}$ *if and only if the same holds in* $\mathbf{K}_\lambda(\mathcal{M})$*.*

(v) $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ *if and only if the same holds in* $\mathbf{K}_\lambda(\mathcal{M})$*.*

(vi) $\mathcal{W}$ *(regarded as a class of morphisms in* $\mathcal{M}$*) has the 2-out-of-3 property in* $\mathcal{M}$ *if and only if the same is true in* $\mathbf{K}_\lambda(\mathcal{M})$*.*

(vii) *The weak factorisation systems induced by* (L', R) *and* (L, R') *underlie a model structure on* $\mathcal{M}$ *if and only if the restrictions to* $\mathbf{K}_\lambda(\mathcal{M})$ *underlie a model structure on* $\mathbf{K}_\lambda(\mathcal{M})$*.*

Proof. (i). It is clear that we can restrict (L', R) and (L, R') to obtain functorial factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$, and these are functorial *weak* factorisation systems by theorem A.3.35.

(ii). Since $R, R' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible, we may use proposition 0.5.28 to deduce that the inclusions $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible.

(iii). Since $\mathcal{F}'$ is a replete subcategory of $[\mathbb{2}, \mathcal{M}]$, we may use proposition 0.3.30 to deduce that $\mathcal{W}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$ and that the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.

(iv). The endofunctor $L' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ is strongly λ-accessible, and $\mathcal{W}$ is closed under colimits for small λ-filtered diagrams, so (recalling propositions 0.2.44 and 0.2.47) if L' sends the subcategory $\left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$ to $\mathcal{W}$, then the entirety of the image of L' must be contained in $\mathcal{W}$. Proposition A.3.37 implies every object in $\mathcal{C}'$ is a retract of an object in the image of L', and claim (iii) implies $\mathcal{W}$ is closed under retracts, so we may deduce that $\mathcal{C}' \subseteq \mathcal{W}$ if and only if $\mathcal{C}' \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right] \subseteq \mathcal{W} \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$.

(v). Claims (ii) and (iii) and proposition 0.3.30 imply the inclusion $\mathcal{W} \cap \mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is strongly λ-accessible; but by propositions 0.2.47 and 0.3.29,

$$\mathbf{K}_\lambda(\mathcal{F}') = \mathcal{F}' \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right] \qquad \mathbf{K}_\lambda(\mathcal{W} \cap \mathcal{F}) = (\mathcal{W} \cap \mathcal{F}) \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$$

so $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ if and only if $\mathcal{F}' \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right] = (\mathcal{W} \cap \mathcal{F}) \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$.

(vi). Consider the three full subcategories $\Lambda^2_i(\mathcal{W})$ (where $i \in \{0, 1, 2\}$) of $[\mathbb{3}, \mathcal{M}]$ spanned (respectively) by the diagrams of the form below:

$$\begin{tikzcd} \bullet \arrow[r, "\in \mathcal{W}"] \arrow[dr, "\in \mathcal{W}"'] & \bullet \arrow[d] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "\in \mathcal{W}"] \arrow[dr] & \bullet \arrow[d, "\in \mathcal{W}"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[dr, "\in \mathcal{W}"'] & \bullet \arrow[d, "\in \mathcal{W}"] \\ & \bullet \end{tikzcd}$$

By proposition 0.3.15, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbb{3}, \mathcal{M}]$ is the pullback of a strongly λ-accessible inclusion of a full subcategory of $[\mathbb{2}, \mathcal{M}]^{\times 3}$ along the evident projection functor $[\mathbb{3}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]^{\times 3}$; thus, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbb{3}, \mathcal{M}]$ is a strongly λ-accessible functor. We may then use proposition 0.3.29 as above to prove the claim.

(vii). Apply lemmas 4.1.10 and 4.1.11 and theorem 4.1.12. ■

Corollary 5.2.16. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category. Then the full subcategory $\mathcal{W}$ of $[\mathbb{2}, \mathcal{M}]$ spanned by the weak equivalences is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

Proof. Combine proposition 5.2.14 and theorem 5.2.15. ■

Proposition 5.2.17. *Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{I}$ be a set of cofibrations in $\mathcal{M}$. If every $\mathcal{I}$-injective morphism in $\mathcal{M}$ is a weak equivalence, then there exists a unique model structure on $\mathcal{M}$ with the same weak equivalences and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ as the class of cofibrations, and this makes $\mathcal{M}$ a combinatorial model category.*

Proof. Recalling proposition 5.2.13 and corollary 5.2.16, we see that the full subcategory of $[\mathbb{2}, \mathcal{M}]$ spanned by the weak equivalences in $\mathcal{M}$ is an accessible subcategory. Furthermore, by theorem 4.1.12 and proposition A.3.17, the class of trivial cofibrations in $\mathcal{M}$ is closed under pushouts and transfinite composition, and the class of $\mathcal{I}$-cofibrations that are weak equivalences is the intersection of $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ and the class of trivial cofibrations in $\mathcal{M}$. Thus, the class of $\mathcal{I}$-cofibrations that are weak equivalences is also closed under pushouts and transfinite composition, so we may apply Smith's recognition principle (theorem 5.2.10) to complete the proof. ∎

Definition 5.2.18. Let κ and λ be regular cardinals. A (κ, λ)**-compact model category** is a model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a (κ, λ)-compactly generated category, and $\kappa \lhd \lambda$.
- $\mathcal{M}$ has limits for finite diagrams and colimits for λ-small diagrams.
- Each hom-set in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ is λ-small.
- There exist λ-small sets of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

Proposition 5.2.19. *If $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, then $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-compact model category (with the weak equivalences, cofibrations, and fibrations inherited from $\mathcal{M}$).*

Proof. By proposition 0.3.7, $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-compactly generated category, and lemma 0.2.18 implies it is closed under colimits for λ-small diagrams in $\mathcal{M}$. Now, choose a pair of functorial factorisation systems as in proposition 5.2.14, and recall that theorem A.3.35 says a morphism is in the left (resp. right) class of a functorial weak factorisation system if and only if it is a retract of the left (resp. right) half of its functorial factorisation. Since we chose factorisation functors that are strongly λ-accessible, it follows that the weak factorisation systems on $\mathcal{M}$ restricts to weak factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$. It is then clear that $\mathbf{K}_\lambda(\mathcal{M})$ inherits a model structure from $\mathcal{M}$, and lemma 0.5.30 implies the model structure on $\mathbf{K}_\lambda(\mathcal{M})$ can be cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$. The remaining axioms for a λ-compact model category are easily verified. ∎

Proposition 5.2.20. *Let $\mathcal{K}$ be a (κ, λ)-compact model category and let $\mathcal{M}$ be the free λ-ind-completion $\mathbf{Ind}^\lambda(\mathcal{K})$. Then there is a unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category such that the canonical embedding $\mathcal{K} \to \mathcal{M}$ preserves and reflects the model structure.*

Proof. We will regard $\mathcal{K}$ as a full subcategory of $\mathcal{M}$ via the canonical embedding $\mathcal{K} \to \mathcal{M}$. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be a λ-small set of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{K})$ that generate the cofibrations (resp. trivial cofibrations) in $\mathcal{K}$. Let (L', R) and (L, R') be functorial weak factorisation systems cofibrantly generated by $\mathcal{I}'$ and $\mathcal{I}$ respectively;

by corollary 0.5.14, we may assume $R, R' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ preserve colimits for small κ-filtered diagrams and are strongly λ-accessible functors.

Let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the right class of the weak factorisation systems induced by (L', R) and (L, R'), respectively. It is not hard to see that any morphism in $\mathcal{K}$ is an object in $\mathcal{F}$ (resp. $\mathcal{F}'$) if and only if it is a fibration (resp. trivial fibration) in $\mathcal{K}$. Corollary 0.5.27 says $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, so we may now apply theorem 5.2.15 to deduce that $\mathcal{F}$ and $\mathcal{F}'$ induce a model structure on $\mathcal{M}$. It is clear that $\mathcal{M}$ equipped with this model structure is then a strongly (κ, λ)-combinatorial model category in a way that is compatible with the canonical embedding $\mathcal{K} \to \mathcal{M}$.

Finally, to see that the above construction is the unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category satisfying the given conditions, we simply have to observe that the model structure of a strongly (κ, λ)-combinatorial model category is necessarily cofibrantly generated by the cofibrations and trivial cofibrations in (a small skeleton of) $\mathbf{K}_\kappa(\mathcal{M})$ (independently of the choice of $\mathcal{I}$ and $\mathcal{I}'$). ■

Remark 5.2.21. Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model $\mathbf{U}$-category, and let $\mathcal{M} \hookrightarrow \mathcal{M}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-extension. By combining propositions 5.2.19 and 5.2.20, we may deduce that there is a unique way of making $\mathcal{M}^+$ into a strongly (κ, λ)-combinatorial model $\mathbf{U}^+$-category such that the embedding $\mathcal{M} \hookrightarrow \mathcal{M}^+$ preserves and reflects the model structure. In other words, combinatorial model categories are stable under universe enlargement.

5.3 Algebraic model categories

Prerequisites. §§0.2, 0.3, 0.5, 4.1, 5.2, A.3.

Though model categories equipped with functorial factorisations are better-behaved than general model categories, one can often extract a bit more structure by using Garner's small object argument (theorem 0.5.24). This leads to the notion of 'algebraic model structure', due to Riehl [2011a,b].

Definition 5.3.1. Let $\mathcal{M}$ be a category. An **algebraic model structure** on $\mathcal{M}$ consists of a pair of algebraic factorisation systems $(\mathsf{L}', \mathsf{R})$ and $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{M}$ and a morphism $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$ satisfying the following condition:

- There exists a model structure on $\mathcal{M}$ such that the cofibrations are the left class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$ and the fibrations are the right class of the weak factorisation system induced by $(\mathsf{L}', \mathsf{R})$.

An **algebraic model category** is a category with limits and colimits for all finite diagrams and equipped with an algebraic model structure.

The following lemma, originally part of Theorem 3.8 in [Riehl, 2011b], is useful in the construction of algebraic model structures:

Lemma 5.3.2. *Let $\mathcal{M}$ be a category with a model structure, let $(\mathsf{L}, \mathsf{R}')$ be an algebraic factorisation system on $\mathcal{M}$, and suppose $\mathcal{I}'$ is a generating set of trivial cofibrations in $\mathcal{M}$. If the left class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$ is the class of cofibrations, then there exists a subset $\tilde{\mathcal{I}}'$ with the following properties:*

- *$\tilde{\mathcal{I}}'$ has at most as many elements as $\mathcal{I}'$.*
- *The weak factorisation system on $\mathcal{M}$ cofibrantly generated by $\mathcal{I}'$ coincides with the one cofibrantly generated by $\mathcal{I}$.*
- *Each morphism in $\tilde{\mathcal{I}}'$ can be equipped with an L-coalgebra structure.*

In particular, if $(\mathsf{L}', \mathsf{R})$ is a free algebraic factorisation system cofibrantly generated by $\tilde{\mathcal{I}}'$, then there must exist a morphism $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$.

Proof. Let $\tilde{\mathcal{I}}' = \{ Le \mid e \in \mathcal{I}' \}$. Since L is a comonad, every morphism in $\tilde{\mathcal{I}}'$ admits an L-coalgebra structure. Consider the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} Z \arrow[r, "Le"] \arrow[d, "e"'] & W' \arrow[d, "R'e"] \\ W \arrow[r, "\mathrm{id}"'] & W \end{tikzcd}$$

Since $e : Z \to W$ is a trivial cofibration and $Re : W' \to W$ is a trivial fibration, there exists a morphism $i : W \to W'$ filling in the diagram. Hence, every morphism in $\mathcal{I}'$ is a retract of one in $\tilde{\mathcal{I}}'$, so by propositions A.3.3 and A.3.17, we have $\tilde{\mathcal{I}}'^{\boxslash} \subseteq \mathcal{I}'^{\boxslash}$. On the other hand, axiom CM2 implies $Le : Z \to W'$ is a trivial cofibration, and so $\tilde{\mathcal{I}}' \subseteq {}^{\boxslash}(\mathcal{I}'^{\boxslash})$. Thus, we have $\mathcal{I}'^{\boxslash} \subseteq \tilde{\mathcal{I}}'^{\boxslash}$ as well. ■

Proposition 5.3.3. *Let $\mathcal{M}$ be a combinatorial model category and let $\mathcal{I}$ be a set of generating cofibrations in $\mathcal{M}$.*

(i) *$\mathcal{I}$ cofibrantly generates an algebraically free algebraic factorisation system $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{M}$.*

(ii) *There exists a set $\tilde{\mathcal{I}}'$ of generating trivial cofibrations in $\mathcal{M}$ such that $\tilde{\mathcal{I}}'$ cofibrantly generates an algebraically free algebraic factorisation system $(\mathsf{L}', \mathsf{R})$ on $\mathcal{M}$ with a morphism $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$.*

In particular, $\mathcal{M}$ is the underlying model category of an algebraic model category.

Proof. (i). Apply Garner's small object argument (theorem 0.5.24).

(ii). Use lemma 5.3.2. ■

Definition 5.3.4. Let κ and λ be regular cardinals. A **strongly** (κ, λ)**-algebraic model category** is an algebraic model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.
- $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.
- The underlying endofunctors of the two given algebraic factorisation systems on $\mathcal{M}$ preserve colimits for small κ-filtered diagrams and are strongly λ-accessible functors.
- The full subcategory $\mathcal{F}$ (resp. $\mathcal{F}'$) of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations (resp. trivial fibrations) in $\mathcal{M}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.

Proposition 5.3.5. *If $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, then there exist algebraic factorisation systems making $\mathcal{M}$ a strongly (κ, λ)-algebraic model category.*

Proof. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be a λ-small set of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ that generate the cofibrations (resp. trivial cofibrations) in $\mathcal{M}$. Replacing $\mathcal{I}$ with $\mathcal{I} \cup \mathcal{I}'$ if necessary, we may assume $\mathcal{I}' \subseteq \mathcal{I}$. Garner's small object argument (0.5.24) says that algebraically free algebraic factorisation systems cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$ exist and are free, and since $\mathcal{I}' \subseteq \mathcal{I}$, the universal property of free algebraic factorisation systems ensures we have the required morphism of algebraic factorisation systems. Lemma 0.3.36 and proposition 0.5.23 then say that the underlying endofunctors of the algebraic factorisation systems preserve colimits for small κ-filtered diagrams and are strongly λ-accessible. Finally, by corollary 0.5.27, the two full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations and trivial fibrations are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$. ■

Proposition 5.3.6. *Let $\mathcal{M}$ be a strongly (κ, λ)-algebraic model category.*

(i) *The algebraic model structure on $\mathcal{M}$ restricts to an algebraic model structure on $\mathbf{K}_\lambda(\mathcal{M})$.*

(ii) *The inclusions $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible functors.*

(iii) *$\mathcal{W}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

Proof. (i). By definition, the underlying endofunctors of the given algebraic factorisation systems are strongly λ-accessible and so send morphisms in $\mathbf{K}_\lambda(\mathcal{M})$ back to $\mathbf{K}_\lambda(\mathcal{M})$. Thus, we obtain algebraic factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$, and it is clear that the given morphism of algebraic factorisation systems on $\mathcal{M}$ restricts to a morphism of algebraic factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$. Since $\mathbf{K}_\lambda(\mathcal{M})$ is a full subcategory of $\mathcal{M}$, it follows that the restricted data define an algebraic model structure on $\mathbf{K}_\lambda(\mathcal{M})$.

(ii) and (iii). Apply theorem 5.2.15. ■

5.4 Cisinski model categories

Prerequisites. §0.5, 3.1, 3.5, 4.1, 5.2, A.3, A.4.

In this section we follow [Cisinski, 2002] and [Cisinski, 2006, Ch. 1].

Definition 5.4.1. A **Cisinski model category** is a combinatorial model category whose underlying category is a Grothendieck topos and whose cofibrations are the monomorphisms.

Remark 5.4.2. Grothendieck toposes are always locally presentable categories, so we may replace 'combinatorial' with 'cofibrantly generated' in the above definition.

Example 5.4.3. The Kan–Quillen model structure on **sSet** makes it into a Cisinski model category.

Remark 5.4.4. In any topos, the unique morphism $0 \to X$ is always a monomorphism; thus, in a Cisinski model category, every object is cofibrant.

Proposition 5.4.5. *Let $\mathcal{M}$ be a Grothendieck topos and let $\mathcal{M}_{\mathrm{f}}$ be a class of objects in $\mathcal{M}$. There is at most one model structure on $\mathcal{M}$ making it a Cisinski model category with $\mathcal{M}_{\mathrm{f}}$ as the class of fibrant objects.*

Proof. This is a special case of proposition 4.4.8. ■

Proposition 5.4.6. *Let $f : X \to Y$ and $g : Z \to W$ be cofibrations in a Cisinski model category $\mathcal{M}$. Suppose the square in the diagram below is a pushout square in $\mathcal{M}$:*

$$\begin{tikzcd}
X \times Z \arrow[r, "\mathrm{id}_X \times g"] \arrow[d, "f \times \mathrm{id}_Z"'] & X \times W \arrow[d] \arrow[ddr, bend left, "f \times \mathrm{id}_W"] & \\
Y \times Z \arrow[r] \arrow[drr, bend right, "\mathrm{id}_Y \times g"'] & (X \times W) \cup^{X \times Z} (Y \times Z) \arrow[dr, dashed, "f \mathbin{\square} g"] & \\
& & Y \times W
\end{tikzcd}$$

(i) *The unique morphism $f \mathbin{\square} g$ making the diagram commute is a cofibration.*

(ii) *Assuming the class of trivial cofibrations in $\mathcal{M}$ is closed under binary products, if either f or g is a trivial cofibration, then $f \mathbin{\square} g$ is a trivial cofibration.*

Proof. (i). The claim is certainly true when $\mathcal{M}$ is a presheaf topos, and since the associated sheaf functor preserves colimits and finite limits, the claim holds for all sheaf toposes as well.

(ii). The two cases are symmetrical; we will assume $f : X \to Y$ is a trivial cofibration. Clearly, $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ and $f \times \mathrm{id}_W : X \times W \to Y \times W$ are monomorphisms, so the hypothesis implies they are trivial cofibrations. The class of trivial cofibrations is closed under pushouts (by proposition A.3.17), so the morphism $X \times W \to (X \times W) \cup^{X \times Z} (Y \times Z)$ is also a trivial cofibration. The 2-out-of-3 property of weak equivalences then implies $f \,\square\, g$ must be a weak equivalence as well; hence, by claim (i), it is a trivial cofibration. ■

¶ 5.4.7. We will now see how to build Cisinski model structures. Throughout this section, $\mathcal{M}$ will be a Grothendieck topos, say $\mathcal{M} = \mathbf{Sh}(\mathbb{C}, J)$ for a small category $\mathbb{C}$ equipped with a Grothendieck topology J.

Definition 5.4.8. A **Cisinski cylinder functor** for $\mathcal{M}$ is a quadruple $\left(I, \iota^0, \iota^1, \rho\right)$ where $I : \mathcal{M} \to \mathcal{M}$ is a functor, $\iota^0, \iota^1 : \mathrm{id}_{\mathcal{M}} \Rightarrow I$ and $\rho : I \Rightarrow \mathrm{id}_{\mathcal{M}}$ are natural transformations, such that:

- $\rho \bullet \iota^0 = \rho \bullet \iota^1 = \mathrm{id}_{\mathrm{id}_{\mathcal{M}}}$.
- The induced morphism $\iota_X = \left(\!\left(\iota^0_X, \iota^1_X\right)\!\right) : X \amalg X \to IX$ is a monomorphism for every object X in $\mathcal{M}$.

We will often abuse notation and simply say that I is a cylinder functor, with the natural transformations ι^0, ι^1, and ρ understood.

Remark 5.4.9. By symmetry, $\left(I, \iota^0, \iota^1, \rho\right)$ is a Cisinski cylinder functor if and only if $\left(I, \iota^1, \iota^0, \rho\right)$ is a Cisinski cylinder functor.

Definition 5.4.10. Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor for $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$. An **I-homotopy** in $\mathcal{M}$ from f_0 to f_1 is a morphism $H : IX \to Y$ such that $H \circ \iota^0_X = f_0$ and $H \circ \iota^1_X = f_1$. We say f_0 and f_1 are **I-homotopic** if there is a zigzag of I-homotopies connecting f_0 to f_1.

Proposition 5.4.11. *Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor for $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$.*

(i) *For any morphism $g : Y \to Z$ in $\mathcal{M}$, if f_0 and f_1 are I-homotopic, then so are $g \circ f_0$ and $g \circ f_1$.*

(ii) *For any morphism $g : W \to X$ in $\mathcal{M}$, if f_0 and f_1 are I-homotopic, then so are $f_0 \circ g$ and $f_1 \circ g$.*

Proof. Obvious. ◆

Definition 5.4.12. Let $(I, \iota^0, \iota^1, \rho)$ be a Cisinski cylinder functor for $\mathcal{M}$. The **I-homotopy category** of $\mathcal{M}$ is the category $\operatorname{Ho}_I \mathcal{M}$ defined below:

- The objects of $\operatorname{Ho}_I \mathcal{M}$ are those of $\mathcal{M}$.
- The hom-set $\operatorname{Ho}_I \mathcal{M}(X, Y)$ is $\mathcal{M}(X, Y)$ modulo I-homotopy.
- Composition and identities are inherited from $\mathcal{M}$.

Proposition 5.4.13. *Let $(I, \iota^0, \iota^1, \rho)$ be a Cisinski cylinder functor and let $\gamma : \mathcal{M} \to \operatorname{Ho}_I \mathcal{M}$ be the functor that sends a morphism in $\mathcal{M}$ to its I-homotopy class.*

(i) *The functor $\gamma : \mathcal{M} \to \operatorname{Ho}_I \mathcal{M}$ is full.*

(ii) *Let $\mathcal{H}$ be the class of morphisms in $\mathcal{M}$ that γ sends to isomorphisms. If $\gamma\rho : \gamma I \Rightarrow \gamma$ is a natural isomorphism, then $\gamma : \mathcal{M} \to \operatorname{Ho}_I \mathcal{M}$ exhibits $\operatorname{Ho}_I \mathcal{M}$ as a localisation of $\mathcal{M}$ at $\mathcal{H}$.*

Proof. (i). Obvious.

(ii). Consider any functor $F : \mathcal{M} \to \mathcal{C}$ such that $F\rho : FI \Rightarrow F$ is a natural isomorphism. Then, we have $F\iota^0 = F\iota^1$, so F factors through $\gamma : \mathcal{M} \to \operatorname{Ho}_I \mathcal{M}$ in a unique way. In particular, if $\gamma\rho : \gamma I \Rightarrow \gamma$ itself is a natural isomorphism, then $\operatorname{Ho}_I \mathcal{M}$ has the universal property of a localisation of $\mathcal{M}$ at $\mathcal{H}$. ■

Definition 5.4.14. A **Cisinski trivial fibration** in $\mathcal{M}$ is a morphism that has the right lifting property with respect to all monomorphisms.

Proposition 5.4.15. *Let $p : X \to Y$ be a Cisinski trivial fibration in $\mathcal{M}$.*

(i) *There exists a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$.*

(ii) *For any such $s : Y \to X$ and any Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$ for $\mathcal{M}$, there exists an I-homotopy from id_X to $s \circ p$.*

(iii) *The morphism $p : X \to Y$ becomes an isomorphism in $\operatorname{Ho}_I \mathcal{M}$.*

Proof. (i). The unique morphism $0 \to Y$ is a monomorphism in any topos, so the right lifting property of $p : X \to Y$ guarantees the existence of a section.

(ii). Consider the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} X \amalg X \arrow[r, "(\!(\mathrm{id}_X, s \circ p)\!)"] \arrow[d, "\iota_X"'] & X \arrow[d, "p"] \\ IX \arrow[r, "p \circ \rho_X"'] & Y \end{tikzcd}$$

By definition, $\iota_X : X \amalg X \to IX$ is a monomorphism, so the right lifting property of $p : X \to Y$ yields a morphism $H : IX \to X$ such that $H \circ \iota_X = (\!(\mathrm{id}_X, s \circ p)\!)$ and $p \circ H = p \circ \rho_X$; in particular, H is an I-homotopy from id_X to $s \circ p$.

(iii). Clearly, the morphisms $p : X \to Y$ and $s : Y \to X$ become mutual inverses in $\mathrm{Ho}_I \, \mathcal{M}$. ■

¶ 5.4.16. Let Ω be a subobject classifier for $\mathcal{M}$ and let $\top, \bot : 1 \to \Omega$ be the morphisms classifying the top and bottom subobjects of 1, respectively. Then the following diagram is a pullback square by definition,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\ 1 \arrow[r, "\bot"'] & \Omega \end{tikzcd}$$

so the induced morphism $(\!(\top, \bot)\!) : 1 \amalg 1 \to \Omega$ is a monomorphism. Since monomorphisms are stable under pullback, the following definition is legitimate:

Definition 5.4.17. The **Lawvere cylinder functor** for $\mathcal{M}$ is the cylinder functor $(I, \iota^0, \iota^1, \rho)$ defined below:

- $I : \mathcal{M} \to \mathcal{M}$ is the functor $\Omega \times -$.
- The morphism $\iota^0_X : X \to \Omega \times X$ corresponds to $\top \times \mathrm{id}_X$.
- The morphism $\iota^1_X : X \to \Omega \times X$ corresponds to $\bot \times \mathrm{id}_X$.
- The morphism $\rho_X : \Omega \times X \to X$ is the product projection.

Proposition 5.4.18. *Let X be any object in $\mathcal{M}$ and let Ω be the subobject classifier for $\mathcal{M}$.*

(i) *The product projection $p_X : \Omega \times X \to X$ is a Cisinski trivial fibration.*

(ii) *For any Cisinski cylinder functor* $(I, \iota^0, \iota^1, \rho)$, *there exists a commutative diagram of the following form:*

$$\begin{tikzcd} X \amalg X \arrow[r, "(\!(\top,\bot)\!)\times \mathrm{id}_X"] \arrow[d, "\iota_X"'] & \Omega \times X \arrow[d, "p_X"] \\ IX \arrow[r, "\rho_X"'] \arrow[ur, dashed] & X \end{tikzcd}$$

Proof. (i). Since the class of Cisinski trivial fibrations is closed under pullbacks (by proposition A.3.17), it suffices to show that the morphism $p_1 : \Omega \times 1 \to 1$ is a trivial fibration. However, Ω is canonically an injective object in $\mathcal{M}$ (with respect to the class of monomorphisms), i.e. the unique morphism $\Omega \to 1$ has the right lifting property with respect to all monomorphisms, so p_1 is indeed a Cisinski trivial fibration.

(ii). This follows from claim (i) and the requirement that $\iota_X : X \amalg X \to IX$ be a monomorphism. ■

Remark 5.4.19. Thus, any pair of morphisms that are homotopic with respect to the Lawvere cylinder functor must also be I-homotopic for any Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$.

Proposition 5.4.20. *Let* $(I, \iota^0, \iota^1, \rho)$ *be a cylinder functor for* $\mathcal{M}$ *and let* $\mathcal{V}$ *be the class of Cisinski trivial fibrations in* $\mathcal{M}$.

(i) *There is a canonical identity-on-objects functor* $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I\,\mathcal{M}$ *compatible with the localising functors, and it is a full functor.*

(ii) *If the natural morphism* $\rho_X : IX \to X$ *is a Cisinski trivial fibration for all objects* X *in* $\mathcal{M}$, *then the canonical functor* $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I\,\mathcal{M}$ *is an isomorphism of categories.*

(iii) *If* $(I, \iota^0, \iota^1, \rho)$ *is the Lawvere cylinder functor for* $\mathcal{M}$, *then the canonical functor* $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I\,\mathcal{M}$ *is an isomorphism of categories.*

Proof. (i). Recall that Cisinski trivial fibrations are I-homotopy equivalences (by proposition 5.4.15), so there is indeed a canonical identity-on-objects functor $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I\,\mathcal{M}$ compatible with the localising functors. Since the localising functor $\mathcal{M} \to \mathrm{Ho}_I\,\mathcal{M}$ is full, the functor $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I\,\mathcal{M}$ must also be full.

(ii). The hypothesis implies any two I-homotopic morphisms in $\mathcal{M}$ are equal as morphisms in $\mathcal{M}[\mathcal{V}^{-1}]$, so the canonical functor $\mathcal{M}[\mathcal{V}^{-1}] \to \mathrm{Ho}_I \mathcal{M}$ is indeed fully faithful and bijective on objects, as required.

(iii). Proposition 5.4.18 says that the natural morphism $\rho_X : IX \to X$ is a Cisinski trivial fibration for all objects X in $\mathcal{M}$ when $(I, \iota^0, \iota^1, \rho)$ is the Lawvere cylinder for $\mathcal{M}$. ■

Definition 5.4.21. An **elementary Cisinski homotopy structure** on $\mathcal{M}$ is a Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$ satisfying these axioms:

DH1. The functor $I : \mathcal{M} \to \mathcal{M}$ preserves monomorphisms and colimits for all small diagrams.

DH2. For all monomorphisms $g : Z \to W$ in $\mathcal{M}$, the following diagrams are pullback squares:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\iota^0_Z"'] & W \arrow[d, "\iota^0_W"] \\ IZ \arrow[r, "Ig"'] & IW \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\iota^1_Z"'] & W \arrow[d, "\iota^1_W"] \\ IZ \arrow[r, "Ig"'] & IW \end{tikzcd}$$

Proposition 5.4.22. *The Lawvere cylinder functor is an elementary Cisinski homotopy structure.*

Proof. The functor $A \times -$ always preserves monomorphisms, and toposes are cartesian closed, so for any object A in $\mathcal{M}$, the functor $A \times -$ preserves colimits. Thus the Lawvere cylinder functor satisfies axiom DH1. It is clear that axiom DH2 is also satisfied. ■

Definition 5.4.23. Let $(I, \iota^0, \iota^1, \rho)$ be an elementary Cisinski homotopy structure on $\mathcal{M}$. A **class of I-anodyne extensions** is a class $\mathcal{A}$ of morphisms in $\mathcal{M}$ satisfying these axioms:

An0. There exists a subset $\Lambda \subseteq \mathcal{A}$ such that the members of Λ are monomorphisms in $\mathcal{M}$ and $\mathcal{A} = {}^{\boxslash}(\Lambda^{\boxslash})$. We say Λ is a **generating set** for $\mathcal{A}$.

An1. If $g : Z \to W$ is a monomorphism in $\mathcal{M}$ and $e \in \{0, 1\}$, then given a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\iota^e_Z"] \arrow[d, "g"'] & IZ \arrow[d] \arrow[ddr, bend left, "Ig"] & \\ W \arrow[r] \arrow[drr, bend right, "\iota^e_W"'] & V_e(g) \arrow[dr, dashed, "j_e"] & \\ & & IW \end{tikzcd}$$

where the top-left square is a pushout square, $j_e : V_e(g) \to IW$ is in $\mathcal{A}$.

An2. If $g : Z \to W$ is in $\mathcal{A}$, then given a commutative diagram

$$\begin{tikzcd} Z \amalg Z \arrow[r, "\iota_Z"] \arrow[d, "g"'] & IZ \arrow[d] \arrow[ddr, bend left, "Ig"] & \\ W \amalg W \arrow[r] \arrow[drr, bend right, "\iota_W"'] & U(g) \arrow[dr, dashed, "j"] & \\ & & IW \end{tikzcd}$$

where the top-left square is a pushout square, $j : U(g) \to IW$ is in $\mathcal{A}$.

Remark 5.4.24. Since I preserves colimits for all small diagrams, $I0$ must be an initial object in $\mathcal{M}$. Thus, by taking $Z = 0$, we see that the morphisms $\iota^0_W, \iota^1_W : W \to IW$ are always in any class of I-anodyne extensions.

Proposition 5.4.25. *Let $(I, \iota^0, \iota^1, \rho)$ be an elementary Cisinski homotopy structure on $\mathcal{M}$, let $\mathcal{A}$ be a class of I-anodyne extensions, and let Λ be a generating set for $\mathcal{A}$.*

(i) *There exists a functorial factorisation system on $\mathcal{M}$ with $\mathcal{A}$ as its left class.*

(ii) *$\mathcal{A}$ is the smallest class of morphisms containing Λ that is closed under pushouts, transfinite composition, and retracts.*

(iii) *Every morphism that is in $\mathcal{A}$ is a monomorphism.*

Proof. (i). Apply Quillen's small object argument (theorem 0.5.12).

(ii). This is corollary 0.5.13.

(iii). The class of monomorphisms in a Grothendieck topos is closed under pushouts, transfinite composition, and retracts because the class of injections in **Set** is closed under the same operations. Since Λ is a collection of monomorphisms, so too is $\mathcal{A}$. ■

Definition 5.4.26. A **Cisinski homotopy structure** on $\mathcal{M}$ is an elementary Cisinski homotopy structure on $\mathcal{M}$ together with a class of anodyne extensions.

Definition 5.4.27. Let $\mathcal{A}$ be the class of anodyne extensions of a Cisinski homotopy structure on $\mathcal{M}$. An **$\mathcal{A}$-fibrant object** in $\mathcal{M}$ is an object X such that the unique morphism $X \to 1$ has the right lifting property with respect to $\mathcal{A}$.

Definition 5.4.28. Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. A **weak equivalence** with respect to $(I, \mathcal{A})$ is a morphism $f : W \to Z$ in $\mathcal{M}$ such that, for every $\mathcal{A}$-fibrant object X, the induced map

$$\mathrm{Ho}_I\,\mathcal{M}(f, X) : \mathrm{Ho}_I\,\mathcal{M}(Z, X) \to \mathrm{Ho}_I\,\mathcal{M}(W, X)$$

is a bijection of sets.

Proposition 5.4.29. *$\mathcal{M}$ together with the class of weak equivalences with respect to a Cisinski homotopy structure $(I, \mathcal{A})$ constitute a saturated homotopical category.*

Proof. Obvious. ♦

Proposition 5.4.30. *Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. Then every morphism in $\mathcal{A}$ is a weak equivalence with respect to $(I, \mathcal{A})$.*

Proof. See Proposition 2.23 in [Cisinski, 2002]. □

Corollary 5.4.31. *Let $\mathcal{W}$ be the class of weak equivalences with respect to $(I, \mathcal{A})$ and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$. If f_0 and f_1 are I-homotopic, then f_0 and f_1 become equal in* $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$.

Proof. It suffices to verify the case where there is an I-homotopy $H : IX \to Y$ from f_0 to f_1. By remark 5.4.24, the morphisms $\iota_X^0, \iota_X^1 : X \to IX$ are anodyne extensions, and so are invertible in $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$. We have $\rho_X \circ \iota_X^0 = \rho_X \circ \iota_X^1 = \mathrm{id}_X$ by definition, so ι_X^0 and ι_X^1 must be equal in $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$; but $H \circ \iota_X^0 = f_0$ and $H \circ \iota_X^1 = f_1$, so f_0 and f_1 must be equal in $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$. ■

Theorem 5.4.32. *Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. Then $\mathcal{M}$ is a combinatorial model category where*

- *the cofibrations are the monomorphisms in $\mathcal{M}$,*
- *the weak equivalences are the weak equivalences with respect to $(I, \mathcal{A})$, and*
- *the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.*

This is the **Cisinski model structure** *on $\mathcal{M}$ defined by $(I, \mathcal{A})$.*

Proof. See Théorème 2.13 in [Cisinski, 2002]. □

Definition 5.4.33. An **$\mathcal{M}$-localiser** is a class $\mathcal{W}$ of morphisms in $\mathcal{M}$ satisfying the following axioms:

L1. $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$.

L2. Every Cisinski trivial fibration is in $\mathcal{W}$.

L3. The class of monomorphisms that are in $\mathcal{W}$ is closed under pushout and transfinite composition.

A **generating set** for $\mathcal{W}$ is a set S such that $\mathcal{W}$ is the smallest $\mathcal{M}$-localiser containing S. An **accessible $\mathcal{M}$-localiser** is an $\mathcal{M}$-localiser that admits a generating set.

Proposition 5.4.34. *Let $\mathcal{W}$ be a class of morphisms in $\mathcal{M}$ satisfying the following axioms:*

FS1. *For any object X in $\mathcal{M}$, the morphism* $\mathrm{id} : X \to X$ *is in $\mathcal{W}$.*

FS2. *$\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$.*

FS3. *$\mathcal{W}$ has the special 2-out-of-4 property in $\mathcal{M}$.*

Then the following are equivalent:

(i) *Every Cisinski trivial fibration is in $\mathcal{W}$.*

(ii) *Let $\left(I, \iota^0, \iota^1, \rho\right)$ be the Lawvere cylinder functor for $\mathcal{M}$. For all objects X in $\mathcal{M}$, the morphism $\rho_X : IX \to X$ is in $\mathcal{W}$.*

(iii) *There exists a Cisinski cylinder functor* $(I, \iota^0, \iota^1, \rho)$ *for* $\mathcal{M}$ *such that the morphism* $\rho_X : IX \to X$ *is in* $\mathcal{W}$ *for all objects* X *in* $\mathcal{M}$.

Proof. (i) ⇒ (ii). This was shown in proposition 5.4.18.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Let $p : X \to Y$ be a Cisinski trivial fibration in $\mathcal{M}$. Proposition 5.4.15 then says that there exists a morphism $s : Y \to X$ and an I-homotopy from id_X to $s \circ p$, i.e. a morphism $H : IX \to X$ such that $H \circ \iota^0_X = \mathrm{id}_X$ and $H \circ \iota^1_X = s \circ p$. Since $\rho_X : IX \to X$ is in $\mathcal{W}$ and $\rho_X \circ \iota^0_X = \rho_X \circ \iota^1_X = \mathrm{id}_X$, axioms FS1 and FS2 imply that $\iota^0_X, \iota^1_X : X \to IX$ are in $\mathcal{W}$, and so $H : IX \to X$ is also in $\mathcal{W}$, and hence $s \circ p : X \to X$ is in $\mathcal{W}$ as well. We may now use axiom FS3 to deduce that $p : X \to Y$ is in $\mathcal{W}$. ■

Proposition 5.4.35. *Let* $(I, \mathcal{A})$ *be a Cisinski homotopy structure on* $\mathcal{M}$. *Then the class of weak equivalences with respect to* $(I, \mathcal{A})$ *is an accessible* $\mathcal{M}$*-localiser.*

Proof. See Proposition 3.8 in [Cisinski, 2002]. □

Theorem 5.4.36. *Let* $\mathcal{W}$ *be any accessible* $\mathcal{M}$*-localiser. Then* $\mathcal{M}$ *is a combinatorial model category where*

- *the cofibrations are the monomorphisms in* $\mathcal{M}$,
- *the weak equivalences are the morphisms that are in* $\mathcal{W}$, *and*
- *the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.*

This is the **Cisinski model structure** *on* $\mathcal{M}$ *associated with* $\mathcal{W}$.

Proof. See Théorème 3.9 in [Cisinski, 2002]. □

Corollary 5.4.37. *If* $\mathcal{W}$ *is any* $\mathcal{M}$*-localiser (not necessarily accessible), then* $\mathcal{W}$ *is closed under retracts.*

Proof. See Corollaire 3.10 in [Cisinski, 2002]. □

5.5 Monoidal model categories

Prerequisites. §§4.1, 4.2, 4.3, B.1, B.4.

Proposition 5.5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pullbacks, let $\mathcal{E}$ be a category with pushouts, and let $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, $\mathcal{J} \subseteq \operatorname{mor} \mathcal{D}$ and $\mathcal{K} \subseteq \operatorname{mor} \mathcal{E}$ be subensembles. Suppose we have the following functors*

$$\oslash : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$
$$\pitchfork : \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$
$$\multimapinv : \mathcal{E} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

and natural bijections:

$$\mathcal{E}(C \oslash D, E) \cong \mathcal{C}(C, D \pitchfork E)$$
$$\mathcal{E}(C \oslash D, E) \cong \mathcal{D}(D, E \multimapinv C)$$
$$\mathcal{C}(C, D \pitchfork E) \cong \mathcal{D}(D, E \multimapinv C)$$

Then the following are equivalent:

(i) *If $f : C \to C'$ is in $\mathcal{I}$, $g : D \to D'$ is in $\mathcal{J}$, and the square in the diagram below is a pushout square in $\mathcal{E}$,*

$$\begin{tikzcd}
C \oslash D \arrow[r, "\mathrm{id}_C \oslash g"] \arrow[d, "f \oslash \mathrm{id}_D"'] & C \oslash D' \arrow[d] \arrow[ddr, bend left, "f \oslash \mathrm{id}_{D'}"] & \\
C' \oslash D \arrow[r] \arrow[drr, bend right, "\mathrm{id}_{C'} \oslash g"'] & (C \oslash D') \cup^{C \oslash D} (C' \oslash D) \arrow[dr, dashed, "f \square g"] & \\
& & C' \oslash D'
\end{tikzcd}$$

then the unique morphism $f \square g$ making the diagram commute is in ${}^{\boxslash}\mathcal{K}$.

(ii) *If $g : D \to D'$ is in $\mathcal{J}$, $h : E \to E'$ is in $\mathcal{K}$, and the square in the diagram below is a pullback square in $\mathcal{C}$,*

$$\begin{tikzcd}
D' \pitchfork E \arrow[dr, dashed, "g \square h"] \arrow[drr, bend left, "g \pitchfork \mathrm{id}_E"] \arrow[ddr, bend right, "\mathrm{id}_{D'} \pitchfork h"'] & & \\
& (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E) \arrow[r] \arrow[d] & D \pitchfork E \arrow[d, "\mathrm{id}_D \pitchfork h"] \\
& D' \pitchfork E' \arrow[r, "g \pitchfork \mathrm{id}_{E'}"'] & D \pitchfork E'
\end{tikzcd}$$

then the unique morphism $g \boxdot h$ making the diagram commute is in $\mathcal{I}^{\boxslash}$.

(iii) *If $h : E \to E'$ is in $\mathcal{K}$, $f : C \to C'$ is in $\mathcal{I}$ and the square in the diagram below is a pullback square in $\mathcal{D}$,*

$$\begin{tikzcd}
E \multimapinv C' \arrow[drr, bend left, "\mathrm{id}_E \multimapinv f"] \arrow[dr, dashed, "h \boxdot f"] \arrow[ddr, bend right, "h \multimapinv \mathrm{id}_C"'] & & \\
& (E' \multimapinv C') \times_{E' \multimapinv C} (E \multimapinv C) \arrow[r] \arrow[d] & E \multimapinv C \arrow[d, "h \multimapinv \mathrm{id}_C"] \\
& E' \multimapinv C' \arrow[r, "\mathrm{id}_{E'} \multimapinv f"'] & E' \multimapinv C
\end{tikzcd}$$

then the unique morphism $h \boxdot f$ making the diagram commute is in $\mathcal{J}^{\boxslash}$.

Proof. (i) $\Rightarrow$ (ii). Let $f : C \to C'$ be in $\mathcal{I}$, let $g : D \to D'$ be in $\mathcal{J}$, let $h : E \to E'$ be in $\mathcal{K}$, and suppose we have a commutative diagram of the following form:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & D' \pitchfork E \arrow[d, "g \boxdot h"] \\
C' \arrow[r] \arrow[ur, dashed] & (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E)
\end{tikzcd}$$

By the universal property of pullbacks, this corresponds to a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & D' \pitchfork E \arrow[dr, "\mathrm{id}_{D'} \pitchfork h"] \arrow[dd, "g \pitchfork \mathrm{id}_E" near end] & \\
C' \arrow[ur, dashed] \arrow[rr, bend left] \arrow[dr] & & D' \pitchfork E' \arrow[d, "g \pitchfork \mathrm{id}_{E'}"'] \\
& D \pitchfork E \arrow[r, "\mathrm{id}_D \pitchfork h"'] & D \pitchfork E'
\end{tikzcd}$$

and, by adjoint transposition, to a commutative diagram in $\mathcal{E}$ of the form

$$\begin{tikzcd}
C \oslash D \arrow[r, "\mathrm{id}_C \oslash g"] \arrow[d, "f \oslash \mathrm{id}_D"'] & C \oslash D' \arrow[dr] \arrow[dd, "f \otimes \mathrm{id}_{D'}" near start] & \\
C' \oslash D \arrow[rr, bend right] \arrow[dr, "\mathrm{id}_{C'} \oslash g"'] & & E \arrow[d, "h"] \\
& C' \oslash D' \arrow[ur, dashed] \arrow[r] & E'
\end{tikzcd}$$

whence, by the universal property of pushouts, commutative diagram in $\mathcal{E}$ of the following form:

$$\begin{tikzcd}
(C \oslash D') \cup^{C \oslash D} (C' \oslash D) \arrow[r] \arrow[d, "f \mathbin{\square_{\lrcorner}} g"'] & E \arrow[d, "h"] \\
C' \oslash D' \arrow[r] \arrow[ur, dashed] & E'
\end{tikzcd}$$

But $(f \mathbin{\square_{\lrcorner}} g) \boxslash h$, so we conclude that $f \boxslash (g \mathbin{\square^{\ulcorner}} h)$.

(ii) ⇒ (iii), (i) ⇒ (ii). A similar argument works. ■

Definition 5.5.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be three model categories. A **Quillen adjunction of two variables** consists of three functors $\oslash, \pitchfork, \multimapinv$ with natural bijections as in the proposition satisfying the following (equivalent) axioms:

(a) If $h : E \to E'$ is a fibration in $\mathcal{E}$ and $f : C \to C'$ is a cofibration in $\mathcal{C}$, then the morphism $h \mathbin{\square^{\ulcorner}} f : E \multimapinv C' \to (E' \multimapinv C') \times_{E' \multimapinv C} (E \multimapinv C)$ is a fibration in $\mathcal{D}$, which is a weak equivalence if either h or f is.

(b) If $f : C \to C'$ is a cofibration in $\mathcal{C}$ and $g : D \to D'$ is a cofibration in $\mathcal{D}$, then the morphism $f \mathbin{\square_{\lrcorner}} g : C \oslash D \to (C \oslash D') \cup^{C \oslash D} (C' \oslash D)$ is a cofibration in $\mathcal{E}$, which is a weak equivalence if either f or g is.

(c) If $g : D \to D'$ is a cofibration in $\mathcal{C}$ and $h : E \to E'$ is a fibration in $\mathcal{D}$, then the morphism $g \mathbin{\square^{\ulcorner}} h : D' \pitchfork E \to (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E)$ is a fibration in $\mathcal{C}$, which is a weak equivalence if either g or h is.

Proposition 5.5.3. *Let $(\oslash, \pitchfork, \multimapinv)$ be a Quillen adjunction of two variables as above.*

(i) *For each cofibrant object C in $\mathcal{C}$, the adjunction*

$$C \oslash (-) \dashv (-) \multimapinv C : \mathcal{E} \to \mathcal{D}$$

is a Quillen adjunction.

(ii) *For each cofibrant object D in $\mathcal{D}$, the adjunction*

$$(-) \oslash D \dashv D \pitchfork (-) : \mathcal{E} \to \mathcal{C}$$

is a Quillen adjunction.

(iii) *For each fibrant object E in $\mathcal{E}$, the adjunction*

$$E \multimapinv (-) \dashv (-) \pitchfork E : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$$

is a Quillen adjunction.

Proof. Immediate from the definitions. ♦

Corollary 5.5.4.

(i) *For each object C in $\mathcal{C}$, $C \oslash (-)$ preserves weak equivalences between cofibrant objects, and $(-) \oslash C$ preserves weak equivalences between fibrant objects.*

(ii) *For each object D in $\mathcal{D}$, $(-) \oslash D$ preserves weak equivalences between cofibrant objects, and $D \pitchfork (-)$ preserves weak equivalences between fibrant objects.*

(iii) *For each object E in $\mathcal{E}$, $E \multimapinv (-)$ sends weak equivalences between cofibrant objects in $\mathcal{C}$ to weak equivalences between fibrant objects in $\mathcal{D}$, and $(-) \pitchfork E$ sends weak equivalences between cofibrant objects in $\mathcal{D}$ to weak equivalences between fibrant objects in $\mathcal{D}$.*

Proof. Apply Ken Brown's lemma (4.3.6). ■

Lemma 5.5.5. *Let $\mathcal{V}$ be a monoidal category, let $\mathcal{M}$ be a model category with fibrant and cofibrant replacement functors, and let $p : \tilde{I} \to I$ be a morphism in $\mathcal{V}$, where I is the monoidal unit of $\mathcal{V}$.*

If $\mathcal{M}$ has a left $\mathcal{V}$-action $\oslash$ and right adjoint right $\mathcal{V}^{\mathrm{op}}$-action $\multimapinv$ such that the adjunction

$$\tilde{I} \otimes (-) \dashv (-) \multimapinv \tilde{I} : \mathcal{M} \to \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

(i) *For all cofibrant objects X in $\mathcal{M}$, $p \oslash \mathrm{id}_X : \tilde{I} \oslash X \to I \oslash X$ is a weak equivalence.*

(ii) *For all fibrant objects Y in $\mathcal{M}$, $\mathrm{id}_Y \multimapinv p : Y \multimapinv I \to Y \multimapinv \tilde{I}$ is a weak equivalence.*

If $\mathcal{M}$ has a right $\mathcal{V}$-action $\obslash$ and a right adjoint left $\mathcal{V}^{\mathrm{op}}$-action $\multimap$ such that the adjunction

$$(-) \obslash \tilde{I} \dashv \tilde{I} \multimap (-) : \mathcal{M} \to \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

(i′) *For all cofibrant objects X in $\mathcal{M}$,* $\mathrm{id}_X \obslash p : X \obslash \tilde{I} \to X \obslash I$ *is a weak equivalence.*

(ii′) *For all fibrant objects Y in $\mathcal{M}$,* $p \multimap \mathrm{id}_Y : I \multimap Y \to \tilde{I} \multimap Y$ *is a weak equivalence.*

Proof. Since $\boldsymbol{\eta}_X : X \to I \oslash X$ is a natural isomorphism, the adjunction

$$I \oslash (-) \dashv (-) \multimapinv I : \mathcal{M} \to \mathcal{M}$$

is an adjoint equivalence of categories, and *a fortiori* a Quillen equivalence, and the natural transformations $p \oslash (-)$ and $(-) \multimapinv p$ constitute a conjugate pair. Theorem 3.3.24 says that the derived natural transformations for $p \oslash (-)$ and $(-) \multimapinv p$ constitute a conjugate pair of natural transformations between the derived adjunctions. Applying proposition 3.3.28 to theorem 4.3.13, we deduce that the following are equivalent:

- For all cofibrant objects X, $p \oslash \mathrm{id}_X$ is a weak equivalence.
- The left derived natural transformation for $p \oslash (-)$ is a natural isomorphism.
- The right derived natural transformation for $(-) \oslash p$ is a natural isomorphism.
- For all fibrant objects Y, $\mathrm{id}_Y \multimapinv p$ is a weak equivalence. ■

The following definition is due to Hovey [1999, §4.2]:

Definition 5.5.6. A **monoidal model category** is a biclosed monoidal category $\mathcal{M}$ equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The right $\mathcal{M}$-hom system $(\otimes, \multimap, \multimapinv)$, where $\multimap$ (resp. $\multimapinv$) is the right (resp. left) internal hom functor of $\mathcal{M}$, is a Quillen adjunction of two variables.
- **Unit axiom.** For each cofibrant replacement $(\tilde{I}, p)$ of the monoidal unit I and each cofibrant object X in $\mathcal{M}$, the morphisms $p \otimes \mathrm{id}_X : \tilde{I} \otimes X \to I \otimes X$ and $\mathrm{id}_X \otimes p : X \otimes \tilde{I} \to X \otimes I$ are weak equivalences in $\mathcal{M}$.

Lemma 5.5.7. *Let $\mathcal{M}$ be a biclosed monoidal category equipped with a model structure satisfying the pushout–product axiom, and let X be any object in $\mathcal{M}$. The following are equivalent:*

(i) *There exists a cofibrant replacement* $(\tilde{I}, p)$ *of the monoidal unit* I *such that* $p \otimes \mathrm{id}_X$ *and* $\mathrm{id}_X \otimes p$ *are weak equivalences in* $\mathcal{M}$*.*

(ii) *There exists a fibrant cofibrant replacement* (QI, q) *of the monoidal unit* I *such that* $q \otimes \mathrm{id}_X$ *and* $\mathrm{id}_X \otimes q$ *are weak equivalences in* $\mathcal{M}$*.*

(iii) *For any cofibrant replacement* $(\tilde{I}, p)$ *of the monoidal unit* I*, both* $p \otimes \mathrm{id}_X$ *and* $\mathrm{id}_X \otimes p$ *are weak equivalences in* $\mathcal{M}$*.*

Proof. (i) ⇒ (ii). Let (QI, q) be a fibrant cofibrant replacement of I; such exists by proposition 4.1.24. Since $\tilde{I}$ is cofibrant, axiom CM5 implies there is a morphism $w : \tilde{I} \to QI$ such that $q \circ w = p$, and the 2-out-of-3 property implies w is a weak equivalence. Corollary 5.5.4 says $w \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes w$ are weak equivalences, thus by the 2-out-of-3 property again $q \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes q$ must be weak equivalences.

(ii) ⇒ (iii). A similar argument works.

(iii) ⇒ (i). Obvious, given the existence of cofibrant replacements. ■

Corollary 5.5.8. *Let* $\mathcal{M}$ *be a biclosed monoidal category equipped with a model structure. If the monoidal unit* I *is a cofibrant object in* $\mathcal{M}$*, then the following are equivalent:*

(i) $\mathcal{M}$ *is a monoidal model category.*

(ii) $\mathcal{M}$ *satisfies the pushout–product axiom.* ■

TODO: State the version without the assumption that the unit is cofibrant.

Proposition 5.5.9. *Let* $\mathcal{M}$ *be a monoidal model category, let* I *be the monoidal unit, and let* $\multimap : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ *be the right internal hom functor. If* I *is a cofibrant object and* (J, i_0, i_1, p) *is a cylinder object for* I*, then* $(J \multimap X, i, p_0, p_1)$ *is a path object for all fibrant* X*, where* $i : X \to [J, X]$ *is the morphism induced by* $p : J \to I$*, and* $p_0, p_1 : [J, X] \to X$ *are (respectively) the morphisms induced by* $i_0, i_1 : I \to J$*.*

Proof. Since I is a cofibrant object, $I + I$ is cofibrant (by proposition A.3.17), and hence J itself is cofibrant. Corollary 5.5.4 says the functor $(-) \multimap X : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}$ sends weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences between fibrant objects in $\mathcal{M}$ when X is fibrant, so it follows that the morphism $i : X \to [J, X]$ is a weak equivalence. Similarly, since the morphism $I + I \to J$

induced by i_0 and i_1 is a cofibration, the morphism $[J, X] \to X \times X$ induced by p_0 and p_1 is a fibration, so $\big([J, X], i, p_0, p_1\big)$ is indeed a path object for X. ■

The following definition can be found in [Rezk, 2010, §2] and [Simpson, 2012, §7.7].

Definition 5.5.10. A **cartesian model category** is a cartesian closed category $\mathcal{M}$ equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The left $\mathcal{M}$-hom system $(\times, [-,-], [-,-])$ is a Quillen adjunction of two variables.
- **Cofibrant unit axiom.** Every terminal object in $\mathcal{M}$ is cofibrant.

Example 5.5.11. The Kan–Quillen model structure on **sSet** makes it a cartesian model category: **sSet** is a cartesian closed combinatorial model category (*a fortiori* a DHK model category), all simplicial sets are cofibrant, and the pushout–product axiom is just proposition 1.4.15.

Proposition 5.5.12. *Let $\mathcal{M}$ be a Cisinski model category.*[3] *The following are equivalent:*

(i) *$\mathcal{M}$ is a cartesian model category.*

(ii) *The class of weak equivalences in $\mathcal{M}$ is closed under binary products.*

(iii) *The class of trivial cofibrations in $\mathcal{M}$ is closed under binary products.*

Proof. (i) ⇒ (ii). Since all objects in $\mathcal{M}$ are cofibrant, corollary 5.5.4 implies that, for any object Y in $\mathcal{M}$, the functor $(-) \times Y : \mathcal{M} \to \mathcal{M}$ preserves weak equivalences. Thus, the class of weak equivalences in $\mathcal{M}$ is closed under binary products.

(ii) ⇒ (iii). The class of monomorphisms is always closed under binary products, so the class of trivial cofibrations (i.e. monic weak equivalences) is closed under binary products if the class of weak equivalences is.

(iii) ⇒ (i). This is the content of proposition 5.4.6. ■

[3] See definition 5.4.1.

Theorem 5.5.13. *If $\mathcal{M}$ is a monoidal model category, then there is an induced monoidal biclosed structure on* $\operatorname{Ho}\mathcal{M}$ *where the monoidal product is the left derived functor of the monoidal product in $\mathcal{M}$ and the coherence data is inherited from $\mathcal{M}$.*

Proof. See Theorem 4.3.2 in [Hovey, 1999]. □

Proposition 5.5.14. *Let $\mathcal{M}$ be a cartesian model category and let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects.*

(i) *$\mathcal{M}_{\mathrm{f}}$ is closed under products of small families of objects in $\mathcal{M}$, and $[X, Y]$ is fibrant if X is cofibrant and Y is fibrant.*

(ii) *The localising functor $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho}\mathcal{M}$ preserves products of small families of objects; in particular,* $\operatorname{Ho}\mathcal{M}$ *has products for all small families of objects.*

(iii) $\operatorname{Ho}\mathcal{M}$ *is a cartesian closed category, and $\gamma[X, Y]$ is naturally isomorphic to $[\gamma X, \gamma Y]$ when X is cofibrant and Y is fibrant.*

(iv) *Let $\Gamma : \mathcal{M} \to$ **Set** be the functor $\mathcal{M}(1, -)$ and let $\tau_0 : \mathcal{M} \to$ **Set** be the functor $\operatorname{Ho}\mathcal{M}(\gamma 1, \gamma -)$. The functor τ_0 preserves small products in $\mathcal{M}_{\mathrm{f}}$, and the component $\chi_Y : \Gamma Y \Rightarrow \tau_0 Y$ of the natural transformation $\chi : \Gamma \Rightarrow \tau_0$ induced by the functor γ is surjective for all fibrant objects Y in $\mathcal{M}$.*

Proof. (i). That $\mathcal{M}_{\mathrm{f}}$ is closed in $\mathcal{M}$ under small products is a straightforward consequence of proposition A.3.17, and pushout–product axiom for cartesian model structures implies the other half of the claim.

(ii). Proposition 4.3.18 says $\operatorname{Ho}[I, \mathcal{M}] \to [I, \operatorname{Ho}\mathcal{M}]$ is an equivalence of categories for all sets I, so products in $\operatorname{Ho}\mathcal{M}$ coincide with homotopy products. Homotopy products in $\mathcal{M}_{\mathrm{f}}$ coincide with ordinary products, hence the localising functor $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho}\mathcal{M}$ preserves small products. Since every object in $\mathcal{M}$ is weakly equivalent to one in $\mathcal{M}_{\mathrm{f}}$, it follows that $\operatorname{Ho}\mathcal{M}$ has products for all small families of objects.

(iii). Apply theorem 5.5.13.

(iv). As a representable functor, $\operatorname{Ho}\mathcal{M}(\gamma 1, -) : \operatorname{Ho}\mathcal{M} \to$ **Set** preserves small products, and by claim (ii), $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho}\mathcal{M}$ preserves small products, so

$\tau_0 : \mathcal{M}_\mathrm{f} \to \mathbf{Set}$ indeed preserves small products. Theorem 4.1.31 says that the localising functor induces hom-set maps $\mathcal{M}(X, Y) \to \operatorname{Ho}\mathcal{M}(\gamma X, \gamma Y)$ that are surjective when X is cofibrant and Y is fibrant; since 1 is cofibrant by hypothesis, it follows that the map $\chi_Y : \Gamma Y \to \tau_0 Y$ is surjective for all cofibrant objects Y. ■

Under stronger hypotheses, the homotopy category of a cartesian model category admits a description à la Hurewicz:

Proposition 5.5.15. *Let $\mathcal{M}$ be a cartesian model category, let $\mathcal{M}_\mathrm{f}$ be the full subcategory of fibrant objects, and let $\operatorname{Ho}\mathcal{M}_\mathrm{f}$ be the localisation of $\mathcal{M}_\mathrm{f}$ at the weak equivalences. If all fibrant objects in $\mathcal{M}$ are cofibrant, then:*

(i) *$\mathcal{M}_\mathrm{f}$ is a cartesian closed category.*

(ii) *The natural transformation $\chi : \Gamma \Rightarrow \tau_0$ induces a functor $\mathcal{M}_\mathrm{f} \to \tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right]$ that is a bijection on objects, full, and preserves small products and exponential objects.*

(iii) *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}_\mathrm{f}$. Then f_0 and f_1 are (right) homotopic if and only if they are sent to the same morphism in $\tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right]$.*

(iv) *The canonical functor $\operatorname{Ho}\mathcal{M}_\mathrm{f} \to \tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right]$ is an isomorphism of categories.*

Proof. (i). Since all fibrant objects are cofibrant, the exponential object $[X, Y]$ is fibrant for all X and Y in $\mathcal{M}_\mathrm{f}$; and by proposition 5.5.14, $\mathcal{M}_\mathrm{f}$ is closed under products of small objects in $\mathcal{M}$, so it follows that $\mathcal{M}_\mathrm{f}$ is a cartesian closed category.

(ii). This is a straightforward consequence of the fact that $\tau_0 : \mathcal{M}_\mathrm{f} \to \mathbf{Set}$ preserves small products, that we have a natural bijection $\Gamma[X, Y] \cong \mathcal{M}(X, Y)$ for all objects X and Y, and that $\chi_Z : \Gamma Z \to \tau_0 Z$ is a surjection for all fibrant objects Z.

(iii). Suppose $f_0, f_1 : X \to Y$ are related by a right homotopy, i.e. there exists a path object (P, i, p_0, p_1) for Y and a morphism $f : X \to P$ such that $p_0 \circ f = f_0$ and $p_1 \circ f = f_1$. Since $p_0, p_1 : P \to Y$ are retractions of the weak equivalence $i : Y \to P$, the two maps $\tau_0[X, P] \to \tau_0[X, Y]$ induced by p_0 and p_1 must be equal. In particular, $\chi_{[X,Y]} : \Gamma[X, Y] \to \tau_0[X, Y]$ must map f_0 and f_1 to the same element.

Conversely, if f_0 and f_1 are sent to the same morphism in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$, then there must exist a cylinder object $\left(J, i_0, i_1, p\right)$ for 1 and a morphism $h : J \to [X, Y]$ such that $h \circ i_0$ (resp. $h \circ i_1$) is the exponential transpose of f_0 (resp. f_1). Taking exponential transposes again and using the fact that $[J, Y]$ is a path object for Y, we deduce that f_0 and f_1 are right homotopic.

(iv). The formal Whitehead theorem implies that weak equivalences in $\mathcal{M}_{\mathrm{f}}$ are mapped to isomorphisms in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$, so the functor $\mathcal{M} \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$ induces a functor $\operatorname{Ho}\mathcal{M}_{\mathrm{f}} \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$. A standard argument then shows that it is an isomorphism: see e.g. theorem 4.4.1. ■

Proposition 5.5.16. *Let $\mathcal{M}$ be a cartesian model category. If all objects in $\mathcal{M}$ are cofibrant, then:*

(i) *The functors $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ and $\tau_0 : \mathcal{M} \to$ **Set** both preserve finite products.*

(ii) *A morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence if and only if the induced maps*

$$\tau_0[f, Z] : \tau_0[Y, Z] \to \tau_0[X, Z]$$

are bijections for all fibrant objects Z in $\mathcal{M}$.

(iii) *The inclusion $\mathcal{M}_{\mathrm{f}} \hookrightarrow \mathcal{M}$ induces a fully faithful functor $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right] \to \tau_0[\underline{\mathcal{M}}]$ with a left adjoint.*

Proof. (i). It suffices to to show that $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ preserves finite products; that $\tau_0 : \mathcal{M} \to \mathbf{Set}$ preserves finite products will follow automatically. It is not hard to check that $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ preserves terminal objects for all model categories $\mathcal{M}$, and we will now show that γ preserves binary products.

The pushout–product axiom implies that, for all cofibrant objects Y, the functor $- \times Y : \mathcal{M} \to \mathcal{M}$ is a left Quillen functor. Since we are assuming that all objects are cofibrant, corollary 5.5.4 implies that $- \times Y$ preserves weak equivalences. We may then deduce that $- \times - : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves all weak equivalences, and hence that it is a homotopical left approximation for itself. Thus, the localising functor $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ indeed preserves binary products.

(ii). If $f : X \to Y$ is a weak equivalence, then $[f, Z] : [Y, Z] \to [X, Z]$ is a weak equivalence for all fibrant objects Z, and hence $\tau_0[f, Z]$ must be a bijection. Conversely, suppose $\tau_0[f, Z]$ is a bijection for all fibrant objects Z. Let $R : \mathcal{M} \to \mathcal{M}$ be a fibrant replacement functor for $\mathcal{M}$. Then, the morphism $Rf : RX \to RY$ also induces bijections $\tau_0[Rf, Z]$ for all fibrant objects Z, and since RX and RY are in $\mathcal{M}_{\mathrm{f}}$, the Yoneda lemma implies that $Rf : RX \to RY$ is sent to an isomorphism in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$, and hence must be a weak equivalence in $\mathcal{M}_{\mathrm{f}}$. The 2-out-of-3 property of weak equivalences then implies $f : X \to Y$ is a weak equivalence in $\mathcal{M}$.

(iii). It is clear that the induced functor $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right] \to \tau_0[\underline{\mathcal{M}}]$ is indeed fully faithful, and it is not hard to check that a fibrant replacement functor provides the required left adjoint $\tau_0[\underline{\mathcal{M}}] \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$. ■

Definition 5.5.17. An **isocofibration** is a functor that is injective on objects. An **isofibration** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that, for every object C in $\mathcal{C}$ and every isomorphism $f : FC \to D$ in $\mathcal{D}$, there exists an isomorphism $\tilde{f} : C \to \tilde{D}$ in $\mathcal{C}$ such that $F\tilde{f} = f$.

Proposition 5.5.18. *Let* **Cat** *be the category of small categories. The following data constitute a model structure on* **Cat***:*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isocofibrations.*
- *The fibrations are the isofibrations.*

Moreover, the factorisations for axiom CM5 may be chosen functorially, so that **Cat** *becomes a DHK model category. This model structure is called the* **canonical model structure** *on* **Cat**.

Proof. It is not hard to show that **Cat** has limits and colimits for all small diagrams, so axiom CM1* is satisfied. It is also clear that the announced class of weak equivalences has the 2-out-of-3 property, so by theorem 4.1.12, it is enough to show that we have a pair of compatible weak factorisation systems.

Let $I : \mathbb{A} \to \mathbb{B}$ be an isocofibration and $P : \mathbb{C} \to \mathbb{D}$ be an isofibration, and suppose we have a commutative diagram of the following form:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "I"'] & \mathbb{C} \arrow[d, "P"] \\ \mathbb{B} \arrow[r, "G"'] & \mathbb{D} \end{tikzcd}$$

First, suppose P is a weak equivalence. Then, P must be surjective on objects, so we may define a map $H : \operatorname{ob} \mathbb{B} \to \operatorname{ob} \mathbb{C}$ by taking $HB = FA$ if $B = IA$ for some A, and if B is not in the image of A, define HB to be any object in $\mathbb{C}$ such that $PHB = GB$; there is then a unique way of extending H to a functor $\mathbb{B} \to \mathbb{C}$ making the evident diagram commute.

Next, instead suppose I is a weak equivalence. Then, I may be regarded as the inclusion of a full subcategory that is essentially surjective on objects. For each object B in $\mathbb{B}$ that is not in the image of I, fix an object A in $\mathbb{A}$ and an isomorphism $IA \xrightarrow{\cong} B$. Since P is an isofibration, for each such B we may also choose an object C in $\mathbb{C}$ and an isomorphism $FA \xrightarrow{\cong} C$ whose image under P is $GIA \xrightarrow{\cong} GB$. There is then a unique functor $H : \mathbb{B} \to \mathbb{C}$ that makes the evident diagram commute and sends B to the chosen C and $IA \xrightarrow{\cong} B$ to $FA \xrightarrow{\cong} C$.

It remains to be shown that every functor can be factorised in the required manner. Let $F : \mathbb{C} \to \mathbb{D}$ be any functor. Consider the iso-comma category $(F \wr \mathbb{D})$:

- The objects are triples (C, D, α), where C is an object in $\mathbb{C}$, D is an object in $\mathbb{D}$, and $\alpha : FC \to D$ is an *isomorphism* in $\mathbb{D}$.
- The morphisms $(C, D, \alpha) \to (C', D', \alpha')$ is a morphism $f : C \to C'$ is in $\mathbb{C}$ together with a morphism $g : D \to D'$ in $\mathbb{D}$ such that $g \circ \alpha = \alpha' \circ Ff$.[4]
- Composition and identities are inherited from $\mathbb{C}$ and $\mathbb{D}$.

There is an evident isocofibration $I : \mathbb{C} \to (F \wr \mathbb{D})$ sending an object C in $\mathbb{C}$ to the object $\left(C, FC, \mathrm{id}_{FC}\right)$, and it is easy to see that I is a weak equivalence. On the other hand, the projection $P : (F \wr \mathbb{D}) \to \mathbb{D}$ is an isofibration by construction, and obviously $F = PI$. Thus, we have factored F as a trivial isocofibration followed by an isofibration, and it is clear that this construction is functorial in F.

Now, consider instead the category $\mathbf{M}(F)$ defined below:

[4] However, because α and α' are isomorphisms, f freely and uniquely determines g.

- $\mathrm{ob}\,\mathbf{M}(F) = \mathrm{ob}\,\mathbb{C} \amalg \mathrm{ob}\,\mathbb{D}$.
- If C and C' are objects in $\mathbb{C}$, while D and D' are objects in $\mathbb{D}$, then:

$$\begin{aligned}
\mathrm{Hom}(C, C') &= \mathbb{D}(FC, FC') \\
\mathrm{Hom}(C, D') &= \mathbb{D}(FC, D') \\
\mathrm{Hom}(D, C') &= \mathbb{D}(D, FC') \\
\mathrm{Hom}(D, D') &= \mathbb{D}(D, D')
\end{aligned}$$

- Composition and identities are inherited from $\mathbb{D}$.

There is an evident isocofibration $I : \mathbb{C} \to \mathbf{M}(F)$ that sends an object C in $\mathbb{C}$ to the corresponding object in $\mathbf{M}(F)$ and sends a morphism $f : C \to C'$ in $\mathbb{C}$ to the morphism in $\mathbf{M}(F)$ corresponding to $Ff : FC \to FC'$ in $\mathbb{D}$. On the other hand, there is an evident projection $P : \mathbf{M}(F) \to \mathbb{D}$ that is fully faithful and surjective on objects, i.e. P is a trivial isofibration. Of course, $F = PI$, so this is a factorisation of F as an isocofibration followed by a trivial isofibration, and it is clear that this construction is functorial in F. ■

Theorem 5.5.19. *Let* **Cat** *be considered as a model category via the canonical model structure.*

(i) *Every object in* **Cat** *is both cofibrant and fibrant.*

(ii) **Cat** *is a combinatorial model category.*

(iii) **Cat** *is a cartesian model category.*

Proof. (i). The unique functor $\varnothing \to \mathbb{C}$ is vacuously an isocofibration, and the unique functor $\mathbb{C} \to \mathbb{1}$ is certainly an isofibration.

(ii). **Cat** is a locally finitely presentable category,[5] and it remains to be shown that the canonical model structure is a cofibrantly generated model structure.

By the very definition of isofibration, the set $\{\{0\} \hookrightarrow \mathbf{I}\mathbb{2}\}$ is a generating set of trivial isocofibrations, where $\mathbf{I}\mathbb{2}$ is the groupoid containing only a pair of non-trivial isomorphisms. It is also straightforward to see that a functor is …

[5] — because e.g. **Cat** is the category of models for a finite limit sketch; see Proposition 1.51 in [LPAC], or Proposition 5.6.4 in [Borceux, 1994b], or theorem 0.5.34.

… surjective on objects if and only if it has the right lifting property with respect to the unique functor $\varnothing \to \mathbb{1}$;

… full if and only if it has the right lifting property with respect to the inclusion $\operatorname{disc} 2 \to \mathbb{2}$; and

… faithful if and only if it has the right lifting property with respect the surjective functor $\mathbb{E} \to \mathbb{2}$, where $\mathbb{E}$ is the category with a parallel pair of non-trivial morphisms.

However, a functor is a trivial isofibration if and only if it is fully faithful and surjective on objects, so $\{\varnothing \to \mathbb{1}, \operatorname{disc} 2 \to \mathbb{2}, \mathbb{E} \to \mathbb{2}\}$ is a set of generating isocofibrations.

(iii). Let $F : \mathbb{C} \to \mathbb{C}'$ and $G : \mathbb{D} \to \mathbb{D}'$ be isocofibrations, and consider the functor $F \mathbin{\Box} F'$ defined by the diagram below:

$$\begin{tikzcd}
\mathbb{C} \times \mathbb{D} \arrow[r, "\mathrm{id}_{\mathbb{C}} \times G"] \arrow[d, "F \times \mathrm{id}_{\mathbb{D}}"'] & \mathbb{C} \times \mathbb{D}' \arrow[d] \arrow[ddr, bend left, "F \times \mathrm{id}_{\mathbb{D}'}"] & \\
\mathbb{C}' \otimes \mathbb{D} \arrow[r] \arrow[drr, bend right, "\mathrm{id}_{\mathbb{C}'} \times G"'] & (\mathbb{C} \times \mathbb{D}') \cup^{\mathbb{C} \times \mathbb{D}} (\mathbb{C}' \times \mathbb{D}) \arrow[dr, dashed, "F \Box G"] & \\
& & \mathbb{C}' \times \mathbb{D}'
\end{tikzcd}$$

The functor $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ has both left and right adjoints, so it is easy to see that $F \mathbin{\Box} G$ is an isocofibration. Moreover, if $F : \mathbb{C} \to \mathbb{C}'$ is a trivial isocofibration, one may directly verify that $F \times \mathrm{id}_{\mathbb{D}} : \mathbb{C} \times \mathbb{D} \to \mathbb{C}' \times \mathbb{D}$ and $F \times \mathrm{id}_{\mathbb{D}'} : \mathbb{C} \times \mathbb{D}' \to \mathbb{C}' \times \mathbb{D}'$ are trivial isocofibrations; but trivial isocofibrations are closed under pushout, so applying the 2-out-of-3 property of weak equivalences, we conclude that $F \mathbin{\Box} G$ is a trivial isocofibration if F is. The symmetrical argument shows that $F \mathbin{\Box} G$ is a trivial isocofibration if G is.

Having shown that **Cat** satisfies the pushout–product axiom, we must now verify that **Cat** is cartesian closed and has a cofibrant unit; but the former is a very well-known fact, and the latter follows from claim (i). ■

Theorem 5.5.20. *Let* **Grpd** *be the category of small groupoids.*

(i) *The following data constitute a model structure on* **Grpd***:*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isocofibrations.*
- *The fibrations are the isofibrations.*

This model structure is called the **canonical model structure** *on* **Grpd**.

(ii) *Every object in* **Grpd** *is both cofibrant and fibrant.*

(iii) **Grpd** *is a combinatorial model category.*

(iv) **Grpd** *is a cartesian model category.*

(v) *The inclusion* und : **Grpd** → **Cat** *preserves* and *reflects weak equivalences, isocofibrations, and isofibrations; moreover, it is both a left Quillen functor and a right Quillen functor.*

Proof. (i). The proof of proposition 5.5.18 goes through for **Grpd** without modifications.

(ii)–(iv). These can be proven in essentially the same way as proposition 5.5.18, though one should note that the generating isocofibrations and generating trivial isocofibrations for **Grpd** are different.

(v). It is clear that und : **Grpd** → **Cat** has the announced preservation and reflection properties. One may check that und has a left adjoint **I** : **Cat** → **Grpd** and a right adjoint iso : **Cat** → **Grpd**, so und is both a left Quillen functor and a right Quillen functor. ■

5.6 Bousfield localisation

Prerequisites. §§4.1, 4.3, 4.8, 5.1, 5.2.

Definition 5.6.1. Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.

- An **$\mathcal{S}$-local object** in $\mathcal{M}$ is a fibrant object X in $\mathcal{M}$ such that, for every morphism $g : Z \to W$ that is in $\mathcal{S}$, the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(g, X) : \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, X) \to \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(Z, X)$$

is an isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$.

- An **$\mathcal{S}$-colocal object** in $\mathcal{M}$ is a cofibrant object W in $\mathcal{M}$ such that, for every morphism $f : X \to Y$ that is in $\mathcal{S}$, the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, f) : \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, X) \to \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, Y)$$

is an isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$.

Lemma 5.6.2. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *If $f : X \to Y$ is a weak equivalence between fibrant objects in $\mathcal{M}$, then X is an $\mathcal{S}$-local object in $\mathcal{M}$ if and only if Y is $\mathcal{S}$-local.*
- *If $g : Z \to W$ is a weak equivalence between cofibrant objects in $\mathcal{M}$, then Z is an $\mathcal{S}$-colocal object in $\mathcal{M}$ if and only if W is $\mathcal{S}$-colocal.*

Proof. Immediate, because $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}$ is a functor $(\mathrm{Ho}\,\mathcal{M})^{\mathrm{op}} \times \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathbf{sSet}$. ■

Proposition 5.6.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be model categories and let*

$$F \dashv G : \mathcal{N} \to \mathcal{M}$$

be a Quillen adjunction.

- *If $\mathcal{S}$ is a class of morphisms in $\mathcal{M}$ and $\mathbf{L}F : \mathrm{Ho}\,\mathcal{M} \to \mathrm{Ho}\,\mathcal{N}$ sends morphisms in (the image of) $\mathcal{S}$ to isomorphisms in $\mathrm{Ho}\,\mathcal{N}$, then $G : \mathcal{N} \to \mathcal{M}$ sends fibrant objects in $\mathcal{N}$ to $\mathcal{S}$-local objects in $\mathcal{M}$.*
- *If $\mathcal{T}$ is a class of morphisms in $\mathcal{N}$ and $\mathbf{R}G : \mathrm{Ho}\,\mathcal{N} \to \mathrm{Ho}\,\mathcal{M}$ sends morphisms in (the image of) $\mathcal{T}$ to isomorphisms in $\mathrm{Ho}\,\mathcal{M}$, then $F : \mathcal{M} \to \mathcal{N}$ sends cofibrant objects in $\mathcal{M}$ to $\mathcal{T}$-colocal objects in $\mathcal{N}$.*

Proof. The two claims are formally dual; we will prove the first version.

By proposition 4.3.4, G sends fibrant objects in $\mathcal{N}$ to fibrant objects in $\mathcal{M}$, and by theorem 4.8.37, we have natural isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$ of the form below:

$$\mathbf{R}\mathrm{Hom}_{\mathcal{N}}((\mathbf{L}F)Z, B) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(Z, (\mathbf{R}G)B)$$

But $(\mathbf{R}G)B$ is isomorphic to GB in $\mathrm{Ho}\,\mathcal{M}$ when B is fibrant (by theorem 4.3.12), so GB is indeed an $\mathcal{S}$-local object in $\mathcal{M}$. ■

Definition 5.6.4. Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.

- An **$\mathcal{S}$-local equivalence** in $\mathcal{M}$ is a morphism $g : Z \to W$ in $\mathcal{M}$ such that the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(g, X) : \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, X) \to \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(Z, X)$$

  is an isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$ for all $\mathcal{S}$-local objects X in $\mathcal{M}$.

- An **$\mathcal{S}$-colocal equivalence** in $\mathcal{M}$ is a morphism $f : X \to Y$ in $\mathcal{M}$ such that the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, f) : \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, X) \to \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(W, Y)$$

  is an isomorphism in $\mathrm{Ho}\,\mathbf{sSet}$ for all $\mathcal{S}$-colocal objects W in $\mathcal{M}$.

Remark 5.6.5. For any given model category $\mathcal{M}$ and any class $\mathcal{S}$ of morphisms in $\mathcal{M}$, the class of $\mathcal{S}$-local equivalences (resp. $\mathcal{S}$-colocal equivalences) is saturated (by lemma 3.1.8). Note that every morphism that is in $\mathcal{S}$ is automatically an $\mathcal{S}$-local equivalence (resp. $\mathcal{S}$-colocal equivalence), as is every weak equivalence in $\mathcal{M}$.

Lemma 5.6.6. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *A morphism between $\mathcal{S}$-local objects in $\mathcal{M}$ is an $\mathcal{S}$-local equivalence if and only if it is a weak equivalence in $\mathcal{M}$.*

- *A morphism between $\mathcal{S}$-colocal objects in $\mathcal{M}$ is an $\mathcal{S}$-colocal equivalence if and only if it is a weak equivalence in $\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that every weak equivalence in $\mathcal{M}$ is an $\mathcal{S}$-local equivalence. Conversely, suppose $g : Z \to W$ is an $\mathcal{S}$-local equivalence between $\mathcal{S}$-local objects in $\mathcal{M}$. Corollary 4.8.15 then implies that $g : Z \to W$ induces bijections

$$\mathrm{Ho}\,\mathcal{M}(g, X) : \mathrm{Ho}\,\mathcal{M}(W, X) \to \mathrm{Ho}\,\mathcal{M}(Z, X)$$

for all $\mathcal{S}$-local objects X, so $g : Z \to W$ must be an isomorphism in $\mathrm{Ho}\,\mathcal{M}$. Since $\mathcal{M}$ is a saturated homotopical category (by theorem 4.4.1), we may then deduce that $g : Z \to W$ is a weak equivalence in $\mathcal{M}$. ■

Definition 5.6.7. Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$ and let X be an object in $\mathcal{M}$.

- An **$\mathcal{S}$-local replacement** for X is a pair $(\hat{X}, i)$ where $\hat{X}$ is an $\mathcal{S}$-local object in $\mathcal{M}$ and $i : X \to \hat{X}$ is an $\mathcal{S}$-local equivalence.
- An **$\mathcal{S}$-colocal replacement** for X in $\mathcal{M}$ is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is an $\mathcal{S}$-colocal object in $\mathcal{M}$ and $p : \tilde{X} \to X$ is an $\mathcal{S}$-colocal equivalence.
- A **cofibrant $\mathcal{S}$-local replacement** for X in $\mathcal{M}$ is an $\mathcal{S}$-local replacement $(\hat{X}, i)$ where $i : X \to \hat{X}$ is cofibration in $\mathcal{M}$ (and also an $\mathcal{S}$-local equivalence).
- A **fibrant $\mathcal{S}$-colocal replacement** for X in $\mathcal{M}$ is an $\mathcal{S}$-colocal replacement $(\tilde{X}, p)$ where $p : \tilde{X} \to X$ is a fibration in $\mathcal{M}$ (and also an $\mathcal{S}$-local equivalence).

Proposition 5.6.8. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *If every cofibrant–fibrant object in $\mathcal{M}$ admits an $\mathcal{S}$-local replacement, then the full subcategory of* $\operatorname{Ho}\mathcal{M}$ *spanned by the $\mathcal{S}$-local objects is a reflective subcategory of* $\operatorname{Ho}\mathcal{M}$.
- *If every cofibrant–fibrant object in $\mathcal{M}$ admits an $\mathcal{S}$-colocal replacement, then the full subcategory of* $\operatorname{Ho}\mathcal{M}$ *spanned by the $\mathcal{S}$-colocal objects is a coreflective subcategory of* $\operatorname{Ho}\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

Let $\mathcal{L}$ be the full subcategory of $\operatorname{Ho}\mathcal{M}$ spanned by the $\mathcal{S}$-local objects. If $g : Z \to W$ is an $\mathcal{S}$-local equivalence in $\mathcal{M}$, then corollary 4.8.15 implies that $g : Z \to W$ induces bijections

$$\operatorname{Ho}\mathcal{M}(g, X) : \operatorname{Ho}\mathcal{M}(W, X) \to \operatorname{Ho}\mathcal{M}(Z, X)$$

for all $\mathcal{S}$-local objects X. Thus, for any $\mathcal{S}$-local replacement $(\hat{Z}, i)$, we have natural bijections

$$\operatorname{Ho}\mathcal{M}(\hat{Z}, X) \cong \operatorname{Ho}\mathcal{M}(Z, X)$$

where X varies among the $\mathcal{S}$-local objects in $\operatorname{Ho}\mathcal{M}$. Proposition 4.1.24 implies every object in $\mathcal{M}$ is weakly equivalent to a cofibrant–fibrant object, so this implies that the inclusion $\mathcal{L} \hookrightarrow \operatorname{Ho}\mathcal{M}$ has a left adjoint, as required. ■

Lemma 5.6.9. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *If $\mathcal{M}$ is a left proper model category, then every $\mathcal{S}$-local object is injective with respect to the class of cofibrations in $\mathcal{M}$ that are also $\mathcal{S}$-local equivalences.*

- *If $\mathcal{M}$ is a right proper model category, then every $\mathcal{S}$-colocal object is projective with respect to the class of fibrations in $\mathcal{M}$ that are also $\mathcal{S}$-colocal equivalences.*

Proof. The two claims are formally dual; we will prove the first version.

Let $i : Z \to W$ be a morphism in $\mathcal{M}$ that is a cofibration and also an $\mathcal{S}$-local equivalence. Using axiom CM5 and theorem 4.6.15, we may choose cosimplicial resolutions $(\tilde{Z}^\bullet, p_Z^\bullet)$ and $(\tilde{W}^\bullet, p_W^\bullet)$ for Z and W (respectively) and a Reedy cofibration $\tilde{\imath}^\bullet : \tilde{Z}^\bullet \to \tilde{W}^\bullet$ such that the following diagram in $\mathcal{M}$ commutes:

$$\begin{tikzcd} \tilde{Z}^0 \arrow[r, "p_Z^0"] \arrow[d, "\tilde{\imath}^0"'] & Z \arrow[d, "i"] \\ \tilde{W}^0 \arrow[r, "p_W^0"'] & W \end{tikzcd}$$

Let X be an $\mathcal{S}$-local object. We wish to show that the unique morphism $X \to 1$ has the right lifting property with respect to $i : Z \to W$. Since $i : Z \to W$ is an $\mathcal{S}$-local equivalence, the induced morphism of left homotopy function complexes

$$\mathcal{H}om_{\mathcal{M}}(\tilde{\imath}, X) : \mathcal{H}om_{\mathcal{M}}(\tilde{W}, X) \to \mathcal{H}om_{\mathcal{M}}(\tilde{Z}, X)$$

is a homotopy equivalence of Kan complexes; but $\tilde{\imath}^\bullet : \tilde{Z}^\bullet \to \tilde{W}^\bullet$ is a Reedy cofibration between cosimplicial resolutions in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, so lemma 4.8.38 says the unique morphism $X \to 1$ has the right lifting property with respect to $\tilde{\imath}^0 : \tilde{Z}^0 \to \tilde{W}^0$.

Now, suppose the square in the diagram below is a pushout diagram in $\mathcal{M}$:

$$\begin{tikzcd} \tilde{Z}^0 \arrow[r, "p_Z^0"] \arrow[d, "\tilde{\imath}^0"'] & Z \arrow[d, "i'"] \arrow[ddr, bend left, "i"] & \\ \tilde{W}^0 \arrow[r] \arrow[drr, bend right, "p_W^0"'] & W' \arrow[dr, dashed, "g"] & \\ & & W \end{tikzcd}$$

Since $\mathcal{M}$ is a left proper model category, axiom CM2 implies $g : W' \to W$ is a weak equivalence in $\mathcal{M}$. Proposition A.3.17 says the unique morphism $X \to 1$ also has the right lifting property with respect to $i' : Z \to W'$, so we may use lemma 4.1.20 to deduce that $X \to 1$ has the right lifting property with respect to $i : Z \to W$. ∎

Definition 5.6.10. Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.

- The **left Bousfield localisation** of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ whose underlying category and cofibrations are the same as $\mathcal{M}$ and whose weak equivalences are the $\mathcal{S}$-local equivalences.

- The **right Bousfield localisation** of $\mathcal{M}$ with respect to $\mathcal{S}$ is a model category $\mathbf{R}_{\mathcal{S}}\mathcal{M}$ whose underlying category and fibrations are the same as $\mathcal{M}$ and whose weak equivalences are the $\mathcal{S}$-colocal equivalences.

Remark 5.6.11. The left (resp. right) Bousfield localisation of $\mathcal{M}$ with respect to $\mathcal{S}$ is unique *if it exists*, by theorem 4.1.12. Note that the theorem also implies that the trivial fibrations in $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ (resp. trivial cofibrations in $\mathbf{R}_{\mathcal{S}}\mathcal{M}$) are the same as the trivial fibrations (resp. trivial cofibrations) in $\mathcal{M}$.

Proposition 5.6.12. *Let $\mathcal{M}$ and $\mathcal{M}'$ be model categories with the same underlying category.*

- *If $\mathcal{M}$ and $\mathcal{M}'$ have the same cofibrations and* $\operatorname{weq}\mathcal{M} \subseteq \operatorname{weq}\mathcal{M}'$, *then the model structure on $\mathcal{M}'$ is the left Bousfield localisation of the model structure of $\mathcal{M}$ with respect to* $\operatorname{weq}\mathcal{M}'$.

- *If $\mathcal{M}$ and $\mathcal{M}'$ have the same fibrations and* $\operatorname{weq}\mathcal{M} \subseteq \operatorname{weq}\mathcal{M}'$, *then the model structure on $\mathcal{M}'$ is the right Bousfield localisation of the model structure of $\mathcal{M}$ with respect to* $\operatorname{weq}\mathcal{M}'$.

Proof. The two claims are formally dual; we will prove the first version.

Let $\mathcal{S} = \operatorname{weq}\mathcal{M}'$. It suffices to prove that every $\mathcal{S}$-local equivalence is a weak equivalence in $\mathcal{M}'$. The hypotheses (plus proposition 4.3.2) imply that the trivial adjunction

$$\mathrm{id} \dashv \mathrm{id} : \mathcal{M}' \to \mathcal{M}$$

is a Quillen adjunction. Let Y be a fibrant object in $\mathcal{M}'$ and let $\hat{Y}_\bullet$ be a simplicial resolution of Y in $\mathcal{M}'$. Then Y is a fibrant object in $\mathcal{M}$ and $\hat{Y}_\bullet$ is a simplicial resolution of Y in $\mathcal{M}$, by (proposition 4.3.4 and) lemma 4.8.36. On the other hand, given an object X in $\mathcal{M}$, any fibrant cofibrant replacement for X in $\mathcal{M}$ is also a fibrant cofibrant replacement for X in $\mathcal{M}'$. Thus, if $(\tilde{X}, p)$ is a cofibrant replacement for X, then the right hom-complex $\mathcal{H}om_{\mathcal{M}}(\tilde{X}, \hat{Y})$ computes both $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(X, Y)$ and $\mathbf{R}\mathrm{Hom}_{\mathcal{M}'}(X, Y)$. Hence, by proposition 4.8.22, every fibrant object in $\mathcal{M}'$ is an $\mathcal{S}$-local object in $\mathcal{M}$, and every $\mathcal{S}$-local equivalence in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}'$. ■

Proposition 5.6.13. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *If the left Bousfield localisation $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ exists, then the trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : \mathbf{L}_{\mathcal{S}}\mathcal{M} \to \mathcal{M}$$

  is a Quillen adjunction, and the right derived functor $\mathrm{Ho}\,\mathbf{L}_{\mathcal{S}}\mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ *is fully faithful.*

- *If the right Bousfield localisation $\mathbf{R}_{\mathcal{S}}\mathcal{M}$ exists, then the trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : \mathcal{M} \to \mathbf{R}_{\mathcal{S}}\mathcal{M}$$

  is a Quillen adjunction, and the left derived functor $\mathrm{Ho}\,\mathbf{R}_{\mathcal{S}}\mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ *is fully faithful.*

Proof. The two claims are formally dual; we will prove the first version.

By definition, $\mathrm{id} : \mathcal{M} \to \mathbf{L}_{\mathcal{S}}\mathcal{M}$ preserves cofibrations, and since every weak equivalence in $\mathcal{M}$ is an $\mathcal{S}$-local equivalence, $\mathrm{id} : \mathcal{M} \to \mathbf{L}_{\mathcal{S}}\mathcal{M}$ also preserves trivial cofibrations. Proposition 4.3.2 then says the trivial adjunction is indeed a Quillen adjunction. We then use theorem 4.3.12 and proposition 3.3.28 to deduce that the derived counit is an isomorphism; thus, by proposition A.1.3, the right derived functor is fully faithful. ■

Proposition 5.6.14. *Let $\mathcal{S}$ be a class of morphisms in a model category $\mathcal{M}$.*

- *If the left Bousfield localisation $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ exists, then fibrant objects in $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ are $\mathcal{S}$-local objects in $\mathcal{M}$; in addition, if $\mathcal{M}$ is left proper, then $\mathcal{S}$-local objects in $\mathcal{M}$ are fibrant objects in $\mathbf{L}_{\mathcal{S}}\mathcal{M}$.*

- *If the right Bousfield localisation $\mathbf{R}_{\mathcal{S}}\mathcal{M}$ exists, then cofibrant objects in $\mathbf{R}_{\mathcal{S}}\mathcal{M}$ are $\mathcal{S}$-colocal objects in $\mathcal{M}$; in addition, if $\mathcal{M}$ is right proper, then $\mathcal{S}$-colocal objects in $\mathcal{M}$ are cofibrant objects in $\mathbf{R}_{\mathcal{S}}\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

Consider the Quillen adjunction of proposition 5.6.13:

$$\mathrm{id} \dashv \mathrm{id} : \mathbf{L}_{\mathcal{S}}\mathcal{M} \to \mathcal{M}$$

Since every morphism that is in $\mathcal{S}$ is an $\mathcal{S}$-local equivalence, proposition 5.6.3 says every fibrant object in $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ is an $\mathcal{S}$-local object in $\mathcal{M}$. Conversely, if $\mathcal{M}$ is left proper, then lemma 5.6.9 says every $\mathcal{S}$-local object in $\mathcal{M}$ is a fibrant object in $\mathbf{L}_{\mathcal{S}}\mathcal{M}$. ■

Theorem 5.6.15 (Existence of left Bousfield localisations)**.** *Let $\mathcal{M}$ be a left proper combinatorial model category and let $\mathcal{S}$ be a set of morphisms in $\mathcal{M}$.*

(i) *The left Bousfield localisation $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ exists.*

(ii) *$\mathbf{L}_{\mathcal{S}}\mathcal{M}$ is a left proper combinatorial model category.*

(iii) *If $\mathcal{M}$ is the underlying model category of a simplicial model category $\underline{\mathcal{M}}$,*[6] *then the left Bousfield localisation $\mathbf{L}_{\mathcal{S}}\mathcal{M}$ is also a simplicial model structure on $\underline{\mathcal{M}}$.*

Proof. This theorem is due to Smith. See Theorem 2.11 and Theorem 3.18 in [Barwick, 2007b], or Theorem 4.7 and Theorem 4.46 in [Barwick, 2010]. (By remark 2.4.13, the weak equivalences in the left Bousfield localisation of a *simplicial* model category $\underline{\mathcal{M}}$ with respect to $\mathcal{S}$ are precisely the $\mathcal{S}$-local equivalences in $\mathcal{M}$.) □

[6] See § 2.4.

— VI —

Quasicategories

Quasicategories were first defined by Boardman and Vogt [BV] as simplicial classes that satisfy the “restricted Kan condition”. The modern name is due to Joyal [2002], who worked out much of the basic theory.

As the word itself suggests, a quasicategory is a structure that is like a category. More precisely, it is a model for an $(\infty, 1)$-category, i.e. a weak higher category with n-morphisms for all $n \geq 0$, such that every n-morphism with $n > 1$ is (weakly) invertible; alternatively, one may think of quasicategories as being homotopy-coherent categories, i.e. a structure which is like a category but only up to a specified, coherent system of homotopies.

6.1 Basics

Prerequisites. §§ 0.1, 1.1, 1.2, A.2.

In this section we use the explicit universe convention.

Definition 6.1.1. An **inner horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where $n \geq 2$ and $0 < k < n$, where Λ^n_k is the union of the faces of Δ^n that include the k-th vertex. (See also definition 1.3.24.)

Definition 6.1.2. A **quasicategory** is a simplicial set X such that the unique morphism $X \to 1$ has the right lifting property with respect to all inner horn inclusions.

¶ 6.1.3. Quasicategories are also called **∞-categories** (by e.g. Lurie [HTT]) or **weak Kan complexes** (by e.g. Cordier and Porter [1986]). We will usually use *bold upright* calligraphic letters to denote quasicategories, e.g. $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \ldots$. As with 'category', the word 'quasicategory' always means a quasicategory that is not necessarily small, even when we are using the implicit universe convention.

Proposition 6.1.4. *Let C be a category and let* $\mathrm{N}(C)$ *be its nerve. For $n \ge 2$ and $0 < k < n$, the unique morphism* $\mathrm{N}(C) \to 1$ *is* right orthogonal *to the inner horn inclusion* $\Lambda^n_k \hookrightarrow \Delta^n$*; in particular,* $\mathrm{N}(C)$ *is a quasicategory.*

Proof. This is a straightforward exercise using induction on n. ◊

¶ 6.1.5. We will often refer to vertices of a quasicategory as **objects**, and edges as **morphisms**. The **domain** of a morphism f in a quasicategory is the object $d_1(f)$, and the **codomain** of f is the object $d_0(f)$. An **identity morphism** is a degenerate edge; we define $f : x \to y$ to mean that x is the domain of f and y is the codomain of f. The **identity morphism** of an object x in a quasicategory is the degenerate edge $s_0(x)$, which we also denote by id_x. Note that all this terminology is compatible with the identification of categories C with their nerves $\mathrm{N}(C)$.

It is not hard to check that a simplicial set X is a quasicategory if and only if the simplicial set X^{op} is a quasicategory.[1] Thus, we may make the following definition:

Definition 6.1.6. The **opposite** of a quasicategory $\mathcal{C}$ is the simplicial set $\mathcal{C}^{\mathrm{op}}$ (regarded as a quasicategory).

Definition 6.1.7. Let f_0 and f_1 be a parallel pair of morphisms in a quasicategory.

- We say f_0 and f_1 are **left homotopic** if there exists a 2-simplex α such that $d_1(\alpha) = f_0$, $d_0(\alpha) = f_1$, and $d_2(\alpha) = s_0\big(d_0(f_0)\big)$.
- We say f_0 and f_1 are **right homotopic** if there exists a 2-simplex α such that $d_2(\alpha) = f_0$, $d_1(\alpha) = f_1$, and $d_0(\alpha) = s_0\big(d_0(f_0)\big)$.
- We say f_0 and f_1 are **homotopic** if they are both left and right homotopic, and we write $f_0 \sim f_1$ in this case.

Obviously, two edges are left homotopic in a quasicategory $\mathcal{C}$ if and only if they are right homotopic in $\mathcal{C}^{\mathrm{op}}$. In fact:

[1] Recall definition 1.1.8.

Lemma 6.1.8. *Let f_0 and f_1 be a parallel pair of morphisms in a quasicategory $\mathcal{C}$. The following are equivalent:*

(i) *f_0 and f_1 are left homotopic.*

(ii) *f_0 and f_1 are right homotopic.*

(iii) *f_0 and f_1 are homotopic.*

Proof. (i) ⇔ (ii). By duality, it suffices to show that (i) ⇒ (ii). Let α be a 2-simplex of $\mathcal{C}$ such that $d_1(\alpha) = f$, $d_0(\alpha) = f'$, and $d_2(\alpha) = s_0\big(d_0(f)\big)$. Using the right lifting property of $\mathcal{C} \to 1$ with respect to the inner horn inclusion $\Lambda^3_1 \to \mathcal{C}$, it is straightforward to obtain a 3-simplex ξ such that $d_2(\xi) = \alpha$, $d_3(\xi) = s_0\big(f_1\big)$, and $d_0(\xi) = s_1\big(f_1\big)$; thus the 2-simplex $d_1(\xi)$ is the required witness for the claim that f_0 and f_1 are right homotopic.

(i) and (ii) ⇔ (iii). This is by definition. ■

Lemma 6.1.9. *Let $\mathcal{C}$ be a quasicategory. The relation of homotopy is an equivalence relation on the set of edges of $\mathcal{C}$.*

Proof. See Proposition 1.2.3.5 in [HTT], or Lemma 4.11 in [BV]. □

Definition 6.1.10. The **homotopy category** of a quasicategory $\mathcal{C}$ is the category $\operatorname{Ho}\mathcal{C}$ defined below:

- The objects are the objects in $\mathcal{C}$.
- A morphism $x \to y$ is a homotopy class of morphisms $f : x \to y$ in $\mathcal{C}$.
- The identity morphism $x \to x$ is the homotopy class of the morphism id_x.
- Composition is induced by the existence of fillers for the inner horn Λ^2_1: if α is a 2-simplex of $\mathcal{C}$, then we have $d_0(\alpha) \circ d_2(\alpha) = d_1(\alpha)$.

Lemma 6.1.11. *The above construction is indeed a category.*

Proof. See Proposition 1.2.3.8 in [HTT]. □

Definition 6.1.12. Let $\mathbf{U}$ be a universe. A **$\mathbf{U}$-small quasicategory** is a quasicategory whose underlying simplicial set is $\mathbf{U}$-small.

Proposition 6.1.13. *Let* **U** *be a universe, let* **sSet** *be the category of simplicial* **U***-sets, and let* **Cat** *be the category of* **U***-small categories.*

(i) *The functor* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *that sends a* **U***-small category* $\mathbb{C}$ *to its nerve* $\mathrm{N}(\mathbb{C})$ *has a left adjoint* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *that sends a simplicial* **U***-set* X *to its fundamental category* $\tau_1 X$.

(ii) *The functor* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *preserves finite products.*

(iii) *For each quasicategory* $\mathfrak{C}$*, there is a canonical isomorphism* $\tau_1\mathfrak{C} \cong \mathrm{Ho}\,\mathfrak{C}$.

Proof. Claims (i) and (ii) were previously proven in proposition 1.2.1, and claim (iii) is essentially a consequence of the fact that $\tau_1 X$ can be presented explicitly in terms of generators and relations as in remark 1.2.3. □

¶ 6.1.14. Henceforth, we will regard all ordinary categories as quasicategories by implicitly identifying a category C with its nerve $\mathrm{N}(C)$. Continuing the terminological conventions in paragraph 6.1.5, we now define functors and natural transformations in the context of quasicategories.

Definition 6.1.15. A **functor** between quasicategories is a morphism of simplicial sets whose domain and codomain are quasicategories.

Recall that theorem A.2.22 implies that the category of simplicial **U**-sets is cartesian closed for all universes **U**. For brevity, we will identify morphisms $X \to Y$ with vertices of the exponential object $[X, Y]$; thus, a functor $\mathfrak{C} \to \mathfrak{D}$ will be both a morphism between simplicial sets and an vertex in $[\mathfrak{C}, \mathfrak{D}]$.

Definition 6.1.16. Let $f_0, f_1 : \mathfrak{C} \to \mathfrak{D}$ be functors between quasicategories.

- A **natural transformation** $\alpha : f_0 \Rightarrow f_1$ is an edge $\alpha : f_0 \to f_1$ in the exponential object $[\mathfrak{C}, \mathfrak{D}]$.
- Two natural transformations are **homotopic** if they are *isomorphic* in the fundamental category $\tau_1[\mathfrak{C}, \mathfrak{D}]$.

Remark 6.1.17. It is a fact that the exponential object $[X, Y]$ is a quasicategory when Y is quasicategory: see corollary 6.2.14. Thus the fundamental category $\tau_1[\mathfrak{C}, \mathfrak{D}]$ can be computed using the homotopy category construction.

Definition 6.1.18. Let $\mathcal{C}$ be a quasicategory. An **equivalence** in $\mathcal{C}$ is a morphism f whose homotopy class is invertible in $\operatorname{Ho}\mathcal{C}$, and a **quasi-inverse** for f is a morphism in $\mathcal{C}$ whose homotopy class is an inverse for (the homotopy class of) f in $\operatorname{Ho}\mathcal{C}$.

One of the requirements for a model of the theory of $(\infty, 1)$-categories is that the groupoid-like instances should be models of ∞-groupoids. If by '∞-groupoid' one means a (weak) homotopy type of Kan complexes, then the following result is relevant:

Proposition 6.1.19. *Let $\mathcal{C}$ be a quasicategory. The following are equivalent:*

(i) $\mathcal{C}$ *(as a simplicial set) is a Kan complex.*

(ii) *Every morphism in $\mathcal{C}$ is an equivalence.*

(iii) $\operatorname{Ho}\mathcal{C}$ *is a groupoid.*

Proof. See Corollary 1.5 in [Joyal, 2002]. □

There is also a homotopy-coherent notion of equivalence. Let $\mathbf{I}\mathbb{2}$ be the groupoid obtained by freely inverting the arrows in the category $\mathbb{2}$ freely generated by a morphism $0 \to 1$.

Definition 6.1.20. A **homotopy-coherent equivalence** in a quasicategory $\mathcal{C}$ is a functor $\mathbf{I}\mathbb{2} \to \mathcal{C}$.

Remark 6.1.21. More explicitly, a homotopy-coherent equivalence in $\mathcal{C}$ consists of the following data:

- A pair of objects in $\mathcal{C}$, say x and y.
- A pair of morphisms in $\mathcal{C}$, say $f : x \to y$ and $g : y \to x$.
- A pair of 2-simplices, say α and β, witnessing the fact that $\mathrm{id}_x \sim g \circ f$ and $f \circ g \sim \mathrm{id}_y$.
- A pair of 3-simplices witnessing the fact that α and β satisfy (versions of) the triangle identities for adjunctions.
- etc.

That is, for each natural number n, we have a pair of $(n+1)$-simplices witnessing a coherence axiom for the given pair of n-simplices. Note that the data for $n \leq 2$ already determine a mutually quasi-inverse pair of equivalences in $\mathfrak{C}$; we will refer to $f : x \to y$ as the **underlying morphism** of the homotopy-coherent equivalence.

When $\mathfrak{C}$ is an ordinary category, the 2-simplices are unique *if* they exist, and given the 2-simplices, the required n-simplices exist and are unique for $n \geq 2$.

In other words, every isomorphism in an ordinary category can be equipped with the structure of a homotopy-coherent equivalence. It turns out the same is true for quasicategories:

Proposition 6.1.22. *Let $\mathfrak{C}$ be a quasicategory. If f is an equivalence in $\mathfrak{C}$, then there is a homotopy-coherent equivalence whose underlying morphism is f.*

Proof. See Corollary 1.6 in [Joyal, 2002], or Theorem 4.14 in [TQA]. □

Definition 6.1.23. Let **U** be a universe. The **homotopy 2-category of U-small quasicategories** is the following 2-category $\mathfrak{Qcat}$:

- The objects are **U**-small quasicategories.
- The category of morphisms $\mathfrak{C} \to \mathfrak{D}$ is the fundamental category $\tau_1[\mathfrak{C}, \mathfrak{D}]$, which we also denote by $\mathbf{QFun}(\mathfrak{C}, \mathfrak{D})$.
- Composition and identity morphisms are induced by τ_1 from the cartesian closed structure of **sSet**.

The construction of the 2-category $\mathfrak{Qcat}$ enables us to apply definitions from formal category theory to the context of quasicategories.

Definition 6.1.24. Let $f_0, f_1 : \mathfrak{C} \to \mathfrak{D}$ be functors between quasicategories. A **natural equivalence** is a natural transformation $\alpha : f_0 \Rightarrow f_1$ whose image in the fundamental category $\tau_1[\mathfrak{C}, \mathfrak{D}]$ is an isomorphism.

As with natural transformations of functors between ordinary categories, natural transformations of functors between quasicategories have components. It is a non-trivial fact that a natural transformation is a natural equivalence if and only if its components are equivalences:

Theorem 6.1.25. *Let $f_0, f_1 : \mathfrak{C} \to \mathfrak{D}$ be functors between quasicategories and let $\alpha : f_0 \Rightarrow f_1$ be a natural transformation. Then α is a natural equivalence if and only if, for every object c in $\mathfrak{C}$, the morphism $\alpha_c : f_0(x) \to f_1(x)$ is an equivalence in $\mathfrak{D}$.*

Proof. See Theorem 5.14 in [TQA], or Lemma 2.3.8 in [Riehl and Verity, 2013a] □

Definition 6.1.26. An **equivalence of quasicategories** is an equivalence in the 2-category $\mathfrak{Qcat}$, i.e. a tuple $(f, g, \eta, \varepsilon)$ where:

- $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{C}$ are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathfrak{C}} \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathfrak{D}}$ are natural equivalences.

We will often abuse notation and say that f is an equivalence of quasicategories, omitting mention of the other data.

Definition 6.1.27. An **adjunction of quasicategories** is an adjunction in the 2-category $\mathfrak{Qcat}$, i.e. a tuple $(f, g, \eta, \varepsilon)$ where:

- $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{C}$ are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathfrak{C}} \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathfrak{D}}$ are natural transformations.
- The triangle identities are satisfied:

$$\begin{aligned} \left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) &= \mathrm{id}_f \qquad && \text{in } \mathbf{QFun}(\mathfrak{C}, \mathfrak{D}) \\ \left(\mathrm{id}_g \circ \varepsilon\right) \bullet \left(\eta \circ \mathrm{id}_g\right) &= \mathrm{id}_g \qquad && \text{in } \mathbf{QFun}(\mathfrak{D}, \mathfrak{C}) \end{aligned}$$

Remark 6.1.28. There also exist homotopy-coherent versions of the above definitions, but it is a theorem of Riehl and Verity [2013b] that every adjunction of quasicategories can be extended to a homotopy-coherent adjunction.

Lemma 6.1.29. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{Cat}$ *be the category of* $\mathbf{U}$*-small categories, and let* $\mathbf{Qcat}$ *be the category of* $\mathbf{U}$*-small quasicategories. The functor* $\mathrm{Ho} : \mathbf{Qcat} \to \mathbf{Cat}$ *is isomorphic to (the underlying functor of) a representable 2-functor* $\mathfrak{Qcat} \to \mathfrak{Cat}$.

Proof. This is an immediate consequence of the natural isomorphism $[\mathbb{1}, -] \cong \mathrm{id}_{\mathbf{Qcat}}$ and the fact that $\mathbf{QFun}(\mathbb{1}, -)$ is a 2-functor $\mathfrak{Qcat} \to \mathfrak{Cat}$. ■

6.2 The Joyal model structure

Prerequisites. §§0.5, 1.4, 1.5, 4.1, 5.5, 6.1, A.2, A.4.

Just as there is a model structure on **Cat** whose homotopy category is the category of small categories modulo natural isomorphism of functors, there is a model structure on **sSet**, due to Joyal [TQ1], whose homotopy category is the category of small quasicategories modulo natural equivalence of functors.

¶ 6.2.1. Throughout this section, τ_0 denotes the functor $\mathbf{sSet} \to \mathbf{Set}$ that sends a simplicial set X to the set of isomorphism classes of objects in the fundamental category $\tau_1 X$. Note that it can be factored as $\pi_0 \circ \mathrm{iso} \circ \tau_1$, where $\mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$ is the *right* adjoint of the inclusion $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

Definition 6.2.2. A **weak categorical equivalence** is a morphism $f : Z \to W$ of simplicial sets such that the induced map

$$\tau_0[f, \mathcal{K}] : \tau_0[W, \mathcal{K}] \to \tau_0[Z, \mathcal{K}]$$

is a bijection for all small quasicategories $\mathcal{K}$.

Lemma 6.2.3. *Every weak categorical equivalence is also a weak homotopy equivalence.*

Proof. Let $f : Z \to W$ be a morphism in **sSet**. Every Kan complex is a quasicategory, so if the map

$$\tau_0[f, \mathcal{K}] : \tau_0[W, \mathcal{K}] \to \tau_0[W, \mathcal{K}]$$

is a bijection for all small quasicategories $\mathcal{K}$, then

$$\pi_0[f, K] : \pi_0[W, K] \to \pi_0[Z, K]$$

is a bijection for all Kan complexes K: indeed, corollary 1.4.16 says $[W, K]$ and $[Z, K]$ are Kan complexes if K is a Kan complex, and lemma 1.4.4 implies that π_0 and τ_0 are naturally isomorphic for Kan complexes. ∎

Lemma 6.2.4. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small quasicategories. The following are equivalent:*

(i) *f is (part of) an equivalence of quasicategories.*

(ii) *f is a weak categorical equivalence.*

(iii) *For all small quasicategories $\mathcal{K}$, the induced map*

$$\tau_0[\mathcal{K}, f] : \tau_0[\mathcal{K}, \mathcal{C}] \to \tau_0[\mathcal{K}, \mathcal{D}]$$

is a bijection.

Proof. (i) ⇒ (ii). It is not hard to see that $f : \mathcal{C} \to \mathcal{D}$ is (part of) an equivalence of quasicategories if and only if the induced functor

$$\tau_1[f, \mathcal{K}] : \tau_1[\mathcal{D}, \mathcal{K}] \to \tau_1[\mathcal{C}, \mathcal{K}]$$

is (part of) an equivalence of categories for all small quasicategories $\mathcal{K}$. The functor $\pi_0 \circ \mathrm{iso} : \mathbf{Cat} \to \mathbf{Set}$ sends equivalences to bijections, so we may deduce that $f : \mathcal{C} \to \mathcal{D}$ is a weak categorical equivalence.

(i) ⇒ (iii). The proof is similar to that of (i) ⇒ (ii).

(ii) ⇒ (i), (iii) ⇒ (i). These are straightforward exercises in chasing identity morphisms. ■

Proposition 6.2.5. **sSet** *with the class of weak categorical equivalences constitute a saturated relative category; in particular, the class of weak categorical equivalences has the 2-out-of-6 property.*

Proof. The collection of functors $\tau_0[-, \mathcal{K}] : \mathbf{sSet} \to \mathbf{Set}$, as $\mathcal{K}$ varies over the small quasicategories, *jointly* reflect isomorphisms as weak categorical equivalences, so the class of weak categorical equivalences must be saturated. For the 2-out-of-6 property, see corollary A.4.15. ■

Definition 6.2.6. An **inner fibration** of simplicial sets is a morphism $f : X \to Y$ with the right lifting property with respect to the inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ for all $n \geq 2$ and $0 < i < n$.

Remark 6.2.7. It is clear that a simplicial set X is a quasicategory if and only if the unique morphism $X \to 1$ is an inner fibration. Unfortunately, these are *not* the fibrations in the Joyal model structure.

Definition 6.2.8. An **inner anodyne extension of simplicial sets** is a member of the smallest class $\mathcal{A} \subset \mathbf{sSet}$ satisfying the following conditions:

- Every inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is in $\mathcal{A}$.

- $\mathcal{A}$ is closed under pushouts.
- $\mathcal{A}$ is closed under (finite and) transfinite composition.
- $\mathcal{A}$ is closed under retracts.

Proposition 6.2.9. *Let $\mathcal{I}'$ be the set of inner horn inclusions.*

(i) *The inner anodyne extensions are precisely the $\mathcal{I}'$-cofibrations, i.e. the morphisms in* **sSet** *that have the left lifting property with respect to all inner fibrations.*

(ii) *Every inner anodyne extension is a retract of a relative $\mathcal{I}'$-cell complex.*

(iii) *Every inner anodyne extension is a monomorphism in* **sSet** *and bijective on vertices.*

Proof. (i) and (ii). Proposition A.3.17 implies that every inner anodyne extension has the left lifting property with respect to all inner fibrations; for the converse, see corollary 0.5.13.

(iii). The functor $(-)_0 : \mathbf{sSet} \to \mathbf{Set}$ preserves colimits, so the class of morphisms that are bijective on vertices is closed under pushouts, transfinite composition, and retracts. Similarly, the class of monomorphisms in **sSet** is closed under the same operations. It is clear that the inner horn inclusions are monomorphisms that are bijective on vertices, so we deduce the same is true for inner anodyne extensions. ■

Remark 6.2.10. The proposition above implies that the class of monomorphisms that are weak categorical equivalences *strictly* contains the class of inner anodyne extensions: indeed, the inclusion $\{0\} \hookrightarrow \mathbf{I}\mathbb{2}$ is both a monomorphism and a categorical equivalence but *not* bijective on vertices.

Proposition 6.2.11. *There exist an $\aleph_0$-accessible functor $M : [\mathbb{2}, \mathbf{sSet}] \to \mathbf{sSet}$ and two natural transformations $i : \mathrm{dom} \Rightarrow M$ and $p : M \Rightarrow \mathrm{codom}$ such that, for all objects f in $[\mathbb{2}, \mathbf{sSet}]$:*

- $f = p_f \circ i_f$.
- *i_f is a relative $\mathcal{I}'$-cell complex, where $\mathcal{I}'$ is the set of inner horn inclusions.*
- *p_f is an inner fibration of simplicial sets.*

Proof. Using proposition 0.2.46, it is not hard to see that the inner horn inclusions are $\aleph_0$-compact as objects in $[\mathbb{2}, \mathbf{sSet}]$. We then apply corollary 0.5.14. ■

Corollary 6.2.12. *There exist an $\aleph_0$-accessible functor $R : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ such that, for all objects X in $\mathbf{sSet}$:*

- *RX is a small quasicategory.*
- *$i_X : X \to RX$ is an inner anodyne extension.* ■

Theorem 6.2.13. *Let $i : Z \to W$ be a monomorphism in $\mathbf{sSet}$ and let $p : X \to Y$ be an inner fibration. Suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[i,X]}"] \arrow[ddr, bend right, "{[W,p]}"'] & & \\
& L(i,p) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,p]}"] \\
& {[W,Y]} \arrow[r, "{[i,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *The unique morphism $q : [W, X] \to L(i, p)$ making the diagram commute is an inner fibration.*

(ii) *If $i : Z \to W$ is an inner anodyne extension, then $q : [W, X] \to L(i, p)$ is a trivial Kan fibration.*

(iii) *If $p : Z \to W$ is a trivial Kan fibration, then so is $q : [W, X] \to L(i, p)$.*

Proof. (i) and (ii). See Theorem 2.18 in [TQA], or Propositions 2.5 and 2.6 in [Dugger and Spivak, 2011a].

(iii). This is a special case of proposition 1.4.15. □

Corollary 6.2.14.

(i) *If $p : X \to Y$ is an inner fibration, then for all simplicial sets W, the morphism $[W, p] : [W, X] \to [W, Y]$ is also an inner fibration.*

(ii) *If $i : Z \to W$ is a monomorphism (resp. inner anodyne extension) and $\mathcal{K}$ is a small quasicategory, then the morphism $[i, \mathcal{K}] : [W, \mathcal{K}] \to [Z, \mathcal{K}]$ is an inner fibration (resp. trivial Kan fibration).*

(iii) *If W is any simplicial set and $\mathcal{K}$ is a small quasicategory, then $[W,\mathcal{K}]$ is also a small quasicategory.*

Proof. The proof is similar to that of corollary 1.4.16. ∎

Corollary 6.2.15. **Qcat** *is an exponential ideal of* **sSet***; in particular,* **Qcat** *is a cartesian closed category.* ∎

Proposition 6.2.16. *Let $f : W \to Z$ be a morphism in* **sSet**. *The following are equivalent:*

(i) *For all small quasicategories $\mathcal{K}$, the induced functor*
$$[f,\mathcal{K}] : [Z,\mathcal{K}] \to [W,\mathcal{K}]$$
is (part of) an equivalence of quasicategories.

(ii) *For all small quasicategories $\mathcal{K}$, the induced functor*
$$\operatorname{Ho}[f,\mathcal{K}] : \operatorname{Ho}[Z,\mathcal{K}] \to \operatorname{Ho}[W,\mathcal{K}]$$
is (part of) an equivalence of categories.

(iii) *The morphism $f : W \to Z$ is a weak categorical equivalence.*

Proof. (i) ⇒ (ii). This is a corollary of lemma 6.1.29.

(ii) ⇒ (iii). Any equivalence of categories must induce a bijection on isomorphism classes of objects.

(iii) ⇒ (i). Suppose $f : W \to Z$ is a weak categorical equivalence, i.e. that the induced map
$$\tau_0[f,\mathcal{K}] : \tau_0[Z,\mathcal{K}] \to \tau_0[W,\mathcal{K}]$$
is a bijection of sets for all small quasicategories $\mathcal{K}$. Then, for all simplicial sets X and all small quasicategories $\mathcal{K}$, the induced map
$$\tau_0[f,[X,\mathcal{K}]] : \tau_0[Z,[X,\mathcal{K}]] \to \tau_0[W,[X,\mathcal{K}]]$$
is a bijection, because $[X,\mathcal{K}]$ is a quasicategory by corollary 6.2.15. Proposition A.2.11 then implies that the induced map
$$\tau_0[X,[f,\mathcal{K}]] : \tau_0[X,[Z,\mathcal{K}]] \to \tau_0[X,[W,\mathcal{K}]]$$
is a bijection for all simplicial sets X and all small quasicategories $\mathcal{K}$. Thus, by lemma 6.2.4, the induced functor $[f,\mathcal{K}] : [Z,\mathcal{K}] \to [W,\mathcal{K}]$ is an equivalence of quasicategories. ∎

Proposition 6.2.17. *The class of weak categorical equivalences is closed under binary products.*

Proof. Let $f : X \to Y$ and $g : Z \to W$ be weak categorical equivalences. Since $f \times g = \left(\mathrm{id}_Y \times g\right) \circ \left(f \times \mathrm{id}_Z\right)$, it suffices (by symmetry) to show that $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a weak categorical equivalence, i.e. that the induced map

$$\tau_0\left[f \times \mathrm{id}_Z, \mathcal{K}\right] : \tau_0[Y \times Z, \mathcal{K}] \to \tau_0[X \times Z, \mathcal{K}]$$

is a bijection for all small quasicategories $\mathcal{K}$. By proposition A.2.11, it is the same to show that

$$\tau_0[f, [Z, \mathcal{K}]] : \tau_0[Y, [Z, \mathcal{K}]] \to \tau_0[X, [Z, \mathcal{K}]]$$

is a bijection for all small quasicategories $\mathcal{K}$; but corollary 6.2.15 says that the exponential object $[Z, \mathcal{K}]$ is a small quasicategory and f is a weak categorical equivalence, so the maps are indeed bijections. ■

Proposition 6.2.18. *Trivial Kan fibrations are weak categorical equivalences.*

Proof. See Proposition 1.22 in [TQA]. □

Corollary 6.2.19. *Inner anodyne extensions are weak categorical equivalences.*

Proof. Let $f : Z \to W$ be an inner anodyne extension. By corollary 6.2.14, the morphism $[f, \mathcal{K}] : [W, \mathcal{K}] \to [Z, \mathcal{K}]$ is a trivial Kan fibration for all small quasicategories $\mathcal{K}$; hence, by propositions 6.2.16 and 6.2.18, $f : Z \to W$ is a weak categorical equivalence. ■

Remark 6.2.20. It is *a priori* not clear whether the notion of weak categorical equivalence is stable under universe enlargement, but in fact it is. First, notice that the notion of weak categorical equivalence *between quasicategories* is stable under universe enlargement, by lemma 6.2.4. Given any morphism $f : X \to Y$ in **sSet**, we may apply the functor R of corollary 6.2.12 to get a commutative diagram of the form below,

$$\begin{array}{ccc} X & \xrightarrow{i_X} & RX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ Y & \xrightarrow[i_Y]{} & RY \end{array}$$

and proposition 6.2.5 implies that the class of weak categorical equivalences has the 2-out-of-3 property, so $f : X \to Y$ is a weak categorical equivalence if and only if $Rf : RX \to RY$ is an equivalence of quasicategories. Since R and i are stable under universe enlargement, it follows that the property of f being a weak categorical equivalence is also stable.

Definition 6.2.21. An **isofibration** of quasicategories is a functor $f : \mathfrak{C} \to \mathfrak{D}$ with the following properties:

- f (as a morphism of simplicial sets) is an inner fibration.
- f has the right lifting property with respect to the inclusion $\{0\} \hookrightarrow \mathbf{I}\mathbb{2}$.

Proposition 6.2.22. *Let $f : \mathfrak{C} \to \mathfrak{D}$ be a functor between small quasicategories.*

(i) *If f (as a morphism of simplicial sets) has the right lifting property with respect to all monomorphisms in* **sSet**, *then f is an isofibration.*

(ii) *If $\mathfrak{D}$ is an ordinary category, then f is an inner fibration.*

(iii) *Assuming f (as a morphism of simplicial sets) is an inner fibration, $f : \mathfrak{C} \to \mathfrak{D}$ is an isofibration if and only if $\operatorname{Ho} f : \operatorname{Ho}\mathfrak{C} \to \operatorname{Ho}\mathfrak{D}$ is an isofibration.*

Proof. (i). This is an immediate consequence of the fact that isofibrations are morphisms that have the right lifting property with respect to certain monomorphisms in **sSet**.

(ii). See Proposition 2.2 in [TQA].

(iii). See Proposition 4.5 in [TQA]. ☐

Theorem 6.2.23 (Joyal)**.** *Let $f : \mathfrak{C} \to \mathfrak{D}$ be a functor between small quasicategories. The following are equivalent:*

(i) *f is an isofibration of quasicategories.*

(ii) *f (as a morphism of simplicial sets) has the right lifting property with respect to all monomorphisms in* **sSet** *that are weak categorical equivalences.*

Proof. See Theorem 6.11 in [TQA], or combine Proposition 2.2.5.8 and Corollary 2.4.6.5 in [HTT]. ☐

Theorem 6.2.24 (Joyal)**.** *The following data constitute a cofibrantly generated model structure on* **sSet***:*

- *The weak equivalences are the weak categorical equivalences.*
- *The cofibrations are the monomorphisms.*
- *The fibrations are the morphisms that have the right lifting property with respect to monomorphisms that are weak categorical equivalences.*

This model structure is called the **Joyal model structure for quasicategories***, and the fibrant objects are the quasicategories.*

Proof. See Theorem 6.12 in [TQA] or Theorem 2.13 in [Dugger and Spivak, 2011a], or combine Proposition 2.2.5.8 with Theorems 2.2.5.1 and 2.4.6.1 in [HTT]. □

Remark 6.2.25. Joyal's determination principle (proposition 4.4.8) implies the Joyal model structure is stable under universe enlargement. Indeed, the claim is obvious for the class of cofibrations, the class of fibrant objects, and lemma 6.2.4 implies that the class of weak equivalences between fibrant objects is stable under universe enlargement; but this is enough data to uniquely determine a model structure.

Proposition 6.2.26. *The Joyal model structure for quasicategories is cartesian.*

Proof. The Joyal model structure for quasicategories is a Cisinski model structure, so we may apply proposition 5.5.12 to proposition 6.2.17 to deduce the claim. ■

Proposition 6.2.27. *Let* **Cat** *be the category of small categories, let* **Qcat** *be the full subcategory of* **sSet** *spanned by the small quasicategories, and let* Ho **Qcat** *be the localisation of* **Qcat** *at the weak categorical equivalences.*

(i) *The adjunction*

$$\tau_1 \dashv \mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$$

is a Quillen adjunction with respect to the canonical model structure on **Cat** *and the Joyal model structure on* **sSet***.*

(ii) *The functors* τ_1 *and* N *preserve weak equivalences, and the induced adjunction*

$$\operatorname{Ho}\tau_1 \dashv \operatorname{Ho}\mathrm{N} : \operatorname{Ho}\mathbf{Cat} \to \operatorname{Ho}\mathbf{Qcat}$$

exhibits $\operatorname{Ho}\mathbf{Cat}$ *as a reflective exponential ideal of* $\operatorname{Ho}\mathbf{Qcat}$.

Proof. (i). See Proposition 6.14 in [TQA].

(ii). Apply theorem 5.5.19, Ken Brown's lemma (4.3.6), propositions 5.5.14 and A.2.13, and the 2-functoriality of Ho (corollary A.4.20). ■

Corollary 6.2.28. *If* $f : X \to Y$ *is an inner anodyne extension of simplicial sets, then* $\tau_1 f : \tau_1 X \to \tau_1 Y$ *is an isomorphism of categories.*

Proof. Proposition 6.2.9 says that $f : X \to Y$ is bijective on vertices, so $\tau_1 f : \tau_1 X \to \tau_1 Y$ is bijective on objects; but corollary 6.2.19 and proposition 6.2.27 imply that $\tau_1 f : \tau_1 X \to \tau_1 Y$ is fully faithful, so we may deduce that it is an isomorphism of categories. ■

Proposition 6.2.29. *Let* $f : X \to Y$ *be a morphism in* $\mathbf{sSet}$*. If* X *and* Y *are Kan complexes, the following are equivalent:*

(i) $f : X \to Y$ *is a weak homotopy equivalence.*

(ii) $f : X \to Y$ *is (part of) an intrinsic homotopy equivalence.*

(iii) $f : X \to Y$ *is (part of) a categorical equivalence.*

(iv) $f : X \to Y$ *is a weak categorical equivalence.*

Proof. (i) ⇔ (ii). See proposition 1.5.3.

(ii) ⇔ (iii). Lemma 1.4.4 and corollary 1.4.16 imply that $\tau_1[X, X]$ and $\tau_1[Y, Y]$ are groupoids; thus, the two notions of equivalence coincide.

(iii) ⇔ (iv). See lemma 6.2.4. ■

Proposition 6.2.30. *Let* $\mathbf{sSet}_{\mathrm{Jo}}$ *be the category* $\mathbf{sSet}$ *equipped with the Joyal model structure, let* $\mathbf{sSet}_{\mathrm{KQ}}$ *be the category* $\mathbf{sSet}$ *equipped with the Kan–Quillen model structure, let* $\mathbf{Qcat}$ *and* $\mathbf{Kan}$ *be the respective full subcategories of fibrant objects, and let* $\operatorname{Ho}\mathbf{Qcat}$ *and* $\operatorname{Ho}\mathbf{Kan}$ *be the respective localisations.*

(i) *The Kan–Quillen model structure on* $\mathbf{sSet}$ *is a left Bousfield localisation of the Joyal model structure on* $\mathbf{sSet}$*.*[2]

[2] See definition 5.6.10.

(ii) *The trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : \mathbf{sSet}_{\mathrm{KQ}} \to \mathbf{sSet}_{\mathrm{Jo}}$$

is a Quillen adjunction between the Kan–Quillen model structure (on the left) and the Joyal model structure (on the right), and the right derived functor is fully faithful.

(iii) *There is an adjunction*

$$\operatorname{Ho} \mathrm{Ex}^{\infty} \dashv \operatorname{Ho} U : \operatorname{Ho} \mathbf{Kan} \to \operatorname{Ho} \mathbf{Qcat}$$

where $U : \mathbf{Kan} \hookrightarrow \mathbf{Qcat}$ *is the inclusion, and* $\operatorname{Ho} U : \operatorname{Ho} \mathbf{Kan} \to \operatorname{Ho} \mathbf{Qcat}$ *is fully faithful.*

Proof. (i). By definition, the Kan–Quillen model structure and the Joyal model structure have the same cofibrations; and lemma 6.2.3 says every weak categorical equivalence is a weak homotopy equivalence, so by proposition 5.6.12, the Kan–Quillen model structure is a left Bousfield localisation of the Joyal model structure.

(ii). Apply proposition 5.6.13.

(iii). Recalling the explicit construction afforded by proposition 3.3.14 and theorem 4.3.13, consider the derived adjunction:

$$\mathbf{L} \dashv \mathbf{R} : \operatorname{Ho} \mathbf{sSet}_{\mathrm{KQ}} \to \operatorname{Ho} \mathbf{sSet}_{\mathrm{Jo}}$$

Given a simplicial set Y, we may compute $\mathbf{R}Y$ as a fibrant replacement for Y in the Kan–Quillen model structure, regarded as an object in the Joyal model structure. In particular, for Kan complexes Y, $\mathbf{R}Y$ is naturally isomorphic to Y (as objects in $\operatorname{Ho} \mathbf{sSet}_{\mathrm{Jo}}$); thus, $\mathbf{R} : \operatorname{Ho} \mathbf{Kan} \to \operatorname{Ho} \mathbf{Qcat}$ is isomorphic to $\operatorname{Ho} U : \operatorname{Ho} \mathbf{Kan} \to \operatorname{Ho} \mathbf{Qcat}$. In particular, $\operatorname{Ho} U : \operatorname{Ho} \mathbf{Kan} \to \operatorname{Ho} \mathbf{Qcat}$ is fully faithful.

On the other hand, for any simplicial set X, we may compute $\mathbf{L}X$ as X itself, regarded as an object in the Kan–Quillen model structure. But (by theorem 1.7.14) $\mathrm{Ex}^{\infty} : \mathbf{sSet} \to \mathbf{sSet}$ is a fibrant replacement functor for the Kan–Quillen model structure, so $\mathbf{L}X$ is naturally isomorphic to $\operatorname{Ho} \mathrm{Ex}^{\infty}(X)$. Thus, $\mathbf{L} : \operatorname{Ho} \mathbf{sSet}_{\mathrm{Jo}} \to \operatorname{Ho} \mathbf{sSet}_{\mathrm{Jo}}$ is isomorphic to $\operatorname{Ho} \mathrm{Ex}^{\infty} : \operatorname{Ho} \mathbf{sSet}_{\mathrm{Jo}} \to \operatorname{Ho} \mathbf{sSet}_{\mathrm{KQ}}$. We thus have the required adjunction. ■

— VII —

Derivators

7.1 Basics

Prerequisites. §§3.1, 3.6, A.1, A.5.

The notion of derivator has a somewhat complicated history; the name and the original idea are due to Grothendieck [1983, 1991], but Heller [1988] studied essentially the same thing independently. The distinguishing characteristic of the theory of derivators is its agnosticism: a derivator is a way of studying homotopy-coherent diagrams and their limits/colimits without using any particular model for homotopical algebra.

In this section, we use the explicit universe convention, all 2-categories and 2-functors will be strict unless otherwise stated, and for simplicity, we say 'coproduct', 'product', 'pullback', etc. instead of '2-coproduct', '2-product', '2-pullback' etc., i.e. we tacitly assume that these have the relevant 2-dimensional universal property in addition to the usual 1-dimensional universal property.

Definition 7.1.1. A **derivator domain** is 2-category $\mathfrak{K}$ satisfying these axioms:

D0. $\mathfrak{K}$ has an initial object 0, a terminal object 1, and tensors with the category $\mathbb{2} = \{0 \to 1\}$.

D1. $\mathfrak{K}$ has finite coproducts and pullbacks.

D2. $\mathfrak{K}$ has comma objects of the form $(u \downarrow b)$ and $(b \downarrow u)$ for all morphisms $u : A \to B$ and $b : 1 \to B$.

A **subdomain** of a derivator domain is a 2-full 2-subcategory that is closed under constructions specified in the above axioms.

Definition 7.1.2. Let $\mathbf{U}$ be a universe. A **$\mathbf{U}$-small prederivator** on $\mathfrak{K}$ is a 2-functor $\mathscr{D} : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$, where $\mathfrak{K}$ is a derivator domain and $\mathfrak{Cat}$ is the 2-category of $\mathbf{U}$-small categories. A **prederivator** is a 2-functor that is a $\mathbf{U}$-small prederivator for some universe $\mathbf{U}$.

We write $\mathscr{D}^A$ for the value of $\mathscr{D}$ at an object A in $\mathfrak{K}$, and we write either $\mathscr{D}^u$ or u^* for the functor $\mathscr{D}^B \to \mathscr{D}^A$ induced by a morphism $u : A \to B$ in $\mathfrak{K}$. If $f : x \to y$ is a morphism in $\mathscr{D}^B$, then we may sometimes write $f \restriction u : x \restriction u \to y \restriction u$ instead of $u^*(f) : u^*(x) \to u^*(y)$. The **underlying category** of a prederivator $\mathscr{D}$ is the category $\mathscr{D}^1$, where 1 is any terminal object of $\mathfrak{K}$.

Remark 7.1.3. While it is true that $\mathfrak{K}$ is a derivator domain if and only if $\mathfrak{K}^{\mathrm{co}}$ is a derivator domain, the duality principle for general prederivators is somewhat subtle: because $(-)^{\mathrm{op}}$ is a 2-functor $\mathfrak{Cat}^{\mathrm{co}} \to \mathfrak{Cat}$, the opposite of a prederivator on $\mathfrak{K}$ is a prederivator on $\mathfrak{K}^{\mathrm{co}}$, which is in general not isomorphic or even equivalent to $\mathfrak{K}$.

One should be aware that some authors (e.g. Cisinski [2003]) define prederivators to be 2-functors $\mathfrak{K}^{\mathrm{coop}} \to \mathfrak{Cat}$; readers should take care to dualise results appropriately when translating between the two conventions.

Definition 7.1.4. A **semiderivator** on $\mathfrak{K}$ is prederivator $\mathscr{D} : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$ satisfying the following axioms:

Der1. $\mathscr{D}$ sends coproducts of finite families of objects in $\mathfrak{K}$ to products in $\mathfrak{Cat}$.

Der2. Let A be an object in $\mathfrak{K}$ and let $f : x \to y$ be a morphism in $\mathscr{D}^A$. Then, f is an isomorphism in $\mathscr{D}^A$ if and only if, for all morphisms $a : 1 \to A$ in $\mathfrak{K}$, the morphism $f \restriction a : x \restriction a \to y \restriction a$ is an isomorphism in $\mathscr{D}^1$.

Example 7.1.5. If B is an object in $\mathfrak{K}$ and $\mathfrak{K}$ is a locally $\mathbf{U}$-small 2-category, then the 2-functor $\mathfrak{K}(-, B) : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$ is a prederivator. We say $\mathfrak{K}(-, B)$ is the **prederivator represented by** B.

Definition 7.1.6. Let C be a $\mathbf{U}$-small relative category. The **prederivator** of C, denoted by $\mathscr{D}(C)$, is the $\mathbf{U}$-small prederivator on $\mathfrak{RelCat}$ (or any subdomain thereof) defined by $\mathscr{D}(C)^{\mathcal{A}} = \mathrm{Ho}\,[\mathcal{A}, C]_{\mathrm{h}}$.

Proposition 7.1.7. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. If A is an object in $\mathfrak{K}$ and $\mathbb{C}$ is a category for which the tensor $\mathbb{C} \odot A$ exists, then there is a canonical comparison functor $\mathscr{D}^{\mathbb{C} \odot A} \to [\mathbb{C}, \mathscr{D}^A]$.*

Proof. By definition, the object $\mathbb{C} \odot A$ in $\mathfrak{K}$ induces isomorphisms

$$\mathfrak{K}(\mathbb{C} \odot A, B) \cong [\mathbb{C}, \mathfrak{K}(A, B)]$$

that are 2-natural in B. Since $\mathscr{D}$ is a prederivator on $\mathfrak{K}$, it induces a functor $\mathfrak{K}(A, B) \to \left[\mathscr{D}^B, \mathscr{D}^A\right]$ that is 2-natural in A and in B, so we obtain a 2-natural functor $\mathfrak{K}(\mathbb{C} \odot A, B) \to \left[\mathbb{C}, \left[\mathscr{D}^B, \mathscr{D}^A\right]\right]$ by composition; but we have 2-natural isomorphisms

$$\left[\mathbb{C}, \left[\mathscr{D}^B, \mathscr{D}^A\right]\right] \cong \left[\mathbb{C} \times \mathscr{D}^B, \mathscr{D}^A\right] \cong \left[\mathscr{D}^B, \left[\mathbb{C}, \mathscr{D}^A\right]\right]$$

so, taking $B = \mathbb{C} \odot A$, we obtain the required functor $\mathscr{D}^{\mathbb{C} \odot A} \to \left[\mathbb{C}, \mathscr{D}^A\right]$. ■

Definition 7.1.8. A **strong semiderivator** on $\mathfrak{K}$ is a semiderivator that satisfies the additional axiom below:

Der5. For any object A in $\mathfrak{K}$, the canonical functor $\mathscr{D}^{\mathbb{2} \odot A} \to \left[\mathbb{2}, \mathscr{D}^A\right]$ is full and essentially surjective on objects (but not necessarily faithful).

Remark 7.1.9. If $\mathscr{D}$ is the prederivator represented by an object in $\mathfrak{K}$, then $\mathscr{D}$ automatically satisfies axioms Der1 and Der5; and if $\mathfrak{K}$ is a 2-full 2-subcategory of $\mathfrak{Cat}$ with the same terminal object, then $\mathscr{D}$ will also satisfy axiom Der2.

Lemma 7.1.10. *If $\mathcal{C}$ is a uni-fractionable category, then the canonical comparison functor* $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \operatorname{Ho} \mathcal{C}]$ *is full and essentially surjective on objects.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\operatorname{weq} \mathcal{C}$ such that $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, and let $\bar{f} : X \to Y$ be any morphism in $\operatorname{Ho} \mathcal{C}$. By the fundamental theorem of three-arrow calculi (3.6.9), there exist $u : Y \to \hat{Y}$ in $\mathcal{U}$, $v : \tilde{X} \to X$ in $\mathcal{V}$, and $f : \tilde{X} \to \hat{Y}$ such that $\bar{f} = u^{-1} \circ f \circ v^{-1}$ in $\operatorname{Ho} \mathcal{C}$, i.e. such that the following diagram in $\operatorname{Ho} \mathcal{C}$ commutes:

$$\begin{tikzcd} X \arrow[r, "v^{-1}"] \arrow[d, "\bar{f}"'] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, "u"'] & \hat{Y} \end{tikzcd}$$

It immediately follows that $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is essentially surjective on objects.

It remains to be shown that $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is a full functor. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms in $\mathcal{C}$, let $\bar{f}_1 : X_1 \to Y_1$ and $\bar{f}_2 : X_2 \to Y_2$ be morphisms in $\operatorname{Ho}\mathcal{C}$, and suppose we have $\bar{f}_2 \circ x = y \circ \bar{f}_1$; note this constitutes a morphism in $[\mathbb{2}, \mathcal{C}]$ between objects in the image of the functor $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$. As before, we may choose $u_1 : Y_1 \to \hat{Y}_1$ and $u_2 : Y_2 \to \hat{Y}_2$ in $\mathcal{U}$, $v_1 : \tilde{X}_1 \to X_1$ and $v_2 : \tilde{X}_2 \to X_2$ in $\mathcal{V}$, and $f_1 : \tilde{X}_1 \to \hat{Y}_1$ and $f_2 : \tilde{X}_2 \to \hat{Y}_2$ in $\mathcal{C}$ such that the equations below hold in $\operatorname{Ho}\mathcal{C}$:

$$\bar{f}_1 = u_1^{-1} \circ f_1 \circ v_1^{-1} \qquad \bar{f}_2 = u_2^{-1} \circ f_2 \circ v_2^{-1}$$

Using axioms A2 and A3, there exist $u_2' : Y_2 \to Z$ in $\mathcal{U}$, $v_1' : W \to X_1$ in $\mathcal{V}$, and $z : \hat{Y}_1 \to Z$ and $w : W \to \tilde{X}_2$ making the following diagrams in $\mathcal{C}$ commute,

$$\begin{tikzcd}
X_1 \arrow[d, equal] & \tilde{X}_1 \arrow[l, "v_1"'] \arrow[r, "f_1"] \arrow[d, equal] & \hat{Y}_1 \arrow[d, dashed, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_1 & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "z \circ f_1"'] & Z & Y_2 \arrow[l, dashed, "u_2'"]
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & W \arrow[l, dashed, "v_1'"'] \arrow[r, "f_2 \circ w"] \arrow[d, dashed, "w"] & \hat{Y}_2 \arrow[d, equal] & Y_2 \arrow[l, "u_2"'] \arrow[d, equal] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & Y_2 \arrow[l, "u_2"]
\end{tikzcd}$$

and since $\bar{f}_2 \circ x = y \circ \bar{f}_1$, the fundamental theorem says there exist a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
X_1 \arrow[d, equal] & W \arrow[l, "v_1'"'] \arrow[r, "f_2 \circ w"] & \hat{Y}_2 & Y_2 \arrow[l, "u_2"'] \arrow[d, equal] \\
X_1 \arrow[d, equal] & \bullet \arrow[l, "v_5"'] \arrow[r, "f_5"] \arrow[u, "v_3"] \arrow[d, "w_3"'] & \bullet \arrow[u, "v_4"'] \arrow[d, "w_4"] & Y_2 \arrow[l, "u_6"'] \arrow[d, equal] \\
X_1 \arrow[d, equal] & \bullet \arrow[l, "v_6"] \arrow[r, "f_6"'] & \bullet & Y_2 \arrow[l, "u_6"] \arrow[d, equal] \\
X_1 & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "z \circ f_1"'] \arrow[u, "u_3"] & Z \arrow[u, "u_4"'] & Y_2 \arrow[l, "u_2'"]
\end{tikzcd}$$

where u_3, u_4, u_5, u_6 are in $\mathcal{U}$, v_3, v_4, v_5, v_6 are in $\mathcal{V}$, and w_3, w_4 are weak equivalences in $\mathcal{C}$.

It is easy to verify that the following diagram in $\mathcal{C}$ commutes,

$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & W \arrow[l, "v_1'"'] \arrow[r, equal] \arrow[d, "w"'] & W \arrow[d, "f_2 \circ w"'] & \bullet \arrow[l, "v_3"'] \arrow[r, "w_3"] \arrow[d, "f_5"'] & \bullet \arrow[d, "f_6"] & \tilde{X}_1 \arrow[l, "u_3"'] \arrow[r, "f_1"] \arrow[d, "z \circ f_1"] & \hat{Y}_1 \arrow[d, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & \bullet \arrow[l, "v_4"] \arrow[r, "w_4"'] & \bullet & Z \arrow[l, "u_4"] \arrow[r, equal] & Z & Y_2 \arrow[l, "u_2'"]
\end{tikzcd}$$

and this is the required lift of $(\bar{f}_1, \bar{f}_2)$ to $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}}$, because the diagram in $\mathcal{C}$ shown below commutes:

$$\begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 & \tilde{X}_1 \arrow[l, "v_1"'] \arrow[r, "f_1"] \arrow[d, equal] & \hat{Y}_1 \arrow[d, equal] & Y_1 \arrow[l, "u_1"'] \arrow[d, equal] \\
X_1 \arrow[d, "x"'] & W \arrow[l, "v_1'"'] \arrow[r, equal] \arrow[u, "v_1'"] \arrow[d, "w"'] & W \arrow[u, "v_1'"] \arrow[d, "f_2 \circ w"'] & \bullet \arrow[l, "v_3"'] \arrow[r, "w_3"] \arrow[u, "v_5"] \arrow[d, "f_5"'] & \bullet \arrow[u, "v_6"'] \arrow[d, "f_6"] & \tilde{X}_1 \arrow[l, "u_3"'] \arrow[r, "f_1"] \arrow[d, "z \circ f_1"] & \hat{Y}_1 \arrow[d, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & \bullet \arrow[l, "v_4"] \arrow[r, "w_4"'] & \bullet & Z \arrow[l, "u_4"] \arrow[r, equal] & Z & Y_2 \arrow[l, "u_2'"] \\
X_2 \arrow[u, equal] & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] \arrow[u, equal] & \hat{Y}_2 \arrow[u, equal] & Y_2 \arrow[l, "u_2"] \arrow[r, equal] \arrow[u, "u_5"] & Y_2 \arrow[r, equal] \arrow[u, "u_6"'] & Y_2 \arrow[r, equal] \arrow[u, "u_2'"'] & Y_2 \arrow[r, equal] \arrow[u, "u_2'"'] & Y_2 \arrow[u, equal]
\end{tikzcd}$$

We may therefore conclude that $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is indeed full. ■

Proposition 7.1.11. *Let $\mathscr{D}$ be the prederivator of a $\mathbf{U}$-small relative category $\mathcal{M}$.*

(i) *$\mathscr{D}$ satisfies axiom Der1.*

(ii) *Moreover, if $\mathcal{M}$ is a (necessarily saturated) homotopical category and each homotopical functor category $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ admits a three-arrow calculus, then $\mathscr{D}$ is a strong semiderivator.*

Proof. (i). Proposition A.4.19 implies $\mathscr{D}$ sends finite coproducts in $\mathfrak{RelCat}$ to products in $\mathfrak{Cat}^{+}$, so axiom Der1 is satisfied.

(ii). Suppose $f : X \to Y$ is a morphism in $\operatorname{Ho}[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ such that all its components are isomorphisms in $\operatorname{Ho}\mathcal{M}$. The fundamental theorem of three-arrow calculi (3.6.9) says $f : X \to Y$ may be represented by a zigzag in $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ of the form below,

$$X \xleftarrow{\psi} \tilde{X} \xrightarrow{\theta} \hat{Y} \xleftarrow{\varphi} Y$$

where ψ and φ are natural weak equivalences. Thus, if A is an object in $\mathcal{A}$, then the following zigzag represents an isomorphism in $\operatorname{Ho}\mathcal{M}$:

$$XA \xleftarrow{\psi_A} \tilde{X}A \xrightarrow{\theta_A} \hat{Y}A \xleftarrow{\varphi_A} YA$$

However, proposition 3.6.10 says $\mathcal{M}$ is a saturated homotopical category, so θ_A must be a weak equivalence in $\mathcal{M}$ as well; hence, $f : X \to Y$ is an isomorphism in $\operatorname{Ho}[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$. This shows that $\mathscr{D}$ satisfies axiom Der2.

Finally, observe that $[\min \mathbb{2} \times \mathcal{A}, \mathcal{M}]_{\mathrm{h}} \cong \left[\min \mathbb{2}, [\mathcal{A}, \mathcal{M}]_{\mathrm{h}}\right]_{\mathrm{h}}$, and the hypothesis says $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ admits a three-arrow calculus, so we apply lemma 7.1.10 to deduce that axiom Der5 is satisfied. ■

Definition 7.1.12. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, let $u : A \to B$ be a morphism in $\mathfrak{K}$, and let X be an object in $\mathscr{D}^A$.

- A **left $\mathscr{D}$-extension** of X along u is an initial object in the comma category $(X \downarrow u^*)$.
- A **right $\mathscr{D}$-extension** of X along u is a terminal object in the comma category $(u^* \downarrow X)$.
- We say $\mathscr{D}$ has **left extensions** along u if the functor $u^* : \mathscr{D}^B \to \mathscr{D}^A$ has a left adjoint, which we denote by $u_! : \mathscr{D}^A \to \mathscr{D}^B$.
- We say $\mathscr{D}$ has **right extensions** along u if the functor $u^* : \mathscr{D}^B \to \mathscr{D}^A$ has a right adjoint, which we denote by $u_* : \mathscr{D}^A \to \mathscr{D}^B$.

We may refer to left and right $\mathscr{D}$-extensions generically as **homotopy Kan extensions** in $\mathscr{D}$.

Remark 7.1.13. It is straightforward to check that $\mathscr{D}$ has left (resp. right) extensions along u if and only if, for every object X in $\mathscr{D}^A$, there exists a left (resp. right) $\mathscr{D}$-extension of X along u.

Example 7.1.14. If $\mathfrak{K}$ is a 2-full 2-subcategory of $\mathfrak{Cat}$ and $\mathscr{D}$ is the prederivator represented by an object in $\mathfrak{K}$, then $\mathscr{D}$-extensions are exactly the same thing as Kan extensions in the usual sense.

As we saw in theorem A.5.15, pointwise left (resp. right) Kan extensions can be computed as colimits (resp. limits) of certain diagrams whose shapes are comma categories. We shall shortly see that more is true.

Definition 7.1.15. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$ and suppose we have a diagram in $\mathfrak{K}$ of the following form:

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

- We say the square is a **left $\mathscr{D}$-exact square** if $\mathscr{D}$ has left extensions along $u : A \to C$ and $q : D \to B$ and the induced diagram shown below satisfies the left Beck–Chevalley condition:

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\theta^*}{\nearrow}"] & \mathscr{D}^B \arrow[d, "q^*"] \\ \mathscr{D}^A \arrow[r, "p^*"'] & \mathscr{D}^D \end{tikzcd}$$

- We say the square is a **right $\mathscr{D}$-exact square** if $\mathscr{D}$ has right extensions along $v : B \to C$ and $p : D \to A$ and the induced diagram shown below satisfies the right Beck–Chevalley condition:

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "u^*"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\underset{\theta^*}{\swarrow}"] & \mathscr{D}^A \arrow[d, "p^*"] \\ \mathscr{D}^B \arrow[r, "q^*"'] & \mathscr{D}^D \end{tikzcd}$$

- A **$\mathscr{D}$-exact square** in $\mathfrak{K}$ is a diagram in $\mathfrak{K}$ that is both left $\mathscr{D}$-exact and right $\mathscr{D}$-exact.

Proposition 7.1.16. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. Given the following diagram in $\mathfrak{K}$,*

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

if $\mathscr{D}$ has left extensions along $u : A \to C$ and $q : D \to B$, and $\mathscr{D}$ has right extensions along $v : B \to C$ and $p : D \to A$, then the following are equivalent:

(i) *The diagram is a $\mathscr{D}$-exact square.*

(ii) *The diagram is a left $\mathscr{D}$-exact square.*

(iii) *The diagram is a right $\mathscr{D}$-exact square.*

Proof. Statement (i) is just the conjunction of statements (ii) and (iii), and when the required left and right adjoints exist, proposition A.1.12 implies that statements (ii) and (iii) are equivalent. ■

Lemma 7.1.17 (Pasting exact squares)**.** *Let $\mathscr{D}$ be a prederivator, and consider pasting diagrams of the following forms in $\mathfrak{K}$:*

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet \arrow[r] & \bullet
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\
\bullet \arrow[r] & \bullet
\end{tikzcd}
$$

In either diagram, if both squares are left (resp. right) $\mathscr{D}$-exact squares, then the rectangle obtained by pasting the two squares is also a left (resp. right) $\mathscr{D}$-exact square.

Proof. Apply lemma A.1.11. ■

Lemma 7.1.18. *Let* **Set** *be the category of* **U***-sets. If $\mathscr{D}$ is the prederivator of* **Set** *restricted to the subdomain $\mathfrak{Cat}$, then every comma square in $\mathfrak{Cat}$ is a right $\mathscr{D}$-exact square.*

Proof. Suppose we have the following comma square in $\mathfrak{Cat}$:

$$
\begin{tikzcd}
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\theta\Nearrow"] & \mathbb{B} \arrow[d, "v"] \\
\mathbb{A} \arrow[r, "u"'] & \mathbb{C}
\end{tikzcd}
$$

Let $Y : \mathbb{B} \to \mathbf{Set}$ be a functor and let (Z, ε) be a right Kan extension of Y along v, i.e. a terminal object in the comma category $(v^* \downarrow Y)$. In view of lemma A.1.10, to deduce the claim, it is enough to show that $\left(u^*(Z), q^*(\varepsilon) \bullet \theta^*_Z\right)$ is a terminal object in the comma category $(p^* \downarrow q^*(Y))$, i.e. a right Kan extension of $Y q$ along p; but this was done in lemma A.5.8. ■

Proposition 7.1.19. *Let $\mathcal{M}$ be a locally* $\mathbf{U}$*-small category, and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted to $\mathfrak{Cat}$.*

- *If $\mathcal{M}$ has colimits for all* $\mathbf{U}$*-small diagrams, then every comma square in $\mathfrak{Cat}$ is a left $\mathscr{D}$-exact square.*
- *If $\mathcal{M}$ has limits for all* $\mathbf{U}$*-small diagrams, then every comma square in $\mathfrak{Cat}$ is a right $\mathscr{D}$-exact square.*

Proof. The two claims are formally dual; we will prove the first version.

Consider a comma square in $\mathfrak{Cat}$:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & \mathbb{B} \arrow[d, "v"] \\ \mathbb{A} \arrow[r, "u"'] & \mathbb{C} \end{tikzcd}$$

If $\mathcal{M}$ has colimits for all $\mathbf{U}$-small diagrams, then theorem A.5.15 implies that, for any functor $X : \mathbb{A} \to \mathcal{M}$, the left Kan extension of X along u exists and is pointwise, and same is true for the left Kan extension of $p^*(X)$ along q. Thus, for any object M in $\mathcal{M}$, if $\hbar_M : \mathcal{M}^{\mathrm{op}} \to \mathbf{Set}^+$ is the representable functor $\mathcal{M}(-, M)$, we may use lemma A.1.10 to deduce that the following (commutative!) diagrams satisfy the *right* Beck–Chevalley condition:

$$\begin{tikzcd} {[\mathbb{C}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{C}^{\mathrm{op}}, \hbar_M]}"] \arrow[d, "(u^{\mathrm{op}})^*"'] & {[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[d, "(u^{\mathrm{op}})^*"] \\ {[\mathbb{A}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{A}^{\mathrm{op}}, \hbar_M]}"'] & {[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}^+]} \end{tikzcd} \qquad \begin{tikzcd} {[\mathbb{B}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{A}^{\mathrm{op}}, \hbar_M]}"] \arrow[d, "(q^{\mathrm{op}})^*"'] & {[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[d, "(q^{\mathrm{op}})^*"] \\ {[\mathbb{D}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{B}^{\mathrm{op}}, \hbar_M]}"'] & {[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}^+]} \end{tikzcd}$$

On the other hand, lemma 7.1.18 says the diagram below satisfies the right Beck–Chevalley condition,

$$\begin{tikzcd} {[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[r, "(v^{\mathrm{op}})^*"] \arrow[d, "(u^{\mathrm{op}})^*"'] \arrow[dr, phantom, "\overset{\Leftarrow}{(\theta^{\mathrm{op}})^*}"] & {[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[d, "(q^{\mathrm{op}})^*"] \\ {[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[r, "(p^{\mathrm{op}})^*"'] & {[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}^+]} \end{tikzcd}$$

and the family $\left\{ \hbar_M : \mathcal{M}^{\mathrm{op}} \to \mathbf{Set}^+ \,\middle|\, M \in \operatorname{ob} \mathcal{M} \right\}$ is jointly conservative, so we deduce that the right Beck–Chevalley condition for the following diagram is satisfied,

$$\begin{tikzcd} {[\mathbb{C}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "(v^{\mathrm{op}})^*"] \arrow[d, "(u^{\mathrm{op}})^*"'] & {[\mathbb{B}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[d, "(q^{\mathrm{op}})^*"] \\ {[\mathbb{A}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "(p^{\mathrm{op}})^*"'] \arrow[ur, phantom, "\Leftarrow (\theta^{\mathrm{op}})^*"] & {[\mathbb{D}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \end{tikzcd}$$

and therefore this diagram satisfies the *left* Beck–Chevalley condition:

$$\begin{tikzcd} {[\mathbb{C}, \mathcal{M}]} \arrow[r, "v^*"] \arrow[d, "u^*"'] & {[\mathbb{B}, \mathcal{M}]} \arrow[d, "q^*"] \\ {[\mathbb{A}, \mathcal{M}]} \arrow[r, "p^*"'] \arrow[ur, phantom, "\theta^* \Rightarrow"] & {[\mathbb{D}, \mathcal{M}]} \end{tikzcd}$$

We then conclude that every comma square in $\mathfrak{Cat}$ is a left $\mathscr{D}$-exact square. ■

Definition 7.1.20. A **$\mathfrak{K}$-cocomplete semiderivator** is a semiderivator $\mathscr{D}$ on $\mathfrak{K}$ satisfying these additional axioms:

Der3L. $\mathscr{D}$ has left extensions along every morphism $u : A \to B$ in $\mathfrak{K}$.

Der4L. Every comma square in $\mathfrak{K}$ of the form below is a left $\mathscr{D}$-exact square:

$$\begin{tikzcd} (u \downarrow c) \arrow[r] \arrow[d] & 1 \arrow[d, "c"] \\ A \arrow[r, "u"'] \arrow[ur, phantom, "\Rightarrow"] & C \end{tikzcd}$$

Dually, a **$\mathfrak{K}$-complete U-semiderivator** is one satisfying these axioms:

Der3R. $\mathscr{D}$ has right extensions along every morphism $u : A \to B$ in $\mathfrak{K}$.

Der4R. Every comma square in $\mathfrak{K}$ of the form below is a right $\mathscr{D}$-exact square:

$$\begin{tikzcd} (c \downarrow v) \arrow[r] \arrow[d] & B \arrow[d, "v"] \\ 1 \arrow[r, "c"'] \arrow[ur, phantom, "\Rightarrow"] & C \end{tikzcd}$$

Theorem 7.1.21. *Let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$, let $\mathcal{M}$ be a $\mathbf{U}^+$-small category, and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted to $\mathfrak{Cat}$.*

(i) *$\mathscr{D}$ is a strong semiderivator.*

(ii) *$\mathscr{D}$ is $\mathfrak{Cat}$-cocomplete (resp. $\mathfrak{Cat}$-complete) if and only if $\mathcal{M}$ is $\mathbf{U}$-complete (resp. $\mathbf{U}$-complete).*

Proof. (i). This can be shown using the same arguments as remark 7.1.9.

(ii). This is the content of proposition 7.1.19. ■

Finally, we come to the definition of the subject of this chapter:

Definition 7.1.22. A **derivator** on $\mathfrak{K}$ is a semiderivator that is $\mathfrak{K}$-cocomplete and $\mathfrak{K}$-complete, and a **strong derivator** is one that satisfies axiom Der5.

Remark 7.1.23. The definition of 'subdomain' ensures that the restriction of any derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator) on $\mathfrak{K}$ to any subdomain of $\mathfrak{K}$ is again a derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator).

Proposition 7.1.24. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, and let $u \dashv v : B \to A$ be an adjunction in $\mathfrak{K}$, with unit $\eta : \mathrm{id}_A \Rightarrow v \circ u$ and counit $\varepsilon : u \circ v \Rightarrow \mathrm{id}_B$.*

(i) *We have an adjunction $v^* \dashv u^* : \mathscr{D}^B \to \mathscr{D}^A$, with unit $\eta^* : \mathrm{id}_{\mathscr{D}^A} \Rightarrow u^* \circ v^*$ and counit $\varepsilon^* : v^* \circ u^* \Rightarrow \mathrm{id}_{\mathscr{D}^B}$; in particular, $\mathscr{D}$ has left extensions along $u : A \to B$ and right extensions along $v : B \to A$.*

(ii) *Consider the following commutative diagrams in $\mathfrak{K}$:*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ 1 \arrow[r, "\mathrm{id}"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "q"] \arrow[d, "v"'] & 1 \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "p"'] & 1 \end{tikzcd}$$

The diagram on the left is a left $\mathscr{D}$-exact square, and the diagram on the right is a right $\mathscr{D}$-exact square.

(iii) *Moreover, if $\mathscr{D}$ has left extensions along $p : A \to 1$ and $q : A \to 1$, then the diagram on the right is a left $\mathscr{D}$-exact square; and if $\mathscr{D}$ has right extensions along $p : A \to 1$ and $q : A \to 1$, then the diagram on the left is a right $\mathscr{D}$-exact square.*

Proof. (i). Since $\mathscr{D}$ is a 2-functor, it preserves the triangle identities; thus $v^* \dashv u^*$ is indeed an adjunction. (The left and right adjoints are exchanged because $\mathscr{D}$ is contravariant.)

(ii). The two halves of the claim are formally dual; we will prove the first version. By claim (i), we may take $u_! = v^*$; but the left Beck–Chevalley transformation

$$u_! p^* \Rightarrow u_! p^* \mathrm{id}^* \mathrm{id}_! \Rightarrow u_! u^* q^* \mathrm{id}_! \Rightarrow q^* \mathrm{id}_!$$

is then equal to $\varepsilon^* q^* : v^* u^* q^* \Rightarrow q^*$, and $\varepsilon^* q^* = (q\varepsilon)^* = \mathrm{id}$, because 1 is a terminal object in $\mathfrak{K}$. Thus the left Beck–Chevalley condition is satisfied.

(iii). This is a special case of proposition 7.1.16. ■

Theorem 7.1.25. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$ that satisfies axioms Der3L and Der3R, and let 1 be a terminal object in $\mathfrak{K}$. The following are equivalent:*

(i) *$\mathscr{D}$ is a derivator.*

(ii) *$\mathscr{D}$ satisfies axiom Der4L.*

(iii) *Every comma square in $\mathfrak{K}$ is left $\mathscr{D}$-exact.*

(iv) *$\mathscr{D}$ satisfies axiom Der4R.*

(v) *Every comma square in $\mathfrak{K}$ is right $\mathscr{D}$-exact.*

Proof. Obviously, statement (i) implies statements (ii)–(v), and the conjunction of statements (iii) and (v) implies statement (i). We are assuming that $\mathscr{D}$ has left and right extensions along all morphisms in $\mathfrak{K}$, so the equivalence of statements (iii) and (v) is just proposition 7.1.16. It remains to be shown that (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v), but the two implications are formally dual, so it is enough to prove just one; we prove the former.

Consider a general comma square in $\mathfrak{K}$:

$$\begin{tikzcd}
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}$$

Let $b : 1 \to B$ be a morphism in $\mathfrak{K}$, and let $c = v \circ b$, and consider the following pasting diagrams,

$$\begin{tikzcd}
(q \downarrow b) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\tau}{\Nearrow}"] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(u \downarrow c) \arrow[r, "r"] \arrow[d, "j"'] \arrow[dr, phantom, "//"] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}$$

where the upper square of the diagram on the left is a comma square, and the upper square of the diagram on the right is a 2-pullback square; note that the pasting lemma for comma squares implies that the outer rectangle of the diagram on the right is also a comma square.

Let $\pi = p \circ j$ and let $\lambda = \theta \circ \mathrm{id}_j$. By the universal property of comma objects, there is a unique morphism $f : (q \downarrow b) \to (u \downarrow c)$ such that $\pi \circ f = p \circ s$, $r \circ f = t$, and $\lambda \circ \mathrm{id}_f = \left(\mathrm{id}_v \circ \tau\right) \bullet \left(\theta \circ \mathrm{id}_s\right)$; and similarly there is a unique morphism $g : (u \downarrow c) \to (q \downarrow b)$ such that $s \circ g = j$, $t \circ g = r$, and $\tau \circ \mathrm{id}_g = \mathrm{id}_{q \circ j} = \mathrm{id}_{b \circ r}$. Then,

$$\pi \circ (f \circ g) = p \circ s \circ g = p \circ j = \pi \qquad\qquad r \circ (f \circ g) = r$$

so $f \circ g = \mathrm{id}_{(u \downarrow c)}$; and since $p \circ s = p \circ s \circ g \circ f$, we may think of $\mathrm{id}_{p \circ s}$ as a 2-cell $\beta : p \circ s \Rightarrow p \circ s \circ g \circ f$, whereas $b \circ t = q \circ s \circ g \circ f$, so $\tau : q \circ s \Rightarrow b \circ t$ is also a 2-cell $\gamma : q \circ s \Rightarrow q \circ s \circ g \circ f$, but then

$$\left(\theta \circ \mathrm{id}_{s \circ g \circ f}\right) \bullet \left(\mathrm{id}_u \circ \beta\right) = \theta \circ \mathrm{id}_j \circ \mathrm{id}_f = \lambda \circ \mathrm{id}_f = \left(\mathrm{id}_v \circ \gamma\right) \bullet \left(\theta \circ \mathrm{id}_s\right)$$

so by the 2-universal property of $(u \downarrow v)$, there is a unique 2-cell $\alpha : s \Rightarrow s \circ g \circ f$ such that $\mathrm{id}_p \circ \alpha = \beta$ and $\mathrm{id}_q \circ \alpha = \gamma$; and furthermore,

TODO: Justify this more carefully...

$$\left(\tau \circ \mathrm{id}_{g \circ f}\right) \bullet \left(\mathrm{id}_q \circ \alpha\right) = \left(\mathrm{id}_b \circ \mathrm{id}_{t \circ g \circ f}\right) \bullet \tau$$

therefore there is a unique 2-cell $\eta : \mathrm{id}_{(q \downarrow b)} \Rightarrow g \circ f$ such that $\mathrm{id}_s \circ \eta = \alpha$ and $\mathrm{id}_t \circ \eta = \mathrm{id}_{t \circ g \circ f}$.

We will now show that we have an adjunction $f \dashv g : (u \downarrow c) \to (q \downarrow b)$ in $\mathfrak{K}$; since $f \circ g = \mathrm{id}_{(u \downarrow c)}$, it is enough to check that $\mathrm{id}_f \circ \eta = \mathrm{id}_f$ and $\eta \circ \mathrm{id}_g = \mathrm{id}_g$. By construction, $\mathrm{id}_\pi \circ \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_p \circ \mathrm{id}_s \circ \eta = \mathrm{id}_{p \circ s}$, and $\mathrm{id}_r \circ \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_t$, so indeed $\mathrm{id}_f \circ \eta = \mathrm{id}_f$; and $\mathrm{id}_s \circ \left(\eta \circ \mathrm{id}_g\right) = \mathrm{id}_s$ and $\mathrm{id}_t \circ \left(\eta \circ \mathrm{id}_g\right) = \mathrm{id}_t$, so

$\eta \circ \mathrm{id}_g = \mathrm{id}_g$ as well. Thus, by proposition 7.1.24, the commutative diagram in $\mathfrak{K}$ shown below on the left is a left $\mathscr{D}$-exact square,

$$\begin{tikzcd} (u \downarrow c) \arrow[r, "r"] \arrow[d, "g"'] & 1 \arrow[d, "\mathrm{id}"] \\ (q \downarrow b) \arrow[r] & 1 \end{tikzcd} \qquad \begin{tikzcd} (q \downarrow b) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\tau}{\nearrow}"] & 1 \arrow[d, "b"] \\ (u \downarrow v) \arrow[r, "q"'] & B \end{tikzcd}$$

and the diagram on the right is a left $\mathscr{D}$-exact square by hypothesis, so by the pasting lemma (7.1.17), the following commutative diagram is also a left $\mathscr{D}$-exact square:

$$\begin{tikzcd} (u \downarrow c) \arrow[r] \arrow[d, "j"'] & 1 \arrow[d, "b"] \\ (u \downarrow v) \arrow[r, "q"'] & B \end{tikzcd}$$

The hypothesis also implies that this diagram satisfies the left Beck–Chevalley condition,

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\lambda^*}{\nearrow}"] & \mathscr{D}^1 \arrow[d, "r^*"] \\ \mathscr{D}^A \arrow[r, "\pi^*"'] & \mathscr{D}^{(u\downarrow c)} \end{tikzcd}$$

but the pasting lemma (A.1.11) says that the left Beck–Chevalley transformation $r_! \pi^* \Rightarrow c^* u_!$ is obtained by pasting together the left Beck–Chevalley transformations of the squares in the diagram below,

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\underset{\theta^*}{\swarrow}"] & \mathscr{D}^B \arrow[r, "b^*"] \arrow[d, "q^*"] \arrow[dr, phantom, "="] & \mathscr{D}^1 \arrow[d, "r^*"] \\ \mathscr{D}^A \arrow[r, "p^*"'] & \mathscr{D}^{(u\downarrow v)} \arrow[r, "j^*"'] & \mathscr{D}^{(u\downarrow c)} \end{tikzcd}$$

and so, allowing $b : 1 \to B$ to vary, we deduce that every component of the left Beck–Chevalley transformation $v^* u_! \Rightarrow q_! p^*$ is an isomorphism in $\mathscr{D}^1$. We may then apply axiom Der2 to conclude that the comma square we started with is a left $\mathscr{D}$-exact square. ■

7.2 Homotopy limits and colimits

Prerequisites. §§ 3.3, 4.1, 4.10, 7.1.

¶ 7.2.1. In this section, we use the **two-universe convention**: we assume that there are two universes $\mathbf{U}$ and $\mathbf{U}^+$, with $\mathbf{U} \in \mathbf{U}^+$. We refer to $\mathbf{U}$-sets, $\mathbf{U}$-small categories, etc. as 'small', and we refer to $\mathbf{U}^+$-sets, $\mathbf{U}^+$-small categories, etc. as 'moderate'.

Definition 7.2.2. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, let A be an object in $\mathfrak{K}$, let 1 be a terminal object in $\mathfrak{K}$, let $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ be the functor induced by the unique morphism $A \to 1$ in $\mathfrak{K}$, and let X be an object in $\mathscr{D}^A$.

- A $\mathscr{D}$**-colimit** for X is an initial object in the comma category $(X \downarrow \Delta_A)$.
- A $\mathscr{D}$**-limit** for X is a terminal object in the comma category $(\Delta_A \downarrow X)$.
- We say $\mathscr{D}$ **has colimits for diagrams of shape** A if $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ has a left adjoint, which we denote by $\operatorname{ho}\varinjlim_A : \mathscr{D}^A \to \mathscr{D}^1$.
- We say $\mathscr{D}$ **has limits for diagrams of shape** A if $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ has a right adjoint, which we denote by $\operatorname{ho}\varprojlim_A : \mathscr{D}^A \to \mathscr{D}^1$.

We may refer to $\mathscr{D}$-colimits (resp. $\mathscr{D}$-limits) generically as **homotopy colimits** (resp. homotopy limits) in $\mathscr{D}$.

Remark 7.2.3. Of course, homotopy colimits (resp. homotopy limits) in $\mathscr{D}$ are a special case of homotopy left (resp. right) Kan extensions in $\mathscr{D}$; in particular, $\mathscr{D}$ has colimits (resp. limits) for diagrams of shape A if and only if, for every object X in $\mathscr{D}^A$, there exists a $\mathscr{D}$-colimit (resp. $\mathscr{D}$-limit) for X.

Proposition 7.2.4. *Let $\mathcal{M}$ be a moderate model category and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted along* $\min : \mathfrak{Cat} \to \mathfrak{RelCat}$.

(i) *$\mathscr{D}$ satisfies axiom Der1.*

(ii) *$\mathscr{D}$ satisfies axiom Der5 at the terminal category $\mathbb{1}$, i.e. the canonical comparison functor $\mathscr{D}^{\mathbb{2}} \to [\mathbb{2}, \mathscr{D}^{\mathbb{1}}]$ is full and essentially surjective on objects.*

(iii) *Moreover, if $\mathcal{M}$ satisfies axiom CM5*, then $\mathscr{D}$ is a strong semiderivator.*

Proof. (i). Proposition A.4.19 implies $\mathscr{D}$ sends finite coproducts in $\mathfrak{RelCat}$ to products in $\mathfrak{Cat}^{+}$, and the embedding $\min : \mathfrak{Cat} \to \mathfrak{RelCat}$ preserves finite coproducts, so axiom Der1 is satisfied.

(ii). By theorem 4.1.31, $\mathcal{M}$ admits a three-arrow calculus, so the claim follows from lemma 7.1.10.

(iii). Moreover, if $\mathcal{M}$ satisfies axiom CM5*, then $\mathcal{M}$ admits a *functorial* three-arrow calculus, so by proposition 3.6.8, each $[\mathbb{A}, \mathcal{M}]_{\mathrm{h}}$ admits a componentwise three-arrow calculus. Theorem 4.4.1 implies $\mathcal{M}$ is a saturated homotopical category, so we deduce that $\mathscr{D}$ is a strong semiderivator using proposition 7.1.11. ■

Theorem 7.2.5. *If $\mathcal{M}$ is a locally small DHK model category, then the restriction of $\mathscr{D}(\mathcal{M})$ to $\mathfrak{Cat}$ is a strong derivator.*

Proof. Let $\mathscr{D}$ be the restriction of $\mathscr{D}(\mathcal{M})$ to $\mathfrak{Cat}$. We have already shown in proposition 7.2.4 that $\mathscr{D}$ is a strong semiderivator, so it remains to be proven that $\mathscr{D}$ is cocomplete and complete. Cocompleteness and completeness are formally dual, so it suffices to demonstrate just one half of the claim; we will show that $\mathscr{D}$ is cocomplete.

By theorem 4.10.16, for every functor $u : \mathbb{A} \to \mathbb{B}$ between small categories, the functor $\mathrm{Lan}_u : [\mathbb{A}, \mathcal{M}] \to [\mathbb{B}, \mathcal{M}]$ is left deformable, so theorem 3.3.24 implies the functor $\mathrm{Ho}\, u^* : \mathrm{Ho}\,[\mathbb{B}, \mathcal{M}] \to \mathrm{Ho}\,[\mathbb{A}, \mathcal{M}]$ has a left adjoint, namely the total left derived functor $\mathbf{L}\big(\mathrm{Lan}_u\big) : \mathrm{Ho}\,[\mathbb{A}, \mathcal{M}] \to \mathrm{Ho}\,[\mathbb{B}, \mathcal{M}]$. Thus $\mathscr{D}$ satisfies axiom Der3L.

Finally, to conclude, we note that proposition 4.10.18 is precisely the statement that axiom Der4L is satisfied. This completes the proof that $\mathscr{D}$ is cocomplete. ■

Theorem 7.2.6 (Cisinski)**.** *Let $\mathcal{M}$ be a locally small model category and let $\mathscr{D}(\mathcal{M})$ be its associated prederivator. If $\mathcal{M}$ has colimits and limits for all small diagrams, then the restriction of $\mathscr{D}(\mathcal{M})$ to the 2-category of small categories is a derivator.*

Proof. See Theorem 6.11 in [Cisinski, 2003]. □

Definition 7.2.7. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$.

- A **$\mathscr{D}$-cofinal morphism** is a morphism $v : B \to A$ in $\mathfrak{K}$ such that the diagram below is a left $\mathscr{D}$-exact square,

$$\begin{tikzcd} B \arrow[r] \arrow[d, "v"'] & 1 \arrow[d, "\mathrm{id}"] \\ A \arrow[r] & 1 \end{tikzcd}$$

  i.e. such that the left Beck–Chevalley transformation

$$\underrightarrow{\operatorname{holim}}_B \circ v^* \Rightarrow \underrightarrow{\operatorname{holim}}_A$$

  is a natural isomorphism.

- A **$\mathscr{D}$-coinitial morphism** is a morphism $u : A \to B$ in $\mathfrak{K}$ such that the diagram below is a right $\mathscr{D}$-exact square,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r, "\mathrm{id}"'] & 1 \end{tikzcd}$$

  i.e. such that the right Beck–Chevalley transformation

$$\underleftarrow{\operatorname{holim}}_B \Rightarrow \underleftarrow{\operatorname{holim}}_A \circ u^*$$

  is a natural isomorphism.

Example 7.2.8. For any derivator $\mathscr{D}$ on $\mathfrak{K}$, every right adjoint (resp. left adjoint) in $\mathfrak{K}$ is a $\mathscr{D}$-cofinal (resp. $\mathscr{D}$-coinitial) morphism: this is the content of proposition 7.1.24.

Example 7.2.9. A category $\mathbb{A}$ has a terminal object if and only if the unique functor $\mathbb{A} \to \mathbb{1}$ has a right adjoint $t : \mathbb{1} \to \mathbb{A}$; thus, for any derivator on $\mathfrak{Cat}$, if $\mathbb{A}$ is a small category with a terminal object, then the left Beck–Chevalley transformation $t^* \Rightarrow \underrightarrow{\operatorname{holim}}_{\mathbb{A}}$ is a natural isomorphism.

Definition 7.2.10. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. A **$\mathscr{D}$-equivalence** is a morphism $u : A \to B$ in $\mathfrak{K}$ satisfying the following condition:

- For all X and Y in $\mathscr{D}^1$, the map $\mathscr{D}^B\big(\Delta_B X, \Delta_B Y\big) \to \mathscr{D}^A\big(\Delta_A X, \Delta_A Y\big)$ induced by $u^* : \mathscr{D}^B \to \mathscr{D}^A$ is a bijection.

Proposition 7.2.11. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$ and let $u : A \to B$ be a morphism in $\mathfrak{K}$. If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then the following are equivalent:*

(i) *The morphism $u : A \to B$ is a $\mathscr{D}$-equivalence.*

(ii) *For η^B the unit of $\operatorname{ho}\varinjlim_B \dashv \Delta_B$ and ε^A the counit of $\operatorname{ho}\varinjlim_A \dashv \Delta_A$, the natural transformation*

$$\left(\varepsilon^A \circ \operatorname{ho}\varinjlim_B \circ \Delta_B\right) \bullet \left(\operatorname{ho}\varinjlim_A \circ u^* \circ \eta^B \circ \Delta_B\right) : \operatorname{ho}\varinjlim_A \circ \Delta_A \Rightarrow \operatorname{ho}\varinjlim_B \circ \Delta_B$$

is a natural isomorphism.

(iii) *For ε^u the counit of $u_! \dashv u^*$, the natural transformation*

$$\operatorname{ho}\varinjlim_B \circ\, \varepsilon^u \circ \Delta_B : \operatorname{ho}\varinjlim_A \circ \Delta_A \Rightarrow \operatorname{ho}\varinjlim_B \circ \Delta_B$$

is a natural isomorphism.

Dually, if $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then the following are equivalent:

(i′) *The morphism $u : A \to B$ is a $\mathscr{D}$-equivalence.*

(ii′) *For η^A the unit of $\Delta_A \dashv \operatorname{ho}\varprojlim_A$ and ε^B the counit of $\Delta_B \dashv \operatorname{ho}\varprojlim_B$, the natural transformation*

$$\left(\operatorname{ho}\varprojlim_A \circ u^* \circ \varepsilon^B \circ \Delta_B\right) \bullet \left(\eta^A \circ \operatorname{ho}\varprojlim_B \circ \Delta_B\right) : \operatorname{ho}\varprojlim_B \circ \Delta_B \Rightarrow \operatorname{ho}\varprojlim_A \circ \Delta_A$$

is a natural isomorphism.

(iii′) *For η^u the unit of $u^* \dashv u_*$, the natural transformation*

$$\operatorname{ho}\varprojlim_B \circ\, \eta^u \circ \Delta_B : \operatorname{ho}\varprojlim_B \circ \Delta_B \Rightarrow \operatorname{ho}\varprojlim_A \circ \Delta_A$$

is a natural isomorphism.

Proof. The two sets of claims are formally dual; we will prove the first version.

Observe that every morphism $u : A \to B$ in $\mathfrak{K}$ induces a commutative diagram of the following form:

$$\begin{tikzcd}
\mathscr{D}^1\left(\varinjlim_B \Delta_B X, Y\right) \arrow[r, "\cong"] \arrow[d] & \mathscr{D}^B\left(\Delta_B X, \Delta_B Y\right) \arrow[d] \\
\mathscr{D}^1\left(\varinjlim_A \Delta_A X, Y\right) \arrow[r, "\cong"'] & \mathscr{D}^A\left(\Delta_A X, \Delta_A Y\right)
\end{tikzcd}$$

Thus, a morphism $u : A \to B$ in $\mathfrak{K}$ satisfies condition (ii) if and only if it is a $\mathscr{D}$-equivalence. By factoring the counit $\varepsilon^A : \operatorname{ho}\varinjlim_A \Delta_A \Rightarrow \mathrm{id}_{\mathscr{D}^1}$ in terms of the counit $\varepsilon^u : u_! u^* \Rightarrow \mathrm{id}_{\mathscr{D}^B}$ and using the left triangle identity, we deduce that

$$\left(\varepsilon^A \circ \operatorname{ho}\varinjlim_B \circ \Delta_B\right) \bullet \left(\operatorname{ho}\varinjlim_A \circ u^* \circ \eta^B \circ \Delta_B\right) = \operatorname{ho}\varinjlim_B \circ \varepsilon^u \circ \Delta_B$$

and so condition (ii) is satisfied if and only if condition (iii) is satisfied. ■

Corollary 7.2.12.

- *If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then every $\mathscr{D}$-cofinal morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.*

- *If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then every $\mathscr{D}$-coinitial morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.* ■

Remark 7.2.13. In particular:

- If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then every right adjoint morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.

- If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then every left adjoint morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.

Proposition 7.2.14. *Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{K}$ be the underlying 1-category of $\mathfrak{K}$. For any prederivator $\mathscr{D}$ on $\mathfrak{K}$, the category $\mathcal{K}$ with the class of $\mathscr{D}$-equivalences in $\mathfrak{K}$ constitute a saturated homotopical category.*

Proof. We will assume that, for every object A in $\mathfrak{K}$, the category $\mathscr{D}^A$ is locally small, but there is no loss of generality in doing so because we may always enlarge the universe.

Observe that, for all objects X and Y in $\mathscr{D}^1$, the functor $\mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ defined by $C \mapsto \mathscr{D}^C(\Delta_C X, \Delta_C Y)$ sends every $\mathscr{D}$-equivalence in $\mathfrak{K}$ to a bijection. Thus, if $u : A \to B$ is a morphism in $\mathfrak{K}$ that becomes invertible in the localisation of $\mathcal{K}$ at $\mathscr{D}$-equivalences, then for all objects X and Y in $\mathscr{D}^1$, the map $\mathscr{D}^B(\Delta_B X, \Delta_B Y) \to \mathscr{D}^A(\Delta_A X, \Delta_A Y)$ induced by u must be a bijection, so u must be a $\mathscr{D}$-equivalence. ■

Proposition 7.2.15. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$.*

- *Given a commutative triangle in $\mathfrak{K}$ as below,*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

if $\mathscr{D}$ is $\mathfrak{K}$-cocomplete and, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ induced by $u : A \to B$ is a $\mathscr{D}$-equivalence, then $u : A \to B$ is itself a $\mathscr{D}$-equivalence.

- *Given a commutative triangle in $\mathfrak{K}$ as below,*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

if $\mathscr{D}$ is $\mathfrak{K}$-complete and, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism ${}^c u : (c \downarrow p) \to (c \downarrow q)$ induced by $u : A \to B$ is a $\mathscr{D}$-equivalence, then $u : A \to B$ is itself a $\mathscr{D}$-equivalence.

Proof. We will use the characterisation of $\mathscr{D}$-equivalences afforded by proposition 7.2.11. We wish to show that the natural transformation defined by the following pasting diagram is a natural isomorphism:

(1)
$$\begin{tikzcd} \mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "\underrightarrow{\mathrm{holim}}"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "="] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "\sslash"] & \mathscr{D}^1 \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}"] & \\ & \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "\underrightarrow{\mathrm{holim}}"'] \arrow[u, phantom, shift left=4, "\Nearrow"] & \mathscr{D}^1 \end{tikzcd}$$

By factoring $A \to 1$ and $B \to 1$ through $C \to 1$ and applying the left triangle identity, we see that it is enough to show that the natural transformation defined below is a natural isomorphism:

$$\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "="] & \mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\mathbin{/\!\!/}"] & \mathscr{D}^C \arrow[d, "p^*"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "p_!"'] \arrow[u, phantom, shift left=4, "\Nearrow"] & \mathscr{D}^C
\end{tikzcd} \tag{2}$$

Axiom Der4L says that the following comma square in $\mathfrak{K}$ is left $\mathscr{D}$-exact,

$$\begin{tikzcd}
(p \downarrow c) \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\Nearrow"] & 1 \arrow[d, "c"] \\
A \arrow[r, "p"'] & C
\end{tikzcd}$$

i.e. the left Beck–Chevalley transformation it induces is a natural isomorphism:

$$\begin{tikzcd}
\mathscr{D}^A \arrow[r, "p_!"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^A \arrow[r, "r^*"'] & \mathscr{D}^{(p\downarrow c)} \arrow[r, "\varinjlim{\mathrm{holim}}"'] \arrow[u, phantom, shift left=4, "\Nearrow"] & \mathscr{D}^1
\end{tikzcd} \tag{3}$$

Similarly, the comma square in $\mathfrak{K}$ shown below

$$\begin{tikzcd}
(q \downarrow c) \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\Nearrow"] & 1 \arrow[d, "c"] \\
B \arrow[r, "q"'] & C
\end{tikzcd}$$

induces a left Beck–Chevalley transformation that is a natural isomorphism:

$$\begin{tikzcd}
\mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "q^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^B \arrow[r, "s^*"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "\varinjlim{\mathrm{holim}}"'] \arrow[u, phantom, shift left=4, "\Nearrow"] & \mathscr{D}^1
\end{tikzcd} \tag{4}$$

Our hypothesis is that unique morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ making the following diagram commute is a $\mathscr{D}$-equivalence,

$$\begin{tikzcd}
(p \downarrow c) \arrow[r, "u_c"] \arrow[d, "r"'] & (q \downarrow c) \arrow[d, "s"] \\
A \arrow[r, "u"'] & B
\end{tikzcd}$$

i.e. the natural transformation defined below is a natural isomorphism:

$$
\text{(5)} \qquad
\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "\underrightarrow{\operatorname{holim}}"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "\mathbin{/\!/}"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\Nearrow"'] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "(u_c)^*"'] & \mathscr{D}^{(p\downarrow c)} \arrow[r, "\underrightarrow{\operatorname{holim}}"'] & \mathscr{D}^1
\end{tikzcd}
$$

However, the natural transformations defined by the following pasting diagrams are equal,

$$
\begin{tikzcd}
\mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\mathbin{/\!/}"] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \\
\mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "r^*"'] & \mathscr{D}^{(p\downarrow c)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\mathbin{/\!/}"] & \mathscr{D}^1 \arrow[d, "\Delta"] \\
\mathscr{D}^B \arrow[r, "s^*"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "(u_c)^*"'] & \mathscr{D}^{(p\downarrow c)}
\end{tikzcd}
$$

so, the natural transformation obtained by pasting together (2) and (3) is equal to the natural transformation obtained by pasting together (4) and (5); but the latter is a natural isomorphism, so we deduce that the former is a natural isomorphism as well. Thus,

$$
\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"'] \arrow[dl, phantom, "\Nwarrow"] \arrow[dr, phantom, "\mathbin{/\!/}"] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\Nearrow"'] & \mathscr{D}^1 \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "p_!"'] & \mathscr{D}^C \arrow[r, "c^*"'] & \mathscr{D}^1
\end{tikzcd}
$$

defines a natural isomorphism. Since $c : 1 \to C$ was arbitrary, we may use axiom Der2 to deduce that (1) itself defines a natural isomorphism, as claimed. ■

Lemma 7.2.16. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$. Consider a diagram of the following form in $\mathfrak{K}$:*

$$
\begin{tikzcd}
E \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & 1 \arrow[d, "b"] \\
1 \arrow[r, "a"'] & C
\end{tikzcd}
$$

If $\mathscr{D}$ is $\mathfrak{K}$-cocomplete (resp. $\mathfrak{K}$-cocomplete), then the following are equivalent:

(i) *The diagram above is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).*

(ii) *The morphism $w : E \to (a \downarrow b)$ induced by the universal property of $(a \downarrow b)$ is a $\mathscr{D}$-equivalence.*

Proof. The two claims are formally dual; we will prove the first version.

By definition, the diagram above is a left $\mathscr{D}$-exact square if and only if the left Beck–Chevalley transformation

$$\underrightarrow{\operatorname{holim}}_E \Delta_E \Rightarrow \underrightarrow{\operatorname{holim}}_E \Delta_E a^* a_! \Rightarrow \underrightarrow{\operatorname{holim}}_E \Delta_E b^* a_! \Rightarrow b^* a_!$$

is a natural isomorphism. However, $\Delta_E = w^* \Delta_{(a \downarrow b)}$, and axiom Der4L says the left Beck–Chevalley transformation

$$\underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} \Rightarrow \underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} a^* a_! \Rightarrow \underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} b^* a_! \Rightarrow b^* a_!$$

is a natural isomorphism, so using the counit of the adjunction $w_! \dashv w^*$, proposition 7.2.11, and the 2-out-of-3 property of natural isomorphisms, we may deduce that conditions (i) and (ii) are equivalent. ■

Theorem 7.2.17. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$. Consider the following diagram in $\mathfrak{K}$:*

(□)
$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

If $\mathscr{D}$ is $\mathfrak{K}$-cocomplete (resp. $\mathfrak{K}$-cocomplete), then the following are equivalent:

(i) *Diagram (□) is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).*

(ii) *For all morphisms $a : 1 \to A$ and $b : 1 \to B$, for all diagrams of the form below in $\mathfrak{K}$,*

(∗)
$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & (q \downarrow b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\beta}{\nearrow}"] & 1 \arrow[d, "b"] \\ (a \downarrow p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{\alpha}{\nearrow}"] & D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ 1 \arrow[r, "a"'] & A \arrow[r, "u"'] & C \end{tikzcd}$$

where the top-left square is a pullback square and the squares inhabited by α and β are comma squares, the outer square is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).

(iii) *For all diagrams of the form* $(*)$ *in* $\mathfrak{K}$, *the morphism* $E \to (u \circ a \downarrow v \circ b)$ *induced by the universal property of* $(u \circ a \downarrow v \circ b)$ *is a* $\mathscr{D}$*-equivalence.*

Proof. (i) ⇒ (ii). The pasting lemma for comma diagrams implies the left rectangle of $(*)$ is a comma diagram, and we may apply lemma 7.1.17 and theorem 7.1.25 to deduce that the outer square of $(*)$ is a left $\mathscr{D}$-exact square.

(ii) ⇔ (iii). This is a special case of the previous lemma.

(ii) ⇒ (i). Using axioms Der2 and Der4L as well as the 2-out-of-3 property for natural isomorphisms, we may deduce that diagram $(\square)$ is left $\mathscr{D}$-exact if every diagram of the form $(*)$ is left $\mathscr{D}$-exact. ■

Corollary 7.2.18. *Let* $\mathscr{D}$ *be a semiderivator on* $\mathfrak{K}$. *If* $\mathscr{D}$ *is* $\mathfrak{K}$*-cocomplete, then the following are equivalent for a morphism* $v : B \to A$ *in* $\mathfrak{K}$*:*

(i) *The morphism* $v : B \to A$ *is a* $\mathscr{D}$*-cofinal morphism.*

(ii) *For every morphism* $a : 1 \to A$ *in* $\mathfrak{K}$, *the unique morphism* $(a \downarrow v) \to 1$ *is a* $\mathscr{D}$*-equivalence.*

Dually, if $\mathscr{D}$ *is* $\mathfrak{K}$*-complete, then the following are equivalent for a morphism* $u : A \to B$ *in* $\mathfrak{K}$*:*

(i) *The morphism* $u : A \to B$ *is a* $\mathscr{D}$*-coinitial morphism.*

(ii) *For every morphism* $b : 1 \to B$ *in* $\mathfrak{K}$, *the unique morphism* $(u \downarrow b) \to 1$ *is a* $\mathscr{D}$*-equivalence.* ■

7.3 Basic localisers

Prerequisites. §§3.1, 7.1.

Definition 7.3.1. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{K}$ be its underlying 1-category. A **basic right localiser** (resp. **basic left localiser**) for $\mathfrak{K}$ is a subcategory $\mathcal{W}$ of $\mathcal{K}$ satisfying these axioms:

LF1. Every identity morphism in $\mathcal{K}$ is also in $\mathcal{W}$, $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{K}$, and $\mathcal{W}$ is closed under retracts in $\mathcal{K}$.

LF2. For any object A in $\mathfrak{K}$, if the unique morphism $A \to 1$ has a right adjoint (resp. left adjoint), then $A \to 1$ is in $\mathcal{W}$.

LF3. Given a commutative triangle in $\mathfrak{K}$,

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

if, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ (resp. ${}^c u : (c \downarrow p) \to (c \downarrow q)$) induced by $u : A \to B$ is in $\mathcal{W}$, then $u : A \to B$ itself is in $\mathcal{W}$.

A **basic localiser** for $\mathfrak{K}$ is a subcategory of $\mathcal{K}$ that is both a basic left localiser and a basic right localiser.

Definition 7.3.2. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$. A **$\mathcal{W}$-equivalence** is a morphism that is in $\mathcal{W}$. A **$\mathcal{W}$-aspherical object** is an object A in $\mathfrak{K}$ such that the unique morphism $A \to 1$ is a $\mathcal{W}$-equivalence.

¶ 7.3.3. The above terminology is non-standard: it is more conventional to refer to basic right localisers as 'basic localisers' and ignore basic left localisers; cf. [Cisinski, 2004]. However, this is unproblematic in the case where $\mathfrak{K} = \mathfrak{Cat}$: one can show that all three notions coincide then. The chirality of the above terminology is chosen to agree with the chirality of the induced asphericity structures (cf. [Maltsiniotis, 2005]).

Proposition 7.3.4. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then the class of $\mathscr{D}$-equivalences is a basic right localiser for $\mathfrak{K}$.*
- *If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then the class of $\mathscr{D}$-equivalences is a basic left localiser for $\mathfrak{K}$.*

Proof. The two claims are formally dual; we will prove the first version.

Proposition 7.2.14 implies that the class of $\mathscr{D}$-equivalences satisfies axiom LF1, and proposition 7.2.15 says that axiom LF3 is satisfied. Axiom LF2 remains to be verified, so suppose A is an object in $\mathfrak{K}$ such that the unique morphism p :

$A \to 1$ has a right adjoint, say $t : 1 \to A$. By remark 7.2.13, t is a $\mathscr{D}$-equivalence; but $p \circ t = \mathrm{id}_1$ since 1 is a terminal object in $\mathfrak{K}$, so we may deduce that $p : A \to 1$ is also a $\mathscr{D}$-equivalence by using axiom LF1. ■

Corollary 7.3.5. *If $\mathscr{D}$ is a derivator on $\mathfrak{K}$, then the class of $\mathscr{D}$-equivalences is a basic localiser for $\mathfrak{K}$.* ■

Example 7.3.6. Let $\mathscr{D}$ be the prederivator of **Set** (restricted to $\mathfrak{Cat}$). By theorem 7.1.21, $\mathscr{D}$ is a derivator, and it is straightforward to verify that the $\mathscr{D}$-equivalences are precisely the functors $u : \mathbb{A} \to \mathbb{B}$ that induce bijections $\pi_0 u : \pi_0 \mathbb{A} \to \pi_0 \mathbb{B}$, where $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ is the connected components functor.[1]

Remark 7.3.7. It is not hard to see that the intersection of any family of basic localisers (resp. basic left localisers, basic right localisers) for a derivator domain $\mathfrak{K}$ is automatically a basic localiser (resp. basic left localiser, basic right localiser) for $\mathfrak{K}$; thus, there is a unique minimal basic localiser (resp. basic left localiser, basic right localiser) for $\mathfrak{K}$.

Definition 7.3.8. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$.

- A **right $\mathcal{W}$-aspherical morphism** is a morphism $u : A \to B$ in $\mathfrak{K}$ such that, for all morphisms $b : 1 \to B$ in $\mathfrak{K}$, the unique morphism $(u \downarrow b) \to 1$ is a $\mathcal{W}$-equivalence.

- A **left $\mathcal{W}$-aspherical morphism** is a morphism $v : B \to A$ in $\mathfrak{K}$ such that, for all morphisms $a : 1 \to A$ in $\mathfrak{K}$, the unique morphism $(a \downarrow v) \to 1$ is a $\mathcal{W}$-equivalence.

Remark 7.3.9. In view of corollary 7.2.18, one might also call right (resp. left) $\mathcal{W}$-aspherical morphisms **$\mathcal{W}$-coinitial** (resp. **$\mathcal{W}$-cofinal**).

Lemma 7.3.10. *Let $\mathfrak{K}$ be a derivator domain.*

- *If a morphism $u : A \to B$ in $\mathfrak{K}$ has a right adjoint, then for any morphism $b : 1 \to B$, the unique morphism $(u \downarrow b) \to 1$ has a right adjoint.*

- *If a morphism $v : B \to A$ in $\mathfrak{K}$ has a left adjoint, then for any morphism $a : 1 \to A$, the unique morphism $(a \downarrow v) \to 1$ has a left adjoint.*

[1] Recall proposition A.2.15.

Proof. The two claims are formally dual; we will prove the first version.

Suppose the following diagram is a comma square in $\mathfrak{K}$:

$$\begin{tikzcd}
(u \downarrow b) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & 1 \arrow[d, "b"] \\
A \arrow[r, "u"'] & B
\end{tikzcd}$$

Let $v : B \to A$ be a right adjoint of $u : A \to B$, say with counit $\varepsilon : u \circ v \Rightarrow \mathrm{id}_B$. Consider the morphism $t : 1 \to (u \downarrow b)$ induced by the diagram in $\mathfrak{K}$ shown below:

$$\begin{tikzcd}
1 \arrow[r, "v \circ b"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\overset{\varepsilon \circ b}{\Nearrow}"] & 1 \arrow[d, "b"] \\
A \arrow[r, "u"'] & B
\end{tikzcd}$$

Via the 2-dimensional universal property of $(u \downarrow b)$, $\theta : u \circ p \Rightarrow b \circ q$ induces a 2-cell $\eta : \mathrm{id}_{(u \downarrow b)} \Rightarrow t \circ q$, and using the 2-dimensional Yoneda lemma, it is straightforward to check that η is the unit of an adjunction $q \dashv t : 1 \to (u \downarrow b)$. Thus, the unique morphism $(u \downarrow b) \to 1$ indeed has a right adjoint. ■

Corollary 7.3.11. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every morphism in $\mathfrak{K}$ that has a right adjoint is a right $\mathcal{W}$-aspherical morphism.*
- *If $\mathcal{W}$ is a basic left localiser for $\mathfrak{K}$, then every morphism in $\mathfrak{K}$ that has a left adjoint is a left $\mathcal{W}$-aspherical morphism.* ■

Proposition 7.3.12. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every right $\mathcal{W}$-aspherical morphism is a $\mathcal{W}$-equivalence; in particular every morphism in $\mathfrak{K}$ that has a right adjoint is a $\mathcal{W}$-equivalence.*
- *If $\mathcal{W}$ is a basic left localiser for $\mathfrak{K}$, then every left $\mathcal{W}$-aspherical morphism is a $\mathcal{W}$-equivalence; in particular every morphism in $\mathfrak{K}$ that has a left adjoint is a $\mathcal{W}$-equivalence.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $u : A \to B$ is a right $\mathcal{W}$-aspherical morphism. Consider the following commutative triangle in $\mathfrak{K}$:

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "\mathrm{id}"] \\ & B & \end{tikzcd}$$

Let $b : 1 \to B$ be a morphism in $\mathfrak{K}$. Since the unique morphism $(u \downarrow b) \to 1$ is a $\mathcal{W}$-equivalence, axioms LF1 and LF2 and lemma 7.3.10 imply the induced morphism $u_b : (u \downarrow b) \to \left(\mathrm{id}_B \downarrow b\right)$ is also a $\mathcal{W}$-equivalence. We may then apply axiom LF3 to deduce that $u : A \to B$ itself is a $\mathcal{W}$-equivalence. ∎

Lemma 7.3.13. *Let A be an object in a derivator domain $\mathfrak{K}$. If $\mathcal{W}$ is a basic left or right localiser for $\mathfrak{K}$, then the morphism $p : \mathbb{2} \odot A \to \mathbb{1} \odot A \cong A$ induced by the unique functor $\mathbb{2} \to \mathbb{1}$ is a $\mathcal{W}$-equivalence.*

Proof. The unique functor $\mathbb{2} \to \mathbb{1}$ has both a left adjoint and a right adjoint, so the induced morphism $p : \mathbb{2} \odot A \to A$ has both a left adjoint and a right adjoint. Proposition 7.3.12 then implies that it is a $\mathcal{W}$-equivalence. ∎

Proposition 7.3.14. *Let $u_0, u_1 : A \to B$ be a parallel pair of morphisms in a derivator domain $\mathfrak{K}$ and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$. If there exists a 2-cell $\alpha : u_0 \Rightarrow u_1$, then the following are equivalent:*

(i) *The morphism $u_0 : A \to B$ is a $\mathcal{W}$-equivalence.*

(ii) *The morphism $u_1 : A \to B$ is a $\mathcal{W}$-equivalence.*

Proof. Let $i_0, i_1 : A \to \mathbb{2} \odot A$ be the morphisms induced by the left and right adjoints of the unique functor $\mathbb{2} \to \mathbb{1}$; note that functoriality yields $p \circ i_0 = \mathrm{id}_A = p \circ i_1$. The previous lemma says that p is a $\mathcal{W}$-equivalence, so we may then use axiom LF1 to deduce that i_0 and i_1 are both $\mathcal{W}$-equivalences.

By definition, there is a bijection

$$\mathcal{K}(\mathbb{2} \odot A, B) \cong \mathrm{Fun}(\mathbb{2}, \mathfrak{K}(A, B))$$

that is natural in B; thus, the 2-cell $\alpha : u_0 \Rightarrow u_1$ corresponds to a morphism $h : \mathbb{2} \odot A \to B$ such that $h \circ i_0 = u_0$ and $h \circ i_1 = u_1$. Axiom LF1 then implies that u_0 is a $\mathcal{W}$-equivalence if and only if u_1 is a $\mathcal{W}$-equivalence. ∎

Corollary 7.3.15. *If $\mathcal{W}$ is a basic left or right localiser for a derivator domain $\mathfrak{K}$, then every left or right adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence.*

Proof. One half of the claim was proved in proposition 7.3.12; it now suffices to show that, if $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every right adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence. We already know that every left adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence, so axiom LF1 and the above proposition together imply that right adjoints are also $\mathcal{W}$-equivalences. ■

7.4 The minimal basic localiser

Prerequisites. §§1.1, 1.2, 1.3, 1.5, 1.9, 1.11, 7.1, 7.2, 7.3.

In this section, we follow [Cisinski, 2004, §2.2].

Proposition 7.4.1. *Let $\mathcal{W}$ be a basic left or right localiser for $\mathfrak{Cat}$. For any functor $u : \mathbb{A} \to \mathbb{B}$, the following are equivalent:*

(i) *The functor $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}$-equivalence.*

(ii) *The functor $u^{\mathrm{op}} : \mathbb{A}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is a $\mathcal{W}$-equivalence.*

Proof. See Proposition 1.2.6 in [Cisinski, 2004]. □

Corollary 7.4.2. *Let $\mathcal{W}$ be a subcategory of* **Cat**. *The following are equivalent:*

(i) *$\mathcal{W}$ is a basic localiser for $\mathfrak{Cat}$.*

(ii) *$\mathcal{W}$ is a basic right localiser for $\mathfrak{Cat}$.*

(iii) *$\mathcal{W}$ is a basic left localiser for $\mathfrak{Cat}$.* ■

¶ 7.4.3. Throughout this section, let $\mathcal{W}$ be any basic localiser for $\mathfrak{Cat}$. We write $\mathcal{W}_\infty$ for the class of weak homotopy equivalences of categories.

Theorem 7.4.4. *$\mathcal{W}_\infty$ is a basic localiser for $\mathfrak{Cat}$.*

Proof. By lemma 1.11.3 (resp. remark 1.11.6, theorem 1.11.14), $\mathcal{W}_\infty$ satisfies axiom LF1 (resp.LF2, LF3). ■

Lemma 7.4.5. *If $\mathbb{A}$ is a small category with a terminal object, then the category $\mathbf{\Delta}(\mathbb{A})$ is $\mathcal{W}$-aspherical.*

Proof. Straightforward. (This is Lemme 2.2.2 in [Cisinski, 2004].) □

Lemma 7.4.6 (Grothendieck)**.** *For all small categories $\mathbb{A}$, the right projection $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{A}) \to \mathbb{A}$ is right $\mathcal{W}$-aspherical; in particular, it is a $\mathcal{W}$-equivalence.*

Proof. Let a be an object in $\mathbb{A}$. Lemma 4.10.12 says that the canonical comparison functor $\mathbf{\Delta}\big(\mathbb{A}_{/a}\big) \to \big(\pi_{\mathrm{R}} \downarrow a\big)$ is an isomorphism, and lemma 7.4.5 implies $\mathbf{\Delta}\big(\mathbb{A}_{/a}\big)$ is $\mathcal{W}$-aspherical, so the induced functor $\big(\pi_{\mathrm{R}} \downarrow a\big) \to \mathbb{A}_{/a}$ is a $\mathcal{W}$-equivalence. Thus, $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{A}) \to \mathbb{A}$ is right $\mathcal{W}$-aspherical. ■

Corollary 7.4.7. *A functor $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}$-equivalence if and only if the functor $\mathbf{\Delta}(u) : \mathbf{\Delta}(\mathbb{A}) \to \mathbf{\Delta}(\mathbb{B})$ is a $\mathcal{W}$-equivalence.*

Proof. Use the naturality of π_{R} and axiom LF1. ■

¶ 7.4.8. Now, let $\mathcal{W}_{\mathbf{\Delta}}$ be the subcategory of **sSet** consisting of those morphisms $f : X \to Y$ such that $\mathbf{\Delta}(f) : \mathbf{\Delta}(X) \to \mathbf{\Delta}(Y)$ are $\mathcal{W}$-equivalences.

Proposition 7.4.9. *For all simplicial sets X and all natural numbers n, the projection $\pi : X \times \Delta^n \to X$ is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. Since $\Delta^m \times \Delta^n \cong \mathrm{N}([m] \times [n])$, lemma 7.4.5 implies $\mathbf{\Delta}(\Delta^m \times \Delta^n)$ is $\mathcal{W}$-aspherical. Now, let x be an m-simplex of X, and consider the comma category $(\mathbf{\Delta}(\pi) \downarrow x)$. It is not hard to see that $(\mathbf{\Delta}(\pi) \downarrow x)$ is isomorphic to $\mathbf{\Delta}(\Delta^m \times \Delta^n)$, and so the induced functor $(\mathbf{\Delta}(\pi) \downarrow x) \to \mathbf{\Delta}(X)_{/x}$ is a $\mathcal{W}$-equivalence. Thus, $\mathbf{\Delta}(\pi) : \mathbf{\Delta}(X \times \Delta^n) \to \mathbf{\Delta}(X)$ is right $\mathcal{W}$-aspherical, and in particular $\pi : X \times \Delta^n \to X$ is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence. ■

Corollary 7.4.10. *Every trivial Kan fibration is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. Apply proposition 1.5.19. ■

Proposition 7.4.11. *Every trivial cofibration in* **sSet** *is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. See Proposition 2.2.9 in [Cisinski, 2004]. □

Theorem 7.4.12 (Cisinski)**.** *Any $\mathcal{W}_\infty$-equivalence is also a $\mathcal{W}$-equivalence.*

Proof. Propositions 1.4.7 and 1.5.10 together imply that every weak homotopy equivalence in **sSet** can be factored as a trivial cofibration followed by a trivial Kan fibration, so applying corollaries 7.4.7 and 7.4.10 and proposition 7.4.11, we deduce that every $\mathcal{W}_\infty$-equivalence is a $\mathcal{W}$-equivalence. ■

We thus obtain a proof of Grothendieck's conjecture ([1983, § 81]):

Corollary 7.4.13. *The minimal basic localiser for* $\mathfrak{Cat}$ *is* $\mathcal{W}_\infty$. ■

— VIII —

Homotopy toposes

8.1 Internal Kan complexes

Prerequisites. §§1.1, 1.4, 1.7, 2.3, A.7.

To do homotopy theory inside a topos, we need a model for homotopy types. One of the simplest options is to internalise the theory of Kan complexes. This was first done in the case of sheaves on a topological space by Brown [1973], then extended to the general case of an effective regular category by van Osdol [1977].

Definition 8.1.1. An **internal Kan fibration** (resp. **internal trivial Kan fibration**) in a regular category $\mathcal{S}$ is a morphism $p : X \to Y$ in $\mathbf{s}\mathcal{S}$ with the following property:

- If $i : Z \to W$ is a horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ (resp. a boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$) and the square in the diagram below is a weak pullback square in $\mathcal{S}$:

$$\begin{tikzcd} \{W, X\} \arrow[r, "\{W,p\}"] \arrow[d, "\{i,X\}"'] & \{W, Y\} \arrow[d, "\{i,Y\}"] \\ \{Z, X\} \arrow[r, "\{Z,p\}"'] & \{Z, Y\} \end{tikzcd}$$

Remark 8.1.2. If $\mathcal{S} = \mathbf{Set}$, then an internal Kan fibration (resp. internal trivial Kan fibration) is just a Kan fibration (resp. trivial Kan fibration) in the usual sense, by lemma A.3.2. If $\mathcal{S} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ for a small category $\mathbb{C}$, then an internal Kan fibration (resp. internal trivial Kan fibration) in $\mathcal{S}$ is the same thing as a componentwise Kan fibration (resp. trivial Kan fibration), because limits and colimits in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ are computed componentwise.

Definition 8.1.3. An **internal Kan complex** in a regular category $\mathcal{S}$ is an object X in $\mathbf{s}\mathcal{S}$ such that the unique morphism $X \to 1$ in $\mathbf{s}\mathcal{S}$ is an internal Kan fibration. We write $\mathbf{Kan}(\mathcal{S})$ for the full subcategory of $\mathbf{s}\mathcal{S}$ spanned by the internal Kan complexes in $\mathcal{S}$.

Proposition 8.1.4. *Let $\mathcal{S}$ be a regular category.*

(i) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in $\mathcal{S}$ contains all isomorphisms in $\mathbf{s}\mathcal{S}$.*

(ii) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in $\mathcal{S}$ is closed under composition.*

(iii) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in $\mathcal{S}$ is closed under pullbacks.*

(iv) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in $\mathcal{S}$ is closed under retracts.*

(v) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in $\mathcal{S}$ is closed under finite products.*

Proof. (i). Obvious.

(ii) and (iii). Apply the weak pullback lemma (A.7.17).

(iv) and (v). By proposition A.7.15, the class of regular epimorphisms in $\mathcal{S}$ is closed under retracts (resp. finite products), so the class of weak pullback squares in $\mathcal{S}$ is also closed under retracts (resp. finite products). ■

Proposition 8.1.5. *Let $\mathcal{S}$ be a regular category, let $i : Z \to W$ be a morphism between finite simplicial sets, and let $p : X \to Y$ be a morphism in $\mathbf{s}\mathcal{S}$. Consider the following commutative diagram in $\mathcal{S}$:*

$$\begin{tikzcd} \{W, X\} \arrow[r, "{\{W,p\}}"] \arrow[d, "{\{i,X\}}"'] & \{W, Y\} \arrow[d, "{\{i,Y\}}"] \\ \{Z, X\} \arrow[r, "{\{Z,p\}}"'] & \{Z, Y\} \end{tikzcd}$$

(i) *If $i : Z \to W$ is an anodyne extension and $p : X \to Y$ is an internal Kan fibration, then the diagram is a weak pullback square in $\mathcal{S}$.*

(ii) *If $i : Z \to W$ is a monomorphism and $p : X \to Y$ is an internal trivial Kan fibration, then the diagram is a weak pullback square in $\mathcal{S}$.*

Proof. The proofs of the two claims are similar; we will prove claim (i).

By proposition 1.4.12, the class of anodyne extensions between finite simplicial sets is the smallest class of morphisms containing the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ that is closed under composition, pushouts, and retracts; but the class of regular epimorphisms $\mathcal{S}$ is closed under composition, pullbacks, and retracts (by proposition A.7.15), and $\{-, X\}$ sends colimits in **sSet** to limits in $\mathcal{S}$, so we are done. ■

Proposition 8.1.6. *Let $\mathcal{S}$ be a regular category, let $i : Z \to W$ be a monomorphism between finite simplicial sets, and let $p : X \to Y$ be an internal Kan fibration in $\mathcal{S}$. Consider the following commutative diagram in $\mathbf{s}\mathcal{S}$,*

$$\begin{tikzcd}
W \pitchfork X \arrow[dr, dashed, "i \boxslash p"] \arrow[drr, bend left, "\mathrm{id}_W \pitchfork p"] \arrow[ddr, bend right, "i \pitchfork \mathrm{id}_X"'] & & \\
& (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) \arrow[r] \arrow[d] & W \pitchfork Y \arrow[d, "i \pitchfork \mathrm{id}_Y"] \\
& Z \pitchfork X \arrow[r, "\mathrm{id}_Z \pitchfork p"'] & Z \pitchfork Y
\end{tikzcd}$$

where the square is a pullback.

(i) *The morphism $i \boxslash p : W \pitchfork X \to (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)$ is an internal Kan fibration.*

(ii) *If $i : Z \to W$ is an anodyne extension, then $i \boxslash p$ is an internal trivial Kan fibration.*

(iii) *If $p : X \to Y$ is an internal trivial Kan fibration, then $i \boxslash p$ is an internal trivial Kan fibration.*

Proof. The proofs of the three claims are similar; we will prove claim (i).

Let $j : K \to L$ be a morphism of finite simplicial sets, and consider the following commutative diagram in $\mathcal{S}$,

$$\begin{tikzcd}
& \{L, W \pitchfork X\} \arrow[rr] \arrow[dl] \arrow[dd] & & \{L, W \pitchfork Y\} \arrow[dl] \arrow[dd] \\
\{K, W \pitchfork X\} \arrow[rr] \arrow[dd] & & \{K, W \pitchfork Y\} \arrow[dd] & \\
& \{L, Z \pitchfork X\} \arrow[rr] \arrow[dl] & & \{L, Z \pitchfork Y\} \arrow[dl] \\
\{K, Z \pitchfork X\} \arrow[rr] & & \{K, Z \pitchfork Y\} &
\end{tikzcd}$$

where the horizontal arrows are induced by $p : X \to Y$, the vertical arrows are induced by $i : Z \to W$, and the diagonal arrows are induced by $j : K \to L$. We wish to show that the following diagram is a weak pullback square in $\mathcal{S}$ when $j : K \to L$ is any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$:

$$\begin{tikzcd}
\{L, W \pitchfork X\} \arrow[r, "{\{L, i \square p\}}"] \arrow[d, "{\{j, W \pitchfork X\}}"'] & \left\{L, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\right\} \arrow[d, "{\{j, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\}}"] \\
\{K, W \pitchfork X\} \arrow[r, "{\{K, i \square p\}}"'] & \left\{K, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\right\}
\end{tikzcd}$$

It is not hard to see that this amounts to showing that the comparison morphism

$$\{L, W \pitchfork X\} \to \{K, W \pitchfork X\} \times_{\{K, Z \pitchfork Y\}} \{L, Z \pitchfork X\} \times_{\{K, Z \pitchfork Y\}} \{L, W \pitchfork Y\}$$

is a regular epimorphism in $\mathcal{S}$. Via the natural isomorphism $\{K, Z \pitchfork X\} \cong \{K \times Z, X\}$, this is in turn equivalent to showing that the following diagram is a weak pullback square in $\mathcal{S}$,

$$\begin{tikzcd}
\{L \times W, X\} \arrow[r, "{\{j \square i, X\}}"] \arrow[d, "{\{L \times W, p\}}"'] & \left\{(K \times W) \cup^{K \times Z} (L \times Z), X\right\} \arrow[d, "{\{(K \times W) \cup^{K \times Z} (L \times Z), p\}}"] \\
\{L \times W, Y\} \arrow[r, "{\{j \square i, Y\}}"'] & \left\{(K \times W) \cup^{K \times Z} (L \times Z), Y\right\}
\end{tikzcd}$$

where $j \square i : (K \times W) \cup^{K \times Z} (L \times Z) \to L \times W$ is the evident monomorphism of finite simplicial sets. But propositions 1.4.15 and 2.4.4 say $j \square i$ is an anodyne extension, so we may apply proposition 8.1.5 to deduce the claim. ■

Corollary 8.1.7 (Internal path spaces)**.** *Let $\mathcal{S}$ be a regular category and let X be an internal Kan complex in $\mathcal{S}$.*

(i) $\Delta^1 \pitchfork X$ *is an internal Kan complex in* $\mathcal{S}$.

(ii) *The morphism* $\Delta^1 \pitchfork X \to \partial\Delta^1 \pitchfork X$ *induced by the boundary inclusion* $\partial\Delta^1 \hookrightarrow \Delta^1$ *is an internal Kan fibration in* $\mathcal{S}$.

(iii) *The morphisms* $\Delta^1 \pitchfork X \to \Delta^0 \pitchfork X$ *induced by the two vertex inclusions* $\Delta^0 \to \Delta^1$ *are internal trivial Kan fibrations in* $\mathcal{S}$. ■

Lemma 8.1.8. *Let $\mathcal{S}$ be a regular category and let $f : X \to Y$ be a morphism of internal Kan complexes in $\mathcal{S}$. Given a commutative square of finite simplicial sets, say*

$$\begin{tikzcd} K' \arrow[r] \arrow[d] & K \arrow[d] \\ L' \arrow[r] & L \end{tikzcd}$$

if $K' \to K$ and $K \cup^{K'} L' \to L$ are monomorphisms, then the induced morphism

$$(K \pitchfork X) \times_{K \pitchfork Y} (L \pitchfork Y) \to (K' \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y)$$

is an internal Kan fibration.

Proof. Let M be the pushout $K \cup^{K'} L'$. Since X (resp. Y) is an internal Kan complex and $K' \to K$ (resp. $M \to L$) is a monomorphism, the induced morphism $K \pitchfork X \to K' \pitchfork X$ (resp. $L \pitchfork Y \to M \pitchfork Y$) is an internal Kan fibration, by proposition 8.1.6. Since $(-) \pitchfork Y$ sends pushout squares to pullback squares, we have a canonical isomorphism $M \pitchfork Y \cong (K \pitchfork Y) \times_{K' \pitchfork Y} (L' \pitchfork Y)$; and in the following commutative diagrams,

$$\begin{tikzcd} (K \pitchfork X) \times_{K \pitchfork Y} (L \pitchfork Y) \arrow[rr] \arrow[d] & & L \pitchfork Y \arrow[d] & \\ (K \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y) \arrow[r] \arrow[d] & & (K \pitchfork Y) \times_{K' \pitchfork Y} (L' \pitchfork Y) \arrow[r] \arrow[d] & L' \pitchfork Y \arrow[d] \\ K \pitchfork X \arrow[rr] & & K \pitchfork Y \arrow[r] & K' \pitchfork Y \end{tikzcd}$$

$$\begin{tikzcd}
(K \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y) \arrow[r] \arrow[d] & K \pitchfork X \arrow[d] \\
(K' \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y) \arrow[r] \arrow[d] & K' \pitchfork X \arrow[d] \\
L' \pitchfork Y \arrow[r] & K' \pitchfork Y
\end{tikzcd}$$

every square is a pullback square, so by proposition 8.1.4, the two morphisms

$$(K \pitchfork X) \times_{K \pitchfork Y} (L \pitchfork Y) \to (K \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y)$$

$$(K \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y) \to (K' \pitchfork X) \times_{K' \pitchfork Y} (L' \pitchfork Y)$$

are internal Kan fibrations, and their composite is the internal Kan fibration we seek. ■

Let D^{n+1} be the relative cylinder $C(\Delta^n, \partial\Delta^n)$, as in definition 1.3.32, and let $j_0, j_1 : \Delta^n \to D^{n+1}$ be the two canonical embeddings.

Definition 8.1.9. A **Dugger–Isaksen weak equivalence** in a regular category $\mathcal{S}$ is a morphism $f : X \to Y$ in $\mathbf{Kan}(\mathcal{S})$ such that the morphism

$$\{\Delta^n, X\} \times_{\{\Delta^n, Y\}} \left\{D^{n+1}, Y\right\} \longrightarrow \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

induced by the commutative diagram

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & \left\{D^{n+1}, Y\right\} \arrow[dr, "\{j_0, Y\}"] \arrow[dd, "\{j_1, Y\}" near end] & \\
& \bullet \arrow[rr] \arrow[dd] & & \{\Delta^n, Y\} \arrow[dd] \\
\{\Delta^n, X\} \arrow[rr, "\{\Delta^n, f\}" near start] \arrow[dr] & & \{\Delta^n, Y\} \arrow[dr] & \\
& \{\partial\Delta^n, X\} \arrow[rr, "\{\partial\Delta^n, f\}"'] & & \{\partial\Delta^n, Y\}
\end{tikzcd}$$

is a regular epimorphism in $\mathcal{S}$.

Remark 8.1.10. If $\mathcal{S} = \mathbf{Set}$, then a Dugger–Isaksen weak equivalence is precisely a weak homotopy equivalence of Kan complexes in the usual sense, by theorem 1.4.35. If $\mathcal{S} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ for a small category $\mathbb{C}$, then an Dugger–Isaksen weak equivalence of internal Kan complexes in $\mathcal{S}$ is the same thing as a componentwise weak homotopy equivalence between componentwise Kan complexes, because limits and colimits in $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ are computed componentwise.

Proposition 8.1.11. *Let $\mathcal{S}$ be a regular category.*

(i) *The class of Dugger–Isaksen weak equivalences in $\mathcal{S}$ contains all isomorphisms in $\mathbf{Kan}(\mathcal{S})$.*

(ii) *The class of Dugger–Isaksen weak equivalences in $\mathcal{S}$ is closed under retracts.*

(iii) *The class of Dugger–Isaksen weak equivalences in $\mathcal{S}$ is closed under finite products.*

Proof. The class of Dugger–Isaksen weak equivalences (considered as a class of *objects* in the category $[\mathbb{2}, \mathbf{Kan}(\mathcal{S})]$) is defined by an internal right lifting property, so we may use the same methods used in the proof of proposition 8.1.4. ■

Proposition 8.1.12. *Let $F : \mathcal{S} \to \mathcal{T}$ be a regular functor.*

(i) *The induced functor $\mathbf{s}F : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{T}$ preserves internal Kan fibrations and internal trivial Kan fibrations.*

(ii) *The induced functor $\mathbf{Kan}(F) : \mathbf{Kan}(\mathcal{S}) \to \mathbf{Kan}(\mathcal{T})$ preserves Brown factorisations and Dugger–Isaksen weak equivalences.*

(iii) *If $F : \mathcal{S} \to \mathcal{T}$ is conservative, then $\mathbf{s}F : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{T}$ reflects internal Kan fibrations and internal trivial Kan fibrations, and $\mathbf{Kan}(F) : \mathbf{Kan}(\mathcal{S}) \to \mathbf{Kan}(\mathcal{T})$ reflects Dugger–Isaksen weak equivalences.*

Proof. These are immediate consequences of the fact that these definitions can be phrased in terms of properties of constructions using only finite limits and regular epimorphisms. ■

Theorem 8.1.13. *Let $\mathcal{S}$ be a regular category.*

(i) *$\mathbf{Kan}(\mathcal{S})$, equipped with the class of Dugger–Isaksen weak equivalences, is a saturated homotopical category.*

(ii) *An internal Kan fibration of internal Kan complexes in $\mathcal{S}$ is an internal trivial Kan fibration if and only if it is also a Dugger–Isaksen weak equivalence.*

(iii) $\mathbf{Kan}(\mathcal{S})$ *is a category of fibrant objects where the weak equivalences are the Dugger–Isaksen weak equivalences and the fibrations are the internal Kan fibrations.*

Proof. (i) and (ii). The claims are known in the case where $\mathcal{S} = \mathbf{Set}$, by remark 8.1.2 and theorems 1.4.27 and 1.4.31. Clearly, the same is true for $\mathcal{S} = \mathbf{Set}^B$ for any set B. On the other hand, if $\mathcal{S}$ is any small regular category, we may apply the classical completeness theorem (A.7.22) and proposition 8.1.12 to reduce to the case of $\mathbf{Set}^B$, so we are done in this case. In general, we may assume $\mathcal{S}$ is small by appealing to the universe axiom[1] and the fact that the properties of being an internal Kan fibration, internal trivial Kan fibration, or Dugger–Isaksen weak equivalence are defined without reference to a choice of universe.

(iii). We have just verified axiom A, proposition 8.1.4 implies axioms B and C are satified, corollary 8.1.7 is axiom D, and axiom E is satisfied by definition. ∎

Lemma 8.1.14. *Let $i : Z \to W$ be a monomorphism of finite simplicial sets and let $f : X \to Y$ be a morphism of internal Kan complexes in a regular category $\mathcal{S}$. Consider the following commutative diagram in $\mathbf{s}\mathcal{S}$,*

$$\begin{tikzcd}
W \pitchfork X \arrow[dr, dashed, "q"] \arrow[drr, bend left, "i \pitchfork \mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_W \pitchfork f"'] & & \\
& L(i, f) \arrow[r] \arrow[d] & Z \pitchfork X \arrow[d, "\mathrm{id}_Z \pitchfork f"] \\
& W \pitchfork Y \arrow[r, "i \pitchfork \mathrm{id}_Y"'] & Z \pitchfork Y
\end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *If $f : X \to Y$ is a Dugger–Isaksen weak equivalence, then so is $q : [W, X] \to L(i, f)$.*

(ii) *If $i : Z \to W$ is an anodyne extension of finite simplicial sets, then $q : [W, X] \to L(i, f)$ is a Dugger–Isaksen weak equivalence.*

Proof. The claim is known in the case where $\mathcal{S} = \mathbf{Set}$, by theorem 1.4.35 and lemma 1.4.33; and in general, we appeal to proposition 8.1.12 and the classical completeness theorem (A.7.22). ∎

[1] See § 0.1.

We have so far avoided discussing homotopy groups for internal Kan complexes. This is because they need not exist in a general regular category: clearly, if we are able to take quotients of internal equivalence relations, then we can construct π_0; and conversely, because internal equivalence relations define internal Kan complexes, being able to construct π_0 implies we can take quotients of internal equivalence relations. This suggests that the right setting for these constructions is an effective regular category.

Proposition 8.1.15. *If X is an internal Kan complex in a regular category $\mathcal{S}$, then the regular image of the morphism $\langle d_1, d_0 \rangle : X_1 \to X_0 \times X_0$ defines an equivalence relation on X_0.*

Proof. For $\mathcal{S} = \mathbf{Set}$, this is a special case of proposition 1.4.17; and in general, we appeal to proposition 8.1.12 and the classical completeness theorem (A.7.22). ■

Proposition 8.1.16. *Let $\mathcal{S}$ be a regular category and let $(-)_0 : \mathbf{s}\mathcal{S} \to \mathcal{S}$ be the functor that sends a simplicial object X in $\mathcal{S}$ to the object X_0 in $\mathcal{S}$.*

(i) *The functor $(-)_0 : \mathbf{s}\mathcal{S} \to \mathcal{S}$ has both a left adjoint* $\operatorname{disc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$ *and a right adjoint* $\operatorname{codisc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$.

(ii) *The functor* $\operatorname{disc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$ *is fully faithful.*

(iii) *For each morphism $f : X \to Y$ in $\mathcal{S}$, the morphism* $\operatorname{disc} f : \operatorname{disc} X \to \operatorname{disc} Y$ *is an internal Kan fibration in $\mathcal{S}$. In particular, each* $\operatorname{disc} X$ *is an internal Kan complex in $\mathcal{S}$.*

Proof. (i). Let $\operatorname{disc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$ be the functor that sends an object X to the constant simplicial object defined by X and let $\operatorname{codisc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$ be the functor that sends an object X to the simplicial object defined by the formula $[n] \mapsto X^{n+1}$. It is straightforward to check that disc is a left adjoint for $(-)_0$ and codisc is a right adjoint for $(-)_0$.

(ii). The functor $\operatorname{disc} : \mathcal{S} \to \mathbf{s}\mathcal{S}$ is fully faithful because $\mathbf{\Delta}$ is connected.

(iii). Since the face and degeneracy operators of $\operatorname{disc} Y$ are isomorphisms, the morphisms $\{\Delta^n, \operatorname{disc} Y\} \to \{\Lambda^n_k, \operatorname{disc} Y\}$ induced by the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ must also be isomorphisms. Thus, the induced morphism

$$\{\Delta^n, \operatorname{disc} X\} \to \{\Lambda^n_k, X\} \times_{\{\Lambda^n_k, Y\}} \{\Delta^n, Y\}$$

is an isomorphism (and a regular epimorphism *a fortiori*). ■

Proposition 8.1.17. *Let $\mathcal{S}$ be an effective regular category and let* $\mathrm{disc} : \mathcal{S} \to \mathbf{Kan}(\mathcal{S})$ *be a left adjoint for the functor* $(-)_0 : \mathbf{Kan}(\mathcal{S}) \to \mathcal{S}$.

(i) *The functor* $\mathrm{disc} : \mathcal{S} \to \mathbf{Kan}(\mathcal{S})$ *has a left adjoint,* $\pi_0 : \mathbf{Kan}(\mathcal{S}) \to \mathcal{S}$.

(ii) *If $f : X \to Y$ is a Dugger–Isaksen weak equivalence in $\mathcal{S}$, then the morphism $\pi_0 f : \pi_0 X \to \pi_0 Y$ is an isomorphism.*

Proof. (i). We define the functor $\pi_0 : \mathbf{Kan}(\mathcal{S}) \to \mathcal{S}$ by the following coequaliser diagram:

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X$$

Note that such coequalisers exist, by proposition 8.1.15 and lemma A.7.25. It is straightforward to verify that $\pi_0 : \mathbf{Kan}(\mathcal{S}) \to \mathcal{S}$ is indeed a left adjoint for $\mathrm{disc} : \mathcal{S} \to \mathbf{Kan}(\mathcal{S})$.

(ii). In the case $\mathcal{S} = \mathbf{Set}$, we may apply corollary 1.3.16 and theorem 1.4.35; and in general, we appeal to proposition 8.1.12 and the classical completeness theorem (A.7.22). ■

Definition 8.1.18. Let n be a positive integer and let X be an internal Kan complex in an effective regular category $\mathcal{S}$.

- The **internal based n-loop fibration** on X is the internal Kan fibration $\Omega^n(X) \to X$ defined by the following pullback diagram in $\mathbf{Kan}(\mathcal{S})$,

$$\begin{tikzcd}
\Omega^n(X) \arrow[r] \arrow[d] & \Delta^n \pitchfork X \arrow[d] \\
X \arrow[r] & \partial\Delta^n \pitchfork X
\end{tikzcd}$$

  where $\Delta^n \pitchfork X \to \partial\Delta^n \pitchfork X$ is the internal Kan fibration induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ and $X \to \partial\Delta^n \pitchfork X$ is the morphism induced by $\partial\Delta^n \to \Delta^0$.

- Let x be a morphism $\mathrm{disc}\, T \to X$ in $\mathbf{Kan}(\mathcal{S})$. The **internal based n-loop space** of (X, x) is the internal Kan complex $\Omega^n(X, x)$ in $\mathcal{S}_{/T}$ defined by the following pullback diagram in $\mathbf{Kan}(\mathcal{S})$:

$$\begin{tikzcd}
\Omega^n(X, x) \arrow[r] \arrow[d] & \Omega^n(X) \arrow[d] \\
\operatorname{disc} T \arrow[r, "x"'] & X
\end{tikzcd}$$

The **internal n-th homotopy group** of (X, x) is the object $\pi_n(X, x) = \pi_0\Omega^n(X, x)$ in $\mathcal{S}_{/T}$.

Remark 8.1.19. It is clear that the above constructions are functorial. Moreover, $\pi_n(X, x)$ admits a natural internal group structure (in $\mathcal{S}_{/T}$); but we do not need this fact.

Lemma 8.1.20. *Let $\mathcal{S}$ be an effective regular category and let $f : X \to Y$ be a morphism in* $\mathbf{Kan}(\mathcal{S})$*. If $f : X \to Y$ is a Dugger–Isaksen weak equivalence, then the induced morphism $\Omega^n(f) : \Omega^n(X) \to \Omega^n(Y)$ is also a Dugger–Isaksen weak equivalence.*

Proof. Since $\mathbf{Kan}(\mathcal{S})$ is a category of fibrant objects (theorem 8.1.13), by Ken Brown's lemma (3.7.8), it suffices to prove that claim in the special case where $f : X \to Y$ is an internal trivial Kan fibration of internal Kan complexes. In that case, we have the following commutative diagram in $\mathbf{Kan}(\mathcal{S})$,

$$\begin{tikzcd}
\Delta^n \pitchfork X \arrow[r] \arrow[d, "\mathrm{id}_{\Delta^n} \pitchfork f"'] & \partial\Delta^n \pitchfork X \arrow[d, "\mathrm{id}_{\partial\Delta^n} \pitchfork f"] & X \arrow[l] \arrow[d, "f"] \\
\Delta^n \pitchfork Y \arrow[r] & \partial\Delta^n \pitchfork Y & Y
\end{tikzcd}$$

and by proposition 8.1.6, all the vertical arrows are internal trivial Kan fibrations in $\mathcal{S}$. Thus, by lemma 3.7.28, the induced morphism $\Omega^n(f) : \Omega^n(X) \to \Omega^n(Y)$ is a Dugger–Isaksen weak equivalence. ■

Theorem 8.1.21. *Let $\mathcal{S}$ be an effective regular category and let $f : X \to Y$ be a morphism in* $\mathbf{Kan}(\mathcal{S})$*. The following are equivalent:*

(i) *$f : X \to Y$ is a Dugger–Isaksen weak equivalence.*

(ii) *$\pi_0 f : \pi_0 X \to \pi_0 Y$ is an isomorphism in $\mathcal{S}$ and, for all positive integers n, all objects T in $\mathcal{S}$, and all morphisms $x : \operatorname{disc} T \to X$ in* $\mathbf{Kan}(\mathcal{S})$*, $\pi_n f : \pi_n(X, x) \to \pi_n(Y, f \circ x)$ is an isomorphism in $\mathcal{S}_{/T}$.*

(iii) $\pi_0 f : \pi_0 X \to \pi_0 Y$ *is an isomorphism in* $\mathcal{S}$ *and, for all positive integers* n, $\pi_n f : \pi_n(X, \bar{x}) \to \pi_n(Y, f \circ \bar{x})$ *is an isomorphism in* $\mathcal{S}_{/X_0}$, *where* $\bar{x} : \operatorname{disc} X_0 \to X$ *is the component of the adjunction counit.*

Proof. Apply the classical completeness theorem (A.7.22) to theorem 1.4.35. ∎

Remark 8.1.22. It should be emphasised that the weak equivalences in $\mathbf{Kan}(\mathcal{S})$ really are *weak* equivalences: Bezem and Coquand [2013] have constructed an internal Kan fibration over $\Delta^1 \odot 1$ in the presheaf topos $\mathcal{S} = [\mathbb{J}, \mathbf{Set}]$ such that the two canonical fibres are not homotopy equivalent. (Of course, the two fibres are weakly homotopy equivalent, because internal weak homotopy equivalences are closed under pullbacks along internal Kan fibrations, by proposition 3.7.12 and theorem 8.1.13.) Furthermore, since 1 is projective in $\mathcal{S}$, (non-)existence in the naïve sense coincides with (non-)existence in sheaf semantics.

We now move on to the problem of internal fibrant replacement. Recall the simplicial sets $\mathrm{N}(P^n)$ defined in paragraph 1.7.1.

Definition 8.1.23. Let $\mathcal{S}$ be a (locally small) category with finite limits. An **extension** of a simplicial object X in $\mathcal{S}$ is a simplicial object $\operatorname{Ex}(X)$ equipped with bijections

$$\mathcal{S}\big(T, \operatorname{Ex}(X)_n\big) \cong \operatorname{Ex}(\mathcal{S}(T, X))_n = \mathbf{sSet}(\mathrm{N}(P^n), \mathcal{S}(T, X))$$

that are natural in both n and T. The **canonical embedding** $i_X : X \to \operatorname{Ex}(X)$ is the unique morphism in $\mathbf{s}\mathcal{S}$ making the diagram below commute:

$$\begin{tikzcd} \mathcal{S}(T, X) \arrow[r, equal] \arrow[d, "\mathcal{S}(T, i_X)"'] & \mathcal{S}(T, X) \arrow[d, "i"] \\ \mathcal{S}(T, \operatorname{Ex}(X)) \arrow[r, "\cong"'] & \operatorname{Ex}(\mathcal{S}(T, X)) \end{tikzcd}$$

where $i : \mathcal{S}(T, X) \to \operatorname{Ex}(\mathcal{S}(T, X))$ is the canonical embedding in $\mathbf{sSet}$.

Remark 8.1.24. The above definition makes sense because finite weighted limits exist in $\mathcal{S}$ and each $\mathrm{N}(P^n)$ is a finite simplicial set. Note that $\operatorname{Ex}(X)$ is unique up to unique isomorphism, so we obtain a functor $\operatorname{Ex} : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$; and the natural transformation $i : \mathrm{id} \Rightarrow \operatorname{Ex}$ in $\mathbf{sSet}$ induces a natural transformation of the same form in $\mathbf{s}\mathcal{S}$. Also note that $\operatorname{Ex} : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$ preserves all degreewise limits that exist in $\mathcal{S}$.

Proposition 8.1.25. *Let $\mathcal{S}$ and $\mathcal{T}$ be categories with finite limits. If $F : \mathcal{S} \to \mathcal{T}$ is a functor that preserves finite limits, then the induced functor $\mathbf{s}F : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{T}$ preserves extensions and canonical embeddings.*

Proof. Obvious. ♦

Lemma 8.1.26. *Let X be a simplicial object in a regular category $\mathcal{S}$. Consider the following pullback diagram in $\mathcal{S}$,*

$$\begin{tikzcd}
\{\Lambda^n_k, \mathrm{Ex}(X)\} \times_{\{\Lambda^n_k, \mathrm{Ex}^2(X)\}} \{\Delta^n, \mathrm{Ex}^2(X)\} \arrow[r] \arrow[d] & \{\Delta^n, \mathrm{Ex}^2(X)\} \arrow[d] \\
\{\Lambda^n_k, \mathrm{Ex}(X)\} \arrow[r, "\{\Lambda^n_k, i_{\mathrm{Ex}(X)}\}"'] & \{\Lambda^n_k, \mathrm{Ex}^2(X)\}
\end{tikzcd}$$

where the morphism $\{\Delta^n, \mathrm{Ex}^2(X)\} \to \{\Lambda^n_k, \mathrm{Ex}^2(X)\}$ is induced by the horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. Then $\{\Lambda^n_k, \mathrm{Ex}(X)\} \times_{\{\Lambda^n_k, \mathrm{Ex}^2(X)\}} \{\Delta^n, \mathrm{Ex}^2(X)\} \to \{\Lambda^n_k, \mathrm{Ex}(X)\}$ is a regular epimorphism in $\mathcal{S}$.

Proof. In the case where $\mathcal{S} = \mathbf{Set}$, the claim is a reformulation of lemma 1.7.6; and in general, we appeal to proposition 8.1.25 and the classical completeness theorem (A.7.22). ■

Lemma 8.1.27. *Let $\mathcal{S}$ be a regular category. The functor $\mathrm{Ex} : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$ preserves internal Kan fibrations and internal trivial Kan fibrations. In particular, it preserves internal Kan complexes.*

Proof. The claim is known in the case where $\mathcal{S} = \mathbf{Set}$, by lemma 1.7.7 and corollary 1.7.10; and in general, we appeal to propositions 8.1.12 and 8.1.25 and the classical completeness theorem (A.7.22). ■

Lemma 8.1.28. *For any internal Kan complex X in a regular category $\mathcal{S}$, the canonical embedding $i_X : X \to \mathrm{Ex}(X)$ is a Dugger–Isaksen weak equivalence in $\mathcal{S}$.*

Proof. The claim is known in the case where $\mathcal{S} = \mathbf{Set}$, by lemma 1.7.9; and in general, we appeal to propositions 8.1.12 and 8.1.25 and the classical completeness theorem (A.7.22). ■

¶ 8.1.29. Let $\mathcal{S}$ be a (locally small) category with limits for finite diagrams and colimits for ω-sequences. For each simplicial object X in $\mathcal{S}$, we define $\operatorname{Ex}^\infty(X)$ to be the colimit of the diagram below:

$$X \xrightarrow{i_X} \operatorname{Ex}(X) \xrightarrow{i_{\operatorname{Ex}(X)}} \operatorname{Ex}^2(X) \xrightarrow{i_{\operatorname{Ex}^2(X)}} \operatorname{Ex}^3(X) \longrightarrow \cdots$$

The above defines a functor $\operatorname{Ex}^\infty : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$ and a natural transformation $i^\infty : \mathrm{id}_{\mathbf{s}\mathcal{S}} \Rightarrow \operatorname{Ex}^\infty$.

Proposition 8.1.30. *Let $\mathcal{S}$ and $\mathcal{T}$ be categories with limits for finite diagrams and colimits for ω-sequences. If $F : \mathcal{S} \to \mathcal{T}$ is a functor that preserves finite limits, then:*

(i) *For each simplicial object X in $\mathcal{S}$, there is a natural comparison morphism of the form* $\operatorname{Ex}(\mathbf{s}F(X)) \to \mathbf{s}F(\operatorname{Ex}(X))$*, and it is compatible with the natural transformation* $i^\infty : \mathrm{id} \Rightarrow \operatorname{Ex}^\infty$.

(ii) *Moreover, if $F : \mathcal{S} \to \mathcal{T}$ preserves colimits for ω-sequences, then the above morphism is an isomorphism.*

Proof. Obvious. ◆

Lemma 8.1.31. *Let $\mathcal{S}$ be a regular category with colimits for ω-sequences. If $\varinjlim : [\omega, \mathcal{S}] \to \mathcal{S}$ preserves finite limits, then the following classes of morphisms are closed under colimits for ω-sequences in $\mathbf{s}\mathcal{S}$:*

- *The class of internal Kan fibrations in $\mathcal{S}$.*
- *The class of internal trivial Kan fibrations in $\mathcal{S}$.*
- *The class of Dugger–Isaksen weak equivalences in $\mathcal{S}$.*

Proof. Colimits commute with colimits, so $\varinjlim : [\omega, \mathcal{S}] \to \mathcal{S}$ always preserves regular epimorphisms. Thus, the hypothesis implies $\varinjlim : [\omega, \mathcal{S}] \to \mathcal{S}$ is a regular functor. On the other hand, the functor $[\omega, \mathcal{S}] \to [\operatorname{ob}\omega, \mathcal{S}]$ induced by restriction along the inclusion $\operatorname{ob}\omega \hookrightarrow \omega$ is a conservative regular functor, so proposition 8.1.12 implies the internal Kan fibrations (resp. internal trivial Kan fibrations, Dugger–Isaksen weak equivalences) in $[\omega, \mathcal{S}]$ are just the componentwise ones. We use the same proposition again to deduce that the indicated classes of morphisms in $\mathbf{s}\mathcal{S}$ are closed under colimits for ω-sequences. ■

Theorem 8.1.32. *Let $\mathcal{S}$ be a regular category with colimits for ω-sequences. If $\varinjlim : [\omega, \mathcal{S}] \to \mathcal{S}$ preserves finite limits, then:*

(i) *For any simplicial object X in $\mathcal{S}$, the simplicial object $\mathrm{Ex}^\infty(X)$ is an internal Kan complex.*

(ii) *For any internal Kan complex X in $\mathcal{S}$, the morphism $i_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is a Dugger–Isaksen weak equivalence.*

(iii) *The functor $\mathrm{Ex}^\infty : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$ preserves internal Kan fibrations, internal trivial Kan fibrations, and finite limits.*

Proof. (i). If $\varinjlim : [\omega, \mathcal{S}] \to \mathcal{S}$ preserves finite limits, then for any finite simplicial set Z, the functor $\{Z, -\} : \mathbf{s}\mathcal{S} \to \mathcal{S}$ preserves colimits for ω-sequences. In particular, we have a commutative diagram of the form below,

$$\begin{tikzcd}
\varinjlim_{m:\omega} \left\{\Delta^n, \mathrm{Ex}^{m+2}(X)\right\} \arrow[r, "\cong"] \arrow[d] & \{\Delta^n, \mathrm{Ex}^\infty(X)\} \arrow[d] \\
\varinjlim_{m:\omega} \left\{\Lambda^n_k, \mathrm{Ex}^{m+2}(X)\right\} \arrow[r, "\cong"'] & \left\{\Lambda^n_k, \mathrm{Ex}^\infty(X)\right\}
\end{tikzcd}$$

where the horizontal arrows are the canonical comparisons and the vertical arrows are induced by the horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. Lemma 8.1.26 gives us the following pullback square in $\mathcal{S}$,

$$\begin{tikzcd}
\left\{\Lambda^n_k, \mathrm{Ex}^{m+1}(X)\right\} \times_{\left\{\Lambda^n_k, \mathrm{Ex}^{m+2}(X)\right\}} \left\{\Delta^n, \mathrm{Ex}^{m+2}(X)\right\} \arrow[r] \arrow[d] & \left\{\Delta^n, \mathrm{Ex}^{m+2}(X)\right\} \arrow[d] \\
\left\{\Lambda^n_k, \mathrm{Ex}^{m+1}(X)\right\} \arrow[r, "\left\{\Lambda^n_k, i_{\mathrm{Ex}^{m+1}(X)}\right\}"'] & \left\{\Lambda^n_k, \mathrm{Ex}^{m+2}(X)\right\}
\end{tikzcd}$$

where $\left\{\Lambda^n_k, \mathrm{Ex}^{m+1}(X)\right\} \times_{\left\{\Lambda^n_k, \mathrm{Ex}^{m+2}(X)\right\}} \left\{\Delta^n, \mathrm{Ex}^{m+2}(X)\right\} \to \left\{\Lambda^n_k, \mathrm{Ex}^{m+1}(X)\right\}$ is a regular epimorphism in $\mathcal{S}$. It is easy to see that

$$\varinjlim_{m:\omega} \left\{\Lambda^n_k, i_{\mathrm{Ex}^{m+1}(X)}\right\} : \varinjlim_{m:\omega} \left\{\Lambda^n_k, \mathrm{Ex}^{m+1}(X)\right\} \to \varinjlim_{m:\omega} \left\{\Lambda^n_k, \mathrm{Ex}^{m+2}(X)\right\}$$

is an isomorphism in $\mathcal{S}$, so $\{\Delta^n, \mathrm{Ex}^\infty(X)\} \to \left\{\Lambda^n_k, \mathrm{Ex}^\infty(X)\right\}$ is indeed a regular epimorphism in $\mathcal{S}$, as required.

(ii). Theorem 8.1.13 and lemma 8.1.28 imply that the composite morphism

$$X \xrightarrow{i_X} \mathrm{Ex}(X) \longrightarrow \cdots \longrightarrow \mathrm{Ex}^m(X) \xrightarrow{i_{\mathrm{Ex}^m(X)}} \mathrm{Ex}^{m+1}(X)$$

is a Dugger–Isaksen weak equivalence, and since $i_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is a colimit for the ω-sequence of these composites, we may apply lemma 8.1.31 to deduce that it is also an internal weak homotopy equivalence.

(iii). Recalling remark 8.1.24 and lemma 8.1.27, an argument similar to the above proves the claim. ■

Definition 8.1.33. Let $\mathcal{S}$ be a σ-pretopos. An **internal weak homotopy equivalence** of simplicial objects in $\mathcal{S}$ is a morphism $f : X \to Y$ in $\mathbf{s}\mathcal{S}$ such that $\mathrm{Ex}^\infty(f) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ is a Dugger–Isaksen weak equivalence.

Remark 8.1.34. Recalling the 2-out-of-3 property of Dugger–Isaksen weak equivalences, theorem 8.1.32 implies that the internal weak homotopy equivalences of internal Kan complexes are precisely the Dugger–Isaksen weak equivalences.

Remark 8.1.35. Lemma 3.1.8 and theorem 8.1.13 imply that $\mathbf{s}\mathcal{S}$, equipped with the class of internal weak homotopy equivalences, is a saturated homotopical category.

Remark 8.1.36. Theorems 8.1.13 and 8.1.32 imply that an internal Kan fibration $p : X \to Y$ is an internal weak homotopy equivalence if and only if $\mathrm{Ex}^\infty(p) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$ is an internal trivial Kan fibration (of internal Kan complexes). In particular, if $p : X \to Y$ is an internal trivial Kan fibration, then so is $\mathrm{Ex}^\infty(p) : \mathrm{Ex}^\infty(X) \to \mathrm{Ex}^\infty(Y)$, and therefore $p : X \to Y$ is an internal weak homotopy equivalence.

Remark 8.1.37. If $\mathcal{S} = \mathbf{Set}$, then an internal weak homotopy equivalence of simplicial objects in $\mathcal{S}$ is precisely a weak homotopy equivalence of simplicial sets in the usual sense, by proposition 1.5.5. Similarly, if $\mathcal{S} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ for a small category $\mathbb{C}$, then an internal weak homotopy equivalence of simplicial objects in $\mathcal{S}$ is the same thing as a componentwise weak homotopy equivalence of simplicial presheaves on $\mathbb{C}$.

Proposition 8.1.38. *Let $\mathcal{S}$ and $\mathcal{T}$ be σ-pretoposes and let $F : \mathcal{S} \to \mathcal{T}$ be a regular functor that preserves colimits for countable diagrams.*

(i) *The induced functor $\mathbf{s}F : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{T}$ preserves internal weak homotopy equivalences.*

(ii) *If $F : \mathcal{S} \to \mathcal{T}$ is conservative, then $\mathbf{s}F : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{T}$ reflects internal weak homotopy equivalences.*

Proof. Combine propositions 8.1.12 and 8.1.30. ■

Theorem 8.1.39. *Let $\mathcal{S}$ be a σ-pretopos. For any simplicial object X in $\mathcal{S}$, the morphism $i_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is an internal weak homotopy equivalence.*

Proof. First, we prove the claim in the case of a Grothendieck topos $\mathcal{E}$. Let $\mathcal{C}$ be the *opposite* of the category of finite simplicial sets. It is not hard to see that the functor $[\mathcal{C}, \mathcal{E}] \to \mathbf{s}\mathcal{E}$ obtained by restricting along $\Delta^\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ itself restricts to an equivalence between the full subcategory of finite-limit-preserving functors $\mathcal{C} \to \mathcal{E}$ and the category $\mathbf{s}\mathcal{E}$ itself. Thus by Diaconescu's theorem,[2] for each simplicial object X in $\mathcal{E}$, there is a functor $x^* : [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ such that x^* is a finite-limit-preserving left adjoint and $X \cong x^* M$, where M is the simplicial object in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ obtained by composing $\Delta^\bullet : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$ and the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$.

Recalling proposition 8.1.38, we see that the canonical embedding $i_X^\infty : X \to \mathrm{Ex}^\infty(X)$ is an internal weak homotopy equivalence of simplicial objects in $\mathcal{E}$ as soon as the universal canonical embedding $i_M^\infty : M \to \mathrm{Ex}^\infty(M)$ is an internal weak homotopy equivalence in $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$. But by remark 8.1.37, this is a special case of theorem 1.7.14. Thus the claim is proved in the case where $\mathcal{E}$ is a Grothendieck topos.

Now, let $\mathcal{S}$ be small σ-pretopos. Then there exist a Grothendieck topos $\mathcal{E}$ and a fully faithful functor $\mathcal{S} \to \mathcal{E}$ that preserves limits for all diagrams and colimits for countable diagrams: see proposition A.7.31. Applying proposition 8.1.38, we then deduce the claim for $\mathcal{S}$ from the result for $\mathcal{E}$ proved above. In general, we may assume $\mathcal{S}$ is small by appealing to the universe axiom.[3] ■

[2] See Lemma 3.2.5 and Theorem 3.2.7 in [Johnstone, 2002, Part B], or Corollary 3 in [ML–M, Ch. VII, §9].

[3] See §0.1.

Corollary 8.1.40. *Let $\mathcal{S}$ be a σ-pretopos. There exist a functor $R : \mathbf{s}\mathcal{S} \to \mathbf{s}\mathcal{S}$ and a natural transformation $i : \mathrm{id}_{\mathbf{s}\mathcal{S}} \Rightarrow R$ such that, for all simplicial objects X in $\mathcal{S}$, RX is an internal Kan complex and $i_X : X \to RX$ is an internal weak homotopy equivalence. Moreover, any such functor R preserves and reflects weak homotopy equivalences.*

Proof. By theorems 8.1.32 and 8.1.39, we may take (R, i) to be $(\mathrm{Ex}^\infty, i^\infty)$. The 2-out-of-3 property of internal weak homotopy equivalences (remark 8.1.35) implies the remainder of the claim. ∎

8.2 Hypercovers

Prerequisites. §§0.5, 1.1, 1.2, 1.3, 8.1.

¶ 8.2.1. Let $\mathcal{C}$ be a small category, let J be a Grothendieck topology on $\mathcal{C}$, let $\mathbf{Psh}(\mathcal{C}) = [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ be the category of presheaves (of sets) on $\mathcal{C}$, let $\mathbf{Sh}(\mathcal{C}, J)$ be the full subcategory of J-sheaves (of sets) on $\mathcal{C}$, and let

$$j^* \dashv j_* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Psh}(\mathcal{C})$$

be the induced adjunction, where $j_* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Psh}(\mathcal{C})$ is the inclusion. Note that the left adjoint $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits. In addition, we define the following subset $\mathcal{I} \subset \operatorname{mor} \mathbf{sPsh}(\mathcal{C})$:

$$\mathcal{I} = \left\{ \partial\Delta^n \odot \mathit{h}_C \hookrightarrow \Delta^n \odot \mathit{h}_C \,\middle|\, n \ge 0, C \in \operatorname{ob} \mathcal{C} \right\}$$

Definition 8.2.2. A **cellular simplicial presheaf** on $\mathcal{C}$ is an object X in $\mathbf{sPsh}(\mathcal{C})$ for which the unique morphism $0 \to X$ is a relative $\mathcal{I}$-cell complex.

Remark 8.2.3. It is not hard to see that every relative $\mathcal{I}$-cell complex is a projective cofibration in $\mathbf{sPsh}(\mathcal{C})$ (in the sense of definition 4.3.15). In particular, every cellular simplicial presheaf on $\mathcal{C}$ is cofibrant in the projective model structure on $\mathbf{sPsh}(\mathcal{C})$.

Remark 8.2.4. If V is a representable presheaf on $\mathcal{C}$, then $\operatorname{disc} V$ is a cellular simplicial presheaf on $\mathcal{C}$.

Lemma 8.2.5. *Let $\mathbf{\Delta}_{\to}$ be the subcategory of $\mathbf{\Delta}$ consisting of the injective maps and let $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Psh}(\mathcal{C})$ be the left Kan extension of a functor $I : {\mathbf{\Delta}_{\to}}^{\mathrm{op}} \to \mathbf{Psh}(\mathcal{C})$ along the inclusion $\mathbf{\Delta}_{\to} \hookrightarrow \mathbf{\Delta}$. If I is degreewise a coproduct of representable presheaves on $\mathcal{C}$, then X is a cellular simplicial presheaf on $\mathcal{C}$ and is also degreewise a coproduct of representable presheaves.*

Proof. By considering the simplicial identities and the formula for left Kan extension given in theorem A.5.15, we find that there are isomorphisms

$$X_n \cong \coprod_{m \le n} D^n_m \odot I_m$$

where D^n_m is the set of surjections $[n] \to [m]$ in $\mathbf{\Delta}$. This proves that X is degreewise a coproduct of representable presheaves if I is. Furthermore, this decomposition yields a sequence of simplicial subpresheaves of X, say

$$0 = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq X^{(2)} \subseteq \cdots$$

such that, for each natural number n, we have a pushout diagram in $\mathbf{sPsh}(\mathcal{C})$ of the form below:

$$\begin{tikzcd}
\partial\Delta^n \odot I_n \arrow[r, hook] \arrow[d] & \Delta^n \odot I_n \arrow[d] \\
X^{(n-1)} \arrow[r, hook] & X^{(n)}
\end{tikzcd}$$

Thus X is indeed an $\mathcal{I}$-cell complex. ■

Definition 8.2.6. A **J-local epimorphism** is a morphism $f : X \to Y$ in $\mathbf{Psh}(\mathcal{C})$ such that $j^* f : j^* X \to j^* Y$ is an epimorphism in $\mathbf{Sh}(\mathcal{C}, J)$.

Proposition 8.2.7.

(i) *The class of J-local epimorphisms contains all presheaf epimorphisms.*

(ii) *The class of J-local epimorphisms is closed under composition.*

(iii) *The class of J-local epimorphisms is closed under pullbacks.*

(iv) *The class of J-local epimorphisms is closed under retracts.*

(v) *The class of J-local epimorphisms is closed under finite products.*

Proof. (i). Since $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves epimorphisms, every presheaf epimorphism is a J-local epimorphism.

(ii). The class of epimorphisms in $\mathbf{Sh}(\mathcal{C}, J)$ is closed under composition, so the class of J-local epimorphisms is also closed under composition.

(iii), (iv), and (v). These are consequences of the fact that $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits and the fact that epimorphisms in $\mathbf{Sh}(\mathcal{C}, J)$ are closed under the same operations. ■

Definition 8.2.8.

- A J-local isomorphism $\mathbf{sPsh}(\mathcal{C})$ is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that $j^* f : j^* X \to j^* Y$ is an isomorphism in $\mathbf{sSh}(\mathcal{C}, J)$.

- A **J-local weak homotopy equivalence** in $\mathbf{sPsh}(\mathcal{C})$ is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that $j^* f : j^* X \to j^* Y$ is an internal weak homotopy equivalence of simplicial objects in $\mathbf{Sh}(\mathcal{C}, J)$.

Lemma 8.2.9. *Componentwise weak homotopy equivalences are J-local weak homotopy equivalences.*

Proof. The functor $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits and all colimits, so we may apply proposition 8.1.38 to remark 8.1.37. ■

Definition 8.2.10.

- A **J-local Kan fibration** is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that $j^* f : j^* X \to j^* Y$ is an internal Kan fibration of simplicial objects in $\mathbf{Sh}(\mathcal{C}, J)$.

- A **J-local trivial Kan fibration** is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that $j^* f : j^* X \to j^* Y$ is an internal trivial Kan fibration of simplicial objects in $\mathbf{Sh}(\mathcal{C}, J)$.

Definition 8.2.11. A **J-locally fibrant simplicial presheaf** on $\mathcal{C}$ is an object X in $\mathbf{sPsh}(\mathcal{C})$ such that the unique morphism $X \to 1$ is a J-local Kan fibration.

Proposition 8.2.12.

(i) *The class of J-local Kan fibrations (resp. J-local trivial Kan fibrations) contains all J-local isomorphisms.*

(ii) *The class of J-local Kan fibrations (resp. J-local trivial Kan fibrations) is closed under composition.*

(iii) *The class of J-local Kan fibrations (resp. J-local trivial Kan fibrations) is closed under pullbacks.*

(iv) *The class of J-local Kan fibrations (resp. J-local trivial Kan fibrations) is closed under retracts.*

(v) *The class of J-local Kan fibrations (resp. J-local trivial Kan fibrations) is closed under finite products.*

Proof. These are immediate consequences of proposition 8.1.4. ■

Lemma 8.2.13.

(i) *Every componentwise Kan fibration in* $\mathbf{sPsh}(C)$ *is a* J*-local Kan fibration.*

(ii) *Every componentwise trivial Kan fibration in* $\mathbf{sPsh}(C)$ *is a* J*-local trivial Kan fibration.*

Proof. (i). Let $f : X \to Y$ be an componentwise Kan fibration in $\mathbf{sPsh}(C)$, let n be a natural number, and let $0 \le k \le n$. We wish to show that the following diagram is a weak pullback square in $\mathbf{sSh}(C, J)$,

$$\begin{tikzcd} \{\Delta^n, j^*X\} \arrow[r, "\{\Delta^n, j^*f\}"] \arrow[d] & \{\Delta^n, j^*Y\} \arrow[d] \\ \left\{\Lambda^n_k, j^*X\right\} \arrow[r, "\{\Lambda^n_k, f\}"'] & \left\{\Lambda^n_k, j^*Y\right\} \end{tikzcd}$$

where the vertical arrows are induced by the horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, and since $j^* : \mathbf{Psh}(C) \to \mathbf{sSh}(C, J)$ preserves finite limits and epimorphisms, it suffices to verify that the diagram below is a weak pullback square in $\mathbf{sPsh}(C)$:

$$\begin{tikzcd} \{\Delta^n, X\} \arrow[r, "\{\Delta^n, f\}"] \arrow[d] & \{\Delta^n, Y\} \arrow[d] \\ \left\{\Lambda^n_k, X\right\} \arrow[r, "\{\Lambda^n_k, f\}"'] & \left\{\Lambda^n_k, Y\right\} \end{tikzcd}$$

But limits and colimits in $\mathbf{Psh}(C)$ can be computed componentwise, so the claim can be reduced to lemma A.3.2.

(ii). An analogous proof works. ■

Theorem 8.2.14 (Dugger–Isaksen)**.** *Let* $f : X \to Y$ *be a morphism in* $\mathbf{sPsh}(C)$*. If* X *and* Y *are* J*-locally fibrant, then the following are equivalent:*

(i) *The morphism* $f : X \to Y$ *is a* J*-local weak homotopy equivalence.*

(ii) *For any presheaf* V *on* C *and any natural number* n*, given a commutative diagram in* $\mathbf{sPsh}(C)$ *of the form below,*

$$\begin{tikzcd} \partial\Delta^n \odot V \arrow[r, "\partial x"] \arrow[d, hook] & X \arrow[d, "f"] \\ \Delta^n \odot V \arrow[r, "y"'] & Y \end{tikzcd}$$

there exist a J-local epimorphism $p : U \to V$ in $\mathbf{Psh}(\mathcal{C})$ and morphisms $x : \Delta^n \odot U \to X$ and $h : D^{n+1} \odot U \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that the following diagrams commute,

$$\begin{tikzcd} \Delta^n \odot U \arrow[r, "x"] \arrow[d, "j_1 \odot \mathrm{id}_U"'] & X \arrow[d, "f"] \\ D^{n+1} \odot U \arrow[r, "h"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} \partial\Delta^n \odot U \arrow[r, "\mathrm{id}_{\partial\Delta^n} \odot p"] \arrow[d, hook] & \partial\Delta^n \odot V \arrow[d, "\partial x"] \\ \Delta^n \odot U \arrow[r, "x"'] & X \end{tikzcd} \qquad \begin{tikzcd} \Delta^n \odot U \arrow[r, "\mathrm{id}_{\Delta^n} \odot p"] \arrow[d, "j_0 \odot \mathrm{id}_U"'] & \Delta^n \odot V \arrow[d, "y"] \\ D^{n+1} \odot U \arrow[r, "h"'] & Y \end{tikzcd}$$

where D^{n+1} is the relative cylinder $C(\Delta^n, \partial\Delta^n)$ and $j_0, j_1 : \Delta^n \to D^{n+1}$ are the two canonical embeddings.

Proof. (i) $\Rightarrow$ (ii). By adjointness, the given commutative diagram defines a morphism

$$V \to \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

and since $j^* f : j^* X \to j^* Y$ is a Dugger–Isaksen weak equivalence of internal Kan complexes in $\mathbf{Sh}(\mathcal{C}, J)$ (by remark 8.1.34), the canonical morphism

$$\{\Delta^n, X\} \times_{\{\Delta^n, Y\}} \left\{D^{n+1}, Y\right\} \longrightarrow \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

is a J-local epimorphism; but by proposition 8.2.7, the class of J-local epimorphisms is closed under pullbacks, so the required J-local epimorphism $p : U \to V$ and morphisms $x : \Delta^n \odot U \to X$ and $h : D^{n+1} \odot U \to Y$ indeed exist.

(ii) $\Rightarrow$ (i). By considering the universal instance where $V = \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$, we see that $j^* f : j^* X \to j^* Y$ must be a Dugger–Isaksen weak equivalence of internal Kan complexes in $\mathbf{Sh}(\mathcal{C}, J)$, and so $f : X \to Y$ is a J-local weak homotopy equivalence. ■

Definition 8.2.15. A **J-hypercover** of a presheaf V on $\mathcal{C}$ is a simplicial presheaf U on $\mathcal{C}$ equipped with a J-local trivial Kan fibration $U \to \operatorname{disc} V$, such that each U_n is a coproduct (in $\mathbf{Psh}(\mathcal{C})$) of representable presheaves on $\mathcal{C}$.

Remark 8.2.16. Propositions 8.1.16 and 8.2.12 imply that the underlying simplicial presheaf of a J-hypercover is J-locally fibrant.

¶ 8.2.17. Given a presheaf X on $\mathcal{C}$, we define $|X| = \sum_{C \in \operatorname{ob} \mathcal{C}} |X(C)|$; and for a simplicial presheaf X on $\mathcal{C}$, we define $|X| = \sum_{n \geq 0} |X_n|$.

Lemma 8.2.18. *Let $f : X \to Y$ be a J-local epimorphism of presheaves on $\mathcal{C}$. For each morphism $y : V \to Y$, there exist a projective object U in $\mathbf{Psh}(\mathcal{C})$, a J-local epimorphism $p : U \to V$, and a morphism $x : U \to X$ making the following diagram commute:*

$$\begin{tikzcd} U \arrow[r, dashed, "x"] \arrow[d, dashed, "p"'] & X \arrow[d, "f"] \\ V \arrow[r, "y"'] & Y \end{tikzcd}$$

Moreover, we may choose U so that it is a coproduct of $\leq |V|$ representable presheaves.

Proof. Let S be the set of all pairs (C, v) where C is an object in $\mathcal{C}$ and v is an element of $V(C)$ such that $y_C(c)$ is in the presheaf image of $f : X \to Y$. Clearly, $|S| \leq |V|$. Let $U = \coprod_{(C,v) \in S} h_C$ and let $p : U \to V$ be the evident projection. By construction, the presheaf image of $p : U \to V$ is the preimage under $y : V \to Y$ of the presheaf image of $f : X \to Y$, so we may apply proposition 8.2.7 to deduce that $p : U \to V$ is a J-local epimorphism. The construction also ensures the existence of a presheaf morphism $x : U \to X$ making the given diagram commute. ■

Lemma 8.2.19. *If $f : X \to Y$ is a J-local trivial Kan fibration, then the component $f_0 : X_0 \to Y_0$ is a J-local epimorphism of presheaves on $\mathcal{C}$.*

Proof. Consider the induced morphism $j^* f : j^* X \to j^* Y$ in $\mathbf{sSh}(\mathcal{C}, J)$. By definition, $j^* f$ is an internal trivial Kan fibration, so the diagram

$$\begin{tikzcd} \left\{\Delta^0, j^* X\right\} \arrow[r, "\{\Delta^0, j^* f\}"] \arrow[d, "\{i, j^* X\}"'] & \left\{\Delta^0, j^* Y\right\} \arrow[d, "\{i, j^* Y\}"] \\ \left\{\partial\Delta^0, j^* X\right\} \arrow[r, "\{\partial\Delta^0, j^* f\}"'] & \left\{\partial\Delta^0, j^* Y\right\} \end{tikzcd}$$

induced by the boundary inclusion $i : \partial\Delta^0 \hookrightarrow \Delta^0$ is a weak pullback square in $\mathbf{Sh}(\mathcal{C}, J)$; but $\partial\Delta^0$ is the initial object in $\mathbf{sSet}$, so $\left\{\partial\Delta^0, j^* X\right\} \to \left\{\partial\Delta^0, j^* Y\right\}$ is an isomorphism. Hence, $\left\{\Delta^0, j^* X\right\} \to \left\{\Delta^0, j^* Y\right\}$ is an epimorphism in $\mathbf{Sh}(\mathcal{C}, J)$, and therefore $f_0 : X_0 \to Y_0$ is a J-local epimorphism. ■

Lemma 8.2.20. *Let $f : X \to Y$ be a J-local trivial Kan fibration and let $y : V \to Y$ be any morphism in $\mathbf{sPsh}(\mathcal{C})$. There exist a cellular simplicial presheaf U on $\mathcal{C}$, a J-local trivial Kan fibration $p : U \to V$, and a morphism $x : U \to X$ making the following diagram commute:*

$$\begin{tikzcd} U \arrow[r, dashed, "x"] \arrow[d, dashed, "p"'] & X \arrow[d, "f"] \\ V \arrow[r, "y"'] & Y \end{tikzcd}$$

Moreover, we may choose U so that each U_n is a coproduct of representable presheaves and $|U| \le \kappa$, where κ is an infinite cardinal such that $|V| \le \kappa$ and $\left|h_C\right| \le \kappa$ for all C in $\mathcal{C}$.

Proof. We construct U, p, and x by induction. To begin, observe that lemmas 8.2.18 and 8.2.19 imply we have a diagram of form below,

$$\begin{tikzcd} U_0' \arrow[r, dashed, "x_0'"] \arrow[d, dashed, "p_0'"'] & X_0 \arrow[d, "f_0"] \\ V_0 \arrow[r, "y_0"'] & Y_0 \end{tikzcd}$$

where U_0' is a coproduct of $\le \kappa$ representable presheaves, hence $\left|U_0'\right| \le \kappa$, and $p' : U_0' \to V_0$ is a J-local epimorphism. Let $U_0 = U_0'$, $p_0 = p_0'$, and $x_0 = x_0'$.

Now, suppose we have defined U, p, and x up to degree $n-1$. It is not hard to verify that $\{\partial\Delta^n, X\}$ depends on X only up to degree $n-1$, so $\{\partial\Delta^n, U\}$ is well-defined. Since $f : X \to Y$ is a J-local trivial Kan fibration, the canonical morphism $\{\Delta^n, X\} \to \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$ is a J-local epimorphism, so we may use lemma 8.2.18 to choose a presheaf U_n' and morphisms $x_n' : U_n' \to X_n$ and $p_n' : U_n' \to \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$ making the diagram below commute,

$$\begin{tikzcd} U_n' \arrow[rr, dashed, "x_n'"] \arrow[d, dashed, "p_n'"'] & & \{\Delta^n, X\} \arrow[d] \\ \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, V\}} \{\Delta^n, V\} \arrow[rr] & & \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\} \end{tikzcd}$$

where $p_n' : U_n' \to \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, V\}} \{\Delta^n, V\}$ is a J-local epimorphism. Clearly, there is a monomorphism $\{\partial\Delta^n, U\} \to \left(U_{n-1}\right)^{n+1}$, so $|\{\partial\Delta^n, U\}| \le \left|U_{n-1}\right|^{n+1} \le \kappa$; and $\left|V_n\right| \le \kappa$ by hypothesis, so U_n' can be chosen to be a coproduct of $\le \kappa$ representable presheaves; note we then have $\left|U_n'\right| \le \kappa$.

Let $U_n = \coprod_{m \le n} D^n_m \odot U'_m$, where D^n_m is the set of surjections $[n] \to [m]$ in $\mathbf{\Delta}$. Since each D^n_m is finite and $\left|U'_m\right| \le \kappa$ for each m, we also have $\left|U_n\right| \le \kappa$. There are evident degeneracy operators $U_{n-1} \to U_n$ induced by composition in $\mathbf{\Delta}$, and we define the face operators $U_n \to U_{n-1}$ using the simplicial identities on the degenerate part and the composite of $p'_n : U'_n \to \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, V\}} \{\Delta^n, V\}$ and the evident projections on the non-degenerate part. We also define $p_n : U_n \to V_n$ in a similar fashion. Using the various $x'_m : U'_m \to \{\Delta^m, X\}$, we can define a morphism $x_n : U_n \to X_n$ such that the following diagram commutes:

$$\begin{tikzcd} U_n \arrow[r, dashed, "x_n"] \arrow[d, dashed, "p_n"'] & X_n \arrow[d, "f_n"] \\ V_n \arrow[r, "y_n"'] & Y_n \end{tikzcd}$$

In the above, note that we can choose $p_n : U_n \to V_n$ and $x_n : U_n \to X_n$ compatible with all the face and degeneracy operators so far.

We thus obtain simplicial presheaf morphisms $p : U \to V$ and $x : U \to X$ making the announced diagram commute, and by construction, the simplicial presheaf U satisfies the required conditions. (Note that U is cellular by the proof of lemma 8.2.5.) Moreover, the induced morphism $U_n \to \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, V\}} \{\Delta^n, V\}$ is a J-local epimorphism because $p'_n : U'_n \to \{\partial\Delta^n, U\} \times_{\{\partial\Delta^n, V\}} \{\Delta^n, V\}$ is, so $p : U \to V$ is indeed a J-local trivial Kan fibration. ■

Definition 8.2.21. Let V be a presheaf on $\mathcal{C}$. A **refinement** of a J-hypercover $f : X \to \operatorname{disc} V$ is a J-hypercover $p : U \to \operatorname{disc} V$ and a morphism $x : U \to X$ in $\mathbf{sPsh}(\mathcal{C})$ such that $p = f \circ x$. We say that $p : U \to \operatorname{disc} V$ **refines** $f : X \to \operatorname{disc} V$ if there exists such a morphism $x : U \to X$.

Remark 8.2.22. Unlike the topos-theoretic situation with 'J-dense subobjects' instead of 'J-hypercovers', the morphism $x : U \to X$ need not be unique (if it exists). However, we can prove something weaker: see proposition 8.5.7.

Proposition 8.2.23. *Let V be a presheaf on $\mathcal{C}$. If $f : X \to \operatorname{disc} V$ is a J-hypercover of V, then there exist a cellular simplicial presheaf U and a J-hypercover $p : U \to \operatorname{disc} V$ that refines $f : X \to \operatorname{disc} V$, such that $|U| \le \kappa$, where κ is an infinite cardinal such that $|V| \le \kappa$ and $\left|h_C\right| \le \kappa$ for all C in $\mathcal{C}$.*

Proof. This is a special case of lemma 8.2.20. ■

Proposition 8.2.24. *Let $f : X \to Y$ be a J-local trivial Kan fibration and let $i : Z \to W$ be a monomorphism between finite simplicial sets. For any simplicial presheaf V on $\mathcal{C}$, given a commutative square in* **sPsh**$(\mathcal{C})$ *of the form below,*

$$\begin{tikzcd} Z \odot V \arrow[r, "z"] \arrow[d, "i \odot \mathrm{id}_V"'] & X \arrow[d, "f"] \\ W \odot V \arrow[r, "w"'] & Y \end{tikzcd}$$

there exist a cellular simplicial presheaf U, a J-local trivial Kan fibration $p : U \to V$ and a morphism $h : W \odot U \to X$ making the diagram below commute:

$$\begin{tikzcd} Z \odot U \arrow[r, "\mathrm{id}_Z \odot p"] \arrow[d, "i \odot \mathrm{id}_U"'] & Z \odot V \arrow[r, "z"] & X \arrow[d, "f"] \\ W \odot U \arrow[r, "\mathrm{id}_W \odot p"'] \arrow[urr, dashed, "h"] & W \odot V \arrow[r, "w"'] & Y \end{tikzcd}$$

Moreover, we may choose U so that each U_n is a coproduct of representable presheaves and $|U| \leq \kappa$, where κ is an infinite cardinal such that $|V| \leq \kappa$ and $\left|\mathit{h}_C\right| \leq \kappa$ for all C in $\mathcal{C}$.

Proof. By adjunction, the we obtain a commutative diagram in **sPsh**$(\mathcal{C})$ of the following form:

$$\begin{tikzcd} V \arrow[dr, dashed, "l"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) \arrow[r] \arrow[d] & W \pitchfork Y \arrow[d, "i \pitchfork \mathrm{id}_Y"] \\ & Z \pitchfork X \arrow[r, "\mathrm{id}_Z \pitchfork f"'] & Z \pitchfork Y \end{tikzcd}$$

Thus, it suffices to find a J-local trivial Kan fibration $p : U \to V$ and a morphism $\tilde{l} : U \to W \pitchfork X$ making the diagram below commute;

$$\begin{tikzcd} U \arrow[r, dashed, "\tilde{l}"] \arrow[d, dashed, "p"'] & W \pitchfork X \arrow[d] \\ V \arrow[r, "l"'] & (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) \end{tikzcd}$$

but proposition 8.1.6 implies that $W \pitchfork X \to (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)$ is a J-local trivial Kan fibration, so we may apply lemma 8.2.20. ■

Proposition 8.2.25. *Let X and Y be J-locally fibrant simplicial presheaves on $\mathcal{C}$, let L be a finite simplicial set, and let K be a simplicial subset of L. Given a J-local weak homotopy equivalence $f : X \to Y$, for any simplicial presheaf V and any commutative diagram in* $\mathbf{sPsh}(\mathcal{C})$ *of the form below,*

$$\begin{tikzcd} K \odot V \arrow[r, "\partial x"] \arrow[d, hook] & X \arrow[d, "f"] \\ L \odot V \arrow[r, "y"'] & Y \end{tikzcd}$$

there exist a cellular simplicial presheaf U, a J-local trivial Kan fibration $p : U \to V$, and morphisms $x : L \odot U \to X$ and $h : C(L, K) \odot U \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ *such that the following diagrams commute,*

$$\begin{tikzcd} L \odot U \arrow[r, "x"] \arrow[d, "j_1 \odot \mathrm{id}_U"'] & X \arrow[d, "f"] \\ C(L, K) \odot U \arrow[r, "h"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} K \odot U \arrow[r, "\mathrm{id}_K \odot p"] \arrow[d, hook] & K \odot V \arrow[d, "\partial x"] \\ L \odot U \arrow[r, "x"'] & X \end{tikzcd} \qquad \begin{tikzcd} L \odot U \arrow[r, "\mathrm{id}_{\Delta^n} \odot p"] \arrow[d, "j_0 \odot \mathrm{id}_U"'] & L \odot V \arrow[d, "y"] \\ C(L, K) \odot U \arrow[r, "h"'] & Y \end{tikzcd}$$

where $C(L, K)$ is the relative cylinder and $j_0, j_1 : L \to C(L, K)$ are the two canonical embeddings. Moreover, we may choose U so that each U_n is a coproduct of representable presheaves and $|U| \le \kappa$, where κ is an infinite cardinal such that $|V| \le \kappa$ and $\left|\hbar_C\right| \le \kappa$ for all C in $\mathcal{C}$.

Proof. Consider the commutative diagram in $\mathbf{sPsh}(\mathcal{C})$ shown below,

$$\begin{tikzcd} C(L, K) \pitchfork X \arrow[r] \arrow[d, "j_0 \pitchfork \mathrm{id}_X"'] & (L \pitchfork X) \times_{L \pitchfork Y} (C(L, K) \pitchfork Y) \arrow[d] \\ L \pitchfork X \arrow[r] & (K \pitchfork X) \times_{K \pitchfork Y} (L \pitchfork Y) \end{tikzcd}$$

where the top horizontal arrow is induced by $j_1 : \Delta^n \to D^{n+1}$, the bottom horizontal arrow is induced by the inclusion $K \hookrightarrow L$, and the vertical arrows are induced by the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & L \arrow[d, "j_1"] \\ L \arrow[r, "j_0"'] & C(L, K) \end{tikzcd}$$

Note that the canonical comparison $L \cup^K L \to C(L, K)$ is a monomorphism, so by lemma 8.1.8, the right vertical arrow is a J-local Kan fibration; moreover, proposition 8.1.6 implies that all the objects in the diagram are J-locally fibrant simplicial presheaves. Since $f : X \to Y$ is a J-local weak homotopy equivalence, we may use lemma 8.1.14 to deduce that both horizontal arrows are J-local weak homotopy equivalences; and $j_0 : L \to C(L, K)$ is an anodyne extension, so $j_0 \pitchfork \mathrm{id}_X : C(L, K) \pitchfork X \to L \pitchfork X$ is a J-local trivial Kan fibration. Thus, the 2-out-of-3 property of J-local weak homotopy equivalences (remark 8.1.35) and theorem 8.1.13 imply that the morphism

$$(L \pitchfork X) \times_{L \pitchfork Y} (C(L, K) \pitchfork Y) \to (K \pitchfork X) \times_{K \pitchfork Y} (L \pitchfork Y)$$

is a J-local trivial Kan fibration. Thus, by adjointness and lemma 8.2.20, we can obtain the required $p : U \to V$, $x : L \odot U \to X$, and $h : C(L, K) \odot U \to Y$. ■

Observe that we can regard a morphism of simplicial presheaves of the form $X \to \operatorname{disc} Y$ as a simplicial object in the slice category $\mathbf{Psh}(C)_{/Y}$.

Definition 8.2.26. A **bounded J-hypercover** of a presheaf V on C is a J-hypercover $p : U \to \operatorname{disc} V$ for which there exists a natural number n such that, for all $k > n$, the morphism

$$\{\Delta^k, p\} \to \{\partial\Delta^k, p\}$$

induced by the boundary inclusion $\partial\Delta^k \hookrightarrow \Delta^k$ is an isomorphism in the slice category $\mathbf{Psh}(C)_{/V}$. The **height** of a bounded hypercover $p : U \to \operatorname{disc} V$ is the least such n.

Lemma 8.2.27. *Let X be a presheaf on C and let $\check{U}(X)$ be the simplicial presheaf on C defined by the formula below:*

$$\check{U}(X)_n = [n] \pitchfork X$$

If V is the presheaf image of the unique morphism $X \to 1$, then there is a unique morphism $\check{U}(X) \to \operatorname{disc} V$ in $\mathbf{sPsh}(C)$ and it is a componentwise trivial Kan fibration.

Proof. Since limits in $\mathbf{Psh}(C)$ can be computed componentwise, it suffices to prove the claim in the case where C is the terminal category $\mathbb{1}$; so we may as well replace $\mathbf{Psh}(C)$ with $\mathbf{Set}$ and $\mathbf{sPsh}(C)$ with $\mathbf{sSet}$.

There are two cases. If X is empty, then the claim is trivial. Otherwise, X is non-empty, so V is a singleton and $\check{U}(X)$ is a contractible Kan complex (because it is the nerve of a contractible groupoid). Either way, there is a unique morphism $\check{U}(X) \to V$ and it is a trivial Kan fibration (by proposition 1.5.8), as required. ■

Definition 8.2.28. The **Čech nerve** of a morphism $f : X \to Y$ in $\mathbf{Psh}(\mathcal{C})$ is the morphism $p : \check{U}(f) \to \operatorname{disc} Y$ in $\mathbf{sPsh}(\mathcal{C})$ corresponding to the simplicial object in the slice category $\mathbf{Psh}(\mathcal{C})_{/Y}$ defined as in lemma 8.2.27.

Lemma 8.2.29. *Let $f : X \to Y$ be a morphism in $\mathbf{Psh}(\mathcal{C})$, let V be the presheaf image of $f : X \to Y$, and let $\overline{V}$ be the J-image of $f : X \to Y$, i.e. the subpresheaf of Y where $y \in \overline{V}(C)$ if and only if the sieve*

$$\{c \in \mathcal{C}(C', C) \mid \exists x \in X(C').\, c^*(y) = f(x)\}$$

is J-covering. Then $V \subseteq \overline{V}$, and the inclusion $V \hookrightarrow \overline{V}$ is a J-local isomorphism.

Proof. It is clear that $V \subseteq \overline{V}$. We know that $j^*V \to j^*\overline{V}$ must be a monomorphism in $\mathbf{Sh}(\mathcal{C}, J)$, so it suffices to prove that it is also an epimorphism; for this, we can use proposition A.3.26 and the fact that there is an orthogonal (epi, mono)-factorisation system on $\mathbf{Sh}(\mathcal{C}, J)$. ■

Proposition 8.2.30. *Let $p : \check{U}(f) \to \operatorname{disc} Y$ be the Čech nerve of a morphism $f : X \to Y$ in $\mathbf{Psh}(\mathcal{C})$ and let $\bar{y} : \overline{V} \to Y$ be the J-image of $f : X \to Y$.*

(i) *We have $p = (\operatorname{disc} \bar{y}) \circ q$ for a unique morphism $\bar{q} : X \to \operatorname{disc} \overline{V}$, and $\bar{q} : \check{U}(f) \to \operatorname{disc} \overline{V}$ is a J-local trivial Kan fibration.*

(ii) *Assuming $\mathcal{C}$ has pullbacks, if Y is a representable presheaf and X is a coproduct of representable presheaves, then $\bar{q} : \check{U}(f) \to \operatorname{disc} \overline{V}$ is a J-hypercover of height 0.*

Proof. (i). Let $y : V \to Y$ be the presheaf image of $f : X \to Y$. By lemma 8.2.29, the inclusion $V \hookrightarrow \overline{V}$ is a J-local isomorphism, hence a J-local trivial Kan fibration *a fortiori* (by proposition 8.2.12). So, recalling lemma 8.2.13, it suffices to show that $p : \check{U}(f) \to \operatorname{disc} Y$ factors as $p = (\operatorname{disc} y) \circ q$ for a componentwise trivial Kan fibration $q : \check{U}(f) \to \operatorname{disc} V$. But the property of being a componentwise trivial Kan fibration is stable under slicing, so we may assume that Y is terminal; the claim then reduces to lemma 8.2.27.

(ii). Since pullbacks distribute over coproducts in $\mathbf{Psh}(\mathcal{C})$, each U_n is the coproduct of an iterated fibred product of representable presheaves; but the Yoneda embedding $\mathcal{C} \to \mathbf{Psh}(\mathcal{C})$ is fully faithful and preserves pullbacks, so (iterated) fibred products of representable presheaves are themselves representable. Thus $\bar{q} : \check{U}(f) \to \operatorname{disc} \overline{V}$ is a J-hypercover, and by construction it is of height 0. ■

8.3 Stacks and hypersheaves

Prerequisites. §§1.1, 1.5, 2.4, 4.1, 5.6, 8.1, 8.2, A.3.

¶ 8.3.1. We continue use the notation set up in paragraph 8.2.1. In addition, we will often refer the Bousfield–Kan (i.e. projective) and Heller (i.e. injective) model structures on $\mathbf{sPsh}(\mathcal{C})$ (tacitly identified with $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]$): see theorems 1.9.13 and 1.9.14.

Definition 8.3.2. A simplicial presheaf F on $\mathcal{C}$ satisfies the **descent condition** for a morphism $p : U \to V$ in $\mathbf{sPsh}(\mathcal{C})$ if the induced morphism of derived hom-spaces[4]

$$\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(p, F) : \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(V, F) \to \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(U, F)$$

is an isomorphism in $\operatorname{Ho} \mathbf{sSet}$.

Remark 8.3.3. Assuming F is an injective-fibrant simplicial presheaf on $\mathcal{C}$, F satisfies the descent condition for $p : U \to V$ if and only if the induced morphism of hom-spaces

$$\underline{\mathbf{sPsh}_{\mathcal{C}}}(p, F) : \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, F) \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(U, F)$$

is (half of) a homotopy equivalence of Kan complexes: this follows from theorem 2.4.9, because all simplicial presheaves are injective-cofibrant.

Similarly, assuming F is a projective-fibrant simplicial presheaf on $\mathcal{C}$, if U and V are projective-cofibrant simplicial presheaves (e.g. cellular simplicial presheaves, by remark 8.2.3), then F satisfies the descent condition with respect to $p : U \to V$ if and only if if the induced morphism of hom-spaces is (half of) a homotopy equivalence of Kan complexes.

Definition 8.3.4. A **J-stack of ∞-groupoids** on $\mathcal{C}$ is a simplicial presheaf on $\mathcal{C}$ that satisfies the descent condition for all morphisms of the form $\operatorname{disc} p : \operatorname{disc} U \to \operatorname{disc} h_C$ where $p : U \to h_C$ is a monomorphism in $\mathbf{Psh}(\mathcal{C})$ that is a J-local epimorphism and C is an object in $\mathcal{C}$.

[4] — with respect to either the Bousfield–Kan or Heller model structure on $\mathbf{sPsh}(\mathcal{C})$; since the weak equivalences in both model structures coincide, so do the derived hom-spaces.

¶ 8.3.5. What we have defined above is perhaps more properly called a '*split J-stack of ∞-groupoids*', as we are requiring it to be a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ in the ordinary (i.e. strict) sense. Nonetheless, for brevity, we will simply refer to them as 'J-stacks'.

Remark 8.3.6. Given a componentwise weak homotopy equivalence $X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$, it is clear that X is a J-stack if and only if Y is a J-stack. Thus, by proposition 4.1.28, the full subcategory of $\mathbf{sPsh}(\mathcal{C})$ spanned by the J-stacks inherits the injective and projective model structures on $\mathbf{sPsh}(\mathcal{C})$ (and hence admits the structure of a derivable category in two ways).

Remark 8.3.7. A simplicial presheaf of the form $\operatorname{disc} F$ is a J-stack if and only if F is a J-sheaf (of sets): indeed, $\operatorname{disc} F$ is an injective-fibrant simplicial presheaf, so by applying remark 8.3.3, we find that $\operatorname{disc} F$ is a J-stack if and only if the induced morphism of hom-sets

$$\mathbf{Psh}_{\mathcal{C}}(p, F) : \mathbf{Psh}_{\mathcal{C}}\left(h_C, F\right) \to \mathbf{Psh}_{\mathcal{C}}(U, F)$$

is a bijection for all monomorphisms $p : U \to h_C$ that are J-local epimorphisms and all objects C in $\mathcal{C}$, which is precisely the J-sheaf condition.

Definition 8.3.8. A **weak J-stack equivalence** in $\mathbf{sPsh}(\mathcal{C})$ is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(f, F) : \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(Y, F) \to \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(X, F)$$

is an isomorphism in $\operatorname{Ho}\mathbf{sSet}$ for all J-stacks F.

Remark 8.3.9. Clearly, every componentwise weak homotopy equivalence is also a weak J-stack equivalence. Conversely, (the proof of) lemma 5.6.6 shows that every weak J-stack equivalence between J-stacks is a componentwise weak homotopy equivalence.

Theorem 8.3.10. *Let S be the class of morphisms of the form* $\operatorname{disc} p : \operatorname{disc} U \to \operatorname{disc} h_C$ *where $p : U \to h_C$ is a monomorphism in* $\mathbf{Psh}(\mathcal{C})$ *that is a J-local epimorphism and C is an object in $\mathcal{C}$.*

(i) *The left Bousfield localisation of the Heller model structure on* $\mathbf{sPsh}(\mathcal{C})$ *with respect to S exists. The localised model structure is called the* **injective model structure for J-stacks of ∞-groupoids**, *and the fibrant objects are the injective-fibrant J-stacks.*

(ii) *The left Bousfield localisation of the Bousfield–Kan model structure on* $\mathbf{sPsh}(\mathcal{C})$ *with respect to* $\mathcal{S}$ *exists. The localised model structure is called the* **projective model structure for J-stacks of ∞-groupoids**, *and the fibrant objects are the projective-fibrant J-stacks.*

In either case:

- *The localised model structure is left proper, combinatorial, and simplicial.*
- *The weak equivalences are the weak J-stack equivalences.*

Proof. It is easy to see that there is a set of representatives of isomorphism classes of elements in $\mathcal{S}$, and the Heller (resp. Bousfield–Kan) model structure on $\mathbf{sPsh}(\mathcal{C})$ is left proper (by propositions 5.1.8 and 5.1.9) and combinatorial, so we may apply proposition 5.6.14 and theorem 5.6.15. ∎

Proposition 8.3.11. *The adjunction unit* $\eta : \mathrm{id}_{\mathbf{sPsh}(\mathcal{C})} \to \mathbf{s}j_* \mathbf{s}j^*$ *is a natural weak J-stack equivalence.*

Proof. See (the proof of) Proposition A.2 in [Dugger, Hollander, and Isaksen, 2004]. (The cited proof does not require $\mathcal{C}$ to have pullbacks.) □

Corollary 8.3.12. *Every J-local isomorphism of simplicial presheaves on $\mathcal{C}$ is a weak J-stack equivalence.* ∎

Theorem 8.3.13. *Let F be a simplicial presheaf on $\mathcal{C}$. If $\mathcal{C}$ has pullbacks, then the following are equivalent:*

(i) *F is a J-stack.*

(ii) *F satisfies the descent condition for all J-hypercovers of height 0.*

(iii) *F satisfies the descent condition for all bounded J-hypercovers.*

(iv) *F satisfies the descent condition for all components of the adjunction unit* $\eta : \mathrm{id}_{\mathbf{sPsh}(\mathcal{C})} \to \mathbf{s}j_* \mathbf{s}j^*$.

Proof. See Theorem A.6 in [Dugger, Hollander, and Isaksen, 2004]. □

TODO: Does this really need $\mathcal{C}$ to have pullbacks?

Definition 8.3.14. A **strong J-stack of ∞-groupoids** on $\mathcal{C}$ is a simplicial J-sheaf on $\mathcal{C}$ that is also a J-stack of ∞-groupoids.

Theorem 8.3.15. *The following data constitute a cofibrantly generated model structure on* $\mathbf{sSh}(\mathcal{C}, J)$*:*

- *The weak equivalences are the weak* J*-stack equivalences.*
- *The cofibrations are the monomorphisms.*
- *The fibrations are the morphisms in* $\mathbf{sSh}(\mathcal{C}, J)$ *that are fibrations in the injective model structure (on* $\mathbf{sPsh}(\mathcal{C})$*) for* J*-stacks of* ∞*-groupoids.*

This model structure is called the **injective model structure for strong J-stacks of ∞-groupoids***, and the adjunction*

$$\mathbf{s}j^* \dashv \mathbf{s}j_* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$$

is a Quillen equivalence between this model structure and the injective model structure for J*-stacks of* ∞*-groupoids.*

Proof. We have the following facts:

- The functor $\mathbf{s}j_* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$ is fully faithful.
- Both $\mathbf{s}j^* : \mathbf{sPsh}(\mathcal{C}) \to \mathbf{sSh}(\mathcal{C}, J)$ and $\mathbf{s}j_* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$ preserve monomorphisms, and the class of monomorphisms in $\mathbf{sSh}(\mathcal{C}, J)$ is closed under pushouts, transfinite composition, and retracts.
- The adjunction unit $\eta : \mathrm{id}_{\mathbf{sPsh}(\mathcal{C})} \Rightarrow \mathbf{s}j_*\mathbf{s}j^*$ is a natural weak J-stack equivalence, by proposition 8.3.11.

We may thus apply corollary 5.2.6 to theorem 8.3.10. In particular, note that every monomorphism is a cofibration in the model structure so constructed: indeed, if $f : X \to Y$ is a monomorphism in $\mathbf{sSh}(\mathcal{C}, J)$, then $j_* f : j_* X \to j_* Y$ is an injective cofibration in $\mathbf{sPsh}(\mathcal{C})$, so $j^* j_* f : j^* j_* X \to j^* j_* Y$ is a cofibration in $\mathbf{sSh}(\mathcal{C}, J)$; but the adjunction counit $\epsilon : \mathbf{s}j^*\mathbf{s}j_* \Rightarrow \mathrm{id}_{\mathbf{sSh}(\mathcal{C}, J)}$ is a natural isomorphism (by proposition A.1.3), so $f : X \to Y$ itself must be a cofibration. ■

Theorem 8.3.16. *The following data constitute a cofibrantly generated model structure on* $\mathbf{sSh}(\mathcal{C}, J)$*:*

- *The weak equivalences are the weak* J*-stack equivalences.*
- *The trivial fibrations are the morphisms in* $\mathbf{sSh}(\mathcal{C}, J)$ *that are componentwise trivial Kan fibrations.*

- *The cofibrations are the morphisms that have the left lifting property with respect to the trivial fibrations.*

This model structure is called the **projective model structure for strong J-stacks of ∞-groupoids**, *and the adjunction*

$$\mathbf{s}j^* \dashv \mathbf{s}j_* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$$

is a Quillen equivalence between this model structure and the projective model structure for J-stacks of ∞-groupoids.

Proof. Let $\mathcal{I}$ be the following subset of $\operatorname{mor} \mathbf{sSh}(\mathcal{C}, J)$:

$$\mathcal{I} = \left\{ j^* \left(\partial\Delta^n \odot \mathcal{h}_C \hookrightarrow \Delta^n \odot \mathcal{h}_C \right) \,\middle|\, n \geq 0, C \in \operatorname{ob} \mathcal{C} \right\}$$

By proposition A.3.26, the $\mathcal{I}$-injective morphisms in $\mathbf{sSh}(\mathcal{C}, J)$ are precisely the morphisms that are componentwise trivial Kan fibrations; and by theorem 8.3.15, each element of $\mathcal{I}$ is a trivial cofibration (in $\mathbf{sSh}(\mathcal{C}, J)$) for the injective model structure for strong J-stacks of ∞-groupoids. We may then obtain the required model structure by applying proposition 5.2.17. The construction ensures that $\mathbf{s}j^* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$ preserves cofibrations, so by proposition 4.3.8, the displayed adjunction is indeed a Quillen equivalence. ■

Definition 8.3.17. A **J-hypersheaf** on $\mathcal{C}$ is a simplicial presheaf on $\mathcal{C}$ that satisfies the descent condition for all J-hypercovers of presheaves of the form $\mathcal{h}_C$ where C is an object in $\mathcal{C}$.

Remark 8.3.18. Given a componentwise weak homotopy equivalence $X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$, it is clear that X is a J-hypersheaf if and only if Y is a J-hypersheaf. Thus, by proposition 4.1.28, the full subcategory of $\mathbf{sPsh}(\mathcal{C})$ spanned by the J-hypersheaves inherits the injective and projective model structures on $\mathbf{sPsh}(\mathcal{C})$ (and hence admits the structure of a derivable category in two ways).

Definition 8.3.19. A **weak J-hypersheaf equivalence** in $\mathbf{sPsh}(\mathcal{C})$ is a morphism $f : X \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ such that the induced morphism

$$\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(f, F) : \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(Y, F) \to \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(X, F)$$

is an isomorphism in $\operatorname{Ho} \mathbf{sSet}$ for all J-hypersheaves F.

Remark 8.3.20. Clearly, every componentwise weak homotopy equivalence is also a weak J-hypersheaf equivalence. Conversely, (the proof of) lemma 5.6.6 shows that every weak J-hypersheaf equivalence between J-hypersheaves is a componentwise weak homotopy equivalence.

Proposition 8.3.21. *Every J-hypersheaf is also a J-stack.*

Proof. Let F be a J-hypersheaf, let C be an object in $\mathcal{C}$, and let $q : V \to \hbar_C$ be a monomorphism in $\mathbf{Psh}(\mathcal{C})$ that is a J-local epimorphism. We wish to show that F satisfies the descent condition for $\operatorname{disc} q : \operatorname{disc} V \to \operatorname{disc} \hbar_C$. To begin, observe that $q : V \to \hbar_C$ is a J-local isomorphism: indeed, since $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits, $j^* q : j^* V \to j^* \hbar_C$ is both a (regular) monomorphism and an epimorphism, hence must be an isomorphism. In particular, $\operatorname{disc} q : \operatorname{disc} V \to \operatorname{disc} \hbar_C$ is a J-local trivial Kan fibration.

Now, using lemma 8.2.20 (for the minimal topology, not J), we may obtain a componentwise trivial Kan fibration $U \to \operatorname{disc} V$ whose composite with $\operatorname{disc} q : \operatorname{disc} V \to \operatorname{disc} \hbar_C$ yields (by proposition 8.2.12 and lemma 8.2.13) a J-hypercover $p : U \to \operatorname{disc} \hbar_C$. Thus, we have a commutative diagram in $\operatorname{Ho} \mathbf{sSet}$ of the form below,

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}\left(\operatorname{disc} \hbar_C, F\right) \arrow[r, "q^*"] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(\operatorname{disc} V, F) \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}\left(\operatorname{disc} \hbar_C, F\right) \arrow[r, "p^*"'] & \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(U, F)
\end{tikzcd}$$

where $\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(\operatorname{disc} V, F) \to \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(U, F)$ is an isomorphism. Thus, F satisfies the descent condition for $p : U \to \operatorname{disc} \hbar_C$ if and only if F satisfies the descent condition for $\operatorname{disc} q : \operatorname{disc} V \to \operatorname{disc} \hbar_C$. ■

Corollary 8.3.22. *Every weak J-stack equivalence is also a weak J-hypersheaf equivalence.* ■

Remark 8.3.23. The converse of proposition 8.3.21 is not true in general: there exist a Grothendieck site $(\mathcal{C}, J)$ and a J-stack F such that F is *not* a J-hypersheaf. For details, see Example A.10 in [Dugger, Hollander, and Isaksen, 2004].

Lemma 8.3.24. *Let κ be an infinite cardinal such that $\left|\mathcal{h}_C\right| \le \kappa$ for all C in $\mathcal{C}$. For any simplicial presheaf F on $\mathcal{C}$, the following are equivalent:*

(i) *F is a J-hypersheaf.*

(ii) *F satisfies the descent condition for all J-hypercovers $U \to \operatorname{disc} \mathcal{h}_C$ where U is a cellular simplicial presheaf on $\mathcal{C}$ and C is an object in $\mathcal{C}$.*

(iii) *F satisfies the descent condition for all J-hypercovers $U \to \operatorname{disc} \mathcal{h}_C$ where U is a cellular simplicial presheaf on $\mathcal{C}$, $\left|U_n\right| \le \kappa$ for all n, and C is an object in $\mathcal{C}$.*

Proof. (i) ⇒ (ii), (ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Let $f : X \to \operatorname{disc} \mathcal{h}_C$ be a J-hypercover. By proposition 8.2.23, there is a J-hypercover $p : U \to \operatorname{disc} \mathcal{h}_C$ such that F satisfies the descent condition for p and p refines f. We then have the following commutative diagram,

$$\begin{tikzcd}
\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}\left(\operatorname{disc} \mathcal{h}_C, F\right) \arrow[r, "f^*"] \arrow[d, equal] & \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(X, F) \arrow[d] \\
\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}\left(\operatorname{disc} \mathcal{h}_C, F\right) \arrow[r, "p^*"'] & \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(U, F)
\end{tikzcd}$$

where $\mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(X, F) \to \mathbf{R}\mathrm{Hom}_{\mathbf{sPsh}(\mathcal{C})}(U, F)$ is an isomorphism in $\operatorname{Ho} \mathbf{sSet}$. Thus, F satisfies the descent condition for $f : X \to \operatorname{disc} \mathcal{h}_C$ if (and only if) F satisfies the descent condition for $p : U \to \operatorname{disc} \mathcal{h}_C$. ∎

Theorem 8.3.25. *Let $\mathcal{S}$ be the class of J-hypercovers of presheaves of the form* $\operatorname{disc} \mathcal{h}_C$ *where C is an object in $\mathcal{C}$.*

(i) *The left Bousfield localisation of the Heller model structure on* $\mathbf{sPsh}(\mathcal{C})$ *with respect to $\mathcal{S}$ exists. The localised model structure is called the* **injective model structure for J-hypersheaves**, *and the fibrant objects in the localised model structure are the injective-fibrant J-hypersheaves.*

(ii) *The left Bousfield localisation of the Bousfield–Kan model structure on* $\mathbf{sPsh}(\mathcal{C})$ *with respect to $\mathcal{S}$ exists. The localised model structure is called the* **projective model structure for J-hypersheaves**, *and the fibrant objects in the localised model structure are the projective-fibrant J-hypersheaves.*

In either case:

- *The localised model structure is left proper, combinatorial, and simplicial.*
- *The weak equivalences are the weak J-hypersheaf equivalences.*

Proof. Let κ be an infinite cardinal such that $\left|\mathfrak{k}_C\right| \leq \kappa$ for all C in $\mathcal{C}$ and let $\mathcal{S}'$ be the class of all J-hypercovers of the form $U \to \operatorname{disc} \mathfrak{k}_C$ where U is a cellular simplicial presheaf on $\mathcal{C}$, $\left|U_n\right| \leq \kappa$ for all n, and C is an object in $\mathcal{C}$. It is clear that there is a set of representatives of isomorphism classes of elements of $\mathcal{S}'$, and lemma 8.3.24 implies that $\mathcal{S}'$-local objects are the same thing as $\mathcal{S}$-local objects. Thus, it suffices to construct the left Bousfield localisation with respect to $\mathcal{S}'$. But the Heller (resp. Bousfield–Kan) model structure on $\mathbf{sPsh}(\mathcal{C})$ is left proper (by propositions 5.1.8 and 5.1.9) and combinatorial, so we may apply proposition 5.6.14 and theorem 5.6.15. ■

Lemma 8.3.26. *Let X and Y be projective-fibrant J-hypersheaves and let $f : X \to Y$ be a morphism in $\mathbf{sPsh}(\mathcal{C})$ that has the right lifting property with respect to all projective cofibrations that are weak J-hypersheaf equivalences. If $f : X \to Y$ is a J-local weak homotopy equivalence, then $f : X \to Y$ is a componentwise trivial Kan fibration in $\mathbf{sPsh}(\mathcal{C})$.*

Proof. It suffices to show that $f : X \to Y$ has the right lifting property with respect to $\partial\Delta^n \odot V \to \Delta^n \odot V$, where $V = \operatorname{disc} \mathfrak{k}_C$, for all natural numbers n and all objects C in $\mathcal{C}$. Consider a lifting problem of the form below:

$$\begin{tikzcd}
\partial\Delta^n \odot V \arrow[r, "z"] \arrow[d, hook] & X \arrow[d, "f"] \\
\Delta^n \odot V \arrow[r, "w"'] & Y
\end{tikzcd}$$

Remark 8.3.20 implies that $f : X \to Y$ is a fibration in the Bousfield–Kan model structure on $\mathbf{sPsh}(\mathcal{C})$, so it is a J-local Kan fibration, by lemma 8.2.13. Similarly, X and Y are J-locally fibrant simplicial presheaves, so we may use theorem 8.1.13 to deduce that $f : X \to Y$ is a J-local trivial Kan fibration. Proposition 8.2.24 then yields a cellular simplicial presheaf U and J-hypercover $p : U \to V$ and a morphism $h : \Delta^n \odot U \to X$ making the following diagram commute:

$$\begin{tikzcd}
\partial\Delta^n \odot U \arrow[r, "\mathrm{id}_{\partial\Delta^n} \odot p"] \arrow[d, hook] & \partial\Delta^n \odot V \arrow[r, "z"] & X \arrow[d, "f"] \\
\Delta^n \odot U \arrow[r, "\mathrm{id}_{\Delta^n} \odot p"'] \arrow[urr, dashed, "h"] & \Delta^n \odot V \arrow[r, "w"'] & Y
\end{tikzcd}$$

Now, consider the commutative diagram of hom-spaces induced by the two lifting problems:

$$\begin{tikzcd} \underline{\mathbf{sPsh}_{C}}(V, \Delta^n \pitchfork X) \arrow[r] \arrow[d, "p^*"'] & \underline{\mathbf{sPsh}_{C}}\big(V, (\partial\Delta^n \pitchfork X) \times_{\partial\Delta^n \pitchfork Y} (\Delta^n \pitchfork Y)\big) \arrow[d, "p^*"] \\ \underline{\mathbf{sPsh}_{C}}(U, \Delta^n \pitchfork X) \arrow[r] & \underline{\mathbf{sPsh}_{C}}\big(U, (\partial\Delta^n \pitchfork X) \times_{\partial\Delta^n \pitchfork Y} (\Delta^n \pitchfork Y)\big) \end{tikzcd}$$

The horizontal arrows are Kan fibrations, because the Bousfield–Kan model structure on $\mathbf{sPsh}(C)$ is simplicial and both U and V are projective-cofibrant (by remarks 8.2.3 and 8.2.4). Since the left Bousfield localisation with respect to J-hypercovers is a simplicial model structure (by theorem 8.3.25), $\Delta^n \pitchfork X$ and $(\partial\Delta^n \pitchfork X) \times_{\partial\Delta^n \pitchfork Y} (\Delta^n \pitchfork Y)$ are J-hypersheaves, and hence the vertical arrows in the above diagram are weak homotopy equivalences. Thus, we may deduce that the image of the upper horizontal arrow meets the connected component of the vertex defined by our original lifting problem, and the path lifting property of Kan fibrations implies that there is a solution for that lifting problem. ■

Theorem 8.3.27. *Let $f : X \to Y$ be a morphism in* $\mathbf{sPsh}(C)$*. The following are equivalent:*

(i) *$f : X \to Y$ is a J-local weak homotopy equivalence.*

(ii) *$f : X \to Y$ is a weak J-hypersheaf equivalence.*

Proof. The following proof is due to Dugger, Hollander, and Isaksen [2004].

Remarks 8.1.35, 8.3.18, and 8.3.20 plus lemma 8.2.9 imply that we may assume that X and Y are projective-fibrant J-hypersheaves (by applying a functorial fibrant replacement in $\mathbf{sSet}$). Moreover, theorem 8.3.25 implies that there is a factorisation of the form $f = q \circ i$ where $i : X \to E$ is a weak J-hypersheaf equivalence between J-hypersheaves and $q : E \to Y$ is a morphism that has the right lifting property with respect to all projective cofibrations that are weak J-hypersheaf equivalences. By remark 8.3.20, $i : X \to E$ is a componentwise weak homotopy equivalence, so $q : E \to Y$ is a J-local weak homotopy equivalence (resp. weak J-hypersheaf equivalence) if and only if $f : X \to Y$ is a J-local weak homotopy equivalence (resp. weak J-hypersheaf equivalence). It is therefore enough to prove that the two conditions are equivalent for $q : E \to Y$.

(i) ⇒ (ii). If $q : E \to Y$ is a J-local weak homotopy equivalence, then we can use lemma 8.3.26 to deduce that $q : E \to Y$ is a componentwise trivial Kan fibration, hence a weak J-hypersheaf equivalence *a fortiori*.

(ii) ⇒ (i). If $q : E \to Y$ is a weak J-hypersheaf equivalence, then we can use remark 8.3.20 to deduce that $q : E \to Y$ is a componentwise weak homotopy equivalence, hence a J-local weak homotopy equivalence by lemma 8.2.9. ■

Corollary 8.3.28. *Let F be a simplicial presheaf on $\mathcal{C}$. The following are equivalent:*

(i) *F is a J-hypersheaf.*

(ii) *F satisfies the descent condition for weak J-hypersheaf equivalences.*

(iii) *F satisfies the descent condition for J-local weak homotopy equivalences.*

(iv) *F satisfies the descent condition for J-local trivial Kan fibrations.*

Proof. (i) ⇒ (ii). Every hypersheaf satisfies the descent condition for weak J-hypersheaf equivalences (by definition).

(ii) ⇒ (iii). Every J-local weak homotopy equivalence is a weak J-hypersheaf equivalence (by theorem 8.3.27).

(iii) ⇒ (iv). Every J-local trivial Kan fibration is a J-local weak homotopy equivalence (by remark 8.1.36).

(iv) ⇒ (i). Every J-hypercover is a J-local trivial Kan fibration (by definition). ■

8.4 Hypersheaf model structures

Prerequisites. §§0.5, 1.4, 1.5, 4.1, 4.3, 5.2, 8.1, A.7.

In this section, we study model structures for hypersheaves on a Grothendieck site. The first such model structure was constructed by Joyal [1984] and generalises the injective model structure of Heller [1988] on the category of simplicial presheaves.[5] We will mostly follow Joyal's original proof, but it should

[5] See also theorem 1.9.14.

be noted that there is another proof due to Jardine [1987], who also constructs a Quillen-equivalent model structure on the category of simplicial *presheaves*.

We will *not* use the theory of hypersheaves in this section; instead we work with the homotopy theory of internal Kan complexes. The two approaches are equivalent by a theorem of Dugger, Hollander, and Isaksen [2004]: see theorem 8.3.27. In particular, the Jardine (resp. Blander) local model structure on simplicial presheaves can be constructed as the left Bousfield localisation of the Heller (resp. Bousfield–Kan) model structure with respect to the class of J-hypercovers of representable presheaves.

Definition 8.4.1. Let $\mathcal{E}$ be a Grothendieck topos and let $\mathbf{s}\mathcal{E}$ be the category of simplicial objects in $\mathcal{E}$.

- A **weak homotopy equivalence** in $\mathbf{s}\mathcal{E}$ is an internal weak homotopy equivalence of simplicial objects in $\mathcal{E}$.
- An **injective cofibration** in $\mathbf{s}\mathcal{E}$ is a monomorphism in $\mathbf{s}\mathcal{E}$.
- An **injective trivial cofibration** in $\mathbf{s}\mathcal{E}$ is an injective cofibration in $\mathbf{s}\mathcal{E}$ that is also a weak homotopy equivalence.
- An **injective fibration** in $\mathbf{s}\mathcal{E}$ is a morphism in $\mathbf{s}\mathcal{E}$ with the right lifting property with respect to the injective trivial cofibrations.
- An **injective trivial fibration** in $\mathbf{s}\mathcal{E}$ is a morphism in $\mathbf{s}\mathcal{E}$ with the right lifting property with respect to the injective cofibrations.

Remark 8.4.2. It is well known that $\mathbf{s}\mathcal{E}$ is a Grothendieck topos if $\mathcal{E}$ is, so $\mathbf{s}\mathcal{E}$ with the Heller–Joyal model structure (once it is shown to exist) is a Cisinski model category.[6] In a particular, injective trivial fibrations in $\mathbf{s}\mathcal{E}$ are (by definition) the same thing as Cisinski trivial fibrations in $\mathbf{s}\mathcal{E}$.

Proposition 8.4.3. *Let $\mathcal{E}$ and $\mathcal{F}$ be Grothendieck toposes and let $u : \mathcal{E} \to \mathcal{F}$ be a geometric morphism. Then the inverse image functor $u^* : \mathcal{F} \to \mathcal{E}$ induces a functor $\mathbf{s}u^* : \mathbf{s}\mathcal{F} \to \mathbf{s}\mathcal{E}$ that preserves injective cofibrations, injective trivial cofibrations, and weak homotopy equivalences.*

[6] See definition 5.4.1.

Proof. By definition, $u^* : \mathcal{F} \to \mathcal{E}$ is a functor that preserves finite limits and has a right adjoint. Thus, $\mathbf{s}u^* : \mathbf{s}\mathcal{F} \to \mathbf{s}\mathcal{E}$ preserves injective cofibrations (i.e. monomorphisms), and to complete the proof, it suffices to show that $\mathbf{s}u^* : \mathbf{s}\mathcal{F} \to \mathbf{s}\mathcal{E}$ preserves weak homotopy equivalences. But by proposition 8.1.30, $\mathbf{s}u^*$ commutes with Ex^∞, so we may use the argument in the proof of lemma 8.1.31 to deduce that $\mathbf{s}u^*$ indeed preserves weak homotopy equivalences. ■

Lemma 8.4.4. *Let $\mathcal{E}$ be a Grothendieck topos. The class of weak homotopy equivalences in $\mathbf{s}\mathcal{E}$ is closed under colimits for small filtered diagrams (in $[\mathbb{2}, \mathbf{s}\mathcal{E}]$).*

Proof. Let $\mathcal{J}$ be a small filtered category. Since $\mathcal{E}$ is a Grothendieck topos, $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathcal{E}] \to \mathcal{E}$ preserves finite limits. Thus, by proposition 8.4.3, $\varinjlim_{\mathcal{J}}$ preserves weak homotopy equivalences, and it is not hard to see that a weak homotopy equivalence of simplicial objects in $[\mathcal{J}, \mathcal{E}]$ is precisely a morphism that is componentwise a weak homotopy equivalence of simplicial objects in $\mathcal{E}$. ■

Lemma 8.4.5. *Let $\mathcal{E}$ be a Grothendieck topos.*

(i) *The class of injective cofibrations (resp. injective trivial cofibrations) in $\mathbf{s}\mathcal{E}$ contains all isomorphisms in $\mathbf{s}\mathcal{E}$.*

(ii) *The class of injective cofibrations (resp. injective trivial cofibrations) in $\mathbf{s}\mathcal{E}$ is closed under (finite and) transfinite composition.*

(iii) *The class of injective cofibrations (resp. injective trivial cofibrations) in $\mathbf{s}\mathcal{E}$ is closed under retracts.*

(iv) *The class of injective cofibrations (resp. injective trivial cofibrations) in $\mathbf{s}\mathcal{E}$ is closed under pushouts.*

(v) *The class of injective cofibrations (resp. injective trivial cofibrations) in $\mathbf{s}\mathcal{E}$ is closed under coproducts.*

Proof. The claims for the class of injective cofibrations are well known; we will prove the claims for the class of injective trivial cofibrations.

(i). Obvious.

(ii) and (iii). It suffices to show that the class of weak homotopy equivalences is closed under the relevant operations; but this was done in remark 8.1.35 and lemma 8.4.4.

TODO: Fill in the details...

(iv). Use the classical completeness theorem for separable theories.

(v). By lemma 0.5.6, a class of morphisms that is closed under transfinite composition and pushouts is also closed under coproducts. ■

Lemma 8.4.6. *Let $\mathcal{E}$ be a Grothendieck topos. Every injective trivial fibration in $\mathbf{s}\mathcal{E}$ is an internal trivial Kan fibration of simplicial objects in $\mathcal{E}$.*

Proof. Let $p : X \to Y$ be an injective trivial fibration in $\mathbf{s}\mathcal{E}$ and let n be a natural number. We wish to show that the following diagram is a weak pullback square in $\mathcal{E}$,

$$\begin{tikzcd} \{\Delta^n, X\} \arrow[r, "\{\Delta^n, p\}"] \arrow[d] & \{\Delta^n, Y\} \arrow[d] \\ \{\partial\Delta^n, X\} \arrow[r, "\{\partial\Delta^n, p\}"'] & \{\partial\Delta^n, Y\} \end{tikzcd}$$

where the vertical arrows are induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, so it suffices to verify that the morphism

$$\{\Delta^n, X\} \to \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

is a *split* epimorphism in $\mathcal{E}$. Let L denote the codomain of this morphism. By adjointness, we find that $\{\Delta^n, X\} \to L$ splits if and only if there is a morphism $\Delta^n \odot L \to X$ in $\mathbf{s}\mathcal{E}$ making the diagram below commute,

$$\begin{tikzcd} \partial\Delta^n \odot L \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \odot L \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where $\partial\Delta^n \odot L \to \Delta^n \odot L$ is induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, $\partial\Delta^n \odot L \to X$ is the left adjoint transpose of the projection $L \to \partial\Delta^n \pitchfork X$, and $\Delta^n \odot L \to Y$ is the left adjoint transpose of the projection $L \to \Delta^n \pitchfork Y$. But $\partial\Delta^n \odot L \to \Delta^n \odot L$ is an injective cofibration and $p : X \to Y$ is an injective trivial cofibration, hence the required morphism $\Delta^n \odot L \to X$ indeed exists. ■

Lemma 8.4.7. *Let $\mathcal{S}$ be a Grothendieck topos. There exist a set $\mathcal{I}$ of monomorphisms in $\mathcal{S}$ such that the relative $\mathcal{I}$-cell complexes are precisely the monomorphisms in $\mathcal{S}$.*

Proof. We may assume without loss of generality that $\mathcal{S} = \mathbf{Sh}(\mathcal{C}, J)$, where $\mathcal{C}$ is a small category and J is a subcanonical Grothendieck topology on $\mathcal{C}$. Consider the class of all monomorphisms $U \to V$ in $\mathcal{S}$ where V is a quotient (in $\mathcal{S}$) of some representable sheaf: since $\mathcal{S}$ is well-powered and well-copowered, we may choose a set $\mathcal{I}$ of representatives of isomorphism classes of such monomorphisms. The argument in the proof of proposition 0.5.20 then shows that every monomorphism in $\mathcal{S}$ is a relative $\mathcal{I}$-cell complex. ■

Lemma 8.4.8. *Let $\mathcal{E}$ be a Grothendieck topos. The full subcategory of $[\mathbb{2}, \mathbf{s}\mathcal{E}]$ spanned by the weak homotopy equivalences in $\mathbf{s}\mathcal{E}$ is an accessible subcategory of $[\mathbb{2}, \mathbf{s}\mathcal{E}]$.*

Proof. It is not hard to see that $\mathrm{Ex} : \mathbf{s}\mathcal{E} \to \mathbf{s}\mathcal{E}$ is an accessible functor, so the same is true for $\mathrm{Ex}^\infty : \mathbf{s}\mathcal{E} \to \mathbf{s}\mathcal{E}$. Thus, by applying proposition 0.3.30, it suffices to show that the full subcategory of $[\mathbb{2}, \mathbf{s}\mathcal{E}]$ spanned by the Dugger–Isaksen weak equivalences is an accessible subcategory of $[\mathbb{2}, \mathbf{s}\mathcal{E}]$. Clearly, this subcategory is the category of models (in $\mathcal{E}$) of a small (geometric) sketch, so it is closed in $[\mathbb{2}, \mathbf{s}\mathcal{E}]$ under colimits for small filtered diagrams, and we may apply Theorem 2.60 in [LPAC] to deduce that it is indeed an accessible category. □

Theorem 8.4.9 (Heller, Joyal)**.** *Let $\mathcal{E}$ be a Grothendieck topos. The following data constitute a cofibrantly generated model structure on $\mathbf{s}\mathcal{E}$:*

- *The weak equivalences are the internal weak homotopy equivalences of simplicial objects in $\mathcal{E}$.*
- *The cofibrations are the injective cofibrations, i.e. the monomorphisms.*
- *The fibrations are the injective fibrations, i.e. the morphisms that have the right lifting property with respect to monomorphisms that are weak homotopy equivalences.*

This model structure is called the **Heller–Joyal model structure**.

Proof. We have shown the following:

- The class of weak homotopy equivalences in $\mathbf{s}\mathcal{E}$ has the 2-out-of-3 property and is closed under retracts, by remark 8.1.35.
- The class of injective cofibrations is the class of relative $\mathcal{I}$-cell complexes for a subset $\mathcal{I} \subset \operatorname{mor} \mathbf{s}\mathcal{E}$, by lemma 8.4.7, and every injective trivial fibration is a weak homotopy equivalence, by remark 8.1.36 and lemma 8.4.6.

- The class of injective trivial cofibrations is closed under pushouts and transfinite composition, by lemma 8.4.5.

- The full subcategory of $[\mathbb{2}, \mathbf{s}\mathcal{E}]$ spanned by the weak homotopy equivalences in $\mathbf{s}\mathcal{E}$ is accessible, by lemma 8.4.8.

Thus, we may apply Smith's recognition principle (theorem 5.2.10) to deduce that we have the required cofibrantly generated model structure. ■

Corollary 8.4.10. *Let $\mathcal{E}$ and $\mathcal{F}$ be Grothendieck toposes. Given a geometric morphism $u : \mathcal{E} \to \mathcal{F}$, the induced adjunction*

$$\mathbf{s}u^* \dashv \mathbf{s}u_* : \mathbf{s}\mathcal{E} \to \mathbf{s}\mathcal{F}$$

is a Quillen adjunction (with respect to the Heller–Joyal model structures on $\mathbf{s}\mathcal{E}$ and $\mathbf{s}\mathcal{F}$).

Proof. Apply proposition 4.3.2 to proposition 8.4.3. ■

Remark 8.4.11. The fact that we have a model structure implies that the injective trivial fibrations in $\mathbf{s}\mathcal{E}$ are precisely the injective fibrations in $\mathbf{s}\mathcal{E}$ that are also weak homotopy equivalences.

Next, we transfer the Heller–Joyal model structure to the category of simplicial presheaves, obtaining the Jardine local model structure.

Definition 8.4.12. In the notation of paragraph 8.2.1, recalling definition 8.2.8:

- An **J-local injective trivial cofibration** in $\mathbf{sPsh}(\mathcal{C})$ is an injective cofibration in $\mathbf{sPsh}(\mathcal{C})$ that is also a J-local weak homotopy equivalence.

- An **J-local injective fibration** in $\mathbf{sPsh}(\mathcal{C})$ is a morphism in $\mathbf{sPsh}(\mathcal{C})$ that has the right lifting property with respect to the J-local injective trivial fibrations.

Remark 8.4.13. By remark 8.1.37, the componentwise weak homotopy equivalences in $\mathbf{sPsh}(\mathcal{C})$ are the internal weak homotopy equivalences of simplicial objects in $\mathbf{Psh}(\mathcal{C})$; thus, they are the weak equivalences in the Heller–Joyal model structure on $\mathbf{sPsh}(\mathcal{C})$, and the Heller–Joyal model structure on $\mathbf{sPsh}(\mathcal{C})$ can be identified with the injective model structure on $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]$.

Lemma 8.4.14. *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$.*

(i) *The class of J-local injective trivial cofibrations in* $\mathbf{sPsh}(\mathcal{C})$ *contains all J-local isomorphisms in* $\mathbf{sPsh}(\mathcal{C})$.

(ii) *The class of J-local injective trivial cofibrations in* $\mathbf{sPsh}(\mathcal{C})$ *is closed under (finite and) transfinite composition.*

(iii) *The class of J-local injective trivial cofibrations in* $\mathbf{sPsh}(\mathcal{C})$ *is closed under retracts.*

(iv) *The class of J-local injective trivial cofibrations in* $\mathbf{sPsh}(\mathcal{C})$ *is closed under pushouts.*

(v) *The class of J-local injective trivial cofibrations in* $\mathbf{sPsh}(\mathcal{C})$ *is closed under coproducts.*

Proof. As noted in paragraph 8.2.1, the associated sheaf functor preserves finite limits, so it preserves monomorphisms in particular. On the other hand, it is a left adjoint, so it also preserves all colimits. We may therefore deduce the announced properties of J-local injective trivial cofibrations in $\mathbf{sPsh}(\mathcal{C})$ from the corresponding properties of injective trivial cofibrations in $\mathbf{sSh}(\mathcal{C}, J)$, which were established in lemma 8.4.5. ∎

Lemma 8.4.15. *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$. The full subcategory of* $[\mathbb{2}, \mathbf{sPsh}(\mathcal{C})]$ *spanned by the J-local weak homotopy equivalences in* $\mathbf{sPsh}(\mathcal{C})$ *is an accessible subcategory of* $[\mathbb{2}, \mathbf{sPsh}(\mathcal{C})]$.

Proof. The associated sheaf functor is automatically an accessible functor, so we may apply proposition 0.3.30 to lemma 8.4.8 to deduce the claim. ∎

Theorem 8.4.16 (Jardine)**.** *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$. The following data constitute a cofibrantly generated model structure on* $\mathbf{sPsh}(\mathcal{C})$*:*

- *The weak equivalences are the J-local weak homotopy equivalences.*
- *The cofibrations are the injective cofibrations, i.e. the monomorphisms.*
- *The fibrations are the J-local injective fibrations, i.e. the morphisms that ahve the right lifting property with respect to monomorphisms that are J-local weak homotopy equivalences.*

This model structure is called the **J-local Jardine model structure**.

Proof. We have the following facts:

- The class of J-local weak homotopy equivalences has the 2-out-of-3 property and is closed under retracts, by remark 8.1.35 and lemma A.4.14.
- The class of injective cofibrations is the class of relative $\mathcal{I}$-cell complexes for a subset $\mathcal{I} \subset \operatorname{mor} \mathbf{sPsh}(\mathcal{C})$, by proposition 0.5.20, and every injective trivial fibration is a J-local weak homotopy equivalence, by lemma 8.2.9.
- The class of J-local injective trivial cofibrations is closed under pushouts and transfinite composition, by lemma 8.4.14.
- The full subcategory of $[\mathbb{2}, \mathbf{sPsh}(\mathcal{C})]$ spanned by the J-local weak homotopy equivalences is accessible, by lemma 8.4.8.

Thus, we may apply Smith's recognition principle (theorem 5.2.10) to deduce that we have the required cofibrantly generated model structure. ∎

Remark 8.4.17. The J-local Jardine model structure on $\mathbf{sPsh}(\mathcal{C})$ is the left Bousfield localisation of the Heller model structure with respect to the class of J-local weak homotopy equivalences, and it makes $\mathbf{sPsh}(\mathcal{C})$ into a Cisinski model category.

Proposition 8.4.18. *In the notation of paragraph 8.2.1, the induced adjunction*

$$\mathbf{s}j^* \dashv \mathbf{s}j_* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sPsh}(\mathcal{C})$$

is a Quillen equivalence between the Heller–Joyal model structure on $\mathbf{sSh}(\mathcal{C}, J)$ *and the* J*-local Jardine model structure on* $\mathbf{sPsh}(\mathcal{C})$*.*

Proof. By construction, the associated sheaf functor $\mathbf{s}j^* : \mathbf{sPsh}(\mathcal{C}) \to \mathbf{sSh}(\mathcal{C}, J)$ is a left Quillen functor, so we indeed have a Quillen adjunction (by proposition 4.3.2). To complete the proof, we simply appeal to propositions 4.3.8 and A.1.3). ∎

Proposition 8.4.19. *Let* $(\mathcal{C}, J)$ *and* $(\mathcal{D}, K)$ *be small Grothendieck sites. If* $u : \mathcal{D} \to \mathcal{C}$ *is a flat functor that sends* K*-covering sieves to* J*-covering sieves, then:*

(i) *The induced adjunction*

$$\mathrm{sLan}_u \dashv \mathbf{s}u^* : \mathbf{sPsh}(\mathcal{C}) \to \mathbf{sPsh}(\mathcal{D})$$

is a Quillen adjunction with respect to the local Jardine model structures.

(ii) *Moreover, it is a Quillen equivalence if the corresponding geometric morphism*

$$u_! \dashv u^* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$$

is an adjoint equivalence of toposes.

Proof. (i). The hypotheses imply that $u^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Psh}(\mathcal{D})$ sends J-sheaves to K-sheaves, so we have the following (strictly) commutative diagram of right adjoints:

$$\begin{tikzcd}
\mathbf{sSh}(\mathcal{C}, J) \arrow[r, hook, "\mathbf{s}j_*"] \arrow[d, "\mathbf{s}u^*"'] & \mathbf{sPsh}(\mathcal{C}) \arrow[d, "\mathbf{s}u^*"] \\
\mathbf{sSh}(\mathcal{D}, K) \arrow[r, hook, "\mathbf{s}k_*"'] & \mathbf{sPsh}(\mathcal{D})
\end{tikzcd}$$

Thus, the diagram of left adjoints commutes up to a canonical natural isomorphism. We then use proposition 8.4.3 to deduce that $\mathbf{s}\,\mathrm{Lan}_u : \mathbf{sPsh}(\mathcal{D}) \to \mathbf{sPsh}(\mathcal{C})$ is a left Quillen functor. This completes the proof (by proposition 4.3.2).

(ii). If $u^* : \mathbf{Sh}(\mathcal{C}, J) \to \mathbf{Sh}(\mathcal{D}, K)$ is (half of) an equivalence of categories, then $\mathbf{s}u^* : \mathbf{sSh}(\mathcal{C}, J) \to \mathbf{sSh}(\mathcal{D}, K)$ must have the same property. Since the Heller–Joyal model structure is invariant under equivalence of categories, proposition 8.4.18 implies that the derived adjunction

$$\mathbf{L}\mathbf{s}\mathrm{Lan}_u \dashv \mathbf{R}\mathbf{s}u^* : \operatorname{Ho}\mathbf{sPsh}(\mathcal{C}) \to \operatorname{Ho}\mathbf{sPsh}(\mathcal{D})$$

is an adjoint equivalence of categories; thus, we may apply proposition 3.3.28 and deduce that

$$\mathbf{s}\mathrm{Lan}_u \dashv \mathbf{s}u^* : \mathbf{sPsh}(\mathcal{C}) \to \mathbf{sPsh}(\mathcal{D})$$

is a Quillen equivalence. ■

Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$. Recall that the Bousfield–Kan model structure on $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]$ has weak equivalences and fibrations that are defined componentwise. We will now construct the left Bousfield localisation of this model structure with respect to the J-local weak homotopy equivalences.

Definition 8.4.20.

- A ***J*-local projective trivial cofibration** in $\mathbf{sPsh}(\mathcal{C})$ is a projective cofibration that is also a J-local weak homotopy equivalence.

- A **J-local projective fibration** in $\mathbf{sPsh}(\mathcal{C})$ is a morphism in $\mathbf{sPsh}(\mathcal{C})$ that has the right lifting property with respect to the J-local projective trivial cofibrations.

Remark 8.4.21. By corollary 4.3.21, every projective cofibration is an injective cofibration; hence, every local injective fibration is a local projective fibration. Similarly, by lemma 8.2.9, every projective trivial cofibration is a local projective trivial cofibration, so every local projective fibration is a projective fibration.

Lemma 8.4.22. *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$.*

(i) *The class of projective cofibrations (resp. J-local projective trivial cofibrations) in $\mathbf{sPsh}(\mathcal{C})$ contains all isomorphisms in $\mathbf{sPsh}(\mathcal{C})$.*

(ii) *The class of projective cofibrations (resp. J-local projective trivial cofibrations) in $\mathbf{sPsh}(\mathcal{C})$ is closed under (finite and) transfinite composition.*

(iii) *The class of projective cofibrations (resp. J-local projective trivial cofibrations) in $\mathbf{sPsh}(\mathcal{C})$ is closed under retracts.*

(iv) *The class of projective cofibrations (resp. J-local projective trivial cofibrations) in $\mathbf{sPsh}(\mathcal{C})$ is closed under pushouts.*

(v) *The class of projective cofibrations (resp. J-local projective trivial cofibrations) in $\mathbf{sPsh}(\mathcal{C})$ is closed under coproducts.*

Proof. The claims for the class of projective cofibrations are consequences of theorem 5.2.7 and proposition A.3.17. Moreover, remark 8.4.21 implies that a J-local projective trivial cofibration is precisely a projective cofibration that is also a J-local injective trivial cofibration, so we may deduce the claims for the class of J-local projective trivial cofibrations from lemma 8.4.14 (plus the claims for the class of projective cofibrations). ■

Theorem 8.4.23 (Blander)**.** *Let $\mathcal{C}$ be a small category and let J be a Grothendieck topology on $\mathcal{C}$. The following data constitute a cofibrantly generated model structure on $\mathbf{sPsh}(\mathcal{C})$:*

- *The weak equivalences are the J-local weak homotopy equivalences.*

- *The cofibrations are the projective cofibrations, i.e. the morphisms that have the left lifting property with respect to componentwise trivial Kan fibrations.*
- *The fibrations are the J-local projective fibrations, i.e. the morphisms that have the right lifting property with respect to the J-local projective trivial cofibrations.*

This model structure is called the **J-local Blander model structure**.

Proof. We have the following facts:

- The class of J-local weak homotopy equivalences has the 2-out-of-3 property and is closed under retracts, by remark 8.1.35 and lemma A.4.14.
- The class of projective cofibrations is the left class of a cofibrantly generated weak factorisation system, by theorem 5.2.7, and the every projective trivial fibration is a J-local weak homotopy equivalence, by lemma 8.2.9.
- The class of J-local injective trivial cofibrations is closed under pushouts and transfinite composition, by lemma 8.4.22.
- The full subcategory of $[\mathbb{2}, \mathbf{sPsh}(C)]$ spanned by the J-local weak homotopy equivalences is accessible, by lemma 8.4.8.

Thus, we may apply Smith's recognition principle (theorem 5.2.10) to deduce that we have the required cofibrantly generated model structure. ∎

Proposition 8.4.24. *Let C be a small category and let J be a Grothendieck topology on C. The trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : \mathbf{sPsh}(C) \to \mathbf{sPsh}(C)$$

is a Quillen equivalence between the J-local Jardine model structure and the J-local Blander model structure.

Proof. Since the weak equivalences in the two model structures are the same, it suffices to prove that we have the announced Quillen adjunction; but this is an immediate consequence of proposition 4.3.2 and remark 8.4.21. ∎

Theorem 8.4.25 (Blander)**.** *Let C be a small category and let J be a Grothendieck topology on C. The following data constitute a cofibrantly generated model structure on* $\mathbf{sSh}(C, J)$*:*

- *The weak equivalences are the internal weak homotopy equivalences of simplicial objects in* $\mathbf{Sh}(\mathcal{C}, J)$.
- *The trivial fibrations are the morphisms in* $\mathbf{sSh}(\mathcal{C}, J)$ *that are componentwise trivial Kan fibrations.*
- *The cofibrations are the morphisms that have the left lifting property with respect to the trivial fibrations.*

This model structure is called the **Blander model structure**.

Proof. Let $\mathcal{I}$ be the following subset of $\operatorname{mor} \mathbf{sSh}(\mathcal{C}, J)$:

$$\mathcal{I} = \left\{ j^* \left(\partial \Delta^n \odot \hbar_C \hookrightarrow \Delta^n \odot \hbar_C \right) \,\middle|\, n \geq 0, C \in \operatorname{ob} \mathcal{C} \right\}$$

By proposition A.3.26, the $\mathcal{I}$-injective morphisms in $\mathbf{sSh}(\mathcal{C}, J)$ are precisely the morphisms that are componentwise trivial Kan fibrations. Since $j^* : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{C}, J)$ preserves finite limits, $\mathcal{I}$ is a set of injective cofibrations in $\mathbf{Sh}(\mathcal{C}, J)$. We may thus apply proposition 5.2.17 and theorem 8.4.9 to construct the required model structure on $\mathbf{sSh}(\mathcal{C}, J)$. ■

Remark 8.4.26. In fact, the fibrations in the Blander model structure are precisely the morphisms in $\mathbf{sSh}(\mathcal{C}, J)$ that are J-local projective fibrations in $\mathbf{sPsh}(\mathcal{C})$: see Theorem 2.1 in [Blander, 2001].

8.5 Verdier's hypercovering theorem

Prerequisites. §§1.1, 1.3, 1.4, 1.5, 2.5, 8.1, 8.2, 8.3.

Although the small object argument provides a functorial choice of what one might call "associated hypersheaves", it is difficult to compute the weak homotopy types of its components (which are well defined, by remark 8.3.20). Indeed, this problem encompasses sheaf cohomology: for instance, if X is the simplicial presheaf constant at $K(A, n)$ and $\hat{X}$ is its associated hypersheaf, then $\pi_0 \hat{X}(C)$ is the sheaf cohomology group $H^n(C, A)$ (where we have identified A with the constant sheaf at A). Verdier's hypercovering theorem gives us a formula for π_0 and the homotopy groups of $\hat{X}(C)$ in terms of X and hypercovers of C. We will follow the proof of Dugger, Hollander, and Isaksen [2004, § 8].

¶ 8.5.1. In this section, we use the notation set up in paragraph 8.2.1.

Lemma 8.5.2. *Let X and Y be J-locally fibrant simplicial presheaves on $\mathcal{C}$, let $f : X \to Y$ be a J-local weak homotopy equivalence, let L be a finite simplicial set, and let K be a simplicial subset of L.*

(i) *For any simplicial presheaf V on $\mathcal{C}$ and any $\partial\alpha : K \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, X)$, if $\beta : L \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Y)$ is a morphism whose restriction along $K \hookrightarrow L$ is $\partial\beta = f_* \circ \partial\alpha$, then there exist a cellular simplicial presheaf U and a J-local trivial Kan fibration $p : U \to V$ such that $p^* \circ \beta$ is in the image of the map*

$$f_* : \pi_{(L,K)}\big(\underline{\mathbf{sPsh}_{\mathcal{C}}}(U, X), \partial\alpha\big) \to \pi_{(L,K)}\big(\underline{\mathbf{sPsh}_{\mathcal{C}}}(U, Y), p^* \circ \partial\beta\big)$$

induced by $f : X \to Y$.

(ii) *Given morphisms $\alpha_0, \alpha_1 : L \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Y)$ whose restrictions along $K \hookrightarrow L$ are $\partial\alpha : K \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Y)$, if*

$$f_* \circ \alpha_0 = f_* \circ \alpha_1 \text{ in } \pi_{(L,K)}\big(\underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Y), \partial\beta\big)$$

where $\partial\beta = f_* \circ \partial\alpha$, then there exist a cellular simplicial presheaf U and a J-local trivial Kan fibration $p : U \to V$ such that:

$$p^* \circ \alpha_0 = p^* \circ \alpha_1 \text{ in } \pi_{(L,K)}\big(\underline{\mathbf{sPsh}_{\mathcal{C}}}(U, X), p^* \circ \partial\alpha\big)$$

Proof. (i). By adjointness, $\partial\alpha : K \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, X)$ and $\beta : L \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Y)$ correspond to morphisms $\partial x : K \odot V \to X$ and $y : L \odot V \to Y$ in $\mathbf{sPsh}(\mathcal{C})$ making the following diagram commute:

$$\begin{tikzcd} K \odot V \arrow[r, "\partial x"] \arrow[d, hook] & X \arrow[d, "f"] \\ L \odot V \arrow[r, "y"'] & Y \end{tikzcd}$$

The claim is then seen to be precisely proposition 8.2.25.

(ii). By adjointness, $\alpha_0, \alpha_1 : L \to \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, X)$ correspond to morphisms $x_0, x_1 : L \odot V \to X$ in $\mathbf{sPsh}(\mathcal{C})$ making the following diagram commute:

$$\begin{tikzcd} K \odot V \arrow[r, hook] \arrow[d, hook] & L \odot V \arrow[d, "x_1"] \\ L \odot V \arrow[r, "x_0"'] & X \end{tikzcd}$$

It suffices to prove the claim in the special case where there is a morphism $l : C(L,K) \odot V \to Y$ such that $l \circ (j_0 \odot \mathrm{id}_V) = f \circ x_0$ and $l \circ (j_1 \odot \mathrm{id}_V) = f \circ x_1$, where $j_0, j_1 : L \to C(L,K)$ are the two canonical embeddings.

Let $\partial x : (L \cup^K L) \odot V \to X$ be the morphism induced by $x_0, x_1 : L \odot V \to X$. Then proposition 8.2.25 gives $p : U \to V$ and $x : C(L,K) \odot U \to X$ such that the following diagram commutes,

$$\begin{tikzcd}
(L \cup^K L) \odot U \arrow[r, "\mathrm{id}_{L \cup^K L} \odot p"] \arrow[d] & (L \cup^K L) \odot V \arrow[d, "\partial x"] \\
C(L,K) \odot U \arrow[r, dashed, "x"'] & X
\end{tikzcd}$$

where $L \cup^K L \to C(L,K)$ is the morphism induced by $j_0, j_1 : L \to C(L,K)$. Thus, we have $p^* \circ \alpha_0 = p^* \circ \alpha_1$ in $\pi_{(L,K)}(\underline{\mathbf{sPsh}_C}(U,X), p^* \circ \partial\alpha)$, as required. ■

Corollary 8.5.3. *Let X and Y be J-locally fibrant simplicial presheaves on C and let $f : X \to Y$ be a J-local weak homotopy equivalence.*

(i) *For any simplicial presheaf V on C and any morphism $y : V \to Y$, there exist a cellular simplicial presheaf U, a J-local trivial Kan fibration $p : U \to V$, a morphism $x : U \to X$, and a simplicial homotopy $y \circ p \Rightarrow f \circ x$, or in other words, a morphism $h : \Delta^1 \odot U \to \hat{X}$ such that the following diagram commutes,*

$$\begin{tikzcd}
U \arrow[r, "j_0"] \arrow[d, dashed, "p"'] & \Delta^1 \odot U \arrow[d, dashed, "h"] & U \arrow[l, "j_1"'] \arrow[d, dashed, "x"] \\
V \arrow[r, "y"'] & Y & X \arrow[l, "f"]
\end{tikzcd}$$

where $j_0, j_1 : U \to \Delta^1 \odot U$ are the morphisms induced by the coface morphisms $\delta^1, \delta^0 : \Delta^0 \to \Delta^1$, respectively.

(ii) *If $x_0, x_1 : V \to X$ are morphisms in $\mathbf{sPsh}(C)$ and there is a simplicial homotopy $f \circ x_0 \Rightarrow f \circ x_1$, then there exist a cellular simplicial presheaf U, a J-local trivial Kan fibration $p : U \to V$, and a simplicial homotopy $x_0 \circ p \Rightarrow x_1 \circ p$.*

Moreover, in each case, we may choose U so that each U_n is a coproduct of representable presheaves and $|U| \leq \kappa$, where κ is an infinite cardinal such that $|V| \leq \kappa$ and $|\mathfrak{h}_C| \leq \kappa$ for all C in C. ■

Definition 8.5.4. Let V be a presheaf on $\mathcal{C}$. The **category of J-hypercovers** of V is the full subcategory $\mathbf{Hc}_J(V)$ of the slice category $\mathbf{sPsh}(\mathcal{C})_{/\operatorname{disc} V}$ spanned by the hypercovers of V.

¶ 8.5.5. Recalling definition 2.1.21, the slice category $\mathbf{sPsh}(\mathcal{C})_{/\operatorname{disc} V}$ has a canonical simplicial enrichment, which is inherited by $\mathbf{Hc}_J(V)$. We define $\operatorname{Ho}\mathbf{Hc}_J(V)$ to be the category $\pi_0\left[\underline{\mathbf{Hc}_J(V)}\right]$: its objects are J-hypercovers of V and its morphisms are simplicial homotopy classes of morphisms in $\mathbf{Hc}_J(V)$. More generally, given any object q in $\mathbf{Hc}_J(V)$, we define $\operatorname{Ho}\mathbf{Hc}_J(V)_{/q}$ to be the category $\pi_0\left[\underline{\mathbf{Hc}_J(V)_{/q}}\right]$.

Remark 8.5.6. If $\mathcal{C}$ has pullbacks, then $\mathbf{Hc}_J\left(h_C\right)$ is contravariantly pseudofunctorial in C: indeed, by proposition 8.2.12, pullbacks of J-local trivial Kan fibrations are again J-local trivial Kan fibrations; and for any pullback diagram in $\mathbf{Psh}(\mathcal{C})$, say

$$\begin{tikzcd} U' \arrow[r] \arrow[d] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

if U, V, and V' are representable presheaves on $\mathcal{C}$, then so is U'; thus, the pullback of a J-hypercover of h_C along any morphism $h_f : h_{C'} \to h_C$ is a J-hypercover of C'. Furthermore, since pullback in $\mathbf{sPsh}(\mathcal{C})$ respects simplicial homotopy, so $\operatorname{Ho}\mathbf{Hc}_J\left(h_C\right)$ is also contravariantly pseudofunctorial in C.

Proposition 8.5.7. *Let V be a presheaf on $\mathcal{C}$, let κ be an infinite cardinal such that $|V| \le \kappa$ and $\left|h_C\right| \le \kappa$ for all C in $\mathcal{C}$, and let $\mathcal{K}$ (resp. $\operatorname{Ho}\mathcal{K}$) be the full subcategory of $\mathbf{Hc}_J(V)$ (resp. $\operatorname{Ho}\mathbf{Hc}_J(V)$) spanned by those hypercovers $p : U \to \operatorname{disc} V$ where U is cellular and $|U| \le \kappa$.*

(i) *V admits a J-hypercover that is in $\mathcal{K}$, and any two J-hypercovers of V admit a common refinement that is in $\mathcal{K}$.*

(ii) *The projection $\pi : \mathbf{Hc}_J(V) \to \operatorname{Ho}\mathbf{Hc}_J(V)$ is a coinitial functor.*[7]

(iii) *$\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}$ is a filtered category,*[8] *and for any object q in $\mathbf{Hc}_J(V)$, $\operatorname{Ho}\mathbf{Hc}_J(V)_{/q}{}^{\mathrm{op}}$ is also filtered.*

(iv) *$\operatorname{Ho}\mathcal{K}^{\mathrm{op}}$ is an essentially small filtered category.*

[7] See definition A.5.31.
[8] See definition 0.2.1.

(v) *$\mathcal{K}$ (resp.* $\operatorname{Ho}\mathcal{K}$*) is a coinitial subcategory of* $\mathbf{Hc}_J(V)$ *(resp.* $\operatorname{Ho}\mathbf{Hc}_J(V)$*).*

Proof. (i). These are immediate consequences of lemma 8.2.20.

(ii). We must show that the comma category $(\pi \downarrow p)$ is connected for every J-hypercover $p : U \to \operatorname{disc} V$. It is inhabited: after all, the projection $\pi : \mathbf{Hc}_J(V) \to \operatorname{Ho}\mathbf{Hc}_J(V)$ is bijective on objects. Thus, the fact that any two J-hypercovers of V admit a common refinement implies that $(\pi \downarrow p)$ is indeed connected.

(iii). We will show that $\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}$ is filtered; similar arguments work for $\operatorname{Ho}\mathbf{Hc}_J(V)_{/q}{}^{\mathrm{op}}$. In view of claim (i), by lemma 0.2.4, it suffices to show the following: for any two J-hypercovers of V, say $p : U \to \operatorname{disc} V$ and $p' : U' \to \operatorname{disc} V$, given a parallel pair of morphisms $f_0, f_1 : U' \to U$ in $\mathbf{sPsh}(\mathcal{C})$ such that $p \circ f_0 = p \circ f_1 = p'$, there exist a morphism $e : U'' \to U$ such that $p'' = p' \circ e$ is a J-hypercover of V that is in $\mathcal{K}$ and $f_0 \circ e = f_1 \circ e$ in $\operatorname{Ho}\mathbf{Hc}_J(V)$.

Consider the following commutative diagram in $\mathbf{sPsh}(\mathcal{C})$,

$$\begin{tikzcd}
\partial\Delta^1 \odot U' \arrow[r, "f"] \arrow[d, hook] & U \arrow[d, "p"] \\
\Delta^1 \odot U' \arrow[r, "q'"'] & \operatorname{disc} V
\end{tikzcd}$$

where $f : \partial\Delta^1 \odot U' \to U$ is induced by the parallel pair $f_0, f_1 : U' \to U$ and the morphism $q' : \Delta^1 \odot U \to \operatorname{disc} V$ is induced by $\Delta^1 \to \Delta^0$ and $p' : U' \to V$. We can then apply proposition 8.2.24 to obtain morphisms $e : U'' \to U'$ and $h : \Delta^1 \odot U'' \to U$ such that the diagram below commutes,

$$\begin{tikzcd}
\partial\Delta^1 \odot U'' \arrow[r, "\mathrm{id}_{\partial\Delta^1} \odot e"] \arrow[d] & \partial\Delta^1 \odot U' \arrow[r, "f"] & U \arrow[d, "p"] \\
\Delta^1 \odot U'' \arrow[r, "\mathrm{id}_{\Delta^1} \odot e"'] \arrow[urr, dashed, "h"] & \Delta^1 \odot U' \arrow[r, "q'"'] & \operatorname{disc} V
\end{tikzcd}$$

and $p'' = p' \circ e : U'' \to \operatorname{disc} V$ is a J-hypercover. Thus, we have a simplicial homotopy $f_0 \circ e \Rightarrow f_1 \circ e$ *in the slice category* $\underline{\mathbf{sPsh}(\mathcal{C})}_{/\operatorname{disc} V}$, as required.

(iv). It is not hard to see that $\mathcal{K}$ is an essentially small category. We know that every J-hypercover of V can be refined by one that is in $\mathcal{K}$, so the filteredness of $\operatorname{Ho}\mathcal{K}^{\mathrm{op}}$ is a consequence of the filteredness of $\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}$.

(v). As with claim (ii), the fact that any two J-hypercovers of V admit a common refinement that is in $\mathcal{K}$ implies that $\mathcal{K}$ (resp. $\operatorname{Ho}\mathcal{K}$) is indeed a coinitial subcategory of $\mathbf{Hc}_J(V)$ (resp. $\operatorname{Ho}\mathbf{Hc}_J(V)$). ■

Lemma 8.5.8. *Let X be a J-locally fibrant simplicial presheaf on $\mathcal{C}$ and let Y be a projective-fibrant J-hypersheaf on $\mathcal{C}$. Given a J-local weak homotopy equivalence $f : X \to Y$, there is an induced bijection*

$$\varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}} \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(U, X) \to \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(\operatorname{disc} V, Y)$$

for each representable presheaf V, where $U : \operatorname{Ho}\mathbf{Hc}_J(V) \to \pi_0[\underline{\mathbf{sPsh}(\mathcal{C})}]$ is the functor that sends a hypercover of V to its domain. Moreover, this bijection is natural in $f : X \to Y$; and if $\mathcal{C}$ has pullbacks, then it is also natural in V (as V varies in the full subcategory of representable presheaves).

Proof. By proposition 1.5.18 and remark 8.3.3, the diagram $\pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(U, Y) : \operatorname{Ho}\mathbf{Hc}_J(V) \to \mathbf{Set}$ sends morphisms in $\operatorname{Ho}\mathbf{Hc}_J(V)$ to bijections; hence, the canonical comparison

$$\varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}} \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(U, Y) \to \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(\operatorname{disc} V, Y)$$

is a bijection. Thus, composition with $f : X \to Y$ induces a map

$$\varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(V)^{\mathrm{op}}} \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(U, X) \to \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(\operatorname{disc} V, Y)$$

and it is clearly natural in $f : X \to Y$ and also in V if $\mathcal{C}$ has pullbacks. It remains to be shown that this map is a bijection, but (recalling the explicit construction of colimits for filtered diagrams in **Set**) this is a straightforward consequence of (lemma 8.2.13 and) corollary 8.5.3. ■

Corollary 8.5.9. *Let X be a J-locally fibrant (resp. projective-fibrant) simplicial presheaf on $\mathcal{C}$ and let Y be a projective-fibrant J-hypersheaf on $\mathcal{C}$. Given a J-local weak homotopy equivalence $f : X \to Y$, there is an induced bijection*

$$\varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(h_C)^{\mathrm{op}}} \pi_0 \underline{\mathbf{sPsh}_\mathcal{C}}(Z \odot U, X) \to \operatorname{Ho}\mathbf{sSet}(Z, Y(C))$$

for any object C in $\mathcal{C}$ and any finite (resp. arbitrary) simplicial set Z, where $U : \operatorname{Ho}\mathbf{Hc}_J(h_C) \to \pi_0[\underline{\mathbf{sPsh}(\mathcal{C})}]$ is the functor that sends a hypercover of h_C to its domain. Moreover, this bijection is natural in $f : X \to Y$ and in Z; and if $\mathcal{C}$ has pullbacks, then it is also natural in C.

Proof. By definition, there are natural isomorphisms

$$\underline{\mathbf{sPsh}_{\mathcal{C}}}(Z \odot V, X) \cong \left[Z, \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, X)\right] \cong \underline{\mathbf{sPsh}_{\mathcal{C}}}(V, Z \pitchfork X)$$

and proposition 8.1.6 (resp. corollary 1.4.16 plus proposition 1.5.15) implies $Z \pitchfork X$ is a J-locally fibrant (resp. projective-fibrant) simplicial presheaf; but

$$\underline{\mathbf{sPsh}_{\mathcal{C}}}\left(\hbar_C, Z \pitchfork Y\right) \cong \left[Z, \underline{\mathbf{sPsh}_{\mathcal{C}}}\left(\hbar_C, Y\right)\right] \cong [Z, Y(C)]$$

by the Yoneda lemma, and remark 1.5.26 implies

$$\pi_0[Z, Y(C)] \cong \operatorname{Ho}\mathbf{sSet}(Z, Y(C))$$

so (using the fact that $Z \pitchfork Y$ is a projective-fibrant J-hypersheaf) we can construct the required natural bijection using lemma 8.5.8. ■

Remark 8.5.10. Given a discrete presheaf X on $\mathcal{C}$, if Y is the associated J-sheaf, then the canonical morphism $X \to Y$ is a J-local isomorphism and hence a J-local weak homotopy equivalence. Since $\operatorname{disc} X$ and $\operatorname{disc} Y$ are projective-fibrant as simplicial presheaves, we can compute the sets $Y(C)$ as follows:

$$Y(C) \cong \varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(\hbar_C)^{\mathrm{op}}} \underline{\mathbf{sPsh}_{\mathcal{C}}}(U, \operatorname{disc} X)$$

Note that $\underline{\mathbf{sPsh}_{\mathcal{C}}}(U, \operatorname{disc} X)$ is discrete as a simplicial set, so applying π_0 is the same as taking the set of vertices.

Proposition 8.5.11. *Let X be a projective-fibrant simplicial presheaf on $\mathcal{C}$ and let C be an object in $\mathcal{C}$. Consider the functor*

$$H_X = \varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(\hbar_C)^{\mathrm{op}}} \pi_0 \underline{\mathbf{sPsh}_{\mathcal{C}}}((-) \odot U, X) : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$$

where $U : \operatorname{Ho}\mathbf{Hc}_J\left(\hbar_C\right) \to \pi_0[\underline{\mathbf{sPsh}(\mathcal{C})}]$ is the functor that sends a hypercover of $\hbar_C$ to its domain.

(i) *$H_X : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$ factors through the localising functor $\mathbf{sSet} \to \operatorname{Ho}\mathbf{sSet}$ (in a unique way).*

(ii) *$H_X : \operatorname{Ho}\mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$ is representable.*

(iii) *For any projective-fibrant J-hypersheaf Y on $\mathcal{C}$, if there is a J-local weak homotopy equivalence $X \to Y$, then (the weak homotopy type of) $Y(C)$ represents $H_X : \operatorname{Ho}\mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$.*

Proof. (i). It is (necessary and) sufficient to show that $H_X : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$ sends weak homotopy equivalences to bijections. By proposition 8.5.7, we can replace $\operatorname{Ho}\mathbf{Hc}_J(\mathcal{h}_C)$ with a full subcategory whose objects are J-hypercovers of $\mathcal{h}_C$ whose domains are cellular simplicial presheaves. But if U is a cellular simplicial presheaf, then (by corollary 2.4.5, proposition 2.4.17, and remark 8.2.4) so is $Z \odot U$; and recalling corollary 2.4.6 and lemma 2.4.8, we may deduce that $\pi_0\underline{\mathbf{sPsh}_C}((-) \odot U, X) : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$ sends weak homotopy equivalences to bijections. Thus, by cofinality, we deduce that $H_X : \mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$ indeed sends weak homotopy equivalences to bijections.

(ii) and (iii). By theorem 8.3.25 (and proposition 4.1.24), there exist a projective-fibrant J-hypersheaf Y and a J-local weak homotopy equivalence $X \to Y$. Thus, we may apply corollary 8.5.9 to deduce that $Y(C)$ indeed represents $H_X : \operatorname{Ho}\mathbf{sSet}^{\mathrm{op}} \to \mathbf{Set}$. ■

Lemma 8.5.12. *Let X be a J-locally fibrant simplicial presheaf on $\mathcal{C}$, let Y be a projective-fibrant J-hypersheaf on $\mathcal{C}$, let $f : X \to Y$ be a J-local weak homotopy equivalence, let C be an object in $\mathcal{C}$, let $q : V \to \operatorname{disc}\mathcal{h}_C$ be a J-hypercover, and let $x : V \to X$ and $y : \operatorname{disc}\mathcal{h}_C \to Y$ be morphisms in $\mathbf{sPsh}(\mathcal{C})$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "x"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ \operatorname{disc}\mathcal{h}_C \arrow[r, "y"'] & Y \end{tikzcd}$$

For each positive integer n, there is an induced bijection

$$\varinjlim_{\operatorname{Ho}\mathbf{Hc}_J(\mathcal{h}_C)_{/q}{}^{\mathrm{op}}} \pi'_n\big(\underline{\mathbf{sPsh}_C}(U, X), x \circ g\big) \to \pi_n\big(\underline{\mathbf{sPsh}_C}\big(\operatorname{disc}\mathcal{h}_C, Y\big), y\big)$$

where π'_n abbreviates $\pi_{(\Delta^n/\partial\Delta^n,*)}$, U is the domain of the underlying J-hypercover of an object of $\operatorname{Ho}\mathbf{Hc}_J(\mathcal{h}_C)_{/q}$, and $g : U \to V$ is its underlying morphism.

Proof. First, note that $\pi'_n\big(\underline{\mathbf{sPsh}_C}(U, X), x \circ g\big)$ does indeed sends simplicially homotopic morphisms in $\underline{\mathbf{Hc}_J(\mathcal{h}_C)}_{/q}$ to equal maps, so we have a diagram of the announced shape. On the other hand, Y is a projective-fibrant J-hypersheaf, so (by remark 8.3.3 and theorem 1.4.29) the diagram

$$\pi_n\big(\underline{\mathbf{sPsh}_C}(U, Y), y \circ q \circ g\big) : \mathbf{Hc}_J(\mathcal{h}_C)_{/q}{}^{\mathrm{op}} \to \mathbf{Set}$$

is isomorphic to the constant diagram at $\pi_n\big(\underline{\mathbf{sPsh}_C\big(}\mathrm{disc}\,\hbar_C, Y\big), y\big)$. Proposition 8.5.7 says $\mathrm{Ho}\,\mathbf{Hc}_J\left(\hbar_C\right)_{/q}{}^{\mathrm{op}}$ is filtered, so (recalling the explicit description of colimits for filtered diagrams in **Set**) this is a straightforward consequence of (lemma 8.2.13 and) lemma 8.5.2. ■

— A —

Generalities

A.1 Adjoints and mates

We begin by recalling a standard definition:

Definition A.1.1. An **adjunction of categories** consists of the following data:

- A functor $F : \mathcal{C} \to \mathcal{D}$, called the **left adjoint**.
- A functor $G : \mathcal{D} \to \mathcal{C}$, called the **right adjoint**.
- A natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, called the **unit**.
- A natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, called the **counit**.

These are moreover required to satisfy the **triangle identities**:

$$\varepsilon F \bullet F\eta = \mathrm{id}_F \qquad\qquad G\varepsilon \bullet \eta G = \mathrm{id}_G$$

If such data exist, we write

$$F \dashv G : \mathcal{D} \to \mathcal{C}$$

and say that F is a left adjoint of G, and G is a right adjoint of F.

Proposition A.1.2. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$. The following are equivalent for an object X in $\mathcal{C}$:*

(i) *The morphism $\eta_X : X \to GFX$ is a monomorphism.*

(ii) *For all objects T in $\mathcal{C}$, the hom-set map $\mathcal{C}(T, X) \to \mathcal{D}(FT, FX)$ induced by $F : \mathcal{C} \to \mathcal{D}$ is injective.*

Dually, the following are equivalent for an object A in $\mathcal{D}$:

(i′) *The morphism $\varepsilon_A : FGA \to A$ is an epimorphism.*

(ii′) *For all objects B in $\mathcal{D}$, the hom-set map $\mathcal{D}(A, B) \to \mathcal{C}(GA, GB)$ is injective.*

Proof. Consider the hom-set map $\mathcal{C}(T, X) \to \mathcal{C}(T, GFX)$ defined by $x \mapsto \eta_X \circ x$. By naturality,

$$\eta_X \circ x = GFx \circ \eta_X$$

but the left triangle identity implies

$$\varepsilon_{FX} \circ F\big(\eta_X \circ x\big) = Fx$$

and so $\eta_X \circ x_0 = \eta_X \circ x_1$ if and only if $Fx_0 = Fx_1$. ■

Proposition A.1.3. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ε. The following are equivalent:*

(i) *The left adjoint $F : \mathcal{C} \to \mathcal{D}$ is fully faithful.*

(ii) *The adjunction unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is a natural isomorphism.*

(iii) *The natural transformation $F\eta G : FG \Rightarrow FGFG$ is a natural isomorphism, $F : \mathcal{C} \to \mathcal{D}$ is conservative, and $G : \mathcal{D} \to \mathcal{C}$ is essentially surjective on objects.*

Dually, the following are equivalent:

(i′) *The right adjoint $G : \mathcal{D} \to \mathcal{C}$ is fully faithful.*

(ii′) *The adjunction counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism.*

(iii′) *The natural transformation $G\varepsilon F : GFGF \Rightarrow GF$ is a natural isomorphism, $G : \mathcal{D} \to \mathcal{C}$ is conservative, and $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects.*

Proof. (i) ⇔ (ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By naturality, we have $\eta_Y \circ f = GFf \circ \eta_X$; but the triangle identities imply the hom-set map $\mathcal{D}(FX, B) \to \mathcal{C}(X, GB)$ given by $g \mapsto Gg \circ \eta_X$ is also a bijection, so we deduce that the hom-set map $\mathcal{C}(X, Y) \to \mathcal{C}(X, GFY)$ given by $f \mapsto \eta_Y \circ f$ is a bijection if and only if the hom-set map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ given by $f \mapsto Ff$ is a bijection because F is fully faithful. We may then deduce that η is a natural isomorphism if and only if F is fully faithful.

(i) ⇒ (iii). We have already shown that $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is a natural isomorphism, so in particular $F\eta G : FG \Rightarrow FGFG$ is a natural isomorphism. Fully faithful functors are conservative, so F is conservative. On the other hand, since η is a natural isomorphism, G is essentially surjective on objects.

(iii) ⇒ (ii). If F is conservative and $F\eta G$ is a natural isomorphism, then ηG is also a natural isomorphism. Since every object in $\mathcal{C}$ is isomorphic to one in the image of G, it follows that η is a natural isomorphism. ■

Proposition A.1.4. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction.*

- *$G : \mathcal{D} \to \mathcal{C}$ is fully faithful if and only if, for all categories $\mathcal{E}$, the induced functor $F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ is fully faithful.*
- *$F : \mathcal{C} \to \mathcal{D}$ is fully faithful if and only if, for all categories $\mathcal{E}$, the induced functor $G^* : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ is fully faithful.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $G : \mathcal{D} \to \mathcal{C}$ is fully faithful. By proposition A.1.3, the adjunction counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ must be a natural isomorphism. On the other hand, we have an induced adjunction $G^* \dashv F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ with counit induced by ε, so the same proposition implies F^* must be fully faithful.

Conversely, suppose $F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ is a fully faithful functor for all categories $\mathcal{E}$. Then the induced adjunction counit $\varepsilon^* : G^* F^* \Rightarrow \mathrm{id}_{[\mathcal{D}, \mathcal{E}]}$ is a natural isomorphism. In particular, this is true when $\mathcal{E} = \mathcal{D}$, so by considering the component of ε^* at $\mathrm{id}_{\mathcal{D}}$, we see that $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ itself is a natural isomorphism. Thus $G : \mathcal{D} \to \mathcal{C}$ must be fully faithful. ■

Proposition A.1.5. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be adjunctions, let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\eta' : \mathrm{id}_{\mathcal{C}'} \Rightarrow G'F'$ be the respective units, and let $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\varepsilon' : F'G' \Rightarrow \mathrm{id}_{\mathcal{D}'}$ be the respective counits. Let $H : \mathcal{C} \to \mathcal{C}'$ and*

$K : \mathcal{D} \to \mathcal{D}'$ be functors, and let φ and ψ be natural transformations as in the diagrams below:

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swArrow_{\varphi}"] & \mathcal{C}' \arrow[d, "F'"] \\
\mathcal{D} \arrow[r, "K"'] & \mathcal{D}'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "{}^{\psi}\!\neArrow"] & \mathcal{D}' \arrow[d, "G'"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{C}'
\end{tikzcd}
$$

Then, the following are equivalent:

(i) $\varepsilon' KF \bullet F'\psi F \bullet F'H\eta = \varphi$.

(ii) $\psi F \bullet H\eta = G'\varphi \bullet \eta' H$.

(iii) $\psi = G'K\varepsilon \bullet G'\varphi G \bullet \eta' HG$.

(iv) $\varepsilon' K \bullet F'\psi = K\varepsilon \bullet \varphi G$.

Proof.

(i) $\Rightarrow$ (ii).
$$
\begin{aligned}
G'\varphi \bullet \eta' H &= G'\varepsilon' KF \bullet G'F'\psi F \bullet G'F'H\eta \bullet \eta' H \\
&= G'\varepsilon' KF \bullet \eta' G'KF \bullet \psi F \bullet H\eta \\
&= \psi F \bullet H\eta
\end{aligned}
$$

(ii) $\Rightarrow$ (iii).
$$
\begin{aligned}
G'K\varepsilon \bullet G'\varphi G \bullet \eta' HG &= G'K\varepsilon \bullet \psi FG \bullet H\eta G \\
&= \psi \bullet HG\varepsilon \bullet H\eta G \\
&= \psi
\end{aligned}
$$

(iii) $\Rightarrow$ (iv).
$$
\begin{aligned}
\varepsilon' K \bullet F'\psi &= \varepsilon' K \bullet F'G'K\varepsilon \bullet F'G'\varphi G \bullet F'\eta' HG \\
&= K\varepsilon \bullet \varphi G \bullet \varepsilon' HG \bullet F'\eta' HG \\
&= K\varepsilon \bullet \varphi G
\end{aligned}
$$

(iv) $\Rightarrow$ (i).
$$
\begin{aligned}
\varepsilon' KF \bullet F'\psi F \bullet F'H\eta &= K\varepsilon F \bullet \varphi GF \bullet F'H\eta \\
&= K\varepsilon F \bullet KF\eta \bullet \varphi \\
&= \varphi
\end{aligned}
$$
∎

Definition A.1.6. A **conjugate pair of natural transformations** is a pair (φ, ψ) satisfying the equivalent conditions of the above proposition. Given such, we say φ is the **left mate** of ψ, and ψ is the **right mate** of φ.

Definition A.1.7. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ be functors, and let φ and ψ be a conjugate pair of natural transformations as in the diagrams below:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow_{\varphi}"] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, "K"'] & \mathcal{D}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "{}^{\psi}\nearrow"] & \mathcal{D}' \arrow[d, "G'"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \end{tikzcd}$$

We say the diagram on the right satisfies the **left Beck–Chevalley condition** if the left mate φ is a natural isomorphism, and we say the diagram on the left satisfies the **right Beck–Chevalley condition** if the right mate ψ is a natural isomorphism. More generally, the **local left Beck–Chevalley condition** is satisfied at an object C in $\mathcal{C}$ if the component $\varphi_C : F'HC \to KFC$ is an isomorphism, and the **local right Beck–Chevalley condition** is satisfied at an object D in $\mathcal{D}$ if the component $\psi_D : HGD \to G'KD$ is an isomorphism.

Remark A.1.8. Unfortunately, the Beck–Chevalley conditions are not vacuous. For example, consider the following (strictly!) commutative diagram of forgetful functors:

$$\begin{tikzcd} \mathbf{CRing} \arrow[r] \arrow[d] & \mathbf{Ab} \arrow[d] \\ \mathbf{Set} \arrow[r, "\mathrm{id}"'] & \mathbf{Set} \end{tikzcd}$$

The left mate of the trivial natural transformation in the above diagram is the group homomorphism $\mathbb{Z}X \to \mathbb{Z}[X]$ that sends a generator in $\mathbb{Z}X$ to the corresponding generator in $\mathbb{Z}[X]$; clearly, this is never an isomorphism. However, this is unsurprising: we do not expect the free abelian group generated by X to be naturally isomorphic to the additive group of free commutative ring generated by X.

Example A.1.9. Let $\mathcal{C}$ be a category with pullbacks, and suppose the following diagram is a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

Let $\Sigma_f : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ etc. be the functor that sends an object $p : E \to X$ in $\mathcal{C}_{/X}$ to the object $f \circ p : E \to Y$ in $\mathcal{C}_{/Y}$, and consider the induced (strictly!) commutative diagram of functors:

$$\begin{tikzcd}
\mathcal{C}_{/Z} \arrow[r, "\Sigma_z"] \arrow[d, "\Sigma_g"'] & \mathcal{C}_{/X} \arrow[d, "\Sigma_f"] \\
\mathcal{C}_{/W} \arrow[r, "\Sigma_w"'] & \mathcal{C}_{/Y}
\end{tikzcd}$$

Since $\mathcal{C}$ has pullbacks, Σ_g and Σ_f have right adjoints,[1] and the pullback pasting lemma then implies that the above square satisfies the right Beck–Chevalley condition.

Lemma A.1.10. *Given a diagram of functors and natural transformations of the form below,*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\psi}{\nearrow}"] & \mathcal{D}' \arrow[d, "G'"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{C}'
\end{tikzcd}$$

where $\psi : HG \Rightarrow G'K$ is a natural isomorphism, $F \dashv G$, and $F' \dashv G'$, for each object C in $\mathcal{C}$, the following are equivalent:

(i) *The diagram satisfies the local left Beck–Chevalley condition at C.*

(ii) *The functor $(C \downarrow G) \to (HC \downarrow G')$ sending an object (D, f) in the comma category $(C \downarrow G)$ to the object $\left(KD, \psi_D \circ Hf\right)$ in $(HC \downarrow G')$ preserves initial objects.*

Proof. We know $\left(FC, \eta_C\right)$ is an initial object of $(C \downarrow G)$ and $\left(F'HC, \eta'_{HC}\right)$ is an initial object of $(HC \downarrow G')$, so there is a unique morphism $\varphi_C : F'HC \to KFC$ such that $G'\varphi_C \circ \eta'_{HC} = \psi_{FC} \circ H\eta_C$. However, we observe that

$$\begin{aligned}
\varphi_C &= \varphi_C \circ \varepsilon'_{F'HC} \circ F'\eta'_{HC} \\
&= \varepsilon'_{KFC} \circ F'G'\varphi_C \circ F'\eta'_{HC} \\
&= \varepsilon'_{KFC} \circ F'\psi_{FC} \circ F'H\eta_C
\end{aligned}$$

so φ_C is precisely the component at C of the left mate of ψ. ■

[1] See lemma A.2.17.

Lemma A.1.11 (Pasting conjugate pairs)**.**

(i) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$, $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$, and $F'' \dashv G'' : \mathcal{D}'' \to \mathcal{C}''$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$, $H' : \mathcal{C}' \to \mathcal{C}''$, $K : \mathcal{D} \to \mathcal{D}'$, and $K' : \mathcal{D}' \to \mathcal{D}''$ be functors, and let $\varphi, \varphi', \psi, \psi'$ be natural transformations as in the following pasting diagrams:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\Swarrow_{\varphi}"] & \mathcal{C}' \arrow[r, "H'"] \arrow[d, "F'"] \arrow[dr, phantom, "\Swarrow_{\varphi'}"] & \mathcal{C}'' \arrow[d, "F''"] \\ \mathcal{D} \arrow[r, "K"'] & \mathcal{D}' \arrow[r, "K'"'] & \mathcal{D}'' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "{}^{\psi}\Nearrow"] & \mathcal{D}' \arrow[r, "K'"] \arrow[d, "G'"] \arrow[dr, phantom, "{}^{\psi'}\Nearrow"] & \mathcal{D}'' \arrow[d, "G''"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \arrow[r, "H'"'] & \mathcal{C}'' \end{tikzcd}$$

Let $\bar{\varphi} = K'\varphi \bullet \varphi' H$ and $\bar{\psi} = \psi' K \bullet H'\psi$. If (φ, ψ) and (φ', ψ') are conjugate pairs, then $(\bar{\varphi}, \bar{\psi})$ is also a conjugate pair.

(ii) *Let $F_1 \dashv G_1 : \mathcal{D} \to \mathcal{C}$, $F_2 \dashv G_2 : \mathcal{E} \to \mathcal{D}$, $F_1' \dashv G_1' : \mathcal{D}' \to \mathcal{C}'$, and $F_2' \dashv G_2' : \mathcal{E}' \to \mathcal{D}'$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$, $K : \mathcal{D} \to \mathcal{D}'$, and $L : \mathcal{E} \to \mathcal{E}'$ be functors, and let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be natural transformations as in the following pasting diagrams:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\Swarrow_{\varphi_1}"] & \mathcal{C}' \arrow[d, "F_1'"] \\ \mathcal{D} \arrow[r, "K"] \arrow[d, "F_2"'] \arrow[dr, phantom, "\Swarrow_{\varphi_2}"] & \mathcal{D}' \arrow[d, "F_2'"] \\ \mathcal{E} \arrow[r, "L"'] & \mathcal{E}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "L"] \arrow[d, "G_2"'] \arrow[dr, phantom, "{}^{\psi_2}\Nearrow"] & \mathcal{E}' \arrow[d, "G_2'"] \\ \mathcal{D} \arrow[r, "K"] \arrow[d, "G_1"'] \arrow[dr, phantom, "{}^{\psi_1}\Nearrow"] & \mathcal{D}' \arrow[d, "G_1'"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \end{tikzcd}$$

Let $\varphi = \varphi_2 F_1 \bullet F_2' \varphi_1$ and $\psi = G_1' \psi_2 \bullet \psi_1 G_2$. If $\big(\varphi_1, \psi_1\big)$ and $\big(\varphi_2, \psi_2\big)$ are conjugate pairs, then (φ, ψ) is also a conjugate pair.

Proof. These are straightforward exercises in using the triangle identities. ⧫

Proposition A.1.12. *Let $u_! \dashv u^* : \mathcal{C} \to \mathcal{A}$, $q_! \dashv q^* : \mathcal{B} \to \mathcal{D}$, $v^* \dashv v_* : \mathcal{B} \to \mathcal{C}$, and $p^* \dashv p_* : \mathcal{D} \to \mathcal{A}$ be adjunctions, and let $\theta : u^* p^* \Rightarrow v^* q^*$ be a natural transformation.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "{}^{\theta}\Nearrow"] & \mathcal{B} \arrow[d, "q^*"] \\ \mathcal{A} \arrow[r, "p^*"'] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "u^*"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\Swarrow_{\theta}"] & \mathcal{A} \arrow[d, "p^*"] \\ \mathcal{B} \arrow[r, "q^*"'] & \mathcal{D} \end{tikzcd}$$

The following are equivalent:

(i) *The diagram on the left satisfies the left Beck–Chevalley condition.*

(ii) *The diagram on the right satisfies the right Beck–Chevalley condition.*

Proof. Let $\varphi : q_! p^* \Rightarrow v^* u_!$ be the left mate of θ, and let $\psi : u^* v_* \Rightarrow p_* q^*$ be the right mate of θ. Then, by proposition A.1.5,

$$\theta u_! \bullet p^* \eta^u = q^* \varphi \bullet \eta^q p^* \qquad \varepsilon^q v^* \bullet q_! \theta = v^* \varepsilon^u \bullet \varphi u^*$$
$$\psi v^* \bullet u^* \eta^v = p_* \theta \bullet \eta^p u^* \qquad \varepsilon^p q^* \bullet p^* \psi = q^* \varepsilon^v \bullet \theta v_*$$

where the η denote the various adjunction units and the ε denote the various adjunction counits, thus:

$$\begin{aligned} \psi v^* u_! \bullet \left(u^* \eta^v u_! \bullet \eta^u\right) &= p_* \theta u_! \bullet \eta^p u^* u_! \bullet \eta^u \\ p_* \theta u_! \bullet p_* p^* \eta^u \bullet \eta^p &= p_* q^* \varphi \bullet \left(p_* \eta^q p^* \bullet \eta^p\right) \\ \left(\varepsilon^q \bullet q_! \varepsilon^p q^*\right) \bullet q_! p^* \psi &= \varepsilon^q \bullet q_! q^* \varepsilon^v \bullet q_! \theta v_* \\ \varepsilon^v \bullet \varepsilon^q v^* v_* \bullet q_! \theta v_* &= \left(\varepsilon^v \bullet v^* \varepsilon^u v_*\right) \bullet \varphi u^* v_* \end{aligned}$$

Thus, (φ, ψ) is a conjugate pair of natural transformations between the adjunctions $v^* u_! \dashv u^* v_*$ and $q_! p^* \dashv p_* q^*$. It follows (by lemma A.1.11) that φ is a natural isomorphism if and only if ψ is a natural isomorphism. ■

A.2 Cartesian closed categories

Definition A.2.1. Let C be a category with binary products, and let Y and Z be objects in C. An **exponential object** for Y and Z is an object $[Y, Z]_C$ in C and a morphism $\mathrm{ev}_{Y,Z} : [Y, Z]_C \times Y \to Z$ with the following universal property:

- For all morphisms $f : X \times Y \to Z$ in C, there exists a unique morphism $\bar{f} : X \to [Y, Z]_C$ such that $\mathrm{ev}_{Y,Z} \circ \left(\bar{f} \times \mathrm{id}_Y\right) = f$.

An **exponentiable object** in C is an object Y such that, for all objects Z in C, the exponential object $[Y, Z]_C$ exists. We may write $[Y, Z]$ or Z^Y instead of $[Y, Z]_C$ if there is no risk of confusion.

Lemma A.2.2. *Let Y be an object in a category C with binary products. The following are equivalent:*

(i) *Y is an exponentiable object in C.*

(ii) *The functor* $- \times Y : \mathcal{C} \to \mathcal{C}$ *has a right adjoint* $[Y, -]_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, *and the counit of this adjunction is* $\mathrm{ev}_{Y,-}$.

Proof. Immediate from the definitions. ♦

Definition A.2.3. A **cartesian closed category** is a category with finite products, in which every object is exponentiable. A **locally cartesian closed category** is a category $\mathcal{C}$ such that, for every object I, the slice category $\mathcal{C}_{/I}$ is a cartesian closed category.

Example A.2.4. Set is cartesian closed category; in fact, it is even a locally cartesian closed category.

Proposition A.2.5. *Let* $\mathcal{C}$ *be a cartesian closed category.*

(i) *The assignment* $(Y, Z) \mapsto [Y, Z]_{\mathcal{C}}$ *extends to a functor* $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

(ii) *For each object* Z, *the functor* $[-, Z]_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ *is a contravariant right adjoint for itself.*

Proof. (i). This is an instance of the parametrised adjunction theorem.[2]

(ii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}(X, [Y, Z]) &\cong \mathcal{C}(X \times Y, Z) \\ &\cong \mathcal{C}(Y \times X, Z) \\ &\cong \mathcal{C}(Y, [X, Z]) \end{aligned}$$

■

Lemma A.2.6. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be cartesian closed categories. If* $F : \mathcal{C} \to \mathcal{D}$ *is a functor that preserves binary products, then:*

(i) *For any two objects* X *and* Y *in* $\mathcal{C}$, *there is a unique morphism* $\varphi_{Y,Z} : F[X, Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ *such that the following diagram commutes:*

$$\begin{tikzcd} F[X,Y]_{\mathcal{C}} \times FX \arrow[r, "\cong"] \arrow[d, "\varphi_{X,Y} \times \mathrm{id}"'] & F\big([X,Y]_{\mathcal{C}} \times X\big) \arrow[d, "F\mathrm{ev}_{X,Y}"] \\ {[FX, FY]_{\mathcal{D}} \times FX} \arrow[r, "\mathrm{ev}_{FX,FY}"'] & FY \end{tikzcd}$$

[2] See Theorem 3 in [CWM, Ch. IV, §7].

(ii) *The morphism $\varphi_{Y,Z}$ is natural in both Y and Z.*

Proof. The existence and uniqueness of $\varphi_{X,Y}$ follows from the universal property of $[FX, FY]_{\mathcal{D}}$ as an exponential object, and a standard argument proves naturality. ⧫

Definition A.2.7. A **cartesian closed functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ between cartesian closed categories such that the canonical comparison morphisms $\varphi_{X,Y} : F[X,Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ described above are isomorphisms.

Proposition A.2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories, and let Y be an object in $\mathcal{C}$ and let Z be an object in $\mathcal{D}$. Suppose we have an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$; then:*

(i) *If $\psi_{FY,Z} : G[FY,Z]_{\mathcal{D}} \to [GFY, GZ]_{\mathcal{C}}$ is the canonical comparison morphism, then $\theta_{Y,Z} = \left[\eta_Y, GZ\right]_{\mathcal{C}} \circ \psi_{FY,Z}$ is the unique morphism in $\mathcal{C}$ making the following diagram commute:*

$$\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\mathrm{id}\times\eta_Y"] \arrow[d, "\theta_{Y,Z}\times\mathrm{id}"'] & G[FY,Z]_{\mathcal{D}} \times GFY \arrow[d, "\cong"] \\
{[Y,GZ]_{\mathcal{C}} \times Y} \arrow[d, "\mathrm{ev}_{Y,GZ}"'] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[d, "G\mathrm{ev}_{FY,Z}"] \\
GZ \arrow[r, equal] & GZ
\end{tikzcd}$$

(ii) *If the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism for all objects X in $\mathcal{C}$, and $\varphi_{Y,GZ} : F[Y,GZ]_{\mathcal{C}} \to [FY, FGZ]_{\mathcal{D}}$ is the canonical comparison morphism, then $\chi_{Y,Z} = \left[FY, \varepsilon_Z\right]_{\mathcal{D}} \circ \varphi_{Y,GZ}$ is the unique morphism in $\mathcal{D}$ making the following diagram commute:*

$$\begin{tikzcd}
F[Y,GZ]_{\mathcal{C}} \times FY \arrow[r, "\cong"] \arrow[d, "\chi_{Y,Z}\times\mathrm{id}"'] & F\big([Y,GZ]_{\mathcal{C}} \times Y\big) \arrow[d, "\mathrm{ev}_{Y,GZ}"] \\
{[FY,Z]_{\mathcal{D}} \times FY} \arrow[d, "\mathrm{ev}_{FY,Z}"'] & FGZ \arrow[d, "\varepsilon_Z"] \\
Z \arrow[r, equal] & Z
\end{tikzcd}$$

Moreover, under this hypothesis, $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$ is a two-sided inverse for $\theta_{Y,Z}$.

(iii) *If $\theta_{Y,Z}$ is an isomorphism for all objects Z in $\mathcal{D}$, then for all objects X in $\mathcal{C}$, the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism.*

Proof. (i). The claim is proven by the commutativity of the following diagram:

$$\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\mathrm{id}\times\eta_Y"] \arrow[d, "\psi_{FY,Z}\times\mathrm{id}"'] & G[FY,Z]_{\mathcal{D}} \times GFY \arrow[r, "\cong"] \arrow[d, "\psi_{FY,Z}\times\mathrm{id}"] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[dd, "G\mathrm{ev}_{FY,Z}"] \\
{[GFY,GZ]_{\mathcal{C}} \times Y} \arrow[r, "\mathrm{id}\times\eta_Y"'] \arrow[d, "{[\eta_Y,Z]_{\mathcal{C}}}"'] & {[GFY,GZ]_{\mathcal{C}} \times GFY} \arrow[dr, "\mathrm{ev}_{GFY,GZ}"] & \\
{[Y,GZ]_{\mathcal{C}}} \arrow[rr, "\mathrm{ev}_{Y,GZ}"'] & & GZ
\end{tikzcd}$$

(ii). To show that $\chi_{Y,Z}$ makes the diagram commute, one uses the fact that $\mathrm{ev}_{FY,Z} : [FY,Z]_{\mathcal{D}} \times FY \to Z$ is natural in Z. Since F preserves products with Y, we have the following natural bijections:

$$\begin{aligned}
\mathcal{C}\big(X, G[FY,Z]_{\mathcal{D}}\big) &\cong \mathcal{D}\big(FX, [FY,Z]_{\mathcal{D}}\big) \cong \mathcal{D}(FX \times FY, Z) \\
&\cong \mathcal{D}(F(X \times Y), Z) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{C}\big(X, [Y,GZ]_{\mathcal{C}}\big)
\end{aligned}$$

One obtains explicit isomorphisms by chasing id_X in both directions. Taking $X = [Y,GZ]_{\mathcal{C}}$, we find that the isomorphism $[Y,GZ]_{\mathcal{C}} \to G[FY,Z]_{\mathcal{D}}$ is precisely $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$, and taking $X = G[FY,Z]_{\mathcal{D}}$, we find that the inverse is the right exponential transpose of

$$G\big(\mathrm{ev}_{FY,Z} \circ \big(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_Y\big)\big) \circ \eta_{G[FY,Z]_{\mathcal{D}} \times Y}$$

where we have suppressed the comparison isomorphism $F\big(G[FY,Z]_{\mathcal{D}} \times Y\big) \cong FG[FY,Z]_{\mathcal{D}} \times FY$; but naturality of the comparison morphisms for binary products gives us the commutative diagram below,

$$\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\eta"] \arrow[dd, equal] & GF\big(G[FY,Z]_{\mathcal{D}} \times Y\big) \arrow[d, "\cong"'] & \\
& G\big(FG[FY,Z]_{\mathcal{D}} \times FY\big) \arrow[r, "G(\varepsilon\times\mathrm{id})"] \arrow[d, "\cong"'] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[d, "\cong"] \\
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\eta\times\eta"] \arrow[rr, bend right, "\mathrm{id}\times\eta"'] & GFG[FY,Z]_{\mathcal{D}} \times GFY \arrow[r, "G\varepsilon\times\mathrm{id}"] & G[FY,Z]_{\mathcal{D}} \times GFY
\end{tikzcd}$$

so, suppressing the comparison isomorphisms, we obtain the following equation:

$$G\big(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_{FY}\big) \circ \eta_{G[FY,Z]_{\mathcal{D}} \times Y} = \mathrm{id}_{G[FY,Z]_{\mathcal{D}}} \times \eta_Y$$

Thus, the isomorphism $G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is indeed $\theta_{Y,Z}$, as claimed.

(iii). Now, suppose $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is an isomorphism for all Z. Then, we have the natural bijections

$$\begin{aligned} \mathcal{D}(FX \times FY, Z) &\cong \mathcal{D}\big(FX, [FY, Z]_{\mathcal{D}}\big) \cong \mathcal{C}\big(X, G[FY, Z]_{\mathcal{D}}\big) \\ &\cong \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{D}(F(X \times Y), Z) \end{aligned}$$

and by chasing id_Z for $Z = FX \times FY$, we conclude that the *canonical* comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism. ■

Definition A.2.9. A **Frobenius adjunction of cartesian closed categories** is an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories, such that the natural morphisms $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ described above are isomorphisms, or equivalently, such that the left adjoint $F : \mathcal{C} \to \mathcal{D}$ preserves binary products.

Remark A.2.10. If $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories and $G : \mathcal{D} \to \mathcal{C}$ is any functor that preserves finite products, then G induces a $\mathcal{C}$-enrichment of $\mathcal{D}$ from the cartesian closed structure of $\mathcal{D}$, and the exponential comparison morphisms $\psi_{Y,Z} : G[Y, Z]_{\mathcal{D}} \to [GY, GZ]_{\mathcal{C}}$ makes $G : \mathcal{D} \to \mathcal{C}$ into a $\mathcal{C}$-enriched functor.

Now, suppose G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$. The adjunction $F \dashv G$ is a Frobenius adjunction precisely when it is compatible with the $\mathcal{C}$-enrichments of $\mathcal{C}$ and $\mathcal{D}$. (Of course, this means F is also a $\mathcal{C}$-enriched functor.)

However, not all enriched adjunctions between cartesian closed categories are of the above form.

Proposition A.2.11. *Let X, Y, and Z be any three objects in a cartesian closed category $\mathcal{C}$.*

(i) *There is a unique morphism $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ making* *the following diagram commute:*

$$\begin{tikzcd}
([X \times Y, Z] \times X) \times Y \arrow[r, "\cong"] \arrow[d, "(\lambda_{X,Y,Z} \times \mathrm{id}_X) \times \mathrm{id}_Y"'] & {[X \times Y, Z] \times (X \times Y)} \arrow[dd, "\mathrm{ev}_{X \times Y, Z}"] \\
([X, [Y, Z]] \times X) \times Y \arrow[d, "\mathrm{ev}_{X,[Y,Z]} \times \mathrm{id}_X"'] & \\
{[Y, Z] \times Y} \arrow[r, "\mathrm{ev}_{Y,Z}"'] & Z
\end{tikzcd}$$

(ii) *The morphisms* $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ *constitute a natural isomorphism.*

Proof. The existence and uniqueness of $\lambda_{X,Y,Z}$ follows from the universal property of $[X, [Y, Z]]$ and $[Y, Z]$ as exponential objects, and a standard argument shows that $\lambda_{X,Y,Z}$ is natural in X, Y, and Z. By the associativity of cartesian products, we have the following natural bijections:

$$\begin{aligned} \mathcal{C}(T, [X \times Y, Z]) &\cong \mathcal{C}(T \times (X \times Y), Z) \\ &\cong \mathcal{C}((T \times X) \times Y, Z) \cong \mathcal{C}(T \times X, [Y, Z]) \cong \mathcal{C}(T, [X, [Y, Z]]) \end{aligned}$$

Chasing id_T for $T = [X \times Y, Z]$, we find that $\lambda_{X,Y,Z}$ is an isomorphism. ■

Definition A.2.12. Let $\mathcal{C}$ be a cartesian closed category. An **exponential ideal** of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that, for all objects Y in $\mathcal{C}$, if Z is in $\mathcal{D}$, then the exponential object $[Y, Z]_{\mathcal{C}}$ is (isomorphic to) an object in $\mathcal{D}$. A **reflective exponential ideal** of $\mathcal{C}$ is an exponential ideal $\mathcal{D}$ such that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint.

Proposition A.2.13. *Let* $\mathcal{C}$ *be a cartesian closed category, let* $G : \mathcal{D} \to \mathcal{C}$ *be the inclusion of a full subcategory, and suppose* G *has a left adjoint* $F : \mathcal{C} \to \mathcal{D}$*. The following are equivalent:*

(i) F *preserves finite products.*

(ii) F *preserves binary products.*

(iii) $\mathcal{D}$ *is a reflective exponential ideal of* $\mathcal{C}$*.*

(iv) $\mathcal{D}$ *is a cartesian closed category,* $G : \mathcal{D} \to \mathcal{C}$ *is a cartesian closed functor, and the canonical morphisms* $G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ *are isomorphisms.*

Proof. (i) ⇒ (ii). Immediate.

(ii) ⇒ (iii). Under our hypotheses, the product of two objects X and Y in $\mathcal{D}$ can be computed as $F(GX \times GY)$. Let $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ be the unit of the adjunction. We have the following natural bijections:

$$\begin{aligned} \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big) &\cong \mathcal{C}(X \times Y, GZ) \\ &\cong \mathcal{D}(FX \times FY, Z) \\ &\cong \mathcal{D}(FGFX \times FY, Z) \\ &\cong \mathcal{C}(GFX \times Y, GZ) \\ &\cong \mathcal{C}\big(GFX, [Y, GZ]_{\mathcal{C}}\big) \end{aligned}$$

By chasing these maps explicitly, we find that every morphism $X \to [Y, GZ]_{\mathcal{C}}$ factors through $\eta_X : X \to GFX$ in a unique way. In particular, we have

$$\mathrm{id}_{[Y,GZ]_{\mathcal{C}}} = r_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$$

for a unique $r_{Y,Z} : GF[Y, GZ]_{\mathcal{C}} \to [Y, GZ]_{\mathcal{C}}$. The triangle identity then implies $Fr_{Y,Z} = \varepsilon_{F[Y,GZ]_{\mathcal{C}}}$, thus,

$$\eta_{[Y,GZ]_{\mathcal{C}}} \circ r_{Y,Z} = GFr_{Y,Z} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = G\varepsilon_{F[Y,GZ]_{\mathcal{C}}} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = \mathrm{id}_{GF[Y,GZ]_{\mathcal{C}}}$$

and therefore $r_{Y,Z}$ is an isomorphism.

(iii) ⇒ (iv). It is a standard fact that a reflective subcategory is closed under all limits that exist in $\mathcal{C}$, so $\mathcal{D}$ must have finite products and $G : \mathcal{D} \to \mathcal{C}$ preserves them. If $\mathcal{D}$ is an exponential ideal, then $\eta_{[Y,GZ]_{\mathcal{C}}} : [Y, GZ]_{\mathcal{C}} \to GF[Y, GZ]_{\mathcal{C}}$ must be an isomorphism, so we obtain natural bijections

$$\begin{aligned} \mathcal{D}(X \times Y, Z) &\cong \mathcal{C}(GX \times GY, GZ) \\ &\cong \mathcal{C}\big(GX, [GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{C}\big(GX, GF[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(FGX, F[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(X, F[GY, GZ]_{\mathcal{C}}\big) \end{aligned}$$

and therefore we may take $[Y, Z]_{\mathcal{D}} = F[GY, GZ]_{\mathcal{C}}$. Obviously, this makes $G : \mathcal{D} \to \mathcal{C}$ into a cartesian closed functor. We also have

$$\mathcal{C}\big(X, G[FY, Z]_{\mathcal{D}}\big) = \mathcal{C}\big(X, GF[GFY, GZ]_{\mathcal{C}}\big)$$

$$
\begin{aligned}
&\cong \mathcal{C}\big(X, [GFY, GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, [X, GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, GF[X, GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, GF[X, GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, [X, GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big)
\end{aligned}
$$

and so the canonical morphism $G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ is an isomorphism.

(iv) ⇒ (i). Since $\mathcal{D}$ has a terminal object and $G : \mathcal{D} \to \mathcal{C}$ preserves it, $F1$ must be a terminal object in $\mathcal{D}$. Now apply proposition A.2.8. ■

Corollary A.2.14. *If $\mathcal{E}$ is a reflective exponential ideal of $\mathcal{D}$, and $\mathcal{D}$ is a reflective exponential ideal of $\mathcal{C}$, then $\mathcal{E}$ is also a reflective exponential ideal of $\mathcal{C}$.* ■

Proposition A.2.15. *Let* **Cat** *be the category of small categories, and let* **Grpd** *be the full subcategory of groupoids.*

(i) *There exist adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Cat}$$

where $\mathrm{ob}\,\mathbb{C}$ *is the set of objects in a category* $\mathbb{C}$*,* $\mathrm{disc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and all arrows trivial, and* $\mathrm{codisc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and a unique arrow between any two objects.*

(ii) *The functor* $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ *is fully faithful and exhibits* **Set** *as a reflective exponential ideal of* **Cat**.

(iii) *The functor* $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ *preserves finite products.*

(iv) *There exist adjunctions*

$$\mathbf{I} \dashv \mathrm{und} \dashv \mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$$

where $\mathrm{und} : \mathbf{Grpd} \to \mathbf{Cat}$ *is the inclusion and* $\mathrm{iso}\,\mathbb{C}$ *is the maximal subgroupoid of a category* $\mathbb{C}$*.*

(v) **Grpd** *is a reflective exponential ideal of* **Cat**.

(vi) *The functor* $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$ *preserves finite products.*

(vii) *The adjunctions in (i) factor through* **Grpd**, *yielding adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Grpd}$$

where $\pi_0 : \mathbf{Grpd} \to \mathbf{Set}$ *again preserves finite products.*

(viii) *The functor* $\mathbf{Cat} \to \mathbf{Set}$ *that sends a category* $\mathbb{C}$ *to the set of isomorphism classes in* $\mathbb{C}$ *preserves finite products.*

Proof. (i). The functor $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ obviously satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$; the existence of the other adjunctions is obvious.

(ii). It is clear that $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ is fully faithful, and direct computation shows that $[\mathbb{C}, \mathrm{disc}\, X]$ is a discrete category for any $\mathbb{C}$, so **Set** is indeed a reflective exponential ideal of **Cat**.

(iii). Thus, by proposition A.2.13, $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ must preserve finite products.

(iv). It is not hard to check that the inclusion $\mathbf{Grpd} \to \mathbf{Cat}$ satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$; the fact that $\mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$ is right adjoint to the inclusion is obvious.

(v). Direct computation shows that $[\mathbb{C}, \mathbb{G}]$ is a groupoid whenever $\mathbb{G}$ is, so **Grpd** is indeed a reflective exponential ideal of **Cat**.

(vi). Thus, $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$ must preserve finite products.

(vii). Clearly, $\mathrm{disc}\, X$ and $\mathrm{codisc}\, X$ are already groupoids, so the adjunctions do indeed factor through **Grpd**.

(viii). The set of isomorphism classes of objects in $\mathbb{C}$ is precisely $\pi_0\, \mathrm{iso}\, \mathbb{C}$. ■

Definition A.2.16. Let $\mathcal{C}$ be any category. The **dependent sum** of an object $p : X \to I$ in $\mathcal{C}_{/I}$ along a morphism $j : I \to J$ in $\mathcal{C}$ is the object $j \circ p : X \to J$ in $\mathcal{C}_{/J}$, and we write $\Sigma_j : \mathcal{C}_{/I} \to \mathcal{C}_{/J}$ for the functor sending an object to its dependent sum along j.

Lemma A.2.17. *Let* $j : I \to J$ *be a morphism in a category* $\mathcal{C}$. *The following are equivalent:*

(i) *C has pullbacks along j.*

(ii) *There exists an adjunction*

$$\Sigma_j \dashv j^* : C_{/J} \to C_{/I}$$

where Σ_j is the dependent sum functor, and the right adjoint $j^* : C_{/J} \to C_{/I}$ is the pullback functor.

Proof. This is just a matter of unwinding the definitions. ♦

Definition A.2.18. Let C be a category with pullbacks. A **dependent product** of an object $p : X \to I$ in $C_{/I}$ along a morphism $j : I \to J$ in C is an object $\Pi_j p$ in $C_{/J}$ and a morphism $\mathrm{ev}_{j,p} : j^*\Pi_j p \to p$ in $C_{/I}$ with the following universal property:

- For all morphisms $f : j^*q \to p$ in $C_{/I}$, there exists a unique morphism $\bar{f} : q \to \Pi_j p$ in $C_{/J}$ such that $\mathrm{ev}_{j,p} \circ j^*\bar{f} = f$.

A **$\Sigma\Pi$-category** is a category C with finite limits such that, for every morphism $j : I \to J$ in C, dependent products along j exist.

Lemma A.2.19. *Let $j : I \to J$ be a morphism in a category C with pullbacks. The following are equivalent:*

(i) *For all objects $p : X \to I$ in C, a dependent product of p along j exists.*

(ii) *The pullback functor $j^* : C_{/J} \to C_{/I}$ has a right adjoint $\Pi_j : C_{/I} \to C_{/J}$ that sends an object to its dependent product along j, and the counit of this adjunction is $\mathrm{ev}_{j,-}$.*

Proof. This is just a matter of unwinding the definitions. ♦

Corollary A.2.20. *If $j : I \to J$ is a morphism in a $\Sigma\Pi$-category C, then the pullback functor $j^* : C_{/J} \to C_{/I}$ preserves all limits and colimits.* ■

Proposition A.2.21. *Let C be a category with a terminal object. The following are equivalent:*

(i) *C is a $\Sigma\Pi$-category.*

(ii) *C is a locally cartesian closed category.*

Proof. See Proposition 9.20 in [Awodey, 2010]. □

Theorem A.2.22. *Let $\mathbb{D}$ be a small category, and let $\mathcal{C} = [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$. Then:*

(i) *$\mathcal{C}$ has limits and colimits for all small diagrams, and these can be constructed componentwise in* **Set***: a cone (resp. cocone) in $\mathcal{C}$ over (resp. under) a diagram in $\mathcal{C}$ is a limiting cone (resp. colimiting cocone) if and only if it is so in every component.*

(ii) *Every internal equivalence relation in $\mathcal{C}$ is the kernel pair of its coequaliser.*

(iii) *For all morphisms $j : I \to J$ in $\mathcal{C}$, the pullback functor $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$ preserves all limits and colimits.*

(iv) *The Yoneda embedding $\hbar_\bullet : \mathbb{D} \to \mathcal{C}$ is a dense functor, i.e. for every presheaf $X : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$, the tautological cocone[3] from the canonical diagram $\left(\hbar_\bullet \downarrow X\right) \to \mathcal{C}$ to X is a colimiting cocone.*

(v) *$\mathcal{C}$ is a locally finitely presentable category.*

(vi) *$\mathcal{C}$ is a $\Sigma\Pi$-category.*

Proof. (i). This is a standard fact about presheaf categories.

(ii) and (iii). The claims are true for **Set**, and hence for $\mathcal{C}$ by claim (i).

(iv). See proposition A.5.25.

(v). See theorem 0.2.40.

(vi). Apply theorem 0.2.50 to construct a right adjoint for $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$. ■

Remark A.2.23. The Yoneda lemma gives us an explicit description of the exponential objects in $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$: given two presheaves $Y, Z : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$, if Z^Y is their exponential object, then we must have

$$Z^Y(d) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\left(\hbar_d, Z^Y\right) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\left(\hbar_d \times Y, Z\right)$$

and so we may *define* Y^Z by $Y^Z(d) = [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\left(\hbar_d \times Y, Z\right)$.

[3] See definition A.5.7.

Definition A.2.24. Let Y and Z be topological spaces, and let $[Y, Z]$ be the set of all *continuous* maps $Y \to Z$. The **compact–open topology** on $[Y, Z]$ is the coarsest topology such that the subsets

$$V(K, U) = \left\{ f \in [Y, Z] \,\middle|\, K \subseteq f^{-1}U \right\}$$

are open in $[Y, Z]$ for all compact subsets $K \subseteq X$ and all open subsets $U \subseteq Y$.

Remark A.2.25. If Y is a discrete space, then the compact–open topology on $[Y, Z]$ coincides with the product topology on Z^Y.

Definition A.2.26. A **compactly generated Hausdorff space** is a Hausdorff topological space X such that a subset $U \subseteq X$ is open if and only if, for every continuous map $f : K \to X$ where K is a compact Hausdorff space, $f^{-1}U$ is an open subset of K. We write **CGHaus** for the category of compactly generated Hausdorff spaces and continuous maps.

Proposition A.2.27.

(i) *If Y is a locally compact Hausdorff space, then for all topological spaces Z, the set of all continuous maps $Y \to Z$, equipped with the compact–open topology, is an exponential object* $[Y, Z]$ *in* **Top**.

(ii) **Top** *is* not *a cartesian closed category.*

(iii) **CGHaus** *is a cartesian closed category.*

Proof. Claim (i) follows from Theorems 46.10 and 46.11 in [Munkres, 2000], and claim (ii) is Proposition 7.1.2 in [Borceux, 1994a], and claim (iii) is proved in [GZ, Ch. III, §2]. □

A.3 Factorisation systems

Definition A.3.1. Let $\mathcal{C}$ be a category.

- Let $f : X \to Y$ and $g : Z \to W$ be morphisms in $\mathcal{C}$. Given a commutative square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

  a **lift** is a morphism $h : W \to X$ such that $f \circ h = w$ and $h \circ g = z$.

- We say g has the **left lifting property** with respect to f and f has the **right lifting property** with respect to g, and we write $g \boxslash f$, if every commutative square in $\mathcal{C}$ of the form above has a lift.

- We say f is **left orthogonal** to g and g is **right orthogonal** to f, and we write $g \perp f$ if lifts exist *and* are unique.

- Given $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, we define the following subensembles of $\operatorname{mor} \mathcal{C}$:

$$
\begin{aligned}
{}^{\boxslash}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\ f \boxslash g \} \\
\mathcal{I}^{\boxslash} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\ f \boxslash g \} \\
{}^{\perp}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\ f \perp g \} \\
\mathcal{I}^{\perp} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\ f \perp g \}
\end{aligned}
$$

Lemma A.3.2. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a locally small category $\mathcal{C}$. Consider the commutative diagram in* **Set** *shown below,*

$$
\begin{tikzcd}
\mathcal{C}(W, X) \arrow[dr, dashed] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "g^*"'] & & \\
& \bullet \arrow[r] \arrow[d] & \mathcal{C}(W, Y) \arrow[d, "g^*"] \\
& \mathcal{C}(Z, X) \arrow[r, "f_*"'] & \mathcal{C}(Z, Y)
\end{tikzcd}
$$

where the inner square is a pullback diagram.

(i) *The dashed arrow is a surjection if and only if $g \boxslash f$.*

(ii) *The dashed arrow is a bijection if and only if $g \perp f$.*

Proof. This is just a restatement of the definition. ∎

Proposition A.3.3. *Let $\mathcal{C}$ be a category.*

(i) *If $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\perp}\mathcal{R} \subseteq {}^{\boxslash}\mathcal{R}$.*

(ii) *If $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\boxslash}\mathcal{R}' \supseteq {}^{\boxslash}\mathcal{R}$.*

(iii) *If $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\perp}\mathcal{R}' \supseteq {}^{\perp}\mathcal{R}$.*

Dually:

(i′) *If* $\mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash}$.

(ii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}'^{\boxslash} \supseteq \mathcal{L}^{\boxslash}$.

(iii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}'^{\perp} \supseteq \mathcal{L}^{\perp}$.

Moreover, we have the following antitone Galois connections:

$$\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R} \textit{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\boxslash}$$
$$\mathcal{L} \subseteq {}^{\perp}\mathcal{R} \textit{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\perp}$$

Proof. Obvious. ♦

Corollary A.3.4. *We have the following identities:*

$$\begin{aligned} {}^{\boxslash}\big(({}^{\boxslash}\mathcal{R})^{\boxslash}\big) &= {}^{\boxslash}\mathcal{R} & \qquad {}^{\perp}\big(({}^{\perp}\mathcal{R})^{\perp}\big) &= {}^{\perp}\mathcal{R} \\ \big({}^{\boxslash}(\mathcal{L}^{\boxslash})\big)^{\boxslash} &= \mathcal{L}^{\boxslash} & \qquad \big({}^{\perp}(\mathcal{L}^{\perp})\big)^{\perp} &= \mathcal{L}^{\perp} \end{aligned}$$

Proof. This is a standard fact about (antitone) Galois connections. ■

Definition A.3.5. A **orthogonality-reflecting functor** is a functor $U : \mathcal{C} \to \mathcal{D}$ with the following property:

- Given a commutative square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

  for each morphism $h : UW \to UX$ in $\mathcal{D}$ making the diagram below commute,

$$\begin{tikzcd} UZ \arrow[r, "Uz"] \arrow[d, "Ug"'] & UX \arrow[d, "Uf"] \\ UW \arrow[r, "Uw"'] \arrow[ur, "h"] & UY \end{tikzcd}$$

  there is a *unique* morphism $\tilde{h} : W \to X$ in $\mathcal{C}$ such that $U\tilde{h} = h$, $f \circ h = w$, and $h \circ g = z$.

Lemma A.3.6. *Let $U : \mathcal{C} \to \mathcal{D}$ be a functor between locally small categories. The following are equivalent:*

(i) *$U : \mathcal{C} \to \mathcal{D}$ is a orthogonality-reflecting functor.*

(ii) *For any morphisms $Z \to W$ and $X \to Y$ in $\mathcal{C}$, the induced commutative diagram*

$$\begin{tikzcd}
\mathcal{C}(W, X) \arrow[r] \arrow[d, "U"'] & \mathcal{C}(Z, X) \times_{\mathcal{C}(Z,Y)} \mathcal{C}(W, Y) \arrow[d, "U"] \\
\mathcal{D}(UW, UX) \arrow[r] & \mathcal{D}(UZ, UX) \times_{\mathcal{D}(UZ,UY)} \mathcal{D}(UW, UY)
\end{tikzcd}$$

is a pullback square in **Set**.

Proof. This is just a restatement of the definition. ■

Lemma A.3.7. *Let $\mathcal{C}$ be a category.*

- *For any object A in $\mathcal{C}$, the projection ${}^{A/}\mathcal{C} \to \mathcal{C}$ is an orthogonality-reflecting functor.*
- *For any object B in $\mathcal{C}$, the projection $\mathcal{C}_{/B} \to \mathcal{C}$ is an orthogonality-reflecting functor.*

Proof. The two claims are formally dual; we will prove the first version.

Since the forgetful functor ${}^{A/}\mathcal{C} \to \mathcal{C}$ is faithful, the uniqueness clause in the definition is automatically satisfied; it thus suffices to verify existence. Suppose we have the following commutative diagrams in $\mathcal{C}$:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& W \arrow[r, "w"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
W \arrow[r, "w"'] \arrow[ur, "h"] & Y
\end{tikzcd}$$

Then the following diagram also commutes:

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& W \arrow[ur, "h"] \arrow[r, "w"'] & Y
\end{tikzcd}$$

This completes the proof. ■

Proposition A.3.8. *Let $U : \mathcal{C} \to \mathcal{D}$ be an orthogonality-reflecting functor and let $f : X \to Y$ and $g : Z \to W$ be morphisms in $\mathcal{C}$.*

1. *If $U g \boxslash U f$ in $\mathcal{D}$, then $g \boxslash f$ in $\mathcal{C}$.*
2. *If $U g \perp U f$ in $\mathcal{D}$, then $g \perp f$ in $\mathcal{C}$.*

Proof. Obvious. ♦

Definition A.3.9. Let $\mathcal{C}$ be a category.

- A **strong monomorphism** in $\mathcal{C}$ is a monomorphism that is right orthogonal to all epimorphisms in $\mathcal{C}$.
- A **strong epimorphism** in $\mathcal{C}$ is a monomorphism that is left orthogonal to all monomorphisms in $\mathcal{C}$.

Lemma A.3.10. *Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. The following are equivalent:*

(i) *f is an isomorphism.*

(ii) *f is right orthogonal to any morphism in $\mathcal{C}$.*

(iii) *f has the right lifting property with respect to any morphism in $\mathcal{C}$.*

(iv) *f has the right lifting property with respect to itself.*

Dually, the following are equivalent:

(i′) *f is an isomorphism.*

(ii′) *f is left orthogonal to any morphism in $\mathcal{C}$.*

(iii′) *f has the left lifting property with respect to any morphism in $\mathcal{C}$.*

(iv′) *f has the left lifting property with respect to itself.*

Proof. (i) ⇒ (ii). Suppose $r : Y \to X$ is a morphism such that $r \circ f = \mathrm{id}_X$. Then, for any commutative square as below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

we have $(r \circ w) \circ g = r \circ f \circ z = z$; but if $f \circ r = \mathrm{id}_Y$ as well, then $f \circ (r \circ w) = w$; thus $r \circ w : W \to X$ is the required lift. It is clearly unique, as f is monic.

(ii) ⇒ (iii), (iii) ⇒ (iv). Obvious.

(iv) ⇒ (i). Consider the following commutative square:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

Since f has the right lifting property with respect to itself, there exists a morphism $h : Y \to X$ such that $h \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. ■

Corollary A.3.11.

- *A morphism is both a monomorphism and a strong epimorphism if and only if it is an isomorphism.*
- *A morphism is both a epimorphism and a strong monomorphism if and only if it is an isomorphism.* ■

Definition A.3.12. A **weak factorisation system** for a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of subensembles of $\operatorname{mor} \mathcal{C}$ satisfying these conditions:

- For each morphism f in $\mathcal{C}$ there exists a pair (g, h) with $g \in \mathcal{L}$ and $h \in \mathcal{R}$ such that $f = h \circ g$. Such a pair is a $(\mathcal{L}, \mathcal{R})$**-factorisation** of f.
- A morphism is in $\mathcal{L}$ if and only if it has the left lifting property with respect to every morphism in $\mathcal{R}$, i.e. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$.

- A morphism is in $\mathcal{R}$ if and only if it has the right lifting property with respect to every morphism in $\mathcal{L}$, i.e. $\mathcal{R} = \mathcal{L}^{\boxslash}$.

An **orthogonal factorisation system** is defined like a weak factorisation system, except for replacing '… has the left/right lifting property with respect to …' with '… is left/right orthogonal to …'.

Remark A.3.13. Obviously, $(\mathcal{L}, \mathcal{R})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}$ if and only if $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}^{\mathrm{op}}$.

Proposition A.3.14. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on $\mathcal{C}$. If either*

- *every morphism in $\mathcal{R}$ is a monomorphism in $\mathcal{C}$, or*
- *every morphism in $\mathcal{L}$ is an epimorphism in $\mathcal{C}$,*

then $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system.

Proof. The two hypotheses are formally dual, so it is enough to check the first case. Observe that, given a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

where $f : X \to Y$ is a monomorphism, for any $h' : W \to X$ such that $f \circ h' = w$, we must have $h = h'$. Thus, for any monomorphism $f : X \to Y$, $g \boxslash f$ if and only if $g \perp f$. Hence, $\mathcal{L} = {}^{\boxslash}\mathcal{R} = {}^{\perp}\mathcal{R}$. On the other hand, $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash} = \mathcal{R}$, so $\mathcal{R} = \mathcal{L}^{\perp}$ as well. ■

Definition A.3.15. A **proper factorisation system** on a category $\mathcal{C}$ is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ such that every morphism in $\mathcal{E}$ is an epimorphism *and* every morphism in $\mathcal{M}$ is a monomorphism.

Example A.3.16. In **Set**, if $\mathcal{E}$ is the class of surjective maps and $\mathcal{M}$ is the class of injective maps, then $(\mathcal{E}, \mathcal{M})$ is a proper factorisation system.

Proposition A.3.17 (Closure properties)**.** *Let $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$ and suppose either $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ or $\mathcal{L} = {}^{\perp}\mathcal{R}$.*

(i) *Given a pushout diagram in $\mathcal{C}$ as below,*

$$\begin{tikzcd} Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ W' \arrow[r, "i_W"'] & W \end{tikzcd}$$

if the morphism g' is in $\mathcal{L}$, then g is also in $\mathcal{L}$.

(ii) *Let I be a set. If $g_i : Z_i \to W_i$ is a morphism in $\mathcal{L}$ for all i in I and the coproduct $\coprod_i g_i : \coprod_i Z_i \to \coprod_i W_i$ exists in $\mathcal{C}$, then $\coprod_i g_i$ is also in $\mathcal{L}$.*

(iii) *Given a commutative diagram of the form*

$$\begin{tikzcd} Z' \arrow[r, "i_Z"'] \arrow[d, "g'"'] \arrow[rr, bend left, "\mathrm{id}"] & Z \arrow[r, "r_Z"'] \arrow[d, "g"'] & Z' \arrow[d, "g'"] \\ W' \arrow[r, "i_W"] \arrow[rr, bend right, "\mathrm{id}"'] & W \arrow[r, "r_W"] & W' \end{tikzcd}$$

if g is in $\mathcal{L}$, then so is g'; in other words, $\mathcal{L}$ is closed under retracts.

(iv) *$\mathcal{L}$ is closed under composition.*

(v) *Let γ be an ordinal and let $Z : \gamma \to \mathcal{C}$ be a colimit-preserving functor. We write Z_α for $Z(\alpha)$, where $\alpha < \gamma$, and $g_{\alpha,\beta} : Z_\alpha \to Z_\beta$ for the morphism $Z(\alpha \to \beta)$, where $\alpha < \beta < \gamma$. If λ is a colimiting cocone from Z to W and each $g_{\alpha,\beta}$ is in $\mathcal{L}$, then each component $\lambda_\alpha : Z_\alpha \to W$ is also in $\mathcal{L}$.*

Proof. (i). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{tikzcd} Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[r, "z"] \arrow[d, "g"] & X \arrow[d, "f"] \\ W' \arrow[r, "i_W"'] & W \arrow[r, "w"'] & Y \end{tikzcd}$$

There exists $h' : W' \to X$ such that $h' \circ g' = z \circ i_Z$ and $f \circ h' = w \circ i_W$. In particular, there exists a unique morphism $h : W \to X$ such that $h \circ g = z$ and

$h \circ i_W = h'$, by the universal property of pullbacks. Thus $f \circ h \circ i_W = f \circ h' = w \circ i_W$ and $f \circ h \circ g = f \circ z = w \circ g$, but i_W and g are jointly epic, so $f \circ h = w$. This shows h is the required lift, and h is unique if h' is.

(ii). We may construct the required lift componentwise.

(iii). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{tikzcd} Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[r, "r_Z"] \arrow[d, "g"'] & Z' \arrow[r, "z"] \arrow[d, "g'"'] & X \arrow[d, "f"] \\ W' \arrow[r, "i_W"'] & W \arrow[r, "r_W"'] & W' \arrow[r, "w"'] & Y \end{tikzcd}$$

There exists $h : W \to X$ such that $h \circ g = z \circ r_Z$ and $f \circ h = w \circ r_W$, and so for $h' = h \circ i_W$:

$$f \circ h' = f \circ h \circ i_W = w \circ r_W \circ i_W = w$$
$$h' \circ g' = h \circ i_W \circ g' = h \circ g \circ i_Z = z \circ r_Z \circ i_Z = z$$

Thus $h' : W' \to X$ is the required lift, and h' is unique if h is (because r_W is split epic).

(iv). Suppose $g' : Z' \to Z$ and $g : Z \to W$ are in $\mathcal{L}$ and $f : X \to Y$ is in $\mathcal{R}$. Consider the following commutative diagram:

$$\begin{tikzcd} Z' \arrow[r, "z'"] \arrow[d, "g'"'] & X \arrow[dd, "f"] \\ Z \arrow[d, "g"'] & \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

There must exist a morphism $z : Z \to X$ such that $z \circ g' = z'$ and $f \circ z' = w \circ g$, and hence a morphism $h : W \to X$ such that $h \circ g = z$ and $f \circ h = w$. Obviously, $h \circ (g' \circ g) = z'$, so h is the required lift. Moreover, h unique if $\mathcal{L} = {}^{\perp}\mathcal{R}$.

(v). We may assume without loss of generality that $\alpha = 0$, since any non-empty terminal segment of γ is cofinal in γ. Suppose $f : X \to Y$ is in $\mathcal{R}$ and consider the following commutative diagram:

$$\begin{tikzcd} Z_0 \arrow[r, "z_0"] \arrow[d, "\lambda_0"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

For each $\alpha < \gamma$, given z_α making the following diagram commute,

$$\begin{tikzcd} Z_\alpha \arrow[rr, "z_\alpha"] \arrow[d, "g_{\alpha,\alpha+1}"'] & & X \arrow[d, "f"] \\ Z_{\alpha+1} \arrow[r, "\lambda_{\alpha+1}"'] & W \arrow[r, "w"'] & Y \end{tikzcd}$$

choose a lift $z_{\alpha+1} : Z_{\alpha+1} \to X$; for each limit ordinal $\beta < \gamma$, let $z_\beta : Z_\beta \to X$ be the unique morphism such that $z_\beta \circ g_{\alpha,\beta} = z_\alpha$ for all $\alpha < \beta$. (Such z_β exist and are unique because $Z_\beta = \varinjlim_{\alpha<\beta} Z_\alpha$.) Note that the universal property of W then guarantees that $w \circ \lambda_\beta = f \circ z_\beta$.

Having constructed morphisms $z_\alpha : Z_\alpha \to X$ for all $\alpha < \gamma$ as above, we may now obtain $h : W \to X$ as the unique morphism such that $h \circ \lambda_\alpha = z_\alpha$ for all $\alpha < \gamma$, and again we automatically have $f \circ h = w$. It is also clear that h is unique if $\mathcal{L} = {}^\perp\mathcal{R}$. ■

Proposition A.3.18 (Cancellation properties)**.** *Let* $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$.

(i) *Let $\mathcal{L}$ be either ${}^\boxslash\mathcal{R}$ or ${}^\perp\mathcal{R}$, let $e : A \to Z$ be an epimorphism in $\mathcal{C}$, and let $g : Z \to W$ be a morphism in $\mathcal{C}$. If $g \circ e$ is in $\mathcal{L}$, then so is g.*

(ii) *Let $\mathcal{L}$ be ${}^\perp\mathcal{R}$, let $f : A \to Z$ be any morphism in $\mathcal{C}$, and let $g : Z \to W$ be a morphism in $\mathcal{C}$. Assuming every morphism that is in $\mathcal{R}$ is a monomorphism in $\mathcal{C}$, if $g \circ f$ is in $\mathcal{L}$, then so is g.*

(iii) *Suppose $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{R}$, and let $e : A \to Z$ be in $\mathcal{L}$. Then, a morphism $g : Z \to W$ is in $\mathcal{L}$ if and only $g \circ e$ is in $\mathcal{L}$.*

Dually, let $\mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$.

(i′) *Let $\mathcal{R}$ be either $\mathcal{L}^\boxslash$ or $\mathcal{L}^\perp$, let $m : Y \to B$ be an monomorphism in $\mathcal{C}$, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If $m \circ f$ is in $\mathcal{R}$, then so is f.*

(ii′) *Let $\mathcal{R}$ be $\mathcal{L}^{\perp}$, let $g : Y \to B$ be any morphism in $\mathcal{C}$, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. Assuming every morphism that is in $\mathcal{L}$ is an epimorphism in $\mathcal{C}$, if $g \circ f$ is in $\mathcal{R}$, then so is f.*

(iii′) *Suppose $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{R}$, and let $m : Y \to B$ be in $\mathcal{L}$. Then, a morphism $f : X \to Y$ is in $\mathcal{L}$ if and only $g \circ e$ is in $\mathcal{L}$.*

Proof. (i). The epimorphism $e : A \to Z$ induces a bijection between solutions of lifting problems in $\mathcal{C}$ of the form

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and solutions of lifting problems of the form

$$\begin{tikzcd} A \arrow[r, "z \circ e"] \arrow[d, "g \circ e"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

so $g \boxslash f$ (resp. $g \perp f$) if and only if $g \circ e \boxslash f$ (resp. $g \circ e \perp f$).

(ii). The proof is similar to that of claim (i).

(iii). By proposition A.3.17, we know $g \circ e$ is in $\mathcal{L}$ if both g and e are in $\mathcal{L}$; the converse remains to be shown. Let $r \circ l$ be an $(\mathcal{L}, \mathcal{R})$-factorisation of g. If $g \circ e$ is in $\mathcal{L}$, then there exists a unique s making the diagram below commute,

$$\begin{tikzcd} A \arrow[r, "l \circ e"] \arrow[d, "g \circ e"'] & M \arrow[d, "r"] \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s"] & W \end{tikzcd}$$

so $r \circ s = \mathrm{id}_W$, but then we also have

$$r \circ (s \circ r) = r$$
$$(s \circ r) \circ (l \circ e) = s \circ (g \circ e) = l \circ e$$

and $l \circ e \perp r$, so we must have $s \circ r = \mathrm{id}_M$. Hence, g is also in $\mathcal{L}$. ■

Proposition A.3.19 (The retract argument)**.** *Let $\mathcal{C}$ be a category and let $(\mathcal{L}, \mathcal{R})$ be a pair of subclasses of* $\operatorname{mor} \mathcal{C}$ *such that $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$. If every morphism in $\mathcal{C}$ admits an $(\mathcal{L}, \mathcal{R})$-factorisation, then the following are equivalent:*

(i) *$(\mathcal{L}, \mathcal{R})$ is a weak factorisation system.*

(ii) *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts in $\mathcal{C}$.*

Proof. (i) $\Rightarrow$ (ii). This is a special case of proposition A.3.17.

(ii) $\Rightarrow$ (i). Suppose $f : X \to Y$ is in $\mathcal{L}^{\boxslash}$. Let $p \circ i$ be an $(\mathcal{L}, \mathcal{R})$-factorisation of f. Then, there must exist a morphism r such that $r \circ i = \mathrm{id}_X$ and $f \circ r = p$, as in the diagram below:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r, "p"'] \arrow[ur, dashed, "r"] & Y \end{tikzcd}$$

Hence, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] & Z \arrow[r, "r"'] \arrow[d, "p"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"] & Y \arrow[r, "\mathrm{id}"] & Y \end{tikzcd}$$

Since $\mathcal{R}$ is closed under retracts, we deduce that f is in $\mathcal{R}$. Thus, $\mathcal{L}^{\boxslash} \subseteq \mathcal{R}$. The dual argument proves that ${}^{\boxslash}\mathcal{R} \subseteq \mathcal{L}$, so $(\mathcal{L}, \mathcal{R})$ is indeed a weak factorisation system. ■

Corollary A.3.20. *Every orthogonal factorisation system is also a weak factorisation system.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorisation system on a category $\mathcal{C}$. Proposition A.3.3 implies $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$, and proposition A.3.17 says $\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts, so we may use the earlier proposition to deduce that $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system. ■

Lemma A.3.21. *Let A be an object in a category $\mathcal{C}$ with a weak (resp. orthgonal) factorisation system $(\mathcal{L}, \mathcal{R})$. Then the slice category $\mathcal{C}_{/A}$ has a weak (resp. orthogonal) factorisation system where a morphism is in the left or right class if and only if it is so in $\mathcal{C}$.*

Proof. Apply lemma A.3.7 and the retract argument (proposition A.3.19). ■

Definition A.3.22. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ is **cofibrantly generated** by a subensemble $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{R} = \mathcal{I}^{\boxslash}$. Dually, $(\mathcal{L}, \mathcal{R})$ is **fibrantly generated** by a subensemble $\mathcal{F} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{L} = {}^{\boxslash}\mathcal{F}$.

Remark A.3.23. Of course, $(\mathcal{L}, \mathcal{R})$ is always cofibrantly generated by $\mathcal{L}$. The condition is most useful when $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by a (small) subset of $\mathcal{L}$, but it is convenient to have the more general definition available.

Definition A.3.24. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on a category $\mathcal{C}$. An **extension** of $(\mathcal{L}, \mathcal{R})$ along a functor $i : \mathcal{C} \to \mathcal{C}^+$ is a weak factorisation system $(\mathcal{L}^+, \mathcal{R}^+)$ on $\mathcal{C}^+$ with the following properties:

- A morphism $f : X \to Y$ in $\mathcal{C}$ is in $\mathcal{R}$ if and only if $i f : i X \to i Y$ is in $\mathcal{R}^+$.
- A morphism $g : Z \to W$ in $\mathcal{C}$ is in $\mathcal{L}$ if and only if $i g : i Z \to i W$ is in $\mathcal{L}^+$.

Proposition A.3.25. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on $\mathcal{C}$, and let $(\mathcal{L}^+, \mathcal{R}^+)$ be a weak factorisation system on $\mathcal{C}^+$.*

(i) *If $\mathcal{L} \subseteq \mathcal{L}^+$, then $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both cofibrantly generated by the same ensemble $\mathcal{I}$, then $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

Dually:

(i′) *If $\mathcal{R} \subseteq \mathcal{R}^+$, then $\mathcal{L} \supseteq \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii′) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both fibrantly generated by the same ensemble $\mathcal{F}$, then $\mathcal{L} = \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

Moreover:

(iii) *If $\mathcal{L} \subseteq \mathcal{L}^+$ and $\mathcal{R} \subseteq \mathcal{R}^+$, then $(\mathcal{L}^+, \mathcal{R}^+)$ is an extension of $(\mathcal{L}, \mathcal{R})$.*

Proof. Since $\mathcal{C}$ is a full subcategory of $\mathcal{C}^+$, if $g : Z \to W$ and $f : X \to Y$ are morphisms in $\mathcal{C}$, then any lifting problem of the following form in $\mathcal{C}^+$ is already in $\mathcal{C}$,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r] & Y \end{tikzcd}$$

and moreover any solution to the above lifting problem in $\mathcal{C}^+$ is also a solution in $\mathcal{C}$. Thus, $g \boxslash f$ in $\mathcal{C}$ if and only if $g \boxslash f$ in $\mathcal{C}^+$.

(i). Suppose f is in $\mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$. Then f has the right lifting property in $\mathcal{C}^+$ with respect to every morphism in $\mathcal{L}^+$, and in particular, f has the right lifting property in $\mathcal{C}$ with respect to every morphism in $\mathcal{L}$; hence f is in $\mathcal{R}$, and therefore $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(ii). A morphism is in $\mathcal{R}$ (resp. $\mathcal{R}^+$) if and only if it has the right lifting property in $\mathcal{C}$ (resp. $\mathcal{C}^+$) with respect to every morphism in $\mathcal{I}$, so by our initial observation, we must have $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(iii). Immediately follows from claims (i) and (i′). ■

Proposition A.3.26. *Let $(\mathcal{L}, \mathcal{R})$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}$, and let $(\mathcal{L}', \mathcal{R}')$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}'$. Given an adjunction*

$$F \dashv U : \mathcal{C}' \to \mathcal{C}$$

the following are equivalent:

(i) *F sends morphisms in $\mathcal{L}$ to morphisms in $\mathcal{L}'$.*

(ii) *U sends morphisms in $\mathcal{R}'$ to morphisms in $\mathcal{R}$.*

Proof. The adjunction induces a bijection between solutions to the two lifting problems shown below:

$$\begin{tikzcd} FZ \arrow[r] \arrow[d, "Fg"'] & X \arrow[d, "f"] \\ FW \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & UX \arrow[d, "Uf"] \\ W \arrow[r] \arrow[ur, dashed, "?"] & UY \end{tikzcd}$$

Thus, $Fg \boxslash f$ (resp. $Fg \perp f$) if and only if $g \boxslash Uf$ (resp. $g \perp Uf$). ■

¶ A.3.27. Let $\mathbb{2}$ be the category $\{0 \to 1\}$ and let $\mathbb{3}$ be $\{0 \to 1 \to 2\}$. Thus, given a category $\mathcal{C}$, the functor category $[\mathbb{2}, \mathcal{C}]$ is the category of arrows and commutative squares in $\mathcal{C}$. There are three embeddings $d^0, d^1, d^2 : \mathbb{2} \to \mathbb{3}$:

$$\begin{aligned} d^0(0) &= 1 & \qquad d^1(0) &= 0 & \qquad d^2(0) &= 0 \\ d^0(1) &= 2 & \qquad d^1(1) &= 2 & \qquad d^2(1) &= 1 \end{aligned}$$

These then induce (by precomposition) three functors $d_0, d_1, d_2 : [\mathbb{3}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$.

Definition A.3.28. A **functorial factorisation system** on a category $\mathcal{C}$ is a pair of functors $L, R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ for which there exists a (necessarily unique) functor $F : [\mathbb{2}, \mathcal{C}] \to [\mathbb{3}, \mathcal{C}]$ satisfying the following equations:

$$d_2 F = L \qquad\qquad d_1 F = \mathrm{id}_{[\mathbb{2}, \mathcal{C}]} \qquad\qquad d_0 F = R$$

A **functorial weak** (resp. **orthogonal**) **factorisation system** on $\mathcal{C}$ is a weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$ together with a functorial factorisation system (L, R) such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$. We will often abuse notation and refer to the functorial factorisation system (L, R) as a functorial weak (resp. orthogonal) factorisation system, omitting mention of the weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$.

Lemma A.3.29. *Let A be an object in a category $\mathcal{C}$ and let $\Sigma_A : \mathcal{C}_{/A} \to \mathcal{C}$ be the projection from the slice category.*

(i) *For each functorial factorisation system (L, R) on $\mathcal{C}$, there exists a unique functorial factorisation system (L_A, R_A) on $\mathcal{C}_{/A}$ such that*

$$[\mathbb{2}, \Sigma_A] \circ L_A = L \circ [\mathbb{2}, \Sigma_A] \qquad\qquad [\mathbb{2}, \Sigma_A] \circ R_A = R \circ [\mathbb{2}, \Sigma_A]$$

where $[\mathbb{2}, \Sigma_A] : [\mathbb{2}, \mathcal{C}_{/A}] \to [\mathbb{2}, \mathcal{C}]$ is the evident induced functor.

(ii) *If (L, R) is part of a functorial weak or orthogonal factorisation system on $\mathcal{C}$, then (L_A, R_A) is compatible with the induced weak or orthogonal factorisation system on $\mathcal{C}_{/A}$ as well.*

Proof. Obvious. ♦

Proposition A.3.30. *Any orthogonal factorisation system can be extended to a functorial one.*

Proof. For each morphism f in a category $\mathcal{C}$ with an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$, choose a factorisation $f = Rf \circ Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$. Given a commutative square in $\mathcal{C}$, say

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

the lifting property ensures that the dashed arrow in the diagram below exists,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[d, "Lf"] \\ \bullet \arrow[r, dashed] \arrow[d, "Rg"'] & \bullet \arrow[d, "Rf"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and orthogonality ensures uniqueness and hence functoriality. ■

Corollary A.3.31. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on a category $\mathcal{C}$, then, for any category $\mathcal{J}$, there exists an orthogonal factorisation system on the functor category $[\mathcal{J}, \mathcal{C}]$ where a natural transformation is in the left (resp. right) class if and only if all its components are in $\mathcal{L}$ (resp. $\mathcal{R}$).*

Proof. Obviously, every morphism in $[\mathcal{J}, \mathcal{C}]$ admits such a factorisation, since $(\mathcal{L}, \mathcal{R})$-factorisations in $\mathcal{C}$ are functorial. By considering a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} Z' \arrow[rr] \arrow[dr] \arrow[dd, "g'"'] & & X' \arrow[dr] \arrow[dd, "f'" near end] & \\ & Z \arrow[rr] \arrow[dd, "g"' near start] & & X \arrow[dd, "f"] \\ W' \arrow[rr] \arrow[dr] & & Y' \arrow[dr] & \\ & W \arrow[rr] & & Y \end{tikzcd}$$

where f and f' are in $\mathcal{R}$ while g and g' are in $\mathcal{L}$, using the fact that $(\mathcal{E}, \mathcal{M})$ is an *orthogonal* factorisation system, one may show that lifting problems in $[\mathcal{J}, \mathcal{C}]$ admit unique solutions, and that these solutions are moreover constructed componentwise. Thus, $(\mathcal{L}, \mathcal{R})$ induces an orthogonal factorisation system on $[\mathcal{J}, \mathcal{C}]$. ■

The following characterisation of functorial orthogonal factorisation systems is due to Grandis and Tholen [2006]:

Theorem A.3.32. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *L is the underlying endofunctor of an idempotent comonad on $[\mathbb{2}, \mathcal{C}]$ with counit given by $\varepsilon_k = (\mathrm{id}_{\mathrm{dom}\,k}, Rk)$, and R is the underlying endofunctor of an idempotent monad on $[\mathbb{2}, \mathcal{C}]$ with unit given by $\eta_h = (h, \mathrm{id}_{\mathrm{codom}\,h})$.*

(ii) *For all morphisms h in $\mathcal{C}$, RLh and LRh are isomorphisms in $\mathcal{C}$.*

(iii) *For any two morphisms in $\mathcal{C}$, say h and k, we have $Lk \perp Rh$.*

(iv) *$(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{C}$ extending (L, R), where:*

$$\mathcal{L} = \{g \in \operatorname{mor}\mathcal{C} \mid Rg \text{ is an isomorphism in } \mathcal{C}\}$$
$$\mathcal{R} = \{f \in \operatorname{mor}\mathcal{C} \mid Lf \text{ is an isomorphism in } \mathcal{C}\}$$

(v) *There exists an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$ extending (L, R).*

Proof. (i) ⇔ (ii). This is a standard fact about idempotent (co)monads.

(ii) ⇒ (iii). Now, consider the following lifting problem:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

Since (L, R) is a functorial factorisation system, we get a commutative diagram of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[d, "Lf"] \\ W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[d, "Rf"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

but Rg and Lf are isomorphisms, so $(Lf)^{-1} \circ t \circ (Rg)^{-1}$ is the required lift $W \to X$. On the other hand, if $s : W \to X$ is any morphism such that $f \circ s = w$

and $s \circ g = z$, then by taking (L, R)-factorisations of the vertical arrows in the following diagram,

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "g"'] & W \arrow[r, "s"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\
W \arrow[r, "\mathrm{id}"'] & W \arrow[r, "s"'] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

we find it must be the case that $Lf \circ s \circ Rg = t$, so we indeed have $g \perp f$.

(iii) ⇒ (iv). In particular, $g \perp Rg$ and $Lf \perp f$, so there must exist morphisms i and r making the diagrams below commute:

$$\begin{tikzcd}
Z \arrow[r, "Lg"] \arrow[d, "g"'] & W' \arrow[d, "Rg"] \\
W \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "i"] & W
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\
X' \arrow[r, "Rf"'] \arrow[ur, dashed, "r"'] & Y
\end{tikzcd}$$

We then obtain the following equations,

$$(i \circ Rg) \circ Lg = Lg \qquad (Lf \circ r) \circ Lf = Lf$$
$$Rg \circ (i \circ Rg) = Rg \qquad Rf \circ (Lf \circ r) = Rf$$

and since $Lg \perp Rg$ and $Lf \perp Rf$, we must have $i \circ Rg = \mathrm{id}_{W'}$ and $Lf \circ r = \mathrm{id}_{X'}$. Thus, $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and the same argument now shows that ${}^{\perp}\mathcal{R} \subseteq \mathcal{L}$ and $\mathcal{L}^{\perp} \subseteq \mathcal{R}$.

It remains to be shown that $\mathcal{L} \subseteq {}^{\perp}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\perp}$. First, suppose $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and consider the following lifting problem:

$$\begin{tikzcd}
Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
W \arrow[r, "w"'] & Y
\end{tikzcd}$$

With r and i as in the previous paragraph, we obtain a commutative diagram of the form below,

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"] & X \arrow[dd, "f"] \\
 & W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[d, "Rf"] \arrow[ur, "r"'] & \\
W \arrow[r, "\mathrm{id}"'] \arrow[ur, "i"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y
\end{tikzcd}$$

where the arrow t is obtained by the functoriality of (L, R)-factorisations. Thus, $r \circ t \circ i$ is the required lift $W \to X$, and it is unique, since Rg and Lf are isomorphisms. (Recall the proof of (ii) ⇒ (iii).) We conclude that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.

(iv) ⇒ (v). Immediate.

(v) ⇒ (iii). If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{C}$ such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$, then we must have $Lk \perp Rh$ for all h and k in $\operatorname{mor} \mathcal{C}$, as required.

(iv) ⇒ (ii). Immediate. ■

Remark A.3.33. It is clear that a functorial factorisation system is associated with *at most one* orthogonal factorisation system: indeed, if $(\mathcal{L}', \mathcal{R}')$ is any orthogonal factorisation system extending a functorial factorisation system (L, R), and $(\mathcal{L}, \mathcal{R})$ is the induced orthogonal factorisation system as in the theorem, then each morphism in $\mathcal{L}$ (resp. $\mathcal{R}$) is a retract of some morphism in in $\mathcal{L}'$ (resp. $\mathcal{R}'$); but by proposition A.3.17, this implies $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{R} \subseteq \mathcal{R}'$, and applying proposition A.3.3, we also get $\mathcal{L} \supseteq \mathcal{L}'$ and $\mathcal{R} \supseteq \mathcal{R}'$.

Corollary A.3.34. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on a category $\mathcal{C}$, then:*

(i) *$\mathcal{L}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is replete and coreflective.*

(ii) *$\mathcal{L}$ is closed under all colimits in $[\mathbb{2}, \mathcal{C}]$.*

(iii) *If a diagram in $\mathcal{L}$ has a limit in $[\mathbb{2}, \mathcal{C}]$, then it also has a limit in $\mathcal{L}$.*

Dually:

(i′) *$\mathcal{R}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is replete and reflective.*

(ii′) *$\mathcal{R}$ is closed under all limits in $[\mathbb{2}, \mathcal{C}]$.*

(iii′) *If a diagram in $\mathcal{R}$ has a colimit in $[\mathbb{2}, \mathcal{C}]$, then it also has a colimit in $\mathcal{R}$.*

Proof. Using proposition A.3.30 and theorem A.3.32, the above claims amount to standard facts about the Eilenberg–Moore category for idempotent (co)monads. ■

There is a similar characterisation of functorial weak factorisation systems, due to Rosický and Tholen [2002]:

Theorem A.3.35. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *For any two morphisms in $\mathcal{C}$, say h and k, $Lk \boxslash Rh$.*

(ii) *$(\mathcal{L}, \mathcal{R})$ is an weak factorisation system on $\mathcal{C}$ extending (L, R), where:*

$$\mathcal{L} = \left\{ g \in \operatorname{mor} \mathcal{C} \,\middle|\, \exists i \in \operatorname{mor} \mathcal{C}.\, i \circ g = Lg \wedge Rg \circ i = \mathrm{id}_{\mathrm{codom}\, g} \right\}$$
$$\mathcal{R} = \left\{ f \in \operatorname{mor} \mathcal{C} \,\middle|\, \exists r \in \operatorname{mor} \mathcal{C}.\, f \circ r = Rf \wedge r \circ Lf = \mathrm{id}_{\mathrm{dom}\, f} \right\}$$

(iii) *There exists a weak factorisation system $(\mathcal{L}, \mathcal{R})$ extending (L, R).*

Proof. The proof is essentially the same as that of theorem A.3.32. ■

Remark A.3.36. As with orthogonal factorisation systems, there is *at most one* weak factorisation system extending any functorial factorisation system.

Proposition A.3.37. *Let (L, R) be a functorial factorisation system on $\mathcal{C}$ and let $\lambda : \mathrm{id}_{[\mathbb{2}, \mathcal{C}]} \Rightarrow R$ and $\rho : L \Rightarrow \mathrm{id}_{[\mathbb{2}, \mathcal{C}]}$ be the natural transformations whose component at an object f in $[\mathbb{2}, \mathcal{C}]$ correspond to the following commutative squares in $\mathcal{C}$:*

$$\begin{tikzcd} \bullet \arrow[r, "Lf"] \arrow[d, "f"'] & \bullet \arrow[d, "Rf"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "Lf"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "Rf"'] & \bullet \end{tikzcd}$$

Suppose (L, R) extends to a functorial weak factorisation system. Then the following are equivalent for a morphism $g : Z \to W$ in $\mathcal{C}$:

(i) *The morphism g is in the left class of the induced weak factorisation system.*

(ii) *There exists a morphism i in $\mathcal{C}$ such that the diagram below commutes:*

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "g"'] & Z \arrow[r, equal] \arrow[d, "Lg"'] & Z \arrow[d, "g"] \\ W \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "Rg"] & W \end{tikzcd}$$

(iii) *The object* g *in* $[\mathbb{2}, \mathcal{C}]$ *admits a coalgebra structure for the copointed endofunctor* (L, ρ)*.*

Dually, the following are equivalent for a morphism $f : X \to Y$ *in* $\mathcal{C}$*:*

(i′) *The morphism* f *is in the right class of the induced weak factorisation system.*

(ii′) *There exists a morphism* r *in* $\mathcal{C}$ *such that the diagram below commutes:*

$$\begin{tikzcd} X \arrow[r, "Lf"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] & \bullet \arrow[r, "r"'] \arrow[d, "Rf"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

(iii′) *The object* f *in* $[\mathbb{2}, \mathcal{C}]$ *admits an algebra structure for the pointed endofunctor* (R, λ)*.*

Proof. (i) ⇒ (ii). Consider the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "Lg"] \arrow[d, "g"'] & \bullet \arrow[d, "Rg"] \\ W \arrow[r, "\mathrm{id}"'] & W \end{tikzcd}$$

Thus, a morphism i of the required form exists in $\mathcal{C}$ as soon as $g \boxslash Rg$.

(ii) ⇔ (iii). This is simply the definition of (L, ρ)-coalgebra.

(ii) ⇒ (i). By definition, the morphism Lf is in the left class of the induced weak factorisation system; but the given diagram exhibits f as a retract of Lf, so we may apply proposition A.3.17 to deduce that f is also in the left class. ■

The results above motivate the following definition:

Definition A.3.38. A **natural weak factorisation system**[4] on a category $\mathcal{C}$ is a pair (L, R) satisfying the following conditions:

- $\mathsf{L} = (L, \varepsilon, \delta)$ is a comonad on $[\mathbb{2}, \mathcal{C}]$, where $\varepsilon_k = \left(\mathrm{id}_{\mathrm{dom}\, k}, Rk\right)$.

[4] — in the sense of Grandis and Tholen [2006], *not* Garner [2009].

- $\mathsf{R} = (R, \eta, \mu)$ is a monad on $[\mathbb{2}, \mathcal{C}]$, where $\eta_h = \left(Lh, \mathrm{id}_{\mathrm{codom}\, h}\right)$.
- (L, R) constitute a functorial factorisation system on $\mathcal{C}$.

Given natural weak factorisation systems $(\mathsf{L}', \mathsf{R})$ and $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{C}$, a **morphism** $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$ is a pair $\left(\theta^L, \theta^R\right)$, where $\theta^L : L' \Rightarrow L$ and $\theta^R : R \Rightarrow R'$ are natural transformations such that the equations below hold,

$$\varepsilon \bullet \theta^L = \varepsilon' \qquad \left(\theta^L \circ \theta^L\right) \bullet \delta' = \delta \bullet \theta^L$$
$$\theta^R \bullet \eta = \eta' \qquad \mu' \bullet \left(\theta^R \circ \theta^R\right) = \theta^R \bullet \mu$$

and furthermore we require $d_0 \theta^L = d_1 \theta^R$.

Remark a.3.39. In other words, a morphism of natural weak factorisation systems is a natural transformation of functors $[\mathbb{2}, \mathcal{C}] \to [\mathbb{3}, \mathcal{C}]$ such that the left half is a morphism of comonads and the right half is a morphism of monads. In particular, we must have $d_1 \theta^L = \mathrm{id}$ and $d_0 \theta^R = \mathrm{id}$; so for every object f in $[\mathbb{2}, \mathcal{C}]$, we obtain a commutative diagram in $\mathcal{C}$ of the form below:

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "L'f"'] & \bullet \arrow[d, "Lf"] \\
\bullet \arrow[r, "\theta_f"] \arrow[d, "Rf"'] & \bullet \arrow[d, "R'f"] \\
\bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

Proposition a.3.40. *Any functorial orthogonal factorisation system extends to a natural weak factorisation system in a unique way; conversely, a natural weak factorisation system induces an orthogonal factorisation system if and only if the underlying comonad and monad are both idempotent.*

Proof. This follows from the definition above and theorem a.3.32. ■

Proposition a.3.41. *Let* (L, R) *be an natural weak factorisation system on a category* $\mathcal{C}$.

(i) *Let* $f : X \to Y$ *and* $g : Z \to W$ *be objects in* $[\mathbb{2}, \mathcal{C}]$. *If* $\alpha : Rf \to f$ *is a* R*-algebra structure and* $\beta : g \to Lg$ *is a* L*-coalgebra structure, then* $d_0(\alpha) : Y \to Y$ *and* $d_1(\beta) : Z \to Z$ *are identity morphisms, and we have the following identities:*

$$d_1(\alpha) \circ Lf = \mathrm{id}_X \qquad Rg \circ d_0(\beta) = \mathrm{id}_W$$
$$f \circ d_1(\alpha) = Rf \qquad d_0(\beta) \circ g = Lg$$

(ii) *If f admits a* L*-coalgebra structure and g admits an* R*-algebra structure, then $f \boxslash g$.*

(iii) *There exists a (unique) weak factorisation system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ such that $Lk \in \mathcal{L}$ and $Rh \in \mathcal{R}$ for all h and k in* $\operatorname{mor} \mathcal{C}$.

Proof. (i). The claim follows from the L-coalgebra counitality axiom and the R-algebra unitality axiom:

$$\alpha \circ \eta_f = \mathrm{id}_f \qquad\qquad \varepsilon_g \circ \beta = \mathrm{id}_g$$

(ii). It then follows that the diagram below commutes,

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"] & X \arrow[dd, "f"] \\
& W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[d, "Rf"] \arrow[ur, "\alpha_0"'] & \\
W \arrow[r, "\mathrm{id}"'] \arrow[ur, "\beta_1"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y
\end{tikzcd}$$

where the arrow t is obtained by the functoriality of (L, R)-factorisations; clearly, $\alpha_0 \circ t \circ \beta_1$ is the required lift.

(iii). Finally, for any two morphisms in $\mathcal{C}$, say h and k, we simply note that $\delta_k : Lk \to LLk$ is an L-coalgebra structure and $\mu_h : RRh \to Rh$ is an R-algebra structure, so we may apply theorem A.3.35 to obtain the conclusion. ■

Proposition A.3.42. *Let* $(\mathsf{L}', \mathsf{R})$ *and* $(\mathsf{L}, \mathsf{R}')$ *be natural weak factorisation systems on a category $\mathcal{C}$. If there exists a morphism* $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$*, then:*

- *Every morphism in the left class of the weak factorisation system induced by* $(\mathsf{L}', \mathsf{R})$ *is also in the left class of the weak factorisation system induced by* $(\mathsf{L}, \mathsf{R}')$.

- *Every morphism in the right class of the weak factorisation system induced by* $(\mathsf{L}, \mathsf{R}')$ *is also in the right class of the weak factorisation system induced by* $(\mathsf{L}', \mathsf{R})$.

Proof. The two claims are formally dual; we will prove the first version.

Let L (resp. L') be the underlying endofunctor of L (resp. L') and let ε (resp. ε') be the counit of L (resp. L'). Suppose we have a morphism $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$. By proposition A.3.37, it suffices to show that every morphism that admits a (L', ε')-coalgebra structure also admits a (L, ε)-coalgebra structure. But if i is a (L', ε')-coalgebra structure on g, then $\theta^L_g \circ i$ is a (L, ε)-coalgebra structure on g, because $\varepsilon_g \circ \theta^L_g = \varepsilon'_g$. ■

Remark A.3.43. Let (L, R) be a natural weak factorisation system. Then, for each morphism $f : X \to Y$, we have a commutative diagram of the following form in $\mathcal{C}$,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "LLf"] \\
\bullet \arrow[r, "d_0(\delta_f)"] \arrow[d, "LRf"'] \arrow[dr, "\mathrm{id}"] & \bullet \arrow[d, "RLf"] \\
\bullet \arrow[r, "d_1(\mu_f)"'] \arrow[d, "RRf"'] & \bullet \arrow[d, "Rf"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

where the upper square corresponds to $\delta_f : Lf \to LLf$ and the lower square corresponds to $\mu_f : RRf \to Rf$; note that the middle square commutes because $(\varepsilon \circ \mathrm{id}_L) \bullet \delta = \mathrm{id}_L$ and $\mu \bullet (\eta \circ \mathrm{id}_R) = \mathrm{id}_R$. Thus, we obtain a canonical natural transformation $\xi : LR \Rightarrow RL$.

The following definition is due to Garner [2009]:

Definition A.3.44. Let $\mathcal{C}$ be a category. An **algebraic factorisation system** on $\mathcal{C}$ is a pair (L, R) satisfying the following conditions:

- (L, R) is a natural weak factorisation system; in particular, $\mathsf{L} = (L, \varepsilon, \delta)$ is a comonad on $[\mathbb{2}, \mathcal{C}]$ and $\mathsf{R} = (R, \eta, \mu)$ is a monad on $[\mathbb{2}, \mathcal{C}]$.
- The canonical natural transformation $\xi : LR \Rightarrow RL$ is a **distributive law**, i.e.

  $$(\mathrm{id}_{d_0} \circ \delta) \bullet (\mathrm{id}_{d_1} \circ \mu) = (\mathrm{id}_{d_1} \circ \mu \circ \mathrm{id}_L) \bullet (\mathrm{id}_M \circ \xi) \bullet (\mathrm{id}_{d_0} \circ \delta \circ \mathrm{id}_R)$$

  where $M = d_0 L = d_1 R$.

¶ A.3.45. Let $\mathcal{C}$ be a category and let $U : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ be a functor. We define a category $\mathbf{RLP}_{\mathcal{C}}(U)$ over $[\mathbb{2}, \mathcal{C}]$ as follows:

- The objects in $\mathbf{RLP}_{\mathcal{C}}(U)$ are morphisms in $\mathcal{C}$ equipped with a coherent choice of liftings, i.e. a pair (f, Φ) where f is a morphism in $\mathcal{C}$ equipped with a chosen morphism $\Phi(e,h) : d_0(Ue) \to d_1(f)$ in $\mathcal{C}$ for each morphism $h : Ue \to f$ in $[\mathbb{2}, \mathcal{C}]$ such that the following diagram in $\mathcal{C}$ commutes,

$$\begin{tikzcd} \bullet \arrow[r, "d_1(h)"] \arrow[d, "Ue"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(h)"'] \arrow[ur, "\Phi(e,h)"] & \bullet \end{tikzcd}$$

  and furthermore, for each morphism $k : e' \to e$ in $\mathcal{I}$, we require that the following diagram commute:

$$\begin{tikzcd} \bullet \arrow[r, "d_1(Uk)"] \arrow[d, "Ue'"'] & \bullet \arrow[r, "d_1(h)"] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(Uk)"'] \arrow[urr, "\Phi(e',h\circ Uk)"] & \bullet \arrow[r, "d_0(h)"'] \arrow[ur, "\Phi(e,h)"'] & \bullet \end{tikzcd}$$

- The morphisms in $\mathbf{RLP}_{\mathcal{C}}(U)$ are commutative squares in $\mathcal{C}$ that are compatible with the chosen liftings, i.e. a morphism $l : (f', \Phi') \to (f, \Phi)$ is a morphism $l : f' \to f$ in $[\mathbb{2}, \mathcal{C}]$ such that, for all morphisms $h' : Ue \to f'$ in $[\mathbb{2}, \mathcal{C}]$, the following diagram commutes:

$$\begin{tikzcd} \bullet \arrow[r, "d_1(h')"] \arrow[d, "Ue"'] & \bullet \arrow[r, "d_1(l)"] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(h')"'] \arrow[ur, "\Phi(e,h')"] \arrow[urr, "\Phi(e,l\circ h')"'] & \bullet \arrow[r, "d_0(l)"'] & \bullet \end{tikzcd}$$

- Composition and identities are inherited from $[\mathbb{2}, \mathcal{C}]$.

- The structure functor $\mathbf{RLP}_{\mathcal{C}}(U) \to [\mathbb{2}, \mathcal{C}]$ is the evident forgetful functor sending (f, Φ) to f.

Note that the construction of $\mathbf{RLP}_{\mathcal{C}}(U)$ is contravariantly functorial in U.

Proposition A.3.46. *Let $\mathcal{C}$ be a category, let (L, R) be a natural weak factorisation system on $\mathcal{C}$, let $\mathcal{L}$ be the category of L-coalgebras, and let $\mathcal{R}$ be the category of R-algebras in $[\mathbb{2}, \mathcal{C}]$. Then there is a natural functor $T : \mathcal{R} \to \mathbf{RLP}_{\mathcal{C}}(U_{\mathsf{L}})$ making the diagram below commute,*

$$\begin{tikzcd} \mathcal{R} \arrow[rr, dashed, "T"] \arrow[dr, "U^{\mathsf{R}}"'] & & \mathbf{RLP}_{\mathcal{C}}(U_{\mathsf{L}}) \arrow[dl, "U"] \\ & {[\mathbb{2}, \mathcal{C}]} & \end{tikzcd}$$

where $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$, $U^{\mathsf{R}} : \mathcal{R} \to [\mathbb{2}, \mathcal{C}]$ and $U : \mathbf{RLP}_{\mathcal{C}}(U_{\mathsf{L}}) \to [\mathbb{2}, \mathcal{C}]$ are the respective forgetful functors.

Proof. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in $\mathcal{C}$, let $(r, \mathrm{id}) : Rf \to f$ be an R-algebra structure on f, and let $(\mathrm{id}, i) : g \to Lf$ be an L-coalgebra structure on g. Given a commutative square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

we choose the lifting $W \to X$ defined by the following commutative diagram,

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"] & X \arrow[dd, "f"] \\ & W' \arrow[r] \arrow[d, "Rg"'] & X' \arrow[ur, "r"'] \arrow[d, "Rf"] & \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, "i"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

where the morphism $W' \to X'$ is the one given by the functorial factorisation. It is not hard to see that this choice of liftings is compatible with the morphisms in $\mathcal{L}$, so we have an object in $\mathbf{RLP}_{\mathcal{C}}(U_{\mathsf{L}})$. Similarly, one may verify that the liftings are compatible with the morphisms in $\mathcal{R}$. Thus, we have the required functor $T : \mathcal{R} \to \mathbf{RLP}_{\mathcal{C}}(U_{\mathsf{L}})$ compatible with the forgetful functors, and it is clearly natural in (L, R). ■

Definition A.3.47. Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. An **algebraically free natural weak factorisation system** on $\mathcal{C}$ cofibrantly generated by U is a natural weak factorisation system (L, R) on $\mathcal{C}$ equipped with a functor $E : \mathcal{I} \to \mathcal{L}$ making the following diagram commute,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, dashed, "E"] \arrow[dr, "U"'] & & \mathcal{L} \arrow[dl, "U_{\mathsf{L}}"] \\ & {[\mathbb{2}, \mathcal{C}]} & \end{tikzcd}$$

where $\mathcal{L}$ is the category of L-coalgebras in $[\mathbb{2}, \mathcal{C}]$ and $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ is the forgetful functor, such that the composite functor shown below is an isomorphism,

$$\mathcal{R} \xrightarrow{T} \mathbf{RLP}_{\mathcal{C}}\left(U_{\mathsf{L}}\right) \xrightarrow{E^*} \mathbf{RLP}_{\mathcal{C}}(U)$$

where $\mathcal{R}$ is the category of R-algebras and $T : \mathcal{R} \to \mathbf{LLP}\left(U_{\mathsf{L}}\right)$ is the canonical functor given in proposition A.3.46.

Remark A.3.48. If $\mathcal{C}$ admits an algebraically free natural weak factorisation system (L, R) cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$, then the forgetful functor $\mathbf{RLP}_{\mathcal{C}}(U) \to [\mathbb{2}, \mathcal{C}]$ is monadic, and the induced monad is isomorphic to R. Garner's small object argument (theorem 0.5.24) gives sufficient conditions for the existence of algebraically free natural weak factorisation systems; note that natural weak factorisation systems so constructed also satisfy the distributive law and are therefore algebraic factorisation systems.

Proposition A.3.49. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subensemble of* $\operatorname{mor} \mathcal{C}$*, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be the evident embedding. If* (L, R) *is an algebraically free natural weak factorisation system cofibrantly generated by U, then the underlying weak factorisation system of* (L, R) *is cofibrantly generated by $\mathcal{I}$.*

Proof. This follows from the definitions and proposition A.3.41. ■

Definition A.3.50. Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. A **free algebraic factorisation system** on $\mathcal{C}$ cofibrantly generated by U is an algebraic factorisation system (L, R) equipped with a functor $E : \mathcal{I} \to \mathcal{L}$ making the following diagram commute,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, dashed, "E"] \arrow[dr, "U"'] & & \mathcal{L} \arrow[dl, "U_{\mathsf{L}}"] \\ & {[\mathbb{2}, \mathcal{C}]} & \end{tikzcd}$$

where $\mathcal{L}$ is the category of L-algebras in $[\mathbb{2}, \mathcal{C}]$ and $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ is the forgetful functor, such that (L, R) and E have the following universal property:

- For all algebraic factorisation systems $(\mathsf{L}', \mathsf{R}')$ and all functors $E' : \mathcal{I} \to \mathcal{L}'$ where $\mathcal{L}'$ is the category of L'-coalgebras and E' is compatible with the forgetful functors, there exists a unique morphism $\theta : (\mathsf{L}, \mathsf{R}) \to (\mathsf{L}', \mathsf{R}')$ such that $E' = \theta^L_* E$, where $\theta^L_* : \mathcal{L} \to \mathcal{L}'$ is the functor induced by the comonad morphism $\theta^L : \mathsf{L} \to \mathsf{L}'$.

Theorem A.3.51. *Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. If (L, R) is an algebraic factorisation system on $\mathcal{C}$ and also an algebraically free natural weak factorisation system cofibrantly generated by U, then (L, R) is a free algebraic factorisation system cofibrantly generated by U.*

Proof. See Theorem A.1 in [Garner, 2009]. □

Remark. The cited proof of the theorem above uses the distributive law for algebraic factorisation systems.

A.4 Relative categories

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.4.1. A **relative category** $\mathcal{C}$ consists of a category $\operatorname{und} \mathcal{C}$ and a subcategory $\operatorname{weq} \mathcal{C}$ such that $\operatorname{ob} \operatorname{und} \mathcal{C} = \operatorname{ob} \operatorname{weq} \mathcal{C}$. We say $\operatorname{und} \mathcal{C}$ is the **underlying category** of $\mathcal{C}$, and that the morphisms in $\operatorname{weq} \mathcal{C}$ are the **weak equivalences** in $\mathcal{C}$. A **relative subcategory** of a relative category $\mathcal{C}$ is a relative category $\mathcal{C}'$ such that $\operatorname{und} \mathcal{C}'$ is a subcategory of $\operatorname{und} \mathcal{C}$, and we further demand that $\operatorname{weq} \mathcal{C}' = \operatorname{weq} \mathcal{C} \cap \operatorname{und} \mathcal{C}'$.

Remark A.4.2. The subcategory $\operatorname{weq} \mathcal{C}$ is entirely determined by $\operatorname{mor} \operatorname{weq} \mathcal{C}$, so a relative category may equivalently be defined as a category equipped with a distinguished subset of morphisms closed under composition and containing all the identity morphisms.

For brevity, we will write $\operatorname{ob} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, $\operatorname{mor} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, and we may occasionally abuse notation and write $\operatorname{weq} \mathcal{C}$ instead of $\operatorname{mor} \operatorname{weq} \mathcal{C}$.

Remark A.4.3. Every category $\mathcal{C}$ can be endowed with the structure of a relative category in two ways: we can make it into a **minimal relative category** $\min \mathcal{C}$ by taking $\operatorname{weq} \min \mathcal{C}$ to be the set of identity morphisms in $\mathcal{C}$; or we could make it into a **maximal relative category** $\max \mathcal{C}$ by taking $\operatorname{weq} \max \mathcal{C} = \operatorname{mor} \mathcal{C}$. We may also define the **minimal saturated relative category** $\min^{+} \mathcal{C}$ by taking $\operatorname{weq} \min^{+} \mathcal{C}$ to be the set of all isomorphisms in $\mathcal{C}$.

Definition A.4.4. Given a relative category $\mathcal{C}$, the **opposite relative category** $\mathcal{C}^{\mathrm{op}}$ is defined by $\operatorname{und} \mathcal{C}^{\mathrm{op}} = (\operatorname{und} \mathcal{C})^{\mathrm{op}}$ and $\operatorname{weq} \mathcal{C}^{\mathrm{op}} = (\operatorname{weq} \mathcal{C})^{\mathrm{op}}$.

Definition A.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **relative functor** $\mathcal{C} \to \mathcal{D}$ is a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ that sends weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. The **relative functor category** $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is the full subcategory of $[\operatorname{und} \mathcal{C}, \operatorname{und} \mathcal{D}]$ spanned by the relative functors, and the weak equivalences in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ are defined to be the natural transformations that are componentwise weak equivalences in $\mathcal{D}$.

Definition A.4.6. Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$. A **localisation of $\mathcal{C}$ at $\mathcal{W}$** is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ with the following universal property:

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that Ff is an isomorphism for all f in $\mathcal{W}$, there exists a unique functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $\overline{F}\gamma = F$.

The functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ is called the **localising functor**.

Remark A.4.7. The universal property in the above definition is strict; as such, $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to unique isomorphism. Nonetheless, $\mathcal{C}[\mathcal{W}^{-1}]$ automatically has a 2-universal property: if $F, G : \mathcal{C} \to \mathcal{D}$ both factor through $\mathcal{C}[\mathcal{W}^{-1}]$, then so do all natural transformations $F \Rightarrow G$.

Proposition A.4.8. *If $\mathcal{C}$ is a* $\mathbf{U}$*-small category, then there exists a* $\mathbf{U}$*-small category with the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$.*

Proof. Use the general adjoint functor theorem. □

Definition A.4.9. The **homotopy category** of a relative category $\mathcal{C}$ is a localisation of $\operatorname{und} \mathcal{C}$ at $\operatorname{weq} \mathcal{C}$ and is denoted $\operatorname{Ho} \mathcal{C}$.

Definition A.4.10.

- A **semi-saturated relative category** is a relative category in which every isomorphism is a weak equivalence.

- A **saturated relative category** is a relative category $\mathcal{C}$ such that the weak equivalences in $\mathcal{C}$ are precisely the ones that become isomorphisms in $\operatorname{Ho}\mathcal{C}$.

Remark A.4.11. Obviously, there is no loss of generality in considering semi-saturated relative categories and their homotopy categories instead of localisations $\mathcal{C}[\mathcal{W}^{-1}]$ for arbitrary subsets $\mathcal{W} \subseteq \operatorname{mor}\mathcal{C}$.

Remark A.4.12. Clearly, every saturated relative category is semi-saturated, and a minimal saturated relative category is indeed saturated in the sense above.

Definition A.4.13. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subset of $\operatorname{mor}\mathcal{C}$. The **2-out-of-3 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, if any two of f, g, or $g \circ f$ are in $\mathcal{W}$, then all of them are.

The **2-out-of-6 property** for $\mathcal{W}$ says:

- Given any three morphisms $f : X \to Y$, $g : Y \to Z$, $h : Y \to Z$ in $\mathcal{C}$, if both $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so too are f, g, h, and $h \circ g \circ f$.

Lemma A.4.14. *Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor}\mathcal{C}$.*

(i) *If $\mathcal{W}$ has the 2-out-of-6 property, then it also has the 2-out-of-3 property.*

(ii) *The set of all isomorphisms in $\mathcal{C}$ has the 2-out-of-6 property.*

(iii) *If $F : \mathcal{C}' \to \mathcal{C}$ is a functor and $\mathcal{W}$ has either the 2-out-of-3 property or the 2-out-of-6 property, then $F^{-1}\mathcal{W}$ has the same property.*

Proof. (i). Consider the three cases $f = \mathrm{id}$, $g = \mathrm{id}$, $h = \mathrm{id}$ in turn.

(ii). If $h \circ g$ and $g \circ f$ are isomorphisms, then g must be split epic and split monic; thus g itself is an isomorphism, hence so too are f and h.

(iii). Obvious. ■

Corollary A.4.15. *If $\mathcal{C}$ is a saturated relative category, then* $\operatorname{weq}\mathcal{C}$ *has the 2-out-of-6 property.* ■

Definition A.4.16. Let C be a category and let $\mathcal{W}$ be a subset of $\operatorname{mor} C$. The **2-out-of-4 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to X$ in C, if $f \circ g$ and $g \circ f$ are in $\mathcal{W}$, then both f and g are in $\mathcal{W}$.

The **special 2-out-of-4 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to X$ in C, if $f \circ g$ is in $\mathcal{W}$ and $g \circ f = \mathrm{id}_X$, then both f and g are in $\mathcal{W}$.

Lemma A.4.17. *Let C be a relative category.*

(i) *If* $\operatorname{weq} C$ *has the 2-out-of-4 property, then* $\operatorname{weq} C$ *has the special 2-out-of-4 property.*

(ii) *If* $\operatorname{weq} C$ *has the 2-out-of-6 property, then* $\operatorname{weq} C$ *has the 2-out-of-4 property.*

(iii) *If* $\operatorname{weq} C$ *has the 2-out-of-3 property and is closed under retracts, then* $\operatorname{weq} C$ *has the special 2-out-of-4 property.*

Proof. (i) and (ii). Obvious.

(iii). Let $f : X \to Y$ and $g : Y \to X$ be morphisms in C such that $f \circ g$ is a weak equivalence and $g \circ f = \mathrm{id}_X$. Consider the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "f\circ g"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

Since $g \circ f = \mathrm{id}_X$, the diagram commutes, so we see that $g : Y \to X$ is a retract of $f \circ g : Y \to Y$. We deduce that g is a weak equivalence in C using the fact that $\operatorname{weq} C$ is closed under retracts, and then we deduce that f is a weak equivalence using the the 2-out-of-3 property of $\operatorname{weq} C$. ■

Proposition A.4.18. *Let* **RelCat** *be the category of* **U**-*small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories, and let* **Cat** *be the category of* **U**-*small categories and functors.*

(i) **RelCat** *is a cartesian closed category, where the product of $\mathcal{C}$ and $\mathcal{D}$ is the cartesian product $\mathcal{C} \times \mathcal{D}$ with weak equivalences taken componentwise, and the exponential of $\mathcal{E}$ by $\mathcal{D}$ is the relative functor category* $[\mathcal{D}, \mathcal{E}]_{\mathrm{h}}$.

(ii) **RelCat** *is a locally finitely presentable* **U***-category,*[5] *and the two functors* $\mathrm{und}, \mathrm{weq} : \mathbf{RelCat} \to \mathbf{Cat}$ *are* $\aleph_0$*-accessible*[6] *and jointly conservative.*

(iii) **SsRelCat** *is a locally finitely presentable* **U***-category, and the inclusion* $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$ *is* $\aleph_0$*-accessible and has a left adjoint.*

(iv) **SsRelCat** *is an exponential ideal in* **RelCat**.

(v) *The full subcategory spanned by the minimal relative categories is an exponential ideal in* **RelCat**.

(vi) *The full subcategory spanned by the minimal saturated relative categories is an exponential ideal in* **SsRelCat**.

Proof. (i). This is straightforward from the definitions.

(ii). Obviously, a relative functor $F : \mathcal{C} \to \mathcal{D}$ such that $\mathrm{und}\, F : \mathrm{und}\, \mathcal{C} \to \mathrm{und}\, \mathcal{D}$ and $\mathrm{weq}\, F : \mathrm{weq}\, \mathcal{C} \to \mathrm{weq}\, \mathcal{D}$ are both isomorphisms is itself an isomorphism, so $\mathrm{und}, \mathrm{weq} : \mathbf{RelCat} \to \mathbf{Cat}$ are indeed jointly conservative.

It is also not hard to check that limits for all **U**-small diagrams and colimits for **U**-small filtered diagrams in **RelCat** exist and can be computed componentwise in **Cat**, so (by theorem 0.2.40) it is enough to show that **RelCat** is a $\aleph_0$-accessible **U**-category. Clearly, a relative category $\mathcal{C}$ such that $\mathrm{und}\, \mathcal{C}$ is finitely presentable in **Cat** and $\mathrm{weq}\, \mathcal{C}$ is a finitely-generated subcategory of $\mathrm{und}\, \mathcal{C}$ is itself finitely presentable in **RelCat**, so **RelCat** is indeed $\aleph_0$-accessible.

(Alternatively, one may appeal to the sketchability theorem[7] and the fact that a relative category is manifestly a model for a certain finite-limit sketch.)

(iii). It is clear that **SsRelCat** is closed in **RelCat** under limits for all **U**-small diagrams and colimits for all **U**-small filtered diagrams, and we know that **RelCat** is a locally finitely presentable category, so (by proposition 0.2.31) it is enough to construct a left adjoint for the inclusion $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$. This may be done using the general adjoint functor theorem.

[5] See definition 0.2.36.
[6] See definition 0.2.28.
[7] See Proposition 1.51 in [LPAC], or Proposition 5.6.4 in [Borceux, 1994b], or theorem 0.5.34.

(iv) – (vi). All straightforward. ■

Proposition A.4.19. *Let* **RelCat** *be the category of* **U***-small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories and relative functors, and let* **Cat** *be the category of* **U***-small categories and functors. We have the following strings of adjoint functors:*

$$\min \dashv \operatorname{und} \dashv \max \dashv \operatorname{weq} : \mathbf{RelCat} \to \mathbf{Cat}$$
$$\operatorname{Ho} \dashv \min^+ \dashv \operatorname{und} \dashv \max \dashv \operatorname{weq} : \mathbf{SsRelCat} \to \mathbf{Cat}$$

The functors min, $\min^+$, *and* max *are moreover fully faithful, and* Ho *preserves finite products.*

Proof. All but the last of the above claims are obvious; for the preservation of finite products under Ho, we refer to proposition A.2.13. ■

Corollary A.4.20. $\operatorname{Ho} : \mathbf{SsRelCat} \to \mathbf{Cat}$ *is 2-functorial.*

Proof. Apply remark A.2.10. ■

Proposition A.4.21. *Let $\mathcal{C}$ be a relative category and let $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ be the localising functor.*

(i) *For all categories $\mathcal{D}$, the induced functor $\gamma^* : [\operatorname{Ho} \mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$ is fully faithful and injective on objects.*

(ii) *Any left or right adjoint for $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ is a fully faithful functor.*

Proof. (i). It is an immediate consequence of the universal property of $\operatorname{Ho} \mathcal{C}$ that $\gamma^* : [\operatorname{Ho} \mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$ is injective on objects. It is moreover fully faithful because we have the following natural isomorphism,

$$[\operatorname{Ho} \mathcal{C}, \mathcal{D}] \cong \operatorname{und} \left[\mathcal{C}, \min^+ \mathcal{D}\right]_{\mathrm{h}}$$

and $\operatorname{und} \left[\mathcal{C}, \min^+ \mathcal{D}\right]_{\mathrm{h}}$ is manifestly a full subcategory of $[\mathcal{C}, \mathcal{D}]$.

(ii). Apply proposition A.1.4. ■

Definition A.4.22. A **zigzag type** is a triple (n, U, V) where n is a natural number and U and V are partial orderings of the set $\{0, \ldots, n, \infty\}$ that satisfy the following axioms:

- $U \cap V$ is the equality relation.
- The partial ordering generated by $U \cup V$ is the standard linear ordering of $\{0, \ldots, n, \infty\}$.
- If $i \leq j \leq k \leq l$ are elements of $\{0, \ldots, n, \infty\}$ and (i, l) is in U (resp. V), then (j, k) is also in U (resp. V).

Remark A.4.23. Let n be a natural number and let U and V be partial orderings of the set $\{0, \ldots, n, \infty\}$. The following are equivalent:

(i) (n, U, V) is a zigzag type.

(ii) U and V are partial orderings generated by some subset of the set of consecutive pairs, i.e. $\{(0, 1), \ldots, (n-1, n), (n, \infty)\}$, and for $0 \leq i \leq n$, either $(i, i+1) \in U$ or $(i, i+1) \in V$ but not both (provided we interpret $n+1$ as ∞).

Thus, we may think of (n, U, V) as a planar graph of the form

$$0 \text{ ——— } \cdots \text{ ——— } n \text{ ——— } \infty$$

where the edge between i and $i+1$ points rightwards (resp. leftwards) if $(i, i+1) \in U$ (resp. $(i, i+1) \in V$). It will be especially convenient to regard zigzag types as relative categories generated by such graphs, with the leftward-pointing arrows generating the weak equivalences.

Definition A.4.24. A **morphism of zigzag types** $(n', U', V') \to (n, U, V)$ is a map $f : \{0, \ldots, n', \infty\} \to \{0, \ldots, n, \infty\}$ that satisfies the following axioms:

- $f(0) = 0$, $f(\infty) = \infty$.
- $(i, j) \in U'$ implies $(f(i), f(j)) \in U$.
- $(i, j) \in V'$ implies $(f(i), f(j)) \in V$.

We write $\mathbf{Z}$ for the category of zigzag types.

Remark. The category $\mathbf{Z}$ defined above is the *opposite* of the category $\mathbf{II}$ defined in [Dwyer and Kan, 1980b, §4] and also the category $\mathbf{T}$ defined in [DHKS, §34].

Remark A.4.25. Following remark A.4.23, if we regard zigzag types as relative categories, then a morphism of zigzag types is a relative functor that preserves the endpoints and *reflects* weak equivalences (in addition to preserving them). Thus $\mathbf{Z}$ can be embedded as a *non-full* subcategory of **RelCat**.

Remark A.4.26. We have the following concrete description of $\mathbf{Z}^{\mathrm{op}}$:

- The objects are triples (n, U_0, V_0) where n is a natural number and (U_0, V_0) is a partition of the set $\{0, \dots, n\}$, i.e. $U_0 \cap V_0 = \varnothing$ and $U_0 \cup V_0 = \{0, \dots, n\}$.
- The morphisms $(n, U_0, V_0) \to (n', U_0', V_0')$ are the monotone maps $f : \{0, \dots, n\} \to \{0, \dots, n'\}$ such that $f U_0 \subseteq U_0'$ and $f V_0 \subseteq V_0'$.
- Identities and composition are inherited from **Set**.

In other words, $\mathbf{Z}^{\mathrm{op}}$ is isomorphic to to the category of positive finite ordinals equipped with a partition into two parts. A zigzag type (n, U, V) corresponds to the object (n, U_0, V_0) where $i \in U_0$ (resp. $i \in V_0$) if $(i, i+1) \in U$ (resp. $(i, i+1) \in V$), and a morphism $f : (n', U', V') \to (n, U, V)$ corresponds to the morphism $(n, U_0, V_0) \to (n', U_0', V_0')$ that sends j to the greatest i such that $f(i) \leq j$.

Lemma A.4.27. *Every morphism of zigzag types whose underlying map is injective is a split monomorphism in* $\mathbf{Z}$.

Proof. This is a straightforward exercise. ◊

Definition A.4.28. The **kernel** of a morphism $f : (n', U', V') \to (n, U, V)$ in $\mathbf{Z}$ is the following subset:

$$\ker f = \{ i \mid f(i) = f(i+1) \} \subseteq \{1, \dots, n'\}$$

Remark A.4.29. It is clear that a morphism of zigzag types whose underlying map is surjective is uniquely determined by its kernel. Moreover, given a zigzag type (n', U', V'), it every subset of $\{1, \dots, n'\}$ occurs as the kernel of some (surjective) morphism $f : (n', U', V') \to (n, U, V)$.

Proposition A.4.30.

(i) *The monomorphisms in* $\mathbf{Z}$ *are the morphisms of zigzag types whose underlying maps are injective.*

(ii) *The epimorphisms in* $\mathbf{Z}$ *is the morphisms of zigzag types whose underlying maps are surjective.*

(iii) *Every morphism in* $\mathbf{Z}$ *factors as an epimorphism followed by a monomorphism, and this factorisation is unique (in the strict sense).*

Proof. This is a straightforward exercise. ◊

Definition A.4.31. Let $T = (n, U, V)$ be a zigzag type.

- An **interior vertex** of T is an element i of $\{1, \ldots, n\}$ such that either
  - both $(i-1, i) \in U$ and $(i, i+1) \in U$, or
  - both $(i-1, i) \in V$ and $(i, i+1) \in V$,

  provided we interpret $n+1$ as ∞.
- An **flex** of T is an element of $\{1, \ldots, n\}$ that is *not* an interior vertex.

Remark A.4.32. As a matter of convention, 0 and ∞ are neither interior vertices nor flexes.

Lemma A.4.33. *Let* $f : S \to T$ *be a morphism in* $\mathbf{Z}$*.*

(i) *Every flex of* T *is in the image of* $f : S \to T$*, and if* $f(i)$ *is a flex of* T*, then* i *is a flex of* S*.*

(ii) *The morphism* $f : S \to T$ *sends flexes of* S *to flexes of* T *if and only if* $f : S \to T$ *is bijective on flexes.*

(iii) *Assuming* $f : S \to T$ *is bijective on flexes,* $f : S \to T$ *is an epimorphism in* $\mathbf{Z}$ *if and only if* $f : S \to T$ *is a split epimorphism in* $\mathbf{Z}$*.*

Proof. This is a straightforward exercise. ◊

Proposition A.4.34.

(i) *Pullbacks of monomorphisms along flex-preserving morphisms (in particular, monomorphisms) exist in* $\mathbf{Z}$*.*

(ii) *Pushouts of epimorphisms along arbitrary morphisms (in particular, epimorphisms) exist in* $\mathbf{Z}$*.*

Proof. (i). By proposition A.4.30 and the pullback pasting lemma, it suffices to prove that pullbacks of monomorphisms along monomorphisms and flex-preserving epimorphisms exist, and this is straightforward.

(ii). Use remark A.4.29. ■

Lemma A.4.35. *Let T be a zigzag type and let $\partial\mathbf{Z}_{\to T}$ be the full subcategory of the slice category $\mathbf{Z}_{/T}$ spanned by the monomorphisms with codomain T* excluding $\mathrm{id} : T \to T$. *Then $\partial\mathbf{Z}_{\to T}$ has at most one connected component.*

Proof. Proposition A.4.34 implies that $\partial\mathbf{Z}_{\to T}$ is closed under binary products in $\mathbf{Z}_{/T}$, so it has at most one connected component. ■

Lemma A.4.36. *Let T be a zigzag type and let $\partial\mathbf{Z}^{\leftarrow T}$ be the full subcategory of the slice category $\mathbf{Z}_{/T}$ spanned by the epimorphisms with domain T* excluding $\mathrm{id} : T \to T$. *Then $\partial\mathbf{Z}^{\leftarrow T}$ has binary coproducts.*

Proof. This is is a straightforward corollary of proposition A.4.34. ■

Definition A.4.37. Let $T' = (n', U', V')$ and $T = (n, U, V)$ be zigzag types. The **concatenation** $T * T'$ is the zigzag type $(n + n' + 1, U'', V'')$ where:

- U'' is the smallest partial ordering of $\{0, \dots, n + n' + 1, \infty\}$ where $(i, i+1) \in U''$ if $(i, i+1) \in U'$ and $(n' + i + 1, n' + i + 2) \in U''$ if $(i, i+1) \in U$.
- V'' is the smallest partial ordering of $\{0, \dots, n + n' + 1, \infty\}$ where $(i, i+1) \in V''$ if $(i, i+1) \in V'$ and $(n' + i + 1, n' + i + 2) \in V''$ if $(i, i+1) \in V$.

Remark A.4.38. Concatenation defines a functor $* : \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, and it is associative as a binary operation. However, it is not unital. To repair this, we introduce the **degenerate zigzag type** $(-1, \varnothing, \varnothing)$: there are no morphisms $(n, U, V) \to (-1, \varnothing, \varnothing)$ when $n \geq 0$, and there is a unique morphism $(-1, \varnothing, \varnothing) \to (n, U, V)$ for all (possibly degenerate) zigzag types (n, U, V). It is clear how to extend $*$ so that $(-1, \varnothing, \varnothing)$ is the unit.

Definition A.4.39. Let C be a relative category, let X and Y be objects in C, and let T be a zigzag type.

- A **zigzag** in C from X to Y of type T is a relative functor $T \to C$ that sends 0 to X and ∞ to Y.

- A **morphism of zigzags** in C from X to Y of type T is a commutative diagram in C of the form below,

$$\begin{tikzcd} X \arrow[r, dash] \arrow[d, equal] & \bullet \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d] & Y \arrow[d, equal] \\ X \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] & Y \end{tikzcd}$$

  where the rows are zigzags in C from X to Y of type T and the unmarked vertical arrows are weak equivalences; the domain is the top row and the codomain is the bottom row.

We write $C^T(X, Y)$ for the category of zigzags in C from X to Y of type T.

REMARK A.4.40. Let C be a relative category, let $\mathcal{W} = \operatorname{weq} C$, and let T be a zigzag type. Then:

- If the leftmost arrow of T points rightwards and the rightmost arrow of T points rightwards, then we have a functor

$$C^T(-,-) : C^{\mathrm{op}} \times C \to \mathbf{Cat}$$

  sending each object (X, Y) to the category of zigzags in C from X to Y of type T.

- If the leftmost arrow of T points leftwards and the rightmost arrow of T points leftwards, then we have a functor

$$C^T(-,-) : \mathcal{W} \times \mathcal{W}^{\mathrm{op}} \to \mathbf{Cat}$$

  sending each object (X, Y) to the category of zigzags in C from X to Y of type T.

- If the leftmost arrow of T points leftwards and the rightmost arrow of T points rightwards, then we have a functor

$$C^T(-,-) : \mathcal{W} \times C \to \mathbf{Cat}$$

  sending each object (X, Y) to the category of zigzags in C from X to Y of type T.

- If the leftmost arrow of T points rightwards and the rightmost arrow of T points leftwards, then we have a functor

$$\mathcal{C}^T(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{W}^{\mathrm{op}} \to \mathbf{Cat}$$

sending each object (X, Y) to the category of zigzags in $\mathcal{C}$ from X to Y of type T.

Remark A.4.41. If $f : X \to Y$ is a weak equivalence in a relative category $\mathcal{C}$, then we have commutative diagrams

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, equal] \\ X \arrow[r, "f"'] & Y & X \arrow[l, "f"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

and these correspond to morphisms of zigzags in $\mathcal{C}$.

Remark A.4.42. It is clear that $\mathcal{C}^T(X, Y)$ is a subcategory of the relative functor category $[T, \mathcal{C}]_{\mathrm{h}}$. In fact, if $\mathcal{C}$ is a $\mathbf{U}$-small relative category, precomposition makes the assignment $T \mapsto \mathcal{C}^T(X, Y)$ into a functor $\mathbf{Z}^{\mathrm{op}} \to \mathbf{Cat}$, which we denote by $\mathcal{C}^*(X, Y)$. A Grothendieck construction applied to this functor yields the following $\mathbf{U}$-small category $\mathcal{C}^{(\mathbf{Z})}(X, Y)$:

- Its objects are pairs (T, f), where T is a zigzag type and f is a zigzag of type T in $\mathcal{C}$ from X to Y.
- A morphism $(T', f') \to (T, f)$ is a pair (α, β) where $\alpha : T \to T'$ is a morphism in $\mathbf{Z}$ and $\beta : \alpha^* f' \to f$ is a morphism in $\mathcal{C}^T(X, Y)$.
- The composite of a pair of morphisms $(\alpha', \beta') : (T'', f'') \to (T', f')$ and $(\alpha, \beta) : (T', f') \to (T, f)$ is given by $(\alpha' \circ \alpha, \beta \circ \alpha^* \beta')$.

There is an evident projection functor $\mathcal{C}^{(\mathbf{Z})}(X, Y) \to \mathbf{Z}^{\mathrm{op}}$, and by construction it is a Grothendieck opfibration with a canonical splitting.

Lemma A.4.43. *Given a commutative diagram of the form below in a relative category $\mathcal{C}$,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

if a and b are weak equivalences in C, then we obtain the following morphisms of zigzags:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, "b"] \arrow[d, "b"] & Y' \arrow[d, equal] \\ X' \arrow[r, "a"'] & X' \arrow[r, "f'"'] & Y' \arrow[r, equal] & Y' \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[d, equal] & X \arrow[l, "a"'] \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[r, equal] \arrow[d, "b"] & Y \arrow[d, equal] \\ X' \arrow[r, equal] & X' \arrow[r, "f'"'] & Y' & Y \arrow[l, "b"] \end{tikzcd}$$

In particular, $X \xrightarrow{f} Y \xrightarrow{b} Y'$ and $X \xrightarrow{a} X' \xrightarrow{f'} Y'$ are in the same connected component of $C^{(\mathbf{Z})}(X, Y')$; and $X' \xleftarrow{a} X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y' \xleftarrow{b} Y$ are in the same connected component of $C^{(\mathbf{Z})}(X', Y)$. ■

Theorem A.4.44. *Let X and Y be objects in a relative category C.*

(i) *For each zigzag type T, the map that sends an object in $C^T(X, Y)$ to the corresponding composite in* $\operatorname{Ho} C(X, Y)$ *is a functor when the latter is regarded as a discrete category.*

(ii) *The functors described above constitute a jointly surjective cocone from the diagram $C^*(X, Y)$ to* $\operatorname{Ho} C(X, Y)$.

(iii) *The induced functor $C^{(\mathbf{Z})}(X, Y) \to$* $\operatorname{Ho} C(X, Y)$ *is surjective, and moreover two objects in $C^{(\mathbf{Z})}(X, Y)$ become equal in* $\operatorname{Ho} C$ *if and only if they are in the same connected component.*

Proof. All obvious except for the last part of claim (iii), for which we refer to paragraphs 33.8 and 33.10 in [DHKS]. □

A.5 Kan extensions

Prerequisites. §§0.1, A.1.

In this section we use the explicit universe convention.

Definition A.5.1. Let $F : C \to D$ and $G : C \to \mathcal{E}$ be two functors. A **left Kan extension** (resp. **right Kan extension**) of G along F is an initial (resp. terminal) object of the category $(G \downarrow F^*)$ (resp. $(F^* \downarrow G)$) described below:

- The objects are pairs (H, α) where H is a functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).

Remark A.5.2. Clearly, Kan extensions are unique up to unique isomorphism if they exist. We write $\left(\operatorname{Lan}_F G, \eta\right)$ for the left Kan extension of G along F and say η is the **unit** of $\operatorname{Lan}_F G$; dually, we write $\left(\operatorname{Ran}_F G, \varepsilon\right)$ for the right Kan extension of G along F and say ε is the **counit** of $\operatorname{Ran}_F G$.

Lemma A.5.3. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. Let* $\mathcal{B}$ *be a* $\mathbf{U}$*-small category and let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category. Given functors* $F : \mathcal{B} \to \mathcal{C}$ *and* $G : \mathcal{B} \to \mathbf{Set}$*, if* $H : \mathcal{C} \to \mathbf{Set}$ *is the functor defined by the formula below,*

$$H(C) = [\mathcal{B}, \mathbf{Set}](\mathcal{C}(C, F-), G-)$$

and $\varepsilon_B : H(FB) \to G(B)$ *is defined by evaluation at* id_{FB}*, then* (H, ε) *is the right Kan extension of* G *along* F*.*

Proof. Note that $H(C)$ so defined is indeed a $\mathbf{U}$-set, because $\mathcal{B}$ is $\mathbf{U}$-small and $\mathcal{C}$ is locally $\mathbf{U}$-small. The claim amounts to saying that (H, ε) is a terminal object in the comma category $(F^* \downarrow G)$, so that is what we must show.

Let $\varphi : (X, \alpha) \to (H, \varepsilon)$ be a morphism in $(F^* \downarrow G)$, i.e. a natural transformation $\varphi : X \Rightarrow H$ such that $\varepsilon \bullet \varphi F = \alpha$. Let C be an object in $\mathcal{C}$, let x be an element of $X(C)$, and consider the element $\varphi_C(x)$ of $H(C)$. By definition, this is a natural transformation $\mathcal{C}(C, F) \Rightarrow G$, so we may consider its component at an object B in $\mathcal{B}$, which will be a map $\mathcal{C}(C, FB) \to G(B)$. Let $f : C \to FB$ be an arrow in $\mathcal{C}$. By hypothesis,

$$\alpha_C(x) = \varepsilon_C\left(\varphi_C(x)_B \circ \mathcal{C}(f, FB)\right) = \varphi_C(x)_B(f)$$

thus the action of φ is entirely determined by α. Conversely, given any object (X, α) in the comma category $(F^* \downarrow G)$, it is easily verified that the above equation defines a morphism $\varphi : (X, \alpha) \to (H, \varepsilon)$, so (H, ε) is indeed a terminal object in $(F^* \downarrow G)$. ■

Corollary A.5.4. *For any two functors* $F : \mathcal{B} \to \mathcal{C}$ *and* $G : \mathcal{B} \to \mathbf{Set}$*, if* $\mathcal{B}$ *is* $\mathbf{U}$*-small and* $\mathcal{C}$ *is locally* $\mathbf{U}$*-small, then the following are equivalent:*

(i) $\left(\operatorname{Ran}_F G, \varepsilon\right)$ *is a right Kan extension of* G *along* F*.*

(ii) *The maps* $\big(\operatorname{Ran}_F G\big)(C) \to [\mathcal{B}, \mathbf{Set}](\mathcal{C}(C, F), G)$ *defined by* $x \mapsto \varepsilon \bullet \theta_x F$, *where* $\theta_x : \mathcal{C}(C, -) \Rightarrow G$ *is the unique natural transformation such that* $\big(\theta_x\big)_C\big(\mathrm{id}_C\big) = x$, *are bijections that are natural in* C. ∎

Definition A.5.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors.

- A functor $L : \mathcal{E} \to \mathcal{F}$ **preserves** left Kan extensions of G along F if, given any left Kan extension (H, α) of G along F, $(LH, L\alpha)$ is a left Kan extension of LG along F.

- A functor $R : \mathcal{E} \to \mathcal{F}$ **preserves** right Kan extensions of G along F if, given any right Kan extension (H, α) of G along F, $(RH, R\alpha)$ is a right Kan extension of LG along F.

If a Kan extension is preserved by *all* functors, then it is said to be **absolute**.

Definition A.5.6. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-small sets, let $\mathcal{E}$ be a locally $\mathbf{U}$-small category, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors.

- A **pointwise left Kan extension** of G along F is one that is preserved by all functors of the form $\mathcal{E}(-, E) : \mathcal{E} \to \mathbf{Set}^{\mathrm{op}}$.

- A **pointwise right Kan extension** of G along F is one that is preserved by all functors of the form $\mathcal{E}(E, -) : \mathcal{E} \to \mathbf{Set}$.

Definition A.5.7. Let $F : \mathcal{B} \to \mathcal{C}$ be a functor and let C be an object in $\mathcal{C}$.

- The **tautological cocone** to C induced by F is the cocone $\theta : FP_C \Rightarrow \Delta C$, where $P_C : (F \downarrow C) \to \mathcal{B}$ is the projection functor sending an object (B, f) in the comma category $(F \downarrow C)$ to the object B in $\mathcal{B}$, and $\theta_{(B,f)} = f$.

- The **tautological cone** from C induced by F is the cone $\theta : \Delta C \Rightarrow FP^C$, where $P^C : (C \downarrow F) \to \mathcal{C}$ is the projection functor sending an object (B, f) in the comma category $(C \downarrow F)$ to the object B in $\mathcal{B}$, and $\theta_{(B,f)} = f$.

Lemma A.5.8. *Let* $\mathcal{A}$ *be any category, let* $\mathcal{B}$ *be a* $\mathbf{U}$*-small category, let* $\mathcal{C}$ *be locally* $\mathbf{U}$*-small category, and let* $U : \mathcal{A} \to \mathcal{C}$, $V : \mathcal{B} \to \mathcal{C}$, *and* $Y : \mathcal{B} \to \mathbf{Set}$

be functors. Consider the following diagram of functors and natural transformations,

$$\begin{tikzcd}
(U \downarrow V) \arrow[r, "Q"] \arrow[d, "P"'] \arrow[dr, phantom, "\overset{\theta}{\Nearrow}"] & \mathcal{B} \arrow[d, "V"] \\
\mathcal{A} \arrow[r, "U"'] & \mathcal{C}
\end{tikzcd}$$

where $(U \downarrow V)$ is the comma category, $P : (U \downarrow V) \to \mathcal{A}$ and $Q : (U \downarrow V) \to \mathcal{B}$ are the two projections, and $\theta : UP \Rightarrow VQ$ is the tautological natural transformation defined by $\theta_{(A,B,f)} = f$. If (Z, ε) is a right Kan extension of Y along V, then $(ZU, \varepsilon Q \bullet Z\theta)$ is a right Kan extension of YQ along P.

Proof. By lemma A.5.3, we may take $Z : \mathcal{C} \to \mathbf{Set}$ to be the functor defined by the formula below,

$$Z(C) = [\mathbb{B}, \mathbf{Set}](\mathcal{C}(C, F-), Y-)$$

with $\varepsilon : V^*(Z) \Rightarrow Y$ being the natural transformation obtained by evaluating elements of $Z(VB)$ at id_{VB}.

Let $\varphi : (X, \alpha) \to (ZU, \varepsilon Q \bullet Z\theta)$ be a morphism in $(P^* \downarrow YQ)$, i.e. a natural transformation $\varphi : X \Rightarrow ZU$ such that $\varepsilon Q \bullet Z\theta \bullet \varphi P = \alpha$. Let A be an object in $\mathcal{A}$, let x be an element of $X(A)$, and consider the element $\varphi_A(x)$ of $Z(UA)$. By definition, this is a natural transformation $\mathrm{N}^V(C) \Rightarrow Y$, so we may consider its component at an object B in $\mathcal{B}$, which will be a map $\mathcal{C}(UA, VB) \to Y(B)$. Let $f : UA \to VB$ be an arrow in $\mathcal{C}$; then (A, B, f) is an object in the comma category $(U \downarrow V)$, and $\theta_{(A,B,f)} = f$ by definition. By hypothesis,

$$\alpha_{(A,B,f)}(x) = \varepsilon_B\big(\varphi_A(x)_B \circ \mathcal{C}(f, VB)\big) = \varphi_A(x)_B(f)$$

thus the action of φ is entirely determined by α. Conversely, given any object (X, α) in the comma category $(P^* \downarrow YQ)$, it is easily verified that the above equation defines a morphism $\varphi : (X, \alpha) \to (ZU, \varepsilon Q \bullet Z\theta)$, so $(ZU, \varepsilon Q \bullet Z\theta)$ is indeed a terminal object in $(P^* \downarrow YQ)$. ■

Corollary A.5.9. *Let $\mathcal{B}$ be a $\mathbf{U}$-small category and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Given functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{B} \to \mathbf{Set}$, if (H, ε) is a right Kan extension of G along F, then, for each object C in $\mathcal{C}$, the image under H of the tautological cone from C induced by F is a limiting cone in $\mathbf{Set}$.*

Proof. In the lemma, take $\mathcal{A}$ to be the terminal category $\mathbb{1}$, take $U : \mathbb{1} \to \mathcal{C}$ to be the functor sending the unique object in $\mathbb{1}$ to C, and take $V = F$; then $(HU, \varepsilon Q \bullet H\theta)$ is a right Kan extension of $GQ : (C \downarrow F) \to \mathbf{Set}$ along the unique functor $P : (C \downarrow F) \to \mathbb{1}$, but it is clear that a right Kan extension of GQ along P amounts to a limit for the diagram GQ in $\mathbf{Set}$. ■

It is convenient at this juncture to introduce a concept borrowed from enriched category theory. The notation below follows [Kelly, 2005, §3.1].

Definition A.5.10. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Given functors $W : \mathcal{J} \to \mathbf{Set}$ and $A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted limit** of A is an object $\{W, A\}^{\mathcal{J}}$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) \cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A))$$

that are natural in C. We may also write $\varprojlim_{j:\mathcal{J}}^{Wj} Aj$ instead of $\{W, A\}^{\mathcal{J}}$, if we wish to use an explicit variable j.

Dually, given functors $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ and $A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted colimit** of A is an object $W \star_{\mathcal{J}} A$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(W \star_{\mathcal{J}} A, C\big) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}](W, \mathcal{C}(A, C))$$

that are natural in C. We may also write $\varinjlim_{j:\mathcal{J}}^{Wj} Aj$ instead of $W \star_{\mathcal{J}} A$, if we wish to use an explicit variable j.

Remark A.5.11. Clearly, weighted limits and colimits are unique up to unique isomorphism if they exist.

It is also not hard to spell out the above definition in elementary terms; for example, one notes that to give a natural transformation $W \Rightarrow \mathcal{C}(C, A)$, one must give a morphism $\lambda_{j,x} : C \to Aj$ for each object j in $\mathcal{J}$ *and* each element x of Wj, and these are required to make various diagrams commute. This is a **W-weighted cone** from C to A, and $\{W, A\}^{\mathcal{J}}$ is an object equipped with a universal W-weighted cone to A. Similarly, one may define the notion of a **W-weighted cocone** from A to C, and then $W \star_{\mathcal{J}} A$ is an object equipped with a universal W-weighted cocone from A. In particular, if $Wj = 1$ for all j, then W-weighted limits and colimits reduce to ordinary limits and colimits.

The above discussion also shows that the concept of a weighted limit or colimit (within a fixed category!) does not depend on $\mathbf{U}$ in any essential way.

Lemma A.5.12. *Let $\mathcal{J}$ be a* $\mathbf{U}$*-small category. Given functors $F, G : \mathcal{J} \to \mathbf{Set}$, the F-weighted limit of G exists in* $\mathbf{Set}$*, and we have bijections*

$$\{F, G\}^{\mathcal{J}} \cong [\mathcal{J}, \mathbf{Set}](F, G)$$

that are natural in F and G.

Proof. One simply has to check that this works. ♦

Proposition A.5.13. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let $F : \mathcal{C} \to \mathcal{D}$ be any functor where $\mathcal{C}$ and $\mathcal{D}$ are locally* $\mathbf{U}$*-small categories.*

(i) *For each weight $W : \mathcal{J} \to \mathbf{Set}$ and each diagram $A : \mathcal{J} \to \mathcal{C}$, if the weighted limits $\{W, A\}^{\mathcal{J}}$ and $\{W, FA\}^{\mathcal{J}}$ both exist, then there is a canonical comparison morphism*

$$F\{W, A\}^{\mathcal{J}} \to \{W, FA\}^{\mathcal{J}}$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FC, FA))$$

induced by the functor F.

(ii) *For any object C in $\mathcal{C}$, the functor $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ preserves* all *weighted limits.*

(iii) *The functors $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$* jointly *reflect weighted limits.*

(iv) *If F has a left adjoint, then F preserves weighted limits.*

Dually:

(i′) *For each weight $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ and each diagram $A : \mathcal{J} \to \mathcal{C}$, if the weighted colimits $W \star_{\mathcal{J}} A$ and $W \star_{\mathcal{J}} FA$ both exist, then there is a canonical comparison morphism*

$$W \star_{\mathcal{J}} FA \to F\big(W \star_{\mathcal{J}} A\big)$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(A, C)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FA, FC))$$

induced by the functor F.

(ii′) *For any object C in $\mathcal{C}$, the functor $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to$ **Set** sends any weighted colimit in $\mathcal{C}$ to the corresponding weighted limit in* **Set**.

(iii′) *The functors $\mathcal{C}(-, C) : \mathcal{C} \to$ **Set**$^{\mathrm{op}}$* jointly *reflect weighted colimits.*

(iv′) *If F has a right adjoint, then F preserves weighted colimits.*

Proof. All straightforward. ♦

Definition A.5.14. Let **U** be a pre-universe, let **Set** be the category of **U**-sets, and let $\mathcal{D}$ be a locally **U**-small category. Given a functor $F : \mathcal{C} \to \mathcal{D}$, the ***F*-nerve functor** $\mathrm{N}^F : \mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is defined by

$$\mathrm{N}^F(D)(C) = \mathcal{D}(FC, D)$$

i.e. $\mathrm{N}^F = F^* \text{\textit{ん}}_\bullet$, where $\text{\textit{ん}}_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is the usual Yoneda embedding.

Theorem A.5.15. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be locally* **U**-*small categories. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the following are equivalent:*

(i) *(H, α) is a pointwise right Kan extension of G along F.*

(ii) *For each object d in $\mathcal{D}$, the weighted limit $\left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$ exists in $\mathcal{E}$, and there are isomorphisms*

$$Hd \cong \left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$$

natural in d, with $\alpha_c : HFc \to Gc$ corresponding to the element id_{Fc} of $\mathrm{N}^{F^{\mathrm{op}}}(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii) *(Assuming $\mathcal{C}$ is* **U**-*small.) For each object d in $\mathcal{D}$, if $P^d : (d \downarrow F) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(d \downarrow F)$ to c, and $\varphi : \Delta d \Rightarrow FP^d$ is the tautological cone in $\mathcal{D}$, then the cone $\alpha P^d \bullet H\varphi : \Delta Hd \Rightarrow GP^d$ is limiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(d' \downarrow F) \to (d \downarrow F)$ sending (c', f') to $(c', f' \circ g)$. In particular, $\alpha_c : HFc \to Gc$ must be (equal to) the component of the limiting cone $\Delta Fc \Rightarrow GP^d$ at the object $\left(c, \mathrm{id}_{Fc}\right)$ of $(Fc \downarrow F)$.*

In particular, if $\mathcal{C}$ is a **U**-*small category and $\mathcal{E}$ is* **U**-*complete, then the right Kan extension of G along F exists and is pointwise.*

Dually, the following are equivalent:

(i′) *(H, α) is a pointwise left Kan extension of G along F.*

(ii′) *For each object d in $\mathcal{D}$, the weighted colimit $\mathrm{N}^F(d) \star_{\mathcal{C}} G$ exists in $\mathcal{E}$, and there are isomorphisms*

$$Hd \cong \mathrm{N}^F(d) \star_{\mathcal{C}} G$$

natural in d, with $\alpha_c : Gc \to HFc$ corresponding to the element id_{Fc} of $\mathrm{N}^F(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii′) *(Assuming $\mathcal{C}$ is $\mathbf{U}$-small.) For each object d in $\mathcal{D}$, if $P_d : (F \downarrow d) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(F \downarrow d)$ to c, and $\varphi : FP_d \Rightarrow \Delta d$ is the tautological cocone in $\mathcal{D}$, then the cocone $H\varphi \bullet \alpha P_d : GP_d \Rightarrow \Delta Hd$ is colimiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(F \downarrow d) \to (F \downarrow d')$ sending (c, f) to $(c, g \circ f)$. In particular, $\alpha_c : Gc \to HFc$ must be (equal to) the component of the colimiting cocone $GP_d \Rightarrow \Delta Fc$ at the object (c, id_{Fc}) of $(F \downarrow Fc)$.*

In particular, if $\mathcal{C}$ is a $\mathbf{U}$-small category and $\mathcal{E}$ is $\mathbf{U}$-cocomplete, then the left Kan extension of G along F exists and is pointwise.

Proof. (i) ⇔ (ii). This is just a matter of unwinding the definitions.

(i) ⇔ (iii). Corollary A.5.9 implies that the construction in (iii) does indeed define a right Kan extension in the special case $\mathcal{E} = \mathbf{Set}$, so we deduce that statements (i) and (iii) are equivalent by applying the Yoneda lemma; see also [CWM, Ch. X, §§3 and 5]. □

Remark A.5.16. It is possible to extract an elementary characterisation of pointwise Kan extensions from the results above, thereby showing that the property of being pointwise does not depend on the choice of universe $\mathbf{U}$.

Corollary A.5.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $\mathcal{C}$ is $\mathbf{U}$-small and $\mathcal{D}$ is locally $\mathbf{U}$-small, then the functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ has both a left adjoint Lan_F and a right adjoint Ran_F.* ■

Corollary A.5.18. *If (H, α) is a pointwise right Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$, and $R : \mathcal{E} \to \mathcal{F}$ is a functor, then $(RH, R\alpha)$ is a pointwise right Kan extension of RG along F, provided either:*

(i) *R preserves all weighted limits, or*

(ii) *R preserves limits for $\mathbf{U}$-small diagrams and $\mathcal{C}$ is $\mathbf{U}$-small.*

If (H, α) is a pointwise left Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$, and $L : \mathcal{E} \to \mathcal{F}$ is a functor, then $(LH, L\alpha)$ is a pointwise left Kan extension of LG along F, provided either:

(i′) *L preserves all weighted colimits, or*

(ii′) *L preserves colimits for $\mathbf{U}$-small diagrams and $\mathcal{C}$ is $\mathbf{U}$-small.* ■

Corollary A.5.19. *If (H, α) is a pointwise right (resp. left) Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along a fully faithful functor $F : \mathcal{C} \to \mathcal{D}$, then $\alpha : HF \Rightarrow G$ (resp. $\alpha : G \Rightarrow HF$) is a natural isomorphism.*

Proof. If F is fully faithful, then the comma category $(Fc \downarrow F)$ (resp. $(F \downarrow Fc)$) has an initial (resp. terminal) object, namely $\left(c, \mathrm{id}_{Fc}\right)$, so the component $\alpha_c : HFc \to Gc$ (resp. $\alpha_c : Gc \to HFc$) must be an isomorphism. ■

Theorem A.5.20. *Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ be functors, and let $i : \mathcal{C} \to \mathcal{C}^+$ and $j : \mathcal{D} \to \mathcal{D}^+$ be fully faithful functors. Consider the following (not necessarily commutative) diagram:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{D} \arrow[r, "j"] & \mathcal{D}^+ \\
\mathcal{C} \arrow[ur, "H"'] \arrow[d, "i"'] & & \\
\mathcal{C}^+ \arrow[uurr, "H^+"'] & &
\end{tikzcd}$$

(i) *If H^+ is a pointwise right Kan extension of jG along iF, and $H^+ i \cong jH$, then H is a pointwise right Kan extension of G along F.*

(ii) *Suppose jH is a pointwise right Kan extension of jG along F. If H^+ is a pointwise right Kan extension of jH along i, then the counit $H^+ i \Rightarrow jH$ is a natural isomorphism, and H^+ is also a pointwise right Kan extension of jG along iF; conversely, if H^+ is a pointwise right Kan extension of jG along iF, then it is also a pointwise right Kan extension of jH along i.*

(iii) *If* $\mathbf{U}$ *is a pre-universe such that* $\mathcal{A}$ *is* $\mathbf{U}$*-small and* j *preserves limits for all* $\mathbf{U}$*-small diagrams, and* H *is a pointwise right Kan extension of* G *along* F*, then a pointwise right Kan extension of* jG *along* iF *can be computed as a pointwise right Kan extension of* jH *along* i *(if either one exists).*

Dually:

(i′) *If* H^+ *is a pointwise left Kan extension of* jG *along* iF*, and* $H^+ i \cong jH$*, then* H *is a pointwise left Kan extension of* G *along* F*.*

(ii′) *Suppose* jH *is a pointwise left Kan extension of* jG *along* F*. If* H^+ *is a pointwise right Kan extension of* jH *along* i*, then the unit* $jH \Rightarrow H^+ i$ *is a natural isomorphism, and* H^+ *is also a pointwise left Kan extension of* jG *along* iF*; conversely, if* H^+ *is a pointwise left Kan extension of* jG *along* iF*, then it is also a pointwise left Kan extension of* jH *along* i*.*

(iii′) *If* $\mathbf{U}$ *is a pre-universe such that* $\mathcal{A}$ *is* $\mathbf{U}$*-small and* j *preserves colimits for all* $\mathbf{U}$*-small diagrams, and* H *is a pointwise left Kan extension of* G *along* F*, then a pointwise left Kan extension of* jG *along* iF *can be computed as a pointwise left Kan extension of* jH *along* i *(if either one exists).*

Proof. (i). Theorem A.5.15 gives an explicit description of $H^+ : \mathcal{C}^+ \to \mathcal{D}^+$ as a weighted limit:

$$H^+(C') \cong \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}$$

Since i is fully faithful, the weights $\mathcal{C}(C, F)$ and $\mathcal{C}^+(iC, iF)$ are naturally isomorphic, hence,

$$jH(C) \cong H^+(iC) \cong \{\mathcal{C}^+(iC, iF), jG\}^{\mathcal{A}} \cong \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}$$

but, since j is fully faithful, j reflects *all* weighted limits, therefore H must be a pointwise right Kan extension of G along F.

(ii). Let $\mathbf{U}^+$ be a pre-universe such that $\mathcal{A}$ and $\mathcal{C}$ are $\mathbf{U}^+$-small categories and $\mathcal{D}, \mathcal{C}^+, \mathcal{D}^+$ are locally $\mathbf{U}^+$-small categories, and let $\mathbf{Set}^+$ be the category of $\mathbf{U}^+$-sets. Using the interchange law (theorem A.6.17) and propositions A.6.11 and A.6.18, we obtain the following natural bijections:

$$\begin{aligned}\mathcal{D}^+(D', H^+(C')) &\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', i), jH\}^{\mathcal{C}}\big)\\ &\cong \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jHC))\end{aligned}$$

$$
\begin{aligned}
&\cong \int_{C:\mathcal{C}} \mathbf{Set}^+\big(\mathcal{C}^+(C', iC), \mathcal{D}^+\big(D', \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}\big)\big) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathbf{Set}^+(\mathcal{C}(C, FA), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{A:\mathcal{A}} \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iFA), \mathcal{D}^+(D', jGA)) \\
&\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}\big)
\end{aligned}
$$

Thus, H^+ is a pointwise right Kan extension of jG along iF if and only if H^+ is a pointwise right Kan extension of jH along i. The fact that the counit $H^+ i \Rightarrow jH$ is a natural isomorphism is just corollary A.5.19.

(iii). Apply corollary A.5.18 to claim (ii). ■

Proposition A.5.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be any two functors. The following are equivalent:*

(i) *$F \dashv G$, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

(ii) *(F, ε) is an absolute right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G.*

(iii) *(F, ε) is a right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G that is preserved by F.*

(iv) *(G, η) is an absolute left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F.*

(v) *(G, η) is a left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F that is preserved by G.*

Proof. See [CWM, Ch. X, §7]. □

Proposition A.5.22.

- *Left adjoints preserve all left Kan extensions.*
- *Right adjoints preserve all right Kan extensions.*

Proof. See Theorem 1 in [CWM, Ch. X, §5]. □

Definition A.5.23. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. A **dense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the F-nerve functor $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. A **dense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a dense functor.

Dually, a **codense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the opposite functor $F^{\mathrm{op}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is dense, and a **codense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a codense functor.

Example A.5.24. The Yoneda lemma implies $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a dense and codense functor.

One may extract an elementary definition for '(co)dense functor' from the following proposition.

Proposition A.5.25. *With notation as in definition A.5.23, the following are equivalent:*

(i) *$F : \mathcal{B} \to \mathcal{C}$ is a dense functor.*

(ii) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C, C') \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]\big(\mathrm{N}^F(C), \mathcal{C}(F, C')\big)$$

induced by $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted colimit $\mathrm{N}^F(C) \star_{\mathcal{B}} F$ in $\mathcal{C}$.

(iii) *For each object C in $\mathcal{C}$, the tautological cocone to C induced by F is a colimiting cocone.*

(iv) *$\big(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F\big)$ is a pointwise left Kan extension of F along F.*

Dually, the following are equivalent:

(i′) *$F : \mathcal{B} \to \mathcal{C}$ is a codense functor.*

(ii′) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C', C) \to [\mathcal{B}, \mathbf{Set}]\big(\mathrm{N}^{F^{\mathrm{op}}}(C), \mathcal{C}(C', F)\big)$$

induced by $\mathrm{N}^{F^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to [\mathcal{B}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted limit $\big\{\mathrm{N}^{F^{\mathrm{op}}}(C), F\big\}^{\mathcal{B}}$ in $\mathcal{C}$.

(iii′) *For each object C in $\mathcal{C}$, the tautological cone from C induced by F is a limiting cone.*

(iv′) $(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F)$ *is a pointwise right Kan extension of F along F.*

Proof. (i) ⇔ (ii). The indicated maps are bijections for all C and C' if and only if N^F is fully faithful, by definition.

(ii) ⇔ (iii) ⇔ (iv). This is an application of theorem A.5.15. ■

Definition A.5.26. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. A **densely-defined partial left adjoint** for G is a triple (F, i, η), where $F : \mathcal{B} \to \mathcal{D}$ is a functor, $i : \mathcal{B} \to \mathcal{C}$ is a dense functor, and $\eta : i \Rightarrow GF$ is a natural transformation such that the maps

$$\begin{aligned} \mathcal{D}(FB, D) &\to \mathcal{C}(iB, GD) \\ g &\mapsto Gg \circ \eta_B \end{aligned}$$

are bijections that are natural in B and D.

Dually, given a functor $F : \mathcal{C} \to \mathcal{D}$, a **codensely-defined partial right adjoint** for F is a triple (G, j, ε), where $G : \mathcal{B} \to \mathcal{C}$ is a functor, $j : \mathcal{B} \to \mathcal{C}$ is a codense functor, and $\varepsilon : FG \Rightarrow j$ is a natural transformation such that the maps

$$\begin{aligned} \mathcal{C}(C, GB) &\to \mathcal{D}(FC, jB) \\ f &\mapsto \varepsilon_B \circ Ff \end{aligned}$$

are bijections that are natural in B and C.

Example A.5.27. The Yoneda embedding $\mathcal{k}_\bullet : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ has a densely-defined partial left adjoint, namely $(\mathrm{id}_{\mathcal{B}}, \mathcal{k}_\bullet, \mathrm{id}_{\mathcal{k}_\bullet})$.

Remark A.5.28. $(F, \mathrm{id}_{\mathcal{C}}, \eta)$ is a densely-defined partial left adjoint for G if and only if F is a left adjoint for G in the usual sense, with η being the adjunction unit.

Proposition A.5.29. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ and $\mathcal{D}$ be locally $\mathbf{U}$-small categories. Given functors $G : \mathcal{D} \to \mathcal{C}$, $F : \mathcal{B} \to \mathcal{D}$, and $i : \mathcal{B} \to \mathcal{C}$, the following are equivalent:*

(i) (F, i, η) *is a densely-defined partial left adjoint for G.*

(ii) *The functor $i : \mathcal{B} \to \mathcal{C}$ is dense, and there exists a diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\text{\textyen}_\bullet"] \arrow[d, "G"'] \arrow[dr, phantom, "\Swarrow_\alpha"] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\ \mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

where α factors through $\eta^* : \mathrm{N}^{GF} \Rightarrow \mathrm{N}^i$ and is a natural isomorphism.

(iii) *The functor $i : \mathcal{B} \to \mathcal{C}$ is dense, and the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\text{\textyen}_\bullet"] \arrow[d, "G"'] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\ \mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

commutes up to natural isomorphism.

Dually, given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{B} \to \mathcal{C}$, and $j : \mathcal{B} \to \mathcal{D}$, the following are equivalent:

(i′) *(G, j, ε) is a codensely-defined partial right adjoint for F.*

(ii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and there exists a diagram*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\text{\textyen}^\bullet"] \arrow[d, "F^{\mathrm{op}}"'] \arrow[dr, phantom, "\Swarrow_\beta"] & {[\mathcal{C}, \mathbf{Set}]} \arrow[d, "G^*"] \\ \mathcal{D}^{\mathrm{op}} \arrow[r, "\mathrm{N}^{j^{\mathrm{op}}}"'] & {[\mathcal{B}, \mathbf{Set}]} \end{tikzcd}$$

where β factors through $(\varepsilon^{\mathrm{op}})^* : \mathrm{N}^{F^{\mathrm{op}}G^{\mathrm{op}}} \Rightarrow \mathrm{N}^{j^{\mathrm{op}}}$ and is a natural isomorphism.

(iii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and the diagram*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\text{\textyen}^\bullet"] \arrow[d, "F^{\mathrm{op}}"'] & {[\mathcal{C}, \mathbf{Set}]} \arrow[d, "G^*"] \\ \mathcal{D}^{\mathrm{op}} \arrow[r, "\mathrm{N}^{j^{\mathrm{op}}}"'] & {[\mathcal{B}, \mathbf{Set}]} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. (i) ⇒ (ii). This immediately follows from the definition.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). The displayed diagram commutes up to natural isomorphism precisely when there are bijections

$$\alpha_{B,D} : \mathcal{D}(FB, D) \to \mathcal{C}(iB, GD)$$

that are natural in both B and D. Taking $D = FB$, let $\eta_B : iB \to GFB$ be the morphism corresponding to $\mathrm{id}_{FB} : FB \to FB$. Applying the Yoneda lemma, we see that the natural bijection $\alpha_{B,D}$ must be the map $g \mapsto Gg \circ \eta_B$. ■

Corollary A.5.30. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a densely-defined partial left adjoint, then G preserves:*

(i) *limits for all diagrams in $\mathcal{D}$,*

(ii) *weighted limits, and*

(iii) *pointwise right Kan extensions.*

Dually, if a functor $F : \mathcal{C} \to \mathcal{D}$ has a codensely-defined partial right adjoint, then F preserves:

(i′) *colimits for all diagrams in $\mathcal{C}$,*

(ii′) *weighted colimits, and*

(iii′) *pointwise left Kan extensions.*

Proof. Choose a universe $\mathbf{U}$ such that the domain of $i : \mathcal{B} \to \mathcal{C}$ is $\mathbf{U}$-small and both $\mathcal{C}$ and $\mathcal{D}$ are locally $\mathbf{U}$-small, and consider the following diagram:

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "h_\bullet"] \arrow[d, "G"'] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\
\mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]}
\end{tikzcd}$$

Since i is dense, the i-nerve functor $\mathrm{N}^i : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. Corollary A.5.17 implies $(F^{\mathrm{op}})^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is a right adjoint, and the Yoneda embedding $h_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ preserves all limits and weighted limits (see proposition A.5.13), so we use the fact that N^i reflects limits and weighted limits to conclude that G preserves them. We then apply corollary A.5.18. ■

Definition A.5.31. A **cofinal functor** (resp. **coinitial functor**) is a functor $F : \mathcal{C} \to \mathcal{D}$ such that, for each object D in $\mathcal{D}$, the comma category $(D \downarrow F)$ (resp. $(F \downarrow D)$) is connected.

Theorem A.5.32. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between* $\mathbf{U}$*-small categories. The following are equivalent:*

(i) $F : \mathcal{C} \to \mathcal{D}$ *is a coinitial functor.*

(ii) *The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[d, "F^*"] \\ \mathbf{Set} \arrow[r, "\Delta"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iii) *The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & \mathbf{Set} \arrow[d, "\Delta"] \\ {[\mathcal{D}, \mathbf{Set}]} \arrow[r, "F^*"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iv) *For all locally* $\mathbf{U}$*-small categories* $\mathcal{E}$ *and all diagrams* $G : \mathcal{D} \to \mathcal{E}$, $\varprojlim_{\mathcal{C}} GF$ *exists if and only if* $\varprojlim_{\mathcal{D}} G$ *exists, in which case the canonical comparison morphism* $\varprojlim_{\mathcal{D}} G \to \varprojlim_{\mathcal{C}} GF$ *is an isomorphism.*

Dually, the following are equivalent:

(i′) $F : \mathcal{C} \to \mathcal{D}$ *is a cofinal functor.*

(ii′) *The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[d, "F^*"] \\ \mathbf{Set} \arrow[r, "\Delta"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iii′) *The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & \mathbf{Set} \arrow[d, "\Delta"] \\ {[\mathcal{D}, \mathbf{Set}]} \arrow[r, "F^*"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iv′) *For all locally* $\mathbf{U}$*-small categories* $\mathcal{E}$ *and all diagrams* $G : \mathcal{D} \to \mathcal{E}$, $\varinjlim_{\mathcal{C}} GF$ *exists if and only if* $\varinjlim_{\mathcal{D}} G$ *exists, in which case the canonical comparison morphism* $\varinjlim_{\mathcal{C}} GF \to \varinjlim_{\mathcal{D}} G$ *is an isomorphism.*

Proof. (i) ⇔ (ii). Using the colimit formula for $\mathrm{Lan}_F : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ indicated in theorem A.5.15, it is clear that the comma categories $(F \downarrow D)$ is connected if and only if the left Beck–Chevalley transformation $\Delta \Rightarrow \mathrm{Ran}_F(\Delta -)$ is a natural isomorphism.

(ii) ⇔ (iii). Apply proposition A.1.12.

(iii) ⇒ (iv). We have the following natural bijections:

$$\begin{aligned} [\mathcal{C}, \mathcal{E}](\Delta E, GF) &\cong \varprojlim_{\mathcal{C}} \mathcal{E}(E, GF) \\ &\cong \varprojlim_{\mathcal{D}} \mathcal{E}(E, G) \\ &\cong [\mathcal{D}, \mathcal{E}](\Delta E, G) \end{aligned}$$

Thus, there is a natural bijection between cones from E to GF and cones from E to G; this implies that limits for GF exist in $\mathcal{E}$ if and only if limits for G exist in $\mathcal{E}$ and that they are canonically isomorphic.

(iv) ⇒ (iii). Obvious. ■

Definition A.5.33. A **sifted category** is a category $\mathcal{J}$ with the following property:

- For every finite set of objects in $\mathcal{J}$, say $j_1, \ldots, j_n$, there exist an object k and a cocone $j_\bullet \to k$ in $\mathcal{J}$.

Remark A.5.34. Every filtered category is sifted.

REMARK A.5.35. If $\mathcal{J}$ is a category with an object k such that, for every object j in $\mathcal{J}$, there is a morphism $j \to k$ in $\mathcal{J}$, then $\mathcal{J}$ is a sifted category.

Theorem A.5.36. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathcal{J}$ *be a* $\mathbf{U}$*-small category. The following are equivalent:*

(i) $\mathcal{J}$ *is a sifted category.*

(ii) $\mathcal{J}$ *is (inhabited and) connected, and the diagonal functor* $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ *is cofinal.*

(iii) *The functor* $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ *preserves finite products.*

Proof. (i) ⇒ (ii). If $\mathcal{J}$ is sifted, then there is an object in $\mathcal{J}$. Let (j_0, j_1) be a pair of objects in $\mathcal{J}$. If $\mathcal{J}$ is sifted, then there exist an object k and morphisms $j_0 \to k$ and $j_1 \to k$, so the comma category $((j_0, j_1) \downarrow \Delta)$ is inhabited and $\mathcal{J}$ is connected; repeating this argument, we find that $((j_0, j_1) \downarrow \Delta)$ itself is connected. Thus $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ is indeed a cofinal functor.

(ii) ⇒ (iii). If $\mathcal{J}$ is connected, then $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ preserves terminal objects. Let $X, Y : \mathcal{J} \to \mathbf{Set}$ be diagrams and suppose $\mathcal{J}$ is sifted. Since $(-) \times (-) : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$ preserves colimits in each variable (because $\mathbf{Set}$ is cartesian closed), the canonical map

$$\varinjlim_{\mathcal{J} \times \mathcal{J}} X \boxtimes Y \to \left(\varinjlim_{\mathcal{J}} X \right) \times \left(\varinjlim_{\mathcal{J}} Y \right)$$

is a bijection, where $X \boxtimes Y : \mathcal{J} \times \mathcal{J} \to \mathbf{Set}$ is the diagram defined by $(X \boxtimes Y)(j_0, j_1) = X j_0 \times Y j_1$; and since $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ is cofinal, the canonical map

$$\varinjlim_{\mathcal{J}} X \times Y \to \varinjlim_{\mathcal{J} \times \mathcal{J}} X \boxtimes Y$$

is also a bijection. We then deduce (by induction) that $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite products.

(iii) ⇒ (i). Let $j_0, \ldots, j_n$ be objects in $\mathcal{J}$. For any object j in $\mathcal{J}$, we have $\varinjlim_{\mathcal{J}} \mathcal{J}(j, -) \cong 1$; thus, if $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ preserves finite products, then

$$\varinjlim_{\mathcal{J}} \mathcal{J}(j_0, -) \times \cdots \times \mathcal{J}(j_n, -) \cong 1$$

and in particular, there must exist an object k and a cocone $j_\bullet \to k$ in $\mathcal{J}$. ■

A.6 Ends and coends

Prerequisites. §§0.1, A.5

In this section we use the explicit universe convention.

Definition A.6.1. Let $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A **dinatural transformation** $\alpha : F \overset{\diamond}{\to} G$ is a family $\big(\alpha_C : F(C,C) \to G(C,C) \,\big|\, C \in \operatorname{ob} \mathcal{C}\big)$ such that the diagram

$$\begin{tikzcd}
 & F(C,C) \arrow[r, "\alpha_C"] & G(C,C) \arrow[dr, "G(f,\mathrm{id}_C)"] & \\
F(C,C') \arrow[ur, "F(\mathrm{id}_C,f)"] \arrow[dr, "F(f,\mathrm{id}_{C'})"'] & & & G(C',C) \\
 & F(C',C') \arrow[r, "\alpha_{C'}"'] & G(C',C') \arrow[ur, "G(\mathrm{id}_{C'},f)"'] &
\end{tikzcd}$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$.

Example A.6.2. Let $\mathbf{U}$ be a pre-universe, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets. Consider the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ that sends a pair of objects in $\mathcal{C}$ to their hom-set. For each natural number n, we have an dinatural transformation $\mathrm{Hom}_{\mathcal{C}} \overset{\diamond}{\to} \mathrm{Hom}_{\mathcal{C}}$ defined by $e \mapsto e^n$, where e^n denotes the n-fold iterate of the endomorphism e.

Definition A.6.3. A **wedge** from an object D in $\mathcal{D}$ to a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a dinatural transformation $\Delta D \overset{\diamond}{\to} G$, where $\Delta D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is the constant functor with value D; dually, a **cowedge** from a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to an object D in $\mathcal{D}$ is a dinatural transformation $F \overset{\diamond}{\to} \Delta D$.

Definition A.6.4. An **end** for a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a wedge $\lambda : \Delta E \overset{\diamond}{\to} G$ with the following universal property:

- For each wedge $\varphi : \Delta D \overset{\diamond}{\to} G$, there is a unique morphism $f : D \to E$ in $\mathcal{D}$ such that $\varphi_C = \lambda_C \circ f$ for all objects C in $\mathcal{C}$.

We define the following formula to mean that E is an end for G:

$$E = \int_{C:\mathcal{C}} G(C,C)$$

Dually, a **coend** for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a cowedge $\lambda : F \overset{\diamond}{\to} \Delta E$ with the following universal property:

- For each cowedge $\varphi : F \overset{\diamond}{\to} \Delta D$, there is a unique morphism $f : E \to D$ in $\mathcal{D}$ such that $\varphi_C = f \circ \lambda_C$ for all objects C in $\mathcal{C}$.

We define the following formula to mean that E is a coend for F:

$$E = \int^{C:\mathcal{C}} F(C, C)$$

Remark A.6.5. Let $\mathbf{U}$ be a pre-universe, let $\mathbb{D}$ be a $\mathbf{U}$-small category, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Then, for all functors $F, G : \mathbb{D} \to \mathcal{C}$, we have a bijection

$$[\mathbb{D}, \mathcal{C}](F, G) \cong \int_{d:\mathbb{D}} \mathcal{C}(Fd, Gd)$$

and this is natural in both F and G. (The size restriction ensures that the LHS is a $\mathbf{U}$-set.) See also lemma A.5.12.

Proposition A.6.6. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, let* $\mathbb{C}$ *be a* $\mathbf{U}$*-small category, let* $F : \mathbb{C} \to \mathbf{Set}$ *be a diagram, and let* $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ *be a weight. Then the coend (in* $\mathbf{Set}$*)*

$$\int_{c:\mathbb{C}} G(c) \times F(c)$$

can be identified with the set of connected components of the following category $\mathbf{G}(G, \mathbb{C}, F)$*:*

- *The objects are triples* (y, c, x) *where* c *is an object in* $\mathbb{C}$*,* x *is an element of* $F(c)$*, and* y *is an element of* $G(c)$*.*

- *The morphisms* $(y', c', x') \to (y, c, x)$ *are morphisms* $f : c' \to c$ *in* $\mathbb{C}$ *such that* $F(f)(x') = x$ *and* $G(f)(y) = y'$*.*

- *Identities and composition are inherited from* $\mathbb{C}$*.*

Proof. Observe that a morphism $f : c' \to c$ in $\mathbb{C}$ defines a morphism $(y', c', x') \to (y, c, x)$ in $\mathbf{G}(G, \mathbb{C}, F)$ if and only if both of the following conditions hold:

- $G(f) \times \mathrm{id}_{F(c')} : G(c) \times F(c') \to G(c') \times F(c')$ sends the element (y, x') to (y', x').

- $\mathrm{id}_{G(c)} \times F(f) : G(c) \times F(c') \to G(c) \times F(c)$ sends the element (y, x') to (y, x).

In particular, (y', c', x') and (y, c, x) are in the same connected component of $\mathbf{G}(G, \mathbb{C}, F)$ if and only if (y', x') and (y, x) are mapped to the same element of the coend $\int_{c:\mathbb{C}} G(c) \times F(c)$. Thus, the set of connected components of $\mathbf{G}(G, \mathbb{C}, F)$ can be identified with $\int_{c:\mathbb{C}} G(c) \times F(c)$, as claimed. ■

Definition A.6.7. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Cat}$ be the category of $\mathbf{U}$-small categories, and let $\mathbf{\S}$ be the (non-full) subcategory of $\mathbf{Cat}$ consisting of the two embeddings $\mathbb{1} \to \mathbb{2}$. Given a $\mathbf{U}$-small category $\mathbb{D}$, the **Mac Lane subdivision category** $\mathbb{D}^{\S}$ is the comma category $(\mathbf{\S} \downarrow \mathbb{D})$, where we regard $\mathbb{D}$ as an object in $\mathbf{Cat}$.

Remark A.6.8. More explicitly, $\mathbb{D}^{\S}$ is the following category:

- The objects are either objects in $\mathbb{D}$ or morphisms in $\mathbb{D}$.
- The non-identity morphisms are of the form $X \to f$ or $Y \to f$, where $f : X \to Y$ is a morphism in $\mathbb{D}$.
- The only composable pairs of morphisms are trivial.

It is clear from this description that $\mathbb{D}^{\S}$ is (isomorphic to) a poset regarded as a category. Note also that, for any regular cardinal κ, the category $\mathbb{D}$ is κ-small if and only if $\mathbb{D}^{\S}$ is κ-small.

Proposition A.6.9. *Let $\mathbb{D}$ be a category and let $\mathbb{D}^{\S}$ be the Mac Lane subdivision category. Then there is a natural coinitial functor $\pi_{\mathrm{R}} : \mathbb{D}^{\S} \to \mathbb{D}$.*

Proof. We define the functor $\pi_{\mathrm{R}} : \mathbb{D}^{\S} \to \mathbb{D}$ as follows: given an object X in $\mathbb{D}$, we set $\pi_{\mathrm{R}} X = X$, and given a morphism $f : X \to Y$ in $\mathbb{D}$, we set $\pi_{\mathrm{R}} f = Y$, $\pi_{\mathrm{R}}(X \to f) = f$, and $\pi_{\mathrm{R}}(Y \to f) = \mathrm{id}_Y$. This functor is clearly natural in $\mathbb{D}$. It remains to be shown that $\pi_{\mathrm{R}} : \mathbb{D}^{\S} \to \mathbb{D}$ is a cofinal functor. Let Y be an object in $\mathbb{D}$ and consider the comma category $\left(\pi_{\mathrm{R}} \downarrow Y\right)$. It is not hard to see that $\left(\pi_{\mathrm{R}} \downarrow Y\right)$ is isomorphic to the Mac Lane subdivision category $\left(\mathbb{D}_{/Y}\right)^{\S}$; but $\mathbb{D}_{/Y}$ has a terminal object (so is a connected category *a fortiori*), therefore $\left(\mathbb{D}_{/Y}\right)^{\S}$ must be a connected category, as required. ■

Proposition A.6.10. *Let $\mathbf{U}$ be a pre-universe and let $\mathbb{D}$ be a $\mathbf{U}$-small category. If $\mathcal{C}$ is a $\mathbf{U}$-complete category, then $\mathcal{C}$ has ends for all functors $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$. Dually, if $\mathcal{C}$ is a $\mathbf{U}$-cocomplete category, then $\mathcal{C}$ has coends for all functors $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$.*

Proof. It is clear from the definition that an end is a special kind of limit, and a coend is a special kind of colimit. To make this precise, one can use the Mac Lane subdivision category $\mathcal{C}^{\S}$: see [CWM, Ch. IX, § 5]. □

Proposition A.6.11. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $F : \mathcal{C} \to \mathcal{D}$ *be any functor where* $\mathcal{C}$ *and* $\mathcal{D}$ *are locally* $\mathbf{U}$*-small categories.*

(i) *For any functor* $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$*, if the ends* $\int_{\mathcal{J}} A$ *and* $\int_{\mathcal{J}} FA$ *both exist, with* λ *being the universal wedge in* $\mathcal{C}$*, then there is a canonical comparison morphism*

$$F \int_{\mathcal{J}} A \to \int_{\mathcal{J}} FA$$

induced by the wedge $F\lambda$*.*

(ii) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ *preserves* all *ends.*

(iii) *The functors* $\mathcal{C}(C, -)$ jointly *reflect ends.*

(iv) *If* F *has a left adjoint, then* F *preserves ends.*

Dually:

(i′) *For any functor* $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$*, if the coends* $\int^{\mathcal{J}} A$ *and* $\int^{\mathcal{J}} FA$ *both exist, with* λ *being the universal cowedge in* $\mathcal{C}$*, then there is a canonical comparison morphism*

$$\int^{\mathcal{J}} FA \to F \int^{\mathcal{J}} A$$

induced by the cowedge $F\lambda$*.*

(ii′) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}$ *sends any coend in* $\mathcal{C}$ *to the corresponding end in* $\mathbf{Set}$*.*

(iii′) *The functors* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ jointly *reflect coends.*

(iv′) *If* F *has a right adjoint, then* F *preserves coends.*

Proof. All straightforward. ◆

Definition A.6.12. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathbb{1}$ be the trivial category with $*$ as its only object. A **tensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted colimit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a tensored $\mathbf{U}$-category, then we write $X \odot C$ for the weighted colimit $W \star_{\mathbb{1}} A$, where $X = W(*)$ and $C = A(*)$.

Dually, a **cotensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted limit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a cotensored $\mathbf{U}$-category, then we write $X \pitchfork C$ for the weighted limit $\{W, A\}^{\mathbb{1}}$, where $X = W(*)$ and $C = A(*)$.

Proposition A.6.13 (Tensor–hom–cotensor adjunction)**.** *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category.*

(i) *If* $\mathcal{C}$ *is a tensored* $\mathbf{U}$*-category, then the assignment* $(X, C) \mapsto X \odot C$ *can be extended to a functor* $\mathbf{Set} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object* C*, we have the following adjunction:*

$$- \odot C \dashv \mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$$

(ii) *If* $\mathcal{C}$ *is a cotensored* $\mathbf{U}$*-category, then the assignment* $(X, C) \mapsto X \pitchfork C$ *can be extended to a functor* $\mathbf{Set}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object* C*, the functors* $- \pitchfork C : \mathbf{Set}^{\mathrm{op}} \to \mathcal{C}$ *and* $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *are contravariantly adjoint on the right.*

(iii) *If* $\mathcal{C}$ *is a tensored and cotensored* $\mathbf{U}$*-category, then for each set* X*, we have the following adjunction:*

$$X \odot - \dashv X \pitchfork - : \mathcal{C} \to \mathcal{C}$$

Proof. Claims (i) and (ii) are formally dual and are straightforward applications of the parametrised adjunction theorem.[8] For claim (iii), simply observe that we have bijections

$$\mathcal{C}(X \odot A, B) \cong \mathbf{Set}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, X \pitchfork B)$$

and these are natural in A, B, and X. ■

[8] See Theorem 3 in [CWM, Ch. IV, §7].

Theorem A.6.14. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a* $\mathbf{U}$*-complete category.*

(ii) *$\mathcal{C}$ is a cotensored* $\mathbf{U}$*-category and, for all* $\mathbf{U}$*-small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, an end for* A *exists in* $\mathcal{C}$*.*

(iii) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted limit for* A*, provided* $\mathbb{D}$ *is a* $\mathbf{U}$*-small category.*

Dually, the following are equivalent:

(i′) *$\mathcal{C}$ is a* $\mathbf{U}$*-cocomplete category.*

(ii′) *$\mathcal{C}$ is a tensored* $\mathbf{U}$*-category and, for all* $\mathbf{U}$*-small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, a coend for* A *exists in* $\mathcal{C}$*.*

(iii′) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted colimit for* A*, provided* $\mathbb{D}$ *is a* $\mathbf{U}$*-small category.*

Proof. (i) ⇒ (ii). It is clear that $X \pitchfork C$ is nothing more than an X-fold product of copies of C, so $\mathcal{C}$ is certainly $\mathbf{U}$-cotensored if it is $\mathbf{U}$-complete, and proposition A.6.10 says $\mathcal{C}$ also has the required ends in that case.

(ii) ⇒ (iii). We have the following natural bijections:

$$\begin{aligned}
\mathcal{C}\big(C, \{W, A\}^{\mathbb{D}}\big) &\cong [\mathbb{D}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\
&\cong \int_{d:\mathbb{D}} \mathbf{Set}(W d, \mathcal{C}(C, A d)) \\
&\cong \int_{d:\mathbb{D}} \mathcal{C}(C, W d \pitchfork A d) \\
&\cong \mathcal{C}\left(C, \int_{d:\mathbb{D}} W d \pitchfork A d\right)
\end{aligned}$$

Thus, using the Yoneda lemma and *assuming $\mathcal{C}$ is a cotensored* $\mathbf{U}$*-category*, the weighted limit $\{W, A\}^{\mathbb{D}}$ exists if and only if the end $\int_{d:\mathbb{D}} W d \pitchfork A d$ exists.

(iii) ⇒ (i). Ordinary limits are a special case of weighted limits, as remarked in A.5.11. ■

Proposition A.6.15. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category, and let* $\mathcal{J}$ *be any category. If* $\mathcal{C}$ *is a tensored* $\mathbf{U}$*-category and has weighted limits for all weights* $W : \mathcal{J} \to \mathbf{Set}$ *and diagrams* $A : \mathcal{J} \to \mathcal{C}$*, then:*

(i) $(W, A) \mapsto \{W, A\}^{\mathcal{J}}$ *extends to a functor* $[\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

(ii) *For each diagram* $A : \mathcal{J} \to \mathcal{C}$*, the functors* $\{-, A\}^{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \to \mathcal{C}$ *and* $\mathcal{C}(-, A) : \mathcal{C}^{\mathrm{op}} \to [\mathcal{J}, \mathbf{Set}]$ *are contravariantly adjoint on the right.*

(iii) *For each weight* $W : \mathcal{J} \to \mathbf{Set}$*, we have the following adjunction:*

$$W \odot - \dashv \{W, -\}^{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$$

Here, $W \odot C : \mathcal{J} \to \mathcal{C}$ *is the diagram* $j \mapsto W j \odot C$.

Dually, if $\mathcal{C}$ *is a cotensored* $\mathbf{U}$*-category and has weighted colimits for all weights* $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ *and diagrams* $A : \mathcal{J} \to \mathcal{C}$*, then:*

(i′) $(W, A) \mapsto W \star_{\mathcal{J}} A$ *extends to a functor* $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}] \times \mathcal{C} \to \mathcal{C}$.

(ii′) *For each diagram* $A : \mathcal{J} \to \mathcal{C}$*, we have the following adjunction:*

$$- \star_{\mathcal{J}} A \dashv \mathcal{C}(A, -) : \mathcal{C} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$$

(iii′) *For each weight* $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$*, we have the following adjunction:*

$$W \star_{\mathcal{J}} - \dashv W \pitchfork - : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$$

Here, $W \pitchfork C : \mathcal{J} \to \mathcal{C}$ *is the diagram* $j \mapsto W j \pitchfork C$.

Proof. Claim (i) is straightforward, and for claims (ii) and (iii), observe that we have bijections

$$\begin{aligned} \mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) &\cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\ &\cong \int_{j:\mathcal{J}} \mathbf{Set}(W j, \mathcal{C}(C, A j)) \\ &\cong \int_{j:\mathcal{J}} \mathcal{C}(W j \odot C, A j) \\ &\cong [\mathcal{J}, \mathcal{C}](W \odot C, A) \end{aligned}$$

and these are natural in W, A, and C. ∎

Lemma A.6.16. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathbb{I}$ *and* $\mathbb{J}$ *be* $\mathbf{U}$*-small categories. For all functors* $A : \mathbb{I}^{\mathrm{op}} \times \mathbb{J}^{\mathrm{op}} \times \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$*:*

(i) *The assignment* $(i', i) \mapsto \int_{j:\mathbb{J}} A(i', j, i, j)$ *extends to a functor* $\mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathbf{Set}$.

(ii) *There is a unique morphism* θ *making the diagram below commute for all* i *and* j*,*

$$\begin{tikzcd}
\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') \arrow[r] \arrow[d, dashed, "\theta"'] & \int_{j':\mathbb{J}} A(i, j', i, j') \arrow[d] \\
\int_{(i',j'):\mathbb{I}\times\mathbb{J}} A(i', j', i', j') \arrow[r] & A(i, j, i, j)
\end{tikzcd}$$

where the unlabelled arrows are the components of the respective universal wedges, and θ *is moreover an isomorphism.*

(iii) *There is a unique morphism* σ *making the diagram below commute for all* i *and* j*,*

$$\begin{tikzcd}
\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') \arrow[r] \arrow[dd, dashed, "\sigma"'] & \int_{j':\mathbb{J}} A(i, j', i, j') \arrow[dr] & \\
& & A(i, j, i, j) \\
\int_{j':\mathbb{J}} \int_{i':\mathbb{I}} A(i', j', i', j') \arrow[r] & \int_{i':\mathbb{I}} A(i, j', i, j') \arrow[ur] &
\end{tikzcd}$$

where the unmarked arrows are the components of the respective universal wedges, and σ *is moreover an isomorphism.*

Proof. See [CWM, Ch. IX, § 8]. □

Theorem A.6.17 (Interchange law for ends and coends)**.** *Let* $\mathcal{C}$ *be any category and let* $A : \mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \times \mathcal{I} \times \mathcal{J} \to \mathbf{Set}$ *be any functor. If the end* $\int_{i:\mathcal{I}} A(i, j', i, j)$ *exists in* $\mathcal{C}$ *for all* j' *and* j *in* $\mathcal{J}$*, and the end* $\int_{j:\mathcal{J}} A(i', j, i, j)$ *exists in* $\mathcal{C}$ *for all* i' *and* i *in* $\mathcal{I}$*, then the following are equivalent:*

(i) *The end* $\int_{(i,j):\mathcal{I}\times\mathcal{J}} A(i, j, i, j)$ *exists in* $\mathcal{C}$.

(ii) *The iterated end* $\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j)$ *exists in* $\mathcal{C}$.

(iii) *The iterated end* $\int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$ *exists in* $\mathcal{C}$.

In this case, we have a canonical isomorphism in $\mathcal{C}$*:*

$$\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j) \cong \int_{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j) \cong \int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$$

Dually, if the coend $\int^{i:\mathcal{I}} A(i,j',i,j)$ *exists in* $\mathcal{C}$ *for all* j' *and* j *in* $\mathcal{J}$*, and the coend* $\int^{j:\mathcal{J}} A(i',j,i,j)$ *exists in* $\mathcal{C}$ *for all* i' *and* i *in* $\mathcal{I}$*, then the following are equivalent:*

(i′) *The coend* $\int^{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j)$ *exists in* $\mathcal{C}$.

(ii′) *The iterated coend* $\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j)$ *exists in* $\mathcal{C}$.

(iii′) *The iterated coend* $\int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$ *exists in* $\mathcal{C}$.

In this case, we have a canonical isomorphism in $\mathcal{C}$*:*

$$\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j) \cong \int^{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j) \cong \int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$$

Proof. Choose a pre-universe $\mathbf{U}$ such that $\mathcal{I}$ and $\mathcal{J}$ are $\mathbf{U}$-small categories and $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and use the Yoneda lemma to reduce the claims to the previous lemma. ■

Proposition A.6.18. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathcal{C}$ *and* $\mathcal{J}$ *be locally* $\mathbf{U}$*-small categories.*

(i) *For all* j *in* $\mathcal{J}$ *and all functors* $A : \mathcal{J} \to \mathcal{C}$*, the Yoneda bijection*

$$\mathcal{C}(C, Aj) \cong [\mathcal{J}, \mathbf{Set}]\big(\hbar^j, \mathcal{C}(C, A)\big)$$

exhibits Aj *as the weighted limit* $\left\{\hbar^j, A\right\}^{\mathcal{J}}$ *in* $\mathcal{C}$.

(ii) *If* $\mathcal{C}$ *is a cotensored* $\mathbf{U}$*-category, then the end* $\int_{j':\mathcal{J}} \mathcal{J}(j,j') \pitchfork Aj'$ *exists in* $\mathcal{C}$ *and can be canonically identified with* Aj.

(iii) *For all functors* $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$*, the weighted limit* $\left\{\mathrm{Hom}_{\mathcal{J}}, H\right\}^{\mathcal{J}^{\mathrm{op}}\times\mathcal{J}}$ *exists in* $\mathcal{C}$ *if and only if the end* $\int_{j:\mathcal{J}} H(j,j)$ *exists in* $\mathcal{C}$*, and there is a canonical identification of the two.*

Dually:

(i′) *For all j in $\mathcal{J}$ and all functors $A : \mathcal{J} \to \mathcal{C}$, the Yoneda bijection*

$$\mathcal{C}(Aj, C) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]\big(\text{\usefont{U}{min}{m}{n}\symbol{'110}}_j, \mathcal{C}(A, C)\big)$$

exhibits Aj as the weighted colimit $\text{\usefont{U}{min}{m}{n}\symbol{'110}}_j \star_{\mathcal{J}} A$ in $\mathcal{C}$.

(ii′) *If $\mathcal{C}$ is a tensored* **U***-category, then the coend $\int^{j':\mathcal{J}} \mathcal{J}(j', j) \odot Aj'$ exists in $\mathcal{C}$ and can be canonically identified with Aj.*

(iii′) *For all functors $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, the weighted colimit $\mathrm{Hom}_{\mathcal{J}^{\mathrm{op}}} \star_{\mathcal{J}^{\mathrm{op}} \times \mathcal{J}} H$ exists in $\mathcal{C}$ if and only if the coend $\int^{j:\mathcal{J}} H(j, j)$ exists in $\mathcal{C}$, and there is a canonical identification of the two.*

Proof. (i). This is an immediate consequence of the Yoneda lemma and the definition of weighted limit.

(ii). Use the identification constructed in the proof of theorem A.6.14.

(iii). For all objects C in $\mathcal{C}$, using claim (ii) and the interchange law for ends (theorem A.6.17), there are bijections

$$\begin{aligned}
[\mathcal{J}^{\mathrm{op}} \times \mathcal{J}, \mathbf{Set}]\big(\mathrm{Hom}_{\mathcal{J}}, \mathcal{C}(C, H)\big) &\cong \int_{(j',j):\mathcal{J}^{\mathrm{op}} \times \mathcal{J}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(C, H(j', j))) \\
&\cong \int_{j:\mathcal{J}} \int_{j':\mathcal{J}^{\mathrm{op}}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(C, H(j', j))) \\
&\cong \int_{j:\mathcal{J}} \mathcal{C}(C, H(j, j))
\end{aligned}$$

and these are natural in C; now apply propositions A.5.13 and A.6.11. ■

A.7 Familial regularity and exactness

Prerequisites. § A.3.

Definition A.7.1. A **strict initial object** in a category $\mathcal{C}$ is an initial object 0 in $\mathcal{C}$ such that every morphism $X \to 0$ in $\mathcal{C}$ is an isomorphism.

Example A.7.2. The empty set is a strict initial object in **Set**.

Proposition A.7.3. *Let $\mathcal{C}$ be a category with a strict initial object* 0.

(i) *For any object Y in $\mathcal{C}$, the unique morphism $0 \to Y$ is a monomorphism.*

(ii) *For any object Y in $\mathcal{C}$, $0 \to Y$ is a strict initial object in the slice category $\mathcal{C}_{/Y}$.*

Proof. Obvious. ♦

Definition A.7.4. Let κ be a (not necessarily small) regular cardinal. A **κ-ary extensive category** is a category $\mathcal{E}$ satisfying the following axioms:

- $\mathcal{E}$ has coproducts for κ-small families of objects (including the empty family).
- Given a κ-small family of objects in $\mathcal{E}$, say $\left(A_i \,\middle|\, i \in I\right)$, a morphism $f : B \to \coprod_{i \in I} A_i$, and commutative diagrams as below,

$$\begin{array}{ccc} B_j & \longrightarrow & B \\ \downarrow & & \downarrow f \\ A_j & \longrightarrow & \coprod_{i \in I} A_i \end{array}$$

  where the morphisms $A_j \to \coprod_{i \in I} A_i$ are the coproduct insertions, the family of morphisms $B_j \to B$ is a coproduct cocone if and only if each of the above commutative diagrams are pullback squares.

An **extensive category** is an $\aleph_0$-ary extensive category, and an **infinitary extensive category** is a κ-ary extensive category where κ is the cardinality of the universe.

Examples A.7.5.

(a) The category **Set** is an infinitary extensive category.

(b) More generally, if $\mathcal{E}$ is a locally cartesian closed category with coproducts for κ-small families of objects, then $\mathcal{E}$ is a κ-ary extensive category.

(c) The category **Top** is an infinitary extensive category but is neither cartesian closed nor locally cartesian closed.

Proposition A.7.6. *Let κ be a (not necessarily small) regular cardinal. If $\mathcal{E}$ is a κ-ary extensive category, then for any object A in $\mathcal{E}$, the slice category $\mathcal{E}_{/A}$ is also a κ-ary extensive category.*

Proof. This is an immediate consequence of the fact that the projection functor $\mathcal{E}_{/A} \to \mathcal{E}$ creates all colimits and pullbacks. ∎

Theorem A.7.7. *Let κ be a (not necessarily small) regular cardinal and let $\mathcal{C}$ be a category with coproducts for κ-small families of objects. The following are equivalent:*

(i) *$\mathcal{C}$ is a κ-ary extensive category.*

(ii) *For any κ-small family of objects in $\mathcal{C}$, say $\left(X_i \,\middle|\, i \in I\right)$, the functor*

$$\prod_{i \in I} \mathcal{C}_{/X_i} \to \mathcal{C}_{/\coprod_{i \in I} X_i}$$

induced by $\coprod_{i \in I}$ is fully faithful and essentially surjective on objects.

Proof. See Proposition 2.14 in [Carboni, Lack, and Walters, 1993]. □

Proposition A.7.8. *Let $\mathcal{C}$ be a category with an initial object 0. The following are equivalent:*

(i) *0 is a strict initial object in $\mathcal{C}$.*

(ii) *The slice category $\mathcal{C}_{/0}$ is equivalent to the terminal category $\mathbb{1}$.*

(iii) *For any morphism $f : X \to Y$, the following diagram is a pullback square:*

$$\begin{tikzcd} 0 \arrow[r, "\mathrm{id}"] \arrow[d] & 0 \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. (i) ⇒ (ii). If $f : X \to 0$ is an object in $\mathcal{C}_{/0}$, then f is an isomorphism in $\mathcal{C}$, and X is also an initial object in $\mathcal{C}$. Thus, any two objects in $\mathcal{C}_{/0}$ are connected by a unique isomorphism, and the unique functor $\mathcal{C}_{/0} \to \mathbb{1}$ is fully faithful and surjective on objects, with quasi-inverse the functor $\mathbb{1} \to \mathcal{C}_{/0}$ sending the unique object in $\mathbb{1}$ to the object $\mathrm{id} : 0 \to 0$.

(ii) ⇒ (i), (i) ⇒ (iii). Obvious.

(iii) ⇒ (i). Take $Y = 0$; then $0 \to X$ must be an isomorphism with $f : X \to 0$ as its inverse. ∎

Corollary A.7.9. *Initial objects in extensive categories are strict.*

Proof. The second axiom in the case where $I = \varnothing$ says that the initial object is preserved by pullbacks. ∎

Definition A.7.10. Let $\mathcal{C}$ be a category.

- An **extremal epimorphism** in $\mathcal{C}$ is an epimorphism e in $\mathcal{C}$ such that, for any morphisms m and z in $\mathcal{C}$, such that $e = m \circ z$, if m is a monomorphism in $\mathcal{C}$, then m is an isomorphism.
- A **regular epimorphism** in $\mathcal{C}$ is a morphism e in $\mathcal{C}$ for which there exist morphisms f_0, f_1 in $\mathcal{C}$ such that e is their coequaliser in $\mathcal{C}$.
- An **effective epimorphism** in $\mathcal{C}$ is a morphism e in $\mathcal{C}$ such that e has a kernel pair in $\mathcal{C}$ and is their coequaliser in $\mathcal{C}$.

Proposition A.7.11. *Let $\mathcal{C}$ be a category and let e be a morphism in $\mathcal{C}$. Consider the following statements:*

(i) *e is an effective epimorphism.*

(ii) *e is a regular epimorphism.*

(iii) *e is a strong epimorphism.*

(iv) *e is an extremal epimorphism.*

(v) *e is an epimorphism.*

We always have the implications (i) ⇒ (ii), (ii) ⇒ (iii), (iii) ⇒ (iv), *and* (iv) ⇒ (v)*; if $\mathcal{C}$ has kernel pairs, then* (ii) ⇒ (i)*; and if $\mathcal{C}$ has pullbacks of monomorphisms, then* (iv) ⇒ (iii).

Proof. (i) ⇒ (ii), (iv) ⇒ (v). Immediate.

(ii) ⇒ (iii). Suppose $e : Z \to W$ is a coequaliser for $f_0, f_1 : T \to Z$ in $\mathcal{C}$, and consider a commutative diagram of the form below in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "e"'] & X \arrow[d, "m"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

It is clear that $e : Z \to W$ is an epimorphism in $\mathcal{C}$. If $m : X \to Y$ is a monomorphism in $\mathcal{C}$, then $z \circ f_0 = z \circ f_1$, so there must exist a unique morphism $h : W \to X$ in $\mathcal{C}$ such that $z = h \circ e$; and $m \circ z = w \circ e$, so we must have $m \circ h = w$ as well. Thus $e : Z \to W$ has the required orthogonality property.

(iii) ⇒ (iv). This is the special case where $w = \mathrm{id}$; the existence of h such that $m \circ h = \mathrm{id}$ implies m is both a monomorphism and a split epimorphism, so m must be an isomorphism in this case.

(ii) ⇒ (i). Suppose $e : Z \to W$ is a coequaliser for $f_0, f_1 : T \to Z$ in $\mathcal{C}$. Let $k_0, k_1 : K \to Z$ be a kernel pair for $e : Z \to W$. There is then a unique morphism $r : T \to K$ in $\mathcal{C}$ such that $k_0 \circ r = f_0$ and $k_1 \circ r = f_1$. Consider any morphism $z : Z \to X$ in $\mathcal{C}$ such that $z \circ k_0 = z \circ k_1$. Then $z \circ f_0 = z \circ f_1$ as well, so there is a unique morphism $h : W \to X$ such that $z = h \circ e$. By definition, we have $e \circ k_0 = e \circ k_1$, so it follows that $e : Z \to W$ is a coequaliser for $k_0, k_1 : K \to Z$ in $\mathcal{C}$ as well.

(iv) ⇒ (iii). Suppose $e : Z \to W$ is a strong epimorphism in $\mathcal{C}$, and consider a commutative diagram of the form below in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "e"'] & X \arrow[d, "m"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

There is then a comparison morphism $Z \to W \times_Y X$, and if $m : X \to Y$ is a monomorphism in $\mathcal{C}$, then so is the projection $W \times_Y X \to W$. Since $e : Z \to W$ is a strong epimorphism, the projection $W \times_Y X \to W$ must be an isomorphism, so we obtain a (unique) morphism $h : W \to X$ in $\mathcal{C}$ such that $h \circ e = z$ and $m \circ h = w$, as required. ■

Definition A.7.12. A **regular category** is a category $\mathcal{C}$ that satisfies the following axioms:

- $\mathcal{C}$ has finite limits.
- $\mathcal{C}$ has coequalisers for kernel pairs.
- The class of regular epimorphisms in $\mathcal{C}$ is closed under pullbacks.

Example A.7.13. **Set** is a regular category. More generally, any ΣΠ-category[9] with coequalisers for kernel pairs is a regular category.

Theorem A.7.14. *Let $\mathcal{C}$ be a regular category.*

(i) *Every morphism in $\mathcal{C}$ admits a (regular epi, mono)-factorisation.*

(ii) *Every extremal epimorphism in $\mathcal{C}$ is a regular epimorphism.*

(iii) *The class of regular epimorphisms in $\mathcal{C}$ is closed under composition.*

Proof. (i). See Theorem 2.1.3 in [Borceux, 1994b].

(ii). Let f be an extremal epimorphism in $\mathcal{C}$, and suppose $f = m \circ e$ where m is a monomorphism and e is a regular epimorphism. Then m must be an isomorphism, so f is indeed a regular epimorphism. □

Proposition A.7.15. *In a regular category:*

(i) *The class of regular epimorphisms is closed under composition, pullbacks, and retracts.*

(ii) *If $g \circ f$ is a regular epimorphism, then g is also a regular epimorphism.*

(iii) *The class of regular epimorphisms is closed under finite products.*

Proof. (i). Recalling proposition A.7.11, theorem A.7.14 implies that strong epimorphisms in a regular category are the same as regular epimorphisms, so we may apply proposition A.3.17.

(ii). Similarly, we may apply proposition A.3.18.

(iii). Let $f : X \to Y$ and $g : Z \to W$ be regular epimorphisms in a regular category. Then $f \times g = \left(f \times \mathrm{id}_W\right) \circ \left(\mathrm{id}_X \times g\right)$, and both $f \times \mathrm{id}_W$ and $\mathrm{id}_X \times g$ are regular epimorphisms (because the class of regular epimorphisms is closed under pullbacks), so their composite is a regular epimorphism as well. ■

[9] See definition A.2.18.

Definition A.7.16. A **weak pullback square** in a regular category $\mathcal{C}$ is a commutative diagram in $\mathcal{C}$, say

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd}$$

such that the comparison morphism $Z \to W \times_Y X$ is a regular epimorphism.

Lemma A.7.17. *Let $\mathcal{C}$ be a regular category, and consider a commutative diagram in $\mathcal{C}$ of the form below:*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

(i) *If the two squares are weak pullback squares in $\mathcal{C}$, then the outer rectangle is a weak pullback diagram in $\mathcal{C}$.*

(ii) *If the right square is an ordinary pullback square and outer rectangle is weak pullback diagram in $\mathcal{C}$, then the left square is a weak pullback square in $\mathcal{C}$.*

Proof. (i). First, form the following pullback diagram in $\mathcal{C}$:

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y'' \times_Y X \arrow[d] \\ X' \arrow[r] & Y' \times_Y X \end{tikzcd}$$

Since $X' \to Y' \times_Y X$ is a regular epimorphism, so is $T \to Y'' \times_Y X$. Next, form a pullback diagram in $\mathcal{C}$ of the form below:

$$\begin{tikzcd} S \arrow[r] \arrow[d] & T \arrow[d] \\ X'' \arrow[r] & Y'' \times_{Y'} X' \end{tikzcd}$$

Since $X'' \to Y'' \times_{Y'} X'$ is a regular epimorphism, so is $S \to T$. We thus obtain a regular epimorphism $S \to Y'' \times_Y X$ that factors through the comparison morphism $X'' \to Y'' \times_Y X$, so we may use proposition A.7.15 to deduce that $X'' \to Y'' \times_Y X$ is a regular epimorphism.

(ii). Start by forming the following pullback diagram in $\mathcal{C}$:

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y'' \times_{Y'} X' \arrow[d] \\ X'' \arrow[r] & Y'' \times_Y X \end{tikzcd}$$

Since $X'' \to Y'' \times_Y X$ is a regular epimorphism, so is $T \to Y'' \times_{Y'} X'$. On the other hand, we have a commutative diagram of the form below in $\mathcal{C}$,

$$\begin{tikzcd} Y'' \times_Y X \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y'' \arrow[r] & Y' \arrow[r] & Y \end{tikzcd}$$

so $T \to Y'' \times_{Y'} X'$ factors through the comparison morphism $X'' \to Y'' \times_{Y'} X'$. Thus, $X'' \to Y'' \times_{Y'} X'$ is a regular epimorphism, as required. ■

Definition A.7.18. A **regular functor** (or **Barr-exact functor**) is a functor between regular categories that preserves finite limits and regular epimorphisms.

Remark A.7.19. By proposition A.7.11, a regular functor is the same thing as a functor between regular categories that preserves finite limits and coequalisers of kernel pairs. Thus, a regular functor between abelian categories automatically preserves finite colimits.

Definition A.7.20. An **exact fork** in a category $\mathcal{C}$ is a diagram in $\mathcal{C}$ of the form below,

$$X \underset{f_1}{\overset{f_0}{\rightrightarrows}} Y \xrightarrow{g} Z$$

where $g : Y \to Z$ is a coequaliser for $f_0, f_1 : X \to Y$ in $\mathcal{C}$ and $f_0, f_1 : X \to Y$ is a kernel pair for $g : Y \to Z$ in $\mathcal{C}$.

Theorem A.7.21 (Regular embedding theorem)**.** *Let $\mathcal{C}$ be a small regular category and let $\mathcal{E}$ be the full subcategory of* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *spanned by those presheaves* $\mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *that send exact forks in $\mathcal{C}$ to equaliser diagrams in* **Set**.

(i) *$\mathcal{E}$ is a reflective subcategory of* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*, and the reflector* $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{E}$ *preserves finite limits.*

(ii) *The Yoneda embedding* $\hbar_\bullet : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ *factors through the inclusion* $\mathcal{E} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$*, and the resulting functor* $\mathcal{C} \to \mathcal{E}$ *is fully faithful, regular, and preserves all limits.*

Proof. (i). This follows from the fact that $\mathcal{E}$ is the category of sheaves for a Grothendieck topology on $\mathcal{C}$: see Lemma 2.7.2 in [Borceux, 1994b] and Theorem 3.3.12 in [Borceux, 1994c].

(ii). See Theorem 2.7.3 in [Borceux, 1994b]. □

Theorem A.7.22 (Barr)**.** *For each small regular category* $\mathcal{C}$*, there exist a set* B *and a conservative regular functor* $\mathcal{C} \to \mathbf{Set}^B$*.*

Proof. See Theorem 1.6 in [Barr, 1971, Ch. III] or Corollary 1.5.4 in [Johnstone, 2002, Part D]. □

Definition A.7.23.

- An **effective equivalence relation** in a category $\mathcal{C}$ is an (internal) equivalence relation in $\mathcal{C}$ that appears as part of an exact fork in $\mathcal{C}$, i.e. is a kernel pair for an effective epimorphism in $\mathcal{C}$.

- An **effective regular category** (or **Barr-exact category**) is a regular category in which all (internal) equivalence relations are effective.

Remark A.7.24. A regular category with coequalisers for all parallel pairs of morphisms is automatically an effective regular category, but effective regular categories need not have coequalisers in general.

Lemma A.7.25. *Let* $\mathcal{C}$ *be an effective regular category. Given a parallel pair of morphisms in* $\mathcal{C}$*, say* $p_0, p_1 : X \to Y$*, if the regular image of the morphism* $\langle p_0, p_1 \rangle : X \to Y \times Y$ *defines an equivalence relation* R *on* Y*, then* $p_0, p_1 : X \to Y$ *have a coequaliser in* $\mathcal{C}$*, and the kernel pair of the coequaliser is the equivalence relation* R*.*

Proof. By definition of R, there exist a regular epimorphism $e : X \to R$ and two projections $r_0, r_1 : R \to Y$ such that $p_0 = r_0 \circ e$ and $p_1 = r_1 \circ e$. Let $q : Y \to Z$ be the coequaliser of r_0 and r_1 in $\mathcal{C}$; such exists because $\mathcal{C}$ is an effective regular category. Note that the kernel pair of $q : Y \to Z$ is $r_0, r_1 : R \to Y$. Now, $q \circ r_0 = q \circ r_1$, so we must have $q \circ p_0 = q \circ p_1$ as well; but if $f : Y \to T$ is any morphism in $\mathcal{C}$ such that $f \circ p_0 = f \circ p_1$, then we must have $f \circ q_0 = f \circ q_1$

(because $e : X \to R$ is an epimorphism), and so there is a unique morphism $\bar{f} : Z \to T$ such that $\bar{f} \circ q = f$. Thus, $q : Y \to Z$ is also the coequaliser of p_0 and p_1 in $\mathcal{C}$. ■

Definition A.7.26. Let κ be a (not necessarily small) regular cardinal. A **κ-ary pretopos** is a category that is both κ-ary extensive and effective regular. A **σ-pretopos** is an $\aleph_1$-ary pretopos.

Proposition A.7.27. *Let κ be a (not necessarily small) regular cardinal. A κ-ary pretopos is a (positive) κ-ary coherent category.*

Proof. See Theorem 5.15 in [Shulman, 2012]. □

Remark A.7.28. The above proposition implies that our definition of 'pretopos' agrees with the one given by Johnstone [2002, Part A, § 1.4].

Proposition A.7.29. *Any epimorphism in a pretopos is a regular epimorphism.*

Proof. See Corollary 1.4.9 in [Johnstone, 2002, Part A]. □

Proposition A.7.30.

(i) *Every σ-pretopos has coequalisers for all parallel pairs; hence, they have colimits for all countable diagrams.*

(ii) *Every regular functor between σ-pretoposes that preserves coproducts for countable families also preserves coequalisers.*

Proof. (i). See Lemma 1.4.19 in [Johnstone, 2002, Part A].

(ii). See Lemma 2.5.7 in [Johnstone, 2002, Part A]. □

Proposition A.7.31. *Let κ be a small regular cardinal. If $\mathcal{C}$ is a small κ-ary pretopos, then there exist a Grothendieck topos $\mathcal{E}$ and a fully faithful regular functor $\mathcal{C} \to \mathcal{E}$ that preserves coproducts for κ-small families of objects. Moreover, if κ is uncountable, then the embedding $\mathcal{C} \to \mathcal{E}$ also preserves coequalisers.*

Proof. By Theorem 5.15 in [Shulman, 2012], or Example 2.1.11(b) in [Johnstone, 2002, Part A], we may take $\mathcal{E}$ to be the category of sheaves for the κ-ary coherent topology on $\mathcal{C}$; then apply proposition A.7.30. □

Theorem A.7.32 (Deligne)**.** *For each small pretopos $\mathcal{C}$, there exist a set B and a conservative regular functor $\mathcal{C} \to \mathbf{Set}^B$ that preserves coproducts for finite families of objects.*

Proof. See Proposition 9.0 in [SGA 4b, Exposé VI], Corollary 3 in [ML–M, Ch. IX, § 11], or Proposition 3.3.13 in [Johnstone, 2002, Part D]. □

— B —

Higher generalities

B.1 Monoidal categories

Standard references for monoidal categories include [CWM, Ch. VII and Ch. XI] and [Kelly, 2005, Ch. 1]. To fix notation, we will quickly review the main definitions in the theory of monoidal categories.

Definition B.1.1. A **strict monoidal category** is a category $\mathcal{C}$ together with an object I and a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the following axioms:

- (Left unit). $I \otimes (-) = \mathrm{id}_{\mathcal{C}}$.
- (Right unit). $(-) \otimes I = \mathrm{id}_{\mathcal{C}}$.
- (Associativity). For all objects X, Y, and Z in $\mathcal{C}$,

  $$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

  and similarly for morphisms in $\mathcal{C}$.

I is called the **monoidal unit**, and $\otimes$ is called the **monoidal product**.

In short, a strict monoidal category is an internal monoid in the metacategory of all categories.

Example B.1.2. For any category $\mathcal{C}$, the endofunctor category $[\mathcal{C}, \mathcal{C}]$ is a strict monoidal category with $\mathrm{id}_{\mathcal{C}}$ as the monoidal unit and endofunctor composition as the monoidal product.

Despite the above example, strict monoidal categories turn out to be less useful than one might hope: not even **Set** equipped with the usual cartesian product is a strict monoidal category.[1] The problem is in the *equations* we have imposed in the axioms above: in naturally-occurring examples, we do not get *identities* but only natural isomorphisms. This observation led Bénabou [1963] to propose the following notion instead:

Definition B.1.3. A **monoidal category** is a category $\mathcal{C}$ together with an object I, a functor $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and three natural isomorphisms $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$,[2] of type

$$\boldsymbol{\lambda}_X : I \otimes X \xrightarrow{\cong} X$$
$$\boldsymbol{\rho}_X : X \otimes I \xrightarrow{\cong} X$$
$$\boldsymbol{\alpha}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

such that the following diagrams commute for all choices of objects in $\mathcal{C}$:

$$\begin{tikzcd}
& (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\boldsymbol{\alpha}_{W,X\otimes Y,Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\boldsymbol{\alpha}_{W,X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "\boldsymbol{\alpha}_{W\otimes X,Y,Z}"'] & & (W \otimes ((X \otimes Y) \otimes Z)) \arrow[d, "\mathrm{id}_W \otimes \boldsymbol{\alpha}_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y\otimes Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\boldsymbol{\alpha}_{X,I,Y}"] \arrow[dr, "\boldsymbol{\rho}_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\mathrm{id}_X \otimes \boldsymbol{\lambda}_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

The natural isomorphisms $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ are called, respectively, the **left unitor**, **right unitor**, and **associator** of the monoidal category $\mathcal{C}$.

Remark B.1.4. Since $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ are natural *isomorphisms*, a monoidal structure on $\mathcal{C}$ induces a monoidal structure on $\mathcal{C}^{\mathrm{op}}$. Less obviously, we can define a monoidal category $\mathcal{C}^{\mathrm{rev}}$ whose underlying category is the same as $\mathcal{C}$, but $X \otimes^{\mathrm{rev}} Y = Y \otimes X$, $\boldsymbol{\lambda}^{\mathrm{rev}} = \boldsymbol{\rho}$, $\boldsymbol{\rho}^{\mathrm{rev}} = \boldsymbol{\lambda}$, and $\boldsymbol{\alpha}^{\mathrm{rev}} = \boldsymbol{\alpha}^{-1}$.

[1] In fact, even if we identify all isomorphic objects, there is still a problem: see the closing remarks in [CWM, Ch. VII, § 1].

[2] Beware: Mac Lane [CWM, Ch. VII] uses the opposite convention for $\boldsymbol{\alpha}$.

¶ B.1.5. A fairly non-trivial theorem of Mac Lane [1963] and Kelly [1964] essentially states that these two axioms are enough to prove that "all diagrams involving only $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ commute". For example, using the pentagon axiom and the triangle axiom, we may derive

$$\begin{tikzcd}
(I \otimes X) \otimes Y \arrow[rr, "\boldsymbol{\alpha}_{I,X,Y}"] \arrow[dr, "\boldsymbol{\lambda}_X \otimes \mathrm{id}_Y"'] & & I \otimes (X \otimes Y) \arrow[dl, "\boldsymbol{\lambda}_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}$$

from which the equation (!) below can be obtained:

$$\boldsymbol{\lambda}_I = \boldsymbol{\rho}_I$$

Definition B.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories.

- A **lax monoidal functor** $\mathcal{C} \to \mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ of the underlying categories, together with a morphism $\boldsymbol{\eta} : I_\mathcal{D} \to FI_\mathcal{C}$ in $\mathcal{D}$ and a natural transformation $\boldsymbol{\mu}$ of type $F(-) \otimes_\mathcal{D} F(-) \Rightarrow F\big(- \otimes_\mathcal{C} -\big)$ making these diagrams commute:

$$\begin{tikzcd}
I_\mathcal{D} \otimes_\mathcal{D} FX \arrow[r, "\boldsymbol{\eta} \otimes_\mathcal{D} \mathrm{id}_{FX}"] \arrow[d, "\boldsymbol{\lambda}_{FX}"'] & FI_\mathcal{C} \otimes_\mathcal{D} FX \arrow[d, "\boldsymbol{\mu}_{I_\mathcal{C}, X}"] \\
FX & F\big(I_\mathcal{C} \otimes_\mathcal{C} X\big) \arrow[l, "F\boldsymbol{\lambda}_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes_\mathcal{D} I_\mathcal{D} \arrow[r, "\mathrm{id}_{FX} \otimes_\mathcal{D} \boldsymbol{\eta}"] \arrow[d, "\boldsymbol{\rho}_{FX}"'] & FX \otimes_\mathcal{D} FI_\mathcal{C} \arrow[d, "\boldsymbol{\mu}_{X, I_\mathcal{C}}"] \\
FX & F\big(X \otimes_\mathcal{C} I_\mathcal{C}\big) \arrow[l, "F\boldsymbol{\rho}_X"]
\end{tikzcd}$$

$$\begin{tikzcd}
\big(FX \otimes_\mathcal{D} FY\big) \otimes_\mathcal{D} FZ \arrow[r, "\boldsymbol{\alpha}_{FX,FY,FZ}"] \arrow[d, "\boldsymbol{\mu}_{X,Y} \otimes_\mathcal{D} \mathrm{id}_{FZ}"'] & FX \otimes_\mathcal{D} \big(FY \otimes_\mathcal{D} FZ\big) \arrow[d, "\mathrm{id}_{FX} \otimes_\mathcal{D} \boldsymbol{\mu}_{Y,Z}"] \\
F\big(X \otimes_\mathcal{C} Y\big) \otimes_\mathcal{D} FZ \arrow[d, "\boldsymbol{\mu}_{X \otimes_\mathcal{C} Y, Z}"'] & FX \otimes_\mathcal{D} F\big(Y \otimes_\mathcal{C} Z\big) \arrow[d, "\boldsymbol{\mu}_{X, Y \otimes_\mathcal{C} Z}"] \\
F\big(\big(X \otimes_\mathcal{C} Y\big) \otimes_\mathcal{C} Z\big) \arrow[r, "F\boldsymbol{\alpha}_{X,Y,Z}"'] & F\big(X \otimes_\mathcal{C} \big(Y \otimes_\mathcal{C} Z\big)\big)
\end{tikzcd}$$

- An **oplax monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax monoidal functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

- A **strong monoidal functor** is a lax monoidal functor such that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are isomorphisms.

- A **strict monoidal functor** is a lax monoidal functor such that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are *identities*.

Definition B.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F, F' : \mathcal{C} \to \mathcal{D}$ be lax monoidal functors. A **monoidal natural transformation** $\varphi : F \Rightarrow F'$ is a natural transformation making the following diagrams commute:

$$
\begin{tikzcd}
I_{\mathcal{D}} \arrow[r, "\eta"] \arrow[d, "\mathrm{id}"'] & F I_{\mathcal{C}} \arrow[d, "\varphi_{I_{\mathcal{C}}}"] \\
I_{\mathcal{D}} \arrow[r, "\eta'"'] & F' I_{\mathcal{C}}
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes_{\mathcal{D}} FY \arrow[r, "\mu_{X,Y}"] \arrow[d, "\varphi_X \otimes_{\mathcal{D}} \varphi_Y"'] & F(X \otimes_{\mathcal{C}} Y) \arrow[d, "\varphi_{X \otimes_{\mathcal{C}} Y}"] \\
F'X \otimes_{\mathcal{D}} F'Y \arrow[r, "\mu'_{X,Y}"'] & F'(X \otimes_{\mathcal{C}} Y)
\end{tikzcd}
$$

Remark B.1.8. Note that if $\mathcal{C}$ and $\mathcal{D}$ are both strict monoidal categories, then the diagrams above simplify to more familiar ones:

$$
\begin{tikzcd}
FX \arrow[r, "\eta \otimes_{\mathcal{D}} \mathrm{id}_{FX}"] \arrow[dr, "\mathrm{id}_{FX}"'] & F I_{\mathcal{C}} \otimes_{\mathcal{D}} FX \arrow[d, "\mu_{I_{\mathcal{C}},X}"] \\
& FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \eta"] \arrow[dr, "\mathrm{id}_{FX}"'] & FX \otimes_{\mathcal{D}} F I_{\mathcal{C}} \arrow[d, "\mu_{X,I_{\mathcal{C}}}"] \\
& FX
\end{tikzcd}
$$

$$
\begin{tikzcd}
& FX \otimes_{\mathcal{D}} FY \otimes_{\mathcal{D}} FZ \arrow[dl, "\mu_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{FZ}"'] \arrow[dr, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \mu_{Y,Z}"] & \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} FZ \arrow[dr, "\mu_{X \otimes_{\mathcal{C}} Y, Z}"'] & & FX \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \arrow[dl, "\mu_{X, Y \otimes_{\mathcal{C}} Z}"] \\
& F(X \otimes_{\mathcal{C}} Y \otimes_{\mathcal{C}} Z) &
\end{tikzcd}
$$

Thus, we see one reason for defining lax monoidal functors as we have done: if $\mathbb{1}$ is the terminal category, then a lax monoidal functor $\mathbb{1} \to \mathcal{D}$ is the same thing as an internal monoid[3] in $\mathcal{D}$, and a monoidal natural transformation of such lax monoidal functors is the same thing as a homomorphism of internal monoids.

Many natural examples of monoidal categories have a "commutative" monoidal product. For example, the cartesian product in any category satisfies $X \times Y \cong Y \times X$. As usual, to do anything useful, we must demand not only the existence of such isomorphisms but also that they be natural and coherent in the following sense:

[3] — in the monoidal category sense, of course.

Definition B.1.9. A **braided monoidal category** is a monoidal category $\mathcal{C}$ together with a natural isomorphism $\boldsymbol{\gamma}$ of type

$$\boldsymbol{\gamma}_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

such that the following diagrams commute for all choices of objects in $\mathcal{C}$:

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[dr, "\boldsymbol{\gamma}_{X,Y\otimes Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\boldsymbol{\alpha}_{X,Y,Z}"] \arrow[d, "\boldsymbol{\gamma}_{X,Y}\otimes \mathrm{id}_Z"'] & & (Y \otimes Z) \otimes X \arrow[d, "\boldsymbol{\alpha}_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[dr, "\boldsymbol{\alpha}_{Y,X,Z}"'] & & Y \otimes (Z \otimes X) \\
 & Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes \boldsymbol{\gamma}_{X,Z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[dr, "\boldsymbol{\gamma}_{X\otimes Y,Z}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\boldsymbol{\alpha}^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X\otimes \boldsymbol{\gamma}_{Y,Z}"'] & & Z \otimes (X \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{Y,Z,X}"] \\
X \otimes (Z \otimes Y) \arrow[dr, "\boldsymbol{\alpha}^{-1}_{Y,X,Z}"'] & & (Z \otimes X) \otimes Y \\
 & (X \otimes Z) \otimes Y \arrow[ur, "\boldsymbol{\gamma}_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes X \arrow[rr, "\boldsymbol{\gamma}_{I,X}"] \arrow[dr, "\boldsymbol{\lambda}_X"'] & & X \otimes I \arrow[dl, "\boldsymbol{\rho}_X"] \\
 & X &
\end{tikzcd}$$

The natural isomorphism $\boldsymbol{\gamma}$ is called the **braiding** of $\mathcal{C}$. A **symmetric monoidal category** is a braided monoidal category $\mathcal{C}$ satisfying the following additional axiom:

$$\boldsymbol{\gamma} \bullet \boldsymbol{\gamma} = \mathrm{id}_{\mathcal{C}}$$

A **braided / symmetric strict monoidal category** is a braided / symmetric monoidal category that is strict as a monoidal category.

There is a coherence theorem for braided and symmetric monoidal categories as well, but in the braided case it is somewhat subtle compared to the coherence theorem for monoidal categories – we cannot be so cavalier as to say that "all diagrams commute" in a braided monoidal category. Instead, just as before, every braided / symmetric monoidal category is equivalent to a strict one via functors respecting the various structural isomorphisms.

Definition b.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories. A **lax / oplax / strong / strict braided monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax / oplax / strong / strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ making the diagram below commute:

$$\begin{tikzcd}
FX \otimes_{\mathcal{D}} FY \arrow[r, "\mu_{X,Y}"] \arrow[d, "\gamma_{FX,FY}"'] & F\big(X \otimes_{\mathcal{C}} Y\big) \arrow[d, "F\gamma_{X,Y}"] \\
FY \otimes_{\mathcal{D}} FX \arrow[r, "\mu_{Y,X}"'] & F\big(Y \otimes_{\mathcal{C}} X\big)
\end{tikzcd}$$

Remark b.1.11. The appropriate notion of natural transformation for lax braided monoidal functors is precisely that of a monoidal natural transformation: we need not impose any extra conditions.

Here is an example of an equation that does *not* necessarily hold in a braided monoidal category, even though they have the same domain and codomain:

$$\gamma_{X,Y} \stackrel{?}{=} \gamma_{Y,X}^{-1}$$

Indeed, if it were true, then every braided monoidal category would be a symmetric monoidal category! On the other hand, in a symmetric strict monoidal category, it is true that any two composites of braiding operations with the same domain and codomain are equal – provided each object is identified with a different letter, so that we do not get absurdities like this:

$$\gamma_{X,X} \stackrel{?}{=} \mathrm{id}_{X \otimes X}$$

A similar restriction applies to our claim that "all diagrams commute" in a monoidal category, so it is not unreasonable to say the same is true in a symmetric monoidal category.

We pause briefly to indicate an important special case of a symmetric monoidal category.

Definition B.1.12. A **cartesian monoidal category** is a category with products for all finite families of objects, and a **cartesian monoidal functor** is a functor between cartesian monoidal categories that preserves all finite products.

Proposition B.1.13.

(i) *A category with all finite products is automatically a symmetric monoidal category, with the terminal object* 1 *as its monoidal unit and the cartesian product* $\times$ *as the monoidal product.*

(ii) *If $\mathcal{C}$ and $\mathcal{D}$ are two categories with finite products regarded as symmetric monoidal categories, then every functor $\mathcal{C} \to \mathcal{D}$ can be equipped with a canonical oplax braided monoidal functor structure.*

(iii) *A cartesian monoidal functor is canonically equipped with the structure of a strong braided monoidal functor.*

Proof. (i). The verification of the axioms is straightforward and left to the reader as an exercise.

(ii). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The universal property of the terminal object gives a unique morphism $\boldsymbol{\varepsilon} : F1 \to 1$ in $\mathcal{D}$, and the universal property of binary products gives a canonical morphism $\boldsymbol{\delta}_{X,Y} : F(X \times Y) \to FX \times FY$. It can be shown that the diagrams below commute,

$$\begin{tikzcd} F(1_{\mathcal{C}} \times_{\mathcal{C}} X) \arrow[r, "\boldsymbol{\delta}_{1_{\mathcal{C}},X}"] \arrow[d, "F\boldsymbol{\lambda}_X"'] & F1_{\mathcal{C}} \times_{\mathcal{D}} FX \arrow[d, "\boldsymbol{\varepsilon} \times_{\mathcal{D}} \mathrm{id}_{FX}"] \\ FX & 1_{\mathcal{D}} \times_{\mathcal{D}} FX \arrow[l, "\boldsymbol{\lambda}_{FX}"] \end{tikzcd} \qquad \begin{tikzcd} F(X \times_{\mathcal{C}} 1_{\mathcal{C}}) \arrow[r, "\boldsymbol{\delta}_{X,1_{\mathcal{C}}}"] \arrow[d, "F\boldsymbol{\rho}_X"'] & FX \times_{\mathcal{D}} F1_{\mathcal{C}} \arrow[d, "\mathrm{id}_{FX} \times_{\mathcal{D}} \boldsymbol{\varepsilon}"] \\ FX & FX \times_{\mathcal{D}} 1_{\mathcal{D}} \arrow[l, "\boldsymbol{\rho}_{FX}"] \end{tikzcd}$$

$$\begin{tikzcd} F\big((X \times_{\mathcal{C}} Y) \times_{\mathcal{C}} Z\big) \arrow[r, "F\boldsymbol{\alpha}_{X,Y,Z}"] \arrow[d, "\boldsymbol{\delta}_{X \times_{\mathcal{C}} Y, Z}"'] & F\big(X \times_{\mathcal{C}} (Y \times_{\mathcal{C}} Z)\big) \arrow[d, "\boldsymbol{\delta}_{X, Y \times_{\mathcal{C}} Z}"] \\ F(X \times_{\mathcal{C}} Y) \times_{\mathcal{D}} FZ \arrow[d, "\boldsymbol{\delta}_{X,Y} \times_{\mathcal{D}} \mathrm{id}_{FZ}"'] & FX \times_{\mathcal{D}} F(Y \times_{\mathcal{C}} Z) \arrow[d, "\mathrm{id}_{FX} \times_{\mathcal{D}} \boldsymbol{\delta}_{Y,Z}"] \\ (FX \times_{\mathcal{D}} FY) \times_{\mathcal{D}} FZ \arrow[r, "\boldsymbol{\alpha}_{FX,FY,FZ}"'] & FX \times_{\mathcal{D}} (FY \times_{\mathcal{D}} FZ) \end{tikzcd}$$

$$\begin{tikzcd}
F\big(X \times_{C} Y\big) \arrow[r, "\delta_{X,Y}"] \arrow[d, "F\gamma_{X,Y}"'] & FX \times_{D} FY \arrow[d, "\gamma_{FX,FY}"] \\
F\big(Y \otimes_{C} X\big) \arrow[r, "\delta_{Y,X}"'] & FY \otimes_{D} FX
\end{tikzcd}$$

making F into an oplax braided monoidal functor $\mathcal{C} \to \mathcal{D}$.

(iii). A functor is cartesian monoidal precisely if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ as defined above are isomorphisms. ◊

Definition B.1.14. Let Y and Z be objects in a monoidal category $\mathcal{C}$.

- A **right internal hom object** for Y and Z is an object $\mathcal{H}om(Y, Z)$ in $\mathcal{C}$ together with a morphism $\mathrm{ev}_{Y,Z} : \mathcal{H}om(Y, Z) \otimes Y \to Z$ having the following universal property: for all morphisms $f : X \otimes Y \to Z$ in $\mathcal{C}$, there is a unique morphism $\tilde{f} : X \to \mathcal{H}om(Y, Z)$ in $\mathcal{C}$ such that $\mathrm{ev}_{Y,Z} \circ \big(\tilde{f} \otimes \mathrm{id}_Y\big) = f$; equivalently, $\mathcal{H}om(Y, Z)$ is an object in $\mathcal{C}$ equipped with bijections

  $$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \mathcal{H}om(Y, Z))$$

  that are natural for each object X in $\mathcal{C}$. We may also write $[Y, Z]$ or $Y \multimap Z$ for a right internal hom object for Y and Z.

- A **left internal hom object** for Y and Z is a right internal hom object $Y \pitchfork Z$ in the reverse monoidal structure on $\mathcal{C}$; equivalently, $Y \pitchfork Z$ is an object equipped with bijections

  $$\mathcal{C}(Y \otimes X, Z) \cong \mathcal{C}(X, Y \pitchfork Z)$$

  that are natural for each object X in $\mathcal{C}$. We may also write Z^Y or $Z \multimapinv Y$ for a left internal hom object for Y and Z.

- A **right-closed monoidal category** is a monoidal category that has right internal hom object for all pairs of objects.

- A **left-closed monoidal category** is a monoidal category that has left internal hom objects for all pairs of objects.

- A **biclosed monoidal category** is a monoidal category that is both left-closed and right-closed.

Note that in a symmetric monoidal category, $Y \pitchfork Z$ and $\mathcal{H}om(Y, Z)$ are naturally isomorphic if they exist; a **symmetric monoidal closed category** is a symmetric monoidal category that is biclosed.

Proposition B.1.15. *Let $\mathcal{C}$ be a right-closed monoidal category.*

(i) *The assignment $(Y, Z) \mapsto \mathcal{H}om(Y, Z)$ extends to a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ making the bijection*

$$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \mathcal{H}om(Y, Z))$$

natural in X, Y, and Z.

(ii) *For each object Y, we have an adjunction*

$$(-) \otimes Y \dashv \mathcal{H}om(Y, -) : \mathcal{C} \to \mathcal{C}$$

whose counit is $\mathrm{ev}_{Y,-} : \mathcal{H}om(Y, -) \otimes Y \Rightarrow \mathrm{id}_{\mathcal{C}}$.

(iii) *If I is the monoidal unit of $\mathcal{C}$, then there is a bijection*

$$\mathcal{C}(Y, Z) \cong \mathcal{C}(I, \mathcal{H}om(Y, Z))$$

that is natural in Y and Z.

Proof. (i). This is a straightforward example of an adjunction with a parameter.[4]

(ii). This is clear from the definition of $\mathcal{H}om(Y, Z)$ and $\mathrm{ev}_{Y,-}$.

(iii). The left unitor $\boldsymbol{\lambda}_Y : Y \xrightarrow{\cong} I \otimes Y$ induces the required bijection. ∎

Remark B.1.16. A cartesian monoidal category is a closed symmetric monoidal category if and only if it is a cartesian closed category (definition A.2.3).

B.2 Enriched categories

Prerequisites. § 0.1, B.1.

In this section, we use the explicit universe convention.

[4] See [CWM, Ch. IV, § 7].

Definition B.2.1. Let $\mathcal{V}$ be a monoidal category. A **$\mathcal{V}$-enriched category** $\underline{C}$ consists of the following data:

- A set of objects, $\operatorname{ob} C$.
- For each pair (A, B) of elements of $\operatorname{ob} C$, an object $\underline{C}(A, B)$ in $\mathcal{V}$.
- For each element A of $\operatorname{ob} C$, a morphism $e_A : I \to \underline{C}(A, A)$ in $\mathcal{V}$.
- For each triple (A, B, C) of elements of $\operatorname{ob} C$, a morphism

$$c_{C,B,A} : \underline{C}(B, C) \otimes \underline{C}(A, B) \to \underline{C}(A, C)$$

  such that the following diagrams in $\mathcal{V}$ commute,

(L)
$$\begin{tikzcd} I \otimes \underline{C}(A, B) \arrow[r, "e_B \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & \underline{C}(B, B) \otimes \underline{C}(A, B) \arrow[d, "c_{B,B,A}"] \\ & \underline{C}(B, A) \end{tikzcd}$$

(R)
$$\begin{tikzcd} \underline{C}(A, B) \otimes I \arrow[r, "\mathrm{id} \otimes e_A"] \arrow[dr, "\rho"'] & \underline{C}(A, B) \otimes \underline{C}(A, A) \arrow[d, "c_{B,A,A}"] \\ & \underline{C}(A, B) \end{tikzcd}$$

(A)
$$\begin{tikzcd} \underline{C}(C, D) \otimes \underline{C}(B, C) \otimes \underline{C}(A, B) \arrow[r, "\mathrm{id} \otimes c_{C,B,A}"] \arrow[d, "c_{D,C,B} \otimes \mathrm{id}"'] & \underline{C}(C, D) \otimes \underline{C}(A, C) \arrow[d, "c_{D,C,A}"] \\ \underline{C}(B, D) \otimes \underline{C}(A, B) \arrow[r, "c_{D,B,A}"'] & \underline{C}(A, D) \end{tikzcd}$$

  where in the last diagram we have suppressed the associator of $\mathcal{V}$.

Definition B.2.2. Let $\mathbf{U}$ be a pre-universe and let $\mathcal{V}$ be a locally $\mathbf{U}$-small monoidal category.

- A **$\mathbf{U}$-small $\mathcal{V}$-enriched category** is a $\mathcal{V}$-category $\underline{C}$ where $\operatorname{ob} C$ is a $\mathbf{U}$-set.
- A **locally $\mathbf{U}$-small $\mathcal{V}$-enriched category** is a $\mathcal{V}$-category where $\operatorname{ob} C$ is a $\mathbf{U}$-class.

Definition B.2.3. Let $\mathcal{V}$ be a monoidal category and let $\underline{C}$ be a $\mathcal{V}$-enriched category. The **underlying ordinary category** of $\underline{C}$ is the category C where:

- The objects in C are the objects in $\underline{C}$.
- The morphisms $A \to B$ in C are the morphisms $I \to \underline{C}(A, B)$ in $\mathcal{V}$.
- For each object A, $\mathrm{id}_A : A \to A$ is $e_A : I \to \underline{C}(A, A)$ regarded as a morphism in C.
- Given morphisms $f : A \to B$ and $g : B \to C$ in C corresponding to morphisms $f : I \to \underline{C}(A, B)$ and $g : I \to \underline{C}(B, C)$ in $\mathcal{V}$, the composite $g \circ f : A \to C$ is the morphism in C corresponding to $c_{C,B,A} \circ (g \otimes f) \circ \boldsymbol{\lambda}_I^{-1}$ in $\mathcal{V}$.

We refer to $\underline{C}$ as a **$\mathcal{V}$-enrichment** of C if C is (isomorphic to) the underlying ordinary category of $\underline{C}$.

Remark B.2.4. Given a $\mathcal{V}$-enriched category $\underline{C}$, there is an evident $\mathcal{V}^{\mathrm{rev}}$-enriched category $\underline{C}^{\mathrm{op}}$ whose underlying ordinary category is C^{op}. If we assume $\mathcal{V}$ is a *symmetric* monoidal category, we can also identify $\underline{C}^{\mathrm{op}}$ with a $\mathcal{V}$-enriched category.

Proposition B.2.5. *Let $\mathcal{V}$ be a* right-closed *monoidal category. Then $\mathcal{V}$ is (isomorphic to) the underlying category of a $\mathcal{V}$-enriched category $\underline{\mathcal{V}}$ where:*

- *The objects are the objects in $\mathcal{V}$.*
- *We have $\underline{\mathcal{V}}(A, B) = \mathcal{H}om(A, B)$.*
- *For each object A in $\mathcal{V}$, $e_A : I \to \underline{\mathcal{V}}(A, A)$ is the right adjoint transpose of $\boldsymbol{\lambda}_A : I \otimes A \to A$.*
- *For each triple (A, B, C) of objects in $\mathcal{V}$,*

$$c_{C,B,A} : \underline{\mathcal{V}}(B, C) \otimes \underline{\mathcal{V}}(A, B) \to \underline{\mathcal{V}}(A, C)$$

is the right adjoint transpose of the following morphism in $\mathcal{V}$:

$$\mathrm{ev}_{B,C} \circ \left(\mathrm{id} \otimes \mathrm{ev}_{A,B}\right) \circ \boldsymbol{\alpha} : (\underline{\mathcal{V}}(B, C) \otimes \underline{\mathcal{V}}(A, B)) \otimes A \to C$$

Proof. Straightforward, if tedious. ♦

Definition b.2.6. Let $\mathcal{V}$ be a monoidal category and let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-enriched categories. A **$\mathcal{V}$-enriched functor** $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ consists of the following data:

- A map $F : \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$.
- For each pair (A, B) of objects in $\mathcal{C}$, a morphism $\underline{F}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$, such that the following diagrams in $\mathcal{V}$ commute:

(U)
$$\begin{tikzcd} I \arrow[r, "e_A"] \arrow[d, equal] & \underline{\mathcal{C}}(A, A) \arrow[d, "\underline{F}_{A,A}"] \\ I \arrow[r, "e_{FA}"'] & \underline{\mathcal{D}}(FA, FA) \end{tikzcd}$$

(M)
$$\begin{tikzcd} \underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B) \arrow[r, "c_{C,B,A}"] \arrow[d, "\underline{F}_{B,C} \otimes \underline{F}_{A,B}"'] & \underline{\mathcal{C}}(A, C) \arrow[d, "\underline{F}_{A,C}"] \\ \underline{\mathcal{D}}(FB, FC) \otimes \underline{\mathcal{D}}(FA, FB) \arrow[r, "c_{FC,FB,FA}"'] & \underline{\mathcal{D}}(FA, FC) \end{tikzcd}$$

Remark b.2.7. Every $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ defines an **underlying ordinary functor** $F : \mathcal{C} \to \mathcal{D}$ in the obvious way.

Proposition b.2.8. *Let $\mathcal{V}$ be a* right-closed *monoidal category and let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category. For each object A in $\mathcal{C}$, there is a $\mathcal{V}$-enriched functor $\underline{\mathcal{C}}(A, -) : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ where:*

- *The map of objects is given by $B \mapsto \underline{\mathcal{C}}(A, B)$.*
- *The morphism $\underline{\mathcal{C}}(A, -)_{B,C} : \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, B), \underline{\mathcal{C}}(A, C))$ is the right adjoint transpose of $c_{C,B,A} : \underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{C}}(A, C)$.*

Dually, assuming $\mathcal{V}$ is a left-closed *monoidal category, for each object C in $\mathcal{C}$, there is a $\mathcal{V}^{\mathrm{rev}}$-enriched functor $\underline{\mathcal{C}}(-, C) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathcal{V}^{\mathrm{rev}}}$ where:*

- *The map of objects is given by $B \mapsto \underline{\mathcal{C}}(B, C)$.*
- *The morphism $\underline{\mathcal{C}}(-, C)_{B,A} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(B, C), \underline{\mathcal{C}}(A, C))$ is the right adjoint transpose of $c_{A,B,C} : \underline{\mathcal{C}}(A, B) \otimes^{\mathrm{rev}} \underline{\mathcal{C}}(B, C) \to \underline{\mathcal{C}}(A, C)$.*

Proof. By adjointness, axiom U corresponds to axiom L, and axiom M corresponds to axiom A. ∎

Definition B.2.9. Let $\mathcal{V}$ be a monoidal category and let $\underline{F}, \underline{G} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a parallel pair of $\mathcal{V}$-enriched functors. A **$\mathcal{V}$-enriched natural transformation** $\varphi : \underline{F} \Rightarrow \underline{G}$ consists of the following data:

- For each object A in $\mathcal{C}$, a morphism $\varphi_A : I \to \underline{\mathcal{D}}(FA, GA)$ in $\mathcal{V}$, such that the following diagram in $\mathcal{V}$ commutes for all pairs (A, B) of objects in $\mathcal{C}$:

$$\begin{tikzcd}
& I \otimes \underline{\mathcal{C}}(A,B) \arrow[r, "\varphi_A \otimes \underline{F}_{A,B}"] & \underline{\mathcal{D}}(FA,GA) \otimes \underline{\mathcal{D}}(FA,FB) \arrow[dr, "c_{FA,FB,GA}"] & \\
\underline{\mathcal{C}}(A,B) \arrow[ur, "\lambda^{-1}"] \arrow[dr, "\rho^{-1}"'] & & & \underline{\mathcal{D}}(FA,GB) \\
& \underline{\mathcal{C}}(A,B) \otimes I \arrow[r, "\underline{G}_{A,B} \otimes \varphi_B"'] & \underline{\mathcal{D}}(GA,GB) \otimes \underline{\mathcal{D}}(FA,GA) \arrow[ur, "c_{FA,GA,GB}"'] &
\end{tikzcd}$$

REMARK B.2.10. Every $\mathcal{V}$-enriched natural transformation $\varphi : \underline{F} \Rightarrow \underline{G}$ defines an **underlying ordinary natural transformation** $\varphi : F \Rightarrow G$ in the obvious way. Furthermore, there is *at most one* $\mathcal{V}$-enriched natural transformation $\varphi : \underline{F} \Rightarrow \underline{G}$ whose underlying ordinary natural transformation is a given natural transformation $\varphi : F \Rightarrow G$, so being a $\mathcal{V}$-enriched natural transformation is really just a property of an ordinary natural transformation. Henceforth, we will identify $\mathcal{V}$-enriched natural transformations with their underlying ordinary natural transformations; in particular, we will think of φ_A as a morphism $FA \to GA$ in $\mathcal{C}$, not a morphism $I \to \underline{\mathcal{D}}(FA, GA)$ in $\mathcal{V}$.

Definition B.2.11. A **$\mathcal{V}$-enriched natural isomorphism** is a $\mathcal{V}$-enriched natural transformation whose underlying ordinary natural transformation is a natural isomorphism (in the usual sense).

REMARK B.2.12. Let $\mathbf{U}$ be a pre-universe and let $\mathcal{V}$ be a locally $\mathbf{U}$-small monoidal category. With the definitions above, there is an evident (locally $\mathbf{U}$-small) 2-category $\mathfrak{Cat}(\mathcal{V})$ where:

- The objects are the $\mathbf{U}$-small $\mathcal{V}$-enriched categories.
- The morphisms are the $\mathcal{V}$-enriched functors.
- The 2-cells are the $\mathcal{V}$-enriched natural transformations.
- Identities and composition are defined in the obvious way.

This is the **2-category of U-small $\mathcal{V}$-enriched categories**. There is then an evident 2-functor $\mathfrak{Cat}(\mathcal{V}) \to \mathfrak{Cat}$ sending each $\mathcal{V}$-enriched category (resp. functor, natural transformation) to its underlying ordinary category (resp. functor, natural transformation).

Remark B.2.13. It is not hard to verify that a $\mathcal{V}$-enriched natural isomorphism is the same thing as an invertible 2-cell in $\mathfrak{Cat}(\mathcal{V})$.

Lemma B.2.14 (Weak Yoneda lemma)**.** *Let $\mathcal{V}$ be a* right-closed *monoidal category and let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category. For each object A in $\mathcal{C}$ and each $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$, the map $\varphi \mapsto \varphi_A \circ e_A$ is a bijection between the set of $\mathcal{V}$-enriched natural transformations $\varphi : \underline{\mathcal{C}}(A, -) \Rightarrow \underline{F}$ and the set of morphisms $I \to FA$ in $\mathcal{V}$.*

Proof. The proof is similar to that of the classical Yoneda lemma.

First, we show existence. Let $x : I \to FA$ be given. For each object B in $\mathcal{C}$, let $\varphi_C : \underline{\mathcal{C}}(A, B) \to FB$ be the composite

$$\underline{\mathcal{C}}(A, B) \xrightarrow{\rho^{-1}} \underline{\mathcal{C}}(A, B) \otimes I \xrightarrow{\underline{F}_{A,B} \otimes x} \underline{\mathcal{V}}(FA, FB) \otimes FA \xrightarrow{\mathrm{ev}_{FA,FB}} FA$$

and observe that axiom U (plus the definition of e_{FA}) guarantees that $\varphi_A \circ e_A = x$, while axiom M implies $\mathcal{V}$-enriched naturality.

For uniqueness, we note that the morphism

$$c \circ \left(\ulcorner \varphi_B \urcorner \otimes \underline{\mathcal{C}}(A, -)_{A,B}\right) : I \otimes \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, A), FB)$$

where $\ulcorner \varphi_B \urcorner$ denotes the morphism $I \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, B), FB)$ corresponding to $\varphi_B : \underline{\mathcal{C}}(A, B) \to FB$, corresponds under adjunction to the morphism

$$\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \left(\ulcorner \varphi_B \urcorner \otimes \underline{\mathcal{C}}(A, -)_{A,B} \otimes \mathrm{id}\right) : I \otimes \underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(A, A) \to FB$$

which after a computation is seen to be equal to

$$\varphi_B \circ c \circ (\boldsymbol{\lambda} \otimes \mathrm{id}) : I \otimes \underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{C}}(A, A) \to FB$$

but we also have the morphism

$$c \circ \left(\underline{F} \otimes \ulcorner \varphi_A \urcorner\right) : \underline{\mathcal{C}}(A, B) \otimes I \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, A), FB)$$

corresponding under adjunction to

$$\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{ev}) \circ \left(\underline{F} \otimes \ulcorner \varphi_A \urcorner \otimes \mathrm{id}\right) : \underline{\mathcal{C}}(A, B) \otimes I \otimes \underline{\mathcal{C}}(A, A) \to FB$$

which is equal to

$$\mathrm{ev} \circ \left(\underline{F} \otimes \varphi_A\right) \circ (\mathrm{id} \otimes \boldsymbol{\lambda}) : \underline{C}(A, B) \otimes I \otimes \underline{C}(A, A) \to FB$$

and thus $\mathcal{V}$-enriched naturality of φ implies that $\varphi_B : \underline{C}(A, B) \to FB$ must be defined as in the previous paragraph. ■

Definition B.2.15. Let $\mathcal{V}$ be a *symmetric* monoidal category and let $\underline{C}$ and $\underline{D}$ be $\mathcal{V}$-enriched categories. The **tensor product** $\underline{C} \otimes \underline{D}$ is the following $\mathcal{V}$-enriched category:

- The objects in $\underline{C} \otimes \underline{D}$ are pairs (A, D) where A is an object in C and D is an object in $\mathcal{D}$.
- For each pair $((A, D), (B, E))$ of objects in $\underline{C} \otimes \underline{D}$,

$$(\underline{C} \otimes \underline{D})((A, D), (B, E)) = \underline{C}(A, B) \otimes \underline{D}(B, E)$$

- For each object (A, D) in $\underline{C} \otimes \underline{D}$:

$$e_{(A,D)} = \left(e_A \otimes e_D\right) \circ \boldsymbol{\rho}$$

- For each triple $((A, D), (B, E), (C, F))$ of objects in $\underline{C} \otimes \underline{D}$,

$$c_{(C,F),(B,E),(A,D)} = \left(c_{C,B,A} \otimes c_{F,E,D}\right) \circ (\mathrm{id} \otimes \boldsymbol{\gamma} \otimes \mathrm{id})$$

  where we have suppressed the associator of $\mathcal{V}$.

We will often abuse notation and write $C \otimes \mathcal{D}$ for the underlying ordinary category of $\underline{C} \otimes \underline{D}$.

Remark B.2.16. Using the fact that $\mathcal{V}(I, -) : \mathcal{V} \to \mathbf{Set}$ is a lax monoidal functor, it is not hard to see that there is a canonical functor $C \times \mathcal{D} \to C \otimes \mathcal{D}$, which is an isomorphism if $\mathcal{V}$ is a cartesian monoidal category.

Remark B.2.17. If $\mathcal{V}$ is a symmetric monoidal category, then $\mathfrak{Cat}(\mathcal{V})$ is also a symmetric monoidal category where the monoidal product is the tensor product defined above and the monoidal unit is the $\mathcal{V}$-enriched category $\underline{\mathbb{I}}$ with a unique object $*$ and $\underline{\mathbb{I}}(*, *) = I$. Note also that there is a natural bijection between the set of $\mathcal{V}$-enriched functors $\underline{\mathbb{I}} \to \underline{C}$ and the set of objects in C.

Proposition B.2.18. *Let $\mathcal{V}$ be a* symmetric *monoidal category and let $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$, and $\underline{\mathcal{E}}$ be $\mathcal{V}$-enriched categories.*

(i) *Given a $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \to \underline{\mathcal{E}}$, for each object A in $\mathcal{C}$, there is a $\mathcal{V}$-enriched functor $\underline{F}(A, -) : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ defined by the following composite,*

$$\underline{\mathcal{D}} \xrightarrow{\lambda^{-1}} \underline{\mathbb{1}} \otimes \underline{\mathcal{D}} \xrightarrow{\ulcorner A \urcorner \otimes \mathrm{id}} \underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \xrightarrow{\underline{F}} \underline{\mathcal{E}}$$

where $\ulcorner A \urcorner : \underline{\mathbb{1}} \to \underline{\mathcal{C}}$ is the unique $\mathcal{V}$-enriched functor sending $*$ in $\mathbb{1}$ to A in $\mathcal{C}$; and similarly, for each object D in $\mathcal{D}$, there is a $\mathcal{V}$-enriched functor $\underline{F}(-, D) : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ defined by the following composite:

$$\underline{\mathcal{C}} \xrightarrow{\rho^{-1}} \underline{\mathcal{C}} \otimes \underline{\mathbb{1}} \xrightarrow{\mathrm{id} \otimes \ulcorner D \urcorner} \underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \xrightarrow{\underline{F}} \underline{\mathcal{E}}$$

Moreover, the following diagram commutes:

$$\begin{tikzcd}
\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{D}}(D, E) \arrow[r, "\underline{F}(-,E) \otimes \underline{F}(A,-)"] \arrow[dr, "\underline{F}"'] \arrow[dd, "\gamma"'] & \underline{\mathcal{E}}(F(A, E), F(B, E)) \otimes \underline{\mathcal{E}}(F(A, D), F(A, E)) \arrow[d, "c"] \\
 & \underline{\mathcal{E}}(F(A, D), F(B, E)) \\
\underline{\mathcal{D}}(D, E) \otimes \underline{\mathcal{C}}(A, B) \arrow[r, "\underline{F}(B,-) \otimes \underline{F}(-,D)"'] & \underline{\mathcal{E}}(F(B, D), F(B, E)) \otimes \underline{\mathcal{E}}(F(A, D), F(A, E)) \arrow[u, "c"']
\end{tikzcd}$$

(ii) *Conversely, given $\mathcal{V}$-enriched functors $\underline{G}_A : \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ and $\underline{H}_D : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ for all objects A in $\mathcal{C}$ and all objects D in $\mathcal{D}$, if $G_A D = F(A, D) = H_D A$ for all pairs (A, D) and the following diagram commutes for all (A, D) and (B, E),*

$$\begin{tikzcd}
\underline{\mathcal{C}}(A, B) \otimes \underline{\mathcal{D}}(D, E) \arrow[r, "\underline{H}_E \otimes \underline{G}_A"] \arrow[dd, "\gamma"'] & \underline{\mathcal{E}}(F(A, E), F(B, E)) \otimes \underline{\mathcal{E}}(F(A, D), F(A, E)) \arrow[d, "c"] \\
 & \underline{\mathcal{E}}(F(A, D), F(B, E)) \\
\underline{\mathcal{D}}(D, E) \otimes \underline{\mathcal{C}}(A, B) \arrow[r, "\underline{G}_B \otimes \underline{H}_D"'] & \underline{\mathcal{E}}(F(B, D), F(B, E)) \otimes \underline{\mathcal{E}}(F(A, D), F(A, E)) \arrow[u, "c"']
\end{tikzcd}$$

then there is a unique $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ such that $\underline{G}_A = \underline{F}(A, -)$ and $\underline{H}_D = \underline{F}(-, D)$.

Proof. Straightforward. ◆

Corollary B.2.19. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category and let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category. Then there is a (unique) $\mathcal{V}$-enriched functor $\underline{\mathcal{C}}(-, -) : \underline{\mathcal{C}}^{\mathrm{op}} \otimes \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ with $\underline{\mathcal{C}}(A, -) : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ and $\underline{\mathcal{C}}(-, B) : \underline{\mathcal{C}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ as defined previously.*

Proof. This essentially boils down to axiom A. ■

Proposition B.2.20. *Let $\mathcal{V}$ be a* symmetric *monoidal category and let $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$, and $\underline{\mathcal{E}}$ be $\mathcal{V}$-enriched categories. Given two $\mathcal{V}$-enriched functors $\underline{F}, \underline{G} : \underline{\mathcal{C}} \otimes \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ and a natural transformation $\varphi : F \Rightarrow G$, the following are equivalent:*

(i) *φ is (the underlying ordinary natural transformation of) a $\mathcal{V}$-enriched natural transformation $\underline{F} \Rightarrow \underline{G}$.*

(ii) *$\varphi_{A,\bullet}$ is a $\mathcal{V}$-enriched natural transformation $\underline{F}(A, -) \Rightarrow \underline{G}(A, -)$ for each object A in $\mathcal{C}$, and $\varphi_{\bullet,D}$ is a $\mathcal{V}$-enriched natural transformation $\underline{F}(-, D) \Rightarrow \underline{G}(-, D)$ for each object D in $\mathcal{D}$.*

Proof. Straightforward. ◆

Corollary B.2.21. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category and let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{V}$-enriched functor. Then the morphisms*

$$\underline{F}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{C}}(FA, FB)$$

constitute a $\mathcal{V}$-enriched natural transformation $\underline{\mathcal{C}}(-, -) \Rightarrow \underline{\mathcal{D}}(\underline{F}-, \underline{F}-)$.

Proof. By proposition B.2.20 and duality, it suffices to verify that the indicated morphisms constitute a $\mathcal{V}$-enriched natural transformation

$$\underline{\mathcal{C}}(F, -) \Rightarrow \underline{\mathcal{D}}(FA, \underline{F}-)$$

for each object A in $\underline{\mathcal{C}}$; and by adjointness, this boils down to axiom M. ■

Definition B.2.22. Let $\mathcal{V}$ be a *right-closed* monoidal category.

- Let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category. A **representation** of a $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ is pair (A, x), where A is an object in $\mathcal{C}$ and x is a morphism $I \to FA$ in $\mathcal{V}$ such that the corresponding $\mathcal{V}$-enriched natural transformation $\underline{\mathcal{C}}(A, -) \Rightarrow F$ (as described by the weak Yoneda lemma) is invertible.

- A **representable $\mathcal{V}$-enriched functor** is one that admits a representation.

Lemma B.2.23. *Let $\mathcal{V}$ be a* right-closed *monoidal category, let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category, and let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched functor. Given any two representations of $\underline{F}$, say (A, x) and (B, y), there is a unique morphism $f : A \to B$ in $\mathcal{C}$ such that $Ff \circ x = y$ in $\mathcal{V}$, and it is an isomorphism.*

Proof. Let $\varphi : \underline{\mathcal{C}}(A, -) \Rightarrow \underline{F}$ and $\psi : \underline{\mathcal{C}}(B, -) \Rightarrow \underline{F}$ be the $\mathcal{V}$-enriched natural isomorphisms such that $\varphi_A \circ e_A = x$ and $\psi_B \circ e_B = y$; such exist and are unique by the weak Yoneda lemma (B.2.14). Consider $\varphi^{-1} \bullet \psi : \underline{\mathcal{C}}(B, -) \Rightarrow \underline{\mathcal{C}}(A, -)$. The weak Yoneda lemma implies this corresponds to a morphism $\ulcorner f \urcorner : I \to \underline{\mathcal{C}}(A, B)$ in $\mathcal{V}$, namely $\ulcorner f \urcorner = \varphi_B^{-1} \circ y$; but $\ulcorner f \urcorner : I \to \underline{\mathcal{C}}(A, B)$ corresponds to a morphism $f : A \to B$ in $\mathcal{C}$, and the following diagram in $\mathcal{V}$ commutes,

$$
\begin{tikzcd}
I \arrow[r, "e_A"] \arrow[d, equal] & \underline{\mathcal{C}}(A, A) \arrow[r, "\varphi_A"] \arrow[d, "\underline{\mathcal{C}}(A, f)"'] & FA \arrow[d, "Ff"] \\
I \arrow[r, "\ulcorner f \urcorner"'] & \underline{\mathcal{C}}(A, B) \arrow[r, "\varphi_B"'] & FB
\end{tikzcd}
$$

so we deduce that $Ff \circ x = y$, as required. Reversing the argument shows that $f : A \to B$ is the unique such morphism in $\mathcal{C}$, and it follows that $f : A \to B$ must be an isomorphism: its inverse is the unique morphism $g : B \to A$ in $\mathcal{C}$ such that $Fg \circ y = x$. ■

Proposition B.2.24. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category, let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-enriched categories, let $\underline{H} : \underline{\mathcal{C}}^{\mathrm{op}} \otimes \underline{\mathcal{D}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched functor, and for each object A in $\mathcal{C}$ and each object D in $\mathcal{D}$, let $\varphi_{A,D} : \underline{\mathcal{C}}(FA, D) \to H(A, D)$ be an isomorphism in $\mathcal{V}$. If each $\varphi_{A,\bullet}$ is a $\mathcal{V}$-enriched natural isomorphism $\underline{\mathcal{D}}(FA, -) \Rightarrow \underline{H}(A, -)$, then there is a unique $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that φ is a $\mathcal{V}$-enriched natural isomorphism $\underline{\mathcal{D}}(\underline{F}-, -) \Rightarrow \underline{H}$.*

Proof. See §1.10 in [Kelly, 2005]. □

Definition B.2.25. Let $\mathcal{V}$ be a monoidal category. An **$\mathcal{V}$-enriched adjunction** consists of the following data:

- A $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, called the **left adjoint**.
- A $\mathcal{V}$-enriched functor $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$, called the **right adjoint**.
- A $\mathcal{V}$-enriched natural transformation $\eta : \mathrm{id}_{\underline{\mathcal{C}}} \Rightarrow \underline{GF}$, called the **unit**.
- A $\mathcal{V}$-enriched natural transformation $\varepsilon : \underline{FG} \Rightarrow \mathrm{id}_{\underline{\mathcal{D}}}$, called the **counit**.

These are moreover required to satisfy the **triangle identities**:

$$\varepsilon F \bullet F\eta = \mathrm{id}_F \qquad\qquad G\varepsilon \bullet \eta G = \mathrm{id}_G$$

If such data exist, we write

$$\underline{F} \dashv \underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$$

and say that $\underline{F}$ is a left adjoint of $\underline{G}$, and $\underline{G}$ is a right adjoint of $\underline{F}$.

Proposition B.2.26. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category and let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ be $\mathcal{V}$-enriched functors.*

(i) *Given a pair (η, ε) of $\mathcal{V}$-enriched natural transformations*

$$\eta : \mathrm{id}_{\underline{\mathcal{C}}} \Rightarrow \underline{GF} \text{ and } \varepsilon : \underline{FG} \Rightarrow \mathrm{id}_{\underline{\mathcal{D}}}$$

satisfying the triangle identities, the composites

$$\underline{\mathcal{D}}(FA, D) \xrightarrow{\underline{G}_{FA,D}} \underline{\mathcal{C}}(GFA, GD) \xrightarrow{\underline{\mathcal{C}}(\eta_A, GD)} \underline{\mathcal{C}}(A, GD)$$

$$\underline{\mathcal{C}}(A, GD) \xrightarrow{\underline{F}_{A,GD}} \underline{\mathcal{D}}(FA, FGD) \xrightarrow{\underline{\mathcal{D}}(FA, \varepsilon_D)} \underline{\mathcal{D}}(FA, D)$$

constitute a mutually inverse pair of $\mathcal{V}$-enriched natural isomorphisms of the following form:

$$\underline{\mathcal{D}}(\underline{F}-, -) \cong \underline{\mathcal{C}}(-, \underline{G}-)$$

(ii) *Given a mutually inverse pair of $\mathcal{V}$-enriched natural isomorphisms of the form above, say*

$$\varphi : \underline{\mathcal{D}}(\underline{F}-, -) \Rightarrow \underline{\mathcal{C}}(-, \underline{G}-)$$

$$\psi : \underline{C}(-, \underline{G}-) \Rightarrow \underline{D}(\underline{F}-, -)$$

the morphisms $\eta_A : A \to GFA$ *(in* C*) and* $\varepsilon_D : FGD \to D$ *(in* D*) defined (respectively) by*

$$\varphi_{A,FA} \circ e_{FA} : I \to \underline{C}(A, GFA)$$
$$\psi_{GD,D} \circ e_{GD} : I \to \underline{D}(FGD, D)$$

constitute a pair (η, ε) *of* $\mathcal{V}$*-enriched natural transformations satisfying the triangle identities.*

(iii) *Moreover, the two constructions described above are mutually inverse.*

Proof. See the first paragraph of §1.11 in [Kelly, 2005]. □

Corollary B.2.27. *Let* $\mathcal{V}$ *be a* symmetric *monoidal* closed *category. The following are equivalent for a* $\mathcal{V}$*-enriched functor* $\underline{G} : \underline{D} \to \underline{C}$*:*

(i) $\underline{G} : \underline{D} \to \underline{C}$ *admits a* $\mathcal{V}$*-enriched left adjoint.*

(ii) *For each object* A *in* C*, the* $\mathcal{V}$*-enriched functor* $\underline{C}(A, \underline{G}-) : \underline{D} \to \underline{\mathcal{V}}$ *is representable.*

Dually, the following are equivalent for a $\mathcal{V}$*-enriched functor* $\underline{F} : \underline{C} \to \underline{D}$*:*

(i′) $\underline{F} : \underline{C} \to \underline{D}$ *admits a* $\mathcal{V}$*-enriched right adjoint.*

(ii′) *For each object* D *in* D*, the* $\mathcal{V}$*-enriched functor* $\underline{D}(\underline{F}-, D) : \underline{C}^{\mathrm{op}} \to \underline{\mathcal{V}}$ *is representable.*

Proof. Combine propositions B.2.24 and B.2.26. ■

Proposition B.2.28. *Let* $\mathcal{V}$ *be a* symmetric *monoidal* closed *category.*

(i) *There exist a* $\mathcal{V}$*-enriched functor* $\underline{\otimes} : \underline{\mathcal{V}} \otimes \underline{\mathcal{V}} \to \underline{\mathcal{V}}$ *and isomorphisms*

$$\underline{\mathcal{V}}(X \otimes Y, Z) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z))$$

that constitute a $\mathcal{V}$*-enriched natural isomorphism of* $\mathcal{V}$*-enriched functors* $\underline{\mathcal{V}}^{\mathrm{op}} \otimes \underline{\mathcal{V}}^{\mathrm{op}} \otimes \underline{\mathcal{V}} \to \underline{\mathcal{V}}$.

(ii) *In particular, for each object* Y *in* $\mathcal{V}$*, there is a* $\mathcal{V}$*-enriched adjunction of the form below:*

$$(-) \underline{\otimes} Y \dashv \underline{\mathcal{V}}(Y, -) : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$$

(iii) *The isomorphisms $\gamma_{X,Y} : X \otimes Y \to Y \otimes X$ constitute a $\mathcal{V}$-enriched natural transformation of $\mathcal{V}$-enriched functors $\underline{\mathcal{V}} \otimes \underline{\mathcal{V}} \to \underline{\mathcal{V}}$.*

Proof. (i). By proposition B.2.24, it suffices to show that there is a $\mathcal{V}$-enriched natural isomorphism

$$\underline{\mathcal{V}}(X \otimes Y, -) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, -))$$

for each pair (X, Y) of objects in $\mathcal{V}$. First, let us show that there is an ordinary natural transformation of the required form. There are bijections

$$\begin{aligned} \mathcal{V}(W, \underline{\mathcal{V}}(X \otimes Y, Z)) &\cong \mathcal{V}(W \otimes (X \otimes Y), Z) \\ &\cong \mathcal{V}((W \otimes X) \otimes Y, Z) \\ &\cong \mathcal{V}(W \otimes X, \underline{\mathcal{V}}(Y, Z)) \\ &\cong \mathcal{V}(W, \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z))) \end{aligned}$$

that are natural in W, X, Y, and Z, so by the Yoneda lemma, there are isomorphisms

$$\underline{\mathcal{V}}(X \otimes Y, Z) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z))$$

that are natural in X, Y, and Z. For $\mathcal{V}$-enriched naturality, it suffices (by adjointness) to verify that a certain diagram in $\mathcal{V}$ of the form below commutes,

$$\begin{tikzcd} \underline{\mathcal{V}}(Z, W) \otimes \underline{\mathcal{V}}(X \otimes Y, Z) \arrow[r, "c"] \arrow[d, "\cong"'] & \underline{\mathcal{V}}(X \otimes Y, W) \arrow[d, "\cong"] \\ \underline{\mathcal{V}}(Z, W) \otimes \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, Z)) \arrow[r] & \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, W)) \end{tikzcd}$$

but this is straightforward, given the definition of $\underline{\mathcal{V}}(Y, -)_{Z,W}$.

(ii). Apply proposition B.2.26.

(iii). By proposition B.2.20 and adjointness, it suffices to verify the commutativity of certain diagrams in $\mathcal{V}$ of the forms below,

$$\begin{tikzcd} \underline{\mathcal{V}}(X', X) \otimes X' \otimes Y \arrow[r, "\mathrm{ev}_{X',X} \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \gamma_{X',Y}"'] & X \otimes Y \arrow[d, "\gamma_{X,Y}"] \\ \underline{\mathcal{V}}(X', X) \otimes Y \otimes X' \arrow[r] & Y \otimes X \end{tikzcd}$$

$$\begin{tikzcd}
\underline{\mathcal{V}}(Y', Y) \otimes X \otimes Y' \arrow[r] \arrow[d, "\mathrm{id} \otimes \gamma_{X,Y'}"'] & X \otimes Y \arrow[d, "\gamma_{X,Y}"] \\
\underline{\mathcal{V}}(Y', Y) \otimes Y' \otimes X \arrow[r, "\mathrm{ev}_{Y',Y} \otimes \mathrm{id}"'] & Y \otimes X
\end{tikzcd}$$

where as usual we have suppressed the associator of $\mathcal{V}$; but this is again straightforward. ■

Definition B.2.29. Let $\mathcal{V}$ be a monoidal category.

- A **fully faithful $\mathcal{V}$-enriched functor** is a $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that each $\underline{F}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$ is an isomorphism in $\mathcal{V}$.

- A $\mathcal{V}$-enriched functor is **injective on objects** (resp. **essentially surjective on objects**) if its underlying ordinary functor is injective on objects (resp. essentially surjective on objects).

- A **full $\mathcal{V}$-enriched subcategory** of a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ is a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}'$ equipped with a fully faithful $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}}' \to \underline{\mathcal{C}}$ that is injective on objects, called the **inclusion**, such that $FA = A$ for all objects A in $\mathcal{C}'$ and $\underline{F}_{A,B} = \mathrm{id}$ for all pairs (A, B) of objects in $\mathcal{C}'$.

Remark B.2.30. If $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a fully faithful $\mathcal{V}$-enriched functor, then the underlying ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is also fully faithful. The converse is true when $\mathcal{V}(I, -) : \mathcal{V} \to \mathbf{Set}$ is a conservative functor.

Definition B.2.31. Let $\mathcal{V}$ be a monoidal category.

- An **equivalence of $\mathcal{V}$-enriched categories** consists of a pair of $\mathcal{V}$-enriched functors, say $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$, together with $\mathcal{V}$-enriched natural isomorphisms $\mathrm{id}_{\underline{\mathcal{C}}} \cong \underline{GF}$ and $\underline{FG} \cong \mathrm{id}_{\underline{\mathcal{D}}}$.

- Two $\mathcal{V}$-enriched categories are **equivalent** if there is an equivalence of $\mathcal{V}$-enriched categories between them.

Proposition B.2.32. *Let $\mathcal{V}$ be a monoidal category and let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{V}$-enriched functor. The following are equivalent:*

(i) *$\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a fully faithful $\mathcal{V}$-enriched functor and essentially surjective on objects.*

(ii) *$\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ admits a $\mathcal{V}$-enriched left or right adjoint where the unit and counit are $\mathcal{V}$-enriched natural isomorphisms.*

(iii) *There exist $\mathcal{V}$-enriched functors $\underline{L}, \underline{R} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ and $\mathcal{V}$-enriched natural isomorphisms $\mathrm{id}_{\underline{\mathcal{C}}} \cong \underline{LF}$ and $\underline{FR} \cong \mathrm{id}_{\underline{\mathcal{D}}}$.*

Proof. (i) ⇒ (ii). First, choose for every object D in $\mathcal{D}$ an object GD in $\mathcal{C}$ and an isomorphism $\varepsilon_D : FGD \to D$ in $\mathcal{D}$. We may do this because $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects. We then define $\underline{G}_{D,E} : \underline{\mathcal{D}}(D, E) \to \underline{\mathcal{C}}(GD, GE)$ as the following composite,

$$\underline{\mathcal{D}}(D, E) \xrightarrow{\quad\cong\quad} \underline{\mathcal{D}}(FGD, FGE) \xrightarrow{(\underline{F}_{GD,GE})^{-1}} \underline{\mathcal{C}}(GD, GE)$$

where $\underline{\mathcal{D}}(D, E) \to \underline{\mathcal{D}}(FGD, FGE)$ is the isomorphism in $\mathcal{V}$ induced by $\varepsilon_D : FGD \to D$ and $\varepsilon_E^{-1} : E \to FGE$. It is straightforward to see that $\underline{G}$ satisfies axioms U and M, so we have a functor $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$. Moreover, the construction ensures that ε is a $\mathcal{V}$-enriched natural isomorphism $\underline{GF} \Rightarrow \mathrm{id}_{\underline{\mathcal{D}}}$.

Next, we show that there is a $\mathcal{V}$-enriched natural isomorphism $\mathrm{id}_{\underline{\mathcal{C}}} \Rightarrow \underline{GF}$. Let $\eta_A : A \to GFA$ be the unique morphism in $\mathcal{C}$ such that $F\eta_A = \varepsilon_{FA}^{-1}$. We know that $\varepsilon^{-1}F$ is a $\mathcal{V}$-enriched natural isomorphism $\underline{F} \Rightarrow \underline{FGF}$, so it follows that η is a $\mathcal{V}$-enriched natural transformation $\mathrm{id}_{\underline{\mathcal{C}}} \Rightarrow \underline{GF}$. Moreover, by construction, we have the left triangle identity $\varepsilon F \bullet F\eta = \mathrm{id}_F$, and the right triangle identity $G\varepsilon \bullet \eta G = \mathrm{id}_G$ is then a formal consequence. Thus, $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ is a $\mathcal{V}$-enriched right adjoint for $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ where the unit and counit are $\mathcal{V}$-enriched natural isomorphisms.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Observe that the $\mathcal{V}$-enriched natural isomorphism $\mathrm{id}_{\underline{\mathcal{C}}} \cong \underline{LF}$ gives us a retraction for $\underline{F}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$, namely,

$$\underline{\mathcal{D}}(FA, FB) \xrightarrow{\underline{L}_{FA,FB}} \underline{\mathcal{C}}(LFA, LFB) \xrightarrow{\quad\cong\quad} \underline{\mathcal{C}}(A, B)$$

where $\underline{\mathcal{C}}(LFA, LFB) \to \underline{\mathcal{C}}(A, B)$ is the isomorphism in $\mathcal{V}$ induced by $\mathrm{id}_{\underline{\mathcal{C}}} \cong \underline{LF}$. The same construction applied to the $\mathcal{V}$-enriched natural isomorphism $\underline{FR} \cong \mathrm{id}_{\underline{\mathcal{D}}}$ yields a section for $\underline{F}_{A,B} : \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{D}}(FA, FB)$. Thus, we may deduce that $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is fully faithful. Moreover, the existence of a natural isomorphism $FR \cong \mathrm{id}_{\mathcal{D}}$ certainly implies that $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects, so we are done. ■

B.3 Enriched diagrams

Prerequisites. §§B.2

¶ B.3.1. Throughout this section, $\mathcal{V}$ is a locally small symmetric monoidal closed category with limits for all small diagrams.

Definition B.3.2. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-enriched categories and let $\underline{F}, \underline{G} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be $\mathcal{V}$-enriched functors. The **object of $\mathcal{V}$-enriched natural transformations** $\underline{F} \Rightarrow \underline{G}$ consists of the following data:

- An object $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ in $\mathcal{V}$.
- For each object C in $\mathcal{C}$, a morphism $\pi_C : [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \to \underline{\mathcal{D}}(FC, GC)$, such that the following equation in $\mathcal{V}$ is satisfied for every pair (A, B) of objects in $\mathcal{C}$:

$$c_{GB,GA,FA} \circ \left(\underline{G}_{A,B} \otimes \pi_A\right) \circ \boldsymbol{\gamma}_{[\underline{\mathcal{C}},\underline{\mathcal{D}}](\underline{F},\underline{G}),\underline{\mathcal{C}}(A,B)} = c_{GB,FB,FA} \circ \left(\pi_B \otimes \underline{F}_{A,B}\right)$$

Moreover, the morphisms $\pi_C : [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \to \underline{\mathcal{D}}(FC, GC)$ are required to be universal, i.e. for each object X in $\mathcal{C}$, the map $\alpha \mapsto \left(\pi_C \circ \alpha \,\middle|\, C \in \operatorname{ob} \mathcal{C}\right)$ is a bijection between the set of morphisms $X \to [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ and the ensemble of families of morphisms $\varphi_C : X \to \underline{\mathcal{D}}(FC, GC)$ in $\mathcal{V}$ satisfying the equations

$$c_{GB,GA,FA} \circ \left(\underline{G}_{A,B} \otimes \varphi_A\right) \circ \boldsymbol{\gamma}_{X,\underline{\mathcal{C}}(A,B)} = c_{GB,FB,FA} \circ \left(\varphi_B \otimes \underline{F}_{A,B}\right)$$

for every pair (A, B) of objects in $\mathcal{C}$.

Remark B.3.3. Assuming $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ exists, the universality condition implies that there is a bijection between the set of morphisms $I \to [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ and the ensemble of $\mathcal{V}$-enriched natural transformations $\underline{F} \Rightarrow \underline{G}$. In particular, the existence of $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ implies that there are not "too many" $\mathcal{V}$-enriched natural transformations $\underline{F} \Rightarrow \underline{G}$.

Remark B.3.4. By adjointness, it is not hard to see that $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ is the limit of the following diagram in $\mathcal{V}$:

- For each object C in $\mathcal{C}$, there is a vertex with value $\underline{\mathcal{D}}(FC, GC)$.
- For each pair (A, B) of objects in $\mathcal{C}$, we have a vertex and two arrows with values as in the diagram below,

$$\underline{\mathcal{D}}(FA, GA) \longrightarrow \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, B), \underline{\mathcal{D}}(FA, GB)) \longleftarrow \underline{\mathcal{D}}(FB, GB)$$

where $\underline{\mathcal{D}}(FA, GA) \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, B), \underline{\mathcal{D}}(FA, GB))$ is the right adjoint transpose of

$$c_{GB,GA,FA} \circ \left(\underline{G}_{A,B} \otimes \mathrm{id}_{\underline{\mathcal{D}}(FA,GA)}\right) \circ \gamma_{\underline{\mathcal{D}}(FB,GB),\underline{\mathcal{C}}(A,B)}$$

and $\underline{\mathcal{D}}(FB, GB) \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(A, B), \underline{\mathcal{D}}(FA, GB))$ is the right adjoint tranpose of

$$c_{GB,FB,FA} \circ \left(\mathrm{id}_{\underline{\mathcal{D}}(FB,GB)} \otimes \underline{F}_{A,B}\right)$$

In particular, this diagram is small if $\underline{\mathcal{C}}$ is, so $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ exists whenever $\underline{\mathcal{C}}$ is a small $\mathcal{V}$-enriched category.

More generally, if $\underline{\mathcal{C}}'$ is a small full $\mathcal{V}$-enriched subcategory of $\underline{\mathcal{C}}$ such that the inclusion $\underline{\mathcal{C}}' \hookrightarrow \underline{\mathcal{C}}$ is essentially surjective on objects, then $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ exists and is naturally isomorphic to $\left[\underline{\mathcal{C}}', \underline{\mathcal{D}}\right](\underline{F}, \underline{G})$.

Definition B.3.5. Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-enriched categories. Assuming $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ exists for all $\mathcal{V}$-enriched functors $\underline{F}, \underline{G} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, the **$\mathcal{V}$-enriched functor category** $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ is defined as follows:

- The objects are $\mathcal{V}$-enriched functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$.
- For each pair $(\underline{F}, \underline{G})$ of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$, $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$ is as defined previously.
- For each $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, $e_{\underline{F}} : I \to [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{F})$ is the unique morphism in $\mathcal{V}$ making the following diagram in $\mathcal{V}$ commute for all objects C in $\mathcal{C}$:

$$\begin{tikzcd} I \arrow[r, "e_{\underline{F}}"] \arrow[d, equal] & {[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{F})} \arrow[d, "\pi_C"] \\ I \arrow[r, "e_{FC}"'] & \underline{\mathcal{D}}(FC, FC) \end{tikzcd}$$

- For each triple $(\underline{F}, \underline{G}, \underline{H})$ of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$,

$$c_{\underline{H},\underline{G},\underline{F}} : [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{G}, \underline{H}) \otimes [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \to [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{H})$$

is the unique morphism making the following diagram in $\mathcal{V}$ commute for all objects C in $\mathcal{C}$:

$$\begin{tikzcd} {[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{G}, \underline{H}) \otimes [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})} \arrow[r, "c_{\underline{H},\underline{G},\underline{F}}"] \arrow[d, "\pi_C \otimes \pi_C"'] & {[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{H})} \arrow[d, "\pi_C"] \\ \underline{\mathcal{D}}(GC, HC) \otimes \underline{\mathcal{D}}(FC, GC) \arrow[r, "c_{HC,GC,FC}"'] & \underline{\mathcal{D}}(FC, HC) \end{tikzcd}$$

Remark B.3.6. By remark B.3.3, the underlying ordinary category of $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ is the category whose objects are the $\mathcal{V}$-enriched functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ and whose morphisms are the $\mathcal{V}$-enriched natural transformations. To avoid confusion, we write $\mathbf{Fun}_{\mathcal{V}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ for the ordinary category of $\mathcal{V}$-enriched functors, and we reserve $[\mathcal{C}, \mathcal{D}]$ for the ordinary category of ordinary functors. Note that $\mathbf{Fun}_{\mathcal{V}}(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ *always* exists, even when $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ does not.

Theorem B.3.7. *Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-enriched categories such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ exists.*

(i) *For each object C in $\underline{\mathcal{C}}$, there is a $\mathcal{V}$-enriched functor $\underline{C^*} : [\underline{\mathcal{C}}, \underline{\mathcal{D}}] \to \underline{\mathcal{D}}$ where $C^* \underline{F} = FC$ and $\underline{C^*}_{\underline{F},\underline{G}} = \pi_C : [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \to \underline{\mathcal{D}}(FC, GC)$.*

(ii) *There is a (unique) $\mathcal{V}$-enriched functor $\mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}} : [\underline{\mathcal{C}}, \underline{\mathcal{D}}] \otimes \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that $\mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}}(-, C) = \underline{C^*}$ and $\mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}}(\underline{F}, -) = \underline{F}$.*

(iii) *For each $\mathcal{V}$-enriched category $\underline{\mathcal{A}}$, there is an isomorphism*

$$\mathbf{Fun}_{\mathcal{V}}(\underline{\mathcal{A}} \otimes \underline{\mathcal{C}}, \underline{\mathcal{D}}) \cong \mathbf{Fun}_{\mathcal{V}}(\underline{\mathcal{A}}, [\underline{\mathcal{C}}, \underline{\mathcal{D}}])$$

that is 2-natural in $\underline{\mathcal{A}}$ and sends $\mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}} : [\underline{\mathcal{C}}, \underline{\mathcal{D}}] \otimes \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ to $\mathrm{id} : [\underline{\mathcal{C}}, \underline{\mathcal{D}}] \to [\underline{\mathcal{C}}, \underline{\mathcal{D}}]$.

Proof. (i). The definition of e and c in $[\underline{\mathcal{C}}, \underline{\mathcal{D}}]$ ensures that the announced definition satisfies axioms U and M, respectively.

(ii). By proposition B.2.18, it suffices to verify that the following diagram commutes,

$$\begin{tikzcd}
{[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \otimes \underline{\mathcal{C}}(A, B)} \arrow[r, "\pi_B \otimes \underline{F}_{A,B}"] \arrow[dd, "\gamma"'] & \underline{\mathcal{D}}(FB, GB) \otimes \underline{\mathcal{D}}(FA, FB) \arrow[d, "c_{GB,FB,FA}"] \\
 & \underline{\mathcal{D}}(FA, GB) \\
\underline{\mathcal{C}}(A, B) \otimes [\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G}) \arrow[r, "\underline{G}_{A,B} \otimes \pi_A"'] & \underline{\mathcal{D}}(GA, GB) \otimes \underline{\mathcal{D}}(FA, GA) \arrow[u, "c_{GB,FA,FA}"']
\end{tikzcd}$$

but this is guaranteed by the definition of $[\underline{\mathcal{C}}, \underline{\mathcal{D}}](\underline{F}, \underline{G})$.

(iii). See §2.3 in [Kelly, 2005]. ☐

Corollary B.3.8. *The 2-category of* small $\mathcal{V}$*-enriched categories is a symmetric monoidal closed 2-category.* ■

Proposition B.3.9 (Strong Yoneda lemma)**.** *Let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category, let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched functor, and let C be an object in $\mathcal{C}$. Then the object of $\mathcal{V}$-enriched natural transformations $\underline{\mathcal{C}}(C, -) \Rightarrow \underline{F}$ exists: it can be identified with FC, with $\pi_A : FC \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(C, A), FA)$ defined to be the right adjoint transpose of $\underline{F}_{C,A} : \underline{\mathcal{C}}(C, A) \to \underline{\mathcal{V}}(FC, FA)$.*

Proof. Let X be any object in $\mathcal{V}$. It is a straightforward exercise in adjointness to see that there is a natural bijection between the ensemble of $\mathcal{V}$-enriched natural transformations $\underline{\mathcal{C}}(C, -) \Rightarrow \underline{\mathcal{V}}(X, \underline{F}-)$ and the ensemble of families of morphisms $\varphi_A : X \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(C, A), FA)$ satisfying the equations

$$c_{FB,FA,\underline{\mathcal{C}}(C,A)} \circ \left(\underline{F}_{A,B} \otimes \varphi_A\right) \circ \boldsymbol{\gamma} = c_{FB,\underline{\mathcal{C}}(C,B),\underline{\mathcal{C}}(C,A)} \circ \left(\varphi_B \otimes \underline{\mathcal{C}}(C, -)_{A,B}\right)$$

for every pair (A, B) of objects in $\mathcal{C}$. Thus, by the weak Yoneda lemma (B.2.14), the latter can be identified with the set of morphisms $I \to \underline{\mathcal{V}}(X, FC)$ in $\mathcal{V}$, and hence, with the set of morphisms $X \to FC$ in $\mathcal{V}$. By tracing the various bijections, one sees that the family $\pi_A : FC \to \underline{\mathcal{V}}(\underline{\mathcal{C}}(C, A), FA)$ announced above corresponds to $\mathrm{id} : FC \to FC$, and this completes the proof. ■

Corollary B.3.10. *Let $\underline{\mathcal{C}}$ be a small $\mathcal{V}$-enriched category and let $\underline{h}$ be the $\mathcal{V}$-enriched functor $\underline{\mathcal{C}} \to [\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ defined by $\underline{h}_C = \underline{\mathcal{C}}(C, -)$. Then there is a $\mathcal{V}$-enriched natural isomorphism of the form below:*

$$[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{V}}]\left(\underline{h}_\bullet, -\right) \Rightarrow \mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}}(-, \bullet)$$

Proof. The strong Yoneda lemma (proposition B.3.9) tells us that there are isomorphisms

$$[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{V}}]\left(\underline{h}_C, \underline{F}\right) \to FC = \mathrm{ev}_{\underline{\mathcal{C}},\underline{\mathcal{D}}}(\underline{F}, C)$$

for each object C in $\mathcal{C}$ and each $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$; it remains to be shown that these constitute $\mathcal{V}$-enriched natural transformation, and by proposition B.2.20, it suffices to verify $\mathcal{V}$-enriched naturality in C and in $\underline{F}$ separately. But since the structure of $[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ is defined entirely in terms of the universal property of objects of natural transformations, we may as well take $[\underline{\mathcal{C}}^{\mathrm{op}}, \underline{\mathcal{V}}]\left(\underline{h}_C, \underline{F}\right) = FC$; then $\mathcal{V}$-enriched naturality is clear. ■

Corollary B.3.11 (Enriched Yoneda embedding)**.** *Let $\underline{C}$ be a small $\mathcal{V}$-enriched category and let $\underline{\hbar}$ be the $\mathcal{V}$-enriched functor $\underline{C} \to [\underline{C}^{\mathrm{op}}, \underline{\mathcal{V}}]$ defined by $\underline{\hbar}_C = \underline{C}(C, -)$. Then $\underline{\hbar} : \underline{C} \to [\underline{C}^{\mathrm{op}}, \underline{\mathcal{V}}]$ is fully faithful and essentially surjective onto the full $\mathcal{V}$-enriched subcategory spanned by the $\mathcal{V}$-enriched representable functors $\underline{C}^{\mathrm{op}} \to \underline{\mathcal{V}}$.*

Proof. It suffices to prove the following claim: for all pairs (A, B) of objects in $\mathcal{C}$, the unique morphism $\underline{C}(A, B) \to [\underline{C}^{\mathrm{op}}, \underline{\mathcal{V}}]\big(\underline{\hbar}_A, \underline{\hbar}_B\big)$ making the following diagrams in $\mathcal{V}$ commute

$$\begin{tikzcd} \underline{C}(A, B) \arrow[r, dashed] \arrow[d, equal] & {[\mathcal{C}^{\mathrm{op}}, \mathcal{V}]\big(\underline{\hbar}_A, \underline{\hbar}_B\big)} \arrow[d, "\pi_C"] \\ \underline{C}(A, B) \arrow[r, "\underline{C}(C,-)_{A,B}"'] & \underline{\mathcal{V}}(\underline{C}(C, A), \underline{C}(C, B)) \end{tikzcd}$$

for all objects C in $\mathcal{C}$ is an isomorphism in $\mathcal{V}$. But by the strong Yoneda lemma (proposition B.3.9), we may as well take $[\underline{C}^{\mathrm{op}}, \underline{\mathcal{V}}]\big(\underline{\hbar}_A, \underline{\hbar}_B\big) = \underline{C}(A, B)$ and $\pi_C = \underline{C}(C, -)_{A,B}$, so that the morphism in question is $\mathrm{id} : \underline{C}(A, B) \to \underline{C}(A, B)$. This proves the claim. ■

Definition B.3.12. Let $\underline{C}$ and $\underline{\mathcal{J}}$ be $\mathcal{V}$-enriched categories.

- Let $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ and $\underline{F} : \underline{\mathcal{J}} \to \underline{C}$ be $\mathcal{V}$-enriched functors, and assume that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists. A **$\underline{W}$-weighted limit** for $\underline{F}$ is a pair $\big(\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}, \lambda\big)$ where $\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}$ is an object in $\mathcal{C}$ and $\lambda : \underline{W} \Rightarrow \underline{C}\big(\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}, \underline{F}\big)$ is a $\mathcal{V}$-enriched natural transformation such that $\big(\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}, \ulcorner\lambda\urcorner\big)$ is a representation for the following $\mathcal{V}$-enriched functor,

$$[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{C}(-, \underline{F})) : \underline{C}^{\mathrm{op}} \to \underline{\mathcal{V}}$$

  where $\ulcorner\lambda\urcorner : I \to [\underline{\mathcal{J}}, \underline{\mathcal{V}}]\big(\underline{W}, \underline{C}\big(\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}, \underline{F}\big)\big)$ is the morphism in $\mathcal{V}$ corresponding to λ. We refer to $\underline{W}$ as the **weight** and $\underline{F}$ as the **diagram**.

- Let $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ and $\underline{F} : \underline{\mathcal{J}} \to \underline{C}$ be $\mathcal{V}$-enriched functors, and assume that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ exists. A **$\underline{W}$-weighted colimit** for $\underline{F}$ is a pair $\big(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, \lambda\big)$ where $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$ is an object in $\mathcal{C}$ and $\lambda : \underline{W} \Rightarrow \underline{C}\big(\underline{F}, \underline{W} \star_{\underline{\mathcal{J}}} \underline{F}\big)$ is a $\mathcal{V}$-enriched natural transformation such that $\big(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, \ulcorner\lambda\urcorner\big)$ is a representation for the following $\mathcal{V}$-enriched functor,

$$[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{C}(\underline{F}, -)) : \underline{C}^{\mathrm{op}} \to \underline{\mathcal{V}}$$

where $\ulcorner\lambda\urcorner : I \to [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]\big(\underline{W}, \underline{\mathcal{C}}\big(\underline{F}, \underline{W} \star_{\underline{\mathcal{J}}} \underline{F}\big)\big)$ is the morphism in $\mathcal{V}$ corresponding to λ. We refer to $\underline{W}$ as the **weight** and $\underline{F}$ as the **diagram**.

Remark B.3.13. By lemma B.2.23, weighted limits/colimits for a given weight and diagram are unique up to unique isomorphism.

Definition B.3.14. Let $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be a $\mathcal{V}$-enriched functor and let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category.

- Let $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ and $\underline{C} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$ be $\mathcal{V}$-enriched functors, and assume that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists. We say $\underline{F}$ **preserves $\underline{W}$-weighted limits for $\underline{C}$** if, for every $\underline{W}$-weighted limit for $\underline{C}$, say (L, λ), the pair $\big(FL, \underline{F}_*\lambda\big)$ is a $\underline{W}$-weighted limit for $\underline{FC}$, where

$$\underline{F}_*\lambda : \underline{W} \Rightarrow \underline{\mathcal{D}}(FL, \underline{FC})$$

  is the $\mathcal{V}$-enriched natural transformation obtained by vertically composing $\lambda : \underline{W} \Rightarrow \underline{\mathcal{C}}(L, \underline{C})$ and the $\mathcal{V}$-enriched natural transformation $\underline{\mathcal{C}}(L, \underline{C}) \Rightarrow \underline{\mathcal{D}}(FL, \underline{FC})$ induced by $\underline{F} : \underline{\mathcal{C}}(-,-) \Rightarrow \underline{\mathcal{D}}(\underline{F}-, \underline{F}-)$.

- Let $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ and $\underline{C} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$ be $\mathcal{V}$-enriched functors, and assume that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ exists. We say $\underline{F}$ **preserves $\underline{W}$-weighted colimits for $\underline{C}$** if, for every $\underline{W}$-weighted colimit for $\underline{C}$, say (L, λ), the pair $\big(FL, \underline{F}_*\lambda\big)$ is a $\underline{W}$-weighted colimit for $\underline{FC}$, where

$$\underline{F}_*\lambda : \underline{W} \Rightarrow \underline{\mathcal{D}}(\underline{FC}, FL)$$

  is the $\mathcal{V}$-enriched natural transformation obtained by vertically composing $\lambda : \underline{W} \Rightarrow \underline{\mathcal{C}}(\underline{C}, L)$ and the $\mathcal{V}$-enriched natural transformation $\underline{\mathcal{C}}(\underline{C}, L) \Rightarrow \underline{\mathcal{D}}(\underline{FC}, FL)$ induced by $\underline{F} : \underline{\mathcal{C}}(-,-) \Rightarrow \underline{\mathcal{D}}(\underline{F}-, \underline{F}-)$.

Proposition B.3.15. *Let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category and let*

$$\underline{F} \dashv \underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$$

be a $\mathcal{V}$-enriched adjunction.

- *For any $\mathcal{V}$-enriched weight $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$, assuming the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists, $\underline{G} : \underline{\mathcal{D}} \to \underline{\mathcal{C}}$ preserves $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{D}}$.*

- *For any $\mathcal{V}$-enriched weight $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$, assuming the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ exists, $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ preserves $\underline{W}$-weighted colimits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $\underline{D} : \underline{\mathcal{J}} \to \underline{\mathcal{D}}$ be a $\mathcal{V}$-enriched diagram and suppose $\underline{\mathcal{D}}$ has a $\underline{W}$-weighted limit for $\underline{D}$. Proposition B.2.26 implies we have the following $\mathcal{V}$-enriched natural isomorphisms:

$$\underline{\mathcal{D}}\big(\underline{F}-, \{\underline{W}, \underline{D}\}^{\underline{\mathcal{J}}}\big) \cong \underline{\mathcal{C}}\big(-, G\{\underline{W}, \underline{D}\}^{\underline{\mathcal{J}}}\big)$$
$$[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{D}}(\underline{F}-, \underline{D})) \cong [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{C}}(-, \underline{GD}))$$

Thus, $G\{\underline{W}, \underline{D}\}^{\underline{\mathcal{J}}}$ is (the object part of) a $\underline{W}$-weighted limit for $\underline{GD} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$. To complete the proof, we must verify that the universal $\mathcal{V}$-enriched natural transformation $\underline{W} \Rightarrow \underline{\mathcal{D}}\big(\{\underline{W}, \underline{D}\}^{\underline{\mathcal{J}}}, \underline{D}\big)$ is sent to a universal $\mathcal{V}$-enriched natural transformation $\underline{W} \Rightarrow \underline{\mathcal{C}}\big(G\{\underline{W}, \underline{D}\}^{\underline{\mathcal{J}}}, \underline{GD}\big)$, but this is a straightforward application of the right triangle identity. ■

Proposition B.3.16. *Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{J}}$ be $\mathcal{V}$-enriched categories. Assuming the $\mathcal{V}$-enriched functor categories $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ and $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ exist:*

(i) *Let $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched weight. If $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$ exist, then there exist a $\mathcal{V}$-enriched functor $\{\underline{W}, -\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ and isomorphisms in $\mathcal{V}$*

$$\underline{\mathcal{C}}\big(C, \{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}\big) \cong [\underline{\mathcal{J}}, \mathcal{V}](\underline{W}, \underline{\mathcal{C}}(C, \underline{F}))$$

that constitute a $\mathcal{V}$-enriched natural isomorphism of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}}^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{V}}$.

(ii) *If the above condition holds for all $\mathcal{V}$-enriched weights $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$, then there is a $\mathcal{V}$-enriched functor $\{-, -\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ making the above isomorphisms a $\mathcal{V}$-enriched natural isomorphism of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}}^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{V}}$.*

(iii) *In particular, when $\{-, -\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ exists, for each $\mathcal{V}$-enriched diagram $\underline{F} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$, the there is a $\mathcal{V}$-enriched adjunction of the form below:*

$$\underline{\mathcal{C}}(-, \underline{F}) \dashv \{-, \underline{F}\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \to \underline{\mathcal{C}}$$

Dually, assuming the $\mathcal{V}$-enriched functor categories $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ and $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ exist:

(i′) *Let $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched weight. If $\underline{W}$-weighted colimits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$ exist, then there exist a $\mathcal{V}$-enriched functor $\underline{W} \star_{\underline{\mathcal{J}}} (-) : [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ and isomorphisms in $\mathcal{V}$*

$$\underline{\mathcal{C}}\big(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, C\big) \cong [\underline{\mathcal{J}}^{\mathrm{op}}, \mathcal{V}](\underline{W}, \underline{\mathcal{C}}(\underline{F}, C))$$

that constitute a $\mathcal{V}$-enriched natural isomorphism of $\mathcal{V}$-enriched functors $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]^{\mathrm{op}} \otimes \underline{\mathcal{C}} \to \underline{\mathcal{V}}$.

(ii′) *If the above condition holds for all $\mathcal{V}$-enriched weights $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$, then there is a $\mathcal{V}$-enriched functor $(-) \star_{\underline{\mathcal{J}}} (-) : [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}] \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ making the above isomorphisms a $\mathcal{V}$-enriched natural isomorphism of $\mathcal{V}$-enriched functors $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}]^{\mathrm{op}} \otimes \underline{\mathcal{C}} \to \underline{\mathcal{V}}$.*

(iii′) *In particular, when $(-) \star_{\underline{\mathcal{J}}} (-) : [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{C}}] \to \underline{\mathcal{C}}$ exists, for each $\mathcal{V}$-enriched diagram $\underline{F} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$, the there is a $\mathcal{V}$-enriched adjunction of the form below:*

$$- \star_{\underline{\mathcal{J}}} \underline{F} \dashv \underline{\mathcal{C}}(\underline{F}, -) : [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}] \to \underline{\mathcal{C}}$$

Proof. (i) and (ii). Apply proposition B.2.24.

(iii). Apply proposition B.2.26. ∎

Definition B.3.17.

- A **complete $\mathcal{V}$-enriched category** is a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ such that, for all small $\mathcal{V}$-enriched categories $\underline{\mathcal{J}}$, weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$ and all $\mathcal{V}$-enriched weights $\underline{\mathcal{J}} \to \underline{\mathcal{V}}$ exist in $\underline{\mathcal{C}}$.

- A **cocomplete $\mathcal{V}$-enriched category** is a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ such that, for all small $\mathcal{V}$-enriched categories $\underline{\mathcal{J}}$, weighted colimits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$ and all $\mathcal{V}$-enriched weights $\underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ exist in $\underline{\mathcal{C}}$.

Theorem B.3.18. *Let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists.*

(i) *For all $\mathcal{V}$-enriched functors $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ and $\underline{F} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$, there is a $\mathcal{V}$-enriched natural isomorphism of the form below:*

$$[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(-, \underline{F})) \cong \underline{\mathcal{V}}(-, [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{F}))$$

In particular, the weighted limit $\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}}$ exists in $\underline{\mathcal{V}}$.

(ii) *The above extends to a $\mathcal{V}$-enriched natural isomorphism $[\underline{\mathcal{J}}, \underline{\mathcal{V}}](-, -) \cong \{-, -\}^{\underline{\mathcal{J}}}$.*

(iii) *For each $\mathcal{V}$-enriched weight $\underline{W} : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$, $\{\underline{W}, -\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{V}}] \to \underline{\mathcal{V}}$ has a $\mathcal{V}$-enriched left adjoint, namely the $\mathcal{V}$-enriched functor $\underline{\mathcal{V}} \to [\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ that sends an object X in $\mathcal{V}$ to the $\mathcal{V}$-enriched diagram $\underline{W \otimes} X : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$.*

Proof. (i). First, we must establish that there is an ordinary natural isomorphism of the required form. Let X be any object in $\mathcal{V}$. By definition, $\underline{\mathcal{V}}(X, -) : \mathcal{V} \to \mathcal{V}$ is a right adjoint, so it preserves all limits; but remark B.3.4 says that objects of natural transformations are certain limits, and by adjointness, the (ordinary) natural isomorphisms

$$\underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Y, -)) \cong \underline{\mathcal{V}}(Y, \underline{\mathcal{V}}(X, -))$$

induced by $\boldsymbol{\gamma}$ make the following diagram in $\mathcal{V}$ commute,

$$\begin{tikzcd}
\underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Wj, Fj)) \arrow[r] \arrow[d, "\cong"'] & \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), \underline{\mathcal{V}}(Wj, Fk))) \arrow[d, "\cong"] & \underline{\mathcal{V}}(X, \underline{\mathcal{V}}(Wk, Fk)) \arrow[l] \arrow[d, "\cong"] \\
\underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(X, Fj)) \arrow[r] & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), \underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(X, Fk))) & \underline{\mathcal{V}}(Wk, \underline{\mathcal{V}}(X, Fk)) \arrow[l]
\end{tikzcd}$$

so we indeed have an ordinary natural isomorphism

$$\underline{\mathcal{V}}(-, [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{F})) \cong [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(-, \underline{F}))$$

as required. For $\mathcal{V}$-enriched naturality, it suffices (by adjointness) to verify that a certain diagram in $\mathcal{V}$ of the form below commutes:

$$\begin{tikzcd}
\underline{\mathcal{V}}(Y, [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{F})) \otimes \underline{\mathcal{V}}(X, Y) \arrow[r, "c"] \arrow[d, "\cong"'] & \underline{\mathcal{V}}(X, [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{F})) \arrow[d, "\cong"] \\
[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(Y, \underline{F})) \otimes \underline{\mathcal{V}}(X, Y) \arrow[r] & [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(X, \underline{F}))
\end{tikzcd}$$

But the universal property of $[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(X, \underline{F}))$ implies it is enough to check that the above equation is satisfied after composing with every projection π_j : $[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(X, \underline{F})) \to \underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(X, Fj))$, and this is straightforward.

(ii). Since we may take $\{\underline{W}, \underline{F}\}^{\underline{\mathcal{J}}} = [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{F})$, it suffices to prove that the isomorphisms constructed above already define a $\mathcal{V}$-enriched natural transformation of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}}^{\mathrm{op}} \otimes [\underline{\mathcal{J}}, \underline{\mathcal{V}}]^{\mathrm{op}} \otimes [\underline{\mathcal{V}}, \underline{\mathcal{V}}] \to \underline{\mathcal{V}}$; for this, similar arguments work.

(iii). By proposition B.2.28 and theorem B.3.7, there is a $\mathcal{V}$-enriched natural isomorphism

$$\underline{\mathcal{V}}(Wj \otimes (-), \underline{j^*}(-)) \cong \underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(-, \underline{j^*}(-)))$$

and it is straightforward to see that these yield a $\mathcal{V}$-enriched natural isomorphism

$$[\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W} \underline{\otimes} (-), -) \cong [\underline{\mathcal{J}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(-, -))$$

so by proposition B.2.26, we have a $\mathcal{V}$-enriched adjunction

$$\underline{W} \underline{\otimes} (-) \dashv \{\underline{W}, -\}^{\underline{\mathcal{J}}} : [\underline{\mathcal{J}}, \underline{\mathcal{V}}] \to \underline{\mathcal{V}}$$

as required. ■

Theorem B.3.19. *If $\mathcal{V}$ has colimits for all small diagrams, then for any small $\mathcal{V}$-enriched category $\underline{\mathcal{J}}$:*

(i) *For all $\mathcal{V}$-enriched weights $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ and all $\mathcal{V}$-enriched diagrams $\underline{F} : \underline{\mathcal{V}} \to \underline{\mathcal{V}}$, the weighted colimit $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$ exists in $\underline{\mathcal{V}}$, and there is a $\mathcal{V}$-enriched natural isomorphism of the form below:*

$$\underline{\mathcal{V}}\big(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, -\big) \cong \{\underline{W}, \underline{\mathcal{V}}(\underline{F}, -)\}^{\underline{\mathcal{J}}^{\mathrm{op}}}$$

(ii) *There is a $\mathcal{V}$-enriched functor $(-) \star_{\underline{\mathcal{J}}} (-) : [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}] \otimes [\underline{\mathcal{J}}, \underline{\mathcal{V}}] \to \underline{\mathcal{V}}$ making the above a $\mathcal{V}$-enriched natural isomorphism of $\mathcal{V}$-enriched functors $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}] \otimes [\underline{\mathcal{J}}, \underline{\mathcal{V}}] \otimes \underline{\mathcal{V}} \to \underline{\mathcal{V}}$.*

(iii) *For each $\mathcal{V}$-enriched weight $\underline{W} : \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$, $\underline{W} \star_{\underline{\mathcal{J}}} (-) : [\underline{\mathcal{J}}, \underline{\mathcal{V}}] \to \underline{\mathcal{V}}$ has a $\mathcal{V}$-enriched right adjoint, namely the $\mathcal{V}$-enriched functor $\underline{\mathcal{V}} \to [\underline{\mathcal{J}}, \underline{\mathcal{V}}]$ that sends an object X in $\mathcal{V}$ to the $\mathcal{V}$-enriched diagram $\underline{\mathcal{V}}(\underline{W}, X) : \underline{\mathcal{J}} \to \underline{\mathcal{V}}$.*

Proof. (i). Recall that a weighted colimit $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$ is (the same thing as) an object equipped with a $\mathcal{V}$-enriched natural isomorphism of the form below:

$$\underline{\mathcal{V}}\left(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, -\right) \cong [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, -))$$

But by theorem B.3.18, we have a $\mathcal{V}$-enriched natural isomorphism

$$[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, -)) \cong \{\underline{W}, \underline{\mathcal{V}}(\underline{F}, -)\}^{\underline{\mathcal{J}}^{\mathrm{op}}}$$

so it suffices to construct $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$. In view of remark B.3.4, we should define $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$ by the dual colimit. More precisely, consider the following (ordinary) diagram in $\mathcal{V}$:

- For each object j in $\mathcal{J}$, there is a vertex with value $Wj \otimes Fj$.
- For each pair (j, k) of objects in $\mathcal{J}$, we have a vertex and two arrows with values as in the diagram below,

  $$Wj \otimes Fj \longleftarrow \underline{\mathcal{J}}(j, k) \otimes Wk \otimes Fj \longrightarrow Wk \otimes Fk$$

  where $\underline{\mathcal{J}}(j, k) \otimes Wk \otimes Fj \to Wj \otimes Fj$ is

  $$\left(\mathrm{ev}_{Wj,Wk} \circ \left(\underline{W}_{k,j} \otimes \mathrm{id}_{Wk}\right)\right) \otimes \mathrm{id}_{Fj}$$

  and $\underline{\mathcal{J}}(j, k) \otimes Wk \otimes Fj \to Wk \otimes Fk$ is

  $$\boldsymbol{\gamma}_{Fk,Wk} \circ \left(\left(\mathrm{ev}_{Fj,Fk} \circ \left(\underline{F}_{j,k} \otimes \mathrm{id}_{Fj}\right)\right) \otimes \mathrm{id}_{Wk}\right) \circ \left(\mathrm{id}_{\underline{\mathcal{J}}(j,k)} \otimes \boldsymbol{\gamma}_{Wk,Fj}\right)$$

The above diagram is small, so there is a colimit for it in $\mathcal{V}$, say $\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}$. By proposition B.2.28, there us a $\mathcal{V}$-enriched natural isomorphism

$$\underline{\mathcal{V}}((-) \otimes (-), -) \cong \underline{\mathcal{V}}(-, \underline{\mathcal{V}}(-, -))$$

and since $\underline{\mathcal{V}}(-, X)$ sends colimits in $\mathcal{V}$ to limits in $\mathcal{V}$, we obtain isomorphisms

$$\underline{\mathcal{V}}\left(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, X\right) \cong [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, X))$$

that are natural in X. For $\mathcal{V}$-enriched naturality, it suffices (by adjointness) to verify that a certain diagram in $\mathcal{V}$ of the form below commutes:

$$\begin{tikzcd}
\underline{\mathcal{V}}(X, Y) \otimes \underline{\mathcal{V}}\left(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, X\right) \arrow[r, "c"] \arrow[d, "\cong"'] & \underline{\mathcal{V}}\left(\underline{W} \star_{\underline{\mathcal{J}}} \underline{F}, X\right) \arrow[d, "\cong"] \\
\underline{\mathcal{V}}(X, Y) \otimes [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, X)) \arrow[r] & [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, Y))
\end{tikzcd}$$

But the universal property of $[\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, Y))$ implies it is enough to check that the above equation is satisfied after composing with every projection $\pi_j : [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(\underline{F}, Y)) \to \underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(Fj, Y))$, and this is straightforward.

(ii). The existence (and uniqueness) of the $\mathcal{V}$-enriched functor $(-) \star_{\underline{\mathcal{J}}} (-)$ is an instance of proposition B.2.24.

(iii). By proposition B.2.28 and theorem B.3.7, there are $\mathcal{V}$-enriched natural isomorphisms

$$\begin{aligned}
&\underline{\mathcal{V}}(Wj, \underline{\mathcal{V}}(\underline{j}^*(-), -)) \\
&\qquad \cong \underline{\mathcal{V}}(Wj \otimes \underline{j}^*(-), -) \cong \underline{\mathcal{V}}(\underline{j}^*(-) \otimes Wj, -) \\
&\qquad\qquad \cong \underline{\mathcal{V}}(\underline{j}^*(-), \underline{\mathcal{V}}(Wj, -))
\end{aligned}$$

and it is straightforward to see that these yield a $\mathcal{V}$-enriched natural isomorphism

$$\underline{\mathcal{V}}\big(\underline{W} \star_{\underline{\mathcal{J}}} (-), -\big) \cong [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{V}}(-, -)) \cong [\underline{\mathcal{J}}, \underline{\mathcal{V}}](-, \underline{\mathcal{V}}(\underline{W}, -))$$

so by proposition B.2.26, we have a $\mathcal{V}$-enriched adjunction

$$\underline{W} \star_{\underline{\mathcal{J}}} (-) \dashv \underline{\mathcal{V}}(\underline{W}, -) : \underline{\mathcal{V}} \to [\underline{\mathcal{J}}^{\mathrm{op}}, \underline{\mathcal{V}}]$$

as required. ■

Definition B.3.20. Let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category and let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists.

- An **end** for a $\mathcal{V}$-enriched functor $\underline{T} : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{C}}$ is a $\underline{H}$-weighted limit for $\underline{T}$ in $\underline{\mathcal{C}}$, where $\underline{H}$ is the $\mathcal{V}$-enriched functor $\underline{\mathcal{J}}(-, -) : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{V}}$. We write
$$\int_{j:\underline{\mathcal{J}}} \underline{T}(j, j)$$
for the object part of an end for $\underline{T}$.

- A **coend** for a $\mathcal{V}$-enriched functor $\underline{T} : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{C}}$ is a $\underline{H}$-weighted colimit for $\underline{T}$ in $\underline{\mathcal{C}}$, where $\underline{H}$ is the $\mathcal{V}$-enriched functor $\underline{\mathcal{J}}^{\mathrm{op}}(-, -) : \underline{\mathcal{J}} \otimes \underline{\mathcal{J}}^{\mathrm{op}} \to \underline{\mathcal{V}}$. We write
$$\int^{j:\underline{\mathcal{J}}} \underline{T}(j, j)$$
for the object part of a coend for $\underline{T}$.

Lemma B.3.21. *Let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists, and let $T : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ be a $\mathcal{V}$-enriched functor. Then $\int_{j:\underline{\mathcal{J}}} \underline{T}(j,j)$ is the limit of the following diagram in $\mathcal{V}$:*

- *For each object j in $\mathcal{J}$, there is a vertex with value $T(j,j)$.*
- *For each pair (j,k) of objects in $\mathcal{J}$, we have a vertex and two arrows with values as in the diagram below,*

$$T(j,j) \longrightarrow \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j,k)) \longleftarrow T(k,k)$$

where $T(j,j) \to \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j,k))$ is the right adjoint transpose of

$$\mathrm{ev}_{T(j,j),T(j,k)} \circ \left(\underline{T}(j,-)_{j,k} \otimes \mathrm{id}_{T(j,j)}\right) \circ \boldsymbol{\gamma}_{T(j,j),\underline{\mathcal{J}}(j,k)}$$

and $T(k,k) \to \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j,k))$ is the right adjoint tranpose of

$$\mathrm{ev}_{T(k,k),T(j,k)} \circ \left(\underline{T}(-,k)_{k,j} \otimes \mathrm{id}_{T(k,k)}\right) \circ \boldsymbol{\gamma}_{T(k,k),\underline{\mathcal{J}}(j,k)}$$

Proof. By remark B.3.4 and theorem B.3.18, $\int_{j:\underline{\mathcal{J}}} \underline{T}(j,j)$ is the limit of the following diagram in $\mathcal{V}$:

- For each pair (j,k) of objects in $\mathcal{J}$, there is a vertex with value

$$\underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j,k))$$

- For each quadruple (j', j, k, k') of objects in $\mathcal{J}$, we have a vertex and two arrows with values in the diagram below,

$$\begin{tikzcd}
\underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j,k)) \arrow[dr] & & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j',k'), T(j',k')) \arrow[dl] \\
& \underline{\mathcal{V}}(\underline{\mathcal{J}}(j',j) \otimes \underline{\mathcal{J}}(k,k'), \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k), T(j',k'))) &
\end{tikzcd}$$

where the arrow on the left is induced by $\underline{T} : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{V}}$ and the arrow on the right is induced by $\underline{\mathcal{J}}(-,-) : \underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{V}}$.

To prove the claim, it is enough to give a natural bijection between cones over the two diagrams. Observe that there is an evident commutative diagram in $\mathcal{V}$ of the form below,

$$\begin{tikzcd}
T(j,j) \arrow[r] \arrow[d] & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k),T(j,k)) \arrow[d] & T(k,k) \arrow[l] \arrow[d] \\
\underline{\mathcal{V}}(\underline{\mathcal{J}}(j',j),T(j',j)) \arrow[r] & \bullet & \underline{\mathcal{V}}(\underline{\mathcal{J}}(k,k'),T(k,k')) \arrow[l] \\
T(j',j') \arrow[r] \arrow[u] & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j',k'),T(j',k')) \arrow[u] & T(k',k') \arrow[l] \arrow[u]
\end{tikzcd}$$

where the vertex in the middle is $\underline{\mathcal{V}}(\underline{\mathcal{J}}(j',j) \otimes \underline{\mathcal{J}}(k,k'), \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k),T(j',k')))$. Consideration of this diagram shows that every cone over the first diagram induces a cone over the second diagram.

On the other hand, suppose we are given a cone over the second diagram, say with components $\varphi_{j,k} : X \to \underline{\mathcal{V}}(\underline{\mathcal{V}}(j,k),T(j,k))$. Observe that the morphism

$$T(j,j) \to \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j),T(j,j))$$

appearing in the first diagram admits a retraction, namely

$$r_j = \mathrm{ev}_{\underline{\mathcal{J}}(j,j),T(j,j)} \circ \left(\mathrm{id}_{\underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j),T(j,j))} \otimes e_j\right) \circ \boldsymbol{\rho}^{-1}_{\underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j),T(j,j))}$$

which can also be identified with $\underline{\mathcal{V}}\big(e_j, T(j,j)\big)$ if we suppress the canonical isomorphism $T(j,j) \to \underline{\mathcal{V}}(I,T(j,j))$. Moreover, by considering a certain commutative diagram in $\mathcal{V}$ of the following form,

$$\begin{tikzcd}
\underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j),T(j,j)) \arrow[dr] \arrow[dd, "r_j"'] & & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k),T(j,k)) \arrow[dl] \arrow[ddl, bend left, "\mathrm{id}"] \\
& \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j) \otimes \underline{\mathcal{J}}(j,k), \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j),T(j,k))) \arrow[d, "\underline{\mathcal{V}}\left(\underline{\mathcal{J}}(j,j)\otimes\underline{\mathcal{J}}(j,k),\underline{\mathcal{V}}\left(e_j,T(j,k)\right)\right)"] & \\
T(j,j) \arrow[r] & \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,j) \otimes \underline{\mathcal{J}}(j,k),T(j,k)) \arrow[d, "\underline{\mathcal{V}}\left(e_j\otimes\mathrm{id},T(j,k)\right)"] & \\
& \underline{\mathcal{V}}(\underline{\mathcal{J}}(j,k),T(j,k)) &
\end{tikzcd}$$

where we have suppressed various canonical isomorphisms involving I, we see that $r_j \circ \varphi_{j,j} : X \to T(j,j)$ defines a cone over the first diagram.

It is clear that the two constructions given above are mutually inverse, so we have a natural bijection between cones over the first diagram and cones over the second diagram, as required. ■

Corollary B.3.22. *Let $\underline{\mathcal{J}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists. Then, for any $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ and any pair $(\underline{F}, \underline{G})$ of $\mathcal{V}$-enriched functors $\underline{\mathcal{J}} \to \underline{\mathcal{C}}$, the end*

$$\int_{j:\underline{\mathcal{J}}} \underline{\mathcal{C}}(\underline{F}j, \underline{G}j)$$

is (the object part of) an object of $\mathcal{V}$-enriched natural transformations $\underline{F} \Rightarrow \underline{G}$. In particular, the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ exists.

Proof. Simply compare the construction of $[\underline{\mathcal{J}}, \underline{\mathcal{C}}](\underline{F}, \underline{G})$ given in remark B.3.4 with the characterisation of $\int_{j:\underline{\mathcal{J}}} \underline{\mathcal{C}}(\underline{F}j, \underline{G}j)$ given in lemma B.3.21. ■

Proposition B.3.23. *Let $\underline{\mathcal{C}}$ and $\underline{\mathcal{J}}$ be $\mathcal{V}$-enriched categories such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{J}}^{\mathrm{op}} \otimes \underline{\mathcal{J}}, \underline{\mathcal{V}}]$ exists.*

- *Let $\underline{\mathcal{K}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{K}}, \underline{\mathcal{V}}]$ exists. If $\underline{W} : \underline{\mathcal{K}} \to \underline{\mathcal{V}}$ is a $\mathcal{V}$-enriched weight such that $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{K}} \to \underline{\mathcal{C}}$ exist in $\underline{\mathcal{C}}$, then $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{K}} \to [\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ exist in $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ and can be computed componentwise.*

- *Let $\underline{\mathcal{K}}$ be a $\mathcal{V}$-enriched category such that the $\mathcal{V}$-enriched functor category $[\underline{\mathcal{K}}^{\mathrm{op}}, \underline{\mathcal{V}}]$ exists. If $\underline{W} : \underline{\mathcal{K}}^{\mathrm{op}} \to \underline{\mathcal{V}}$ is a $\mathcal{V}$-enriched weight such that $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{K}} \to \underline{\mathcal{C}}$ exist in $\underline{\mathcal{C}}$, then $\underline{W}$-weighted limits for all $\mathcal{V}$-enriched diagrams $\underline{\mathcal{K}} \to [\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ exist in $[\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ and can be computed componentwise.*

Proof. The two claims are formally dual; we will prove the first version.

Let $\underline{G} : \underline{\mathcal{K}} \to [\underline{\mathcal{J}}, \underline{\mathcal{C}}]$ be a $\mathcal{V}$-enriched functor, let $\underline{G}' : \underline{\mathcal{J}} \to [\underline{\mathcal{K}}, \underline{\mathcal{C}}]$ be the $\mathcal{V}$-enriched functor defined by $\underline{G}'(j)(k) = \underline{G}(k)(j)$, and let $\underline{H} : \underline{\mathcal{J}} \to \underline{\mathcal{C}}$ be the $\mathcal{V}$-enriched functor defined by $\underline{H}(j) = \{\underline{W}, \underline{G}'(j)\}^{\underline{\mathcal{K}}}$. We wish to construct a $\mathcal{V}$-enriched natural isomorphism of the form below:

$$[\underline{\mathcal{K}}, \underline{\mathcal{V}}](\underline{W}, [\underline{\mathcal{J}}, \underline{\mathcal{C}}](-, \underline{G})) \cong [\underline{\mathcal{J}}, \underline{\mathcal{C}}](-, \underline{H})$$

Proposition B.3.15 and theorem B.3.18 imply $[\underline{\mathcal{K}}, \underline{\mathcal{V}}](\underline{W}, -)$ preserves weighted limits, and corollary B.3.22 says that objects of natural transformations are ends (hence, weighted limits), so we have the following $\mathcal{V}$-enriched natural isomorphism:

$$[\underline{\mathcal{K}}, \underline{\mathcal{V}}](\underline{W}, [\underline{\mathcal{J}}, \underline{\mathcal{C}}](-, \underline{G})) \cong \int_{j:\underline{\mathcal{J}}} [\underline{\mathcal{K}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{C}}(\underline{j}^*(-), \underline{G}'(j)))$$

On the other hand, by definition, we have a $\mathcal{V}$-enriched natural isomorphism

$$[\underline{\mathcal{K}}, \underline{\mathcal{V}}](\underline{W}, \underline{\mathcal{C}}(-, \underline{G})) \cong \underline{\mathcal{C}}(-, \underline{H})$$

of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}}^{\mathrm{op}} \otimes \underline{\mathcal{J}} \to \underline{\mathcal{V}}$, and

$$\int_{j:\underline{\mathcal{J}}} \underline{\mathcal{C}}(\underline{j^*}(-), \underline{H}(j)) \cong [\underline{\mathcal{J}}, \underline{\mathcal{C}}](-, \underline{H})$$

so we are done. ■

B.4 Categories with actions

Prerequisites. §B.1, B.2, B.3.

Definition B.4.1. Let $\mathcal{V}$ be a monoidal category.

- A **left $\mathcal{V}$-action** on a category $\mathcal{C}$ is a strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]$, where $[\mathcal{C}, \mathcal{C}]$ is regarded as a strict monoidal category under composition.
- A **right $\mathcal{V}$-action** on $\mathcal{C}$ is a strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]^{\mathrm{rev}}$, where $[\mathcal{C}, \mathcal{C}]$ is regarded as a strict monoidal category under composition.

Remark B.4.2. We can unfold the above definition somewhat by taking the left exponential transpose of the strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]$: let $\oslash$ be the corresponding functor $\mathcal{V} \times \mathcal{C} \to \mathcal{C}$. Since the original functor was strong monoidal, we get a natural isomorphism $\boldsymbol{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow I \oslash (-)$ and a natural isomorphism $\boldsymbol{\mu}_{X,Y} : X \oslash (Y \oslash (-)) \Rightarrow (X \otimes Y) \oslash (-)$ for each pair of objects X and Y in $\mathcal{V}$; these moreover satisfy the following coherence laws:

$$\begin{tikzcd}
X \oslash (Y \oslash (-)) \arrow[r, "\boldsymbol{\mu}_{X,Y}"] \arrow[d, "f \oslash (g \oslash \mathrm{id})"'] & (X \otimes Y) \oslash (-) \arrow[d, "(f \otimes g) \oslash \mathrm{id}"] \\
X' \oslash (Y' \oslash (-)) \arrow[r, "\boldsymbol{\mu}_{X',Y'}"'] & (X \otimes Y) \oslash (-)
\end{tikzcd}$$

$$\begin{tikzcd}
X \oslash (-) \arrow[r, "\boldsymbol{\eta}(X \oslash (-))"] \arrow[d, "\mathrm{id}"'] & I \oslash (X \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{I,X}"] \\
X \oslash (-) & (I \otimes X) \oslash (-) \arrow[l, "\boldsymbol{\lambda}_X \oslash \mathrm{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \oslash (-) \arrow[r, "(X \oslash (-))\boldsymbol{\eta}"] \arrow[d, "\mathrm{id}"'] & X \oslash (I \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{X,I}"] \\
X \oslash (-) & (X \otimes I) \oslash (-) \arrow[l, "\boldsymbol{\rho}_X \oslash \mathrm{id}"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& W \oslash (X \oslash (Y \oslash (-))) \arrow[dl, "\boldsymbol{\mu}_{W,X}(Y\oslash(-))"'] \arrow[dr, "\mathrm{id}_W \oslash \boldsymbol{\mu}_{X,Y}"] & \\
(W \otimes X) \oslash (Y \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{W\otimes X,Y}"'] & & W \oslash ((X \otimes Y) \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{W,X\otimes Y}"] \\
((W \otimes X) \otimes Y) \oslash (-) \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y} \oslash \mathrm{id}"'] & & (W \otimes (X \otimes Y)) \oslash (-)
\end{tikzcd}$$

Conversely, any functor $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ equipped with such a collection of natural isomorphisms defines a left $\mathcal{V}$-action on $\mathcal{C}$.

Proposition B.4.3 (Bénabou)**.** *For any monoidal category $\mathcal{C}$, there is a faithful strong monoidal functor $F : \mathcal{C} \to [\mathcal{C}, \mathcal{C}]$ defined by the following data:*

$$FX = X \otimes (-)$$
$$\boldsymbol{\eta} = \boldsymbol{\lambda}^{-1}$$
$$\left(\boldsymbol{\mu}_{X,Y}\right)_Z = \boldsymbol{\alpha}^{-1}_{X,Y,Z}$$

In particular, this defines a left $\mathcal{C}$-action on $\mathcal{C}$, called the **left regular representation** *of $\mathcal{C}$.*

Proof. F is clearly a faithful functor. In this case, the strong monoidal functor axioms become the following diagrams:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\boldsymbol{\lambda}^{-1}_{X\otimes Y}"] \arrow[d, "\mathrm{id}"'] & I \otimes (X \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{I,X,Y}"] \\
X \otimes Y & (I \otimes X) \otimes Y \arrow[l, "\boldsymbol{\lambda}_X \otimes \mathrm{id}_Y"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id}_X \otimes \boldsymbol{\lambda}^{-1}_Y"] \arrow[d, "\mathrm{id}"'] & X \otimes (I \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{X,I,Y}"] \\
X \otimes Y & (X \otimes I) \otimes Y \arrow[l, "\boldsymbol{\rho}_X \otimes \mathrm{id}_Y"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& W \otimes (X \otimes (Y \otimes Z)) \arrow[dl, "\boldsymbol{\alpha}^{-1}_{W,X,Y\otimes Z}"'] \arrow[dr, "\mathrm{id}_W \otimes \boldsymbol{\alpha}^{-1}_{X,Y,Z}"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\boldsymbol{\alpha}^{-1}_{W\otimes X,Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "\boldsymbol{\alpha}^{-1}_{W,X\otimes Y,Z}"] \\
((W \otimes X) \otimes Y) \otimes Z \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y} \otimes \mathrm{id}_Z"'] & & (W \otimes (X \otimes Y)) \otimes Z
\end{tikzcd}$$

The left square commutes by the coherence theorem, while the right square and the pentagon are seen to be immediate consequences of the triangle and pentagon axioms, respectively. ■

Proposition B.4.4. *Let $\mathcal{V}$ be a monoidal category and let $\mathcal{C}$ be a category.*

- *If $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $X \oslash (-)$ has a right adjoint $(-) \multimapinv X$, then the functor $\multimapinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ defines a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$.*
- *If $\oslash : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ defines a right $\mathcal{V}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $(-) \oslash X$ has a right adjoint $X \multimap (-)$, then the functor $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$.*
- *If $\multimapinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ defines a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $X \multimapinv (-)$ has a left adjoint $X \oslash (-)$, then the functor $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}$-action on $\mathcal{C}$.*
- *If $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $X \multimap (-)$ has a left adjoint $(-) \oslash X$, then the functor $\oslash : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ defines a right $\mathcal{V}$-action on $\mathcal{C}$.*

Proof. The four statements are related by applying $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{rev}}$ at the appropriate points, so it suffices to prove the first claim.

First, note that $\multimapinv$ is indeed a functor $\mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$, by the parameter theorem for adjunctions.[5] Let $\mathrm{ev}_{X,A} : X \oslash (A \multimapinv X) \to A$ denote the component of the counit of the adjunction $X \oslash (-) \dashv (-) \multimapinv X$ at an object A in $\mathcal{C}$. For each pair of objects X and Y in $\mathcal{V}$ and each object A in $\mathcal{C}$, we define the morphism $(\boldsymbol{\delta}_{X,Y})_A : A \multimapinv (X \otimes Y) \to (A \multimapinv X) \multimapinv Y$ to be the right adjoint transpose of $\mathrm{ev}_{X \otimes Y, A} \circ (\boldsymbol{\mu}_{X,Y})_{(A \multimapinv X) \multimapinv Y}$, and for each A, we define $\boldsymbol{\varepsilon}_A : A \multimapinv I \to A$ to be the composite $\mathrm{ev}_{I,A} \circ \boldsymbol{\eta}_{A \multimapinv I}$. These are clearly natural in A, and it is straightforward to check that $\boldsymbol{\delta}_{X,Y}$ is also natural in X and Y. One may then use the calculus of mates to show that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}_{X,Y}$ are natural isomorphisms and that they satisfy the axioms for making the right exponential transpose of $\multimapinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ into a strong monoidal functor $\mathcal{V}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]^{\mathrm{rev}}$, i.e. a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$. ■

Example B.4.5. $\mathcal{V}$ is a left-closed (resp. right-closed) monoidal category if and only if the left (resp. right) self-action of $\mathcal{V}$ has a parametrised right adjoint as in the proposition, and the right adjoint right (resp. left) $\mathcal{V}^{\mathrm{op}}$-action so obtained is precisely a left (resp. right) internal hom functor.

[5] See [CWM, Ch. IV, §7].

Definition B.4.6. Let $\mathcal{V}$ be a monoidal category, let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category, let X be an object in $\mathcal{V}$, and let C be an object in $\mathcal{C}$.

- Assuming $\mathcal{V}$ is right-closed with right internal hom functor $\multimap$, a **tensor product** of X and C is a pair $(X \odot C, \lambda)$ where $X \odot C$ is an object in $\mathcal{C}$ and λ is a morphism $X \to \underline{\mathcal{C}}(C, X \odot C)$ in $\mathcal{V}$ such that the $\mathcal{V}$-enriched natural transformation

  $$\underline{\mathcal{C}}(X \odot C, -) \Rightarrow X \multimap \underline{\mathcal{C}}(C, -)$$

  induced (as in the weak Yoneda lemma) by the corresponding morphism $\ulcorner \lambda \urcorner : I \to X \multimap \underline{\mathcal{C}}(C, X \odot C)$ is a $\mathcal{V}$-enriched natural isomorphism.

- Assuming $\mathcal{V}$ is left-closed with left internal hom functor $\multimapinv$, a **cotensor product** of X and C is a pair $(X \pitchfork C, \lambda)$ where $X \pitchfork C$ is an object in $\mathcal{C}$ and λ is a morphism $X \to \underline{\mathcal{C}}(X \pitchfork C, C)$ in $\mathcal{V}$ such that the $\mathcal{V}$-enriched natural transformation

  $$\underline{\mathcal{C}}(-, X \pitchfork C) \Rightarrow \underline{\mathcal{C}}(-, C) \multimapinv X$$

  induced (as in the weak Yoneda lemma) by the corresponding morphism $\ulcorner \lambda \urcorner : I \to \underline{\mathcal{C}}(X \pitchfork C, C) \multimapinv X$ is a $\mathcal{V}$-enriched natural isomorphism. We may also write $C \multimapinv X$ instead of $X \pitchfork C$.

Remark B.4.7. By lemma B.2.23, cotensor products (resp. tensor products) are unique up to unique isomorphism. Moreover, if $\mathcal{V}$ is a *symmetric* monoidal *closed* category, then a cotensor product (resp. tensor product) is just a weighted limit (resp. weighted colimit) for a $\mathcal{V}$-enriched diagram of shape $\mathbb{1}$, where $\mathbb{1}$ is the $\mathcal{V}$-enriched category with only one object $*$ and $\underline{\mathbb{1}}(*, *) = I$.

Definition B.4.8. Let $\mathcal{V}$ be a monoidal category.

- Assuming $\mathcal{V}$ is right-closed, a **$\mathcal{V}$-tensored category** is a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ equipped with a choice of tensor product for each object in $\mathcal{V} \times \mathcal{C}$.

- Assuming $\mathcal{V}$ is left-closed, a **$\mathcal{V}$-cotensored category** is a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ equipped with a choice of cotensor product for each object in $\mathcal{V} \times \mathcal{C}$.

REMARK B.4.9. Suppose $\mathcal{V}$ is a *symmetric* monoidal *closed* category. By proposition B.3.16, if $\underline{\mathcal{C}}$ is a $\mathcal{V}$-tensored category (resp. $\mathcal{V}$-cotensored category), then there is a $\mathcal{V}$-enriched functor $\underline{\odot} : \underline{\mathcal{V}} \otimes \underline{\mathcal{C}} \to \underline{\mathcal{C}}$ (resp. $\underline{\multimapdotinv} : \underline{\mathcal{C}} \otimes \underline{\mathcal{V}}^{\mathrm{op}} \to \underline{\mathcal{C}}$) sending a pair (X, C) to their chosen tensor product $X \odot C$ (resp. cotensor product $C \multimapdotinv X$).

Proposition B.4.10. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category.*

- *If $\underline{\mathcal{C}}$ is a $\mathcal{V}$-tensored category, then the functor $\odot : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}$-action on $\mathcal{C}$.*
- *If $\underline{\mathcal{C}}$ is a $\mathcal{V}$-cotensored category, then the functor $\multimapdotinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ defines a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$.*

Proof. The two claims are formally dual; we will prove the first version.

Following remark B.4.2, we seek natural isomorphisms $\boldsymbol{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow I \odot (-)$ and $\boldsymbol{\mu}_{X,Y} : X \odot (Y \odot (-)) \Rightarrow (X \otimes Y) \odot (-)$ satisfying the relevant coherence laws. To that end, observe that we have the following natural bijections:

$$\begin{aligned} \mathcal{C}(A, B) &\cong \mathcal{V}(I, \underline{\mathcal{C}}(A, B)) \\ &\cong \mathcal{C}(I \odot A, B) \end{aligned}$$

$$\begin{aligned} \mathcal{C}((X \otimes Y) \odot A, B) &\cong \mathcal{V}(X \otimes Y, \underline{\mathcal{C}}(A, B)) \\ &\cong \mathcal{V}(X, \underline{\mathcal{V}}(Y, \underline{\mathcal{C}}(A, B))) \\ &\cong \mathcal{V}(X, \underline{\mathcal{C}}(Y \odot A, B)) \\ &\cong \mathcal{C}(X \odot (Y \odot A), B) \end{aligned}$$

Thus, by the Yoneda lemma, we have natural isomorphisms of the required form. The coherence laws remain to be verified: this is straightforward, if tedious. (See also proposition B.2.5.) ∎

Definition B.4.11. Let $\mathcal{V}$ be a monoidal category and let $\mathcal{C}$ be a category.

- A **right $\mathcal{V}$-hom system** for $\mathcal{C}$ consists of a left $\mathcal{V}$-action $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$, a functor $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, and a right $\mathcal{V}^{\mathrm{op}}$-action $\multimapdotinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ together with natural bijections of the types below,

$$\begin{aligned} \mathcal{V}(X, \underline{\mathcal{C}}(A, B)) &\cong \mathcal{C}(A, B \multimapdotinv X) \\ \mathcal{C}(X \oslash A, B) &\cong \mathcal{C}(A, B \multimapdotinv X) \end{aligned}$$

$$\mathcal{C}(X \oslash A, B) \cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B))$$

where X varies over the objects in $\mathcal{V}$, and A and B vary over the objects in $\mathcal{C}$, such that the cyclic composition of the three bijections is the identity.

- A **left $\mathcal{V}$-hom system** for $\mathcal{C}$ consists of a right $\mathcal{V}$-action $\otimes : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$, a functor $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, and a left $\mathcal{V}^{\mathrm{op}}$-action $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, together with natural bijections of the types below,

$$\mathcal{V}(X, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, X \multimap B)$$
$$\mathcal{C}(A \otimes X, B) \cong \mathcal{C}(A, X \multimap B)$$
$$\mathcal{C}(A \otimes X, B) \cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B))$$

where X varies over the objects in $\mathcal{V}$, and A and B vary over the objects in $\mathcal{C}$, such that the cyclic composition of the three bijections is the identity.

Remark b.4.12. The cyclic composition condition implies it is enough to provide two out of the three natural bijections: the third is then forced to be the inverse of the composite of the other two.

Example b.4.13. If $\mathcal{V}$ is a biclosed monoidal category with right internal hom functor $\mathcal{H}om$ and left internal hom functor $\pitchfork$, then $(\otimes, \pitchfork, \mathcal{H}om)$ is a left $\mathcal{V}$-hom system for $\mathcal{V}$:

$$\mathcal{V}(Y, X \pitchfork Z) \cong \mathcal{V}(X, \mathcal{H}om(Y, Z))$$
$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, \mathcal{H}om(Y, Z))$$
$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(Y, X \pitchfork Z)$$

Proposition b.4.14. *Let $\mathcal{V}$ be a* symmetric *monoidal* closed *category and let $\underline{\mathcal{C}}$ be a $\mathcal{V}$-enriched category that is both $\mathcal{V}$-tensored and $\mathcal{V}$-cotensored.*

(i) *For each object X in $\mathcal{V}$, there exist $\mathcal{V}$-enriched natural isomorphisms*

$$\underline{\mathcal{C}}(X \odot (-), -) \cong \underline{\mathcal{V}}(X, \underline{\mathcal{C}}(-, -)) \cong \underline{\mathcal{C}}(-, (-) \mathbin{⟜} X)$$

and moreover, these constitute $\mathcal{V}$-enriched natural isomorphisms of $\mathcal{V}$-enriched functors $\underline{\mathcal{V}}^{\mathrm{op}} \otimes \underline{\mathcal{C}}^{\mathrm{op}} \otimes \underline{\mathcal{C}} \to \underline{\mathcal{V}}$.

(ii) *In particular, for each object X in $\mathcal{V}$, there is a $\mathcal{V}$-enriched adjunction of the form below:*

$$X \odot (-) \dashv (-) \mathbin{⟜} X : \underline{\mathcal{C}} \to \underline{\mathcal{C}}$$

(iii) $(\odot, \underline{C}, \multimapinv)$ *is a right $\mathcal{V}$-hom system for C.*

Proof. (i). This is a special case of proposition B.3.16.

(ii). Apply proposition B.2.26.

(iii). The $\mathcal{V}$-enriched natural isomorphisms have underlying natural bijections

$$C(X \odot A, B) \cong \mathcal{V}(X, \underline{C}(A, B)) \cong C(A, B \multimapinv X)$$

as required. ∎

Theorem B.4.15. *Let $\mathcal{V}$ be a monoidal category and let C be a category.*

(i) *If $\oslash$ is a left $\mathcal{V}$-action on C and $\underline{C} : C^{\mathrm{op}} \times C \to \mathcal{V}$ is a functor with natural bijections of the form below,*

$$C(X \oslash A, B) \cong \mathcal{V}(X, \underline{C}(A, B))$$

then $\underline{C}$ is the hom functor of a $\mathcal{V}$-enriched category $\underline{C}$ whose underlying ordinary category is isomorphic to C.

(ii) *Moreover, if $\mathcal{V}$ is right-closed, then the hypothesised natural bijection underlies a $\mathcal{V}$-enriched natural isomorphism*

$$C(X \oslash A, -) \cong \mathcal{V}(X, \underline{C}(A, -))$$

for each object X in $\mathcal{V}$ and each object A in C. In particular, $\underline{C}$ is a $\mathcal{V}$-tensored category.

Dually:

(i) *If $\multimapinv$ is a right $\mathcal{V}^{\mathrm{op}}$-action on C and $\underline{C} : C^{\mathrm{op}} \times C \to \mathcal{V}$ is a functor with natural bijections of the form below,*

$$C(A, B \multimapinv X) \cong \mathcal{V}(X, \underline{C}(A, B))$$

then $\underline{C}$ is the hom functor of a $\mathcal{V}$-enriched category $\underline{C}$ whose underlying ordinary category is isomorphic to C.

(ii) *Moreover, if $\mathcal{V}$ is left-closed, then the hypothesised natural bijection underlies a $\mathcal{V}$-enriched natural isomorphism*

$$\underline{\mathcal{C}}(-, B \multimapinv X) \cong \mathcal{V}(X, \underline{\mathcal{C}}(-, X))$$

for each object X in $\mathcal{V}$ and each object B in $\mathcal{C}$. In particular, $\underline{\mathcal{C}}$ is a $\mathcal{V}$-cotensored category.

Proof. (i). The natural isomorphism $A \cong I \oslash A$ induces a family of bijections

$$\mathcal{C}(A, B) \cong \mathcal{V}(I, \underline{\mathcal{C}}(A, B))$$

natural in A and B, so we have a morphism $e_A : I \to \underline{\mathcal{C}}(A, A)$ in $\mathcal{V}$ for every object A in $\mathcal{C}$ corresponding to $\mathrm{id}_A : A \to A$ in $\mathcal{C}$. Let $\mathrm{ev}_{A,B} : \underline{\mathcal{C}}(A, B) \oslash A \to B$ be the component at B of the counit of the adjunction $(-) \oslash A \dashv \underline{\mathcal{C}}(A, -)$, and define $c_{A,B,C} : \underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{C}}(A, C)$ to be the right adjoint transpose of the following morphism in $\mathcal{C}$:

$$\mathrm{ev}_{B,C} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(B,C)} \oslash \mathrm{ev}_{A,B}\right) \circ \left(\boldsymbol{\mu}_{\underline{\mathcal{C}}(B,C),\underline{\mathcal{C}}(A,B)}\right)^{-1}_A : (\underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B)) \oslash A \to C$$

By definition, the left adjoint transpose of e_B is $\boldsymbol{\eta}_B^{-1}$, so the left and right unit axioms are satisfied:

$$c_{A,B,B} \circ \left(e_B \otimes \mathrm{id}_{\underline{\mathcal{C}}(A,B)}\right) = \boldsymbol{\lambda}_{\underline{\mathcal{C}}(A,B)}$$
$$c_{B,B,C} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(B,C)} \otimes e_B\right) = \boldsymbol{\rho}_{\underline{\mathcal{C}}(B,C)}$$

One may similarly verify the associativity axiom:

$$c_{A,B,D} \circ \left(c_{B,C,D} \otimes \mathrm{id}_{\underline{\mathcal{C}}(A,B)}\right) = c_{A,C,D} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(C,D)} \otimes c_{A,B,C}\right) \circ \boldsymbol{\alpha}_{\underline{\mathcal{C}}(C,D),\underline{\mathcal{C}}(B,C),\underline{\mathcal{C}}(A,B)}$$

(ii). See Lemma 2.1 in [Janelidze and Kelly, 2001]. □

Definition B.4.16. Let $\mathcal{V}$ be a monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be categories with left $\mathcal{V}$-actions. A **$\mathcal{V}$-strength** for a functor $F : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\boldsymbol{\sigma} : (-) \oslash F(-) \Rightarrow F(- \oslash -)$ making these diagrams commute:

$$\begin{tikzcd} & FA \arrow[dl, "\boldsymbol{\eta}_{FA}"'] \arrow[dr, "F\boldsymbol{\eta}_A"] & \\ I \oslash FA \arrow[rr, "\boldsymbol{\sigma}_{I,A}"'] & & F(I \oslash A) \end{tikzcd}$$

$$\begin{tikzcd}
& X \oslash (Y \oslash FA) \arrow[dl, "\mathrm{id}_X \oslash \sigma_{Y,A}"'] \arrow[dr, "(\mu_{X,Y})_A"] & \\
X \oslash F(Y \oslash A) \arrow[d, "\sigma_{X,Y \oslash A}"'] & & (X \otimes Y) \oslash FA \arrow[d, "\sigma_{X \otimes Y,A}"] \\
F(X \oslash (Y \oslash A)) \arrow[rr, "F(\mu_{X,Y})_A"'] & & F((X \otimes Y) \oslash A)
\end{tikzcd}$$

A $\mathcal{V}$**-strong functor** is a functor equipped with a $\mathcal{V}$-strength.

Theorem B.4.17 (Kock)**.** *Let $\mathcal{V}$ be a right-closed monoidal category, let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-tensored categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor.*

(i) *Given a $\mathcal{V}$-enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ whose underlying ordinary functor is $F : \mathcal{C} \to \mathcal{D}$, there is a (unique) natural transformation*

$$\mathcal{C}((-) \odot (-), -) \Rightarrow \mathcal{D}((-) \odot F(-), F(-))$$

whose components make the following diagram commute,

$$\begin{tikzcd}
\mathcal{C}(X \odot A, B) \arrow[r, "\cong"] \arrow[d, dashed] & \mathcal{V}(X, \underline{\mathcal{C}}(A,B)) \arrow[d, "\mathcal{V}(X, \underline{F}_{A,B})"] \\
\mathcal{D}(X \odot FA, FB) \arrow[r, "\cong"'] & \mathcal{V}(X, \underline{\mathcal{D}}(FA, FB))
\end{tikzcd}$$

where the horizontal arrows are the components of underlying natural bijections of the canonical $\mathcal{V}$-enriched natural isomorphisms. In particular, for any (ordinary) functors $P : \mathcal{J} \to \mathcal{V}$ and $Q, R : \mathcal{J} \to \mathcal{C}$, there is an induced map from the ensemble of (ordinary) natural transformations $P \odot Q \Rightarrow R$ to the ensemble of (ordinary) natural transformations $P \odot FQ \Rightarrow FR$.

(ii) *Moreover, the natural transformation $\sigma : (-) \odot F(-) \Rightarrow F(- \odot -)$ induced by* $\mathrm{id} : (-) \odot (-) \Rightarrow (-) \odot (-)$ *is a $\mathcal{V}$-strength for $F : \mathcal{C} \to \mathcal{D}$.*

(iii) *This construction defines a bijection between the ensemble of $\mathcal{V}$-strengths for $F : \mathcal{C} \to \mathcal{D}$ and the ensemble of $\mathcal{V}$-enriched functors $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ whose underlying ordinary functor is F.*

Proof. (i). Straightforward; but see also remark A.6.5.

TODO: Give a proper proof; the cited one is incomplete.

(ii) and (iii). See Theorem 1.3 in [Kock, 1972]. □

Definition B.4.18. Let $\mathcal{V}$ be a monoidal category, let $\mathcal{C}$ and $\mathcal{D}$ be categories with left $\mathcal{V}$-actions, and let $F, F' : \mathcal{C} \to \mathcal{D}$ be functors with $\mathcal{V}$-strengths $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$ respectively. A **$\mathcal{V}$-strong natural transformation** $\varphi : F \Rightarrow F'$ is a natural transformation making the following diagram commute:

$$\begin{tikzcd} X \oslash FA \arrow[r, "\sigma_{X,A}"] \arrow[d, "\mathrm{id}_X \oslash \varphi_A"'] & F(X \oslash A) \arrow[d, "\varphi_{X \oslash A}"] \\ X \oslash F'A \arrow[r, "\sigma'_{X,A}"'] & F'(X \oslash A) \end{tikzcd}$$

Theorem B.4.19 (Kock). *Let $\mathcal{V}$ be a right-closed monoidal category, let $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ be $\mathcal{V}$-tensored categories and let $\underline{F}, \underline{F}' : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ be $\mathcal{V}$-enriched functors. The following are equivalent for an ordinary natural transformation $\varphi : F \Rightarrow F'$:*

(i) *$\varphi : F \Rightarrow F'$ is the underlying ordinary natural transformation of a $\mathcal{V}$-enriched natural transformation $\underline{F} \Rightarrow \underline{F}'$.*

(ii) *$\varphi : F \Rightarrow F'$ is a $\mathcal{V}$-strong natural transformation (with respect to the $\mathcal{V}$-strengths induced by $\underline{F}$ and $\underline{F}'$).*

Proof. See Remark 1.4 in [Kock, 1972]. □ TODO: Give a proper proof.

B.5 Indexed categories

Prerequisites. §§ A.1, A.2, A.3.

Definition B.5.1. Let $\mathcal{B}$ be a category. A **$\mathcal{B}$-indexed category** $\mathbb{E}$ consists of the following data:

- For each object B in $\mathcal{B}$, a category $\mathcal{E}^B$, called the **fibre** of $\mathbb{E}$ over B.
- For each morphism $f : B' \to B$ in $\mathcal{B}$, a functor $f^* : \mathcal{E}^B \to \mathcal{E}^{B'}$, called the **reindexing functor** along $f : B' \to B$.
- For each object B in $\mathcal{B}$, a natural isomorphism $\boldsymbol{\eta}_B : \mathrm{id}_{\mathcal{E}^B} \Rightarrow \left(\mathrm{id}_B\right)^*$.
- For each commutative triangle in $\mathcal{B}$ of the form below,

$$\begin{tikzcd} B'' \arrow[dr, "f'"'] \arrow[drr, bend left, "f \circ f'"] & & \\ & B' \arrow[r, "f"'] & B \end{tikzcd}$$

a natural isomorphism $\boldsymbol{\mu}_{f,f'} : f'^* f^* \Rightarrow (f \circ f')^*$.

These data are moreover required to satisfy the equations shown below:

$$\begin{aligned}
\boldsymbol{\mu}_{f,\mathrm{id}_{B'}} \bullet \boldsymbol{\eta}_{B'} f^* &= \mathrm{id}_{f^*} \\
\boldsymbol{\mu}_{\mathrm{id}_B,f} \bullet f^* \boldsymbol{\eta}_B &= \mathrm{id}_{f^*} \\
\boldsymbol{\mu}_{f\circ f',f''} \bullet f''^* \boldsymbol{\mu}_{f,f'} &= \boldsymbol{\mu}_{f,f'\circ f''} \bullet \boldsymbol{\mu}_{f',f''} f^*
\end{aligned}$$

¶ B.5.2. If $\mathbb{E}$ is a $\mathcal{B}^{\mathrm{op}}$-indexed category and $f : B' \to B$ is a morphism in $\mathcal{B}$, then we will usually write either $f_!$ or f_* for the reindexing functor $\mathcal{E}^{B'} \to \mathcal{E}^B$, depending on context.

Definition B.5.3. Let $\mathcal{B}$ be a category and let $\mathbb{D}$ and $\mathbb{E}$ be $\mathcal{B}$-indexed categories. An **oplax $\mathcal{B}$-indexed functor** $F : \mathbb{D} \to \mathbb{E}$ consists of the following data:

- For each object B in $\mathcal{B}$, a functor $F^B : \mathcal{D}^B \to \mathcal{E}^B$.
- For each morphism $f : B' \to B$ in $\mathcal{B}$, a natural transformation $\boldsymbol{\theta}_f$ as in the diagram below:

$$\begin{tikzcd}
\mathcal{D}^B \arrow[r, "F^B"] \arrow[d, "f^*"'] & \mathcal{E}^B \arrow[d, "f^*"] \\
\mathcal{D}^{B'} \arrow[r, "F^{B'}"'] \arrow[ur, phantom, "\boldsymbol{\theta}_f \Nearrow"] & \mathcal{E}^{B'}
\end{tikzcd}$$

These data are required to satisfy the equations shown below:

$$\begin{tikzcd}
\mathcal{D}^B \arrow[r, equal] \arrow[d, "\mathrm{id}"'] & \mathcal{D}^B \arrow[r, "F^B"] \arrow[d] & \mathcal{E}^B \arrow[d, "(\mathrm{id}_B)^*"] \\
\mathcal{D}^B \arrow[r, equal] \arrow[ur, phantom, "\boldsymbol{\eta}_B \Nearrow"] & \mathcal{D}^B \arrow[r, "F^B"'] \arrow[ur, phantom, "\boldsymbol{\theta}_{\mathrm{id}} \Nearrow"] & \mathcal{E}^B
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{D}^B \arrow[r, "F^B"] \arrow[d, "\mathrm{id}"'] & \mathcal{E}^B \arrow[r, equal] \arrow[d, "\mathrm{id}"'] & \mathcal{E}^B \arrow[d, "(\mathrm{id}_B)^*"] \\
\mathcal{D}^B \arrow[r, "F^B"'] & \mathcal{E}^B \arrow[r, equal] \arrow[ur, phantom, "\boldsymbol{\eta}_B \Nearrow"] & \mathcal{E}^B
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathcal{D}^B \arrow[r, "F^B"] \arrow[dl, bend right, "f^*"'] \arrow[dd] & \mathcal{E}^B \arrow[dd, "(f\circ f')^*"] \\
\mathcal{D}^{B'} \arrow[dr, bend right, "f'^*"'] \arrow[r, phantom, "\overset{\boldsymbol{\mu}_{f,f'}}{\Rightarrow}"] & & \\
& \mathcal{D}^{B''} \arrow[r, "F^{B''}"'] \arrow[uur, phantom, "\boldsymbol{\theta}_{f''} \Nearrow"] & \mathcal{E}^{B''}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{D}^B \arrow[r, "F^B"] \arrow[d, "f^*"'] & \mathcal{E}^B \arrow[d, "f^*"] \arrow[dd, bend left=80, "(f\circ f')^*"] \\
\mathcal{D}^{B'} \arrow[r, "F^{B'}"] \arrow[d, "f'^*"'] \arrow[ur, phantom, "\boldsymbol{\theta}_f \Nearrow"] & \mathcal{E}^{B'} \arrow[d, "f'^*"] \arrow[r, phantom, "\overset{\boldsymbol{\mu}_{f,f'}}{\Rightarrow}"] & {} \\
\mathcal{D}^{B''} \arrow[r, "F^{B''}"'] \arrow[ur, phantom, "\boldsymbol{\theta}_{f'} \Nearrow"] & \mathcal{E}^{B''}
\end{tikzcd}$$

A **$\mathcal{B}$-indexed functor** is an oplax $\mathcal{B}$-indexed functor such that the natural transformations $\boldsymbol{\theta}_f$ are natural isomorphisms.

Definition B.5.4. Let $\mathcal{B}$ be a category and let $F, G : \mathbb{D} \to \mathbb{E}$ be a parallel pair of $\mathcal{B}$-indexed functors. A **$\mathcal{B}$-indexed natural transformation** $\alpha : F \Rightarrow G$ consists of a natural transformation $\alpha_B : F^B \Rightarrow G^B$ for each object B in $\mathcal{B}$, such that the following equation holds:

$$
\begin{tikzcd}
\mathcal{D}^B \arrow[r, "G^B"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\overset{\theta_f}{\nearrow}"] & \mathcal{E}^B \arrow[d, "f^*"] \\
\mathcal{D}^{B'} \arrow[r, "G^{B'}"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha_{B'}}{\nearrow}"] & \mathcal{E}^{B'} \arrow[d, equal] \\
\mathcal{D}^{B'} \arrow[r, "F^{B'}"'] & \mathcal{E}^{B'}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{D}^B \arrow[r, "F^B"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\alpha_B}{\nearrow}"] & \mathcal{E}^B \arrow[d, equal] \\
\mathcal{D}^B \arrow[r, "G^B"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\overset{\theta_f}{\nearrow}"] & \mathcal{E}^B \arrow[d, "f^*"] \\
\mathcal{D}^{B'} \arrow[r, "G^{B'}"'] & \mathcal{E}^{B'}
\end{tikzcd}
$$

Remark B.5.5. In other words, a $\mathcal{B}$-indexed category is a contravariant pseudofunctor from the category $\mathcal{B}$ to the meta-2-category of all (not necessarily small) categories, a $\mathcal{B}$-indexed functor is a pseudonatural transformation between such pseudofunctors, and in turn, a $\mathcal{B}$-indexed natural transformation is a modification between such pseudonatural transformations.

Example B.5.6. Any contravariant (strict) functor from $\mathcal{B}$ to the meta-category of all (not necessarily small) categories is a $\mathcal{B}$-indexed category in a trivial way. In particular, every presheaf on $\mathcal{B}$ can be regarded as a $\mathcal{B}$-indexed category.

¶ B.5.7. Given two $\mathcal{B}$-indexed categories, say $\mathbb{D}$ and $\mathbb{E}$, we may form a category $[\mathbb{D}, \mathbb{E}]$ whose objects are $\mathcal{B}$-indexed functors $\mathbb{D} \to \mathbb{E}$ and whose morphisms are $\mathcal{B}$-indexed natural transformations. Of course, these are the hom-categories of the evident meta-2-category of $\mathcal{B}$-indexed categories, $\mathcal{B}$-indexed functors, and $\mathcal{B}$-indexed natural transformations.

Lemma B.5.8. *Let $\mathcal{B}$ be a category, let B be an object in $\mathcal{B}$, and let $\mathbb{E}$ be a $\mathcal{B}$-indexed category. Then the following functor is (half of) an equivalence of categories,*

$$
\begin{aligned}
\left[\hbar_B, \mathbb{E}\right] &\to \mathcal{E}^B \\
F &\mapsto F^B\left(\mathrm{id}_B\right)
\end{aligned}
$$

where $\hbar_B$ is the representable presheaf on $\mathcal{B}$ regarded as a $\mathcal{B}$-indexed category.

Proof. First, we show that the given functor is fully faithful. Let $F, G : \mathcal{h}_B \to \mathbb{E}$ be two $\mathcal{B}$-indexed functors, and let $\alpha : F \Rightarrow G$ be a $\mathcal{B}$-indexed natural transformation. Then, for any morphism $f : B' \to B$ in $\mathcal{B}$, we have the following commutative diagram:

$$\begin{tikzcd} F^{B'}(f) \arrow[r, "(\alpha_{B'})_f"] \arrow[d, "(\boldsymbol{\theta}_f)_{\mathrm{id}}"'] & G^{B'}(f) \arrow[d, "(\boldsymbol{\theta}_f)_{\mathrm{id}}"] \\ f^* F^B(\mathrm{id}_B) \arrow[r, "f^*((\alpha_B)_{\mathrm{id}})"'] & f^* G^B(\mathrm{id}_B) \end{tikzcd}$$

Since $\boldsymbol{\theta}_f : F^{B'} f^* \Rightarrow f^* F^B$ is a natural isomorphism, we may deduce that all the components of α are uniquely determined by $(\alpha_B)_{\mathrm{id}} : F^B(\mathrm{id}_B) \to G^B(\mathrm{id}_B)$. Conversely, given any morphism $g : F^B(\mathrm{id}_B) \to G^B(\mathrm{id}_B)$, we may define a $\mathcal{B}$-indexed natural transformation $\alpha : F \Rightarrow G$ such that $(\alpha_B)_{\mathrm{id}} = g$: the components of α are determined as above, and it is straightforward to check that the various axioms are satisfied.

It now suffices to show that the given functor $[\mathcal{h}_B, \mathbb{E}] \to \mathcal{E}^B$ is essentially surjective on objects. Let E be an object in $\mathcal{E}^B$. We define a $\mathcal{B}$-indexed functor $F : \mathcal{h}_B \to \mathbb{E}$ as follows: given a morphism $f : B' \to B$ in $\mathcal{B}$, set $F^{B'}(f) = f^* E$, and given another morphism $f' : B'' \to B'$ in $\mathcal{B}$, set $(\boldsymbol{\theta}_{f'})_f = (\boldsymbol{\mu}_{f,f'})_E^{-1}$. It is easy to verify that these data indeed constitute a $\mathcal{B}$-indexed functor, and by construction, we have a canonical isomorphism $E \to F^B(\mathrm{id}_B)$, namely $(\boldsymbol{\eta}_B)_E : E \to (\mathrm{id}_B)^* E$. ■

Definition B.5.9. Let $P : \mathcal{E} \to \mathcal{B}$ be a functor.

- A ***P*-vertical morphism** is a morphism in $\mathcal{E}$ whose image in $\mathcal{B}$ is an identity morphism.
- The **fibre** of P over an object B in $\mathcal{B}$ is the subcategory $P^{-1}\{B\} \subseteq \mathcal{E}$ whose objects are those E such that $PE = B$ and whose morphisms are the P-vertical morphisms.

Remark B.5.10. The fibres of a functor $P : \mathcal{E} \to \mathcal{B}$ are usually *not* full subcategories of $\mathcal{E}$. Moreover, the notion of P-vertical morphism is *not* stable under equivalence: indeed, a morphism in $\mathcal{E}$ may be isomorphic to a P-vertical morphism without being a P-vertical morphism.

Definition B.5.11. Let $P : \mathcal{E} \to \mathcal{B}$ be a functor.

- A **weakly P-cartesian morphism** in $\mathcal{E}$ is a morphism $g : E' \to E$ with the following property: for each morphism $h : E'' \to E$ in $\mathcal{E}$, if $PE'' = PE'$ and $Ph = Pg$, then there is a unique P-vertical morphism $g' : E'' \to E'$ such that $h = g \circ g'$.

- A **weakly P-cocartesian morphism** in $\mathcal{E}$ is a morphism $g : E \to E'$ with the following property: for each morphism $h : E \to E''$ in $\mathcal{E}$, if $PE'' = PE'$ and $Ph = Pg$, then there is a unique P-vertical morphism $g' : E' \to E''$ such that $h = g' \circ g$.

Remark. In the French literature (e.g. [SGA 1, Exposé VI]), weakly cartesian morphisms are simply called 'morphismes cartésiens'.

Lemma B.5.12 (Chevalley). *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor, let E be an object in $\mathcal{E}$, let $B = PE$, let $f : B' \to B$ be a morphism in $\mathcal{B}$, and let $Q : P^{-1}\{B'\} \to (B' \downarrow P)$ be the evident functor sending each object E'' in $P^{-1}\{B'\}$ to $\left(B', E'', \mathrm{id}_B\right)$ in the comma category $(\mathcal{B} \downarrow P)$. The following are equivalent for a morphism $g : E' \to E$ in $\mathcal{E}$ such that $Pg = f$:*

(i) *The morphism $g : E' \to E$ is a weakly P-cocartesian morphism.*

(ii) *For every object E'' in $P^{-1}\{B'\}$, the following map is a bijection:*

$$\begin{aligned} P^{-1}\{B'\}(E'', E') &\to (B' \downarrow P)(QE'', (B', E, f)) \\ g' &\mapsto g \circ g' \end{aligned}$$

(iii) *The object (E', g) in the comma category $(Q \downarrow (B', E, f))$ is a terminal object.*

Proof. This is a straightforward exercise. ■

Corollary B.5.13. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. If both $g : E' \to E$ and $h : E'' \to E$ are weakly P-cartesian morphisms with $Pg = Pg'$, then there is a unique P-vertical morphism $g' : E'' \to E'$ such that $h = g \circ g'$, and it is an isomorphism in PE'.* ■

Definition B.5.14. Let $P : \mathcal{E} \to \mathcal{B}$ be a functor.

- A ***P*-prone morphism** is a morphism $g : E' \to E$ in $\mathcal{E}$ with the following property: if $f : B' \to B$ is the image of g under P, and $h : E'' \to E$ is any morphism in $\mathcal{E}$ such that $Ph = f \circ f'$ for some $f' : B'' \to B'$ in $\mathcal{B}$, then there is a unique morphism $g' : E'' \to E'$ in $\mathcal{E}$ such that $Pg' = f'$ and $h = g \circ g'$:

$$\begin{tikzcd}
E'' \arrow[dr, dashed, "g'", "\exists!"'] \arrow[drr, bend left, "h"] \arrow[dd, mapsto] & & \\
& E' \arrow[r, "g"'] \arrow[dd, mapsto] & E \arrow[dd, mapsto] \\
B'' \arrow[dr, "f'"'] \arrow[drr, bend left, "f \circ f'"] & & \\
& B' \arrow[r, "f"'] & B
\end{tikzcd}$$

- A ***P*-supine morphism** is a morphism $g : E \to E'$ in $\mathcal{E}$ with the following property: if $f : B \to B'$ is the image of g under P, and $h : E \to E''$ is any morphism in $\mathcal{E}$ such that $Ph = f' \circ f$ for some $f' : B' \to B''$ in $\mathcal{B}$, then there is a unique morphism $g' : E \to E''$ in $\mathcal{E}$ such that $Pg' = f'$ and $h = g' \circ g$:

$$\begin{tikzcd}
& & E'' \arrow[dd, mapsto] \\
E \arrow[r, "g"'] \arrow[urr, bend left, "h"] \arrow[dd, mapsto] & E' \arrow[ur, dashed, "\exists!", "g'"'] \arrow[dd, mapsto] & \\
& & B'' \\
B \arrow[r, "f"'] \arrow[urr, bend left, "f' \circ f"] & B' \arrow[ur, "f'"'] &
\end{tikzcd}$$

Remark. In the French literature (e.g. [Giraud, 1964]), prone morphisms are called 'morphismes hypercartésiens', whereas in the English literature (e.g. [Bénabou, 1985]), they are simply called 'cartesian morphisms'. To avoid confusion, we will avoid the latter term.

Remark B.5.15. Clearly, every P-prone (resp. P-supine) morphism in $\mathcal{E}$ is also weakly P-cartesian (resp. weakly P-cocartesian).

Example B.5.16. Let $\mathcal{B}$ be a category, let $[\mathbb{2}, \mathcal{B}]$ be the arrow category, and let $P : [\mathbb{2}, \mathcal{B}] \to \mathcal{B}$ be the evident functor that sends an object in $[\mathbb{2}, \mathcal{B}]$ to its codomain (considered as a morphism in $\mathcal{B}$). Then a morphism in $[\mathbb{2}, \mathcal{B}]$ is P-prone if and only if it is a pullback square in $\mathcal{B}$. This is the reason why P-prone morphisms are often called '(strongly) P-cartesian'. However, a morphism in $[\mathbb{2}, \mathcal{B}]$ is P-supine if and only if the top arrow is an isomorphism in $\mathcal{B}$—not a pushout square, as one might expect!

Lemma B.5.17 (Chevalley)**.** *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor, let $Q : \mathcal{E} \to (\mathcal{B} \downarrow P)$ be the evident functor sending each object E in $\mathcal{E}$ to $\left(PE, E, \mathrm{id}_{PE}\right)$ in the comma category $(\mathcal{B} \downarrow P)$, and let $f : B' \to B$ be a morphism in $\mathcal{B}$. The following are equivalent for any morphism $g : E' \to E$ in $\mathcal{E}$ such that $Pg = f$:*

(i) *The morphism $g : E' \to E$ is a P-prone morphism in $\mathcal{E}$.*

(ii) *For every object E'' in $\mathcal{E}$, the following map is a bijection:*

$$\begin{aligned}\mathcal{E}(E'', E') &\to (\mathcal{B} \downarrow P)(QE'', (B', E, f))\\ g' &\mapsto (Pg', g \circ g')\end{aligned}$$

(iii) *The object $\left(E', \left(\mathrm{id}_{B'}, f\right)\right)$ in the comma category $(Q \downarrow (B', E, f))$ is a terminal object.*

Proof. This is a straightforward exercise. ◇

Corollary B.5.18. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. If both $g : E' \to E$ and $h : E'' \to E$ are P-prone morphisms with $Pg = Pg'$, then there is a unique P-vertical morphism $g' : E'' \to E'$ such that $h = g \circ g'$, and it is an isomorphism in the fibre of PE'.* ■

Definition B.5.19.

- A **Grothendieck prefibration** (resp. **Grothendieck fibration**) is a functor $P : \mathcal{E} \to \mathcal{B}$ with the following lifting property: for every object E in $\mathcal{E}$ and every morphism $f : B' \to PE$ in $\mathcal{B}$, there exists a weakly P-cartesian (resp. P-prone) morphism $g : f^*E \to E$ in $\mathcal{E}$ with $Pg = f$.

- A **Grothendieck pre-opfibration** (resp. **Grothendieck opfibration**) is a functor $P : \mathcal{E} \to \mathcal{B}$ with the following lifting property: for every object E in $\mathcal{E}$ and every morphism $f : PE \to B'$ in $\mathcal{B}$, there exists a weakly P-cocartesian (resp. P-supine) morphism $g : E \to f_*E$ in $\mathcal{E}$ with $Pg = f$.

Lemma B.5.20. *Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck prefibration (resp. Grothendieck fibration). Then every morphism in $\mathcal{E}$ can be factored as a P-vertical morphism followed by a weakly P-cartesian (resp. P-prone) morphism, and this factorisation is unique up to unique P-vertical isomorphism.*

Proof. Let $h : E'' \to E$ be a morphism in $\mathcal{E}$ and let $f = Ph$. By hypothesis, there is a weakly P-cartesian (resp. P-prone) morphism $g : E' \to E$ in $\mathcal{E}$ such that $Pg = f$; and since $g : E' \to E$ is weakly P-cartesian (resp. P-prone), there is a unique P-vertical morphism $g' : E'' \to E'$ such that $h = g \circ g'$. Corollary B.5.13 (resp. corollary B.5.18) then implies that this factorisation is unique up to unique P-vertical isomorphism. ∎

Proposition B.5.21. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. The following are equivalent:*

(i) *$P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck prefibration.*

(ii) *For each object B' in $\mathcal{B}$, there exists a right adjoint for the evident functor $Q : P^{-1}\{B'\} \to (B' \downarrow P)$ that sends each object E' in $P^{-1}\{B'\}$ to the object $(E', \mathrm{id}_{B'})$ in the comma category $(B' \downarrow P)$.*

Proof. Apply lemma B.5.12. ∎

Proposition B.5.22. *Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. The following are equivalent for a morphism $g : E' \to E$ in $\mathcal{E}$:*

(i) *$g : E' \to E$ is P-prone.*

(ii) *$g : E' \to E$ is weakly P-cartesian.*

Proof. (i) ⇒ (ii). See remark B.5.15.

(ii) ⇒ (i). Suppose $g : E' \to E$ is weakly P-cartesian. Let $f = Pg$. Since $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration, there is a P-prone morphism $h : f^* E \to E$ in $\mathcal{E}$ such that $Ph = f$; but P-prone morphisms are also weakly P-cartesian, so by corollary B.5.13, there is a P-vertical isomorphism $g' : f^* E \to E'$ such that $h = g \circ g'$, and therefore $g : E' \to E$ is also P-prone. ∎

Lemma B.5.23. *For any functor $P : \mathcal{E} \to \mathcal{B}$, the class of P-prone morphisms in $\mathcal{E}$ is closed under retracts.*

Proof. Suppose we have a commutative diagram in $\mathcal{E}$ of the form below,

$$\begin{tikzcd}
A' \arrow[r, "s'"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "a"'] & E' \arrow[r, two heads, "r'"'] \arrow[d, "g"'] & A' \arrow[d, "a"] \\
A \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & E \arrow[r, two heads, "r"] & A
\end{tikzcd}$$

where $g : E' \to E$ is a P-cartesian morphism. Let $h : A'' \to A$ be a morphism in $\mathcal{E}$ and let $f' : PA'' \to PA'$ be a morphism in $\mathcal{B}$ such that $Ph = Pa \circ Pf'$. There is a unique morphism $g' : A'' \to E'$ in $\mathcal{E}$ such that $s \circ h = g \circ g'$ and $Pg' = Ps' \circ f'$. Let $a' = r' \circ g'$. Then,

$$h = r \circ s \circ h = r \circ g \circ g' = a \circ r' \circ g' = a \circ a'$$
$$Pa' = Pr' \circ Pg' = Pr' \circ Ps' \circ f' = f'$$

and moreover, for any morphism $h' : A'' \to A'$ in $\mathcal{E}$ such that $h = a \circ h'$ and $Ph' = f'$, we must have

$$s \circ h = s \circ a \circ h' = g \circ s' \circ h'$$
$$P(s' \circ h') = Ps' \circ f'$$

so $s' \circ h' = g'$, and therefore $h' = r' \circ s' \circ h' = r' \circ g' = a'$. Thus $a : A' \to A$ is indeed P-cartesian. ■

Lemma B.5.24. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor and let $g : E' \to E$ be a P-prone morphism in $\mathcal{E}$. The following are equivalent for a morphism $g' : E'' \to E'$ in $\mathcal{E}$:*

(i) *$g' : E'' \to E'$ is P-prone.*

(ii) *$g \circ g' : E'' \to E$ is P-prone.*

Proof. (i) ⇒ (ii). Let $g' : E'' \to E'$ is a P-prone morphism in $\mathcal{E}$, let $h : E''' \to E$ be a morphism in $\mathcal{E}$ and let $f'' : PE''' \to PE''$ be a morphism in $\mathcal{B}$ such that $Ph = Pg \circ Pg' \circ f''$. We must show that there is a unique morphism $g'' : E''' \to E''$ in $\mathcal{E}$ such that $h = g \circ g' \circ g''$ and $Pg'' = f''$. To that end, observe that there is a unique morphism $h' : E''' \to E'$ in $\mathcal{E}$ such that $h = g \circ h'$ and $Ph' = Pg' \circ f''$, and thus there is a unique morphism $g'' : E''' \to E''$ in $\mathcal{E}$ such that $h' = g' \circ g''$ and $Pg'' = f''$; in particular, $h = g \circ g' \circ g''$. Moreover, if $h'' : E''' \to E''$ is any morphism in $\mathcal{E}$ such that $h = g \circ g' \circ h''$ and $Ph'' = f''$, then we must have $h' = g' \circ h''$ and hence $g'' = h''$. This proves that $g \circ g' : E'' \to E$ is a P-prone morphism in $\mathcal{E}$.

(ii) ⇒ (i). Suppose $g \circ g' : E'' \to E$ is a P-prone morphism in $\mathcal{E}$. Let $h' : E''' \to E'$ be a morphism in $\mathcal{E}$ and let $f'' : PE''' \to PE''$ be a morphism in $\mathcal{B}$ such that $Ph' = Pg' \circ f''$. We must show that there is a unique morphism $g'' : E''' \to E''$ in $\mathcal{E}$ such that $h' = g' \circ g''$ and $Pg'' = f''$. But since $g \circ g' : E'' \to E$ is P-prone, there is a unique morphism $g'' : E''' \to E''$ in $\mathcal{E}$ such that $g \circ h' = g \circ g' \circ g''$ and $Pg'' = f''$; and since $g : E' \to E$ is P-prone, we must have $h' = g' \circ g''$. Thus, $g' : E'' \to E'$ is a P-prone morphism in $\mathcal{E}$. ■

Remark B.5.25. In the special case where P is $\mathrm{codom} : [\mathbb{2}, \mathcal{B}] \to \mathcal{B}$, we recover the well-known pullback pasting lemma.

Proposition B.5.26. *Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. The following are equivalent for a morphism $g : E' \to E$:*

(i) *The morphism $g : E' \to E$ is a P-prone morphism in $\mathcal{E}$.*

(ii) *The morphism $g : E' \to E$ is right orthogonal to every P-vertical morphism in $\mathcal{E}$.*

(iii) *The morphism $g : E' \to E$ has the right lifting property with respect to every P-vertical morphism in $\mathcal{E}$.*

Proof. (i) ⇒ (ii). Consider a lifting problem in $\mathcal{E}$ of the form below,

$$\begin{tikzcd} A' \arrow[r, "e'"] \arrow[d, "a"'] & E' \arrow[d, "g"] \\ A \arrow[r, "e"'] & E \end{tikzcd}$$

where $a : A' \to A$ is a P-vertical morphism and $g : E' \to E$ is a P-prone morphism. Then, $Pe = P(e \circ a) = Pg \circ Pe'$, so there is a unique morphism $h : A \to E'$ in $\mathcal{E}$ such that $e = g \circ h$ and $Ph = Pe'$. Thus, $g \circ e' = g \circ h \circ a$ and $Pe' = P(h \circ a)$, so we must have $e' = h \circ a$ as well. Moreover, if $l : A \to E'$ is any morphism such that $e' = l \circ a$ and $e = g \circ l$, then $Pl = P(l \circ a) = Pe'$, so we must have $h = l$. Thus, $g : E' \to E$ is right orthogonal to every P-vertical morphism in $\mathcal{E}$.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Let $h : E'' \to E$ be a morphism in $\mathcal{E}$ with the right lifting property with respect to every P-vertical morphism in $\mathcal{E}$. By lemma B.5.20, $h = g \circ g'$ for some P-prone morphism $g : E' \to E$ and P-vertical morphism $g' : E'' \to E'$. Thus, there is a morphism $r : E' \to E''$ making the following diagram commute,

$$\begin{tikzcd}
E'' \arrow[r, "\mathrm{id}"] \arrow[d, "g'"'] & E'' \arrow[d, "h"] \\
E' \arrow[r, "g"'] \arrow[ur, dashed, "r"] & E
\end{tikzcd}$$

and thus $h : E'' \to E$ is a retract of $g : E' \to E$. But proposition B.5.28 says that the class of P-prone morphisms is closed under retracts, so it follows that $h : E'' \to E$ is indeed P-prone. ■

Corollary B.5.27. *Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration. Then there is an orthogonal factorisation system on $\mathcal{E}$ where:*

- *The left class contains all P-vertical morphisms.*
- *The right class is the class of P-prone morphisms.*

Proof. Apply the retract argument (proposition A.3.19) to lemma B.5.20 and proposition B.5.26. ■

Proposition B.5.28. *Let $P : \mathcal{E} \to \mathcal{B}$ be a Grothendieck fibration.*

(i) *Every isomorphism in $\mathcal{E}$ is P-prone.*

(ii) *The class of P-prone morphisms in $\mathcal{E}$ is closed under composition.*

(iii) *The class of P-prone morphisms in $\mathcal{E}$ is closed under retracts.*

(iv) *The class of P-prone morphisms in $\mathcal{E}$ is closed under pullback.*

(v) *The class of P-prone morphisms in $\mathcal{E}$ is closed under limits in $[\mathbb{2}, \mathcal{E}]$.*

Proof. (i)–(v). Apply proposition A.3.17 to proposition B.5.26. ■

Proposition B.5.29. *Let $P : \mathcal{E} \to \mathcal{B}$ be a functor. The following are equivalent:*

(i) *$P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration.*

(ii) *$P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck prefibration such that the class of weakly P-cartesian morphisms in $\mathcal{E}$ is closed under composition.*

(iii) *There exists a right adjoint for the evident functor $Q : \mathcal{E} \to (\mathcal{B} \downarrow P)$ that sends an object E in $\mathcal{E}$ to the object $\left(PE, E, \mathrm{id}_{PE}\right)$ in the comma category $(\mathcal{B} \downarrow P)$.*

Proof. (i) ⇒ (ii). Suppose $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration. Remark B.5.15 then implies $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck prefibration. We then combine propositions B.5.22 and B.5.28 to deduce that the class of weakly P-cartesian morphisms in $\mathcal{E}$ is closed under composition.

(ii) ⇒ (i). Suppose $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck prefibration such that the class of weakly P-cartesian morphisms in $\mathcal{E}$ is closed under composition. To verify that $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck fibration, it suffices to show that every weakly P-cartesian morphism in $\mathcal{E}$ is a P-prone morphism.

Let $g : E' \to E$ be a weakly P-cartesian morphism in $\mathcal{E}$, let $h : E'' \to E$ be a morphism in $\mathcal{E}$, and let $f' : PE'' \to PE'$ be a morphism in $\mathcal{B}$ such that $Ph = Pg \circ f'$. By hypothesis, there is a weakly P-cartesian morphism $\tilde{f}' : \tilde{E}'' \to E'$ such that $P\tilde{f}' = f'$; moreover, $g \circ \tilde{f}'$ is weakly P-cartesian, so there exists a unique P-vertical morphism $k : E'' \to \tilde{E}''$ such that $h = g \circ \tilde{f}' \circ k$. Let $g' = \tilde{f}' \circ k$. Clearly, $h = g \circ g'$ and $Pg' = f'$. Suppose $h' : E'' \to E'$ is a morphism in $\mathcal{E}$ such that $h = g \circ h'$ and $Ph' = f'$. Then there is a unique P-vertical morphism $l : E'' \to \tilde{E}''$ such that $h' = \tilde{f}' \circ l$; but then $g \circ \tilde{f}' \circ k = g \circ \tilde{f}' \circ l$, so we must have $k = l$ and hence $g' = h'$. Thus, $g : E' \to E$ is indeed a P-prone morphism in $\mathcal{E}$.

(i) ⇔ (iii). Apply lemma B.5.17. ■

Definition B.5.30. Let $\mathcal{B}$ be a category, let $\mathbb{E}$ be a $\mathcal{B}$-indexed category, and let $\mathbb{F}$ be a $\mathcal{B}^{\mathrm{op}}$-indexed category. The Grothendieck construction $\mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F})$ is the following category:

- The objects are triples (E, B, F), where B is an object in $\mathcal{B}$, E is an object in $\mathcal{E}^B$, and F is an object in $\mathcal{F}^B$.
- The morphisms $(E', B', F') \to (E, B, F)$ are triples (g, f, h), where $f : B' \to B$ is a morphism in $\mathcal{B}$, $g : E' \to f^* E$ is a morphism in $\mathcal{E}^{B'}$, and $h : f_* F' \to F$ is a morphism in $\mathcal{F}^B$.

- Identities and composition are defined using composition in $\mathcal{B}$ and the fibres of $\mathbb{E}$ and $\mathbb{F}$ as well as the coherence data of $\mathbb{E}$ and $\mathbb{F}$.

Note that there is a canonical projection $\mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \to \mathcal{B}$ whose fibre over B is (isomorphic to) $\mathcal{E}^B \times \mathcal{F}^B$.

Proposition B.5.31. *Let $\mathcal{B}$ be a category.*

- *For any $\mathcal{B}$-indexed category $\mathbb{E}$, the projection $\mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1})$ is a Grothendieck fibration, where $\Delta\mathbb{1}$ is the unique $\mathcal{B}^{\mathrm{op}}$-indexed category whose fibres are $\mathbb{1}$.*

- *For any $\mathcal{B}^{\mathrm{op}}$-indexed category $\mathbb{F}$, the projection $\mathbf{G}(\Delta\mathbb{1}, \mathbb{B}, \mathbb{F})$ is a Grothendieck opfibration, where $\Delta\mathbb{1}$ is the unique $\mathcal{B}$-indexed category whose fibres are $\mathbb{1}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $\mathcal{E} = \mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1})$ and let $P : \mathcal{E} \to \mathcal{B}$ be the canonical projection.

Suppose $(g, f) : (E', B') \to (E, B)$ is a morphism in $\mathcal{E}$ where $g : E' \to f^* E$ is an isomorphism in $\mathcal{E}^{B'}$. Let $h : E'' \to f^* E$ be a morphism in $\mathcal{E}^{B'}$. Then there is a unique morphism $g' : E'' \to \left(\mathrm{id}_{B'}\right)^*$ in $\mathcal{E}^{B'}$ such that

$$g \circ \left(\boldsymbol{\eta}_{B'}\right)^{-1}_{E'} \circ g' = h$$

i.e. such that $(g, f) \circ \left(g', \mathrm{id}_{B'}\right) = (h, f)$ as morphisms in $\mathcal{E}$. Thus $(g, f) : (E', B') \to (E, B)$ is a weakly P-cartesian morphism. In particular, $\left(\mathrm{id}_{f^* E}, f\right) : (f^* E, B') \to (E, B)$ is a weakly P-cartesian morphism in $\mathcal{E}$.

Conversely, suppose $(g, f) : (E', B') \to (E, B)$ is a weakly P-cartesian morphism. Then, by corollary B.5.13 and the above observation, there is a unique isomorphism $g' : E' \to f^* E$ in $\mathcal{E}^{B'}$ such that $g = \mathrm{id}_{f^* E} \circ g'$, i.e. $g : E' \to f^* E$ itself is an isomorphism in $\mathcal{E}^{B'}$.

It now follows that $P : \mathcal{E} \to \mathcal{B}$ is a Grothendieck prefibration where the class of weakly P-cartesian morphisms is closed under composition, so by proposition B.5.29, $P : \mathcal{E} \to \mathcal{B}$ is indeed a Grothendieck fibration. ■

Lemma B.5.32. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories, let $\mathbb{E}$ be a $\mathcal{B}$-indexed category, and let $\mathbb{F}$ be a $\mathcal{B}^{\mathrm{op}}$-indexed category.*

- *We have a natural commutative diagram of the form below,*

$$\begin{tikzcd} \mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F} \times \Delta\mathcal{A}) \arrow[r, "\cong"] \arrow[d] & \mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \times \mathcal{A} \arrow[d] \\ \mathcal{B} \arrow[r, equal] & \mathcal{B} \end{tikzcd}$$

where the two vertical arrows are the evident projections and the top horizontal arrow is defined by $(E, B, (F, A)) \mapsto ((E, B, F), A)$.

- *We have a natural commutative diagram of the form below,*

$$\begin{tikzcd} \mathbf{G}(\Delta\mathcal{A} \times \mathbb{E}, \mathcal{B}, \mathbb{F}) \arrow[r, "\cong"] \arrow[d] & \mathcal{A} \times \mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \arrow[d] \\ \mathcal{B} \arrow[r, equal] & \mathcal{B} \end{tikzcd}$$

where the two vertical arrows are the evident projections and the top horizontal arrow is defined by $((A, E), B, F) \mapsto (A, (E, B, F))$.

Proof. This is a straightforward exercise. ◊

Lemma B.5.33. *Let $U : \mathcal{A} \to \mathcal{B}$ be a functor, let $\mathbb{E}$ be a $\mathcal{B}$-indexed category, and let $\mathbb{F}$ be a $\mathcal{B}^{\mathrm{op}}$-indexed category. Then there is a natural pullback square of the form below,*

$$\begin{tikzcd} \mathbf{G}(\mathbb{E} \circ U, \mathcal{A}, \mathbb{F} \circ U) \arrow[r] \arrow[d] & \mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \arrow[d] \\ \mathcal{A} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

where the vertical arrows are the canonical projections.

Proof. This is a straightforward exercise. ◊

Proposition B.5.34. *Let $\mathcal{B}$ be a category, let $\mathbb{E}$ be a $\mathcal{B}$-indexed category, and let $\mathbb{F}$ be a $\mathcal{B}^{\mathrm{op}}$-indexed category. Then there are natural isomorphisms*

$$\mathbf{G}(\Delta\mathbb{1}, \mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1}), \mathbb{F} \circ P) \cong \mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \cong \mathbf{G}(\mathbb{E} \circ Q, \mathbf{G}(\Delta\mathbb{1}, \mathcal{B}, \mathbb{F}), \Delta\mathbb{1})$$

where $P : \mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1}) \to \mathcal{B}$ *and* $Q : \mathbf{G}(\Delta\mathbb{1}, \mathcal{B}, \mathbb{F}) \to \mathcal{B}$ *are the canonical projections, such that the following diagram commutes,*

$$\begin{tikzcd}
\mathbf{G}(\Delta\mathbb{1}, \mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1}), \mathbb{F} \circ P) \arrow[r] \arrow[d, "\cong"'] & \mathbf{G}(\mathbb{E}, \mathcal{B}, \Delta\mathbb{1}) \arrow[r] & \mathcal{B} \arrow[d, equal] \\
\mathbf{G}(\mathbb{E}, \mathcal{B}, \mathbb{F}) \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{B} \arrow[d, equal] \\
\mathbf{G}(\mathbb{E} \circ Q, \mathbf{G}(\Delta\mathbb{1}, \mathcal{B}, \mathbb{F}), \Delta\mathbb{1}) \arrow[r] \arrow[u, "\cong"] & \mathbf{G}(\Delta\mathbb{1}, \mathcal{B}, \mathbb{F}) \arrow[r] & \mathcal{B}
\end{tikzcd}$$

where the horizontal arrows are the canonical projections and the diagonal arrows are induced by the unique indexed functors $\mathbb{E} \to \Delta\mathbb{1}$ *and* $\mathbb{F} \to \Delta\mathbb{1}$.

Proof. This is a straightforward exercise. $\Diamond$

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