

# Notes on homotopical algebra

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## PREFACE

These notes are intended as a kind of annotated index to the various standard references in homotopical algebra: the focus is on definitions and statements of results, *not* proofs.



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## FOUNDATIONS

### 0.1 Set theory

In category theory it is often convenient to invoke a certain set-theoretic device commonly known as a ‘Grothendieck universe’, but we shall say simply ‘universe’, so as to simplify exposition and proofs by eliminating various circumlocutions involving cardinal bounds, proper classes etc.

**Definition 0.1.1.** A **pre-universe** is a set  $\mathbf{U}$  satisfying these axioms:

1. If  $x \in y$  and  $y \in \mathbf{U}$ , then  $x \in \mathbf{U}$ .
2. If  $x \in \mathbf{U}$  and  $y \in \mathbf{U}$  (but not necessarily distinct), then  $\{x, y\} \in \mathbf{U}$ .
3. If  $x \in \mathbf{U}$ , then  $\mathcal{P}(x) \in \mathbf{U}$ , where  $\mathcal{P}(x)$  denotes the set of all subsets of  $x$ .
4. If  $x \in \mathbf{U}$  and  $f : x \rightarrow \mathbf{U}$  is a map, then  $\bigcup_{i \in x} f(i) \in \mathbf{U}$ .

A **universe** is a pre-universe  $\mathbf{U}$  with this additional property:

5.  $\omega \in \mathbf{U}$ , where  $\omega$  is the set of all finite (von Neumann) ordinals.

**Example 0.1.2.** The empty set is a pre-universe, and with very mild assumptions, so is the set  $\mathbf{HF}$  of all hereditarily finite sets.

¶ 0.1.3. The notion of universe makes sense in any material set theory, but their existence must be postulated. We adopt the following:

- **Grothendieck–Verdier universe axiom.** For each set  $x$ , there exists a universe  $\mathbf{U}$  with  $x \in \mathbf{U}$ .

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For definiteness, we may take our base theory to be Mac Lane set theory, which is a weak subsystem of Zermelo–Fraenkel set theory with choice (ZFC). Readers interested in the details of Mac Lane set theory are referred to [Mathias, 2001], but in practice, as long as one is working at all times *inside some universe*, one may as well be working in ZFC. Indeed:

**Proposition 0.1.4.** *With the assumptions of Mac Lane set theory, any universe is a transitive model of ZFC.*

*Proof.* Let  $\mathbf{U}$  be a universe. By definition,  $\mathbf{U}$  is a transitive set containing pairs, power sets, unions, and  $\omega$ , so the axioms of extensionality, empty set, pairs, power sets, unions, choice, and infinity are all automatically satisfied. We must show that the axiom schemas of separation and replacement are also satisfied, and in fact it is enough to check that replacement is valid; but this is straightforward using axioms 2 and 4. ■

**Definition 0.1.5.** Let  $\mathbf{U}$  be a pre-universe. A **U-set** is a member of  $\mathbf{U}$ , a **U-class** is a subset of  $\mathbf{U}$ , and a **proper U-class** is a U-class that is not a U-set.

**Lemma 0.1.6.** *A U-class  $X$  is a U-set if and only if there exists a U-class  $Y$  such that  $X \in Y$ .* ■

**Proposition 0.1.7.** *If  $\mathbf{U}$  is a universe in Mac Lane set theory, then the collection of all U-classes is a transitive model of Morse–Kelley class–set theory (MK), and so is a transitive model of von Neumann–Bernays–Gödel class–set theory (NBG) in particular.* ■

**Definition 0.1.8.** A **U-small category** is a category  $\mathcal{C}$  such that  $\text{ob } \mathcal{C}$  and  $\text{mor } \mathcal{C}$  are U-sets. A **locally U-small category** is a category  $\mathcal{D}$  satisfying these conditions:

- $\text{ob } \mathcal{D}$  and  $\text{mor } \mathcal{D}$  are U-classes, and
- for all objects  $x$  and  $y$  in  $\mathcal{D}$ , the hom-set  $\mathcal{D}(x, y)$  is a U-set.

An **essentially U-small category** is a category  $\mathcal{D}$  for which there exist a U-small category  $\mathcal{C}$  and a functor  $\mathcal{C} \rightarrow \mathcal{D}$  that is fully faithful and essentially surjective on objects.

**Proposition 0.1.9.** *If  $\mathbb{D}$  is a U-small category and  $\mathcal{C}$  is a locally U-small category, then the functor category  $[\mathbb{D}, \mathcal{C}]$  is locally U-small.*

*Proof.* Strictly speaking, this depends on the set-theoretic implementation of ordered pairs, categories, functors, etc., but at the very least  $[\mathbb{D}, C]$  should be isomorphic to a locally  $\mathbf{U}$ -small category.

In the context of  $[\mathbb{D}, C]$ , we may regard functors  $\mathbb{D} \rightarrow C$  as being the pair consisting of the *graph* of the object map  $\text{ob } \mathbb{D} \rightarrow \text{ob } C$  and the *graph* of the morphism map  $\text{mor } \mathbb{D} \rightarrow \text{mor } C$ , and these are  $\mathbf{U}$ -sets by the  $\mathbf{U}$ -replacement axiom. Similarly, if  $F$  and  $G$  are objects in  $[\mathbb{D}, C]$ , then we may regard a natural transformation  $\alpha : F \Rightarrow G$  as being the triple  $(F, G, A)$ , where  $A$  is the set of all pairs  $(c, \alpha_c)$ . ■

One complication introduced by having multiple universes concerns the existence of (co)limits.

**Theorem 0.I.10** (Freyd). *Let  $C$  be a category and let  $\kappa$  be a cardinal such that  $|\text{mor } C| \leq \kappa$ . If  $C$  has products for families of size  $\kappa$ , then any two parallel morphisms in  $C$  must be equal.*

*Proof.* Suppose, for a contradiction, that  $f, g : X \rightarrow Y$  are distinct morphisms in  $C$ . Let  $Z$  be the product of  $\kappa$ -many copies of  $Y$  in  $C$ . The universal property of products implies there are at least  $2^\kappa$ -many distinct morphisms  $X \rightarrow Z$ ; but  $C(X, Z) \subseteq \text{mor } C$ , so this is an absurdity. ■

**Definition 0.I.11.** Let  $\mathbf{U}$  be a pre-universe. A  **$\mathbf{U}$ -complete** (resp.  **$\mathbf{U}$ -cocomplete**) **category** is a category  $C$  with the following property:

- For all  $\mathbf{U}$ -small categories  $\mathbb{D}$  and all diagrams  $A : \mathbb{D} \rightarrow C$ , a limit (resp. colimit) of  $A$  exists in  $C$ .

We may instead say  $C$  has all **finite limits** (resp. **finite colimits**) in the special case  $\mathbf{U} = \mathbf{HF}$ .

**Proposition 0.I.12.** *Let  $C$  be a category and let  $\mathbf{U}$  be a non-empty pre-universe. The following are equivalent:*

- (i)  $C$  is  $\mathbf{U}$ -complete.
- (ii)  $C$  has all finite limits and products for all families of objects indexed by a  $\mathbf{U}$ -set.

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(iii) For each  $\mathbf{U}$ -small category  $\mathbb{D}$ , there exists an adjunction

$$\Delta \dashv \lim_{\leftarrow \mathbb{D}} : [\mathbb{D}, \mathcal{C}] \rightarrow \mathcal{C}$$

where  $\Delta X$  is the constant functor with value  $X$ .

Dually, the following are equivalent:

(i')  $\mathcal{C}$  is  $\mathbf{U}$ -cocomplete.

(ii')  $\mathcal{C}$  has all finite colimits and coproducts for all families of objects indexed by a  $\mathbf{U}$ -set.

(iii') For each  $\mathbf{U}$ -small category  $\mathbb{D}$ , there exists an adjunction

$$\lim_{\rightarrow \mathbb{D}} \dashv \Delta : \mathcal{C} \rightarrow [\mathbb{D}, \mathcal{C}]$$

where  $\Delta X$  is the constant functor with value  $X$ .

*Proof.* This is a standard result; but we remark that we do require a sufficiently powerful form of the axiom of choice to pass from (ii) to (iii).  $\square$

**¶ 0.1.13.** In the **explicit universe convention**, the words ‘set’, ‘class’, etc. have their usual meanings, and in the **one-universe convention**, these instead abbreviate ‘ $\mathbf{U}$ -set’, ‘ $\mathbf{U}$ -class’, etc. for a fixed (but arbitrary) universe  $\mathbf{U}$ . However, the word ‘category’ always refers to a category that is contained in *some* universe, which may or may not be locally  $\mathbf{U}$ -small, and we shall use the word ‘ensemble’ to refer to sets which may or may not be in  $\mathbf{U}$ . In subsequent chapters, the implicit universe convention should be assumed *unless otherwise stated*.

We now recall some definitions and results about ordinal and cardinal numbers. Readers familiar with axiomatic set theory may wish to skip ahead.

**Definition 0.1.14.** A **von Neumann ordinal** is a set  $\alpha$  with the following properties:

- If  $x \in y$  and  $y \in \alpha$ , then  $x \in \alpha$ .
- The binary relation  $\in$  is strict total ordering of  $\alpha$ .
- If  $S$  is a subset of  $\alpha$  such that

$$- \emptyset \in S,$$

- If  $\beta \in S$  and  $\beta \cup \{\beta\} \in \alpha$ , then  $\beta \cup \{\beta\} \in S$ .
- If  $T \subseteq S$ , then  $\bigcup T \in S$ .

then  $S = \alpha$ .

We identify 0 with the von Neumann ordinal  $\emptyset$ , and by induction, we identify the natural number  $n + 1$  with the von Neumann ordinal  $\{0, \dots, n\}$ .

**Proposition 0.I.15.**

- (i) *If  $\alpha$  is a von Neumann ordinal, then every member of  $\alpha$  is an initial segment of  $\alpha$  and is in particular a von Neumann ordinal.*
- (ii) *If  $\alpha$  is a von Neumann ordinal, so is  $\alpha \cup \{\alpha\}$ . (This is usually denoted by  $\alpha + 1$  and called the **successor** of  $\alpha$ .)*
- (iii) *The union of a set  $S$  of von Neumann ordinals is another von Neumann ordinal. (This is usually denoted by  $\sup S$  and called the **supremum** of  $S$ .)*
- (iv) *If  $\mathbf{U}$  is a pre-universe and  $\kappa(\mathbf{U})$  is the set of von Neumann ordinals in  $\mathbf{U}$ , then  $\kappa(\mathbf{U})$  a von Neumann ordinal, but  $\kappa(\mathbf{U}) \notin \mathbf{U}$ .*

*Proof.* Claims (i) – (iii) are all easy, and claim (iv) is Burali-Forti’s paradox. ◆

**Theorem 0.I.16** (Classification of well-orderings).

- (i) *In Zermelo–Fraenkel set theory, every well-ordered set is isomorphic to a unique von Neumann ordinal.*
- (ii) *In Mac Lane set theory, if  $\mathbf{U}$  is a pre-universe and  $X$  is a well-ordered set in  $\mathbf{U}$ , then  $X$  is isomorphic to a unique von Neumann ordinal in  $\mathbf{U}$ .*

*Proof.* Claim (i) is a standard result in axiomatic set theory, and claim (ii) is an obvious corollary. □

**Definition 0.I.17.** A **transitive set** is a set  $T$  such that, given  $x \in y$ , if  $y \in T$ , then  $x \in T$  as well. The **transitive closure** of a set  $X$  is a set  $\text{tcl}(X)$  such that, for all transitive sets  $T$  with  $X \subseteq T$ , we have  $\text{tcl}(X) \subseteq T$  as well.

**Lemma 0.I.18.** *In Mac Lane set theory, every set has a unique transitive closure.*

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*Proof.* One of the axioms of Mac Lane set theory states that every set  $X$  is a member of some transitive set  $T$ , and so  $X \subseteq T$ . Clearly, the intersection of any family of transitive sets containing  $X$  is again a transitive set containing  $X$ , so  $\text{tcl}(X)$  exists and is unique so long as there is at least one transitive set containing  $X$ . ■

**Definition 0.1.19.** A **partial rank function** from a transitive set  $T$  to a well-ordered set  $W$  is a partial function  $\rho : T \rightarrow W$  with these properties:

- If  $\emptyset \in T$ , then  $\rho(\emptyset)$  is the least element of  $W$ .
- If  $y \in T$  and  $\rho(x)$  is defined for all  $x \in y$ , then

$$\rho(y) = \min \{w \in W \mid \forall x \in y. \rho(x) < w\}$$

provided the RHS is defined.

- Otherwise  $\rho(y)$  is undefined.

A **total rank function** is a partial rank function that is defined on its entire domain. The **rank** of a set  $X$ , if it exists, the least von Neumann ordinal  $\text{rank}(X)$  for which there exists a total rank function  $\text{tcl}(X) \rightarrow \text{rank}(X)$ .

**Proposition 0.1.20.** *In Mac Lane set theory:*

- (i) *If  $T$  is a transitive set and  $W$  is a well-ordered set, then there is a unique partial rank function  $\rho : T \rightarrow W$ .*
- (ii) *If  $\mathbf{U}$  is a pre-universe and  $x \in \mathbf{U}$ , then  $\text{rank}(x)$  can be defined by a  $\Delta_0$ -formula with  $\mathbf{U}$  as a parameter, and for each von Neumann ordinal  $\alpha$  in  $\mathbf{U}$ , the set*

$$\mathbf{V}_\alpha = \{x \in \mathbf{U} \mid \text{rank}(x) < \alpha\}$$

*is a  $\mathbf{U}$ -set.*

- (iii) *Assuming the Grothendieck–Verdier universe axiom,  $\text{rank}(x)$  is defined for all  $x$ .*

*Proof.* (i). This is a straightforward application of well-founded induction.

(ii).  $\mathbf{U}$  is a transitive set and the set  $\kappa(\mathbf{U})$  of all von Neumann ordinals in  $\mathbf{U}$  is well-ordered by inclusion, so by claim (i) there is a partial rank function  $\rho :$

$\mathbf{U} \rightarrow \kappa(\mathbf{U})$ . ZFC proves that every set has a rank, so  $\rho$  must in fact be a total rank function; hence, for any  $x \in \mathbf{U}$ ,  $\text{rank}(x)$  is defined. It is clear that  $\rho$  can be defined by a  $\Delta_0$ -formula with only  $\mathbf{U}$  as a parameter, and the rest of the claim follows.

(iii). Obvious, assuming claim (ii). □

**Definition 0.1.21.** Two sets are **equinumerous** if there exists a bijection between them. A **cardinality class** in a pre-universe  $\mathbf{U}$  is an equivalence class under the relation of equinumerosity.

**Definition 0.1.22.** An  **$\aleph$ -number** is an infinite von Neumann ordinal  $\kappa$  such that, for any von Neumann ordinal  $\lambda$  such that  $\kappa$  and  $\lambda$  are equinumerous, we have  $\kappa \subseteq \lambda$ .

**Example 0.1.23.** The first infinite von Neumann ordinal, i.e.  $\omega = \{0, 1, 2, \dots\}$ , is the  $\aleph$ -number  $\aleph_0$ .

**Lemma 0.1.24.** *If  $\kappa$  is an  $\aleph$ -number, then there exists a unique  $\aleph$ -number  $\kappa^+$  with the following property:*

- *For any  $\aleph$ -number  $\lambda$  such that  $\kappa < \lambda$ , we have  $\kappa^+ \leq \lambda$ .*

*The cardinal successor of  $\kappa$  is  $\kappa^+$ .*

*Proof.* The class of  $\aleph$ -numbers is well-ordered and unbounded, so the class of all  $\aleph$ -numbers  $> \kappa$  has a minimal element  $\kappa^+$ , as required. ■

**Theorem 0.1.25** (Classification of cardinalities).

- (i) *In Zermelo–Fraenkel set theory, for every well-ordered infinite set  $X$ , there exists a unique  $\aleph$ -number  $\kappa$  such that  $X$  and  $\kappa$  are equinumerous.*
- (ii) *In Zermelo–Fraenkel set theory with the axiom of choice, the same is true for any infinite set whatsoever.*
- (iii) *In Mac Lane set theory, if  $\mathbf{U}$  is a universe and  $X$  is an infinite set in  $\mathbf{U}$ , then there exists a unique  $\aleph$ -number  $\kappa$  in the cardinality class of  $X$ .*
- (iv) *In Mac Lane set theory with the Grothendieck–Verdier universe axiom, if  $\mathbf{U}$  is a pre-universe and  $\kappa$  is an  $\aleph$ -number not in  $\mathbf{U}$ , then the cardinality of  $\mathbf{U}$  is at most  $\kappa$ .*

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*Proof.* Claim (i) is a standard fact, whence claims (ii) and (iii), by the well-ordering theorem. Claim (iv) can be proven using axiom 4 for pre-universes.  $\square$

$\P$  0.1.26. Henceforth, we identify the cardinality class of a finite set with the unique von Neumann ordinal contained in that class, and similarly we identify the cardinality class of an infinite set with the unique  $\aleph$ -number in that class. These are the **cardinal numbers**.

**Definition 0.1.27.** A **cofinal subset** of a partially-ordered set  $X$  is a subset  $Y \subseteq X$  such that, for all  $x$  in  $X$ , there exists some  $y$  in  $Y$  such that  $x \leq y$ . A **regular cardinal number** is an  $\aleph$ -number  $\kappa$  such that any cofinal subset of  $\kappa$  has cardinality equal to  $\kappa$ . A **singular cardinal number** is an  $\aleph$ -number that is not regular.

The following helps to motivate the definition of regular cardinal numbers.

**Definition 0.1.28.** Let  $\mathbf{U}$  be a pre-universe. An **arity class** in  $\mathbf{U}$  is a  $\mathbf{U}$ -class  $K$  of cardinal numbers satisfying the following conditions:

- $1 \in K$ .
- If  $\kappa \in K$  and  $\lambda : \kappa \rightarrow K$  is a function, then the cardinal sum  $\sum_{\alpha \in \kappa} \lambda(\alpha)$  is also in  $K$ .
- If  $\kappa \in K$  and  $\lambda : \kappa \rightarrow \mathbf{U}$  is a function such that each  $\lambda(\alpha)$  is a cardinal number and  $\sum_{\alpha \in \kappa} \lambda(\alpha) \in K$ , then  $\lambda(\alpha) \in K$  as well.

**Theorem 0.1.29** (Classification of arity classes). *In Mac Lane set theory, if  $K$  is an arity class in a pre-universe  $\mathbf{U}$ , then  $K$  must be either*

- $\{1\}$ , or
- $\{0, 1\}$ , or
- of the form  $\{\lambda \in \mathbf{U} \mid \lambda \text{ is a cardinal number and } \lambda < \kappa\}$  for some regular cardinal number  $\kappa$  (possibly not in  $\mathbf{U}$ ).

*Proof.* The notion of arity class and this result are due to Shulman [2012].  $\square$

**Definition 0.1.30.** Let  $\kappa$  be a regular cardinal number. A  **$\kappa$ -small category** is a category  $\mathbb{C}$  such that  $\text{mor } \mathbb{C}$  has cardinality  $< \kappa$ . A **finite category** is an  $\aleph_0$ -small



category, i.e. a category  $\mathbb{C}$  such that  $\text{mor } \mathbb{C}$  is finite. A **finite diagram** (resp.  **$\kappa$ -small diagram**,  **$\mathbf{U}$ -small diagram**) in a category  $\mathcal{C}$  is a functor  $\mathbb{D} \rightarrow \mathcal{C}$  where  $\mathbb{D}$  is a finite (resp.  $\kappa$ -small,  $\mathbf{U}$ -small) category.

**Theorem 0.1.31.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{U}^+$  be a universe with  $\mathbf{U} \in \mathbf{U}^+$ , let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathbf{Set}^+$  be the category of  $\mathbf{U}^+$ -sets.*

- (i) *If  $X : \mathbb{D} \rightarrow \mathbf{Set}$  is a  $\mathbf{U}$ -small diagram, then there exist a limit and a colimit for  $X$  in  $\mathbf{Set}$ .*
- (ii) *The inclusion  $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$  is fully faithful and preserves limits and colimits for all  $\mathbf{U}$ -small diagrams.*

*Proof.* One can construct products, equalisers, coproducts, coequalisers, and hom-sets in a completely explicit way, making the preservation properties obvious. ◆

**Corollary 0.1.32.** *The inclusion  $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$  reflects limits and colimits for all  $\mathbf{U}$ -small diagrams.* ■

**Corollary 0.1.33.** *For any  $\mathbf{U}$ -small category  $\mathbb{C}$ :*

- (i) *The functor category  $[\mathbb{C}, \mathbf{Set}]$  is  $\mathbf{U}$ -complete and  $\mathbf{U}$ -cocomplete, with limits and colimits for  $\mathbf{U}$ -small diagrams computed componentwise in  $\mathbf{Set}$ .*
- (ii) *The inclusion  $[\mathbb{C}, \mathbf{Set}] \hookrightarrow [\mathbb{C}, \mathbf{Set}^+]$  is fully faithful and both preserves and reflects limits and colimits for all  $\mathbf{U}$ -small diagrams.* ■

**Definition 0.1.34.** An **strongly inaccessible cardinal number** is a regular cardinal number  $\kappa$  such that, for all sets  $X$  of cardinality less than  $\kappa$ , the power set  $\mathcal{P}(X)$  is also of cardinality less than  $\kappa$ .

**Example 0.1.35.**  $\aleph_0$  is a strongly inaccessible cardinal number and is the only one that can be proven to exist in ZFC. It is more conventional to exclude  $\aleph_0$  from the definition of strongly inaccessible cardinal number by demanding that they be uncountable.

**Proposition 0.1.36.** *In Mac Lane set theory:*

- (i) *If  $\mathbf{U}$  is a non-empty pre-universe, then there exists a strongly inaccessible cardinal number  $\kappa$  such that the members of  $\mathbf{U}$  are all the sets of rank less than  $\kappa$ . Moreover, this  $\kappa$  is the rank and the cardinality of  $\mathbf{U}$ .*

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- (ii) If  $\mathbf{U}$  is a universe and  $\kappa$  is a strongly inaccessible cardinal number such that  $\kappa \in \mathbf{U}$ , then there exists a  $\mathbf{U}$ -set  $\mathbf{V}_\kappa$  whose members are all the sets of rank less than  $\kappa$ , and  $\mathbf{V}_\kappa$  is a pre-universe.
- (iii) If  $\mathbf{U}$  and  $\mathbf{U}'$  are pre-universes, then either  $\mathbf{U} \subseteq \mathbf{U}'$  or  $\mathbf{U}' \subseteq \mathbf{U}$ ; and if  $\mathbf{U} \subsetneq \mathbf{U}'$ , then  $\mathbf{U} \in \mathbf{U}'$ .

*Proof.* (i). Let  $\kappa$  be the set of all von Neumann ordinals in  $\mathbf{U}$ ; this exists by  $\Delta_0$ -separation applied to  $\mathbf{U}$ . Since  $\mathbf{U}$  is closed under power sets and internally-indexed unions,  $\kappa$  must be a strongly inaccessible cardinal.

We can construct the set all of  $\mathbf{U}$ -sets of rank less than  $\kappa$  using transfinite recursion on  $\kappa$  as follows: starting with  $\mathbf{V}_0 = \emptyset$ , for each von Neumann ordinal  $\alpha$  less than  $\kappa$ , we set  $\mathbf{V}_{\alpha+1} = \mathcal{P}(\mathbf{V}_\alpha)$ , and for each ordinal  $\lambda$  that is not a successor, we set  $\mathbf{V}_\lambda = \bigcup_{\alpha < \lambda} \mathbf{V}_\alpha$ . The well-foundedness of  $\in$  (restricted to  $\mathbf{U}$ ) implies that in fact this must be all of  $\mathbf{U}$ .

Clearly, every set of rank less than  $\kappa$  is in fact a  $\mathbf{U}$ -set, and  $\mathbf{U}$  is itself a set of rank  $\kappa$ . The cardinality of  $\mathbf{U}$  is also  $\kappa$ , since  $\kappa$  is a regular cardinal number and any cardinal number less than  $\kappa$  is a member of  $\mathbf{U}$ .

(ii). We may construct  $\mathbf{V}_\kappa$  using the same method as in (i). By construction  $\mathbf{V}_\kappa$  satisfies axiom 1; since  $\kappa$  is infinite,  $\mathbf{V}_\kappa$  satisfies axioms 2 and 3; and since  $\kappa$  is strongly inaccessible,  $\mathbf{V}_\kappa$  satisfies axiom 4. Thus  $\mathbf{V}_\kappa$  is a pre-universe.

(iii). Again, let  $\kappa$  be the rank of  $\mathbf{U}$ . If  $\kappa \in \mathbf{U}'$  then we can show by transfinite induction that  $\mathbf{V}_\kappa \in \mathbf{U}'$  and so  $\mathbf{U} \subsetneq \mathbf{U}'$ ; else we must have  $\mathbf{U}' \subseteq \mathbf{V}_\kappa = \mathbf{U}$ . ■

## 0.2 Accessibility and ind-completions

### Prerequisites. §0.1.

A classical technology for controlling size problems in category theory, due to Gabriel and Ulmer [1971], Grothendieck and Verdier [SGA 4a, Exposé I, §9], and Makkai and Paré [1989], is the notion of accessibility. Though we make use of universes, accessibility remains important and is a crucial tool in verifying the stability of various universal constructions when one passes from one universe to a larger one.

**Definition 0.2.1.** Let  $\kappa$  be a regular cardinal.

- A  **$\kappa$ -filtered category** is a category  $\mathcal{J}$  with the following property:
  - For each  $\kappa$ -small diagram  $A$  in  $\mathcal{J}$ , there exist an object  $j$  and a cocone  $A \Rightarrow \Delta j$ .

A  **$\kappa$ -filtered diagram** in a category  $\mathcal{C}$  is a functor  $\mathcal{J} \rightarrow \mathcal{C}$  where  $\mathcal{J}$  is a  $\kappa$ -filtered category.

- A  **$\kappa$ -directed preorder** is a preordered set  $X$  that is  $\kappa$ -filtered when considered as a category, i.e. a preorder with the following property:
  - For each  $\kappa$ -small subset  $Y \subseteq X$ , there exists an element  $x$  of  $X$  such that  $y \leq x$  for all  $y$  in  $Y$ .

A  **$\kappa$ -directed diagram** in a category  $\mathcal{C}$  is a functor  $\mathcal{J} \rightarrow \mathcal{C}$  where  $\mathcal{J}$  is a  $\kappa$ -directed preorder (considered as a category).

In both cases, it is conventional to omit  $\kappa$  when  $\kappa = \aleph_0$ .

**REMARK 0.2.2.** For any regular cardinal  $\kappa$ , the category with one object and only one non-trivial arrow  $f$  is  $\kappa$ -filtered if and only if  $f = f \circ f$ . In particular, any category that has colimits for small  $\kappa$ -filtered diagrams must also have splittings for idempotents.

**Example 0.2.3.** Let  $X$  be any set. The set of all finite subsets of  $X$ , partially ordered by inclusion, is a directed preorder. More generally, if  $\kappa$  is any regular cardinal, then the set of all subsets of  $X$  of cardinality  $< \kappa$  is a  $\kappa$ -directed preorder.

**Definition 0.2.4.** Let  $\alpha$  be an ordinal. An  **$\alpha$ -chain** in a category  $\mathcal{C}$  is a functor  $\alpha \rightarrow \mathcal{C}$ , where we have identified  $\alpha$  with the well-ordered set of ordinals  $< \alpha$ .

**REMARK 0.2.5.** If  $\alpha$  is an ordinal with cofinality  $\kappa$ , then  $\alpha$  is a  $\kappa$ -directed preorder. In particular,  $\alpha$ -chains are  $\kappa$ -directed diagrams.

**Lemma 0.2.6.** Let  $\mathcal{I}$  be any category and let  $\mathcal{J}$  be a filtered category. Given a full functor  $F : \mathcal{I} \rightarrow \mathcal{J}$ , the following are equivalent:

- (i)  $F : \mathcal{I} \rightarrow \mathcal{J}$  is a cofinal functor.<sup>[1]</sup>

[1] See definition A.5.31.

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(ii) *For each object  $j$  in  $\mathcal{J}$ , there exist an object  $i$  in  $\mathcal{I}$  and a morphism  $j \rightarrow Fi$  in  $\mathcal{J}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Since  $F : \mathcal{I} \rightarrow \mathcal{J}$  is a cofinal functor, the comma category  $(j \downarrow F)$  is connected; in particular, it is inhabited.

(ii)  $\Rightarrow$  (i). The hypothesis says that the comma category  $(j \downarrow F)$  is inhabited for all objects  $j$  in  $\mathcal{J}$ ; it remains to be shown that each  $(j \downarrow F)$  is connected. Suppose we have morphisms  $f : j \rightarrow Fi$  and  $f' : j \rightarrow Fi'$  in  $\mathcal{J}$ . Since  $\mathcal{J}$  is a filtered category, there exist morphisms  $g : Fi \rightarrow j'$  and  $g' : Fi' \rightarrow j'$  such that  $g \circ f = g' \circ f'$ . By hypothesis, there is a morphism  $h : j' \rightarrow Fi''$  in  $\mathcal{J}$ , and since  $F : \mathcal{I} \rightarrow \mathcal{J}$  is full, there exist morphisms  $k : i \rightarrow i''$  and  $k' : i' \rightarrow i''$  in  $\mathcal{I}$  such that  $Fk = h \circ g$  and  $Fk' = h \circ g'$ . Thus, we have  $Fk \circ f = Fk' \circ f'$ , so  $(j \downarrow F)$  is indeed connected.  $\blacksquare$

**Lemma 0.2.7.** *Let  $\mathcal{I}$  be a filtered category and let  $\mathcal{J}$  be any preorder. Given a functor  $F : \mathcal{I} \rightarrow \mathcal{J}$ , the following are equivalent:*

(i)  *$F : \mathcal{I} \rightarrow \mathcal{J}$  is a cofinal functor.*

(ii) *For each object  $j$  in  $\mathcal{J}$ , there exist an object  $i$  in  $\mathcal{I}$  such that  $j \leq Fi$  in  $\mathcal{J}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Since  $F : \mathcal{I} \rightarrow \mathcal{J}$  is a cofinal functor, the comma category  $(j \downarrow F)$  is connected; in particular, it is inhabited.

(ii)  $\Rightarrow$  (i). The hypothesis says that the comma category  $(j \downarrow F)$  is inhabited for all objects  $j$  in  $\mathcal{J}$ ; it remains to be shown that each  $(j \downarrow F)$  is connected. Suppose we have morphisms  $j \leq Fi$  and  $j \leq Fi'$  in  $\mathcal{J}$ . Since  $\mathcal{I}$  is a filtered category, there exist an object  $i''$  in  $\mathcal{I}$  and morphisms  $i \rightarrow i''$  and  $i' \rightarrow i''$ ; thus, we have  $j \leq Fi \leq Fi''$  and  $j \leq Fi' \leq Fi''$ , so  $(j \downarrow F)$  is indeed connected.  $\blacksquare$

**Lemma 0.2.8.** *Let  $\mathcal{J}$  be a  $\kappa$ -filtered diagram. If  $\mathcal{J}$  is also  $\kappa$ -small, then there exist an object  $j$  in  $\mathcal{J}$  and an idempotent morphism  $e : j \rightarrow j$  such that the subcategory of  $\mathcal{J}$  generated by  $e$  is cofinal in  $\mathcal{J}$ .*

*Proof.* Since  $\text{id} : \mathcal{J} \rightarrow \mathcal{J}$  is a  $\kappa$ -small diagram in  $\mathcal{J}$ , there must exist an object  $j$  in  $\mathcal{J}$  and a cocone  $\lambda : \text{id} \Rightarrow \Delta j$ . Let  $e = \lambda_j : j \rightarrow j$ . Since  $\lambda$  is a cocone, we must have  $e = e \circ e$ , i.e.  $e : j \rightarrow j$  is idempotent.

Let  $\mathcal{I}$  be the subcategory of  $\mathcal{J}$  generated by  $e$  and let  $j'$  be any object in  $\mathcal{J}$ . We must show that the comma category  $(j' \downarrow \mathcal{I})$  is connected. It is inhabited:

$\lambda_{j'} : j' \rightarrow j$  is an object in  $(j' \downarrow \mathcal{I})$ . Moreover, given any morphism  $f : j' \rightarrow j$  in  $\mathcal{J}$ , we must have  $\lambda_{j'} = \lambda_j \circ f = e \circ f$ , so  $(j' \downarrow \mathcal{I})$  is indeed connected. Thus,  $\mathcal{I}$  is a cofinal subcategory of  $\mathcal{J}$ . ■

**Lemma 0.2.9.** *Let  $\kappa$  be a regular cardinal and let  $(\mathcal{J}_i \mid i \in I)$  be a set of  $\kappa$ -filtered categories.*

- (i) *The product  $\mathcal{J} = \prod_{i \in I} \mathcal{J}_i$  is a  $\kappa$ -filtered category.*
- (ii) *Each projection  $\pi_i : \mathcal{J} \rightarrow \mathcal{J}_i$  is a cofinal functor.*

*Proof.* (i). We may construct cones over  $\kappa$ -small diagrams in  $\mathcal{J}$  componentwise.

(ii). Similarly, one can show that the comma categories  $(j_i \downarrow \pi_i)$  are connected for all  $j_i$  in  $\mathcal{J}_i$  and all  $i$  in  $I$ . ■

**Theorem 0.2.10.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . If  $\mathcal{J}$  is a  $\mathbf{U}$ -small  $\kappa$ -filtered category, then there exist a  $\mathbf{U}$ -small  $\kappa$ -directed poset  $\mathcal{I}$  and a cofinal functor  $P : \mathcal{I} \rightarrow \mathcal{J}$ .*

*Proof.* See Theorem 1.5 and Remark 1.21 in [LPAC]. □

**Theorem 0.2.11.** *Let  $\mathbf{U}$  be a universe. The following are equivalent for a category  $\mathcal{C}$ :*

- (i)  *$\mathcal{C}$  has colimits for  $\mathbf{U}$ -small  $\aleph_0$ -filtered diagrams.*
- (ii)  *$\mathcal{C}$  has colimits for  $\mathbf{U}$ -small  $\aleph_0$ -directed diagrams.*
- (iii)  *$\mathcal{C}$  has colimits for  $\alpha$ -chains for all infinite ordinals  $\alpha$  in  $\mathbf{U}$ .*

*Proof.* (i)  $\Leftrightarrow$  (ii). This is implied by [theorem 0.2.10](#).

(ii)  $\Rightarrow$  (iii). Immediate.

(iii)  $\Rightarrow$  (ii). See Corollary 1.7 in [LPAC]. □

**Theorem 0.2.12.** *Let  $\mathbf{U}$  be a universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\kappa$  be any regular cardinal in  $\mathbf{U}$ . Given a  $\mathbf{U}$ -small category  $\mathcal{J}$ , the following are equivalent:*

- (i)  *$\mathcal{J}$  is a  $\kappa$ -filtered category.*

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- (ii) *The functor  $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \rightarrow \mathbf{Set}$  preserves limits for all  $\kappa$ -small diagrams.*

*Proof.* The claim (i)  $\Rightarrow$  (ii) is very well known, and the converse is an exercise in using the Yoneda lemma and manipulating limits and colimits for diagrams of representable functors; see Satz 5.2 in [Gabriel and Ulmer, 1971].  $\square$

**Definition 0.2.13.** Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$  and let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets.

- A  $(\kappa, \lambda)$ -**compact object** in a locally  $\mathbf{U}$ -small category  $\mathcal{C}$  is an object  $A$  such that the representable functor  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$  preserves colimits for all  $\lambda$ -small  $\kappa$ -filtered diagrams.
- Let  $\mathbf{U}'$  be a universe with  $\mathbf{U}' \subseteq \mathbf{U}$ . A  $(\kappa, \mathbf{U}')$ -**compact object** in a locally  $\mathbf{U}$ -small category is an object that is  $(\kappa, \lambda)$ -compact for all regular cardinals  $\lambda$  in  $\mathbf{U}'$ .

Though the above definition is stated using a universe  $\mathbf{U}$ , the following lemma shows there is in fact no dependence on  $\mathbf{U}$ .

**Lemma 0.2.14.** *Let  $A$  be an object in a locally  $\mathbf{U}$ -small category  $\mathcal{C}$ . The following are equivalent:*

- (i)  *$A$  is a  $(\kappa, \lambda)$ -compact object in  $\mathcal{C}$ .*
- (ii) *For all  $\lambda$ -small  $\kappa$ -filtered diagrams  $B : \mathcal{J} \rightarrow \mathcal{C}$ , if  $\varepsilon : B \Rightarrow \Delta C$  is a colimiting cocone, then for any morphism  $f : A \rightarrow C$ , there exist an object  $i$  in  $\mathcal{J}$  and a morphism  $f' : A \rightarrow B_i$  in  $\mathcal{C}$  such that  $f = \varepsilon_i \circ f'$ ; and moreover if  $f = \varepsilon_j \circ f''$  for some morphism  $f'' : A \rightarrow B_j$  in  $\mathcal{C}$ , then there exists an object  $k$  and a pair of arrows  $g : i \rightarrow k, h : j \rightarrow k$  in  $\mathcal{J}$  such that  $B_g \circ f' = B_h \circ f''$ .*

*Proof.* Use the explicit description of  $\varinjlim_{\mathcal{J}} \mathcal{C}(A, B)$  as a filtered colimit of sets; see Definition 1.1 in [LPAC], or Proposition 5.1.3 in [Borceux, 1994b].  $\square$

**Corollary 0.2.15.** *Let  $B : \mathcal{J} \rightarrow \mathcal{C}$  be a  $\lambda$ -small  $\kappa$ -filtered diagram, and let  $\lambda : B \Rightarrow \Delta C$  be a colimiting cocone in  $\mathcal{C}$ . If  $C$  is a  $(\kappa, \lambda)$ -compact object in  $\mathcal{C}$ , then  $C$  is a retract of some vertex of  $B$ , i.e. there exists an object  $i$  in  $\mathcal{J}$  such that  $\lambda_i : B_i \rightarrow C$  is a split epimorphism.  $\blacksquare$*

**Lemma 0.2.16.** *Let  $A$  be an object in a category  $C$ .*

- (i) *If  $A$  is a  $(\kappa, \lambda)$ -compact object in  $C$  and  $\lambda'$  is any regular cardinal  $\leq \lambda$ , then  $A$  is  $(\kappa, \lambda')$ -compact as well.*
- (ii) *If  $A$  is  $(\kappa, \lambda)$ -compact and  $\mu$  is any regular cardinal  $\geq \kappa$ , then  $A$  is also  $(\mu, \lambda)$ -compact.*

*Proof.* Obvious. ◆

**Lemma 0.2.17.** *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ . If  $B : \mathbb{D} \rightarrow C$  is a  $\kappa$ -small diagram of  $(\kappa, \lambda)$ -compact objects in a locally  $\mathbf{U}$ -small category, then the colimit  $\varinjlim_{\mathbb{D}} B$ , if it exists, is also a  $(\kappa, \lambda)$ -compact object in  $C$ .*

*Proof.* Use [theorem 0.2.12](#) and the fact that  $C(-, C) : C^{\text{op}} \rightarrow \mathbf{Set}^+$  maps colimits in  $C$  to limits in  $\mathbf{Set}^+$ . ■

**Corollary 0.2.18.** *A retract of a  $(\kappa, \lambda)$ -compact object is also a  $(\kappa, \lambda)$ -compact object.*

*Proof.* Suppose  $r : A \rightarrow B$  and  $s : B \rightarrow A$  are morphisms in  $C$  such that  $r \circ s = \text{id}_B$ . Then  $e = s \circ r$  is an idempotent morphism and the diagram below

$$A \begin{array}{c} \xrightarrow{\text{id}_A} \\ \xrightarrow{e} \end{array} A \xrightarrow{r} B$$

is a (split) coequaliser diagram in  $C$ , so  $B$  is  $(\kappa, \lambda)$ -compact if  $A$  is. ■

**Proposition 0.2.19.** *Let  $\mathbf{U}$  be a pre-universe and let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets. For any  $\mathbf{U}$ -set  $A$ , the following are equivalent:*

- (i)  *$A$  has cardinality less than  $\kappa$ .*
- (ii) *The representable functor  $\mathbf{Set}(A, -) : \mathbf{Set} \rightarrow \mathbf{Set}$  preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.*
- (iii) *The representable functor  $\mathbf{Set}(A, -) : \mathbf{Set} \rightarrow \mathbf{Set}$  preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -directed diagrams.*

*Proof.* The claim (i)  $\Rightarrow$  (ii) follows from [theorem 0.2.12](#), and (ii)  $\Rightarrow$  (iii) is obvious. To see (iii)  $\Rightarrow$  (i), we may use [corollary 0.2.15](#) and the fact that every set is the  $\kappa$ -directed union of its subsets of cardinality  $< \kappa$ . ■

**Corollary 0.2.20.** *A  $\mathbf{U}$ -set  $X$  is  $(\kappa, \mathbf{U})$ -compact if and only if  $|X| < \kappa$ . ■*

**Definition 0.2.21.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . A  $\kappa$ -**accessible  $\mathbf{U}$ -category** is a locally  $\mathbf{U}$ -small category  $\mathcal{C}$  satisfying the following conditions:

- $\mathcal{C}$  has colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.
- There exists a  $\mathbf{U}$ -set  $\mathcal{G}$  whose element are  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  such that, for each object  $B$  in  $\mathcal{C}$ , there exists a  $\mathbf{U}$ -small  $\kappa$ -filtered diagram in  $\mathcal{C}$  whose vertices are in  $\mathcal{G}$  and whose colimit is  $B$ .

We write  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  for the full subcategory of  $\mathcal{C}$  spanned by the  $(\kappa, \mathbf{U})$ -compact objects.

**Example 0.2.22.** The category of  $\mathbf{U}$ -sets is a  $\kappa$ -accessible  $\mathbf{U}$ -category for any regular cardinal  $\kappa$  in  $\mathbf{U}$ .

**Theorem 0.2.23.** *Let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category and let  $\kappa$  be a regular cardinal in  $\mathbf{U}$ . There exist a locally  $\mathbf{U}$ -small category  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  and a functor  $\gamma : \mathcal{C} \rightarrow \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  with the following properties:*

- (i) *The objects of  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  are  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams  $B : \mathbb{D} \rightarrow \mathcal{C}$ , and  $\gamma$  sends an object  $C$  in  $\mathcal{C}$  to the corresponding trivial diagram  $\mathbb{1} \rightarrow \mathcal{C}$  with value  $C$ .*
- (ii) *The functor  $\gamma : \mathcal{C} \rightarrow \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  is fully faithful, injective on objects, preserves all limits that exist in  $\mathcal{C}$ , and preserves all  $\kappa$ -small colimits that exist in  $\mathcal{C}$ .*
- (iii)  *$\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  has colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.*
- (iv) *For every object  $C$  in  $\mathcal{C}$ , the object  $\gamma C$  is  $(\kappa, \mathbf{U})$ -compact in  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$ , and for each  $\mathbf{U}$ -small  $\kappa$ -filtered diagram  $B : \mathbb{D} \rightarrow \mathcal{C}$ , there is a canonical colimiting cocone  $\gamma B \Rightarrow \Delta B$  in  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$ .*
- (v) *If  $\mathcal{D}$  is a category with colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, then for each functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , there exists a functor  $\bar{F} : \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C}) \rightarrow \mathcal{D}$  that preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  such that  $\gamma \bar{F} = F$ , and given any functor  $\bar{G} : \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C}) \rightarrow \mathcal{D}$  whatsoever, the induced map  $\text{Nat}(\bar{F}, \bar{G}) \rightarrow \text{Nat}(F, \gamma \bar{G})$  is a bijection.*

The category  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$  is called the **free  $(\kappa, \mathbf{U})$ -ind-completion** of  $\mathcal{C}$ , or the **category of  $(\kappa, \mathbf{U})$ -ind-objects** in  $\mathcal{C}$ .



*Proof.* If  $B : \mathbb{D} \rightarrow C$  and  $B' : \mathbb{D}' \rightarrow C$  are two  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, then properties (ii) and (iii) together imply that

$$\mathrm{Hom}(B', B) \cong \lim_{\leftarrow \mathbb{D}'} \lim_{\rightarrow \mathbb{D}} C(B', B)$$

and so, taking the RHS as the *definition* of the LHS, we need only find a suitable notion of composition to make  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(C)$  into a locally  $\mathbf{U}$ -small category. However, we observe that, if  $N : C \rightarrow [C^{\mathrm{op}}, \mathbf{Set}]$  is the Yoneda embedding, then

$$\mathrm{Hom}\left(\lim_{\rightarrow \mathbb{D}'} NB', \lim_{\rightarrow \mathbb{D}} NB\right) \cong \lim_{\leftarrow \mathbb{D}'} \lim_{\rightarrow \mathbb{D}} C(B', B)$$

and, assuming property (v), the Yoneda embedding  $N : C \rightarrow [C^{\mathrm{op}}, \mathbf{Set}]$  must extend along  $\gamma$  to a functor  $\bar{N} : \mathbf{Ind}_{\mathbf{U}}^{\kappa}(C) \rightarrow [C^{\mathrm{op}}, \mathbf{Set}]$  that preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagram, so, in consideration of properties (i) and (iv), we may as well *define* the composition in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(C)$  so that  $\bar{N}$  becomes fully faithful. This completes the definition of  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(C)$  as a category.

It remains to be shown that  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(C)$  actually has properties (ii), (iii), (iv), and (v); see Corollary 6.4.14 in [Borceux, 1994a] and Theorem 2.26 in [LPAC]. Note that the fact that  $\gamma$  preserves colimits for  $\kappa$ -small diagrams essentially follows from [theorem 0.2.12](#).  $\square$

**Proposition 0.2.24.** *Let  $\mathbb{B}$  be a  $\mathbf{U}$ -small category and let  $\kappa$  be a regular cardinal in  $\mathbf{U}$ .*

- (i)  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category.
- (ii) Every  $(\kappa, \mathbf{U})$ -compact object in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  is a retract of an object of the form  $\gamma B$ , where  $\gamma : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  is the canonical embedding.
- (iii)  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}))$  is an essentially  $\mathbf{U}$ -small category.

*Proof.* (i). This claim more-or-less follows from the properties of  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  explained in the previous theorem.

(ii). Use [corollary 0.2.18](#).

(iii). Since  $\mathbb{B}$  is  $\mathbf{U}$ -small and  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  is locally  $\mathbf{U}$ -small, claim (ii) implies that  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}))$  must be essentially  $\mathbf{U}$ -small.  $\blacksquare$

**Proposition 0.2.25.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category and let  $C$  be an object in  $\mathcal{C}$ .*

- (i) *The comma category  $(\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \downarrow \mathcal{C})$  is an essentially  $\mathbf{U}$ -small  $\kappa$ -filtered category.*
- (ii) *If  $P^C : (\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \downarrow \mathcal{C}) \rightarrow \mathcal{C}$  is the canonical diagram, then the tautological cocone<sup>[2]</sup>  $P^C \Rightarrow \Delta C$  is a colimiting cocone in  $\mathcal{C}$ .*

*Proof.* See Proposition 2.1.5 in [Makkai and Paré, 1989] or Proposition 2.8 in [LPAC]. □

**Corollary 0.2.26.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category. For any  $\mathbf{U}$ -small  $\kappa$ -filtered diagram  $\mathbb{D}$ ,  $\lim_{\rightarrow \mathbb{D}} : [\mathbb{D}, \mathcal{C}] \rightarrow \mathcal{C}$  preserves componentwise limits for  $\kappa$ -small diagrams.*

*Proof.* The claim is certainly true when  $\mathcal{C} = [\mathbb{B}^{\text{op}}, \mathbf{Set}]$ , by theorem 0.2.12. In general, choose a fully faithful functor  $R : \mathcal{C} \rightarrow [\mathbb{B}^{\text{op}}, \mathbf{Set}]$  that preserves limits for all  $\kappa$ -small diagrams and colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams; then  $R$  reflects limits for  $\kappa$ -small diagrams and colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, so we may deduce the claim from the corresponding fact for  $[\mathbb{B}^{\text{op}}, \mathbf{Set}]$ . Note that such a functor exists: propositions 0.2.25 and A.5.25 imply we may take  $\mathbb{B}$  to be  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  and  $R$  to be the induced Yoneda representation. ■

**Definition 0.2.27.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . A  $(\kappa, \mathbf{U})$ -**accessible functor** is a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that

- $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, and
- $F$  preserves all colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.

We write  $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D})$  for the full subcategory of the functor category  $[\mathcal{C}, \mathcal{D}]$  spanned by the  $(\kappa, \mathbf{U})$ -accessible functors. An **accessible functor** is a functor that is  $(\kappa, \mathbf{U})$ -accessible functor for some regular cardinal  $\kappa$  in some universe  $\mathbf{U}$ .

**Theorem 0.2.28** (Classification of accessible categories). *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$  and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. The following are equivalent:*

- (i)  *$\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category.*

[2] See definition A.5.7.

- (ii) *The inclusion  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \hookrightarrow \mathcal{C}$  extends along  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \rightarrow \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}))$  to a  $(\kappa, \mathbf{U})$ -accessible functor  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})) \rightarrow \mathcal{C}$  that is fully faithful and essentially surjective on objects.*
- (iii) *There exist a  $\mathbf{U}$ -small category  $\mathbb{B}$  and a functor  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{B}) \rightarrow \mathcal{C}$  that is fully faithful and essentially surjective on objects.*

*Proof.* See Theorem 2.26 in [LPAC], or Theorem 5.3.5 in [Borceux, 1994b].  $\square$

**Corollary 0.2.29.** *If  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category and  $\mathcal{D}$  is any category, then:*

- (i) *The restriction  $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \rightarrow [\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}), \mathcal{D}]$  is fully faithful and surjective on objects.*
- (ii) *In particular, if  $\mathcal{D}$  is also locally  $\mathbf{U}$ -small, then  $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D})$  is equivalent to a locally  $\mathbf{U}$ -small category.*
- (iii) *If  $\mathcal{D}$  has colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, then the inclusion  $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \hookrightarrow [\mathcal{C}, \mathcal{D}]$  has a left adjoint.  $\blacksquare$*

**Proposition 0.2.30.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category and let  $\mathcal{D}$  be a locally  $\mathbf{U}$ -small category. Given an adjunction  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$ , if  $G$  is fully faithful and preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, then  $\mathcal{D}$  is also a  $\kappa$ -accessible  $\mathbf{U}$ -category.*

*Proof.* Under our hypotheses, given any  $\mathbf{U}$ -small  $\kappa$ -filtered diagram  $A : \mathcal{J} \rightarrow \mathcal{D}$ , we may take  $F \varinjlim_{\mathcal{J}} GA$  as its colimit in  $\mathcal{C}$ . Our hypotheses also imply that  $F$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{D}$ ; thus if  $\mathcal{G}$  is a  $\mathbf{U}$ -small set of objects that generates  $\mathcal{C}$  under  $\mathbf{U}$ -small  $\kappa$ -filtered colimits, then  $\{FX \mid X \in \mathcal{G}\}$  is a  $\mathbf{U}$ -small set of objects that generates  $\mathcal{D}$  in the same sense.  $\blacksquare$

**Definition 0.2.31.** Let  $\kappa$  and  $\lambda$  be regular cardinals and let  $\mathcal{P}_\kappa(X)$  denote the set of all  $\kappa$ -small subsets of a set  $X$ . We say  $\kappa$  is **sharply less than**  $\lambda$  if

- $\kappa < \lambda$ , and
- for all  $\lambda$ -small sets  $X$ , there exists a  $\lambda$ -small cofinal subposet of the poset  $\mathcal{P}_\kappa(X)$ .

We define  $\kappa \triangleleft \lambda$  to mean that  $\kappa$  is sharply less than  $\lambda$ .

## 0. FOUNDATIONS

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**Example 0.2.32.** Let  $\kappa$  be a regular cardinal and let  $\kappa^+$  be its cardinal successor. Then  $\kappa \triangleleft \kappa^+$ : every  $\kappa^+$ -small set can be mapped bijectively onto an initial segment  $\alpha$  of  $\kappa$  (but possibly all of  $\kappa$ ), and it is clear that the subposet

$$\{\beta \mid \beta \leq \alpha\} \subseteq \mathcal{P}_\kappa(\alpha)$$

is a  $\kappa^+$ -small cofinal subposet of  $\mathcal{P}_\kappa(\alpha)$ : given any  $\kappa$ -small subset  $X \subseteq \alpha$ , we must have  $\sup X \leq \alpha$ , and  $X \subseteq \sup X$  by definition.

**Theorem 0.2.33.** *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ , and suppose  $\kappa < \lambda$ . The following are equivalent:*

- (i)  $\kappa \triangleleft \lambda$ .
- (ii) *For any  $\mathbf{U}$ -small  $\kappa$ -directed poset  $X$  and any  $\lambda$ -small subset  $Y \subseteq X$ , there exists a  $\lambda$ -small  $\kappa$ -directed subposet  $X' \subseteq X$  with  $Y \subseteq X'$ .*
- (iii) *Any  $\kappa$ -accessible  $\mathbf{U}$ -category is also a  $\lambda$ -accessible  $\mathbf{U}$ -category.*

*Proof.* See Theorem 2.11 in [LPAC]. □

**Proposition 0.2.34.**

- (i) *The binary relation  $\triangleleft$  is transitive.*
- (ii) *If  $\kappa \leq \lambda$ , then  $\kappa \triangleleft (2^{<\lambda})^+$ , where  $2^{<\lambda} = \sup \{2^\mu \mid \mu \text{ is a cardinal } < \lambda\}$  and  $2^\mu = |\mathcal{P}(\mu)|$ , and also  $\kappa \triangleleft (2^\lambda)^+$ .*
- (iii) *For any set  $K$  of regular cardinals, there exists a regular cardinal  $\lambda$  such that  $\kappa \triangleleft \lambda$  for all  $\kappa$  in  $K$ .*

*Proof.* (i). See Proposition 2.3.2 in [Makkai and Paré, 1989], or [theorem 0.2.33](#).

(ii). See Proposition 2.3.5 in [Makkai and Paré, 1989], or Example 2.13(5) in [LPAC], or Proposition 5.4.7 in [Borceux, 1994b].

(iii). This follows from claim (ii). □

**Definition 0.2.35.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . A **locally  $\kappa$ -presentable  $\mathbf{U}$ -category** is a  $\kappa$ -accessible  $\mathbf{U}$ -category that is also  $\mathbf{U}$ -cocomplete. A **locally presentable  $\mathbf{U}$ -category** is one that is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category for some regular cardinal  $\kappa$  in  $\mathbf{U}$ , and we often say ‘locally finitely presentable’ instead of ‘locally  $\aleph_0$ -presentable’.

**Example 0.2.36.** The category of  $\mathbf{U}$ -sets is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category for any regular cardinal  $\kappa$  in  $\mathbf{U}$ .

**Lemma 0.2.37.** *Let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*

- (i) *For any regular cardinal  $\lambda$  in  $\mathbf{U}$ , if  $\kappa \leq \lambda$ , then  $\mathcal{C}$  is a locally  $\lambda$ -presentable  $\mathbf{U}$ -category.*
- (ii) *With  $\lambda$  as above, if  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a  $(\kappa, \mathbf{U})$ -accessible functor, then it is also a  $(\lambda, \mathbf{U})$ -accessible functor.*
- (iii) *If  $\mathbf{U}^+$  is any universe with  $\mathbf{U} \in \mathbf{U}^+$ , and  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}^+$ -category, then  $\mathcal{C}$  must be a preorder.*

*Proof.* (i). See the remark after Theorem 1.20 in [LPAC], or Propositions 5.3.2 and 5.2.3 in [Borceux, 1994b].

(ii). A  $\lambda$ -filtered diagram is certainly  $\kappa$ -filtered, so if  $F$  preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{C}$ , it must also preserve colimits for all  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams.

(iii). This is a corollary of [theorem 0.1.10](#). ■

**Corollary 0.2.38.** *A category  $\mathcal{C}$  is a locally presentable  $\mathbf{U}$ -category for at most one universe  $\mathbf{U}$ , provided  $\mathcal{C}$  is not a preorder.*

*Proof.* Use [proposition 0.1.36](#) together with the above lemma. ■

**Theorem 0.2.39** (Classification of locally presentable categories). *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. The following are equivalent:*

- (i)  *$\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*
- (ii) *There exist a  $\mathbf{U}$ -small category  $\mathbb{B}$  that has colimits for  $\kappa$ -small diagrams and a functor  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}) \rightarrow \mathcal{C}$  that is fully faithful and essentially surjective on objects.*
- (iii) *The restricted Yoneda embedding  $\mathcal{C} \rightarrow [\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})^{\text{op}}, \mathbf{Set}]$  is fully faithful,  $(\kappa, \mathbf{U})$ -accessible, and has a left adjoint.*

- (iv) *There exist a  $\mathbf{U}$ -small category  $\mathbb{A}$  and a fully faithful  $(\kappa, \mathbf{U})$ -accessible functor  $R : \mathcal{C} \rightarrow [\mathbb{A}, \mathbf{Set}]$  such that  $\mathbb{A}$  has limits for all  $\kappa$ -small diagrams,  $R$  has a left adjoint, and  $R$  is essentially surjective onto the full subcategory of functors  $\mathbb{A} \rightarrow \mathbf{Set}$  that preserve limits for all  $\kappa$ -small diagrams.*
- (v) *There exist a  $\mathbf{U}$ -small category  $\mathbb{A}$  and a fully faithful functor  $\mathcal{C} \rightarrow [\mathbb{A}, \mathbf{Set}]$  that preserves colimits for small  $\kappa$ -filtered diagrams and has a left adjoint.*
- (vi)  *$\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category and is  $\mathbf{U}$ -complete.*

*Proof.* See Proposition 1.27, Corollary 1.28, Theorem 1.46, and Corollary 2.47 in [LPAC], or Theorems 5.2.7 and 5.5.8 in [Borceux, 1994b].  $\square$

**REMARK 0.2.40.** If  $\mathcal{C}$  is equivalent to  $\mathbf{Ind}_{\kappa}^{\mathbf{U}}(\mathbb{B})$  for some  $\mathbf{U}$ -small category  $\mathbb{B}$  that has colimits for all  $\kappa$ -small diagrams, then  $\mathbb{B}$  must be equivalent to  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$  by [proposition 0.2.24](#). In other words, every locally  $\kappa$ -presentable  $\mathbf{U}$ -category is, up to equivalence, the free  $(\kappa, \mathbf{U})$ -ind-completion of an essentially unique  $\mathbf{U}$ -small  $\kappa$ -cocomplete category.

**Example 0.2.41.** Obviously, for any  $\mathbf{U}$ -small category  $\mathbb{A}$ , the functor category  $[\mathbb{A}, \mathbf{Set}]$  is locally finitely presentable. More generally, one may show that for any  $\kappa$ -ary algebraic theory  $\mathbf{T}$ , possibly many-sorted, the category of  $\mathbf{T}$ -algebras in  $\mathbf{U}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category. The above theorem can also be used to show that  $\mathbf{Cat}$ , the category of  $\mathbf{U}$ -small categories, is a locally finitely presentable  $\mathbf{U}$ -small category.

**Proposition 0.2.42.** *If  $\mathcal{C}$  is an accessible  $\mathbf{U}$ -category and  $\mathbb{D}$  is any  $\mathbf{U}$ -small category, then the functor category  $[\mathbb{D}, \mathcal{C}]$  is also an accessible  $\mathbf{U}$ -category.*

*Proof.* See Theorem 2.39 in [LPAC].  $\square$

**Proposition 0.2.43.** *If  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category and  $\mathbb{D}$  is any  $\mathbf{U}$ -small category, then the functor category  $[\mathbb{D}, \mathcal{C}]$  is also a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*

*Proof.* This can be proven using the classification theorem by noting that the 2-functor  $[\mathbb{D}, -]$  preserves reflective subcategories, but see also Corollary 1.54 in [LPAC].  $\square$

It is commonplace to say ‘ $\lambda$ -presentable object’ instead of ‘ $\lambda$ -compact object’, especially in algebraic contexts. The following propositions justify the alternative terminology.

**Proposition 0.2.44.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category. If  $\lambda$  is a regular cardinal in  $\mathbf{U}$  and  $\kappa \triangleleft \lambda$ , then the following are equivalent for an object  $C$  in  $\mathcal{C}$ :*

- (i)  $C$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ .
- (ii) There exists a  $\lambda$ -small  $\kappa$ -filtered diagram  $A : \mathcal{J} \rightarrow \mathcal{C}$  such that each  $A_j$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$  and  $C \cong \varinjlim_{\mathcal{J}} A$ .
- (iii) There exists a  $\lambda$ -small  $\kappa$ -directed diagram  $A : \mathcal{J} \rightarrow \mathcal{C}$  such that each  $A_j$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$  and  $C$  is a retract of  $\varinjlim_{\mathcal{J}} A$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). See Proposition 2.3.11 in [Makkai and Paré, 1989].

(i)  $\Leftrightarrow$  (iii). See Remark 2.15 in [LPAC]. □

**Proposition 0.2.45.** *Let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, and let  $\lambda$  be a regular cardinal in  $\mathbf{U}$  with  $\lambda \geq \kappa$ . If  $\mathcal{H}$  is a  $\mathbf{U}$ -small full subcategory of  $\mathcal{C}$  such that*

- every  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$  is isomorphic to an object in  $\mathcal{H}$ , and
- $\mathcal{H}$  is closed in  $\mathcal{C}$  under colimits for  $\lambda$ -small diagrams,

*then every  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$  is isomorphic to an object in  $\mathcal{H}$ . In particular,  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$  is the smallest replete full subcategory of  $\mathcal{C}$  containing  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  and closed in  $\mathcal{C}$  under colimits for  $\lambda$ -small diagrams.*

TODO: Simplify this argument.

*Proof.* Let  $C$  be any  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ . Clearly, the comma category  $(\mathcal{H} \downarrow \mathcal{C})$  is a  $\mathbf{U}$ -small  $\lambda$ -filtered category. Let  $\mathcal{G} = \mathcal{H} \cap \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ . One can show that  $(\mathcal{G} \downarrow \mathcal{C})$  is a cofinal subcategory in  $(\mathcal{H} \downarrow \mathcal{C})$ , and the classification theorem (0.2.39) plus proposition A.5.25 implies that the tautological cocone on the diagram  $(\mathcal{G} \downarrow \mathcal{C}) \rightarrow \mathcal{C}$  is colimiting, so the tautological cocone on the diagram  $(\mathcal{H} \downarrow \mathcal{C}) \rightarrow \mathcal{C}$  is also colimiting. Now, by corollary 0.2.15,  $C$  is a retract of an object in  $\mathcal{H}$ , and hence  $C$  must be isomorphic to an object in  $\mathcal{H}$ , because  $\mathcal{H}$  is closed under coequalisers.

For the final claim, note that  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$  is certainly a replete full subcategory of  $\mathcal{C}$  and contained in any replete full subcategory containing  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  and closed in  $\mathcal{C}$  under colimits for  $\lambda$ -small diagrams, so we just have to show that  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$  is also closed in  $\mathcal{C}$  under colimits for  $\lambda$ -small diagrams; for this, we simply appeal to lemma 0.2.17. ■

**Proposition 0.2.46.** *Let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category and let  $\mathbb{D}$  be a  $\kappa$ -small category in  $\mathbf{U}$ .*

- (i) *If  $\lambda$  is a regular cardinal  $\geq \kappa$ ,  $\mathcal{C}$  has colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams, and  $A : \mathbb{D} \rightarrow \mathcal{C}$  is componentwise  $(\lambda, \mathbf{U})$ -compact, then  $A$  is a  $(\lambda, \mathbf{U})$ -compact object in  $[\mathbb{D}, \mathcal{C}]$ .*
- (ii) *If  $\mathcal{C}$  is a  $\lambda$ -accessible  $\mathbf{U}$ -category and has products for  $\kappa$ -small families of objects, then every  $(\lambda, \mathbf{U})$ -compact object in  $[\mathbb{D}, \mathcal{C}]$  is componentwise  $(\lambda, \mathbf{U})$ -compact.*

*Proof.* (i). First, note that the Mac Lane subdivision category<sup>[3]</sup>  $\mathbb{D}^\S$  is also  $\kappa$ -small, so  $[\mathbb{D}, \mathcal{C}](A, B)$  is computed as the limit of a  $\kappa$ -small diagram of hom-sets. More precisely, using end notation,<sup>[4]</sup>

$$[\mathbb{D}, \mathcal{C}](A, B) \cong \int_{d:\mathbb{D}} \mathcal{C}(Ad, Bd)$$

and so if  $\kappa \leq \lambda$  and  $A$  is componentwise  $(\lambda, \mathbf{U})$ -compact, then  $[\mathbb{D}, \mathcal{C}](A, -)$  preserves colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams, hence  $A$  is itself  $(\lambda, \mathbf{U})$ -compact.

(ii). Now, suppose  $A$  is a  $(\lambda, \mathbf{U})$ -compact object in  $[\mathbb{D}, \mathcal{C}]$ . Let  $d$  be an object in  $\mathbb{D}$ , let  $d^* : [\mathbb{D}, \mathcal{C}] \rightarrow \mathcal{C}$  be evaluation at  $d$ , and let  $d_* : \mathcal{C} \rightarrow [\mathbb{D}, \mathcal{C}]$  be the right adjoint, which is explicitly given by

$$(d_* C)(d') = \mathbb{D}(d', d) \pitchfork C$$

where  $\pitchfork$  is defined by following adjunction:

$$\mathbf{Set}(X, \mathcal{C}(C, C')) \cong \mathcal{C}(C, X \pitchfork C')$$

The unit  $\eta_A : A \rightarrow d_* d^* A$  is constructed using the universal property of  $\pitchfork$  in the obvious way, and the counit  $\varepsilon_C : d^* d_* C \rightarrow C$  is the projection  $\mathbb{D}(d, d) \pitchfork C \rightarrow C$  corresponding to  $\text{id}_d \in \mathbb{D}(d, d)$ . Since  $\mathcal{C}$  is a  $\lambda$ -accessible  $\mathbf{U}$ -category, there exist a  $\mathbf{U}$ -small  $\lambda$ -filtered diagram  $B : \mathcal{J} \rightarrow \mathcal{C}$  consisting of  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{C}$  and a colimiting cocone  $\alpha : B \Rightarrow \Delta d^* A$ , and since each  $\mathbb{D}(d', d)$  has cardinality  $< \kappa$ , the cocone  $d_* \alpha : d_* B \Rightarrow \Delta d_* d^* A$  is also colimiting, by

[3] See [definition A.6.6](#).

[4] See [§A.6](#).



corollary 0.2.26. Lemma 0.2.14 then implies  $\eta_A : A \rightarrow d_* d^* A$  factors through  $d_* \alpha_j : d_*(Bj) \rightarrow d_* d^* A$  for some  $j$  in  $\mathcal{J}$ , say

$$\eta_A = d_* \alpha_j \circ \sigma$$

for some  $\sigma : A \rightarrow d_* Bj$ . But then, by the triangle identity,

$$\text{id}_{Ad} = \varepsilon_{Ad} \circ d^* \eta_A = \varepsilon_{Ad} \circ d^* d_* \alpha_j \circ d^* \sigma = \alpha_j \circ \varepsilon_{Bj} \circ d^* \sigma$$

and so  $\alpha_j : Bj \rightarrow Ad$  is a split epimorphism, hence  $Ad$  is a  $(\lambda, \mathbf{U})$ -compact object, by corollary 0.2.18. ■

REMARK 0.2.47. The claim in the above proposition can fail if  $\kappa > \lambda$ . For example, we could take  $\mathcal{C} = \mathbf{Set}$ , with  $\mathbb{D}$  being the set  $\omega$  considered as a discrete category; then the terminal object in  $[\mathbb{D}, \mathbf{Set}]$  is componentwise finite, but is not itself an  $\aleph_0$ -compact object in  $\mathbf{Set}$ .

**Lemma 0.2.48.** *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ , with  $\kappa \leq \lambda$ .*

- (i) *If  $\mathcal{D}$  is a locally  $\lambda$ -presentable  $\mathbf{U}$ -category,  $\mathcal{C}$  is a locally  $\mathbf{U}$ -small category, and  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a  $(\lambda, \mathbf{U})$ -accessible functor that preserves limits for all  $\mathbf{U}$ -small diagrams in  $\mathcal{C}$ , then, for any  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$ , the comma category  $(C \downarrow G)$  has an initial object.*
- (ii) *If  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category,  $\mathcal{D}$  is a locally  $\mathbf{U}$ -small category, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor that preserves colimits for all  $\mathbf{U}$ -small diagrams in  $\mathcal{C}$ , then, for any object  $D$  in  $\mathcal{D}$ , the comma category  $(F \downarrow D)$  has a terminal object.*

*Proof.* (i). Let  $\mathcal{F}$  be the full subcategory of  $(C \downarrow G)$  spanned by those  $(D, g)$  where  $D$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{D}$ .  $G$  preserves colimits for all  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams, so, by lemma 0.2.14,  $\mathcal{F}$  must be a weakly initial family in  $(C \downarrow G)$ . Proposition 0.2.24 implies  $\mathcal{F}$  is an essentially  $\mathbf{U}$ -small category, and since  $\mathcal{D}$  has limits for all  $\mathbf{U}$ -small diagrams and  $G$  preserves them,  $(C \downarrow G)$  is also  $\mathbf{U}$ -complete. Thus, the inclusion  $\mathcal{F} \hookrightarrow (C \downarrow G)$  has a limit, and it can be shown that this is an initial object in  $(C \downarrow G)$ .<sup>[5]</sup>

(ii). Let  $\mathcal{G}$  be the full subcategory of  $(F \downarrow D)$  spanned by those  $(C, f)$  where  $C$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ ; note that proposition 0.2.24 implies  $\mathcal{G}$  is an

[5] See Theorem 1 in [CWM, Ch. X, §2].

essentially  $\mathbf{U}$ -small category. Since  $\mathcal{C}$  has colimits for all  $\mathbf{U}$ -small diagrams and  $F$  preserves them,  $(F \downarrow \mathcal{D})$  is also  $\mathbf{U}$ -cocomplete.<sup>[6]</sup> Let  $(C, f)$  be a colimit for the inclusion  $\mathcal{C} \hookrightarrow (F \downarrow \mathcal{D})$ . It is not hard to check that  $(C, f)$  is a weakly terminal object in  $(F \downarrow \mathcal{D})$ , so the formal dual of Freyd's initial object lemma<sup>[7]</sup> gives us a terminal object in  $(F \downarrow \mathcal{D})$ ; explicitly, it may be constructed as the joint coequaliser of all the endomorphisms of  $(C, f)$ . ■

**Theorem 0.2.49** (Accessible adjoint functor theorem). *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ , with  $\kappa \leq \lambda$ , let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, and let  $\mathcal{D}$  be a locally  $\lambda$ -presentable  $\mathbf{U}$ -category.*

*Given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , the following are equivalent:*

- (i)  *$F$  has a right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$ , and  $G$  is a  $(\lambda, \mathbf{U})$ -accessible functor.*
- (ii)  *$F$  preserves colimits for all  $\mathbf{U}$ -small diagrams and sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{D}$ .*
- (iii)  *$F$  has a right adjoint and sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{D}$ .*

*On the other hand, given a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$ , the following are equivalent:*

- (iv)  *$G$  has a left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and  $F$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{D}$ .*
- (v)  *$G$  is a  $(\lambda, \mathbf{U})$ -accessible functor and preserves limits for all  $\mathbf{U}$ -small diagrams.*
- (vi)  *$G$  is a  $(\lambda, \mathbf{U})$ -accessible functor and there exist a functor  $F_0 : \mathbf{K}_\kappa^\mathbf{U}(\mathcal{C}) \rightarrow \mathcal{D}$  and hom-set bijections*

$$\mathcal{C}(C, GD) \cong \mathcal{D}(F_0C, D)$$

*natural in  $D$  for each  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$ , where  $D$  varies in  $\mathcal{D}$ .*

*Proof.* We will need to refer back to the details of the proof of this theorem later, so here is a sketch of the constructions involved.

[6] See the Lemma in [CWM, Ch. V, §6].

[7] See Theorem I in [CWM, Ch. V, §6].

(i)  $\Rightarrow$  (ii). If  $F$  is a left adjoint, then  $F$  certainly preserves colimits for all  $\mathbf{U}$ -small diagrams. Given a  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$  and a  $\mathbf{U}$ -small  $\lambda$ -filtered diagram  $B : \mathcal{J} \rightarrow \mathcal{D}$ , observe that

$$\begin{aligned} \mathcal{D}\left(FC, \lim_{\rightarrow \mathcal{J}} B\right) &\cong \mathcal{C}\left(C, G \lim_{\rightarrow \mathcal{J}} B\right) \cong \mathcal{C}\left(C, \lim_{\rightarrow \mathcal{J}} GB\right) \\ &\cong \lim_{\rightarrow \mathcal{J}} \mathcal{C}(C, GB) \cong \lim_{\rightarrow \mathcal{J}} \mathcal{C}(FC, B) \end{aligned}$$

and thus  $FC$  is indeed a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{D}$ .

(ii)  $\Rightarrow$  (iii). It is enough to show that, for each object  $D$  in  $\mathcal{D}$ , the comma category  $(F \downarrow D)$  has a terminal object  $(GD, \varepsilon_D)$ ; [8] but this was done in the previous lemma.

(iii)  $\Rightarrow$  (i). Given a  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$  and a  $\mathbf{U}$ -small  $\lambda$ -filtered diagram  $B : \mathcal{J} \rightarrow \mathcal{D}$ , observe that

$$\begin{aligned} \mathcal{C}\left(C, G \lim_{\rightarrow \mathcal{J}} B\right) &\cong \mathcal{D}\left(FC, \lim_{\rightarrow \mathcal{J}} B\right) \cong \lim_{\rightarrow \mathcal{J}} \mathcal{C}(FC, B) \\ &\cong \lim_{\rightarrow \mathcal{J}} \mathcal{C}(C, GB) \cong \mathcal{C}\left(C, \lim_{\rightarrow \mathcal{J}} GB\right) \end{aligned}$$

because  $FC$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{D}$ ; but [theorem 0.2.39](#) says the restricted Yoneda embedding  $\mathcal{C} \rightarrow [\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})^{\text{op}}, \mathbf{Set}]$  is fully faithful, so this is enough to conclude that  $G$  preserves colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams.

(iv)  $\Rightarrow$  (v). If  $G$  is a right adjoint, then  $G$  certainly preserves limits for all  $\mathbf{U}$ -small diagrams; the rest of this implication is just (iii)  $\Rightarrow$  (i).

(v)  $\Rightarrow$  (vi). It is enough to show that, for each  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$ , the comma category  $(C \downarrow G)$  has an initial object  $(F_0C, \eta_C)$ ; but this was done in the previous lemma. It is clear how to make  $F_0$  into a functor  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \rightarrow \mathcal{D}$ .

(vi)  $\Rightarrow$  (iv). We use [theorems 0.2.23](#) and [0.2.39](#) to extend  $F_0 : \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \rightarrow \mathcal{D}$  along the inclusion  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \hookrightarrow \mathcal{C}$  to get  $(\kappa, \mathbf{U})$ -accessible functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ . We then observe that, for any  $\mathbf{U}$ -small  $\kappa$ -filtered diagram  $A : \mathbb{I} \rightarrow \mathcal{C}$  of  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$ ,

$$\mathcal{C}\left(\lim_{\rightarrow \mathbb{I}} A, GD\right) \cong \lim_{\leftarrow \mathbb{I}} \mathcal{C}(A, GD) \cong \lim_{\leftarrow \mathbb{I}} \mathcal{C}(F_0A, D)$$

[8] See Theorem 2 in [CWM, Ch. IV, § 1].

$$\cong \mathcal{C}\left(\varinjlim_{\mathbb{I}} FA, D\right) \cong \mathcal{C}\left(F \varinjlim_{\mathbb{I}} A, D\right)$$

is a series of bijections natural in  $D$ , where  $D$  varies in  $\mathcal{D}$ ; but  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, so this is enough to show that  $F$  is a left adjoint of  $G$ . The remainder of the claim is a corollary of (i)  $\Rightarrow$  (ii). ■

**Corollary 0.2.50.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be locally presentable  $\mathbf{U}$ -categories. If a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a left adjoint, then there exists a regular cardinal  $\mu$  in  $\mathbf{U}$  such that  $G$  is a  $(\mu, \mathbf{U})$ -accessible functor.*

*Proof.* Suppose  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category,  $\mathcal{D}$  is a locally  $\lambda$ -presentable  $\mathbf{U}$ -category, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a left adjoint for  $G$ . Since  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  is an essentially  $\mathbf{U}$ -small category, recalling lemma 0.2.16, there certainly exists a regular cardinal  $\mu$  in  $\mathbf{U}$  such that  $\mu \geq \lambda$  and  $F$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\mu, \mathbf{U})$ -compact objects in  $\mathcal{D}$ . The above theorem, plus lemma 0.2.37, implies  $G$  is an  $(\mu, \mathbf{U})$ -accessible functor. ■

### 0.3 Accessible constructions

**Prerequisites.** §§0.1, 0.2, A.5

**Definition 0.3.1.** Let  $\mathbf{U}$  be a universe and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The  **$\mathbf{U}$ -rank** of  $F$  is the smallest regular cardinal  $\kappa$  in  $\mathbf{U}$  such that  $F$  preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, provided any such cardinal exists.

REMARK 0.3.2. The class of regular cardinals is well-ordered, so the definition above makes sense. Of course, every  $(\kappa, \mathbf{U})$ -accessible functor has  $\mathbf{U}$ -rank  $\leq \kappa$ .

**Definition 0.3.3.** Let  $\mathbf{U}$  be a universe and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. The **compactness  $\mathbf{U}$ -rank** of an object  $A$  in  $\mathcal{C}$  is the  $\mathbf{U}$ -rank of the hom-functor  $\mathcal{C}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ , where  $\mathbf{Set}$  is the category of  $\mathbf{U}$ -sets.

REMARK 0.3.4. Lemma 0.2.17 implies that, for each object  $A$  in an accessible  $\mathbf{U}$ -category, there exists a regular cardinal  $\lambda$  in  $\mathbf{U}$  such that  $A$  is  $(\lambda, \mathbf{U})$ -compact; in particular, every object in an accessible  $\mathbf{U}$ -category has a compactness  $\mathbf{U}$ -rank.

**Definition 0.3.5.** Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ . A  **$(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category** is an essentially  $\mathbf{U}$ -small category  $\mathcal{C}$  that satisfies the following conditions:

- $\mathcal{C}$  has colimits for all  $\lambda$ -small  $\kappa$ -filtered diagrams.
- Every object in  $\mathcal{C}$  is a colimit for some  $\lambda$ -small  $\kappa$ -filtered diagram of  $(\kappa, \lambda)$ -compact objects in  $\mathcal{C}$ .

We write  $\mathbf{K}_\kappa^\lambda(\mathcal{C})$  for the full subcategory of  $\mathcal{C}$  spanned by the  $(\kappa, \lambda)$ -compact objects.

**REMARK 0.3.6.** Lemma 0.2.8 implies an essentially  $\mathbf{U}$ -small category is  $(\kappa, \kappa)$ -compactly generated if and only if it is Cauchy-complete, i.e. if and only if all idempotent endomorphisms in  $\mathcal{C}$  are split.

**Proposition 0.3.7.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category.*

- (i)  $\mathbf{K}_\kappa^\mathbf{U}(\mathcal{C})$  is a  $(\kappa, \kappa)$ -compactly generated  $\mathbf{U}$ -category, and every object in  $\mathbf{K}_\kappa^\mathbf{U}(\mathcal{C})$  is  $(\kappa, \kappa)$ -compact.
- (ii) If  $\lambda$  is a regular cardinal in  $\mathbf{U}$  and  $\kappa \triangleleft \lambda$ , then  $\mathbf{K}_\lambda^\mathbf{U}(\mathcal{C})$  is a  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category, and the  $(\kappa, \lambda)$ -compact objects in  $\mathbf{K}_\lambda^\mathbf{U}(\mathcal{C})$  are precisely the  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$ .

*Proof.* (i). This follows from lemma 0.2.16, corollary 0.2.18, and remark 0.3.6.

(ii). Combine corollary 0.2.15, lemma 0.2.17, and proposition 0.2.44. ■

**Proposition 0.3.8.** *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ , let  $\mathbb{A}$  and  $\mathbb{B}$  be  $\mathbf{U}$ -small categories, and let  $F : \mathbb{A} \rightarrow \mathbb{B}$  be a fully faithful functor. Assume the following hypotheses:*

- $\kappa \leq \lambda$ .
- $\mathbb{A}$  is a Cauchy-complete category and  $\mathbb{B}$  has colimits for  $\lambda$ -small  $\kappa$ -filtered diagrams.
- Each  $FA$  is a  $(\kappa, \lambda)$ -compact object in  $\mathbb{B}$ , and each object in  $\mathbb{B}$  is a colimit for a  $\lambda$ -small  $\kappa$ -filtered diagram of objects in the image of  $F$ .

*Then:*

- (i) Every  $(\kappa, \lambda)$ -compact object in  $\mathbb{B}$  is isomorphic to an object in the image of  $F : \mathbb{A} \rightarrow \mathbb{B}$ .

## 0. FOUNDATIONS

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(ii) *There exists a functor  $U : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  equipped with a natural bijection of the form below,*

$$\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})(A, UB) \cong \mathbb{B}(FA, B)$$

*and it is unique up to unique isomorphism.*

(iii) *Moreover, the functor  $U : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is fully faithful and essentially surjective onto the full subcategory of  $(\lambda, \mathbf{U})$ -compact objects in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$ .*

(iv)  *$F : \mathbb{A} \rightarrow \mathbb{B}$  is a dense functor.*

(v) *If  $\kappa \triangleleft \lambda$ , then the  $(\lambda, \mathbf{U})$ -accessible functor  $\bar{U} : \mathbf{Ind}_{\mathbf{U}}^{\lambda}(\mathbb{B}) \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  induced by  $U : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is fully faithful and essentially surjective on objects.*

*Proof.* (i). Let  $B$  be an object in  $\mathbb{B}$ . By hypothesis, there is a  $\lambda$ -small  $\kappa$ -filtered diagram  $Y : \mathcal{J} \rightarrow \mathbb{B}$  such that each  $Yj$  is in the image of  $F$  and  $B \cong \varinjlim_{\mathcal{J}} Y$ . Thus, if  $B$  is a  $(\kappa, \lambda)$ -compact object in  $\mathbb{B}$ , then  $B$  must be a retract of some  $Yj$  (by [corollary 0.2.15](#)). But  $\mathbb{A}$  is Cauchy-complete and  $F : \mathbb{A} \rightarrow \mathbb{B}$  is fully faithful, so  $B$  must be isomorphic to some object in the image of  $F$ .

(ii). The assumptions imply each functor  $\mathbb{B}(F-, B) : \mathbb{A}^{\text{op}} \rightarrow \mathbf{Set}$  is a colimit for a  $\lambda$ -small  $\kappa$ -filtered diagram of functors of the form  $\mathbb{A}(-, A')$  for various  $A'$  in  $\mathbb{A}$ . Hence, for each object  $B$  in  $\mathbb{B}$ , there exist an object  $UB$  in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  and bijections

$$\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})(A, UB) \cong \mathbb{B}(FA, B)$$

that are natural in  $A$ . Since the canonical embedding  $\mathbb{A} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is dense, we thus obtain a functor  $U : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  with the required property.

(iii). It is clear that  $U$  is a fully faithful functor that preserves colimits for  $\lambda$ -small  $\kappa$ -filtered diagrams. We may then apply [proposition 0.2.44](#) to deduce that every  $(\lambda, \mathbf{U})$ -compact object in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is isomorphic to one in the image of  $U$ .

(iv). This follows from claim (iii) and the fact that the canonical embedding  $\mathbb{A} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is dense.

(v). If  $\kappa \triangleleft \lambda$ , then [theorem 0.2.33](#) says  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is a  $\lambda$ -accessible category, so we may apply the classification theorem ([0.2.28](#)) to deduce that  $\bar{U} : \mathbf{Ind}_{\mathbf{U}}^{\lambda}(\mathbb{B}) \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{A})$  is fully faithful and essentially surjective on objects.  $\blacksquare$

**Corollary 0.3.9** (Classification of compactly generated categories). *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ . If either  $\kappa = \lambda$  or  $\kappa \triangleleft \lambda$ , then the following are equivalent for an essentially small Cauchy-complete category  $C$ :*

- (i)  $C$  is a  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category.
- (ii)  $\mathbf{Ind}_{\mathbf{U}}^{\lambda}(C)$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category.
- (iii)  $C$  is equivalent to  $\mathbf{K}_{\lambda}^{\mathbf{U}}(D)$  for some  $\kappa$ -accessible  $\mathbf{U}$ -category  $D$ .

*Proof.* (i)  $\Rightarrow$  (ii). See [proposition 0.3.8](#).

(ii)  $\Rightarrow$  (iii). Apply [proposition 0.2.24](#).

(iii)  $\Rightarrow$  (i). See [proposition 0.3.7](#). ■

**Definition 0.3.10.** Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ . A  $(\kappa, \lambda)$ -**compactly defined functor** is a functor  $F : C \rightarrow D$  with the following properties:

- $C$  is a  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category.
- $F : C \rightarrow D$  preserves colimits for  $\lambda$ -small  $\kappa$ -filtered diagrams of  $(\kappa, \lambda)$ -compact objects in  $C$ .

**Lemma 0.3.11.** *Let  $C$  be a  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category, let  $D$  be a locally  $\mathbf{U}$ -small category, and let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets. If  $F : C \rightarrow D$  is a  $(\kappa, \lambda)$ -compactly defined functor, then the natural maps*

$$\begin{aligned} D(FC, D) &\rightarrow [\mathbf{K}_{\kappa}^{\lambda}(C)^{\text{op}}, \mathbf{Set}](C(-, C), D(F-, D)) \\ f &\mapsto (c \mapsto f \circ Fc) \end{aligned}$$

are bijections.

*Proof.* Choose a  $\lambda$ -small  $\kappa$ -filtered diagram  $X : \mathcal{J} \rightarrow C$  such that each vertex is  $(\kappa, \lambda)$ -compact in  $C$  and  $C \cong \varinjlim_{\mathcal{J}} X$ . We then have a natural bijection

$$C(A, C) \cong \varinjlim_{\mathcal{J}} C(A, X)$$

as  $A$  varies in  $\mathbf{K}_{\kappa}^{\lambda}(C)$ , so

$$[\mathbf{K}_{\kappa}^{\lambda}(C)^{\text{op}}, \mathbf{Set}](C(-, C), D(-, D)) \cong \varprojlim_{\mathcal{J}} [\mathbf{K}_{\kappa}^{\lambda}(C)^{\text{op}}, \mathbf{Set}](C(-, X), D(F-, D))$$

and by applying the Yoneda lemma, we have

$$\lim_{\leftarrow J} [\mathbf{K}_\kappa^\lambda(C)^{\text{op}}, \mathbf{Set}](C(-, X), \mathcal{D}(F-, D)) \cong \lim_{\leftarrow J} \mathcal{D}(FX, D)$$

but  $F : C \rightarrow D$  preserves colimits for  $\lambda$ -small  $\kappa$ -filtered diagrams of  $(\kappa, \lambda)$ -compact objects in  $C$ , so:

$$\lim_{\leftarrow J} \mathcal{D}(FX, D) \cong \mathcal{D}\left(\lim_{\rightarrow J} FX, D\right) \cong \mathcal{D}(FC, D)$$

We may therefore deduce that the indicated maps are bijections. ■

**Proposition 0.3.12.** *Let  $C$  and  $D$  be  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -categories. If  $F : C \rightarrow D$  is a  $(\kappa, \lambda)$ -compactly defined functor, then the induced functor  $\mathbf{Ind}_{\mathbf{U}}^\lambda(F) : \mathbf{Ind}_{\mathbf{U}}^\lambda(C) \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  is  $(\kappa, \mathbf{U})$ -accessible.*

*Proof.* Let  $\mathcal{A} = \mathbf{K}_\kappa^\lambda(C)$ , let  $\gamma_C : C \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(C)$  and  $\gamma_D : D \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  be the canonical embeddings and let  $\bar{F} = \mathbf{Ind}_{\mathbf{U}}^\lambda(F)$ . Theorems 0.2.23 and A.5.15 imply  $\bar{F} : \mathbf{Ind}_{\mathbf{U}}^\lambda(C) \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  is (the functor part of) a pointwise left Kan extension of  $\gamma_D F : C \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  along  $\gamma_C : C \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(C)$ . By proposition 0.3.8,  $\mathbf{Ind}_{\mathbf{U}}^\lambda(C)$  and  $\mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  are  $\kappa$ -accessible  $\mathbf{U}$ -categories, and to verify that  $\bar{F}$  is a  $(\kappa, \mathbf{U})$ -accessible functor, it suffices to show that  $\bar{F}$  is (the functor part of) a pointwise left Kan extension of  $\gamma_D F|_{\mathcal{A}}$  along  $\gamma_C|_{\mathcal{A}}$ .

Since  $\gamma_D : D \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  preserves colimits for  $\lambda$ -small diagrams, the composite  $\gamma_D F : C \rightarrow \mathbf{Ind}_{\mathbf{U}}^\lambda(D)$  is also a  $(\kappa, \lambda)$ -compactly defined functor, and so  $\gamma_D F$  is (the functor part of) a pointwise left Kan extension of  $\gamma_D F|_{\mathcal{A}}$  along the inclusion  $\mathcal{A} \hookrightarrow C$  (by lemma 0.3.11). We may therefore apply theorem A.5.20 to deduce that  $\bar{F}$  is indeed (the functor part of) a pointwise left Kan extension of  $\gamma_D F|_{\mathcal{A}}$  along  $\gamma_C|_{\mathcal{A}}$ . ■

**Definition 0.3.13.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . A **strongly  $(\kappa, \mathbf{U})$ -accessible functor** is a functor  $F : C \rightarrow D$  with the following properties:

- Both  $C$  and  $D$  are  $\kappa$ -accessible  $\mathbf{U}$ -categories.
- $F$  preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.
- $F$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $C$  to  $(\kappa, \mathbf{U})$ -compact objects in  $D$ .

**Example 0.3.14.** Given any functor  $F : \mathbb{A} \rightarrow \mathbb{B}$ , if  $\mathcal{A}$  and  $\mathcal{B}$  are small categories, then the induced functor  $\mathbf{Ind}_{\mathbf{U}}^\kappa(F) : \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{A}) \rightarrow \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{B})$  is strongly  $(\kappa, \mathbf{U})$ -accessible, by corollaries 0.2.15 and 0.2.18.



**Proposition 0.3.15** (Products of accessible categories). *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . If  $(C_i \mid i \in I)$  is a  $\kappa$ -small family of  $\kappa$ -accessible  $\mathbf{U}$ -categories, then:*

- (i) *The product  $C = \prod_{i \in I} C_i$  is also a  $\kappa$ -accessible  $\mathbf{U}$ -category.*
- (ii) *Moreover, the projection functors  $C \rightarrow C_i$  are strongly  $(\kappa, \mathbf{U})$ -accessible functors.*

*Proof.* It is clear that  $C$  has colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams: indeed, they can be computed componentwise. [Theorem 0.2.12](#) implies that an object in  $C$  is  $(\kappa, \mathbf{U})$ -compact as soon as its components are  $(\kappa, \mathbf{U})$ -compact objects in their respective categories. Recalling [lemma 0.2.9](#), it follows that  $C$  is generated under  $\mathbf{U}$ -small  $\kappa$ -filtered colimits by a  $\mathbf{U}$ -small family of  $(\kappa, \mathbf{U})$ -compact objects, as required of a  $\kappa$ -accessible  $\mathbf{U}$ -category. ■

**Lemma 0.3.16.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathbf{U}^+$  be a universe with  $\mathbf{U} \subseteq \mathbf{U}^+$ , let  $C$  be an accessible  $\mathbf{U}$ -category, let  $\mathcal{D}$  be an accessible  $\mathbf{U}^+$ -category, and let  $F : C \rightarrow \mathcal{D}$  be a  $(\kappa, \mathbf{U})$ -accessible functor.*

- (i) *There is a regular cardinal  $\lambda$  in  $\mathbf{U}^+$  such that  $F$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $C$  to  $(\lambda, \mathbf{U}^+)$ -compact objects in  $\mathcal{D}$ .*
- (ii) *Moreover, if  $\mu$  is a regular cardinal in  $\mathbf{U}^+$  such that  $\kappa \triangleleft \mu$  and  $\lambda \leq \mu$ , then  $F$  sends  $(\mu, \mathbf{U})$ -compact objects in  $C$  to  $(\mu, \mathbf{U}^+)$ -compact objects in  $\mathcal{D}$ .*

*Proof.* (i). Such a regular cardinal exists by [remark 0.3.4](#) and [proposition 0.2.24](#).

(ii). If  $\mu$  is not in  $\mathbf{U}$ , then the claim is trivial; otherwise, [proposition 0.2.44](#) and [lemma 0.2.17](#) imply that  $F$  sends  $(\mu, \mathbf{U})$ -compact objects in  $C$  to  $(\mu, \mathbf{U}^+)$ -compact objects in  $\mathcal{D}$ , as required. ■

**Corollary 0.3.17.** *Let  $C$  and  $\mathcal{D}$  be accessible  $\mathbf{U}$ -categories. If  $F : C \rightarrow \mathcal{D}$  is a  $(\kappa, \mathbf{U})$ -accessible functor, then:*

- (i) *There exists a regular cardinal  $\lambda$  in  $\mathbf{U}$  such that  $F$  is strongly  $(\lambda, \mathbf{U})$ -accessible.*
- (ii) *Moreover, if  $\mu$  is a regular cardinal in  $\mathbf{U}$  and  $\lambda \triangleleft \mu$ , then  $F$  is also strongly  $(\mu, \mathbf{U})$ -accessible.*

*Proof.* Combine [lemma 0.3.16](#), [theorem 0.2.33](#), and [proposition 0.2.34](#). ■

**Lemma 0.3.18.** *Let  $\mathcal{J}$  be a  $\kappa$ -filtered category. If  $\mathbb{A}$  is a  $\kappa$ -small category, then the functor category  $[\mathbb{A}, \mathcal{J}]$  is also a  $\kappa$ -filtered category.*

*Proof.* There is a natural bijection between diagrams  $\mathbb{D} \rightarrow [\mathbb{A}, \mathcal{J}]$  and diagrams  $\mathbb{D} \times \mathbb{A} \rightarrow \mathcal{J}$ ; but if  $\mathbb{D}$  is  $\kappa$ -small, then so is  $\mathbb{D} \times \mathbb{A}$ . Thus, every  $\kappa$ -small diagram in  $[\mathbb{A}, \mathcal{J}]$  has a cocone, as required. ■

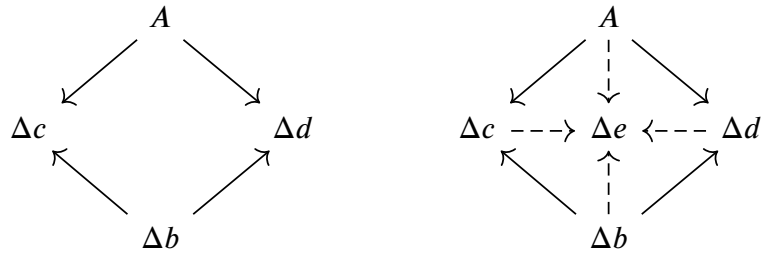
**Lemma 0.3.19.** *Let  $\mathcal{J}$  be a  $\kappa$ -filtered category, let  $A : \mathcal{I} \rightarrow \mathcal{J}$  be a  $\kappa$ -small diagram, let  ${}^A/\mathcal{J}$  be the cocone category ( $A \downarrow \Delta$ ), and let  $P : {}^A/\mathcal{J} \rightarrow \mathcal{J}$  be the projection functor.*

(i) *The cocone category  ${}^A/\mathcal{J}$  is also a  $\kappa$ -filtered category.*

(ii)  *$P : {}^A/\mathcal{J} \rightarrow \mathcal{J}$  is a cofinal functor.*<sup>[9]</sup>

*Proof.* (i). Let  $\mathbb{D}$  be a  $\kappa$ -small category. There exists a  $\kappa$ -small category  $\tilde{\mathbb{D}}$  equipped with a functor  $L : \mathcal{I} \rightarrow \tilde{\mathbb{D}}$  and a natural bijection between diagrams  $X : \mathbb{D} \rightarrow {}^A/\mathcal{J}$  and diagrams  $\tilde{X} : \tilde{\mathbb{D}} \rightarrow \mathcal{J}$  such that  $\tilde{X}L = A$ , and moreover this construction is natural in  $\mathbb{D}$ . Thus, every  $\kappa$ -small diagram in  ${}^A/\mathcal{J}$  admits a cocone, as required.

(ii). We must show that the comma category  $(b \downarrow P)$  is connected for all objects  $b$  in  $\mathcal{J}$ . Since  $\mathcal{J}$  is filtered, there must exist an object  $c$ , a cocone  $A \Rightarrow \Delta c$ , and a morphism  $b \rightarrow c$  in  $\mathcal{J}$ ; thus,  $(b \downarrow P)$  is inhabited. Moreover, any diagram in  $[\mathcal{I}, \mathcal{J}]$  of the form shown below on the left can be completed to one of the form shown below on the right,



so we may conclude that  $(b \downarrow P)$  is indeed connected. ■

**Lemma 0.3.20.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $X : \mathcal{I} \rightarrow \mathcal{C}$  be a  $\kappa$ -small diagram, let  $Y : \mathcal{J} \rightarrow \mathcal{C}$  be a  $\mathbf{U}$ -small  $\kappa$ -filtered diagram, and let*

[9] See [definition A.5.31](#).

$\varepsilon : Y \Rightarrow \Delta B$  be a colimiting cocone in  $\mathcal{C}$ . If each  $X_i$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , then every cocone  $X \Rightarrow \Delta B$  must factor through  $\varepsilon_j : Y_j \rightarrow B$  for some  $j$  in  $\mathcal{J}$ .

*Proof.* Let  $\varphi : X \Rightarrow \Delta B$  be a cocone, and regard it as a morphism in the functor category  $[\mathcal{I}, \mathcal{C}]$ . By [proposition 0.2.46](#),  $X$  is a  $(\kappa, \mathbf{U})$ -compact object in  $[\mathcal{I}, \mathcal{C}]$ ; but  $\Delta\varepsilon : \Delta Y \Rightarrow \Delta\Delta B$  is a colimiting cocone in  $[\mathcal{I}, \mathcal{C}]$ , so we may apply [lemma 0.2.14](#).  $\blacksquare$

**Lemma 0.3.21.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$  and let  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be functors that send  $(\kappa, \mathbf{U})$ -compact objects to  $(\kappa, \mathbf{U})$ -compact objects. Given an object  $(C, D, e)$  in the comma category  $(F \downarrow G)$ , if  $C$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$  and  $D$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{D}$ , then  $(C, D, e)$  is a  $(\kappa, \mathbf{U})$ -compact object in  $(F \downarrow G)$ .*

*Proof.* Let  $\mathcal{B} = (F \downarrow G)$  and let  $\varphi : FP \Rightarrow GQ$  be the canonical natural transformation. Then, given any two objects  $B$  and  $B'$  in  $\mathcal{B}$ , we have the following pullback diagram,

$$\begin{array}{ccc} \mathcal{B}(B, B') & \longrightarrow & \mathcal{D}(QB, QB') \\ \downarrow & & \downarrow \\ \mathcal{C}(PB, PB') & \longrightarrow & \mathcal{E}(FPB, GQB') \end{array}$$

where the map  $\mathcal{C}(PB, PB') \rightarrow \mathcal{E}(FPB, GQB')$  is induced by the functor  $F : \mathcal{C} \rightarrow \mathcal{E}$  and the morphism  $\varphi_{B'} : FPB' \rightarrow GQB'$ , and the map  $\mathcal{D}(QB, QB') \rightarrow \mathcal{E}(FPB, GQB')$  is induced by the functor  $G : \mathcal{D} \rightarrow \mathcal{E}$  and the morphism  $\varphi_B : FPB \rightarrow GQB$ . Thus, if  $PB$  and  $QB$  are  $(\kappa, \mathbf{U})$ -compact objects, then so are  $FPB$  and  $GQB$ , and therefore we may use [theorem 0.2.12](#) deduce that  $B$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{B}$ .  $\blacksquare$

**Theorem 0.3.22** (Accessibility of comma categories). *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$  and let  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be  $(\kappa, \mathbf{U})$ -accessible functors.*

- (i) *The comma category  $(F \downarrow G)$  has colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, created by the projection functor  $(F \downarrow G) \rightarrow \mathcal{C} \times \mathcal{D}$ .*
- (ii) *If  $F$  and  $G$  are strongly  $(\kappa, \mathbf{U})$ -accessible functors, then  $(F \downarrow G)$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, and the projection functors  $P : (F \downarrow G) \rightarrow \mathcal{C}$  and  $Q : (F \downarrow G) \rightarrow \mathcal{D}$  are strongly  $(\kappa, \mathbf{U})$ -accessible.*

*Proof.* See Theorem 2.43 in [LPAC].  $\square$

**Corollary 0.3.23.** *If  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category and  $A$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , then:*

- *The slice category  ${}^A/\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, and the projection functor  ${}^A/\mathcal{C} \rightarrow \mathcal{C}$  is a strongly  $(\kappa, \mathbf{U})$ -accessible functor.*
- *The slice category  $\mathcal{C}/_A$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, and the projection functor  $\mathcal{C}/_A \rightarrow \mathcal{C}$  is a strongly  $(\kappa, \mathbf{U})$ -accessible functor.  $\blacksquare$*

**Corollary 0.3.24.** *If  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, then so is the functor category  $[2, \mathcal{C}]$ , and moreover the  $(\kappa, \mathbf{U})$ -compact objects in  $[2, \mathcal{C}]$  are precisely the componentwise  $(\kappa, \mathbf{U})$ -compact objects.*

*Proof.* The functor category  $[2, \mathcal{C}]$  is isomorphic to the comma category  $(\mathcal{C} \downarrow \mathcal{C})$ , and  $\text{id} : \mathcal{C} \rightarrow \mathcal{C}$  is certainly a strongly  $(\kappa, \mathbf{U})$ -accessible functor.  $\blacksquare$

**Corollary 0.3.25.** *If  $\mathcal{C}$  is a  $(\kappa, \lambda)$ -compactly generated  $\mathbf{U}$ -category, then so is  $[2, \mathcal{C}]$ .*

*Proof.* Combine lemma 0.3.21 and corollaries 0.3.9 and 0.3.24.  $\blacksquare$

**Lemma 0.3.26.** *Let  $\kappa$  and  $\lambda$  be regular cardinals in a universe  $\mathbf{U}$ , with  $\kappa \leq \lambda$ , let  $\mathcal{E}$  be a locally  $\mathbf{U}$ -small category with colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, let  $X : \mathcal{I} \rightarrow \mathcal{E}$  and  $Y : \mathcal{J} \rightarrow \mathcal{E}$  be  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams that are componentwise  $(\lambda, \mathbf{U})$ -compact, let  $C = \varinjlim_{\mathcal{I}} X$  and  $D = \varinjlim_{\mathcal{J}} Y$ , and let  $c_i : Xi \rightarrow C$  and  $d_j : Yj \rightarrow D$  be the components of the respective colimiting cocones.*

- (i) *Given any object  $i_0$  in  $\mathcal{I}$  and any morphism  $e : C \rightarrow D$ , there exist an object  $j_0$  in  $\mathcal{J}$  and a morphism  $f_0 : Xi_0 \rightarrow Yj_0$  such that the following diagram commutes:*

$$\begin{array}{ccc} Xi_0 & \xrightarrow{c_{i_0}} & C \\ f_0 \downarrow & & \downarrow e \\ Yj_0 & \xrightarrow{d_{j_0}} & D \end{array}$$

- (ii) *Given any commutative diagram of the above form, if  $e : C \rightarrow D$  is an isomorphism in  $\mathcal{E}$ , then there exist chains  $I : \kappa \rightarrow \mathcal{I}$  and  $J : \kappa \rightarrow \mathcal{J}$  and*

a factorisation of the form below,

$$\begin{array}{ccccc} Xi_0 & \longrightarrow & C' & \longrightarrow & C \\ f_0 \downarrow & & \downarrow e' & & \downarrow e \\ Yj_0 & \longrightarrow & D' & \longrightarrow & D \end{array}$$

where  $I(0) = i_0$ ,  $J(0) = j_0$ ,  $C' = \varinjlim_{\alpha < \kappa} XI(\alpha)$ ,  $D' = \varinjlim_{\alpha < \kappa} YJ(\alpha)$ ,  $e' : C' \rightarrow D'$  is an isomorphism, and the morphisms  $C' \rightarrow C$  and  $D' \rightarrow D$  are the ones induced by the evident cocones.

*Proof.* (i). Since  $Xi_0$  is  $(\lambda, \mathbf{U})$ -compact and  $Y : \mathcal{J} \rightarrow \mathcal{E}$  is a  $\mathbf{U}$ -small  $\lambda$ -filtered diagram, such a factorisation of  $e \circ c_{i_0}$  must exist, by [lemma 0.2.14](#).

(ii). We will construct  $I$ ,  $J$ , and  $e'$  by transfinite induction on  $\kappa$ .

- Given  $j_\alpha$  and  $f_\alpha$ , choose a morphism  $i_{\alpha \rightarrow \alpha+1} : i_\alpha \rightarrow i_{\alpha+1}$  in  $\mathcal{I}$  and a morphism  $g_\alpha : Yj_\alpha \rightarrow Xi_{\alpha+1}$  in  $\mathcal{E}$  such that the diagram below commutes:

$$\begin{array}{ccccc} Xi_\alpha & \xrightarrow{Xi_{\alpha \rightarrow \alpha+1}} & Xi_{\alpha+1} & \xrightarrow{c_{i_{\alpha+1}}} & C \\ f_\alpha \downarrow & \nearrow g_\alpha & & & \uparrow e^{-1} \\ Yj_\alpha & \xrightarrow{d_{j_\alpha}} & & & D \end{array}$$

Such  $i_{\alpha \rightarrow \alpha+1}$  and  $g_\alpha$  exist because  $f_\alpha : Xi_\alpha \rightarrow Yj_\alpha$  defines a  $(\lambda, \mathbf{U})$ -compact object in the slice category  $^{Xi_\alpha}/\mathcal{E}$  (by [lemma 0.3.21](#)) and there is an evident  $\mathbf{U}$ -small  $\lambda$ -filtered diagram  $^{i_\alpha}/\mathcal{I} : i_\alpha/\mathcal{I} \rightarrow ^{Xi_\alpha}/\mathcal{E}$  with colimit defined by  $c_{i_\alpha} : Xi_\alpha \rightarrow C$  (by [lemma 0.3.19](#)).

- Given  $i_{\alpha+1}$  and  $g_\alpha$ , choose a morphism  $j_{\alpha \rightarrow \alpha+1} : j_\alpha \rightarrow j_{\alpha+1}$  in  $\mathcal{J}$  and a morphism  $f_{\alpha+1} : Xi_{\alpha+1} \rightarrow Yj_{\alpha+1}$  in  $\mathcal{E}$  such that the diagram below commutes:

$$\begin{array}{ccccc} Yj_\alpha & \xrightarrow{Yj_{\alpha \rightarrow \alpha+1}} & Yj_{\alpha+1} & \xrightarrow{d_{j_{\alpha+1}}} & D \\ g_\alpha \downarrow & \nearrow f_{\alpha+1} & & & \uparrow e \\ Xi_{\alpha+1} & \xrightarrow{c_{i_{\alpha+1}}} & & & C \end{array}$$

- Given a limit ordinal  $\beta < \kappa$  and  $i_\alpha$  for all ordinals  $\alpha < \beta$ , choose an object  $i_\beta$  in  $\mathcal{I}$  and a cocone from the chain defined by  $(i_\alpha \mid \alpha < \beta)$  to  $i_\beta$ .

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- Given  $i_\beta$  for a limit ordinal  $\beta < \kappa$  and  $j_\alpha$  for all ordinals  $\alpha < \beta$ , choose an object  $j_\beta$  in  $\mathcal{J}$ , a cocone from the chain defined by  $(j_\alpha \mid \alpha < \beta)$ , and a morphism  $f_\beta : Xi_\beta \rightarrow Yj_\beta$  such that the following diagram commutes for all ordinals  $\alpha < \beta$ :

$$\begin{array}{ccccc}
 Yj_\alpha & \xrightarrow{Yj_{\alpha \rightarrow \beta}} & Yj_\beta & \xrightarrow{d_{j_\beta}} & D \\
 g_\alpha \downarrow & & \uparrow f_\beta & & \uparrow e \\
 Xi_{\alpha+1} & \xrightarrow{Xi_{\alpha+1 \rightarrow \beta}} & Xi_\beta & \xrightarrow{c_{i_\beta}} & C
 \end{array}$$

Such data exist because the chains  $X'$  and  $Y'$  defined by  $(Xi_\alpha \mid \alpha < \beta)$  and  $(Yj_\alpha \mid \alpha < \beta)$  are  $(\lambda, \mathbf{U})$ -compact objects in the category  $[\beta, \mathcal{E}]$  (by [proposition 0.2.46](#)) and there is an evident  $\mathbf{U}$ -small  $\lambda$ -filtered diagram in  $Y'/[\beta, \mathcal{E}]$  with colimit  $\Delta D$  (by [lemma 0.3.19](#)).

Now take  $I : \kappa \rightarrow \mathcal{I}$  and  $J : \kappa \rightarrow \mathcal{J}$  to be the chains defined by  $I(\alpha) = i_\alpha$  and  $J(\alpha) = j_\alpha$ . Let  $C' = \varinjlim_{\alpha < \kappa} Xi_\alpha$  and  $D' = \varinjlim_{\alpha < \kappa} Yj_\alpha$ . The above construction yields commutative diagrams of the form below for all ordinals  $\alpha < \beta < \kappa$ ,

$$\begin{array}{ccc}
 Xi_\alpha & \xrightarrow{Xi_{\alpha \rightarrow \beta}} & Xi_\beta \\
 f_\alpha \downarrow & & \downarrow f_\beta \\
 Yj_\alpha & \xrightarrow{Yj_{\alpha \rightarrow \beta}} & Yj_\beta
 \end{array}
 \qquad
 \begin{array}{ccc}
 Yj_\alpha & \xrightarrow{Yj_{\alpha \rightarrow \beta}} & Yj_\beta \\
 g_\alpha \downarrow & & \downarrow g_\beta \\
 Xi_{\alpha+1} & \xrightarrow{Xi_{\alpha+1 \rightarrow \beta+1}} & Xi_{\beta+1}
 \end{array}$$

so there are induced morphisms  $f : C' \rightarrow D'$  and  $g : D' \rightarrow C'$ ; moreover, since  $g_\alpha \circ f_\alpha = Xi_{\alpha \rightarrow \alpha+1}$  and  $f_{\alpha+1} \circ g_\alpha = Yj_{\alpha \rightarrow \alpha+1}$ , we have  $g \circ f = \text{id}_{C'}$ , and  $f \circ g = \text{id}_{D'}$ . Thus, we have the required isomorphism  $e : C' \rightarrow D'$ .  $\blacksquare$

**Theorem 0.3.27** (Accessibility of iso-comma categories). *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathcal{C}$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  be categories with colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, and let  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be functors of  $\mathbf{U}$ -rank  $\leq \kappa$ .*

- The iso-comma category  $(F \wr G)$  has colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, created by the projection functor  $(F \wr G) \rightarrow \mathcal{C} \times \mathcal{D}$ .*
- Assuming  $F$  and  $G$  are strongly  $\lambda$ -accessible functors, given an object  $(C, D, e)$  in  $(F \wr G)$ , if  $C$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$  and  $D$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{D}$ , then  $(C, D, e)$  is a  $(\lambda, \mathbf{U})$ -compact object in  $(F \wr G)$ .*

(iii) If  $F$  and  $G$  are strongly  $(\lambda, \mathbf{U})$ -accessible functors and  $\kappa < \lambda$ , then  $(F \wr G)$  is a  $\lambda$ -accessible  $\mathbf{U}$ -category, and the projection functors  $P : (F \wr G) \rightarrow C$  and  $Q : (F \wr G) \rightarrow D$  are strongly  $(\lambda, \mathbf{U})$ -accessible.

*Proof.* (i). This is a straightforward consequence of the hypothesis that both  $F : C \rightarrow \mathcal{E}$  and  $G : D \rightarrow \mathcal{E}$  preserve colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.

(ii). Since the iso-comma category  $(F \wr G)$  is a full subcategory of the comma category  $(F \downarrow G)$ , the claim is an immediate corollary of [lemma 0.3.21](#).

(iii). Let  $\mathcal{B} = (F \wr G)$ . First, we must show that there is a  $\mathbf{U}$ -small set of  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{B}$  that generate  $\mathcal{B}$  under colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered colimits. Let  $(C, D, e)$  be an object in  $\mathcal{B}$ . Since  $\kappa \triangleleft \lambda$ , we may choose  $\mathbf{U}$ -small skeletons  $\mathcal{I}$  and  $\mathcal{J}$  of the comma categories  $(\mathbf{K}_\lambda^{\mathbf{U}}(C) \downarrow C)$  and  $(\mathbf{K}_\lambda^{\mathbf{U}}(D) \downarrow D)$  and obtain  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams  $X : \mathcal{I} \rightarrow C$  and  $Y : \mathcal{J} \rightarrow D$  that are componentwise  $(\lambda, \mathbf{U})$ -compact and have  $C \cong \varinjlim_{\mathcal{I}} X$  and  $D \cong \varinjlim_{\mathcal{J}} Y$  (by [proposition 0.2.25](#) and [theorem 0.2.33](#)). Let  $\mathcal{K}$  be full subcategory of the iso-comma category  $(FX \wr GY)$  spanned by those objects  $(i, j, f)$  such that the following diagram commutes,

$$\begin{array}{ccc} FXi & \xrightarrow{Fc_i} & FC \\ f \downarrow & & \downarrow e \\ GYj & \xrightarrow{Gd_j} & GD \end{array}$$

where  $c_i : Xi \rightarrow C$  and  $d_j : Yj \rightarrow D$  are the components of the respective colimiting cocones. Let  $P' : \mathcal{K} \rightarrow \mathcal{I}$  and  $Q' : \mathcal{K} \rightarrow \mathcal{J}$  be the projection functors, and let  $Z : \mathcal{K} \rightarrow \mathcal{B}$  be the evident diagram with  $PZ = FXP'$  and  $QZ = GYQ'$ . It is clear that  $\mathcal{K}$  is a  $\mathbf{U}$ -small category, and we claim  $Z : \mathcal{K} \rightarrow \mathcal{B}$  is  $\lambda$ -filtered diagram with  $(C, D, e)$  as its colimit.

First, we verify that  $(C, D, e)$  is a colimit for the diagram  $Z : \mathcal{K} \rightarrow \mathcal{B}$ . Let  $i$  be any object in  $\mathcal{I}$  and consider the comma category  $(i \downarrow P')$ . [Lemma 0.3.26](#) implies it is inhabited. Suppose we have two objects in  $(i \downarrow P')$ , i.e. two objects  $(i_0, j_0, f_0)$  and  $(i_1, j_1, f_1)$  in  $\mathcal{K}$  and two morphisms  $h_0 : i \rightarrow i_0$  and  $h_1 : i \rightarrow i_1$  in  $\mathcal{I}$ . Since  $\mathcal{I}$  is a filtered category, there exist an object  $i'$  in  $\mathcal{I}$  and morphisms  $h'_0 : i_0 \rightarrow i'$  and  $h'_1 : i_1 \rightarrow i'$  such that  $h'_0 \circ h_0 = h'_1 \circ h_1$ . Similarly,  $\mathcal{J}$  is a filtered category, so there exist an object  $j_2$  in  $\mathcal{J}$  and morphisms  $j_0 \rightarrow j_2$  and  $j_1 \rightarrow j_2$ . By considering a suitable diagram of shape  $j_2/\mathcal{J}$  in the category  $(GY_{j_0}, GY_{j_1})/\mathcal{E} \times \mathcal{E}$  (using the fact that  $f_0 : FXi_0 \rightarrow GY_{j_0}$  and  $f_1 : FXi_1 \rightarrow GY_{j_1}$

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are isomorphisms in  $\mathcal{E}$ ) and applying lemmas 0.3.19 and 0.3.26, we see that there is a commutative diagram in  $\mathcal{E}$  of the form shown below,

$$\begin{array}{ccccc}
 & FXi_0 & \xrightarrow{Fc_{i_0}} & & FC \\
 & \downarrow f_0 & \searrow FXh'_0 & & \downarrow e \\
 & & & FXi' & \xrightarrow{Fc_{i'}} \\
 FXi_1 & \xrightarrow{Fh'_1} & & & \\
 & \downarrow Fc_{i_1} & & & \\
 & & & & \\
 & GYj_0 & \xrightarrow{Gd_{j_0}} & & GD \\
 & \downarrow GYk'_0 & \searrow & & \\
 & & & GYj' & \xrightarrow{Gd_{j'}} \\
 GYj_1 & \xrightarrow{GYk'_1} & & & \\
 & \downarrow Gd_{j_1} & & & \\
 & & & & 
 \end{array}$$

and recalling lemma 0.2.17, we may assume that  $f' : FXi' \rightarrow GYj'$  is an isomorphism in  $\mathcal{E}$ . Thus, the comma category  $(i \downarrow P')$  is connected, and therefore  $P' : \mathcal{K} \rightarrow \mathcal{I}$  is a cofinal functor. The symmetric argument shows that  $Q' : \mathcal{K} \rightarrow \mathcal{J}$  is also a cofinal functor, and since  $F : \mathcal{C} \rightarrow \mathcal{E}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  preserve colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams, we may deduce that the canonical cocone from  $Z$  to  $(C, D, e)$  in  $\mathcal{B}$  is a colimiting cocone.

It remains to be shown that  $\mathcal{K}$  is a  $\mathbf{U}$ -small  $\lambda$ -filtered category. Indeed, suppose  $K : \mathbb{A} \rightarrow \mathcal{K}$  is a  $\lambda$ -small diagram. Since  $\mathcal{I}$  is a  $\lambda$ -filtered category, there is an object  $i_0$  in  $\mathcal{I}$  with a cocone  $P'K \Rightarrow \Delta i_0$ , and by considering a suitable  $\lambda$ -filtered diagram in the category  $^{GQ'K}/[\mathbb{A}, \mathcal{E}]$ , we obtain an object  $j_0$  in  $\mathcal{J}$  and a morphism  $f_0 : FXi_0 \rightarrow GYj_0$  such that the diagram below commutes,

$$\begin{array}{ccc}
 FXi_0 & \xrightarrow{Fc_{i_0}} & FC \\
 f_0 \downarrow & & \downarrow e \\
 GYj_0 & \xrightarrow{Gd_{j_0}} & GD
 \end{array}$$

as well as a cocone from  $K$  to  $(Xi_0, Yj_0, f_0)$  in the comma category  $(F \downarrow G)$  that is compatible with the colimiting cocone  $GY \Rightarrow \Delta GD$ . Combining lemmas 0.2.17 and 0.3.26, we then obtain a cocone under  $P$  in  $\mathcal{K}$ , as required. This shows that every object in  $\mathcal{B}$  is a colimit for a  $\mathbf{U}$ -small  $\lambda$ -filtered diagram of



componentwise  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{B}$ , and since  $\mathcal{C}$  and  $\mathcal{D}$  are  $\lambda$ -accessible  $\mathbf{U}$ -categories, [proposition 0.2.24](#) implies the full subcategory of  $\mathcal{B}$  spanned by such componentwise  $(\lambda, \mathbf{U})$ -compact objects is essentially  $\mathbf{U}$ -small.

Finally, observe that every  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{B}$  is a retract of a componentwise  $(\lambda, \mathbf{U})$ -compact object (because the set of such objects generate  $\mathcal{B}$  under colimits for  $\mathbf{U}$ -small  $\lambda$ -filtered diagrams), and thus we may apply [corollary 0.2.18](#) to deduce that every  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{B}$  is itself componentwise  $(\lambda, \mathbf{U})$ -compact. Thus the projection functors  $P : \mathcal{B} \rightarrow \mathcal{C}$  and  $Q : \mathcal{B} \rightarrow \mathcal{D}$  are strongly  $(\lambda, \mathbf{U})$ -accessible.  $\blacksquare$

**Definition 0.3.28.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ . A  $\kappa$ -accessible  $\mathbf{U}$ -subcategory of a  $\kappa$ -accessible  $\mathbf{U}$ -category  $\mathcal{C}$  is a subcategory  $\mathcal{B} \subseteq \mathcal{C}$  such that  $\mathcal{B}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category and the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is a  $(\kappa, \mathbf{U})$ -accessible functor.

**Proposition 0.3.29.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category and let  $\mathcal{B}$  be a replete and full  $\kappa$ -accessible  $\mathbf{U}$ -subcategory of  $\mathcal{C}$ .*

- (i) *If  $A$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$  and  $A$  is in  $\mathcal{B}$ , then  $A$  is also a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ .*
- (ii) *If the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is strongly  $(\kappa, \mathbf{U})$ -accessible, then  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{B}) = \mathcal{B} \cap \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ .*

*Proof.* (i). This is clear, since hom-sets and colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{B}$  are computed as in  $\mathcal{C}$ .

(ii). Given claim (i), it suffices to show that every  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{B}$  is also  $(\kappa, \mathbf{U})$ -compact in  $\mathcal{C}$ , but this is precisely the hypothesis that the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is strongly  $(\kappa, \mathbf{U})$ -accessible.  $\blacksquare$

**Proposition 0.3.30.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathcal{C}$  and  $\mathcal{E}$  be categories with colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, let  $\mathcal{D}$  be a replete and full subcategory of  $\mathcal{E}$  that is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, let  $F : \mathcal{C} \rightarrow \mathcal{E}$  be a functor of  $\mathbf{U}$ -rank  $\leq \kappa$ , and let  $\mathcal{B}$  be the preimage of  $\mathcal{D}$  under  $F$ , so that we have the following strict pullback diagram:*

$$\begin{array}{ccc} \mathcal{B} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \mathcal{E} \end{array}$$

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- (i)  $\mathcal{B}$  is a replete and full subcategory of  $\mathcal{D}$  and is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{D}$ .
- (ii) If  $F : \mathcal{C} \rightarrow \mathcal{E}$  and the inclusion  $\mathcal{D} \hookrightarrow \mathcal{E}$  are strongly  $(\lambda, \mathbf{U})$ -accessible functors and  $\kappa < \lambda$ , then  $\mathcal{B}$  is a  $\lambda$ -accessible  $\mathbf{U}$ -subcategory of  $\mathcal{C}$  and the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is also strongly  $(\lambda, \mathbf{U})$ -accessible.

*Proof.* (i). This is a straightforward exercise.

(ii). Consider the iso-comma category  $(F \wr \mathcal{D})$  and the induced comparison functor  $K : \mathcal{B} \rightarrow (F \wr \mathcal{D})$ . It is clear that  $\mathcal{B}$  is fully faithful; but since  $\mathcal{D}$  is a replete subcategory of  $\mathcal{C}$ , for every object  $(C, D, e)$  in  $(F \wr \mathcal{D})$ , there is a canonical isomorphism  $KC \rightarrow (C, D, e)$ , namely the one corresponding to the following commutative diagram in  $\mathcal{E}$ :

$$\begin{array}{ccc} FC & \xrightarrow{\text{id}} & FC \\ \text{id} \downarrow & & \downarrow e \\ FC & \xrightarrow{e} & D \end{array}$$

Thus,  $K : \mathcal{B} \rightarrow (F \wr \mathcal{D})$  is (half of) an equivalence of categories. [Theorem 0.3.27](#) says the projection  $P : (F \wr \mathcal{D}) \rightarrow \mathcal{C}$  is a strongly  $(\lambda, \mathbf{U})$ -accessible functor, so we may deduce that the same is true for the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$ . ■

**Proposition 0.3.31.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a strongly  $(\kappa, \mathbf{U})$ -accessible functor, and let  $\mathcal{D}'$  be the full subcategory of  $\mathcal{D}$  spanned by the image of  $F$ .*

- (i) *Every object in  $\mathcal{D}'$  is a colimit for some  $\mathbf{U}$ -small  $\kappa$ -filtered diagram consisting of objects in  $\mathcal{D}'$  that are  $(\kappa, \mathbf{U})$ -compact as objects in  $\mathcal{D}$ .*
- (ii) *Every  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{D}'$  is also  $(\kappa, \mathbf{U})$ -compact as an object in  $\mathcal{D}$ .*
- (iii) *If  $\mathcal{D}'$  is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{D}$ , then  $\mathcal{D}'$  is a  $\kappa$ -accessible  $\mathbf{U}$ -subcategory of  $\mathcal{D}$ .*

*Proof.* (i). Let  $D$  be any object in  $\mathcal{D}'$ . By definition, there is an object  $C$  in  $\mathcal{C}$  such that  $D = FC$ , and since  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, there is a  $\mathbf{U}$ -small  $\kappa$ -filtered diagram  $X : \mathcal{J} \rightarrow \mathcal{C}$  such that each  $Xj$  is a  $(\kappa, \mathbf{U})$ -compact object in

$\mathcal{C}$  and  $\mathcal{C} \cong \varinjlim_{\mathcal{J}} X$ . Since  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a strongly  $(\kappa, \mathbf{U})$ -accessible functor, each  $FXj$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{D}$  and we have  $\mathcal{D} \cong \varinjlim_{\mathcal{J}} FX$ .

(ii). Moreover, if  $D$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{D}'$ , then  $D$  must be a retract of  $FXj$  for some object  $j$  in  $\mathcal{J}$ , and so  $D$  is also  $(\kappa, \mathbf{U})$ -compact as an object in  $\mathcal{D}$ .

(iii). Any object in  $\mathcal{D}'$  that is  $(\kappa, \mathbf{U})$ -compact as an object in  $\mathcal{D}$  must be  $(\kappa, \mathbf{U})$ -compact as an object in  $\mathcal{D}'$ , because  $\mathcal{D}'$  is a full subcategory of  $\mathcal{D}$  that is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.  $\blacksquare$

**Proposition 0.3.32.** *Let  $\mathbb{B}$  be a  $\mathbf{U}$ -small category and let  $\mathbb{D}$  be a  $\kappa$ -small poset. If  $\mathbb{D}$  is well-founded, then:*

(i) *The  $(\kappa, \mathbf{U})$ -accessible functor*

$$\mathbf{Ind}^{\kappa}([\mathbb{D}, \mathbb{B}]) \rightarrow [\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathbb{B})]$$

*obtained by extending the canonical embedding  $[\mathbb{D}, \mathbb{B}] \rightarrow [\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathbb{B})]$  is fully faithful and essentially surjective on objects.*

(ii) *The evaluation functors  $[\mathbb{D}, \mathbf{Ind}^{\kappa}(\mathbb{B})] \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  are strongly  $(\kappa, \mathbf{U})$ -accessible functors.*

*Proof.* Let  $Y : \mathbb{D} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  be a diagram, let  $\gamma : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  be the canonical embedding, and consider the following pullback diagram,

$$\begin{array}{ccc} \mathcal{J} & \longrightarrow & [\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]_{/Y} \\ P \downarrow & & \downarrow \\ [\mathbb{D}, \mathbb{B}] & \xrightarrow{[\mathbb{D}, \gamma]} & [\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})] \end{array}$$

where the functor  $[\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]_{/C} \rightarrow [\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]$  is the projection. The objects of the category  $\mathcal{J}$  are diagrams  $\mathbb{D} \rightarrow \mathbb{B}$  (regarded as diagrams  $\mathbb{D} \rightarrow \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$ ) equipped with a morphism  $X \rightarrow Y$  in  $[\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]$ , so  $\mathcal{J}$  is a  $\mathbf{U}$ -small category. Recalling corollaries 0.2.15 and 0.2.18 and proposition 0.2.46, to prove the claims, it suffices to show that  $\mathcal{J}$  is a  $\kappa$ -filtered category and that the tautological cocone is a colimiting cocone.

Let  $X : \mathcal{I} \rightarrow \mathcal{J}$  be a  $\kappa$ -small diagram. We can then build an object  $\tilde{X}$  in  $\mathcal{J}$  equipped with a cocone under  $X$  by well-founded induction over  $\mathbb{D}$ :

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- Given  $\tilde{X}(d')$  and the cocone components for all  $d' < d$  in  $\mathbb{D}$ , by considering an appropriate diagram in  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  and using [lemma 0.3.20](#), we may choose an object  $\tilde{X}(d)$  in  $\mathbb{B}$  equipped with morphisms  $\tilde{X}(d') \rightarrow \tilde{X}(d)$  for all  $d' < d$ , morphisms  $X(i)(d) \rightarrow \tilde{X}(d)$  for all  $i$  in  $\mathcal{I}$ , and a morphism  $\tilde{X}(d) \rightarrow Y(d)$ , all these making the appropriate diagrams commute.

Thus  $\mathcal{J}$  is indeed a  $\kappa$ -filtered category. To complete the proof, we must check that the tautological cocone to  $Y$  is a colimiting cocone in  $[\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]$ . Let  $d$  be an object in  $\mathbb{D}$  and consider the comma category  $(\gamma \downarrow Yd)$ . There is an evident functor  $P_d : \mathcal{J} \rightarrow (\gamma \downarrow Yd)$  induced by  $\mathcal{J} \rightarrow [\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]_{/Y}$ , and  $P_d$  is a cofinal functor: indeed, by modifying the construction above (at the stage where  $\tilde{X}(d)$  is chosen) in the cases  $\mathcal{I} = \emptyset$  and  $\mathcal{I} = \text{disc } 2$ , one may verify that the comma category  $((B, q) \downarrow P_d)$  is connected for each object  $(B, q)$  in  $(\gamma \downarrow Yd)$ . Thus, the tautological cocone under the canonical diagram  $\mathcal{J} \rightarrow [\mathbb{D}, \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})]$  is a colimiting cocone, as required.  $\blacksquare$

**Corollary 0.3.33.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$  and let  $\mathbb{D}$  be a  $\kappa$ -small well-founded poset.*

- (i) *If  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category, then so is  $[\mathbb{D}, \mathcal{C}]$ , and the evaluation functors  $[\mathbb{D}, \mathcal{C}] \rightarrow \mathcal{C}$  are strongly  $(\kappa, \mathbf{U})$ -accessible.*
- (ii) *If  $\mathbb{B}$  is a  $\mathbf{U}$ -small category with colimits (resp. limits) of shape  $\mathbb{D}$ , then  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  has colimits (resp. limits) of shape  $\mathbb{D}$ .*

*Proof.* (i). The classification theorem for accessible categories ([theorem 0.2.28](#)) says  $\mathcal{C}$  is equivalent to  $\mathbf{Ind}^{\kappa}(\mathbb{B})$  for some small category  $\mathbb{B}$ , so we may apply [proposition 0.3.32](#).

(ii). Recalling [proposition 0.1.12](#), this follows from claim (i) and the fact that  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(-)$  is pseudofunctorial (hence, preserves adjunctions).  $\blacksquare$

**Corollary 0.3.34.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$  and let  $\mathbb{B}$  be a  $\mathbf{U}$ -small Cauchy-complete category. The following are equivalent:*

- (i)  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  has colimits for  $\mathbf{U}$ -small  $\aleph_0$ -filtered diagrams.
- (ii)  $\mathbb{B}$  has colimits for  $\kappa$ -small  $\aleph_0$ -filtered diagrams.
- (iii)  $\mathbb{B}$  has colimits for  $\alpha$ -chains for all ordinals  $\alpha$  of cardinality  $< \kappa$ .

*Proof.* (i)  $\Rightarrow$  (ii). Use [lemma 0.2.17](#) and [proposition 0.2.24](#).

(ii)  $\Rightarrow$  (iii). Immediate.

(iii)  $\Rightarrow$  (i). By [corollary 0.3.33](#),  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  has colimits for  $\alpha$ -chains for all ordinals  $\alpha$  of cardinality  $< \kappa$  if  $\mathbb{B}$  has them; but since  $\alpha$ -chains for ordinals  $\alpha$  of cardinality  $\geq \kappa$  are  $\kappa$ -filtered, it then follows that  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  has colimits for all  $\mathbf{U}$ -small chains. We may then apply [theorem 0.2.11](#) to deduce that  $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$  has colimits for  $\mathbf{U}$ -small  $\aleph_0$ -filtered diagrams.  $\blacksquare$

**Theorem 0.3.35** (The category of algebras for an accessible monad). *Let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, let  $\mathbf{T} = (T, \eta, \mu)$  be a monad on  $\mathcal{C}$ , and let  $\mathcal{C}^{\mathbf{T}}$  be the category of algebras for  $\mathbf{T}$ . If  $T : \mathcal{C} \rightarrow \mathcal{C}$  is a  $(\kappa, \mathbf{U})$ -accessible functor, then:*

(i) *The forgetful functor  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$  creates colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams and creates limits for all  $\mathbf{U}$ -small diagrams.*

(ii)  *$\mathcal{C}^{\mathbf{T}}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*

*Proof.* (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). See Theorem 2.78 and the following remark in [LPAC], or Theorem 5.5.9 in [Borceux, 1994b].  $\square$

**Lemma 0.3.36.** *Let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category and let  $\mathbf{T} = (T, \eta, \mu)$  be a monad on  $\mathcal{C}$ . If the forgetful functor  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$  is strongly  $(\kappa, \mathbf{U})$ -accessible, then so is the functor  $T : \mathcal{C} \rightarrow \mathcal{C}$ .*

*Proof.* The accessible adjoint functor theorem (0.2.49) says the free  $\mathbf{T}$ -algebra functor  $F : \mathcal{C} \rightarrow \mathcal{C}^{\mathbf{T}}$  is strongly  $(\kappa, \mathbf{U})$ -accessible if the forgetful functor  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$  is  $(\kappa, \mathbf{U})$ -accessible; but  $T = UF$ , so  $T$  is strongly  $(\kappa, \mathbf{U})$ -accessible when  $U$  is.  $\blacksquare$

**Theorem 0.3.37** (The category of algebras for a strongly accessible monad). *Let  $\mathcal{C}$  be a locally  $\lambda$ -presentable  $\mathbf{U}$ -category, let  $\mathbf{T} = (T, \eta, \mu)$  be a monad on  $\mathcal{C}$  where  $T : \mathcal{C} \rightarrow \mathcal{C}$  has  $\mathbf{U}$ -rank  $\kappa$ , and let  $\mathcal{C}^{\mathbf{T}}$  be the category of algebras for  $\mathbf{T}$ . If  $T : \mathcal{C} \rightarrow \mathcal{C}$  is a strongly  $(\lambda, \mathbf{U})$ -accessible functor and  $\kappa < \lambda$ , then:*

(i) Given a coequaliser diagram in  $\mathcal{C}^{\mathbf{T}}$  of the form below,

$$(A, \alpha) \rightrightarrows (B, \beta) \longrightarrow (C, \gamma)$$

if  $A$  and  $B$  are  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{C}$ , then so is  $C$ .

(ii) Given a  $\lambda$ -small family  $((A_i, \alpha_i) \mid i \in I)$  of  $\mathbf{T}$ -algebras, if each  $A_i$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ , then so is the underlying object of the  $\mathbf{T}$ -algebra coproduct  $\sum_{i \in I} (A_i, \alpha_i)$ .

(iii) The forgetful functor  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$  is strongly  $(\lambda, \mathbf{U})$ -accessible.

*Proof.* (i). By referring to the explicit construction of coequalisers in  $\mathcal{C}^{\mathbf{T}}$  given in the proof of Proposition 4.3.6 in [Borceux, 1994b] and applying lemma 0.2.17, we see that  $C$  is indeed a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$  when  $A$  and  $B$  are, provided  $T : \mathcal{C} \rightarrow \mathcal{C}$  has  $\mathbf{U}$ -rank  $\kappa$  and is strongly  $(\lambda, \mathbf{U})$ -accessible.

(ii). Let  $F : \mathcal{C} \rightarrow \mathcal{C}^{\mathbf{T}}$  be a left adjoint for  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$ . In the proof of Proposition 4.3.4 in [Borceux, 1994b], we find that the  $\mathbf{T}$ -algebra coproduct  $\sum_{i \in I} (A_i, \alpha_i)$  may be computed by a coequaliser diagram of the following form:

$$F\left(\sum_{i \in I} TA_i\right) \rightrightarrows F\left(\sum_{i \in I} A_i\right) \longrightarrow \sum_{i \in I} (A_i, \alpha_i)$$

Since  $T : \mathcal{C} \rightarrow \mathcal{C}$  is strongly  $(\lambda, \mathbf{U})$ -accessible, the underlying objects of the  $\mathbf{T}$ -algebras  $F\left(\sum_{i \in I} TA_i\right)$  and  $F\left(\sum_{i \in I} A_i\right)$  are  $(\lambda, \mathbf{U})$ -compact objects in  $\mathcal{C}$ . Thus, by claim (i), the underlying object of  $\sum_{i \in I} (A_i, \alpha_i)$  must also be a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ .

(iii). It is shown in the proof of Theorem 5.5.9 in [Borceux, 1994b] that the full subcategory  $\mathcal{F}$  of  $\mathcal{C}^{\mathbf{T}}$  spanned by the image of  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$  under  $F : \mathcal{C} \rightarrow \mathcal{C}^{\mathbf{T}}$  is a dense subcategory. Let  $\mathcal{G}$  be the smallest replete full subcategory of  $\mathcal{C}^{\mathbf{T}}$  that is closed under colimits for  $\lambda$ -small diagrams in  $\mathcal{C}$  and that contains  $\mathcal{F}$ . Observe that claims (i) and (ii) imply that the underlying object of every  $\mathbf{T}$ -algebra that is in  $\mathcal{G}$  must be a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ . To show that the forgetful functor  $U : \mathcal{C}^{\mathbf{T}} \rightarrow \mathcal{C}$  is strongly  $(\lambda, \mathbf{U})$ -accessible, it is enough to verify that every  $(\lambda, \mathbf{U})$ -compact object is in  $\mathcal{G}$ .

It is not hard to see that the comma category  $(\mathcal{G} \downarrow (A, \alpha))$  is then an essentially  $\mathbf{U}$ -small  $\lambda$ -filtered category for any  $\mathbf{T}$ -algebra  $(A, \alpha)$ , and moreover, it can be shown that the tautological cocone for the canonical diagram  $(\mathcal{G} \downarrow (A, \alpha)) \rightarrow \mathcal{C}^{\mathbf{T}}$

is a colimiting cocone. Thus, if  $(A, \alpha)$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}^\top$ , it must be a retract of an object in  $\mathcal{G}$ . But  $\mathcal{G}$  is closed under retracts, so  $(A, \alpha)$  is indeed in  $\mathcal{G}$ .  $\blacksquare$

**Definition 0.3.38.** Let  $\mathcal{C}$  be any category.

- A **pointed endofunctor** on  $\mathcal{C}$  is a functor  $J : \mathcal{C} \rightarrow \mathcal{C}$  equipped with a natural transformation  $\iota : \text{id}_{\mathcal{C}} \Rightarrow J$ .
- An **algebra** for a pointed endofunctor  $(J, \iota)$  on  $\mathcal{C}$  is an object  $A$  in  $\mathcal{C}$  equipped with a morphism  $\alpha : JA \rightarrow A$  such that  $\alpha \circ \iota_A = \text{id}_A$ .
- A **homomorphism of algebras** for a pointed endofunctor  $(J, \iota)$  on  $\mathcal{C}$ , say  $f : (A, \alpha) \rightarrow (B, \beta)$ , is a morphism  $f : A \rightarrow B$  making the following diagram commute:

$$\begin{array}{ccc} JA & \xrightarrow{Jf} & JB \\ \alpha \downarrow & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

We write  $\mathcal{C}^{(J, \iota)}$  for the category of algebras for a pointed endofunctor  $(J, \iota)$  on  $\mathcal{C}$ .

The following result on the existence of free algebras for a pointed endofunctor is a special case of a general construction due to [Kelly, 1980].

**Theorem 0.3.39** (Free algebras for a pointed endofunctor). *Let  $\kappa$  be a regular cardinal, let  $\mathcal{C}$  be a category with pushouts and colimits for chains of length  $\leq \kappa$ , and let  $(J, \iota)$  be a pointed endofunctor on  $\mathcal{C}$  such that  $J : \mathcal{C} \rightarrow \mathcal{C}$  preserves colimits for  $\kappa$ -chains.*

- The forgetful functor  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  has a left adjoint, say  $F : \mathcal{C} \rightarrow \mathcal{C}^{(J, \iota)}$ .*
- Let  $\lambda$  be a regular cardinal in a universe  $\mathbf{U}$ . If  $J : \mathcal{C} \rightarrow \mathcal{C}$  sends  $(\lambda, \mathbf{U})$ -compact objects to  $(\lambda, \mathbf{U})$ -compact objects and  $\kappa < \lambda$ , then the functor  $UF : \mathcal{C} \rightarrow \mathcal{C}$  has the same property.*

*Proof.* Let  $X$  be an object in  $\mathcal{C}$ . We now define a chain  $X_\bullet : \kappa + 2 \rightarrow \mathcal{C}$  by transfinite induction:

- Let  $X_0 = X$ , let  $X_1 = JX_0$ , let  $q_0 = \text{id}_{JX_0}$ , and let  $X_{0 \rightarrow 1} : X_0 \rightarrow X_1$  be  $\iota_{X_0}$ .

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- Given  $q_\alpha : JX_\alpha \rightarrow X_{\alpha+1}$  for an ordinal  $\alpha < \kappa$ , define  $X_{\alpha+2}$  by the following coequaliser diagram in  $\mathcal{C}$ :

$$JX_\alpha \begin{array}{c} \xrightarrow{Jq_\alpha \circ J\iota_{X_\alpha}} \\ \xrightarrow{Jq_\alpha \circ \iota_{JX_\alpha}} \end{array} JX_{\alpha+1} \xrightarrow{q_{\alpha+1}} X_{\alpha+2}$$

Then, for all  $\alpha' < \alpha + 2$ , set  $X_{\alpha' \rightarrow \alpha+2} = q_{\alpha+1} \circ \iota_{X_{\alpha+1}} \circ X_{\alpha' \rightarrow \alpha+1}$ ; note that the diagram below commutes:

$$\begin{array}{ccc} JX_\alpha & \xrightarrow{JX_{\alpha \rightarrow \alpha+1}} & JX_{\alpha+1} \\ q_\alpha \downarrow & & \downarrow q_{\alpha+1} \\ X_{\alpha+1} & \xrightarrow{X_{\alpha+1 \rightarrow \alpha+2}} & X_{\alpha+2} \end{array}$$

- Given a limit ordinal  $\beta \leq \kappa$  and  $q_\alpha$  for all ordinals  $\alpha < \beta$ , define  $X_\beta = \varinjlim_{\alpha < \beta} X_\alpha$  and take  $X_{\beta \rightarrow \alpha} : X_\beta \rightarrow X_\alpha$  to be the component of the colimiting cocone; then define  $X_{\beta+1}$  to be the colimit of the following diagram,

$$\begin{array}{ccccccc} JX_0 & \longrightarrow & JX_1 & \longrightarrow & JX_2 & \longrightarrow & \cdots \longrightarrow JX_\beta \\ q_0 \downarrow & & q_1 \downarrow & & q_2 \downarrow & & \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \longrightarrow X_\beta \end{array}$$

and let  $q_\beta : JX_\beta \rightarrow X_{\beta+1}$  and  $X_{\beta \rightarrow \beta+1} : X_\beta \rightarrow X_{\beta+1}$  be the respective components of the colimiting cocone; note that the following diagram commutes,

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \longrightarrow X_\beta \\ \iota_{X_0} \downarrow & & \iota_{X_1} \downarrow & & \iota_{X_2} \downarrow & & \downarrow \iota_{X_\beta} \\ JX_0 & \longrightarrow & JX_1 & \longrightarrow & JX_2 & \longrightarrow & \cdots \longrightarrow JX_\beta \\ q_0 \downarrow & & q_1 \downarrow & & q_2 \downarrow & & \downarrow q_\beta \\ X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 & \longrightarrow & \cdots \longrightarrow X_\beta \longrightarrow X_{\beta+1} \end{array}$$

so we have  $X_{\beta \rightarrow \beta+1} = q_\beta \circ \iota_{X_\beta}$ .

Our hypothesis is that  $J$  preserves colimits for  $\kappa$ -chains, so the canonical comparison  $\varinjlim_{\alpha < \kappa} JX_\alpha \rightarrow JX_\kappa$  is an isomorphism, as is  $X_{\kappa \rightarrow \kappa+1}$ . However, for all ordinals  $\alpha < \beta < \kappa$ , we have

$$X_{\alpha+1 \rightarrow \beta+1} \circ q_\alpha = q_\beta \circ JX_{\alpha \rightarrow \beta}$$



so there is a unique morphism  $\gamma_X : JX_\kappa \rightarrow X_\kappa$  such that

$$\gamma_X \circ JX_{\alpha \rightarrow \kappa} = X_{\alpha+1 \rightarrow \kappa} \circ q_\alpha$$

for all ordinals  $\alpha < \kappa$ . Moreover, we have

$$\gamma_X \circ \iota_{X_\kappa} \circ X_{\alpha \rightarrow \kappa} = \gamma_X \circ JX_{\alpha \rightarrow \kappa} \circ \iota_{X_\alpha} = X_{\alpha+1 \rightarrow \kappa} \circ q_\alpha \circ \iota_{X_\alpha} = X_{\alpha \rightarrow \kappa}$$

and  $\{X_{\alpha \rightarrow \kappa} \mid \alpha < \kappa\}$  is a jointly epimorphic family, so  $\gamma_X \circ \iota_{X_\kappa} = \text{id}_{X_\kappa}$ , i.e.  $(X_\kappa, \gamma_X)$  is a  $(J, \iota)$ -algebra.

It remains to be shown that  $(X_\kappa, \gamma_X)$  is a free  $(J, \iota)$ -algebra generated by  $X$ . Let  $\eta_X = X_{0 \rightarrow \kappa}$ , let  $(D, \delta)$  be any  $(J, \iota)$ -algebra, and let  $f : X \rightarrow D$  be any morphism in  $\mathcal{C}$ . We construct a cocone  $f_\bullet : X_\bullet \Rightarrow \Delta D$  by transfinite induction:

- Let  $f_0 = f$ , let  $f_1 = \delta \circ Jf_0$ , and note that  $\delta \circ Jf_0 = f_1 \circ q_0$ .
- Given  $f_\alpha : X_\alpha \rightarrow D$  and  $f_{\alpha+1} : X_{\alpha+1} \rightarrow D$  such that  $f_{\alpha+1} \circ q_\alpha = \delta \circ Jf_\alpha$ , let  $f_{\alpha+2} : X_{\alpha+2} \rightarrow D$  be the unique morphism satisfying the following equation:

$$f_{\alpha+2} \circ q_{\alpha+1} = \delta \circ Jf_{\alpha+1}$$

Note that such a morphism exists because the diagrams below commute,

$$\begin{array}{ccc} JX_\alpha & \xrightarrow{\iota_{JX_\alpha}} & J J X_\alpha \\ q_\alpha \downarrow & & \downarrow Jq_\alpha \\ X_{\alpha+1} & \xrightarrow{\iota_{JX_{\alpha+1}}} & J X_{\alpha+1} \\ f_{\alpha+1} \downarrow & & \downarrow Jf_{\alpha+1} \\ D & \xrightarrow{\iota_A} & J D \\ & \searrow \text{id} & \downarrow \delta \\ & & D \end{array} \quad \begin{array}{ccc} JX_\alpha & \xrightarrow{Jf_\alpha} & J D \quad \equiv \quad J D \\ J \iota_{X_\alpha} \downarrow & & \downarrow J \iota_D \\ J J X_\alpha & \xrightarrow{J J f_\alpha} & J J D \\ J q_\alpha \downarrow & & \downarrow J \delta \\ J X_{\alpha+1} & \xrightarrow{J f_{\alpha+1}} & J D \xrightarrow{\delta} D \end{array}$$

i.e. because the equation below holds,

$$(\delta \circ Jf_{\alpha+1}) \circ (Jq_\alpha \circ \iota_{JX_\alpha}) = (\delta \circ Jf_{\alpha+1}) \circ (Jq_\alpha \circ J \iota_{X_\alpha})$$

and  $q_{\alpha+1} : JX_{\alpha+1} \rightarrow X_{\alpha+2}$  is the coequaliser of  $Jq_\alpha \circ \iota_{JX_\alpha}$  and  $Jq_\alpha \circ J \iota_{X_\alpha}$ .

- Given a limit ordinal  $\beta \leq \kappa$ , we define  $f_\beta : X_\beta \rightarrow D$  be the unique morphism such that  $f_\beta \circ X_{\alpha \rightarrow \beta} = f_\alpha$  for all ordinals  $\alpha < \beta$ ; we may do this because the following equation holds:

$$f_{\alpha+1} \circ X_{\alpha \rightarrow \alpha+1} = f_{\alpha+1} \circ q_\alpha \circ \iota_{X_{\alpha+1}} = \delta \circ Jf_\alpha \circ \iota_{X_{\alpha+1}} = \delta \circ \iota_D \circ f_\alpha = f_\alpha$$

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Furthermore,

$$(\delta \circ Jf_\beta) \circ JX_{\alpha \rightarrow \beta} = \delta \circ Jf_\alpha = f_{\alpha+1} \circ q_\alpha$$

so there exists a unique morphism  $f_{\beta+1} : X_{\beta+1} \rightarrow D$  such that  $f_{\beta+1} \circ q_\beta = \delta \circ Jf_\beta$  and  $f_{\beta+1} \circ X_{\alpha \rightarrow \beta+1} = f_\alpha$  for all ordinals  $\alpha < \beta$ .

Now observe that, for all ordinals  $\alpha < \kappa$ ,

$$\begin{aligned} \delta \circ Jf_\kappa \circ JX_{\alpha \rightarrow \kappa} &= \delta \circ Jf_\alpha \\ &= f_{\alpha+1} \circ q_\alpha \\ &= f_\kappa \circ X_{\alpha+1 \rightarrow \kappa} \circ q_\alpha \\ &= f_\kappa \circ \gamma_X \circ JX_{\alpha \rightarrow \kappa} \end{aligned}$$

and  $\{JX_{\alpha \rightarrow \kappa} \mid \alpha < \kappa\}$  is a jointly epimorphic family, so  $\delta \circ Jf_\kappa = f_\kappa \circ \gamma_X$ , i.e.  $f_\kappa$  is a  $(J, \iota)$ -algebra homomorphism  $(X_\kappa, \gamma_X) \rightarrow (D, \delta)$ . Finally, notice that, for any homomorphism  $\bar{f} : (X_\kappa, \gamma_X) \rightarrow (D, \delta)$  such that  $\bar{f} \circ \eta_X = f_0$ , then,

$$\delta \circ J(\bar{f} \circ X_{\alpha \rightarrow \kappa}) = \bar{f} \circ \gamma_X \circ JX_{\alpha \rightarrow \kappa} = (\bar{f} \circ X_{\alpha+1 \rightarrow \kappa}) \circ q_\alpha$$

hence we must have  $\bar{f} = f_\kappa$ , by transfinite induction.

The above argument shows that the comma category  $(X \downarrow U)$  has an initial object, and it is well known that  $U$  has a left adjoint if and only if each comma category  $(X \downarrow U)$  has an initial object, so this completes the proof of claim (i). For claim (ii), we simply observe that  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$  is closed under colimits for  $\lambda$ -small diagrams in  $\mathcal{C}$  (by lemma 0.2.17), so the above construction can be carried out entirely in  $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ . ■

**Theorem 0.3.40** (The category of algebras for a accessible pointed endofunctor). *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category, let  $J : \mathcal{C} \rightarrow \mathcal{C}$  be a  $(\kappa, \mathbf{U})$ -accessible functor, let  $\iota : \text{id}_{\mathcal{C}} \Rightarrow J$  be a natural transformation, and let  $\mathcal{C}^{(J, \iota)}$  be the category of algebras for the pointed endofunctor  $(J, \iota)$ .*

- (i) *The forgetful functor  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  creates colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams; and if  $\mathcal{C}$  is  $\mathbf{U}$ -complete, then  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  also creates limits for all  $\mathbf{U}$ -small diagrams.*
- (ii)  *$\mathcal{C}^{(J, \iota)}$  is an accessible  $\mathbf{U}$ -category.*
- (iii) *If  $\mathcal{C}$  has pushouts and colimits for chains of length  $\leq \kappa$ , then  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  is a monadic functor.*

*Proof.* (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). We may construct  $\mathcal{C}^{(J, \iota)}$  using inserters and equifiers, as in the proof of Theorem 2.78 in [LPAC].

(iii). Since  $\kappa$ -chains are  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, the hypotheses of [theorem 0.3.39](#) are satisfied, and so the forgetful functor  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  has a left adjoint. It is not hard to check that the other hypotheses of Beck's monadicity theorem are satisfied, so  $U$  is indeed a monadic functor.  $\square$

**Theorem 0.3.41** (The category of algebras for a strongly accessible pointed endofunctor). *Let  $\mathcal{C}$  be a locally  $\lambda$ -presentable  $\mathbf{U}$ -category, let  $J : \mathcal{C} \rightarrow \mathcal{C}$  be a functor of  $\mathbf{U}$ -rank  $\leq \kappa$ , let  $\iota : \text{id}_{\mathcal{C}} \Rightarrow J$  be a natural transformation, let  $\mathcal{C}^{(J, \iota)}$  be the category of algebras for the pointed endofunctor  $(J, \iota)$ , and let  $\mathbf{T} = (T, \eta, \mu)$  be the induced monad on  $\mathcal{C}$ . If  $J : \mathcal{C} \rightarrow \mathcal{C}$  is a strongly  $(\lambda, \mathbf{U})$ -accessible functor and  $\kappa < \lambda$ , then:*

- (i) *The functor  $T : \mathcal{C} \rightarrow \mathcal{C}$  has  $\mathbf{U}$ -rank  $\leq \kappa$  and is strongly  $(\lambda, \mathbf{U})$ -accessible.*
- (ii)  *$\mathcal{C}^{(J, \iota)}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*
- (iii) *The forgetful functor  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  is a strongly  $(\lambda, \mathbf{U})$ -accessible functor.*

*Proof.* (i). We know that the forgetful functor  $U : \mathcal{C}^{(J, \iota)} \rightarrow \mathcal{C}$  creates colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams when  $J : \mathcal{C} \rightarrow \mathcal{C}$  has  $\mathbf{U}$ -rank  $\leq \kappa$ , so  $T : \mathcal{C} \rightarrow \mathcal{C}$  must also have  $\mathbf{U}$ -rank  $\leq \kappa$ . Moreover, [theorem 0.3.39](#) implies  $T : \mathcal{C} \rightarrow \mathcal{C}$  is strongly  $(\lambda, \mathbf{U})$ -accessible if  $J : \mathcal{C} \rightarrow \mathcal{C}$  is.

(ii). Apply [theorem 0.3.35](#).

(iii). Apply [theorem 0.3.37](#).  $\blacksquare$

## 0.4 Change of universe

**Prerequisites.** §§0.1, 0.2, A.1, A.5.

Having introduced universes into our ontology, it becomes necessary to ask whether an object with some universal property retains that property when we enlarge the universe. Though it sounds inconceivable, there do exist examples of

badly-behaved constructions that are not stable under change-of-universe; for example, Waterhouse [1975] defined a functor  $F : \mathbf{CRing} \rightarrow \mathbf{Set}^+$ , where  $\mathbf{CRing}$  is the category of commutative rings in a universe  $\mathbf{U}$  and  $\mathbf{Set}^+$  is the category of  $\mathbf{U}^+$ -sets for some universe  $\mathbf{U}^+$  with  $\mathbf{U} \in \mathbf{U}^+$ , such that the value of  $F$  at any given commutative ring in  $\mathbf{U}$  does not depend on  $\mathbf{U}$ , and yet the value of the fpqc sheaf associated with  $F$  at the field  $\mathbb{Q}$  depends on the size of  $\mathbf{U}$ .

**Definition 0.4.1.** Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , and let  $\mathbf{U}^+$  be a universe with  $\mathbf{U} \subseteq \mathbf{U}^+$ . A  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -**accessible extension** is a  $(\kappa, \mathbf{U})$ -accessible functor  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  such that

- $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category,
- $\mathcal{C}^+$  is a  $\kappa$ -accessible  $\mathbf{U}^+$ -category,
- $i$  sends  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$  to  $(\kappa, \mathbf{U}^+)$ -compact objects in  $\mathcal{C}^+$ , and
- the functor  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \rightarrow \mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$  so induced by  $i$  is fully faithful and essentially surjective on objects.

**REMARK 0.4.2.** Let  $\mathbb{B}$  be a  $\mathbf{U}$ -small category in which idempotents split. Then the  $(\kappa, \mathbf{U})$ -accessible functor  $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{B}) \rightarrow \mathbf{Ind}_{\mathbf{U}^+}^\kappa(\mathbb{B})$  obtained by extending the embedding  $\gamma^+ : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}^+}^\kappa(\mathbb{B})$  along  $\gamma : \mathbb{B} \rightarrow \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{B})$  is a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension, by [proposition 0.2.24](#). The classification theorem ([0.2.28](#)) implies all examples of  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extensions are essentially of this form.

**Proposition 0.4.3.** *Let  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  be a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension.*

- (i)  *$\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category if and only if  $\mathcal{C}^+$  is a locally  $\kappa$ -presentable  $\mathbf{U}^+$ -category.*
- (ii) *The functor  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  is fully faithful.*
- (iii) *If  $B : \mathcal{J} \rightarrow \mathcal{C}$  is any diagram (not necessarily  $\mathbf{U}$ -small) and  $\mathcal{C}$  has a limit for  $B$ , then  $i$  preserves this limit.*

*Proof.* (i). If  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, then  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  has colimits for all  $\kappa$ -small diagrams, so  $\mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$  also has colimits for all  $\kappa$ -small diagrams. The classification theorem ([0.2.28](#)) then implies  $\mathcal{C}^+$  is a locally  $\kappa$ -presentable  $\mathbf{U}^+$ -category. Reversing this argument proves the converse.

(ii). Let  $A : \mathbb{I} \rightarrow \mathcal{C}$  and  $B : \mathbb{J} \rightarrow \mathcal{C}$  be two  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams of  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$ . Then,

$$\begin{aligned} \mathcal{C}\left(\lim_{\mathbb{I}} A, \lim_{\mathbb{J}} B\right) &\cong \lim_{\mathbb{I}} \lim_{\mathbb{J}} \mathcal{C}(A, B) \cong \lim_{\mathbb{I}} \lim_{\mathbb{J}} \mathcal{C}^+(iA, iB) \\ &\cong \mathcal{C}^+\left(\lim_{\mathbb{I}} iA, \lim_{\mathbb{J}} iB\right) \cong \mathcal{C}^+\left(i \lim_{\mathbb{I}} A, i \lim_{\mathbb{J}} B\right) \end{aligned}$$

because  $i$  is  $(\kappa, \mathbf{U})$ -accessible and is fully faithful on the subcategory  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ , and therefore  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  itself is fully faithful. Note that this hinges crucially on [theorem 0.1.31](#).

(iii). Let  $B : \mathcal{J} \rightarrow \mathcal{C}$  be any diagram. We observe that, for any  $(\kappa, \mathbf{U})$ -compact object  $C$  in  $\mathcal{C}$ ,

$$\begin{aligned} \mathcal{C}^+\left(iC, i \lim_{\mathcal{J}} B\right) &\cong \mathcal{C}\left(C, \lim_{\mathcal{J}} B\right) && \text{because } i \text{ is fully faithful} \\ &\cong \lim_{\mathcal{J}} \mathcal{C}(C, B) && \text{by definition of limit} \\ &\cong \lim_{\mathcal{J}} \mathcal{C}^+(iC, iB) && \text{because } i \text{ is fully faithful} \end{aligned}$$

but we know the restricted Yoneda embedding  $\mathcal{C}^+ \rightarrow [\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})^{\text{op}}, \mathbf{Set}^+]$  is fully faithful, so this is enough to conclude that  $i \lim_{\mathcal{J}} B$  is the limit of  $iB$  in  $\mathcal{C}^+$ . ■

**REMARK 0.4.4.** Similar methods show that any fully faithful functor  $\mathcal{C} \rightarrow \mathcal{C}^+$  satisfying the four bulleted conditions in the definition above is necessarily  $(\kappa, \mathbf{U})$ -accessible.

**Lemma 0.4.5.** *Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \in \mathbf{U}^+$ , and let  $\kappa$  be a regular cardinal in  $\mathbf{U}$ . Suppose:*

- $\mathcal{C}$  and  $\mathcal{D}$  are locally  $\kappa$ -presentable  $\mathbf{U}$ -categories.
- $\mathcal{C}^+$  and  $\mathcal{D}^+$  are locally  $\kappa$ -presentable  $\mathbf{U}^+$ -categories.
- $i : \mathcal{C} \rightarrow \mathcal{C}^+$  and  $j : \mathcal{D} \rightarrow \mathcal{D}^+$  are  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extensions.

Given a strictly commutative diagram of the form below,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \\ G \downarrow & & \downarrow G^+ \\ \mathcal{C} & \xrightarrow{i} & \mathcal{C}^+ \end{array}$$

where  $G$  is  $(\kappa, \mathbf{U})$ -accessible,  $G^+$  is  $(\kappa, \mathbf{U}^+)$ -accessible, if both have left adjoints, then the diagram satisfies the left Beck–Chevalley condition.

*Proof.* Let  $C$  be a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ . Inspecting the proof of [theorem 0.2.49](#), we see that the functor  $(C \downarrow G) \rightarrow (iC \downarrow G^+)$  induced by  $j$  preserves initial objects. [Lemma A.1.10](#) says the component at  $C$  of the left Beck–Chevalley natural transformation  $F^+i \Rightarrow jF$  is an isomorphism; but  $\mathcal{C}$  is generated by  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  and the functors  $F, F^+, i, j$  all preserve colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams, so in fact  $F^+i \Rightarrow jF$  is a natural isomorphism.  $\blacksquare$

**Proposition 0.4.6.** *If  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  is a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension and  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, then  $i$  preserves colimits for all  $\mathbf{U}$ -small diagrams in  $\mathcal{C}$ .*

*Proof.* It is well-known that a functor preserves colimits for all  $\mathbf{U}$ -small diagrams if and only if it preserves coequalisers for all parallel pairs and coproducts for all  $\mathbf{U}$ -small families, but coproducts for  $\mathbf{U}$ -small families can be constructed in a uniform way using coproducts for  $\kappa$ -small families and colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams. It is therefore enough to show that  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  preserves all colimits for  $\kappa$ -small diagrams, since  $i$  is already  $(\kappa, \mathbf{U})$ -accessible.

Let  $\mathbb{D}$  be a  $\kappa$ -small category. Recalling [proposition 0.1.12](#), our problem amounts to showing that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{C}^+ \\ \Delta \downarrow & & \downarrow \Delta^+ \\ [\mathbb{D}, \mathcal{C}] & \xrightarrow{i_*} & [\mathbb{D}, \mathcal{C}^+] \end{array}$$

satisfies the left Beck–Chevalley condition. It is clear that  $i_*$  is fully faithful. Colimits for  $\mathbf{U}$ -small diagrams in  $[\mathbb{D}, \mathcal{C}]$  and in  $[\mathbb{D}, \mathcal{C}^+]$  are computed componentwise, so  $\Delta$  and  $i_*$  are certainly  $(\kappa, \mathbf{U})$ -accessible, and  $\Delta^+$  is  $(\kappa, \mathbf{U}^+)$ -accessible. Using [proposition 0.2.46](#), we see that  $i_*$  is also a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension, so we apply the lemma above to conclude that the left Beck–Chevalley condition is satisfied.  $\blacksquare$

**Theorem 0.4.7** (Stability of accessible adjoint functors). *Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \in \mathbf{U}^+$ , and let  $\kappa$  and  $\lambda$  be regular cardinals in  $\mathbf{U}$ , with  $\kappa \leq \lambda$ . Suppose:*

- $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.
- $\mathcal{D}$  is a locally  $\lambda$ -presentable  $\mathbf{U}$ -category.
- $\mathcal{C}^+$  is a locally  $\kappa$ -presentable  $\mathbf{U}^+$ -category.
- $\mathcal{D}^+$  is a locally  $\lambda$ -presentable  $\mathbf{U}^+$ -category.

*Let  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  be a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension and let  $j : \mathcal{D} \rightarrow \mathcal{D}^+$  be a fully faithful functor.*

(i) *Given a strictly commutative diagram of the form below,*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \\ G \downarrow & & \downarrow G^+ \\ \mathcal{C} & \xrightarrow{i} & \mathcal{C}^+ \end{array}$$

*where  $G$  is  $(\lambda, \mathbf{U})$ -accessible and  $G^+$  is  $(\lambda, \mathbf{U}^+)$ -accessible, if both have left adjoints and  $j$  is a  $(\lambda, \mathbf{U}, \mathbf{U}^+)$ -accessible extension, then the diagram satisfies the left Beck–Chevalley condition.*

(ii) *Given a strictly commutative diagram of the form below,*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{C}^+ \\ F \downarrow & & \downarrow F^+ \\ \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \end{array}$$

*if both  $F$  and  $F^+$  have right adjoints, then the diagram satisfies the right Beck–Chevalley condition.*

*Proof.* (i). The proof is essentially the same as [lemma 0.4.5](#), though we have to use [proposition 0.4.6](#) to ensure that  $j$  preserves colimits for all  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{C}$ .

(ii). Let  $D$  be any object in  $\mathcal{D}$ . Inspecting the proof of [theorem 0.2.49](#), we see that our hypotheses, plus the fact that  $i$  preserves colimits for all  $\mathbf{U}$ -small diagrams in  $\mathcal{C}$ , imply that the functor  $(F \downarrow D) \rightarrow (F^+ \downarrow jD)$  induced by  $i$  preserves

terminal objects. Thus, [lemma A.1.10](#) implies that the diagram satisfies the right Beck–Chevalley condition. ■

**Theorem 0.4.8.** *Let  $i : C \rightarrow C^+$  be a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension and let  $C$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.*

- (i) *If  $\lambda$  is a regular cardinal in  $\mathbf{U}$  and  $\kappa \leq \lambda$ , then  $i : C \rightarrow C^+$  is also a  $(\lambda, \mathbf{U}, \mathbf{U}^+)$ -accessible extension.*
- (ii) *If  $\mu$  is the cardinality of  $\mathbf{U}$ , then  $i : C \rightarrow C^+$  factors through the inclusion  $\mathbf{K}_\mu^{\mathbf{U}^+}(C^+) \hookrightarrow C^+$  as functor  $C \rightarrow \mathbf{K}_\mu^{\mathbf{U}^+}(C^+)$  that is (fully faithful and) essentially surjective on objects.*
- (iii) *The  $(\mu, \mathbf{U}^+)$ -accessible functor  $\mathbf{Ind}_{\mathbf{U}^+}^\mu(C) \rightarrow C^+$  induced by  $i : C \rightarrow C^+$  is fully faithful and essentially surjective on objects.*

*Proof.* (i). Since  $i : C \rightarrow C^+$  is a  $(\kappa, \mathbf{U})$ -accessible functor, it is certainly also  $(\lambda, \mathbf{U})$ -accessible, by [lemma 0.2.37](#). It is therefore enough to show that  $i$  restricts to a functor  $\mathbf{K}_\kappa^{\mathbf{U}}(C) \rightarrow \mathbf{K}_\kappa^{\mathbf{U}^+}(C^+)$  that is (fully faithful and) essentially surjective on objects.

[Proposition 0.2.45](#) says  $\mathbf{K}_\lambda^{\mathbf{U}}(C)$  is the smallest replete full subcategory of  $C$  that contains  $\mathbf{K}_\kappa^{\mathbf{U}}(C)$  and is closed in  $C$  under colimits for  $\lambda$ -small diagrams, therefore the replete closure of the image of  $\mathbf{K}_\lambda^{\mathbf{U}}(C)$  must be the smallest replete full subcategory of  $C^+$  that contains  $\mathbf{K}_\kappa^{\mathbf{U}^+}(C^+)$  and is closed in  $C^+$  under colimits for  $\lambda$ -small diagrams, since  $i$  is fully faithful and preserves colimits for all  $\mathbf{U}$ -small diagrams. This proves the claim.

(ii). Since every object in  $C$  is  $(\lambda, \mathbf{U})$ -compact for some regular cardinal  $\lambda < \mu$ , claim (i) implies that the image of  $i : C \rightarrow C^+$  is contained in  $\mathbf{K}_\mu^{\mathbf{U}^+}(C)$ . To show  $i$  is essentially surjective onto  $\mathbf{K}_\mu^{\mathbf{U}^+}(C)$ , we simply have to observe that the inaccessibility of  $\mu$  ([proposition 0.1.36](#)) and [proposition 0.2.45](#) imply that, for  $C'$  any  $(\mu, \mathbf{U}^+)$ -compact object in  $C^+$ , there exists a regular cardinal  $\lambda < \mu$  such that  $C'$  is also a  $(\lambda, \mathbf{U}^+)$ -compact object, which reduces the question to claim (i).

(iii). This is an immediate corollary of claim (ii) and the classification theorem ([0.2.28](#)) applied to  $C^+$ , considered as a  $(\mu, \mathbf{U}^+)$ -accessible category. ■

**REMARK 0.4.9.** Although the fact  $i : C \rightarrow C^+$  that preserves limits and colimits for all  $\mathbf{U}$ -small diagrams in  $C$  is a formal consequence of the theorem above (via



e.g. [corollary A.5.30](#)), it is not clear whether the theorem can be proved without already knowing this.

**Corollary 0.4.10.** *If  $\mathbb{B}$  is a  $\mathbf{U}$ -small category and has colimits for all  $\kappa$ -small diagrams, and  $\mu$  is the cardinality of  $\mathbf{U}$ , then the canonical  $(\mu, \mathbf{U}^+)$ -accessible functor  $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}(\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})) \rightarrow \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$  is fully faithful and essentially surjective on objects. ■*

**Proposition 0.4.11.** *Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \in \mathbf{U}^+$ , and let  $\kappa$  and  $\lambda$  be regular cardinals in  $\mathbf{U}$ . Suppose:*

- $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category.
- $\mathcal{D}$  is a locally  $\lambda$ -presentable  $\mathbf{U}$ -category.
- $\mathcal{C}^+$  is a locally  $\kappa$ -presentable  $\mathbf{U}^+$ -category.
- $\mathcal{D}^+$  is a locally  $\lambda$ -presentable  $\mathbf{U}^+$ -category.

Let  $F : \mathcal{A} \rightarrow \mathcal{C}$  and  $G : \mathcal{A} \rightarrow \mathcal{D}$  be functors, let  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  be a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -accessible extension, and let  $j : \mathcal{D} \rightarrow \mathcal{D}^+$  be a  $(\lambda, \mathbf{U}, \mathbf{U}^+)$ -accessible extension. Consider the following (not necessarily commutative) diagram:

$$\begin{array}{ccccc}
 \mathcal{A} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \\
 F \downarrow & \nearrow H & & \nearrow H^+ & \\
 \mathcal{C} & & & & \\
 i \downarrow & & & & \\
 \mathcal{C}^+ & & & & 
 \end{array}$$

- (i) *If  $H$  is a pointwise right Kan extension of  $G$  along  $F$ , then  $jH$  is a pointwise right Kan extension of  $jG$  along  $F$ , and if  $H^+$  is a pointwise right Kan extension of  $jH$  along  $i$ , then  $H^+$  is also a pointwise right Kan extension of  $jG$  along  $iF$ .*
- (ii) *Assuming  $\mathcal{A}$  is  $\mathbf{U}$ -small, if  $H$  is a pointwise left Kan extension of  $G$  along  $F$ , then  $jH$  is a pointwise left Kan extension of  $jG$  along  $F$ , and if  $H^+$  is a pointwise left Kan extension of  $jH$  along  $i$ , then  $H^+$  is also a pointwise left Kan extension of  $jG$  along  $iF$ .*

*Proof.* Use [theorem A.5.20](#) and the fact that  $i$  and  $j$  preserve limits for all diagrams and colimits for  $\mathbf{U}$ -small diagrams. ■

## 0.5 Small object arguments

**Prerequisites.** §§ 0.1, 0.2, 0.3, 0.4, A.3, A.5.

The small object argument is a recurring construction in homotopical algebra, originally due to Quillen [1967, Ch. II, §3] but refined by many authors since—notably by Garner [2009]. Roughly speaking, the small object argument shows that, under certain hypotheses, starting from a small set  $\mathcal{I}$  of morphisms in a cocomplete category  $\mathcal{C}$ , one can define the notions of ‘relative  $\mathcal{I}$ -cell complex’ and ‘ $\mathcal{I}$ -fibration’ so that every morphism in  $\mathcal{C}$  factors as a relative  $\mathcal{I}$ -cell complex followed by an  $\mathcal{I}$ -fibration.

In this section, we will study the small object argument with a view toward questions of stability under change-of-universe.

**Definition 0.5.1.** Let  $\mathcal{C}$  be a category, and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ . A **presentation for a relative  $\mathcal{I}$ -cell complex** in  $\mathcal{C}$  consists of the following data:

- An ordinal  $\alpha$ . (We say the presentation is **indexed over**  $\alpha$ .)
- A colimit-preserving functor  $X_\bullet : [\alpha] \rightarrow \mathcal{C}$ , where  $[\alpha]$  is the well-ordered set  $\{0, \dots, \alpha\}$  considered as a preorder category.
- For each ordinal  $\beta < \alpha$ , a (possibly empty) indexing set  $T_\beta$ ; and for each element  $j$  of  $T_\beta$ , a commutative diagram of the form below,

$$\begin{array}{ccc} U_{\beta,j} & \xrightarrow{u_{\beta,j}} & X_\beta \\ e_{\beta,j} \downarrow & & \downarrow X_{\beta \rightarrow \beta+1} \\ V_{\beta,j} & \xrightarrow{v_{\beta,j}} & X_{\beta+1} \end{array}$$

where  $e_{\beta,j} : U_{\beta,j} \rightarrow V_{\beta,j}$  is a morphism in  $\mathcal{I}$ .

These data are moreover required to satisfy the following condition:

- For each ordinal  $\beta < \gamma$ , the coproducts  $\coprod_{j \in T_\beta} S_{\beta,j}$  and  $\coprod_{j \in T_\beta} D_{\beta,j}$  exist in  $\mathcal{C}$ , and the induced diagram

$$\begin{array}{ccc} \coprod_{j \in T_\beta} U_{\beta,j} & \xrightarrow{u_\beta} & X_\beta \\ \coprod_{j \in T_\beta} e_{\beta,j} \downarrow & & \downarrow X_{\beta \rightarrow \beta+1} \\ \coprod_{j \in T_\beta} V_{\beta,j} & \xrightarrow{v_\beta} & X_{\beta+1} \end{array}$$

is a pushout square in  $\mathcal{C}$ .

The presentation is said to be **U-small** (resp.  $\kappa$ -**small** for a regular cardinal  $\kappa$ ) if  $\alpha$  is an ordinal in  $\mathbf{U}$  (resp.  $|\alpha| < \kappa$ ) and the disjoint union  $\coprod_{\beta < \alpha} T_\beta$  is in  $\mathbf{U}$  (resp. has cardinality less than  $\kappa$ ). A **sequential presentation** is one where each  $T_\beta$  is a singleton, in which case we suppress the index  $j$  in  $e_{\beta,j}$ ,  $u_{\beta,j}$ , and  $v_{\beta,j}$ .

A **relative  $\mathcal{I}$ -cell complex** in  $\mathcal{C}$  is a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  for which there exists a presentation as above with  $f$  equal to  $X_0 \rightarrow X_\alpha$ . Given an initial object  $0$  in  $\mathcal{C}$ , an  **$\mathcal{I}$ -cell complex** in  $\mathcal{C}$  is an object  $Y$  for which the unique morphism  $0 \rightarrow Y$  is a relative  $\mathcal{I}$ -cell complex.

**REMARK 0.5.2.** For any object  $X$  in  $\mathcal{C}$  and any subset  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$ , the morphism  $\text{id} : X \rightarrow X$  is a relative  $\mathcal{I}$ -cell complex in  $\mathcal{C}$  (with the obvious presentation indexed over  $0$ ). More generally, every isomorphism in  $\mathcal{C}$  is a relative  $\mathcal{I}$ -cell complex, with a presentation indexed over  $1$  (and  $T_0 = \emptyset$ ); but in order to get a *sequential* presentation, one must assume that there is an isomorphism in  $\mathcal{I}$ .

**Proposition 0.5.3.** *Let  $\mathcal{C}$  be a category, let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ , let  $\kappa$  be a regular cardinal, and let  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$  be the set of relative  $\mathcal{I}$ -cell complexes in  $\mathcal{C}$  that admit a  $\kappa$ -small presentation.*

- (i) *Every morphism in  $\mathcal{I}$  is also in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ .*
- (ii) *For each object  $X$  in  $\mathcal{C}$ , the morphism  $\text{id} : X \rightarrow X$  is in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ .*
- (iii) *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are both in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ , then so is  $g \circ f$ .*
- (iv) *Let  $\alpha$  be an ordinal and let  $X_\bullet : \alpha \rightarrow \mathcal{C}$  be a colimit-preserving functor. If  $|\alpha| < \kappa$  and  $\lambda$  is a colimiting cocone from  $X_\bullet$  to  $Y$  and, for  $\beta \leq \gamma < \alpha$ , the morphism  $X_{\beta \rightarrow \gamma} : X_\beta \rightarrow X_\gamma$  is in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ , then each component  $\lambda_\beta : X_\beta \rightarrow Y$  is also in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ .*
- (v) *Given a pushout diagram of the form below in  $\mathcal{C}$ ,*

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

*if  $g$  is in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$  and  $\mathcal{C}$  has colimits for all  $\kappa$ -small diagrams, then  $f$  is also in  $\text{cell}_{\mathcal{I},\kappa} \mathcal{C}$ .*

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*Proof.* (i). Given any morphism  $e : U \rightarrow V$  in  $\mathcal{I}$ , we have the following pushout diagram:

$$\begin{array}{ccc} U & \xrightarrow{\text{id}} & U \\ e \downarrow & & \downarrow e \\ V & \xrightarrow{\text{id}} & V \end{array}$$

Thus  $e : U \rightarrow V$  is in  $\text{cell}_{\mathcal{I}} \mathcal{C}$ .

(ii). See [remark 0.5.2](#).

(iii). It is clear that appending any  $\kappa$ -small presentation for  $g$  to any  $\kappa$ -small presentation for  $f$  yields a  $\kappa$ -small presentation of  $g \circ f$ .

(iv). The case  $\alpha = 0$  falls under claim (ii). If  $\alpha = \gamma + 1$ , then the component  $\lambda_{\gamma} : X_{\gamma} \rightarrow Y$  must be an isomorphism, and thus  $\lambda_{\beta} = \lambda_{\gamma} \circ X_{\beta \rightarrow \gamma}$  is also in  $\text{cell}_{\mathcal{I}} \mathcal{C}$ ; and if  $\alpha$  is a positive limit ordinal, since every terminal segment of  $\alpha$  is cofinal in  $\alpha$ , it is clear that concatenating  $\kappa$ -small presentations for  $X_{\gamma \rightarrow \gamma+1}$  for  $\beta \leq \gamma < \alpha$  yields a  $\kappa$ -small presentation for  $\lambda_{\beta} : X_{\beta} \rightarrow Y$ .

(v). Fix a  $\kappa$ -small presentation of  $g : Z \rightarrow W$ . By the pushout pasting lemma, given a commutative diagram of the form below,

$$\begin{array}{ccccc} \coprod_{j \in T_{\beta}} U_{\beta,j} & \xrightarrow{u_{\beta}} & Z_{\beta} & \longrightarrow & X_{\beta} \\ \coprod_{j \in T_{\beta}} e_{\beta,j} \downarrow & & \downarrow Z_{\beta \rightarrow \beta+1} & & \downarrow X_{\beta \rightarrow \beta+1} \\ \coprod_{j \in T_{\beta}} V_{\beta,j} & \xrightarrow{v_{\beta}} & Z_{\beta+1} & \longrightarrow & X_{\beta+1} \end{array}$$

if both squares are pushout diagrams, then the outer rectangle is a pushout diagram as well. Since pushout along  $z : Z \rightarrow X$  is the left adjoint of the evident functor  $z^* : X/\mathcal{C} \rightarrow Z/\mathcal{C}$ , it preserves all colimits, and thus we obtain a  $\kappa$ -small presentation of  $f : X \rightarrow Y$ . ■

**Definition 0.5.4.** Let  $\mathcal{C}$  be a category and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ . An  **$\mathcal{I}$ -injective morphism** in  $\mathcal{C}$  is a morphism that has the right lifting property with respect to every morphism in  $\mathcal{I}$ .<sup>[10]</sup> An  **$\mathcal{I}$ -cofibration** in  $\mathcal{C}$  is a morphism that has the left lifting property with respect to every  $\mathcal{I}$ -injective morphism.

[10] Equivalently, it is a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  that is an  $\mathcal{I}$ -injective object in the slice category  $\mathcal{C}_{/Y}$ .

**Proposition 0.5.5.** *Let  $\mathcal{C}$  be a category, let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ , and let  $\text{cell}_{\mathcal{I}} \mathcal{C}$ ,  $\text{inj}^{\mathcal{I}} \mathcal{C}$ , and  $\text{cof}_{\mathcal{I}} \mathcal{C}$  be the set of relative  $\mathcal{I}$ -cell complexes,  $\mathcal{I}$ -injections, and  $\mathcal{I}$ -cofibrations in  $\mathcal{C}$ , respectively.*

- (i) *We have  $\mathcal{I} \subseteq \text{cell}_{\mathcal{I}} \mathcal{C} \subseteq \text{cof}_{\mathcal{I}} \mathcal{C}$ .*
- (ii) *A morphism is in  $\text{inj}^{\mathcal{I}} \mathcal{C}$  if and only if it has the right lifting property with respect to every  $\mathcal{I}$ -cofibration.*
- (iii) *In particular, a morphism is in  $\text{inj}^{\mathcal{I}} \mathcal{C}$  if and only if it has the right lifting property with respect to every relative  $\mathcal{I}$ -cell complex.*

*Proof.* (i). Follows immediately from the definition of ‘relative  $\mathcal{I}$ -cell complex’ and [proposition A.3.14](#).

(ii) and (iii). See [proposition A.3.3](#). ■

Some authors define ‘relative  $\mathcal{I}$ -cell complex’ so that every such morphism admits a *sequential* presentation. The following lemma and its corollary show that there is no loss of generality in doing so.

**Lemma 0.5.6.** *Let  $\kappa$  be a regular cardinal, let  $\mathcal{C}$  be a category with colimits for all  $\kappa$ -small diagrams, and let  $\alpha$  be an ordinal of cardinality less than  $\kappa$ . For each ordinal  $\beta < \alpha$ , let  $e_{\beta} : U_{\beta} \rightarrow V_{\beta}$  be a morphism in  $\mathcal{C}$ , and for each ordinal  $\beta \leq \alpha$ , let*

$$C_{\beta} = \left( \coprod_{\gamma < \beta} V_{\gamma} \right) \amalg \left( \coprod_{\beta \leq \gamma < \alpha} U_{\gamma} \right)$$

*be a coproduct in  $\mathcal{C}$  with coproduct insertions  $u_{\gamma, \beta} : U_{\gamma} \rightarrow C_{\beta}$  (for  $\beta \leq \gamma < \alpha$ ) and  $v_{\gamma, \beta} : V_{\gamma} \rightarrow C_{\beta}$  (for  $\gamma < \beta$ ).*

*Given ordinals  $\beta < \beta' \leq \alpha$ , there is a unique morphism  $C_{\beta} \rightarrow C_{\beta'}$  such that, for  $\zeta < \beta \leq \zeta' < \beta' \leq \zeta''$ , the following diagrams commute:*

$$\begin{array}{ccc} V_{\zeta} & \xrightarrow{v_{\zeta, \beta}} & C_{\beta} & & U_{\zeta'} & \xrightarrow{u_{\zeta', \beta}} & C_{\beta} & & U_{\zeta''} & \xrightarrow{u_{\zeta'', \beta}} & C_{\beta} \\ \text{id} \downarrow & & \downarrow & & e_{\zeta'} \downarrow & & \downarrow & & \text{id} \downarrow & & \downarrow \\ V_{\zeta} & \xrightarrow{v_{\zeta, \beta'}} & C_{\beta'} & & V_{\zeta'} & \xrightarrow{v_{\zeta', \beta'}} & C_{\beta'} & & U_{\zeta''} & \xrightarrow{u_{\zeta'', \beta'}} & C_{\beta'} \end{array}$$

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This yields a functor  $C_\bullet : [\alpha] \rightarrow \mathcal{C}$ , and it preserves colimits. Moreover, the diagrams below are pushout squares for all ordinals  $\beta < \alpha$ :

$$\begin{array}{ccc} U_\beta & \xrightarrow{u_{\beta,\beta}} & C_\beta \\ e_\beta \downarrow & & \downarrow \\ V_\beta & \xrightarrow{v_{\beta,\beta+1}} & C_{\beta+1} \end{array}$$

*Proof.* This is a straightforward exercise. See Proposition 10.2.7 in [Hirschhorn, 2003].  $\square$

**Corollary 0.5.7.** *Let  $\kappa$  be a regular cardinal, let  $\mathcal{C}$  be a category with colimits for  $\kappa$ -small diagrams, and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ . If  $f : X \rightarrow Y$  is a relative  $\mathcal{I}$ -cell complex in  $\mathcal{C}$  that admits a  $\kappa$ -small presentation, and either*

- $X = Y$  and  $f = \text{id}_X$ , or
- $f$  is an isomorphism and  $\mathcal{I}$  contains an isomorphism, or
- $f$  is not an isomorphism,

*then  $f$  also admits a  $\kappa$ -small sequential presentation.*

*Proof.* We have already commented on the first two cases in [remark 0.5.2](#). The third case is proven by transfinite induction, where in the induction step we may assume that  $f$  is presented by just one pushout diagram:

$$\begin{array}{ccc} \coprod_{j \in T} U_j & \xrightarrow{u} & X \\ \prod_{j \in T} e_j \downarrow & & \downarrow f \\ \coprod_{j \in T} V_j & \xrightarrow{v} & Y \end{array}$$

By decomposing the morphism  $\prod_{j \in T} e_j : \prod_{j \in T} U_j \rightarrow \prod_{j \in T} V_j$  as in the earlier lemma and applying the pushout pasting lemma, we obtain a sequential presentation of  $f$ , which is  $\kappa$ -small precisely if  $|T| < \kappa$ .  $\blacksquare$

**Definition 0.5.8.** Let  $\mathbf{U}$  be a universe, let  $\mathcal{C}$  be a category, let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ , and let  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$  be the set of relative  $\mathcal{I}$ -cell complexes in  $\mathcal{C}$  that have a  $\mathbf{U}$ -small presentation. We say  $(\mathcal{I}, \mathcal{C})$  is **admissible for the  $\mathbf{U}$ -small object argument** when the following conditions are satisfied:

- $\mathcal{I}$  is a  $\mathbf{U}$ -set.
- $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category with colimits for all  $\mathbf{U}$ -small diagrams.
- There is a regular cardinal  $\kappa$  in  $\mathbf{U}$  such that, for every morphism  $e : U \rightarrow V$  in  $\mathcal{I}$ , every ordinal  $\alpha$  in  $\mathbf{U}$ , and every functor  $X_\bullet : \alpha \rightarrow \mathcal{C}$ , if  $|\alpha| \geq \kappa$ , and the morphism  $X_{\beta \rightarrow \gamma} : X_\beta \rightarrow X_\gamma$  is in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$  for all ordinals  $\beta \leq \gamma < \alpha$ , then the canonical comparison map  $\varinjlim_{\beta < \alpha} C(U, X_\beta) \rightarrow C\left(U, \varinjlim_{\beta < \alpha} X_\beta\right)$  is a bijection.

The **sequential  $\mathbf{U}$ -rank** of  $\mathcal{I}$  in  $\mathcal{C}$  is the least cardinal  $\kappa$  with the above property.

**REMARK 0.5.9.** Notice that, if  $|\alpha| \geq \kappa$ , then  $\alpha$  is a  $\kappa$ -directed preorder. Thus, for any locally presentable  $\mathbf{U}$ -category  $\mathcal{C}$  and any  $\mathbf{U}$ -subset  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$  whatsoever,  $(\mathcal{I}, \mathcal{C})$  is admissible for the  $\mathbf{U}$ -small object argument.

**Definition 0.5.10.** Let  $\mathbf{U}$  be a universe. A  **$\mathbf{U}$ -cofibrantly generated factorisation system** on a category  $\mathcal{C}$  on is a weak factorisation system on  $\mathcal{C}$  that is cofibrantly generated by some  $\mathbf{U}$ -subset of  $\text{mor } \mathcal{C}$ .

**Lemma 0.5.11.** *Let  $\mathcal{C}$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category, let  $A$  be a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , and let  $B$  be a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ . If the hom-set  $C(A, A')$  is  $\mu$ -small for all  $(\kappa, \mathbf{U})$ -compact objects  $A'$  in  $\mathcal{C}$  and  $\kappa \triangleleft \lambda$ , then the hom-set  $C(A, B)$  has cardinality  $< \max\{\lambda, \mu\}$ .*

*Proof.* By [proposition 0.2.44](#), there is a  $\lambda$ -small  $\kappa$ -filtered diagram  $Y : \mathcal{J} \rightarrow \mathcal{C}$  with each vertex  $(\kappa, \mathbf{U})$ -compact in  $\mathcal{C}$  and  $B \cong \varinjlim_{\mathcal{J}} Y$ . Since  $A$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , we have

$$C(A, B) \cong \varinjlim_{\mathcal{J}} C(A, Y)$$

and the RHS is a set of cardinality  $< \max\{\lambda, \mu\}$  by [lemma 0.2.17](#). ■

**Theorem 0.5.12** (Quillen's small object argument). *Let  $\mathbf{U}$  be a universe, let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category with colimits for all  $\mathbf{U}$ -small diagrams, and let  $\mathcal{I}$  be a  $\mathbf{U}$ -subset of  $\text{mor } \mathcal{C}$ .*

- (i) *There exist a functor  $M : [2, \mathcal{C}] \rightarrow \mathcal{C}$  and two natural transformations  $i : \text{dom} \Rightarrow M$ ,  $p : M \Rightarrow \text{codom}$  such that, for all morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$ , the morphism  $i_f : X \rightarrow M(f)$  is in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$ , and we have  $f = p_f \circ i_f$ .*

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- (ii) If  $(\mathcal{I}, \mathcal{C})$  is moreover admissible for the  $\mathbf{U}$ -small object argument, then we may choose  $M$ ,  $i$ , and  $p$  so that, for all morphisms  $f : X \rightarrow Y$  in  $\mathcal{C}$ , the morphism  $p_f : M(f) \rightarrow Y$  in  $\text{inj}^{\mathcal{I}} \mathcal{C}$ .
- (iii) In particular, if  $(\mathcal{I}, \mathcal{C})$  is admissible for the  $\mathbf{U}$ -small object argument, then  $(\text{cof}_{\mathcal{I}} \mathcal{C}, \text{inj}^{\mathcal{I}} \mathcal{C})$  is a  $\mathbf{U}$ -cofibrantly generated factorisation system on  $\mathcal{C}$  and extends to a functorial weak factorisation system.

*Proof.* (i). Let  $\kappa$  be any regular cardinal, and let  $\alpha$  be the least ordinal of cardinality  $\kappa$ .<sup>[11]</sup> For each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , we construct by transfinite recursion a colimit-preserving functor  $M_{\bullet}(f) : [\alpha] \rightarrow \mathcal{C}$  and a cocone  $p_{f;\bullet} : M_{\bullet}(f) \rightarrow Y$  satisfying the following conditions:

- $M_0(f) = X$ ,  $p_{f;0} = p$ .
- For each ordinal  $\beta < \alpha$ , if  $T_{\beta}(f)$  is the set of all commutative diagrams in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} U_{\beta,j} & \xrightarrow{u_{\beta,j}} & M_{\beta}(f) \\ e_{\beta,j} \downarrow & & \downarrow p_{f;\beta} \\ V_{\beta,j} & \xrightarrow{v_{\beta,j}} & Y \end{array}$$

where  $e_{\beta,j} : U_{\beta,j} \rightarrow V_{\beta,j}$  is in  $\mathcal{I}$ , then  $T_{\beta}(f)$  is a  $\mathbf{U}$ -set (because  $\mathcal{I}$  is a  $\mathbf{U}$ -set and  $\mathcal{C}$  is a locally  $\mathbf{U}$ -small category), and we have a pushout square of the following form,

$$\begin{array}{ccc} \coprod_{j \in T_{\beta}(f)} U_{\beta,j} & \xrightarrow{u_{\beta}} & M_{\beta}(f) \\ \coprod_{j \in T_{\beta}(f)} e_{\beta,j} \downarrow & & \downarrow p_{f;\beta+1} \\ \coprod_{j \in T_{\beta}(f)} V_{\beta,j} & \xrightarrow{\bar{v}_{\beta}} & M_{\beta+1}(f) \end{array}$$

where  $u_{\beta} : \coprod_{j \in T_{\beta}(f)} U_{\beta,j} \rightarrow M_{\beta}(f)$  is the evident morphism induced by the universal property of coproducts. Observe that there is then a unique morphism  $p_{f;\beta+1} : M_{\beta+1}(f) \rightarrow Y$  such that

$$p_{f;\beta+1} \circ M_{\beta \rightarrow \beta+1}(f) = p_{\beta}$$

and

$$p_{f;\beta+1} \circ \bar{v}_{\beta,j} = v_{\beta,j}$$

for all  $j$  in  $T_{\beta}(f)$ , where  $\bar{v}_{\beta,j} : V_{\beta,j} \rightarrow M_{\beta+1}(f)$  is the evident component of  $\bar{v}_{\beta} : \coprod_{j \in T_{\beta}(f)} V_{\beta,j} \rightarrow M_{\beta+1}(f)$ .

[11] In particular, we could take  $\kappa = 0$ , but then the factorisation so obtained is trivial.



- For limit ordinals  $\gamma \leq \alpha$ ,  $M_\gamma(f) = \varinjlim_{\beta < \gamma} M_\beta(f)$ , and  $p_\gamma : M_\gamma(f) \rightarrow Y$  is defined by the universal property of  $X_\gamma$ .

It is not hard to see that the functor  $M_\bullet(f) : [\alpha] \rightarrow \mathcal{C}$  so defined is itself functorial in  $f$ ; in particular, defining  $M(f) = M_\alpha(f)$ ,  $i_f = M_{0 \rightarrow \alpha}(f)$ ,  $p_f = p_{f;\alpha}$ , we obtain a functor  $M : [2, \mathcal{C}] \rightarrow \mathcal{C}$  with two natural transformations  $i : M \Rightarrow \text{dom}$  and  $p : M \Rightarrow \text{codom}$ ; by construction, we have  $f = p_f \circ i_f$ , and  $i_f : X \rightarrow M(f)$  is in  $\text{cell}_{\mathcal{I}, U} \mathcal{C}$ .

(ii). Now, take  $\kappa$  to be a regular cardinal as in [definition 0.5.8](#). We wish to show that the morphism  $p_f$  constructed above has the right lifting property with respect to all morphisms in  $\mathcal{I}$ . Consider a lifting problem of the form below,

$$\begin{array}{ccc} U & \xrightarrow{u} & M(f) \\ e \downarrow & & \downarrow p_f \\ V & \xrightarrow{v} & Y \end{array}$$

where  $e : U \rightarrow V$  is in  $\mathcal{I}$ . Since  $\mathcal{I}$  is admissible, there must exist an ordinal  $\beta < \alpha$  and a morphism  $u' : U \rightarrow M_\beta(f)$  such that  $u = M_{\beta \rightarrow \alpha}(f) \circ u'$ . We then obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{u'} & M_\beta(f) \\ e \downarrow & & \downarrow p_{f;\beta} \\ V & \xrightarrow{v} & Y \end{array}$$

Since this is one of the diagrams in the set  $T_\beta(f)$ , it must embed in a commutative diagram of the form below,

$$\begin{array}{ccccc} U & \xrightarrow{u'} & M_\beta(f) & \longrightarrow & M_\alpha(f) \\ e \downarrow & & \downarrow & \nearrow & \downarrow p_{f;\alpha} \\ V & \longrightarrow & M_{\beta+1}(f) & & \\ \parallel & & & & \\ V & \xrightarrow{v} & & \longrightarrow & Y \end{array}$$

and thus we have the required lift  $V \rightarrow M(f)$ .

(iii). Finally, apply [proposition 0.5.5](#) and [theorem A.3.3 I](#). ■

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**Corollary 0.5.13.** *With other notation in the theorem, a morphism  $g : Z \rightarrow W$  is in  $\text{cof}_{\mathcal{I}} \mathcal{C}$  if and only if there exists a commutative diagram of the following form in  $\mathcal{C}$ ,*

$$\begin{array}{ccc} Z & \xrightarrow{i} & W' \\ g \downarrow & \nearrow & \downarrow \\ W & \xrightarrow{\text{id}} & W \end{array}$$

where  $i : Z \rightarrow W'$  is in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$ .

*Proof.* (i). If  $g : Z \rightarrow W$  is in  $\text{cof}_{\mathcal{I}} \mathcal{C}$ , then  $g$  has the left lifting property with respect to  $p_g : M(g) \rightarrow W$ , and so there exists a commutative diagram of the required form. Conversely, suppose we have  $g = p \circ i$ ,  $i = j \circ g$ , and  $\text{id}_W = p \circ j$  for some  $i : Z \rightarrow W'$  in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$  and some  $j : W \rightarrow W'$  in  $\mathcal{C}$ . Then  $g$  is a retract of  $i$ ,

$$\begin{array}{ccccc} Z & \xrightarrow{\text{id}} & Z & \xrightarrow{\text{id}} & Z \\ g \downarrow & & i \downarrow & & \downarrow g \\ W & \xrightarrow{j} & W' & \xrightarrow{p} & W \\ & \searrow & \text{id} & \nearrow & \\ & & & & \end{array}$$

but [proposition 0.5.5](#) says  $i$  is in  $\text{cof}_{\mathcal{I}} \mathcal{C}$ , so by [proposition A.3.14](#),  $g$  is also in  $\text{cof}_{\mathcal{I}} \mathcal{C}$ . ■

**Corollary 0.5.14.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathcal{C}$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, and let  $\mathcal{I}$  be a  $\mathbf{U}$ -small subset of  $\text{mor } \mathcal{C}$ . If the morphisms that are in  $\mathcal{I}$  are  $(\kappa, \mathbf{U})$ -compact as objects in  $[\mathbf{2}, \mathcal{C}]$ , then there exist a  $(\kappa, \mathbf{U})$ -accessible functor  $M : [\mathbf{2}, \mathcal{C}] \rightarrow \mathcal{C}$  and two natural transformations  $i : \text{dom} \Rightarrow M$  and  $p : M \Rightarrow \text{codom}$  such that, for all objects  $f$  in  $[\mathbf{2}, \mathcal{C}]$ :*

- $f = p_f \circ i_f$ .
- $i_f$  is in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$ .
- $p_f$  is in  $\text{inj}^{\mathcal{I}} \mathcal{C}$ .

Moreover, if  $\lambda$  is a regular cardinal in  $\mathbf{U}$  such that every hom-set of  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$  is  $\lambda$ -small,  $\mathcal{I}$  is  $\lambda$ -small, and  $\kappa \triangleleft \lambda$ , then  $M : [\mathbf{2}, \mathcal{C}] \rightarrow \mathcal{C}$  is also strongly  $(\lambda, \mathbf{U})$ -accessible.

*Proof.* As observed in [remark 0.5.9](#), under these hypotheses,  $(\mathcal{I}, C)$  is admissible for the  $\mathbf{U}$ -small object argument and the sequential  $\mathbf{U}$ -rank of  $\mathcal{I}$  is  $\leq \kappa$ . By tracing the construction of the functor  $M$  in [theorem 0.5.12](#), we see that  $M$  preserves colimits for  $\kappa$ -filtered  $\mathbf{U}$ -small diagrams, so we are done. Similarly, applying [proposition 0.2.46](#) and lemmas [0.2.17](#) and [0.5.11](#) shows that  $M$  is strongly  $(\lambda, \mathbf{U})$ -accessible.  $\blacksquare$

**Corollary 0.5.15.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $C$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, and let  $\mathcal{I}$  be a  $\mathbf{U}$ -small subset of  $\text{mor } C$ . If the morphisms that are in  $\mathcal{I}$  are  $(\kappa, \mathbf{U})$ -compact as objects in  $[2, C]$ , then there exists a  $(\kappa, \mathbf{U})$ -accessible functor  $L : [2, C] \rightarrow [2, C]$  such that  $\text{cof}_{\mathcal{I}} C$  is the closure of the full subcategory of  $[2, C]$  spanned by the image of  $L$  under the splitting of idempotent endomorphisms.*

*Proof.* Take  $L$  to be the functor that sends a morphism in  $C$  (considered as an object in  $[2, C]$ ) to the left half of its  $(\text{cell}_{L, \kappa} C, \text{inj}^{\mathcal{I}} C)$ -factorisation, and then apply [theorem A.3.31](#).  $\blacksquare$

**Lemma 0.5.16.** *Let  $C$  be a full subcategory of a category  $C^+$ , let  $\mathcal{I}$  be a subset of  $\text{mor } C$ , and let  $\kappa$  be a regular cardinal. If  $C$  is closed in  $C^+$  under colimits for all  $\kappa$ -small diagrams, then  $\text{cell}_{L, \kappa} C = \text{cell}_{L, \kappa} C^+ \cap \text{mor } C$ .*

*Proof.* Obvious.  $\blacklozenge$

**Theorem 0.5.17** (Stability of cofibrantly generated factorisation systems). *Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \in \mathbf{U}^+$ . Suppose:*

- $C$  is a locally  $\mathbf{U}$ -small and  $\mathbf{U}$ -cocomplete category.
- $C^+$  is a locally  $\mathbf{U}^+$ -small and  $\mathbf{U}^+$ -cocomplete category.
- The inclusion  $C \hookrightarrow C^+$  preserves colimits for all  $\mathbf{U}$ -small diagrams.
- $\mathcal{I}$  is a  $\mathbf{U}$ -subset of  $\text{mor } C$ .
- $(\mathcal{I}, C)$  is admissible for the  $\mathbf{U}$ -small object argument, and  $(L, R)$  is the functorial factorisation system on  $C$  constructed by Quillen's small object argument argument.
- $(\mathcal{I}, C^+)$  is admissible for the  $\mathbf{U}^+$ -small object argument, and  $(L^+, R^+)$  is the functorial factorisation system on  $C^+$  constructed by Quillen's small object argument argument.

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Under these hypotheses, if the sequential  $\mathbf{U}$ -rank of  $\mathcal{I}$  in  $\mathcal{C}$  is equal to the sequential  $\mathbf{U}^+$ -rank of  $\mathcal{I}$  in  $\mathcal{C}^+$ , then:

- (i) For each morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ , we have a commutative diagram of the following form in  $\mathcal{C}^+$ ,

$$\begin{array}{ccc}
 & M^+(f) & \\
 L^+f \nearrow & \downarrow \cong & \searrow R^+f \\
 X & & Y \\
 Lf \searrow & & \nearrow Rf \\
 & M(f) & 
 \end{array}$$

and the isomorphism  $M^+(f) \rightarrow M(f)$  is moreover canonical and natural in  $f$ .

- (ii) We have  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C} \subseteq \text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}^+ \subseteq \text{cell}_{\mathcal{I}, \mathbf{U}^+} \mathcal{C}^+$ .
- (iii)  $(\text{cof}_{\mathcal{I}} \mathcal{C}^+, \text{inj}^{\mathcal{I}} \mathcal{C}^+)$  is an extension of  $(\text{cof}_{\mathcal{I}} \mathcal{C}, \text{inj}^{\mathcal{I}} \mathcal{C})$ .

*Proof.* (i). This can be seen by examining the explicit construction in the proof of [theorem 0.5.12](#).

(ii). This is implied by the lemma.

(iii). Since  $(\text{cof}_{\mathcal{I}} \mathcal{C}, \text{inj}^{\mathcal{I}} \mathcal{C})$  and  $(\text{cof}_{\mathcal{I}} \mathcal{C}^+, \text{inj}^{\mathcal{I}} \mathcal{C}^+)$  are both cofibrantly generated by  $\mathcal{I}$ , by [proposition A.3.21](#), we have  $\text{inj}^{\mathcal{I}} \mathcal{C} \subseteq \text{inj}^{\mathcal{I}} \mathcal{C}^+$  and so  $\text{cof}_{\mathcal{I}} \mathcal{C} \supseteq \text{cof}_{\mathcal{I}} \mathcal{C}^+ \cap \text{mor } \mathcal{C}$ . It remains to be shown that  $\text{cof}_{\mathcal{I}} \mathcal{C} \subseteq \text{cof}_{\mathcal{I}} \mathcal{C}^+$ , but this is implied by [corollary 0.5.13](#) applied to claim (ii).  $\blacksquare$

**REMARK 0.5.18.** Let  $\kappa$  be a regular cardinal in  $\mathbf{U}$ , let  $\mathcal{B}$  be a  $\mathbf{U}$ -small category with colimits for all  $\kappa$ -small diagrams, let  $\mathcal{C} = \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{B})$ , and let  $\mathcal{C}^+ = \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathcal{B})$ . Then  $\mathcal{C}$  is a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, the inclusion  $\mathcal{C} \hookrightarrow \mathcal{C}^+$  is an accessible  $(\kappa, \mathbf{U}, \mathbf{U}^+)$  extension, and any  $\mathbf{U}$ -subset  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$  whatsoever will satisfy the hypotheses of the theorem.

**Proposition 0.5.19.** Let  $F \dashv U : \mathcal{D} \rightarrow \mathcal{C}$  be an adjunction of categories, let  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$ , and let  $\mathcal{J} = \{Ff \mid f \in \mathcal{I}\}$ .

- (i)  $F$  sends relative  $\mathcal{I}$ -cell complexes in  $\mathcal{C}$  to relative  $\mathcal{J}$ -cell complexes in  $\mathcal{D}$ .
- (ii)  $U$  sends  $\mathcal{J}$ -injective morphisms in  $\mathcal{D}$  to  $\mathcal{I}$ -injective morphisms in  $\mathcal{C}$ .

(iii)  $F$  sends  $\mathcal{I}$ -cofibrations in  $\mathcal{C}$  to  $\mathcal{J}$ -cofibrations in  $\mathcal{D}$ .

*Proof.* (i). This is a corollary of the fact that  $F$  preserves all colimits.

(ii). As in the proof of [proposition A.3.22](#), a morphism  $f : X \rightarrow Y$  in  $\mathcal{D}$  has the right lifting property with respect to all morphisms in  $\mathcal{J}$  if and only if  $Uf : UX \rightarrow UY$  has the right lifting property with respect to all morphisms in  $\mathcal{I}$ .

(iii). Similarly, a morphism  $g : Z \rightarrow W$  in  $\mathcal{C}$  has the left lifting property with respect to all morphisms of the form  $Uf : UX \rightarrow UY$  where  $f : X \rightarrow Y$  is a  $\mathcal{J}$ -injective morphism  $f : X \rightarrow Y$  in  $\mathcal{D}$  if and only if  $Fg : FZ \rightarrow FW$  is a  $\mathcal{J}$ -cofibration in  $\mathcal{D}$ ; but we know that  $U$  sends  $\mathcal{J}$ -injective morphisms in  $\mathcal{D}$  to  $\mathcal{I}$ -injective morphisms in  $\mathcal{C}$ , so  $F$  must send  $\mathcal{I}$ -cofibrations in  $\mathcal{C}$  to  $\mathcal{J}$ -cofibrations in  $\mathcal{D}$ . ■

**Proposition 0.5.20.** *Let  $\mathbf{U}$  be a universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, let  $\mathbb{B}$  be a  $\mathbf{U}$ -small category, let  $\mathcal{C} = [\mathbb{B}^{\text{op}}, \mathbf{Set}]$ , and let  $\mathcal{I}$  be the subset of  $\text{mor } \mathcal{C}$  consisting of all monomorphisms  $e : U \rightarrow V$  in  $\mathcal{C}$  where  $V$  is a quotient of a representable presheaf.*

- (i)  $(\text{cof}_{\mathcal{I}} \mathcal{C}, \text{inj}^{\mathcal{I}} \mathcal{C})$  is a  $\mathbf{U}$ -cofibrantly generated weak factorisation system.
- (ii)  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$  is precisely the class of all monomorphisms in  $\mathcal{C}$ .
- (iii)  $\text{cof}_{\mathcal{I}} \mathcal{C} = \text{cell}_{\mathcal{I}} \mathcal{C}$ .

*Proof.* (i). Since  $\mathbb{B}$  is small and  $\mathcal{C}$  is well-powered and well-copowered, the full subcategory of  $[\mathbb{2}, \mathcal{C}]$  spanned by  $\mathcal{I}$  is essentially  $\mathbf{U}$ -small. We know that  $\mathcal{C}$  is locally finitely presentable, thus, taking a  $\mathbf{U}$ -set of representatives of the isomorphism classes in  $\mathcal{I}$ , and recalling [remark 0.5.9](#), Quillen's small object argument ([theorem 0.5.12](#)) implies  $(\text{cof}_{\mathcal{I}} \mathcal{C}, \text{inj}^{\mathcal{I}} \mathcal{C})$  is indeed a  $\mathbf{U}$ -cofibrantly generated weak factorisation system.

(ii). It is clear that the class of injective maps is closed under pushout and transfinite composition in  $\mathbf{Set}$ , so the same must be true of monomorphisms in  $\mathcal{C}$ , since colimits in  $\mathcal{C}$  are computed componentwise. Thus every morphism in  $\text{cell}_{\mathcal{I}} \mathcal{C}$  is a monomorphism.

Conversely, suppose  $f : X \rightarrow Y$  is a monomorphism. Fix an ordinal  $\alpha$  and a bijection  $y_{\bullet} : \alpha \rightarrow \coprod_{B \in \text{ob } \mathbb{B}} Y(B)$ , and write  $B_{\beta}$  for the object in  $\mathbb{B}$  such that  $y_{\beta} \in Y(B_{\beta})$ . We will construct a  $\mathbf{U}$ -small presentation for  $f$  by transfinite recursion on  $\alpha$ .

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- To begin, put  $X_0 = X$  and  $f_0 = f$ .
- For each ordinal  $\beta < \alpha$ , the Yoneda lemma implies there is a unique morphism  $a_\beta : \hat{h}_{B_\beta} \rightarrow Y$  in  $\mathcal{C}$  such that  $a_\beta(\text{id}_{B_\beta}) = y_\beta$ ; let  $\bar{v}_\beta : V_\beta \rightarrow Y$  be the image of  $a_\beta$ , and let  $e_\beta : U_\beta \rightarrow V_\beta$  and  $u_\beta : U_\beta \rightarrow X_\beta$  be defined by the pullback square shown below:

$$\begin{array}{ccc} U_\beta & \xrightarrow{u_\beta} & X_\beta \\ e_\beta \downarrow & & \downarrow f_\beta \\ V_\beta & \xrightarrow{\bar{v}_\beta} & Y \end{array}$$

Since  $f_\beta$  is a monomorphism,  $e_\beta$  must also be a monomorphism and hence is in  $\mathcal{I}$ . There is then a commutative diagram in  $\mathcal{C}$  of the following form,

$$\begin{array}{ccccc} U_\beta & \xrightarrow{u_\beta} & X_\beta & & \\ e_\beta \downarrow & & \downarrow & \searrow f_\beta & \\ V_\beta & \xrightarrow{v_\beta} & X_{\beta+1} & & \\ & \searrow \bar{v}_\beta & & \searrow f_{\beta+1} & \\ & & & & Y \end{array}$$

where  $f_{\beta+1} : X_{\beta+1} \rightarrow Y$  is the union of  $f_\beta : X_\beta \rightarrow Y$  and  $\bar{v}_\beta : V_\beta \rightarrow Y$  considered as subobjects of  $Y$ ; note that the inner square of the diagram is then a pushout square.

- Finally, for limit ordinals  $\gamma < \alpha$ , we take  $f_\gamma : X_\gamma \rightarrow Y$  to be the union  $\bigcup_{\beta < \gamma} f_\beta$ .

This completes the presentation of  $f : X \rightarrow Y$  as a relative  $\mathcal{I}$ -cell complex in  $\mathcal{C}$ , and it is clearly  $\mathbf{U}$ -small.

(iii). [Corollary 0.5.13](#) implies that each morphism in  $\text{cof}_{\mathcal{I}} \mathcal{C}$  is a retract of some morphism in  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$ , but the class of monomorphisms is closed under retracts, so in this case we must have  $\text{cof}_{\mathcal{I}} \mathcal{C} = \text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C}$ . Since  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C} \subseteq \text{cell}_{\mathcal{I}} \mathcal{C} \subseteq \text{cof}_{\mathcal{I}} \mathcal{C}$ , we also deduce that  $\text{cell}_{\mathcal{I}, \mathbf{U}} \mathcal{C} = \text{cell}_{\mathcal{I}} \mathcal{C}$ .  $\blacksquare$

We now turn our attention to Garner's small object argument.

**Lemma 0.5.21.** *Let  $\kappa$  be a regular cardinal in a universe  $\mathbf{U}$ , let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category, let  $F : \mathcal{A} \rightarrow \mathcal{C}$  be a functor, and let  $G : \mathcal{C} \rightarrow \mathcal{C}$  be (the functor part of) a pointwise left Kan extension of  $F$  along itself. If each  $FA$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , then:*

- (i)  $G : \mathcal{C} \rightarrow \mathcal{C}$  preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.
- (ii) *In addition, if  $\mathcal{C}$  is a  $\kappa$ -accessible  $\mathbf{U}$ -category,  $\lambda$  is a regular cardinal in  $\mathbf{U}$  such that every hom-set of  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$  is  $\lambda$ -small,  $\mathcal{A}$  is a  $\lambda$ -small category, and  $\kappa \triangleleft \lambda$ , then  $G : \mathcal{C} \rightarrow \mathcal{C}$  is strongly  $(\lambda, \mathbf{U})$ -accessible.*

*Proof.* (i). [Theorem A.5.15](#) says there is a natural bijection of the form below:

$$\mathcal{C}(GX, C) \cong [\mathcal{A}^{\text{op}}, \mathbf{Set}](\mathcal{C}(F-, X), \mathcal{C}(F-, C))$$

Since colimits are computed componentwise in  $[\mathcal{A}^{\text{op}}, \mathbf{Set}]$ , the hypothesis implies  $\mathcal{C}(F, -) : \mathcal{C} \rightarrow [\mathcal{A}^{\text{op}}, \mathbf{Set}]$  preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams. By the Yoneda lemma, the functors  $\mathcal{C}(-, C) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  jointly reflect limits, so it follows that  $G : \mathcal{C} \rightarrow \mathcal{C}$  preserves colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams.

(ii). Now suppose  $X$  is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ . [Lemma 0.5.11](#) then says each hom-set  $\mathcal{C}(FA, X)$  is  $\lambda$ -small, and since  $\mathcal{A}$  is a  $\lambda$ -small category, this shows that the comma category  $(F \downarrow X)$  is also  $\lambda$ -small. Thus,  $GX$  is a colimit for a  $\lambda$ -small diagram of  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{C}$ , and so we may use [lemma 0.2.17](#) to deduce that it is a  $(\lambda, \mathbf{U})$ -compact object in  $\mathcal{C}$ . ■

**Proposition 0.5.22.** *Let  $\mathcal{C}$  be a category with pushouts and let  $U : \mathcal{I} \rightarrow [2, \mathcal{C}]$  be a functor. Suppose a pointwise left Kan extension of  $U$  along itself exists.*

- (i)  $\mathbf{RLP}(U)$  is isomorphic as a category over  $[2, \mathcal{C}]$  to the category of algebras for a pointed endofunctor  $(J, \iota)$  on  $[2, \mathcal{C}]$ .
- (ii) *Moreover, if (the functor part of) the pointwise left Kan extension of  $U$  along itself is a  $(\kappa, \mathbf{U})$ -accessible functor (resp. strongly  $(\kappa, \mathbf{U})$ -accessible functor), then so is  $J$ .*

*Proof.* Let  $G : [2, \mathcal{C}] \rightarrow [2, \mathcal{C}]$  be (the functor part of) a pointwise left Kan extension of  $U$  along itself and let  $\alpha : U \Rightarrow GU$  be the unit. Then there is a unique natural transformation  $\varepsilon : G \Rightarrow \text{id}_{[2, \mathcal{C}]}$  such that  $\varepsilon U \bullet \alpha = \text{id}_{[2, \mathcal{C}]}$ . Let

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$f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . By [theorem A.5.15](#), there is a natural bijection of the form below:

$$[2, \mathcal{C}](Gf, g) \cong [\mathcal{I}^{\text{op}}, \mathbf{Set}]([2, \mathcal{C}](U-, f), [2, \mathcal{C}](U-, g))$$

It is not hard to see that a coherent choice  $\Phi$  of right liftings for  $f$  with respect to  $U : \mathcal{I} \rightarrow [2, \mathcal{C}]$  is the same thing as a natural transformation  $[2, \mathcal{C}](U-, f) \Rightarrow [2, \mathcal{C}](U-, \text{id}_X)$  making the following diagram commute for all objects  $e$  in  $\mathcal{I}$ ,

$$\begin{array}{ccc} & & [2, \mathcal{C}](Ue, \text{id}_X) \\ & \nearrow \text{dashed} & \downarrow \\ [2, \mathcal{C}](Ue, f) & \xrightarrow{\text{id}} & [2, \mathcal{C}](Ue, f) \end{array}$$

where the map  $[2, \mathcal{C}](Ue, \text{id}_X) \rightarrow [2, \mathcal{C}](Ue, f)$  is the one induced by the morphism  $(\text{id}_X, f) : \text{id}_X \rightarrow f$  in  $[2, \mathcal{C}]$ . We may therefore identify choices  $\Phi$  with morphisms  $l : d_0(Gf) \rightarrow X$  in  $\mathcal{C}$  making the diagram below commute:

$$(*) \quad \begin{array}{ccccc} \bullet & \xrightarrow{d_1(\varepsilon_f)} & X & \xrightarrow{\text{id}} & X \\ Gf \downarrow & & \downarrow \text{id} & & \downarrow f \\ \bullet & \xrightarrow{l} & X & \xrightarrow{f} & Y \\ & \searrow & \text{dashed} & \nearrow & \\ & & d_0(\varepsilon_f) & & \end{array}$$

Now, define functors  $J, K : [2, \mathcal{C}] \rightarrow [2, \mathcal{C}]$  so the square in the following diagram is a natural pushout square in  $\mathcal{C}$ :

$$\begin{array}{ccc} \bullet & \xrightarrow{d_1(\varepsilon_f)} & X \\ Gf \downarrow & & \downarrow Kf \\ \bullet & \longrightarrow & Mf \\ & \searrow & \downarrow Jf \\ & & Y \end{array} \quad \begin{array}{c} \nearrow f \\ \searrow Jf \\ \nearrow d_0(\varepsilon_f) \end{array}$$

We then have a natural transformation  $\iota : \text{id}_{[2, \mathcal{C}]} \Rightarrow J$  where  $\iota_f = (\text{id}_X, Kf)$ , and the universal property of pushouts yields a natural bijection between morphisms  $l : d_0(Gf) \rightarrow X$  making the diagram  $(*)$  commute and morphisms  $\tilde{l} : Mf \rightarrow X$  such that  $\tilde{l} \circ Kf = \text{id}_X$  and  $Jf = f \circ \tilde{l}$ , i.e. coalgebra structures on  $f$  for the



pointed endofunctor  $(J, \iota)$ . The naturality of these identifications then ensures that  $\mathbf{RLP}(U)$  is indeed isomorphic to  $[2, C]^{(J, \iota)}$  as categories over  $[2, C]$ . This proves claim (i).

For claim (ii), simply observe that pushouts preserve all colimits, so  $J : [2, C] \rightarrow [2, C]$  is  $(\kappa, \mathbf{U})$ -accessible if  $G : [2, C] \rightarrow [2, C]$  is, and lemmas 0.2.17 and 0.3.21 imply  $J$  is strongly  $(\kappa, \mathbf{U})$ -accessible if  $G$  is. ■

**Proposition 0.5.23.** *Let  $C$  be a locally  $\kappa$ -presentable  $\mathbf{U}$ -category, let  $\mathcal{I}$  be a  $\mathbf{U}$ -small category, and let  $U : \mathcal{I} \rightarrow [2, C]$  be a functor. If each  $Ue$  is a  $(\kappa, \mathbf{U})$ -compact object in  $[2, C]$ , then:*

- (i) *The forgetful functor  $\mathbf{RLP}(U) \rightarrow [2, C]$  is  $(\kappa, \mathbf{U})$ -accessible and monadic.*
- (ii) *In addition, if  $\lambda$  is a regular cardinal in  $\mathbf{U}$  such that each hom-set in  $\mathbf{K}_\kappa^{\mathbf{U}}(C)$  is  $\lambda$ -small,  $\mathcal{I}$  is a  $\lambda$ -small category, and  $\kappa \triangleleft \lambda$ , then the forgetful functor  $\mathbf{RLP}(U) \rightarrow [2, C]$  is strongly  $(\lambda, \mathbf{U})$ -accessible.*

*Proof.* Use theorems 0.3.40 and 0.3.41, lemma 0.5.21, and proposition 0.5.22. ■

**Theorem 0.5.24** (Garner's small object argument). *Let  $C$  be a locally presentable  $\mathbf{U}$ -category, let  $\mathcal{I}$  be a  $\mathbf{U}$ -small category, and let  $U : \mathcal{I} \rightarrow [2, C]$  be a functor:*

- (i) *There exists a free algebraic factorisation system  $(\mathbf{L}, \mathbf{R})$  on  $C$  cofibrantly generated by  $U : \mathcal{I} \rightarrow [2, C]$ .*
- (ii)  *$(\mathbf{L}, \mathbf{R})$  is (part of) an algebraically free natural weak factorisation system on  $C$  cofibrantly generated by  $U : \mathcal{I} \rightarrow [2, C]$ .*
- (iii) *In particular, if  $\mathcal{I}$  is discrete, then there exists a functorial weak factorisation system on  $C$  cofibrantly generated by the image of  $\text{ob } \mathcal{I} \rightarrow \text{mor } C$ .*

*Proof.* (i). See Theorem 4.4 in [Garner, 2009].

(ii). See Theorem 5.4 in [Garner, 2009].

(iii). This is proposition A.3.45. □

**Lemma 0.5.25.** *Let  $\mathcal{C}$  be a category and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ . If  $\kappa$  is a regular cardinal in a universe  $\mathbf{U}$  such that the domains of morphisms in  $\mathcal{I}$  are  $(\kappa, \mathbf{U})$ -compact in  $\mathcal{C}$ , then the class of  $\mathcal{I}$ -injective objects in  $\mathcal{C}$  is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $\mathcal{C}$ .*

*Proof.* Let  $\mathbb{D}$  be a  $\mathbf{U}$ -small  $\kappa$ -filtered category and let  $X : \mathbb{D} \rightarrow \mathcal{C}$  be a diagram such that each  $Xd$  is an  $\mathcal{I}$ -injective object in  $\mathcal{C}$ . Suppose  $\bar{X}$  is a colimit for  $X$  in  $\mathcal{C}$  with colimiting cocone  $\lambda : X \Rightarrow \Delta \bar{X}$ . Let  $g : Z \rightarrow W$  be in  $\mathcal{I}$ , and consider the induced hom-set map  $g^* : \mathcal{C}(W, \bar{X}) \rightarrow \mathcal{C}(Z, \bar{X})$ ; we must show that it is surjective. Since  $Z$  is a  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{C}$ , the canonical comparison  $\lim_{\rightarrow \mathbb{D}} \mathcal{C}(Z, X) \rightarrow \mathcal{C}(Z, \bar{X})$  is a bijection, and so every morphism  $Z \rightarrow \bar{X}$  factors through  $\lambda_d : Xd \rightarrow \bar{X}$  for some  $d$  in  $\mathbb{D}$ . By hypothesis  $Xd$  is  $\mathcal{I}$ -injective, so we obtain an extension of  $Z \rightarrow Xd$  along  $g : Z \rightarrow W$ , and hence, an extension of  $Z \rightarrow \bar{X}$  along  $g$ . Thus  $X$  is also  $\mathcal{I}$ -injective.  $\blacksquare$

**Lemma 0.5.26.** *Let  $\mathcal{C}$  be a category and let  $g : Z \rightarrow W$  be a morphism in  $\mathcal{C}$ . A morphism  $f : X \rightarrow Y$  has the left lifting property with respect to  $g$  if and only if  $f$  is injective as an object in  $[2, \mathcal{C}]$  with respect to the singleton set  $\{(g, \text{id}_W) : g \rightarrow \text{id}_W\}$ .*  $\blacksquare$

**Corollary 0.5.27.** *Let  $\mathcal{C}$  be a category and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{C}$ . If the domains and codomains of morphisms in  $\mathcal{I}$  are  $(\kappa, \mathbf{U})$ -compact in  $\mathcal{C}$ , then  $\text{inj}^{\mathcal{I}} \mathcal{C}$  is closed under colimits for  $\mathbf{U}$ -small  $\kappa$ -filtered diagrams in  $[2, \mathcal{C}]$ .*

*Proof.* Apply [proposition 0.2.46](#) and the two lemmas above.  $\blacksquare$

**Proposition 0.5.28.** *Let  $\mathcal{C}$  be a locally presentable  $\mathbf{U}$ -category, let  $(L, R)$  be a functorial weak factorisation system on  $\mathcal{C}$ , and let  $\lambda : \text{id}_{[2, \mathcal{C}]} \Rightarrow R$  be the natural transformation whose component at an object  $f$  in  $[2, \mathcal{C}]$  corresponds to the following commutative square in  $\mathcal{C}$ :*

$$\begin{array}{ccc} \bullet & \xrightarrow{Lf} & \bullet \\ f \downarrow & & \downarrow Rf \\ \bullet & \xlongequal{\quad} & \bullet \end{array}$$

*Let  $\mathcal{R}$  be the full subcategory of  $[2, \mathcal{C}]$  spanned by the morphisms in  $\mathcal{C}$  that are in the right class of the induced weak factorisation system.*

- (i)  $\mathcal{R}$  is also the full subcategory of  $[2, \mathcal{C}]$  spanned by the image of the forgetful functor  $[2, \mathcal{C}]^{(R, \lambda)} \rightarrow [2, \mathcal{C}]$ , where  $[2, \mathcal{C}]^{(R, \lambda)}$  is the category of algebras for the pointed endofunctor  $(R, \lambda)$ .

- (ii) If  $R : [2, C] \rightarrow [2, C]$  is an accessible functor, then  $[2, C]^{(R, \lambda)}$  is a locally presentable  $\mathbf{U}$ -category, and the forgetful functor  $[2, C]^{(R, \lambda)} \rightarrow [2, C]$  is monadic.
- (iii) If  $R : [2, C] \rightarrow [2, C]$  is strongly  $(\pi, \mathbf{U})$ -accessible and has  $\mathbf{U}$ -rank  $\kappa < \pi$ , and  $\mathcal{R}$  is closed under colimits for  $\mathbf{U}$ -small  $\pi$ -filtered diagrams in  $[2, C]$ , then  $\mathcal{R}$  is a  $\pi$ -accessible  $\mathbf{U}$ -subcategory of  $[2, C]$ .

*Proof.* (i). This is [proposition A.3.33](#).

(ii). Apply [theorem 0.3.40](#).

(iii). By [theorem 0.3.41](#),  $[2, C]^{(R, \lambda)}$  is a locally  $\pi$ -presentable  $\mathbf{U}$ -category, and the forgetful functor  $[2, C]^{(R, \lambda)} \rightarrow [2, C]$  is moreover strongly  $(\pi, \mathbf{U})$ -accessible. Thus, we may apply [proposition 0.3.31](#) to claim (i) and deduce that  $\mathcal{R}$  is a  $\pi$ -accessible  $\mathbf{U}$ -subcategory.  $\blacksquare$

**Proposition 0.5.29.** *Let  $C$  be a locally presentable  $\mathbf{U}$ -category, and let  $\mathcal{I}$  be a  $\mathbf{U}$ -subset of  $\text{mor } C$ . Then  $\text{inj}^{\mathcal{I}} C$ , considered as a full subcategory of  $[2, C]$ , is an accessible  $\mathbf{U}$ -subcategory.*

*Proof.* Combine [corollary 0.5.14](#) and [proposition 0.5.28](#).  $\blacksquare$

**Lemma 0.5.30.** *Let  $C$  be a  $\kappa$ -accessible  $\mathbf{U}$ -category and let  $\mathcal{R}$  be a  $\kappa$ -accessible full subcategory of  $[2, C]$ . If  $g : Z \rightarrow W$  is a morphism in  $C$  and  $Z$  and  $W$  are  $(\kappa, \mathbf{U})$ -compact objects in  $C$ , then:*

- (i) *Given a morphism  $f : X \rightarrow Y$  in  $C$  that is in  $\mathcal{R}$ , any morphism  $g \rightarrow f$  in  $[2, C]$  admits a factorisation of the form  $g \rightarrow f' \rightarrow f$  where  $f'$  is in  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{R})$ .*
- (ii) *The morphism  $g : Z \rightarrow W$  has the left lifting property with respect to  $\mathcal{R}$  if and only if it has the left lifting property with respect to  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{R})$ .*

*Proof.* (i). [Proposition 0.2.46](#) says that  $g$  is a  $(\kappa, \mathbf{U})$ -compact object in  $[2, C]$ ; but every object in  $\mathcal{R}$  is the colimit of a  $\mathbf{U}$ -small  $\kappa$ -filtered diagram of  $(\kappa, \mathbf{U})$ -compact objects in  $\mathcal{R}$ , and the inclusion  $\mathcal{R} \hookrightarrow [2, C]$  is  $(\kappa, \mathbf{U})$ -accessible, so any morphism  $g \rightarrow f$  must factor through some  $(\kappa, \mathbf{U})$ -compact object in  $\mathcal{R}$ .

(ii). If  $g$  has the left lifting property with respect to  $\mathcal{R}$ , then it certainly has the left lifting property with respect to  $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{R})$ . Conversely, by factorising morphisms

## 0. FOUNDATIONS

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$g \rightarrow f$  as in claim (i), we see that  $g$  has the left lifting property with respect to  $\mathcal{R}$  as soon as it has the left lifting property with respect to  $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{R})$ . ■

## SIMPLICIAL SETS

Simplicial sets, like simplicial complexes, are combinatorial models for spaces built up by gluing standard  $n$ -simplices together; unlike simplicial complexes, an  $n$ -simplex in a simplicial set need not be uniquely determined by its vertices. It is for this reason that simplicial sets were once known by the unwieldy name ‘complete semi-simplicial (c.s.s.) complex’.

In the 1960s, it was discovered that one can mimic the definitions and constructions of classical homotopy theory by combinatorial means using simplicial sets, and that the resulting theory is moreover equivalent to the classical theory in a natural, functorial way. More recently, it has been shown that the homotopy theory of simplicial sets is *universal* in a precise sense,<sup>[1]</sup> so it seems fitting that we begin here.

### I.1 Basics

**Definition I.1.1.** The **simplex category** is the category  $\mathbf{\Delta}$  whose objects are the positive finite ordinals and whose morphisms are the monotone maps. We use the geometer’s convention:  $[n]$  denotes the ordinal  $\{0, 1, \dots, n\}$ .

**Definition I.1.2.** A **simplicial object** in a category  $\mathcal{C}$  is a functor  $\mathbf{\Delta}^{\text{op}} \rightarrow \mathcal{C}$ , and a **morphism of simplicial objects** in  $\mathcal{C}$  is a natural transformation of such functors. The **category of simplicial objects** in  $\mathcal{C}$  is the functor category  $[\mathbf{\Delta}^{\text{op}}, \mathcal{C}]$  and is denoted by  $\text{s}\mathcal{C}$ .

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[1] See [Dugger, 2001a].

**Definition 1.1.3.** The **coface maps** in  $\mathbf{\Delta}$  are the morphisms  $\delta_n^i : [n-1] \rightarrow [n]$ , where  $\delta_n^i$  is the unique injective monotone map that misses  $i$ ; and the **codegeneracy maps** in  $\mathbf{\Delta}$  are the morphisms  $\sigma_n^i : [n+1] \rightarrow [n]$ , where  $\sigma_n^i$  is the unique surjective monotone map with  $\sigma_n^i(i) = \sigma_n^i(i+1) = i$ .

**Theorem 1.1.4** (Cosimplicial identities). *The following equations hold in  $\mathbf{\Delta}$ :*

$$\begin{aligned}
 \delta_{n+1}^{j+1} \circ \delta_n^i &= \delta_{n+1}^i \circ \delta_n^j && \text{if } 0 \leq i \leq j \leq n \\
 \sigma_n^j \circ \sigma_{n+1}^i &= \sigma_n^i \circ \sigma_{n+1}^{j+1} && \text{if } 0 \leq i \leq j \leq n \\
 \sigma_{n+1}^{j+1} \circ \delta_{n+1}^i &= \delta_n^i \circ \sigma_n^j && \text{if } 0 \leq i \leq j \leq n \\
 \delta_n^{j+1} \circ \sigma_n^i &= \sigma_{n+1}^i \circ \delta_{n+1}^{j+2} && \text{if } 0 \leq i < j < n \\
 \sigma_n^i \circ \delta_n^i &= \text{id} && \text{if } 0 \leq i \leq n \\
 \sigma_n^{i+1} \circ \delta_n^i &= \text{id} && \text{if } 0 \leq i < n
 \end{aligned}$$

*Equivalently, the following diagrams commute:*

$$\begin{array}{ccc}
 [n-1] & \xrightarrow{\delta^i} & [n] \\
 \delta^j \downarrow & & \downarrow \delta^{j+1} \\
 [n] & \xrightarrow{\delta^i} & [n+1]
 \end{array} \quad \text{for } 0 \leq i \leq j \leq n$$

$$\begin{array}{ccc}
 [n+1] & \xrightarrow{\sigma^i} & [n] \\
 \sigma^{j+1} \downarrow & & \downarrow \sigma^j \\
 [n] & \xrightarrow{\sigma^i} & [n-1]
 \end{array} \quad \text{for } 0 \leq i \leq j \leq n$$

$$\begin{array}{ccc}
 [n] & \xrightarrow{\delta^i} & [n+1] \\
 \sigma^j \downarrow & & \downarrow \sigma^{j+1} \\
 [n-1] & \xrightarrow{\delta^i} & [n]
 \end{array} \quad \text{for } 0 \leq i \leq j \leq n$$

$$\begin{array}{ccc}
 [n] & \xrightarrow{\sigma^i} & [n-1] \\
 \delta^{j+2} \downarrow & & \downarrow \delta^{j+1} \\
 [n+1] & \xrightarrow{\sigma^i} & [n]
 \end{array} \quad \text{for } 0 \leq i < j < n$$

$$\begin{array}{ccc}
 [n-1] & \xrightarrow{\delta^i} & [n] \\
 \delta^i \downarrow & \searrow \text{id} & \downarrow \sigma^{i+1} \\
 [n] & \xrightarrow{\sigma^i} & [n-1]
 \end{array} \quad \text{for } 0 \leq i \leq n$$

Moreover, every morphism  $[n] \rightarrow [m]$  in  $\Delta$  is uniquely a composite of the form

$$\delta_m^{j_1} \circ \dots \circ \delta_k^{j_{m-k}} \circ \sigma_k^{i_{n-k}} \circ \dots \circ \sigma_n^{i_1}$$

where  $k \leq \min \{n, m\}$ , and

$$\begin{aligned} 0 \leq i_{n-k} \leq \dots \leq i_1 \leq n \\ 0 \leq j_{m-k} \leq \dots \leq j_1 \leq m \end{aligned}$$

The category  $\Delta$  is uniquely characterised by these properties.

*Proof.* See [May, 1967, §2], [GZ, Ch. II, §2], or [Weibel, 1994, §8.1].  $\square$

**Definition 1.1.5.** Let  $A$  be a simplicial object in a category  $\mathcal{C}$ . A **face operator** for  $A$  is a morphism of the form  $A(\delta_n^i) : A([n]) \rightarrow A([n-1])$ , and a **degeneracy operator** for  $A$  is a morphism of the form  $A(\sigma_n^i) : A([n]) \rightarrow A([n+1])$ . For brevity, we will usually write  $A_n$  instead of  $A([n])$ ,  $d_i^n$  instead of  $A(\delta_n^i)$ , and  $s_i^n$  instead of  $A(\sigma_n^i)$ .

**Corollary 1.1.6** (Simplicial identities). *The face and degeneracy operators of a simplicial object satisfy the formal duals of the equations in theorem 1.1.4.*  $\blacksquare$

**Corollary 1.1.7.** *A simplicial object  $A$  is uniquely determined by the sequence of objects  $A_0, A_1, A_2, \dots$  together with the face and degeneracy operators. Conversely, any sequence of objects equipped with face and degeneracy operators satisfying the simplicial identities defined a simplicial object.*  $\blacksquare$

**Definition 1.1.8.** A **simplicial set** is a simplicial object in **Set**, and the **category of simplicial sets** is denoted by **sSet**.

**Lemma 1.1.9.**

- (i) *Limits (resp. colimits) in sSet are constructed degreewise: a cone (resp. cocone) in sSet over a diagram is limiting (resp. colimiting) if and only if it is so in every degree.*
- (ii) *A morphism of sSet is monic (resp. epic) if and only if it is degreewise injective (resp. surjective).*

*Proof.* These are standard facts about functor categories.  $\square$

**Definition 1.1.10.** The **standard  $n$ -simplex** in **sSet**, denoted by  $\Delta^n$ , is the representable presheaf  $\Delta(-, [n])$ .

**Theorem I.I.II.** *Let  $\Delta^\bullet : \Delta \rightarrow \mathbf{sSet}$  be the functor  $[n] \mapsto \Delta^n$ .*

- (i) *For any simplicial set  $X$ , the map  $\mathbf{sSet}(\Delta^n, X) \rightarrow X_n$  defined by  $f \mapsto f_n(\text{id}_{[n]})$  is a bijection and is moreover natural in  $[n]$  and  $X$ .*
- (ii)  *$\mathbf{sSet}$  has limits and colimits for all small diagrams, every epimorphism is effective, and for all morphisms  $f : X \rightarrow Y$  in  $\mathbf{sSet}$ , the pullback functor  $f^* : \mathbf{sSet}_{/Y} \rightarrow \mathbf{sSet}_{/X}$  preserves colimits.*
- (iii)  *$\Delta^\bullet : \Delta \rightarrow \mathbf{sSet}$  is a dense functor, i.e. for any simplicial set  $X$ , the tautological cocone<sup>[2]</sup> from the canonical diagram  $(\Delta^\bullet \downarrow X) \rightarrow \mathbf{sSet}$  to  $X$  is colimiting.*
- (iv) *Let  $\mathcal{E}$  be a locally small category with colimits for all small diagrams. If  $F : \mathbf{sSet} \rightarrow \mathcal{E}$  is a functor that preserves small colimits, then it is left adjoint to the functor  $\mathcal{E} \rightarrow \mathbf{sSet}$  defined by  $E \mapsto \mathcal{E}(F\Delta^\bullet, E)$ .*
- (v) *With  $\mathcal{E}$  as above, the functor  $F \mapsto F\Delta^\bullet$  from the category of colimit-preserving functors  $\mathbf{sSet} \rightarrow \mathcal{E}$  to the category of all functors  $\Delta \rightarrow \mathcal{E}$  is fully faithful and essentially surjective on objects.*

*Proof.* Claim (i) is just the Yoneda lemma, claim (ii) follows from the lemma above, and claims (iii)–(v) are just facts about dense functors, pointwise left Kan extensions, weighted colimits: see [proposition A.5.25](#), [theorem A.5.15](#), and [proposition A.6.14](#). ■

**Definition I.I.I2.** Let  $X$  be a simplicial set. An  **$n$ -simplex** of  $X$  is an element of  $X_n$ ; a **vertex** is a 0-simplex, and an **edge** is a 1-simplex. This is justified by statement (i) in the above theorem. Given an edge  $f$  of  $X$ , the **source** of  $f$  is the vertex  $d_1(f)$ , and the **target** of  $f$  is the vertex  $d_0(f)$ ; we write  $f : x \rightarrow y$  to mean  $d_1(f) = x$  and  $d_2(f) = y$ .

**Definition I.I.I3.** A **degenerate  $n$ -simplex** of a simplicial set  $X$  is an  $n$ -simplex  $\alpha$  for which there exist an  $(n - 1)$ -simplex  $\beta$  and  $0 \leq i < n$  such that  $s_i(\beta) = \alpha$ . A **non-degenerate  $n$ -simplex** of  $X$  is an  $n$ -simplex that is not degenerate.

**REMARK I.I.I4.** An  $n$ -simplex of  $X$  can be non-degenerate even when the corresponding morphism  $\Delta^n \rightarrow X$  is not a monomorphism! Similarly, it is possible for all the proper faces of a non-degenerate simplex to be degenerate.

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[2] See [definition A.5.7](#).



**Definition 1.1.15.** A **finite simplicial set** is a simplicial set that has only finitely many *non-degenerate* simplices.

**Proposition 1.1.16.** *Let  $X$  be a simplicial set. The following are equivalent:*

- (i)  $X$  is a finite simplicial set.
- (ii)  $X$  is an  $\aleph_0$ -compact object in  $\mathbf{sSet}$ .<sup>[3]</sup>
- (iii)  $X$  is in the smallest full subcategory of  $\mathbf{sSet}$  that contains the standard simplices and is closed in  $\mathbf{sSet}$  under (isomorphisms and) colimits for finite diagrams.

*Proof.* (i)  $\Rightarrow$  (ii). A morphism  $f : X \rightarrow Y$  is determined uniquely by the images of the non-degenerate simplices of  $X$ , and the faces of any particular simplex can only satisfy finitely many equations, so if  $X$  is a finite simplicial set and  $Y$  is a colimit for a small filtered diagram of simplicial sets, then  $f$  must factor through one of the components of the colimiting cocone. It is straightforward to check that the factorisation of  $f$  is unique up to the appropriate equivalence relation, and we may then deduce that  $X$  is an  $\aleph_0$ -compact object.

(ii)  $\Rightarrow$  (iii). Let  $\mathcal{K}$  be the indicated full subcategory of  $\mathbf{sSet}$ , and consider the comma category  $(\mathcal{K} \downarrow X)$ . Let  $P : (\mathcal{K} \downarrow X) \rightarrow \mathbf{sSet}$  be the projection, and let  $\lambda : P \Rightarrow \Delta X$  be the tautological cocone.<sup>[4]</sup> It is not hard to check that  $\lambda$  is a colimiting cocone. Since  $\mathcal{K}$  has colimits for finite diagrams,  $(\mathcal{K} \downarrow X)$  is filtered; and it is clear that  $\mathcal{K}$  is essentially small, so we deduce that  $X$  is a retract of an object in  $\mathcal{K}$  if  $X$  is  $\aleph_0$ -compact. Noting that  $\mathcal{K}$  is closed under retracts, we conclude that  $X$  is in  $\mathcal{K}$  if it is  $\aleph_0$ -compact.

(iii)  $\Rightarrow$  (i). Now, let  $\mathcal{K}'$  be the full subcategory of  $\mathbf{sSet}$  spanned by the finite simplicial sets. It is easy to see that  $\mathcal{K}'$  is closed in  $\mathbf{sSet}$  under (isomorphisms and) finite colimits, and the standard simplices are all in  $\mathcal{K}'$ , so we must have  $\mathcal{K} \subseteq \mathcal{K}'$ , as required. ■

**Definition 1.1.17.** The **standard  $n$ -simplex** in  $\mathbf{Top}$ , denoted by  $|\Delta^n|$ , is the topological space

$$|\Delta^n| = \{(x_0, \dots, x_n) \in [0, 1]^{n+1} \mid x_0 + \dots + x_n = 1\}$$

[3] See definition 0.2.13.

[4] See definition A.5.7.

where  $[0, 1]$  is the closed unit interval with the standard metric. The functor  $|\Delta^\bullet| : \mathbf{\Delta} \rightarrow \mathbf{Top}$  sends  $[n]$  to  $|\Delta^n|$  and is defined on morphisms by linearly interpolating the obvious map of vertices.

**Corollary 1.1.18.** *There exists an adjunction*

$$|-| \dashv S : \mathbf{Top} \rightarrow \mathbf{sSet}$$

*extending the functor  $|\Delta^\bullet| : \mathbf{\Delta} \rightarrow \mathbf{Top}$  defined above, and this adjunction is unique up to unique isomorphism. Explicitly, we may take*

$$S(Y)_n = \mathbf{Top}(|\Delta^n|, Y)$$

*with the evident face and degeneracy operators induced by the coface and codegeneracy maps in  $\mathbf{\Delta}$ .* ■

**Definition 1.1.19.** The **geometric realisation** of a simplicial set  $X$  is the topological space  $|X|$ , and the **singular set** of a topological space  $Y$  is the simplicial set  $S(Y)$ .

**REMARK 1.1.20.** The geometric realisation  $|X|$  is stable under universe enlargement, by [theorem A.5.20](#).

**Theorem 1.1.21.** *Let  $\mathbf{CGHaus}$  be the category of compactly-generated Hausdorff spaces<sup>[5]</sup> and continuous maps.*

- (i) *The topological standard  $n$ -simplex  $|\Delta^n|$  is a compact Hausdorff space.*
- (ii) *For any simplicial set  $X$ , the geometric realisation  $|X|$  is a compactly-generated Hausdorff space.*
- (iii) *The previously-constructed adjunction  $|-| \dashv S : \mathbf{Top} \rightarrow \mathbf{sSet}$  restricts to an adjunction between  $\mathbf{CGHaus}$  and  $\mathbf{sSet}$ , and moreover the functor  $|-| : \mathbf{sSet} \rightarrow \mathbf{CGHaus}$  preserves finite limits and reflects isomorphisms.*

*Proof.* Claim (i) is a standard fact, while claims (ii) and (iii) are proven in [\[GZ, Ch. III, §3\]](#). □

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[5] See [definition A.2.26](#).

## I.2 Nerves, skeletons, and coskeletons

**Prerequisites.** §§ I.1, A.2.

**Proposition I.2.1.** *Let  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  be the functor defined by the formula*

$$N(\mathbb{C})_n = \text{Fun}([n], \mathbb{C})$$

where  $[n]$  here denotes the preorder category  $\{0 \rightarrow \dots \rightarrow n\}$ .

- (i)  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  has a left adjoint  $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$  such that  $\tau_1 \Delta^n = [n]$ .
- (ii) The functor  $N$  is fully faithful and exhibits  $\mathbf{Cat}$  as a reflective subcategory of  $\mathbf{sSet}$ .
- (iii)  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  is a cartesian closed functor.
- (iv) The functor  $\tau_1$  preserves finite products.

*Proof.* (i). Apply [theorem I.1.11](#).

(ii). A functor is entirely determined by its action on objects, arrows, and composable strings of arrows, so  $N$  is fully faithful.

(iii).  $N$  preserves binary products, so we have the following natural bijections:

$$\begin{aligned} \mathbf{sSet}(\Delta^n, N([\mathbb{C}, \mathbb{D}])) &\cong \text{Fun}([n], [\mathbb{C}, \mathbb{D}]) \\ &\cong \text{Fun}([n] \times \mathbb{C}, \mathbb{D}) \\ &\cong \mathbf{sSet}(N([n] \times \mathbb{C}), N(\mathbb{D})) \\ &\cong \mathbf{sSet}(N([n]) \times N(\mathbb{C}), N(\mathbb{D})) \\ &\cong \mathbf{sSet}(N([n]), [N(\mathbb{C}), N(\mathbb{D})]) \\ &\cong \mathbf{sSet}(\Delta^n, [N(\mathbb{C}), N(\mathbb{D})]) \end{aligned}$$

Thus, by the Yoneda lemma, the canonical morphism  $N([\mathbb{C}, \mathbb{D}]) \rightarrow [N(\mathbb{C}), N(\mathbb{D})]$  is an isomorphism.

(iv). It is clear that  $\tau_1$  preserves terminal objects. Let  $X$  and  $Y$  be simplicial sets. We wish to show that the canonical morphism  $\tau_1(X \times Y) \rightarrow \tau_1 X \times \tau_1 Y$  is an isomorphism; but since  $\tau_1$  is a left adjoint and both  $\mathbf{sSet}$  and  $\mathbf{Cat}$  are cartesian

closed, it is enough to check the claim for  $Y = \Delta^n$ , because  $\mathbf{sSet}$  is generated under colimits by  $\{\Delta^n \mid n \in \mathbb{N}\}$ . We have the following natural bijections:

$$\begin{aligned} \mathrm{Fun}(\tau_1(X \times \Delta^n), \mathbb{C}) &\cong \mathbf{sSet}(X \times \Delta^n, \mathbf{N}(\mathbb{C})) \\ &\cong \mathbf{sSet}(X, \mathbf{N}(\mathbb{C})^{\Delta^n}) \\ &\cong \mathbf{sSet}(X, \mathbf{N}([n], \mathbb{C})) \\ &\cong \mathrm{Fun}(\tau_1 X, [[n], \mathbb{C}]) \\ &\cong \mathrm{Fun}(\tau_1 X \times [n], \mathbb{C}) \\ &\cong \mathrm{Fun}(\tau_1 X \times \tau_1 \Delta^n, \mathbb{C}) \end{aligned}$$

The claim follows by the Yoneda lemma. ■

**Definition 1.2.2.** The **fundamental category** of a simplicial set  $X$  is the small category  $\tau_1 X$ , and the **nerve** of a small category  $\mathbb{C}$  is the simplicial set  $\mathbf{N}(\mathbb{C})$ .

**REMARK 1.2.3.** Given a simplicial set  $X$ , the fundamental category  $\tau_1 X$  admits the following presentation by generators and relations: the objects are the vertices of  $X$ , and the arrows are generated by the edges of  $X$ , modulo the relation  $d_0(\alpha) \bullet d_2(\alpha) = d_1(\alpha)$  for all 2-simplices  $\alpha$  in  $X$ . This shows that  $\tau_1 X$  is stable under universe enlargement.

**Proposition 1.2.4.** *Let  $\mathrm{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$  be the functor defined by the formula*

$$(\mathrm{disc} Y)_n = Y$$

*with  $\mathrm{id}_Y$  for all the face and degeneracy maps.*

- (i)  $\mathrm{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$  has a left adjoint  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  such that  $\pi_0 \Delta^n = 1$ .
- (ii) The functor  $\mathrm{disc}$  is fully faithful and exhibits  $\mathbf{Set}$  as a reflective subcategory of  $\mathbf{sSet}$ .
- (iii)  $\mathbf{N} : \mathbf{Set} \rightarrow \mathbf{sSet}$  is a cartesian closed functor.
- (iv) The functor  $\pi_0$  preserves products.

*Proof.* (i). We could apply [theorem 1.1.11](#), but it is also fairly straightforward to check that this explicit construction works: for each simplicial set  $X$ , we define  $\pi_0 X$  by the coequaliser diagram in  $\mathbf{Set}$  shown below,

$$X_1 \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} X_0 \longrightarrow \pi_0 X$$

and for each morphism  $f : X \rightarrow Y$  in  $\mathbf{sSet}$ , we define  $\pi_0 f$  to be the unique morphism making the evident diagram commute.

(ii). It is clear that  $\text{disc}$  is fully faithful.

(iii). By [proposition A.2.15](#), we have an analogous adjunction  $\pi_0 \dashv \text{disc} : \mathbf{Set} \rightarrow \mathbf{Cat}$ . It is clear that we have a natural isomorphism  $N(\text{disc } Y) \cong \text{disc } Y$  for every set  $Y$ , and we know  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{Cat}$  and  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  are cartesian closed functors, so  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$  must also be cartesian closed.

(iv). Similarly, for any simplicial set  $X$ , we have a natural isomorphism  $\pi_0 X \cong \pi_0 \tau_1 X$ ; but we know that  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$  preserves finite products, and  $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$  preserves finite products by [proposition 1.2.1](#), so  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  must also preserve finite products. ■

**Definition 1.2.5.** The **set of connected components** of a simplicial set  $X$  is the set  $\pi_0 X$ , and a **discrete simplicial set** is one that is isomorphic to  $\text{disc } Y$  for some set  $Y$ .

¶ 1.2.6. We will usually not distinguish between  $Y$  and  $\text{disc } Y$  notationally.

**Proposition 1.2.7.** Let  $N : \mathbf{Grpd} \rightarrow \mathbf{sSet}$  be the functor defined by the formula

$$N(\mathbb{G})_n = \text{Fun}(\mathbf{I}[n], \mathbb{G})$$

where  $\mathbf{I}[n]$  here denotes the groupoid obtained by freely inverting the arrows in the preorder category  $[n]$ .

- (i) For any groupoid  $\mathbb{G}$ , the nerve  $N(\mathbb{G})$  is the same (up to isomorphism) whether computed for  $\mathbb{G}$  as a groupoid or  $\mathbb{G}$  as a category.
- (ii)  $N : \mathbf{Grpd} \rightarrow \mathbf{sSet}$  has a left adjoint  $\pi_1 : \mathbf{sSet} \rightarrow \mathbf{Grpd}$  such that  $\pi_1 \Delta^n = \mathbf{I}[n]$ .
- (iii) The functor  $N$  is fully faithful and exhibits  $\mathbf{Grpd}$  as a reflective subcategory of  $\mathbf{sSet}$ .
- (iv)  $N : \mathbf{Grpd} \rightarrow \mathbf{sSet}$  is a cartesian closed functor.
- (v) The functor  $\pi_1$  preserves finite products.

*Proof.* (i). By the universal property of  $\mathbf{I}[n]$ , there is a natural bijection

$$\mathrm{Fun}(\mathbf{I}[n], \mathbb{G}) \cong \mathrm{Fun}([n], \mathbb{G})$$

for all groupoids  $\mathbb{G}$ , so the two nerve constructions do indeed agree.

(ii) and (iii). These are proven in exactly the same way as in [proposition 1.2.1](#).

(iv) and (v). These are proven in exactly the same way as in [proposition 1.2.4](#). ■

**Definition 1.2.8.** The **fundamental groupoid** of a simplicial set  $X$  is the small groupoid  $\pi_1 X$ .

REMARK 1.2.9. Given a simplicial set  $X$ , the fundamental groupoid  $\pi_1 X$  admits a presentation of the same kind as the fundamental category  $\tau_1 X$ , and in fact  $\pi_1 X$  is isomorphic to the groupoid obtained by freely inverting the arrows in  $\tau_1 X$ :

$$\mathrm{Fun}(\pi_1 X, \mathbb{G}) \cong \mathbf{sSet}(X, \mathbf{N}(\mathbb{G})) \cong \mathrm{Fun}(\tau_1 X, \mathbb{G})$$

This shows that  $\pi_1 X$  is stable under universe enlargement.

**Definition 1.2.10.** Let  $n$  be a natural number, and let  $\mathbf{\Delta}_{\leq n}$  be the full subcategory of  $\mathbf{\Delta}$  spanned by the objects  $[0], \dots, [n]$ . An  $n$ -**truncated simplicial set** is a functor  $\mathbf{\Delta}_{\leq n}^{\mathrm{op}} \rightarrow \mathbf{Set}$ , and we write  $\mathbf{sSet}_{\leq n}$  for the category of  $n$ -truncated simplicial sets. The **brutal  $n$ -truncation** of a simplicial set  $X$  is the  $n$ -truncated simplicial set  $X_{\leq n}$  defined by the evident reduct:

$$X_{\leq n}([m]) = X([m])$$

**Proposition 1.2.11.** Let  $n$  be a natural number, and let  $j : \mathbf{\Delta}_{\leq n} \rightarrow \mathbf{\Delta}$  be the inclusion.

- (i) The functor  $j^* : \mathbf{sSet} \rightarrow \mathbf{sSet}_{\leq n}$  has a left adjoint  $\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \rightarrow \mathbf{sSet}$ .
- (ii) The unit  $\mathrm{id} \Rightarrow j^* \mathrm{Lan}_j$  is a natural isomorphism.
- (iii)  $\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \rightarrow \mathbf{sSet}$  is a fully faithful functor.
- (i') The functor  $j^* : \mathbf{sSet} \rightarrow \mathbf{sSet}_{\leq n}$  has a right adjoint  $\mathrm{Ran}_j : \mathbf{sSet}_{\leq n} \rightarrow \mathbf{sSet}$ .
- (ii') The counit  $j^* \mathrm{Ran}_j \Rightarrow \mathrm{id}$  is a natural isomorphism.

(iii')  $\text{Ran}_j : \mathbf{sSet}_{\leq n} \rightarrow \mathbf{sSet}$  is a fully faithful functor.

*Proof.* (i) and (i'). Use [theorem A.5.15](#).

(ii) and (ii'). The inclusion  $j : \Delta_{\leq n} \rightarrow \Delta$  is fully faithful, so the unit  $\text{id} \Rightarrow j^* \text{Lan}_j$  and the counit  $j^* \text{Ran}_j \Rightarrow \text{id}$  are natural isomorphisms, by [corollary A.5.19](#).

(iii) and (iii'). Use [proposition A.1.3](#). ■

**Definition 1.2.12.** For each natural number  $n$ , with notation as above, let  $\text{sk}_n : \mathbf{sSet} \rightarrow \mathbf{sSet}$  be the composite  $\text{Lan}_j j^*$ , and let  $\text{cosk}_n : \mathbf{sSet} \rightarrow \mathbf{sSet}$  be the composite  $\text{Ran}_j j^*$ . The  $n$ -**skeleton** of a simplicial set  $X$  is the simplicial set  $\text{sk}_n(X)$ , and the  $n$ -**coskeleton** of a simplicial set is the simplicial set  $\text{cosk}_n(X)$ . A  $n$ -**skeletal simplicial set** is one that is isomorphic to the  $n$ -skeleton of some simplicial set, and an  $n$ -**coskeletal simplicial set** is one that is isomorphic to the  $n$ -coskeleton of some simplicial set.

**REMARK 1.2.13.** In the special case  $n = 0$ ,  $\text{Lan}_j$  may be identified with the functor  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$  defined in [proposition 1.2.4](#). Thus, 0-skeletal simplicial sets are precisely the discrete simplicial sets. On the other hand, given a set  $X$ ,  $\text{Ran}_j X$  can be identified with the simplicial set whose  $m$ -simplices are  $(m + 1)$ -tuples of elements of  $X$ , with face and degeneracy maps induced by the appropriate projections.

**Proposition 1.2.14.** *Let  $n$  be a natural number.*

- (i) *The full subcategory of  $n$ -skeletal simplicial sets is a coreflective subcategory of  $\mathbf{sSet}$ , with coreflector  $\text{sk}_n$ .*
- (ii)  *$\text{sk}_n$  is the underlying endofunctor of an idempotent comonad on  $\mathbf{sSet}$ .*
- (iii) *A simplicial set  $X$  is  $n$ -skeletal if and only if the counit  $\text{sk}_n(X) \rightarrow X$  is an isomorphism.*
- (iv) *If  $m \geq n$ , then any  $n$ -skeletal simplicial set is also  $m$ -skeletal.*
- (i') *The full subcategory of  $n$ -coskeletal simplicial sets is a reflective subcategory of  $\mathbf{sSet}$ , with reflector  $\text{cosk}_n$ .*
- (ii')  *$\text{cosk}_n$  is the underlying endofunctor of an idempotent monad on  $\mathbf{sSet}$ .*

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(iii') *A simplicial set  $X$  is  $n$ -coskeletal if and only if the unit  $X \rightarrow \text{cosk}_n(X)$  is an isomorphism.*

(iv') *If  $m \geq n$ , then any  $n$ -coskeletal simplicial set is also  $m$ -coskeletal.*

*Proof.* All straightforward from the definitions. ◆

**Proposition 1.2.15.** *Let  $n$  be a natural number, and let  $X$  be a simplicial set.*

(i) *We have the following adjunction:*

$$\text{sk}_n \dashv \text{cosk}_n : \mathbf{sSet} \rightarrow \mathbf{sSet}$$

(ii) *The counit  $\text{sk}_n(X) \rightarrow X$  is a monomorphism, and  $X$  is  $n$ -skeletal if and only if all  $m$ -simplices of  $X$  are degenerate for  $m > n$ .*

(iii)  *$X$  is  $n$ -coskeletal if and only if, for all natural numbers  $m$ , the map*

$$X_m \cong \mathbf{sSet}(\Delta^m, X) \rightarrow \mathbf{sSet}(\text{sk}_n(\Delta^m), X)$$

*induced by the counit  $\text{sk}_n(\Delta^m) \rightarrow \Delta^m$  is a bijection.*

*Proof.* (i). Immediate from the definition of  $\text{sk}_n$  and  $\text{cosk}_n$ .

(ii). The most straightforward way of seeing this is to construct  $\text{sk}_n(X)$  explicitly as the smallest simplicial subset of  $X$  containing all of its  $n$ -simplices.

(iii). Apply the Yoneda lemma in conjunction with claim (i). ■

**Example 1.2.16.** For any small category  $\mathbb{C}$ , the nerve  $N(\mathbb{C})$  is a 2-coskeletal simplicial set: by definition, an  $m$ -simplex of  $N(\mathbb{C})$  is just a functor  $[m] \rightarrow \mathbb{C}$ , but the property of being a functor can be detected by only inspecting the vertices, edges, and 2-cells.

**Proposition 1.2.17.** *The following full subcategories are exponential ideals of  $\mathbf{sSet}$ :*

(i) *Discrete simplicial sets.*

(ii) *Simplicial sets isomorphic to the nerve of some category.*

(iii) *Simplicial sets isomorphic to the nerve of some groupoid.*



(iv)  $n$ -coskeletal simplicial sets for some natural number  $n$ .

*Proof.* Apply proposition A.2.13 to propositions 1.2.4, 1.2.1, 1.2.7, and 1.2.14. ■

**Definition 1.2.18.** The **boundary** of  $\Delta^n$  is the simplicial subset  $\partial\Delta^n \subseteq \Delta^n$  generated by the images of  $\delta_n^0, \dots, \delta_n^n : \Delta^{n-1} \rightarrow \Delta^n$ .

**REMARK 1.2.19.** The boundary  $\partial\Delta^n$  may be identified with  $\text{sk}_{n-1}\Delta^n$ .

**Proposition 1.2.20** (Relative skeletal filtration). *Let  $f : X \rightarrow Y$  be a monomorphism in  $\mathbf{sSet}$ . There exist simplicial sets  $Y^{(0)}, Y^{(1)}, Y^{(2)}, \dots$  and a chain of monomorphisms*

$$X = Y^{(-1)} \xrightarrow{i^{(0)}} Y^{(0)} \xrightarrow{i^{(1)}} Y^{(1)} \xrightarrow{i^{(2)}} Y^{(2)} \longrightarrow \dots$$

such that the following conditions are satisfied:

- For each natural number  $n$ , there is a pushout diagram of the form below,

$$\begin{array}{ccc} I_n \odot \partial\Delta^n & \hookrightarrow & I_n \odot \Delta^n \\ \downarrow & & \downarrow \\ Y^{(n-1)} & \xrightarrow{i^{(n)}} & Y^{(n)} \end{array}$$

where  $I_n \subseteq Y_n$  is the set of non-degenerate  $n$ -simplices of  $Y$  that are not in the image of  $f : X \rightarrow Y$  and  $I_n \odot \partial\Delta^n \hookrightarrow I_n \odot \Delta^n$  is induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$ .

- There is a colimiting cocone from the above chain to  $Y$  where the component  $Y^{(-1)} \rightarrow Y$  is  $f : X \rightarrow Y$ .

In particular, if  $Y$  is a finite simplicial set, then there is a natural number  $d$  such that  $i^{(n)} : Y^{(n-1)} \rightarrow Y^{(n)}$  is an isomorphism for all  $n > d$ .

*Proof.* We may assume without loss of generality that  $f : X \rightarrow Y$  is the inclusion of a simplicial subset of  $Y$ . Let  $Y^{(n)}$  be the union of  $X$  and the image of counit  $\text{sk}_n(Y) \rightarrow Y$ , i.e. the smallest simplicial subset of  $Y$  containing  $X$  and all the  $n$ -simplices of  $Y$ , and let  $i^{(n)} : Y^{(n-1)} \rightarrow Y^{(n)}$  be the inclusion. Then  $I_n$  is precisely the set of  $n$ -simplices of  $Y^{(n)}$  that are *not* in  $Y^{(n-1)}$ , so we have the desired pushout diagram for each  $n$ . It is clear that the inclusions  $Y^{(n-1)} \hookrightarrow Y$  define the required colimiting cocone. ■

In the language of §0.5, what we have shown is that every monomorphism in  $\mathbf{sSet}$  is a relative  $\mathcal{I}$ -cell complex, where  $\mathcal{I} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}$ . Since the class of monomorphisms is closed under retracts, the following definition is justified:

**Definition 1.2.21.** A **cofibration of simplicial sets** is a monomorphism in  $\mathbf{sSet}$ .

REMARK 1.2.22. Cofibrations of simplicial sets have a homotopy extension property, albeit one that is weaker than what one might expect from the homotopy theory of topological spaces: see theorem 1.3.20.

## 1.3 Intrinsic homotopy

**Prerequisites.** §§ 1.2, 3.1, A.4.

**Definition 1.3.1.** Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathbf{sSet}$ . An **intrinsic homotopy**  $\alpha : f_0 \Rightarrow f_1$  is an edge of the exponential object  $[X, Y]$  such that  $d_1(\alpha) = f_0$  and  $d_0(\alpha) = f_1$ . (Note the subscripts!) We say  $f_0$  and  $f_1$  are **intrinsically homotopic** if there is a zigzag of intrinsic homotopies connecting  $f_0$  and  $f_1$ , and we write  $f_0 \sim f_1$  in this case.

REMARK 1.3.2. By the Yoneda lemma,

$$[X, Y]_1 \cong \mathbf{sSet}(\Delta^1, [X, Y]) \cong \mathbf{sSet}(\Delta^1 \times X, Y)$$

so an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$  is essentially the same thing as a morphism  $\tilde{\alpha} : \Delta^1 \times X \rightarrow Y$  such that  $\tilde{\alpha} \circ (\delta^1 \times \text{id}_X) = f_0$  and  $\tilde{\alpha} \circ (\delta^0 \times \text{id}_X) = f_1$  (where we have suppressed the canonical isomorphism  $X \cong \Delta^0 \times X$ ), just as in classical homotopy theory. Also,

$$\mathbf{sSet}(\Delta^1 \times X, Y) \cong \mathbf{sSet}(X, [\Delta^1, Y])$$

so intrinsic homotopies  $\alpha : f_0 \Rightarrow f_1$  correspond to morphisms  $\hat{\alpha} : X \rightarrow [\Delta^1, Y]$  such that  $[\delta^1, Y] \circ \hat{\alpha} = f_0$  and  $[\delta^0, Y] \circ \hat{\alpha} = f_1$  (where we have suppressed the canonical isomorphism  $[\Delta^0, Y] \cong Y$ ).

REMARK 1.3.3. The notion of intrinsic homotopy is not well behaved for general simplicial sets  $Y$ . For example, the existence of an intrinsic homotopy  $f_0 \Rightarrow f_1$  does not guarantee the existence of an “inverse” intrinsic homotopy  $f_1 \Rightarrow f_0$ , and even if we have intrinsic homotopies  $f_0 \Rightarrow f_1$  and  $f_1 \Rightarrow f_2$ , there need not be an intrinsic homotopy  $f_0 \Rightarrow f_2$ .

¶ 1.3.4. Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms and let  $\alpha : f_0 \Rightarrow f_1$  be an intrinsic homotopy.

- Given a morphism  $g : W \rightarrow X$ , the intrinsic homotopy  $\alpha g : f_0 \circ g \Rightarrow f_1 \circ g$  is the image of  $\alpha$  under the induced morphism  $[g, Y] : [X, Y] \rightarrow [W, X]$ .
- Given a morphism  $g : Y \rightarrow Z$ , the intrinsic homotopy  $g\alpha : g \circ f_0 \Rightarrow g \circ f_1$  is the image of  $\alpha$  under the induced morphism  $[X, g] : [X, Y] \rightarrow [X, Z]$ .

**Lemma 1.3.5.** *The relation of intrinsic homotopy is a congruence on  $\mathbf{sSet}$ , i.e. given morphisms  $f_0, f_1 : X \rightarrow Y$  and  $g_0, g_1 : Y \rightarrow Z$ , if  $f_0 \sim f_1$  and  $g_0 \sim g_1$ , then  $g_0 \circ f_0 \sim g_1 \circ f_1$ . ■*

**Definition 1.3.6.** The **intrinsic homotopy category of simplicial sets** is the category  $\mathrm{Ho}_\Delta \mathbf{sSet}$  obtained by taking the quotient of  $\mathbf{sSet}$  with respect to the congruence of intrinsic homotopy.

REMARK 1.3.7. A parallel pair  $f_0, f_1 : X \rightarrow Y$  in  $\mathbf{sSet}$  are intrinsically homotopic if and only if they are in the same connected component of  $[X, Y]$ . In particular, we have a bijection of the form below,

$$\mathrm{Ho}_\Delta \mathbf{sSet}(X, Y) \cong \pi_0[X, Y]$$

and it is natural as  $X$  and  $Y$  vary in  $\mathbf{sSet}$ .

REMARK 1.3.8. The set  $\pi_0[X, Y]$  can be very far from what one expects geometrically. For instance,  $\pi_0[\partial\Delta^2, \partial\Delta^2]$  contains only two elements, while the set of homotopy classes of continuous endomaps of the circle is countably infinite!

**Lemma 1.3.9.** *Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathbf{sSet}$ . Given an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$ , for each simplicial set  $Z$ , there is an induced intrinsic homotopy  $[\alpha, Z] : [f_0, Z] \Rightarrow [f_1, Z]$ .*

*Proof.* Let  $\tilde{\alpha} : \Delta^1 \times X \rightarrow Y$  be the morphism corresponding to  $\alpha : f_0 \Rightarrow f_1$ . Then we have a morphism  $[\tilde{\alpha}, Z] : [Y, Z] \rightarrow [\Delta^1 \times X, Z]$ . Proposition A.2.11 says there is a natural isomorphism

$$[\Delta^1 \times X, Z] \cong [\Delta^1, [X, Z]]$$

so  $[\tilde{\alpha}, Z]$  corresponds to an intrinsic homotopy  $[\alpha, Z]$  between two morphisms of type  $[Y, Z] \rightarrow [X, Z]$ ; it is not hard to check that it is an intrinsic homotopy of type  $[f_0, Z] \Rightarrow [f_1, Z]$ . ■

**Definition 1.3.10.** Let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{sSet}$ .

- An **intrinsic homotopy left inverse** for  $f$  is a morphism  $g : Y \rightarrow X$  such that  $g \circ f$  and  $\text{id}_X$  are intrinsically homotopic.
- An **intrinsic homotopy right inverse** for  $f$  is a morphism  $g : Y \rightarrow X$  such that  $f \circ g$  and  $\text{id}_Y$  are intrinsically homotopic.

**Definition 1.3.11.** An **intrinsic homotopy equivalence** in  $\mathbf{sSet}$  is a pair  $(f, g)$  where  $g$  (resp.  $f$ ) is both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse for  $f$  (resp.  $g$ ).

**Example 1.3.12.** The pair  $(\sigma_0^0, \delta_1^0)$  is an intrinsic homotopy equivalence between the standard simplices  $\Delta^0$  and  $\Delta^1$ . Multiplying by  $\text{id}_X$ , we deduce that  $X$  and  $\Delta^1 \times X$  are naturally isomorphic in  $\text{Ho}_{\Delta^1} \mathbf{sSet}$ .

**Proposition 1.3.13.** Let  $\gamma : \mathbf{sSet} \rightarrow \text{Ho}_{\Delta^1} \mathbf{sSet}$  be the functor that sends each morphism to its intrinsic homotopy class. For any functor  $F : \mathbf{sSet} \rightarrow \mathcal{C}$ , the following are equivalent:

- (i) For all simplicial sets  $X$ ,  $F(\delta_1^0 \times \text{id}_X) : F(\Delta^0 \times X) \rightarrow F(\Delta^1 \times X)$  is an isomorphism in  $\mathcal{C}$ .
- (ii) For all simplicial sets  $X$ ,  $F(\delta_1^1 \times \text{id}_X) : F(\Delta^0 \times X) \rightarrow F(\Delta^1 \times X)$  is an isomorphism in  $\mathcal{C}$ .
- (iii) For all simplicial sets  $X$ ,  $F(\sigma_0^0 \times \text{id}_X) : F(\Delta^1 \times X) \rightarrow F(\Delta^0 \times X)$  is an isomorphism in  $\mathcal{C}$ .
- (iv) For all parallel pairs  $f_0, f_1 : X \rightarrow Y$  in  $\mathbf{sSet}$ , if  $f_0 \sim f_1$ , then  $Ff_0 = Ff_1$ .
- (v)  $F : \mathbf{sSet} \rightarrow \mathcal{C}$  factors through  $\gamma : \mathbf{sSet} \rightarrow \text{Ho}_{\Delta^1} \mathbf{sSet}$ .

Moreover, the factorisation is unique if it exists.

*Proof.* (i)  $\Leftrightarrow$  (iii), (ii)  $\Leftrightarrow$  (iii). Let  $e \in \{0, 1\}$ . The simplicial identity  $\sigma_0^0 \circ \delta_1^e = \text{id}$  implies that  $F(\sigma_0^0 \times \text{id}_X)$  is an isomorphism in  $\mathcal{C}$  if and only if  $F(\delta_0^e \times \text{id}_X)$  is an isomorphism in  $\mathcal{C}$ .

(iii)  $\Rightarrow$  (iv). It suffices to show that  $Ff_0 = Ff_1$  whenever there is an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$ , where  $f_0, f_1 : X \rightarrow Y$  are an arbitrary parallel pair of morphisms in  $\mathbf{sSet}$ . Let  $\tilde{\alpha} : \Delta^1 \times X \rightarrow Y$  be the morphism corresponding to

$\alpha : f_0 \Rightarrow f_1$ . Since  $F(\sigma_0^0 \times \text{id}_X)$  is an isomorphism, the uniqueness of inverses implies  $F(\delta_1^0 \times \text{id}_X) = F(\delta_1^1 \times \text{id}_X)$ ; so, suppressing the canonical isomorphism  $\Delta^0 \times X \cong X$ , we obtain the required equation:

$$Ff_0 = F\tilde{\alpha} \circ F(\delta_1^1 \times \text{id}_X) = F\tilde{\alpha} \circ F(\delta_1^0 \times \text{id}_X) = Ff_1$$

(iv)  $\Leftrightarrow$  (v). This is the universal property of the quotient by the congruence of intrinsic homotopy.

(v)  $\Rightarrow$  (iii). Since  $\sigma_0^0 \times \text{id}_X : \Delta^1 \times X \rightarrow \Delta^0 \times X$  is (half of) an intrinsic homotopy equivalence,  $\gamma(\sigma_0^0 \times \text{id}_X)$  is an isomorphism in  $\text{Ho}_{\Delta^1} \mathbf{sSet}$ . Hence, if the functor  $F : \mathbf{sSet} \rightarrow \mathcal{C}$  factors through  $\gamma : \mathbf{sSet} \rightarrow \text{Ho}_{\Delta^1} \mathbf{sSet}$ ,  $F(\sigma_0^0 \times \text{id}_X)$  must be an isomorphism in  $\mathcal{C}$ .  $\blacksquare$

**Corollary 1.3.14.** *The functor  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  factors through  $\gamma : \mathbf{sSet} \rightarrow \text{Ho}_{\Delta^1} \mathbf{sSet}$ .*

*Proof.* By [proposition 1.2.4](#),  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  preserves finite products; but  $\pi_0 \Delta^1 \cong \pi_0 \Delta^0 \cong 1$ , so  $\pi_0(\sigma_0^0 \times \text{id}_X) : \pi_0(\Delta^1 \times X) \rightarrow \pi_0(\Delta^0 \times X)$  is a bijection for any simplicial set  $X$ .  $\blacksquare$

**Definition 1.3.15.** Let  $X$  be a simplicial set.

- A **forward contracting homotopy** for  $X$  consists of a set  $X_{-1}$  and maps  $r : X_0 \rightarrow X_{-1}$ ,  $s : X_{-1} \rightarrow X_0$ , and  $h^n : X_n \rightarrow X_{n+1}$  satisfying these identities:

$$\begin{aligned} r \circ d_1^1 &= r \circ d_0^1 \\ r \circ s &= \text{id} \\ d_0^1 \circ h^0 &= s \circ r \\ d_1^1 \circ h^0 &= \text{id} \\ d_i^{n+1} \circ h^n &= h^{n-1} \circ d_i^n && \text{if } 0 \leq i \leq n \\ d_{n+1}^{n+1} \circ h^n &= \text{id} \\ h^{n+1} \circ s_i^n &= s_i^{n+1} \circ h^n && \text{if } 0 \leq i \leq n \\ h^{n+1} \circ h^n &= s_{n+1}^{n+1} \circ h^n \end{aligned}$$

- A **backward contracting homotopy** for  $X$  consists of a set  $X_{-1}$  and maps  $r : X_0 \rightarrow X_{-1}$ ,  $s : X_{-1} \rightarrow X_0$ , and  $h^n : X_n \rightarrow X_{n+1}$  satisfying these identities:

$$\begin{aligned}
 r \circ d_1^1 &= r \circ d_0^1 \\
 r \circ s &= \text{id} \\
 d_0^1 \circ h^0 &= \text{id} \\
 d_1^1 \circ h^0 &= s \circ r \\
 d_0^{n+1} \circ h^n &= \text{id} \\
 d_{i+1}^{n+1} \circ h^n &= h^{n-1} \circ d_i^n && \text{if } 0 \leq i \leq n \\
 h^{n+1} \circ h^n &= s_0^{n+1} \circ h^n \\
 h^{n+1} \circ s_i^n &= s_{i+1}^{n+1} \circ h^n && \text{if } 0 \leq i \leq n
 \end{aligned}$$

**Proposition 1.3.16.** *Let  $X$  be a simplicial set. If  $X$  admits a forward or backward contracting homotopy, then the canonical map  $X \rightarrow \text{disc } \pi_0 X$  has an intrinsic homotopy inverse.*

*Proof.* The two cases are similar; we will assume that  $X$  has a forward contracting homotopy. First, notice that the definition implies that we have the following absolute coequaliser diagram:

$$\begin{array}{ccccc}
 X_1 & \xrightarrow{d_1^1} & X_0 & \xrightarrow{r} & X_{-1} \\
 & \xleftarrow{d_0^1} & & \xleftarrow{s} & \\
 & \xleftarrow{h^0} & & & 
 \end{array}$$

Thus, as remarked in the proof of [proposition 1.2.4](#),  $\pi_0 X \cong X_{-1}$ . As always, there is a unique morphism  $s' : \text{disc } X_{-1} \rightarrow X$  whose degree 0 component is  $s : X_{-1} \rightarrow X_0$ , and the above observation ensures that there also exists a unique morphism  $r' : X \rightarrow \text{disc } X_{-1}$  whose degree 0 component is  $r : X_0 \rightarrow X_{-1}$ .

Clearly,  $r' \circ s' = \text{id}_{\text{disc } X_{-1}}$ ; we must show that  $s' \circ r' \sim \text{id}_X$ . Let  $\chi_n^i : [n] \rightarrow [1]$  denote the map in  $\mathbf{\Delta}$  defined below:

$$\chi_n^i(j) = \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \geq i \end{cases}$$

It is not hard to see that  $\mathbf{\Delta}([n], [1]) = \{\chi_n^i \mid 0 \leq i \leq n+1\}$ , and moreover we have the following identities:

$$\chi_{n+1}^i \circ \delta_{n+1}^j = \chi_n^i \quad \text{if } 0 \leq i \leq j \leq n+1$$

$$\begin{aligned}
 \chi_{n+1}^j \circ \delta_{n+1}^i &= \chi_n^{j-1} && \text{if } 0 \leq i < j \leq n+2 \\
 \chi_n^i \circ \sigma_n^j &= \chi_{n+1}^i && \text{if } 0 \leq i \leq j \leq n \\
 \chi_n^j \circ \sigma_n^i &= \chi_{n+1}^{j+1} && \text{if } 0 \leq i < j \leq n+1
 \end{aligned}$$

We construct by recursion a sequence of maps  $H_n : X_n \times \Delta([n], [1]) \rightarrow X_n$ :

- For all  $x$  in  $X_0$ :

$$\begin{aligned}
 H_0(x, \chi_0^1) &= x \\
 H_0(x, \chi_0^0) &= s(r(x))
 \end{aligned}$$

- For each  $x$  in  $X_{n+1}$ :

$$\begin{aligned}
 H_{n+1}(x, \chi_{n+1}^{n+2}) &= x \\
 H_{n+1}(x, \chi_{n+1}^{n+1}) &= h^n(d_{n+1}^{n+1}(x)) \\
 H_{n+1}(x, \chi_{n+1}^j) &= s_n^n(H_n(d_{n+1}^{n+1}(x), \chi_n^j)) \quad \text{for } 0 \leq j \leq n
 \end{aligned}$$

It is straightforward to check that these equations hold,

$$d_0^1 \circ H_1 = H_0 \circ d_0^1 \quad d_1^1 \circ H_1 = H_0 \circ d_1^1 \quad s_0^0 \circ H_0 = H_1 \circ s_0^0$$

so we assume for induction that these identities hold for some  $n > 0$ :

$$\begin{aligned}
 d_i^k \circ H_k &= H_{k-1} \circ d_i^k && \text{for } 0 < k \leq n, 0 \leq i \leq k \\
 s_i^k \circ H_k &= H_{k+1} \circ s_i^k && \text{for } 0 \leq k < n, 0 \leq i \leq k
 \end{aligned}$$

Then, for  $0 \leq i \leq n+1$ ,

$$\begin{aligned}
 d_i^{n+1}(H_{n+1}(x, \chi_{n+1}^{n+2})) &= d_i^{n+1}(x) \\
 &= H_n(d_i^{n+1}(x), \chi_n^{n+1}) = H_n(d_i^{n+1}(x), \chi_{n+1}^{n+2} \circ \delta_{n+1}^i)
 \end{aligned}$$

and, for  $0 \leq i \leq n$ ,

$$\begin{aligned}
 d_i^{n+1}(H_{n+1}(x, \chi_{n+1}^{n+1})) &= d_i^{n+1}(h^n(d_{n+1}^{n+1}(x))) \\
 &= h^{n-1}(d_i^n(d_{n+1}^{n+1}(x))) = h^{n-1}(d_n^n(d_i^{n+1}(x))) \\
 &= H_n(d_i^{n+1}(x), \chi_n^n) = H_n(d_i^{n+1}(x), \chi_n^n \circ \delta_{n+1}^i)
 \end{aligned}$$

while, for  $i = n+1$ :

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$$\begin{aligned} d_{n+1}^{n+1}(H_{n+1}(x, \chi_{n+1}^{n+1})) &= d_{n+1}^{n+1}(h^n(d_{n+1}^{n+1}(x))) \\ &= d_{n+1}^{n+1}(x) = H_n(d_{n+1}^{n+1}(x), \chi_n^{n+1}) = H_n(d_{n+1}^{n+1}(x), \chi_{n+1}^{n+1} \circ \delta_{n+1}^{n+1}) \end{aligned}$$

Similarly, for  $0 \leq j < n$ ,

$$\begin{aligned} d_{n+1}^{n+1}(H_{n+1}(x, \chi_{n+1}^j)) &= d_{n+1}^{n+1}(s_n^n(H_n(d_{n+1}^{n+1}(x), \chi_n^j))) \\ &= H_n(d_{n+1}^{n+1}(x), \chi_n^j) H_n(d_{n+1}^{n+1}(x), \chi_{n+1}^j \circ \delta_{n+1}^{n+1}) \end{aligned}$$

$$\begin{aligned} d_n^{n+1}(H_{n+1}(x, \chi_{n+1}^j)) &= d_n^{n+1}(s_n^n(H_n(d_{n+1}^{n+1}(x), \chi_n^j))) = H_n(d_{n+1}^{n+1}(x), \chi_n^j) \\ &= s_{n-1}^{n-1}(H_{n-1}(d_n^n(d_{n+1}^{n+1}(x)), \chi_{n-1}^j)) = s_{n-1}^{n-1}(H_{n-1}(d_n^n(d_{n+1}^{n+1}(x)), \chi_{n-1}^j)) \\ &= H_n(d_{n+1}^{n+1}(x), \chi_n^j) = H_n(d_{n+1}^{n+1}(x), \chi_{n+1}^j \circ \delta_{n+1}^j) \end{aligned}$$

and for  $0 \leq i < n$ , we have:

$$\begin{aligned} d_i^{n+1}(H_{n+1}(x, \chi_{n+1}^j)) &= d_i^{n+1}(s_n^n(H_n(d_{n+1}^{n+1}(x), \chi_n^j))) \\ &= s_{n-1}^{n-1}(d_i^n(H_n(d_{n+1}^{n+1}(x), \chi_n^j))) = s_{n-1}^{n-1}(H_{n-1}(d_i^n(d_{n+1}^{n+1}(x)), \chi_n^j \circ \delta_n^i)) \\ &= s_{n-1}^{n-1}(H_{n-1}(d_n^n(d_{n+1}^{n+1}(x)), \chi_n^j \circ \delta_n^i)) = H_n(d_i^{n+1}(x), \chi_{n+1}^j \circ \delta_{n+1}^i) \end{aligned}$$

On the other hand, for  $0 \leq i \leq n$ ,

$$s_i^n(H_n(x, \chi_n^{n+1})) = s_i^n(x) = H_{n+1}(s_i^n(x), \chi_{n+1}^{n+2}) = H_{n+1}(s_i^n(x), \chi_{n+1}^{n+1} \circ \sigma_n^i)$$

and for  $0 \leq i < n$ ,

$$\begin{aligned} s_i^n(H_n(x, \chi_n^n)) &= s_i^n(h^{n-1}(d_n^n(x))) \\ &= h^n(s_i^{n-1}(d_n^n(x))) = h^n(d_{n+1}^{n+1}(s_i^n(x))) \\ &= H_{n+1}(s_i^n(x), \chi_{n+1}^{n+1}) = H_{n+1}(s_i^n(x), \chi_n^n \circ \sigma_n^i) \end{aligned}$$

while for  $i = n$ , we have:

$$\begin{aligned} s_n^n(H_n(x, \chi_n^n)) &= s_n^n(H_n(d_{n+1}^{n+1}(s_n^n(x)), \chi_n^n)) \\ &= H_{n+1}(s_n^n(x), \chi_{n+1}^n) = H_{n+1}(s_n^n(x), \chi_n^n \circ \sigma_n^n) \end{aligned}$$

Finally, for  $0 \leq i \leq n$  and  $0 \leq j < n$ :



$$\begin{aligned}
 s_i^n \left( H_n(x, \chi_n^j) \right) &= s_i^n \left( s_{n-1}^{n-1} \left( H_{n-1} \left( d_n^n(x), \chi_{n-1}^j \right) \right) \right) \\
 &= s_n^n \left( s_i^{n-1} \left( H_{n-1} \left( d_n^n(x), \chi_{n-1}^j \right) \right) \right) = s_n^n \left( H_n \left( s_i^{n-1} \left( d_n^n(x) \right), \chi_{n-1}^j \circ \sigma_{n-1}^i \right) \right) \\
 &= s_n^n \left( H_n \left( d_{n+1}^{n+1} \left( s_i^n(x) \right), \chi_{n-1}^j \circ \sigma_{n-1}^i \right) \right) = H_{n+1} \left( s_i^n(x), \chi_n^j \circ \sigma_n^i \right)
 \end{aligned}$$

We therefore have a morphism  $H : X \times \Delta^1 \rightarrow X$  such that  $H \circ (\text{id}_X \times \delta_1^0) = s' \circ r'$  and  $H \circ (\text{id}_X \times \delta_1^1) = \text{id}_X$ , as required.  $\blacksquare$

**Definition 1.3.17.** Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be morphisms in  $\mathbf{sSet}$ .

- $f$  has the **forward homotopy lifting property** with respect to  $g$  if, for every commutative diagram of the following form,

$$\begin{array}{ccc}
 Z & \xrightarrow{z_0} & X \\
 g \downarrow & \nearrow h_0 & \downarrow f \\
 W & \xrightarrow{w_0} & Y
 \end{array}$$

given intrinsic homotopies  $\alpha : w_0 \Rightarrow w_1$  and  $\beta : z_0 \Rightarrow z_1$  such that  $\alpha \circ \text{id}_g = \text{id}_f \circ \beta$ , there exist a morphism  $h_1 : W \rightarrow X$  and an intrinsic homotopy  $\gamma : h_0 \Rightarrow h_1$  such that  $f \circ h_1 = w_1$ ,  $h_1 \circ g = z_1$ ,  $\text{id}_f \circ \gamma = \alpha$ , and  $\gamma \circ \text{id}_g = \beta$ .

- $f$  has the **backward homotopy lifting property** with respect to  $g$  if, for every commutative diagram of the following form,

$$\begin{array}{ccc}
 Z & \xrightarrow{z_1} & X \\
 g \downarrow & \nearrow h_1 & \downarrow f \\
 W & \xrightarrow{w_1} & Y
 \end{array}$$

given intrinsic homotopies  $\alpha : w_0 \Rightarrow w_1$  and  $\beta : z_0 \Rightarrow z_1$  such that  $\alpha \circ \text{id}_g = \text{id}_f \circ \beta$ , there exist a morphism  $h_0 : W \rightarrow X$  and an intrinsic homotopy  $\gamma : h_0 \Rightarrow h_1$  such that  $f \circ h_0 = w_0$ ,  $h_0 \circ g = z_0$ ,  $\text{id}_f \circ \gamma = \alpha$ , and  $\gamma \circ \text{id}_g = \beta$ .

- $f$  has the **intrinsic homotopy lifting property** with respect to  $g$  if  $f$  has *both* the forward and backward homotopy lifting properties with respect to  $g$ .

## I. SIMPLICIAL SETS

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- $f$  has the **forward** (resp. **backward**, **intrinsic**) **homotopy lifting property** with respect to the object  $W$  if  $f$  has the forward (resp. backward, intrinsic) homotopy lifting property with respect to the unique morphism  $0 \rightarrow W$ .
- $g$  has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to  $f$  if  $f$  has the forward (resp. backward, intrinsic) homotopy lifting property with respect to  $g$ .
- $g$  has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to the object  $X$  if  $g$  has the forward (resp. backward, intrinsic) homotopy extension property with respect to the unique morphism  $X \rightarrow 1$ .

**Proposition 1.3.18.** *Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be morphisms in  $\mathbf{sSet}$ , and suppose we have a commutative diagram*

$$\begin{array}{ccc}
 [W, X] & \xrightarrow{[g, X]} & [Z, X] \\
 \downarrow [W, f] & \searrow q & \downarrow [Z, f] \\
 & L(g, f) \longrightarrow & [Z, X] \\
 & \downarrow & \downarrow [Z, f] \\
 [W, Y] & \xrightarrow{[g, Y]} & [Z, Y]
 \end{array}$$

where the square in the lower right is a pullback square. The following are equivalent:

- $f$  has the forward homotopy lifting property with respect to  $g$ .
- The morphism  $q : [W, X] \rightarrow L(g, f)$  has the right lifting property with respect to the horn inclusion  $\Lambda_0^1 \hookrightarrow \Delta^1$ .
- Suppose we have a commutative diagram

$$\begin{array}{ccc}
 \Lambda_0^1 \times Z & \hookrightarrow & \Delta^1 \times Z \\
 \text{id}_{\Lambda_0^1} \times g \downarrow & & \downarrow \\
 \Lambda_0^1 \times W & \longrightarrow & V_0(g) \\
 & \searrow & \downarrow \text{id}_{\Delta^1} \times g \\
 & & \Delta^1 \times W
 \end{array}$$

where the square in the upper left is a pushout square. Then the morphism  $j : V_0(g) \rightarrow \Delta^1 \times W$  has the left lifting property with respect to  $f : X \rightarrow Y$ .

Symmetrically, the following are equivalent:

- (i')  $f$  has the backward homotopy lifting property with respect to  $g$ .
- (ii') The morphism  $q : [W, X] \rightarrow L(g, f)$  has the right lifting property with respect to the horn inclusion  $\Lambda_1^1 \hookrightarrow \Delta^1$ .
- (iii') Suppose we have a commutative diagram

$$\begin{array}{ccc}
 \Lambda_1^1 \times Z & \hookrightarrow & \Delta^1 \times Z \\
 \text{id}_{\Lambda_1^1} \times g \downarrow & & \downarrow \\
 \Lambda_1^1 \times W & \longrightarrow & V_1(g) \\
 & \searrow & \downarrow j \\
 & & \Delta^1 \times W
 \end{array}$$

$\text{id}_{\Delta^1} \times g$  (curved arrow from  $\Delta^1 \times Z$  to  $\Delta^1 \times W$ )  
 $j$  (dashed arrow from  $V_1(g)$  to  $\Delta^1 \times W$ )

where the square in the upper left is a pushout square. Then the morphism  $j : V_1(g) \rightarrow \Delta^1 \times W$  has the left lifting property with respect to  $f : X \rightarrow Y$ .

*Proof.* This is a special case of [proposition 5.4.1](#): use [remark 1.3.2](#) and the exponential adjunction. ■

**Definition 1.3.19.** A **horn** is a simplicial subset of the form  $\Lambda_k^n \subseteq \Delta^n$ , where  $\Lambda_k^n$  is the union of the images of  $\delta_n^0, \dots, \delta_n^{k-1}, \delta_n^{k+1}, \dots, \delta_n^n : \Delta^{n-1} \rightarrow \Delta^n$  in **sSet**. In other words,  $\Lambda_k^n$  is the union of all the faces of  $\Delta^n$  that include the  $k$ -th vertex.

**Theorem 1.3.20.** Let  $p : X \rightarrow Y$  be a morphism in **sSet**. The following are equivalent:

- (i)  $p : X \rightarrow Y$  has the right lifting property with respect to the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  (for all  $n \geq 1$  and  $0 \leq k \leq n$ ).
- (ii)  $p : X \rightarrow Y$  has the intrinsic homotopy lifting property with respect to the boundary inclusions  $\partial \Delta^n \hookrightarrow \Delta^n$  (for all  $n \geq 0$ ).
- (iii)  $p : X \rightarrow Y$  has the intrinsic homotopy lifting property with respect to any monomorphism in **sSet**.

*Proof.* Combine propositions 1.3.18 and A.3.14 with either Theorem 2.1 in [GZ, Ch. IV] or Proposition 4.2 in [GJ, Ch. I].  $\square$

REMARK. The analogous theorem for cubical sets was announced as Theorem 2 in [Kan, 1955].

REMARK 1.3.21. Let  $B^n$  be the closed unit ball in the euclidean space  $\mathbb{R}^n$ , let  $\partial B^n$  be its boundary, and let  $I$  be the closed unit interval  $[0, 1]$ . It is not hard to see that the inclusion  $B^n \times \{0\} \hookrightarrow B^n \times I$  is isomorphic to the inclusion  $B^n \times \{0\} \cup \partial B^n \times I \hookrightarrow B^n \times I$ . Thus, a continuous map  $p : X \rightarrow Y$  has the homotopy lifting property with respect to all  $B^n$  if and only if it has the homotopy lifting property with respect to all boundary inclusions  $\partial B^n \hookrightarrow B^n$ .

Unfortunately,  $\mathbf{sSet}$  does not have the analogous property. Indeed, for any simplicial set  $X$ , the unique morphism  $X \rightarrow 1$  has the intrinsic homotopy lifting property with respect to the  $n$ -simplices  $\Delta^n$ , but it need not have the intrinsic right lifting property with respect to all boundary inclusions  $\partial \Delta^n \hookrightarrow \Delta^n$ .

**Lemma 1.3.22.** *Let  $p : X \rightarrow Y$  be a morphism in  $\mathbf{sSet}$ .*

- (i) *If  $p : X \rightarrow Y$  has the right lifting property with respect to the boundary inclusion  $\partial \Delta^0 \hookrightarrow \Delta^0$ , then  $p_0 : X_0 \rightarrow Y_0$  is surjective.*
- (ii) *If  $p : X \rightarrow Y$  has the right lifting property with respect to the boundary inclusions  $\partial \Delta^0 \hookrightarrow \Delta^0$  and  $\partial \Delta^1 \hookrightarrow \Delta^1$ , then  $p : X \rightarrow Y$  has the intrinsic homotopy lifting property with respect to  $\Delta^0$ , and  $\pi_0 p : \pi_0 X \rightarrow \pi_0 Y$  is a bijection.*

*Proof.* (i). Let  $y$  be a vertex of  $Y$ . Then the right lifting property of  $p : X \rightarrow Y$  with respect to the boundary inclusion  $\partial \Delta^0 \hookrightarrow \Delta^0$  yields a vertex  $x$  of  $X$  such that  $p_0(x) = y$ , as required.

(ii). By proposition 1.3.18,  $p : X \rightarrow Y$  has the intrinsic homotopy lifting property with respect to  $\Delta^0$  if and only if it has the right lifting property with respect to the horn inclusions  $\Lambda_0^1 \hookrightarrow \Delta^1$  and  $\Lambda_1^1 \hookrightarrow \Delta^1$ . Since  $\Delta^1$  is 1-skeletal, we may apply propositions 1.2.20 and A.3.14 to deduce that  $p : X \rightarrow Y$  does indeed have the aforementioned right lifting properties.

It remains to be shown that  $\pi_0 : \pi_0 X \rightarrow \pi_0 Y$  is a bijection. We already know that  $\pi_0 p : \pi_0 X \rightarrow \pi_0 Y$  is a surjection, so it suffices to show that it is also injective. Let  $x_0$  and  $x_1$  be vertices of  $X$  such that  $y_0 = p(x_0)$  and  $y_1 = p(x_1)$

are in the same connected component. We proceed by induction on the length of the shortest path (i.e. zigzag of edges) in  $Y$  connecting  $y_0$  and  $y_1$ .

If  $y_0$  and  $y_1$  are connected by an edge of  $Y$ , then we may use the right lifting property of  $p : X \rightarrow Y$  with respect to the boundary inclusion  $\partial\Delta^1 \hookrightarrow \Delta^1$  to find an edge of  $X$  connecting  $x_0$  and  $x_1$ . Otherwise, we use the intrinsic homotopy lifting property of  $p : X \rightarrow Y$  with respect to  $\Delta^0$  to reduce to the case where  $y_0$  and  $y_1$  are connected by a strictly shorter path.  $\blacksquare$

**Lemma 1.3.23.** *Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be monomorphisms in  $\mathbf{sSet}$ . Suppose the square in the diagram below is a pushout square in  $\mathbf{sSet}$ :*

$$\begin{array}{ccc}
 X \times Z & \xrightarrow{\text{id}_X \times g} & X \times W \\
 f \times \text{id}_Z \downarrow & & \downarrow \\
 Y \times Z & \xrightarrow{\quad} & (X \times W) \cup^{X \times Z} (Y \times Z) \\
 & \searrow & \downarrow f \boxtimes g \\
 & & Y \times W \\
 & \nearrow \text{id}_Y \times g & \\
 & & 
 \end{array}$$

- (i) *The morphism  $f \boxtimes g : (X \times W) \cup^{X \times Z} (Y \times Z) \rightarrow Y \times W$  is a monomorphism.*
- (ii) *If at least one of  $f : X \rightarrow Y$  or  $g : Z \rightarrow W$  is an anodyne extension, then so is  $f \boxtimes g : (X \times W) \cup^{X \times Z} (Y \times Z) \rightarrow Y \times W$ .*

*Proof.* (i). Using the fact that limits and colimits in  $\mathbf{sSet}$  can be calculated degreewise, this reduces to a well-known fact about  $\mathbf{Set}$ .

(ii). See Proposition 2.2 in [GZ], or Corollary 4.6 in [GJ, Ch. I].  $\square$

**Definition 1.3.24.** Let  $L$  be any simplicial set, let  $K$  be a simplicial subset of  $L$ , and let  $f_0, f_1 : L \rightarrow Y$  be a parallel pair of morphisms in  $\mathbf{sSet}$ . An **intrinsic homotopy**  $f_0 \Rightarrow f_1$  relative to  $K$  is an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$  such that the image of  $\alpha$  under morphism  $[L, Y] \rightarrow [K, Y]$  (induced by the inclusion  $K \hookrightarrow L$ ) is a degenerate edge.

**Definition 1.3.25.** Let  $L$  be any simplicial set and let  $K$  be a simplicial subset of  $L$ . The **relative cylinder** on  $(L, K)$  is the simplicial set  $C(L, K)$  defined by

the following pushout diagram,

$$\begin{array}{ccc} K \times \Delta^1 & \hookrightarrow & L \times \Delta^1 \\ \downarrow & & \downarrow q \\ K & \longrightarrow & C(L, K) \end{array}$$

where  $K \times \Delta^1 \rightarrow K$  is the projection and  $K \times \Delta^1 \hookrightarrow L \times \Delta^1$  is induced by the inclusion  $K \hookrightarrow L$ .

**REMARK 1.3.26.** Let  $j_0, j_1 : L \rightarrow C(L, K)$  be the morphisms obtained by composing with  $q : L \times \Delta^1 \rightarrow C(L, K)$  the two morphisms  $L \rightarrow L \times \Delta^1$  induced by the two vertex inclusions  $\Delta^0 \rightarrow \Delta^1$ . There is a natural bijection between the set of intrinsic homotopies  $f_0 \Rightarrow f_1$  and the set of morphisms  $h : C(L, K) \rightarrow Y$  such that  $h \circ j_0 = f_0$  and  $h \circ j_1 = f_1$ .

**Definition 1.3.27.** Let  $f : X \rightarrow Y$  be a morphism in **sSet**. With other notation as above, we say that  $f$  has the **homotopical right lifting property** with respect to  $K \hookrightarrow L$  if, for each commutative diagram the form below,

$$\begin{array}{ccc} K & \xrightarrow{\partial x} & X \\ \downarrow & & \downarrow f \\ L & \xrightarrow{y} & Y \end{array}$$

there exist a morphism  $x : L \rightarrow X$  and a homotopy  $\alpha : y \Rightarrow f \circ x$  relative to  $K$ , or equivalently, morphisms  $x : L \rightarrow X$  and  $h : C(L, K) \rightarrow Y$  making the following diagram commute:

$$\begin{array}{ccccc} & & \xrightarrow{\partial x} & & \\ & & \curvearrowright & & \\ K & \hookrightarrow & L & \dashrightarrow^x & X \\ \downarrow & & \downarrow j_1 & & \downarrow f \\ L & \xrightarrow{j_0} & C(L, K) & \dashrightarrow^h & Y \\ & & \curvearrowleft & & \\ & & \xrightarrow{y} & & \end{array}$$

**Proposition 1.3.28.** Let  $f : X \rightarrow Y$  be a morphism in **sSet** and let  $\mathcal{A}$  be the class of pairs  $(L, K)$  such that  $f$  has the homotopical right lifting property with respect to  $K \hookrightarrow L$ .

- (i)  $\mathcal{A}$  is closed under coproducts for small families.

(ii)  $\mathcal{A}$  is closed under pushout.

(iii)  $\mathcal{A}$  is closed under retracts.

*Proof.* See Lemma 3.4 in [Dugger and Isaksen, 2004]. □

## I.4 Kan complexes

**Prerequisites.** §§ I.3, 3.I, A.4.

We have seen in the previous section that the notion of intrinsic homotopy is not well behaved for general simplicial sets. To remedy this, we shall (temporarily) restrict our attention to Kan complexes. These are simplicial sets with the so-called “extension property”, and they are named in honour of Kan [1955], who first observed the importance of the aforementioned property.

**Definition I.4.1.** A **Kan fibration** is a morphism  $f : X \rightarrow Y$  in **sSet** that has the right lifting property with respect to the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$ , where  $n \geq 1$  and  $0 \leq k \leq n$ . A **Kan complex** is a simplicial set  $X$  such that the unique morphism  $X \rightarrow 1$  is a Kan fibration.

**REMARK I.4.2.** In other words, a Kan complex is a simplicial set  $X$  satisfying the **Kan condition**: every horn  $\alpha' : \Lambda_k^n \rightarrow X$  has a **filler**, i.e. a morphism  $\alpha : \Delta^n \rightarrow X$  (equivalently, an  $n$ -simplex of  $X$ ) such that  $\alpha'$  is the restriction along the inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ .

**Proposition I.4.3.** *Let  $X$  be a simplicial set. The following are equivalent:*

- (i)  $X$  is a Kan complex.
- (ii)  $X$  has the intrinsic homotopy extension property with respect to the boundary inclusions  $\partial\Delta^n \hookrightarrow \Delta^n$ .
- (iii)  $X$  has the intrinsic homotopy extension property with respect to any monomorphism in **sSet**.

*Proof.* This is a special case of [theorem I.3.20](#). ■

**Lemma I.4.4.** *If  $X$  is a Kan complex, then the fundamental category  $\tau_1 X$  is a groupoid, and the unit  $\eta_X : X \rightarrow \mathbf{N}(\tau_1 X)$  is an epimorphism.*

*Proof.* Let  $x, y,$  and  $z$  be vertices in  $X$ , and let  $f : x \rightarrow y$  and  $g : y \rightarrow z$  be edges in  $X$ .<sup>[6]</sup> Then the pair  $(f, g)$  defines a horn  $\Lambda_1^2 \rightarrow X$ , and so by the Kan condition, there exists a 2-simplex  $\alpha$  of  $X$  such that  $d_2(\alpha) = f$  and  $d_0(\alpha) = g$ . By remark [remark 1.2.3](#), the composite  $g \bullet f$  defined in  $\tau_1 X$  must correspond to the edge  $d_1(\alpha)$ . Since the arrows in  $\tau_1 X$  are generated by the edges of  $X$ , we conclude by induction that  $\eta_X : X \rightarrow N(\tau_1 X)$  is a surjection on vertices and edges.

Similarly, given an edge  $f : x \rightarrow y$ , the Kan condition ensures that there exist two 2-simplices  $\beta$  and  $\gamma$  such that

$$\begin{array}{ll} d_2(\alpha) = f & d_1(\alpha) = \text{id}_x \\ d_0(\alpha) = f & d_1(\alpha) = \text{id}_y \end{array}$$

where  $\text{id}_x : x \rightarrow x$  is the edge  $s_0(x)$ , and  $\text{id}_y : y \rightarrow y$  is the edge  $s_0(y)$ . Together with the argument in the previous paragraph, this shows that  $\tau_1 X$  is a groupoid.

Finally, to show that  $\eta_X : X \rightarrow N(\tau_1 X)$  is a surjection on  $n$ -simplices for  $n \geq 2$ , we simply observe that an  $n$ -simplex of  $N(\tau_1 X)$  is just a string of  $n$  composable edges of  $X$ , so we may appeal to the Kan condition again to obtain the corresponding  $n$ -simplex of  $X$ . ■

**Corollary 1.4.5.** *If  $X$  is a Kan complex, then the unit  $\eta_X : X \rightarrow N(\pi_1 X)$  is an epimorphism.*

*Proof.* Since  $\tau_1 X$  is already a groupoid, the canonical functor  $\tau_1 X \rightarrow \pi_1 X$  must be an isomorphism. (See [remark 1.2.9](#).) ■

**Proposition 1.4.6.** *Let  $X$  be a Kan complex and let  $\alpha_0, \alpha_1 : x_0 \rightarrow x_1$  be edges in  $X$ . The following are equivalent:*

- (i)  $\alpha_0 = \alpha_1$  in the fundamental groupoid  $\pi_1 X$ .
- (ii) There exists a 2-simplex  $\sigma$  of  $X$  such that  $d_0(\sigma) = s_0(x_1)$ ,  $d_1(\sigma) = \alpha_1$ , and  $d_2(\sigma) = \alpha_0$ .
- (iii) There exists an edge  $\beta : \alpha_0 \rightarrow \alpha_1$  in the exponential object  $[\Delta^1, X]$  such that  $[\delta^1, X](\beta) = s_0(x_0)$  and  $[\delta^0, X](\beta) = s_0(x_1)$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). See Proposition 1.2.3.9 in [\[HTT\]](#).

(i)  $\Leftrightarrow$  (iii). See paragraph 5.2 in [\[GZ\]](#). □

[6] Recall [definition 1.1.12](#).



**Proposition 1.4.7.** *Let  $\mathcal{I}$  and  $\mathcal{I}'$  be the following subsets of  $\text{mor sSet}$ :*

$$\begin{aligned}\mathcal{I} &= \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\} \\ \mathcal{I}' &= \{\Lambda_k^n \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n\}\end{aligned}$$

- (i) *There exist a pair of functorial factorisation systems on  $\text{sSet}$ , one inducing a weak factorisation system cofibrantly generated by  $\mathcal{I}$ , and the other inducing a weak factorisation system cofibrantly generated by  $\mathcal{I}'$ .*
- (ii) *A morphism is  $\mathcal{I}'$ -injective if and only if it is a Kan fibration, and every  $\mathcal{I}'$ -cofibration is a monomorphism (but not vice versa).*
- (iii) *A morphism is a  $\mathcal{I}$ -cofibration if and only if it is a monomorphism, and every  $\mathcal{I}$ -injective morphism is a Kan fibration (but not vice versa).*

*Proof.* (i). Since  $\text{sSet}$  is a locally finitely presentable category, we may apply Quillen's small object argument ([theorem 0.5.12](#)).

(ii). The definition of 'Kan fibration' is exactly the definition of ' $\mathcal{I}$ -injective morphism'; on the other hand, the class of monomorphisms is closed under push-out, transfinite composition, and retracts in  $\text{Set}$ , so the same is true for  $\text{sSet}$ , and thus, by [corollary 0.5.13](#), every  $\mathcal{I}$ -cofibration must be a monomorphism.

(iii). To prove that  $\text{inj}^{\mathcal{I}} C \supseteq \text{inj}^{\mathcal{I}'} C$ , it is enough to check that  $\mathcal{I} \subseteq \text{cof}_{\mathcal{I}'} C$ . Since every morphism in  $\mathcal{I}$  is a monomorphism, it will suffice to show that  $\text{cell}_{\mathcal{I}'} C$  contains all monomorphisms; but this is an immediate corollary of [proposition 1.2.20](#). ■

**Corollary 1.4.8.** *Let  $i : Z \rightarrow W$  be a morphism in  $\text{sSet}$ . The following are equivalent:*

- (i)  *$i : Z \rightarrow W$  is an anodyne extension.*
- (ii)  *$i : Z \rightarrow W$  has the left lifting property with respect to any Kan fibration.*
- (iii)  *$i : Z \rightarrow W$  is a retract of a relative  $\mathcal{I}'$ -cell complex.*

*Proof.* (i)  $\Rightarrow$  (ii). Apply [proposition A.3.14](#).

(ii)  $\Rightarrow$  (iii). This is a special case of [corollary 0.5.13](#).

(iii)  $\Rightarrow$  (i). By definition, the class of anodyne extensions is closed under pushout, transfinite composition, and retracts. ■

**Definition 1.4.9.** A **trivial Kan fibration** is a morphism in  $\mathbf{sSet}$  that has the right lifting property with respect to the boundary inclusions  $\partial\Delta^n \hookrightarrow \Delta^n$ , where  $n \geq 0$ .

**REMARK 1.4.10.** **Proposition 1.4.7** implies that a trivial Kan fibration is the same thing as a morphism in  $\mathbf{sSet}$  that has the right lifting property with respect to any monomorphism. In particular, trivial Kan fibrations are Kan fibrations.

**Proposition 1.4.11.** *If  $p : X \rightarrow Y$  is a trivial Kan fibration, then  $p : X \rightarrow Y$  is fibrewise contractible, i.e. there exist a morphism  $s : Y \rightarrow X$  and an intrinsic homotopy  $\alpha : \text{id}_X \Rightarrow s \circ p$  satisfying the following conditions:*

- $p \circ s = \text{id}_Y$ .
- $\alpha s$  is the trivial homotopy  $s \Rightarrow s$ .
- $p\alpha$  is the trivial homotopy  $p \Rightarrow p$ .

Moreover, given a monomorphism  $i : Y' \rightarrow Y$  and any morphism  $s' : Y' \rightarrow X$  such that  $p \circ s' = i'$ , the morphism  $s : Y \rightarrow X$  given above may be chosen so that  $s \circ i = s'$ .

*Proof.* Since  $i : Y' \rightarrow Y$  is a monomorphism, the right lifting property of  $p : X \rightarrow Y$  yields a morphism  $s : Y \rightarrow X$  such that  $p \circ s = \text{id}_Y$  and  $s \circ i = s'$ . We then obtain a commutative diagram of the form below,

$$\begin{array}{ccc} X \cup^Y X & \xrightarrow{(\text{id}_X, s \circ p)} & X \\ (j_0, j_1) \downarrow & & \downarrow p \\ C(X, Y) & \xrightarrow{p \circ r} & Y \end{array}$$

where  $C(X, Y)$  is the relative cylinder,  $X \cup^Y X$  is the pushout of  $s : Y \rightarrow X$  along itself, the morphisms  $j_0, j_1 : X \rightarrow C(X, Y)$  are defined as in [remark 1.3.26](#), and  $r : C(X, Y) \rightarrow X$  is defined by the following commutative diagram:

$$\begin{array}{ccc} Y \times \Delta^1 & \xrightarrow{s \times \text{id}_{\Delta^1}} & X \times \Delta^1 \\ \text{proj} \downarrow & & \downarrow \text{proj} \\ Y & \xrightarrow{\quad} & C(X, Y) \\ & \searrow s & \downarrow r \\ & & X \end{array}$$

It is not hard to see that  $(j_0, j_1) : X \cup^Y X \rightarrow C(X, Y)$  is a monomorphism, so there must exist  $h : C(X, Y) \rightarrow Y$  making the evident triangles commute. The corresponding intrinsic homotopy  $\text{id}_X \Rightarrow s \circ p$  is then the desired  $\alpha$ . ■

**Proposition 1.4.12.** *Let  $\mathcal{K}$  be the full subcategory of  $\mathbf{sSet}$  spanned by the finite simplicial sets.*

- (i) *The class of monomorphisms that are in  $\mathcal{K}$  is the smallest class containing the boundary inclusions  $\partial\Delta^n \hookrightarrow \Delta^n$  that is closed under composition and pushouts.*
- (ii) *The class of anodyne extensions that are in  $\mathcal{K}$  is the smallest class containing the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  that is closed under composition, pushouts, and retracts.*

*Proof.* (i). This is a corollary of [proposition 1.2.20](#).

(ii). [Corollary 1.4.8](#) says that every anodyne extension in  $\mathbf{sSet}$  is a retract of a relative  $\mathcal{I}'$ -cell complex, where  $\mathcal{I}'$  is the set of all horn inclusions. More precisely, if  $g : Z \rightarrow W$  is an anodyne extension, then there is a commutative diagram in  $\mathbf{sSet}$  of the form below,

$$\begin{array}{ccccc}
 Z & \xrightarrow{\text{id}} & Z & \xrightarrow{\text{id}} & Z \\
 g \downarrow & & i \downarrow & & \downarrow g \\
 W & \xrightarrow{i_W} & W' & \xrightarrow{r_W} & W \\
 & \searrow & \text{id} & \nearrow & \\
 & & & & 
 \end{array}$$

where  $i : Z \rightarrow W'$  is a relative  $\mathcal{I}'$ -cell complex. Suppose  $W$  is a finite simplicial set. [Proposition 1.1.16](#) says that finite simplicial sets are  $\aleph_0$ -compact objects in  $\mathbf{sSet}$ , so by considering a sequential presentation for  $i : Z \rightarrow W'$ , we see that  $g : Z \rightarrow W$  is a retract of some relative  $\mathcal{I}'$ -cell complex that admits an  $\aleph_0$ -small presentation. In particular, if  $Z$  is a finite simplicial set, then so is  $W'$  (by [lemma 0.2.17](#)). Hence, the class of anodyne extensions in  $\mathcal{K}$  is the smallest class containing  $\mathcal{I}'$  that is closed under composition, pushouts, and retracts. ■

**Proposition 1.4.13.** *Let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{sSet}$  and, for each  $n$ -simplex  $\varphi : \Delta^n \rightarrow Y$ , let  $f_\varphi : X_\varphi \rightarrow \Delta^n$  be defined by the pullback diagram in*

**sSet** shown below:

$$\begin{array}{ccc} X_\varphi & \longrightarrow & X \\ f_\varphi \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{\varphi} & Y \end{array}$$

- (i)  $f : X \rightarrow Y$  is a Kan fibration if and only if each  $f_\varphi : X_\varphi \rightarrow \Delta^n$  is a Kan fibration.
- (ii)  $f : X \rightarrow Y$  is a trivial Kan fibration if and only if each  $f_\varphi : X_\varphi \rightarrow \Delta^n$  is a trivial Kan fibration.

*Proof.* This is a straightforward exercise. ◇

**Corollary 1.4.14.**

- (i) The coproduct of a small family of Kan complexes is a Kan complex.
- (ii) The coproduct of a small family of Kan fibrations is a Kan fibration.
- (iii) The coproduct of a small family of trivial Kan fibrations is a trivial Kan fibration.

*Proof.* Given the previous proposition and the fact that coproducts in **sSet** are disjoint and stable under pullback, it suffices to observe that any  $\Delta^n \rightarrow \coprod_{i \in I} Y_i$  must factor through one of the coproduct insertions  $Y_j \rightarrow \coprod_{i \in I} Y_i$ . ■

**Proposition 1.4.15.** *Let  $i : Z \rightarrow W$  be a cofibration in **sSet** and let  $p : X \rightarrow Y$  be a Kan fibration. Suppose we have a commutative diagram*

$$\begin{array}{ccccc} [W, X] & & \xrightarrow{[i, X]} & & [Z, X] \\ & \searrow \text{dashed } q & & \searrow & \\ & & L(i, p) & \longrightarrow & [Z, X] \\ & \searrow [W, p] & \downarrow & & \downarrow [Z, p] \\ & & [W, Y] & \xrightarrow{[i, Y]} & [Z, Y] \end{array}$$

where the square in the lower right is a pullback square.

- (i) The unique morphism  $q : [W, X] \rightarrow L(i, p)$  making the diagram commute is a Kan fibration.

- (ii) If  $i : Z \rightarrow W$  is an anodyne extension, then  $q : [W, X] \rightarrow L(i, p)$  is a trivial Kan fibration.
- (iii) If  $p : Z \rightarrow W$  is a trivial Kan fibration, then so is  $q : [W, X] \rightarrow L(i, p)$ .

*Proof.* This is a special case of [proposition 5.4.1](#): use [lemma 1.3.23](#) and the exponential adjunction. ■

**Corollary 1.4.16.**

- (i) If  $p : X \rightarrow Y$  is a Kan fibration (resp. trivial Kan fibration), then for all simplicial sets  $W$ , the morphism  $[W, p] : [W, X] \rightarrow [W, Y]$  is also a Kan fibration (resp. trivial Kan fibration).
- (ii) If  $i : Z \rightarrow W$  is a cofibration (resp. anodyne extension) and  $X$  is a Kan complex, then the morphism  $[i, X] : [W, X] \rightarrow [Z, X]$  is a Kan fibration (resp. trivial Kan fibration).
- (iii) If  $W$  is any simplicial set and  $X$  is a Kan complex, then  $[W, X]$  is also a Kan complex.

*Proof.* (i). Take  $Z = \emptyset$ ; noting that the canonical morphism  $\emptyset \rightarrow W$  is a cofibration, and that  $[\emptyset, p] : [\emptyset, X] \rightarrow [\emptyset, Y]$  is an isomorphism, the proposition above then implies  $[W, p] : [W, X] \rightarrow [W, Y]$  is a Kan fibration (resp. trivial Kan fibration).

(ii). Take  $Y = 1$ ; since  $[W, 1] \rightarrow [Z, 1]$  is an isomorphism, the proposition above implies  $[i, X] : [W, X] \rightarrow [Z, X]$  is a Kan fibration (resp. trivial Kan fibration).

(iii). Noting that  $[\emptyset, X]$  is a terminal object in  $\mathbf{sSet}$ , we apply claim (ii) to the case  $Z = \emptyset$  to obtain the desired conclusion. ■

**Proposition 1.4.17.** For any simplicial set  $X$  and any Kan complex  $Y$ , the relation  $\rightsquigarrow$  on  $\mathbf{sSet}(X, Y)$  defined by

$$f_0 \rightsquigarrow f_1 \text{ if and only if there exists an intrinsic homotopy } f_0 \Rightarrow f_1$$

is an equivalence relation.

*Proof.* The relation  $\rightsquigarrow$  is certainly reflexive whether or not  $Y$  is a Kan complex. By [corollary 1.4.16](#), the exponential object  $[X, Y]$  is a Kan complex; so recalling [lemma 1.4.4](#), the transitivity of  $\rightsquigarrow$  may be deduced from the fact that the unit  $\eta_{[X, Y]} : [X, Y] \rightarrow N(\tau_1[X, Y])$  is an epimorphism, and the symmetry of  $\rightsquigarrow$  corresponds to the fact that  $\tau_1[X, Y]$  is a groupoid. ■

**Proposition 1.4.18.** *Let  $X$  and  $Y$  be Kan complexes. If  $i : X \rightarrow Y$  is an anodyne extension, then there exist a morphism  $r : Y \rightarrow X$  and an intrinsic homotopy  $\alpha : \text{id}_Y \Rightarrow i \circ r$  satisfying the the following conditions:*

- $r \circ i = \text{id}_X$ .
- $\alpha i$  is the trivial homotopy  $i \Rightarrow i$ .

*Proof.* By hypothesis, the unique morphism  $X \rightarrow 1$  is a Kan fibration, so [corollary 1.4.8](#) implies there is a morphism  $r : Y \rightarrow X$  such that  $r \circ i = \text{id}_X$ . We then obtain the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{c \circ i} & [\Delta^1, Y] \\ i \downarrow & & \downarrow \langle p_0, p_1 \rangle \\ Y & \xrightarrow{\langle \text{id}_Y, i \circ r \rangle} & Y \times Y \end{array}$$

where  $p_0, p_1 : [\Delta^1, Y] \rightarrow Y$  are the morphisms induced by the coface morphisms  $\delta_0^1, \delta_0^0 : \Delta^0 \rightarrow \Delta^1$  (respectively) and  $c : Y \rightarrow [\Delta^1, Y]$  is induced by the codegeneracy morphism  $\sigma_0^0 : \Delta^1 \rightarrow \Delta^0$ . Suppressing a canonical isomorphism  $[\partial\Delta^1, Y] \cong Y \times Y$ , we see that [corollary 1.4.16](#) implies  $\langle p_0, p_1 \rangle : [\Delta^1, Y] \rightarrow Y \times Y$  is a Kan fibration. Thus, there exists a morphism  $h : Y \rightarrow [\Delta^1, Y]$  making the evident triangles commute, and the corresponding intrinsic homotopy  $\text{id}_Y \Rightarrow i \circ r$  is then the desired  $\alpha$ . ■

We will now define the homotopy groups of a Kan complex.

**Definition 1.4.19.** Let  $n$  be a positive integer and let  $X$  be a Kan complex.

- The **based  $n$ -loop fibration** on  $X$  is the Kan fibration  $\Omega^n(X) \rightarrow X$  defined by the following pullback diagram in **sSet**,

$$\begin{array}{ccc} \Omega^n(X) & \longrightarrow & [\Delta^n, X] \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\partial\Delta^n, X] \end{array}$$

where  $[\Delta^n, X] \rightarrow [\partial\Delta^n, X]$  is the Kan fibration induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  and  $X \rightarrow [\partial\Delta^n, X]$  is the morphism induced by  $\partial\Delta^n \rightarrow \Delta^0$ .

- Let  $x$  be a vertex of  $X$ . The **based  $n$ -loop space** of  $(X, x)$  is the Kan complex  $\Omega^n(X, x)$  defined by the following pullback diagram in  $\mathbf{sSet}$ ,

$$\begin{array}{ccc} \Omega^n(X, x) & \longrightarrow & \Omega^n(X) \\ \downarrow & & \downarrow \\ \Delta^0 & \longrightarrow & X \end{array}$$

where  $\Delta^0 \rightarrow X$  is the morphism corresponding to the vertex  $x$ . The  **$n$ -th homotopy group** of  $(X, x)$  is defined by  $\pi_n(X, x) = \pi_0\Omega^n(X, x)$ .

REMARK 1.4.20. In other words,  $\pi_n(X, x)$  is the set of morphisms  $\Delta^n \rightarrow X$  whose restriction along  $\partial\Delta^n \hookrightarrow \Delta^n$  factors through the morphism  $\Delta^0 \rightarrow X$  corresponding to  $x$ , modulo the equivalence relation that identifies two morphisms  $\Delta^n \rightarrow X$  if they are intrinsically homotopic relative to  $\partial\Delta^n$ .

**Proposition 1.4.21.** *Let  $n$  be a positive integer.*

- (i) *The assignment  $(X, x) \mapsto \pi_n(X, x)$  extends to a functor  $\pi_n : \Delta^0/\mathbf{Kan} \rightarrow \mathbf{Grp}$ , and  $\pi_n(X, x)$  is abelian for  $n > 1$ .*
- (ii) *The functor  $\pi_n : \Delta^0/\mathbf{Kan} \rightarrow \mathbf{Grp}$  preserves finite products and colimits for small filtered diagrams.*
- (iii) *Let  $(X, x)$  and  $(Y, y)$  be pointed Kan complexes. If  $f_0, f_1 : (X, x) \rightarrow (Y, y)$  are a parallel pair of morphisms for which there exists an intrinsic homotopy  $f_0 \Rightarrow f_1$  relative to  $x$  (considered as a subcomplex of  $X$ ), then  $\pi_n f_0 = \pi_n f_1$ .*

*Proof.* (i). See Lemma 7.1 and Theorem 7.2 in [GJ, Ch. I]. Functoriality is straightforward.

(ii). It is not hard to check that the functor  $\Omega^n : \Delta^0/\mathbf{Kan} \rightarrow \mathbf{sSet}$  preserves all limits and colimits for small filtered diagrams, and  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  preserves finite products and all colimits by proposition 1.2.4, so  $\pi_n : \Delta^0/\mathbf{Kan} \rightarrow \mathbf{Set}$  preserves finite products and colimits for small filtered diagrams. But the forgetful functor

$\mathbf{Grp} \rightarrow \mathbf{Set}$  creates finite products and colimits for small filtered diagrams, so the claim follows.

(iii). Use paragraph 1.3.4 and remark 1.4.20.  $\square$

**Definition 1.4.22.** The **homotopy category of Kan complexes** is the full subcategory  $\mathbf{H} \subseteq \mathbf{Ho}_{\Delta^1} \mathbf{sSet}$  spanned by the Kan complexes.

**Proposition 1.4.23.** *Let  $\mathbf{Kan}$  be the full subcategory of  $\mathbf{sSet}$  spanned by the Kan complexes and let  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$  be the functor that sends each morphism to its intrinsic homotopy class. For any functor  $F : \mathbf{Kan} \rightarrow \mathbf{C}$ , the following are equivalent:*

- (i) *For all Kan complexes  $X$ ,  $F[\delta_1^0, X] : F[\Delta^1, X] \rightarrow F[\Delta^0, X]$  is an isomorphism in  $\mathbf{C}$ .*
- (ii) *For all Kan complexes  $X$ ,  $F[\delta_1^1, X] : F[\Delta^1, X] \rightarrow F[\Delta^0, X]$  is an isomorphism in  $\mathbf{C}$ .*
- (iii) *For all simplicial sets  $X$ ,  $F[\sigma_0^0, X] : F[\Delta^1, X] \rightarrow F[\Delta^0, X]$  is an isomorphism in  $\mathbf{C}$ .*
- (iv) *For all parallel pairs  $f_0, f_1 : X \rightarrow Y$  in  $\mathbf{Kan}$ , if  $f_0 \sim f_1$ , then  $Ff_0 = Ff_1$ .*
- (v)  *$F : \mathbf{Kan} \rightarrow \mathbf{C}$  factors through  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$ .*

Moreover, the factorisation is unique if it exists.

*Proof.* The proof is similar to that of proposition 1.3.13. (Use corollary 1.4.16 to deduce that  $[\Delta^1, X]$  is a Kan complex if  $X$  is.)  $\diamond$

**Proposition 1.4.24.** *Let  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$  be the functor that sends a morphism of Kan complexes to its intrinsic homotopy class.*

- (i) *The functor  $\pi$  is full, surjective on objects, and preserves finite products and finite coproducts.*
- (ii)  *$\mathbf{Kan}$  is closed under products for all small families in  $\mathbf{sSet}$ , and  $\mathbf{H}$  has products for finite families.*
- (iii)  *$\mathbf{Kan}$  and  $\mathbf{H}$  are cartesian closed categories, and  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$  is a cartesian closed functor.*



(iv) A morphism  $f : X \rightarrow Y$  in  $\mathbf{Kan}$  admits an intrinsic homotopy inverse if and only if  $\pi f : \pi X \rightarrow \pi Y$  is an isomorphism in  $\mathbf{H}$ .

*Proof.* (i). The construction of  $\mathbf{H}$  as  $\pi_0[\mathbf{Kan}]$  ensures that  $\pi$  is indeed a functor.

(ii). It is clear from the construction of  $\pi_0 Z$  as a coequaliser that  $\chi_Z : Z_0 \rightarrow \pi_0 Z$  is a surjection; thus  $\pi$  is a full functor. It is obviously surjective on objects, and it preserves finite products and finite coproducts because  $\pi_0$  preserves finite products.

(iii). By [proposition A.3.14](#), the class of Kan fibrations is closed under products for small families, so  $\mathbf{Kan}$  is as well. By claim (ii),  $\mathbf{H}$  inherits finite products from  $\mathbf{Kan}$ .

(iv). By [proposition 1.4.15](#),  $[Y, K]$  is a Kan complex whenever  $K$  is, which combined with claim (iii) implies  $\mathbf{Kan}$  is cartesian closed. [Proposition A.2.11](#) says we have natural isomorphisms  $[X \times Y, K] \cong [X, [Y, K]]$ , so it follows that we have natural bijections

$$\pi_0[X \times Y, K] \cong \pi_0[X, [Y, K]]$$

for all  $X, Y$ , and  $K$  in  $\mathbf{Kan}$ , and this descends along  $\pi$  to make  $\mathbf{H}$  cartesian closed.

(v). The definition says  $f : X \rightarrow Y$  is a weak homotopy equivalence if and only if  $\pi_0[f, K] : \pi_0[Y, K] \rightarrow \pi_0[X, K]$  is a bijection for all Kan complexes  $K$ ; but this is natural in  $K$ , so the Yoneda lemma implies this happens if and only if  $\pi f : \pi X \rightarrow \pi Y$  is an isomorphism in  $\mathbf{H}$ . ■

**Definition 1.4.25.** Let  $n$  be an integer,  $n \geq -2$ . An  $n$ -connected morphism of Kan complexes is a morphism  $f : X \rightarrow Y$  in  $\mathbf{sSet}$ , where  $X$  and  $Y$  are Kan complexes, such that the following conditions are satisfied:

- If  $n \geq -1$ , then  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is a surjection.
- If  $n \geq 0$ , then  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is a bijection and, for all vertices  $x$  of  $X$ , the homomorphism  $\pi_1 f : \pi_1(X, x) \rightarrow \pi_1(Y, f(x))$  is a surjection.
- If  $n \geq 1$ , then for all  $1 \leq m \leq n$  and all vertices  $x$  of  $X$ , the homomorphism  $\pi_m f : \pi_m(X, x) \rightarrow \pi_m(Y, f(x))$  is an isomorphism, and  $\pi_n f : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$  is a surjection.

An  $\infty$ -connected morphism of Kan complexes is one that is  $n$ -connected for all  $n \geq -2$ .

**Proposition 1.4.26.** *The class of  $\infty$ -connected morphisms of Kan complexes has the 2-out-of-3 and 2-out-of-6 properties.*<sup>[7]</sup>

*Proof.* This is a straightforward check (using lemma A.4.14). ◆

**Theorem 1.4.27.** *Let  $p : X \rightarrow Y$  be a Kan fibration. If  $X$  and  $Y$  are Kan complexes, then the following are equivalent:*

- (i)  $p : X \rightarrow Y$  is a trivial Kan fibration.
- (ii)  $p : X \rightarrow Y$  is an  $\infty$ -connected morphism of Kan complexes.

*Proof.* (i)  $\Rightarrow$  (ii). Lemma 1.3.22 says  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is a bijection. Fix a positive integer  $n$  and a vertex  $x$  of  $X$ . Then proposition 1.4.11 implies that there exist a morphism  $s : Y \rightarrow X$  such that  $p \circ s = \text{id}_Y$  and an intrinsic homotopy  $\alpha : \text{id}_X \Rightarrow s \circ p$  relative to  $x$  (considered as a subcomplex of  $X$ ), so we may apply proposition 1.4.21 to deduce that  $\pi_n p : \pi_n(X, x) \rightarrow \pi_n(Y, p(x))$  is an isomorphism.

(ii)  $\Rightarrow$  (i). See Theorem 7.10 in [GJ, Ch. I]. ■

**Corollary 1.4.28.** *Let  $X$  and  $Y$  be Kan complexes. If  $f_0, f_1 : X \rightarrow Y$  are intrinsically homotopic, then for all positive integers  $n$  and all vertices  $x$ , there exists a commutative diagram of the form below:*

$$\begin{array}{ccc} \pi_n(X, x) & \xrightarrow{\pi_n f_0} & \pi_n(Y, f_0(x)) \\ \parallel & & \downarrow \cong \\ \pi_n(X, x) & \xrightarrow{\pi_n f_1} & \pi_n(Y, f_1(x)) \end{array}$$

*Proof.* We may assume without loss of generality that there is an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$ . Let  $h : X \rightarrow [\Delta^1, Y]$  be the corresponding morphism. It is clear that the coface morphisms  $\delta_0^e : \Delta^0 \rightarrow \Delta^1$  are isomorphic to the horn inclusions  $\Lambda_k^1 \hookrightarrow \Delta^1$  (where  $k = 0$  if  $e = 1$  and  $k = 1$  if  $e = 0$ ), so by corollary 1.4.16,

---

[7] See definition A.4.13.

the morphisms  $[\delta_0^e, X] : [\Delta^1, Y] \rightarrow [\Delta^0, Y]$  are trivial Kan fibrations. Thus, we have the following commutative diagram,

$$\begin{array}{ccc}
 \pi_n(X, x) & \xrightarrow{\pi_n f_0} & \pi_n(Y, f_0(x)) \\
 \parallel & & \uparrow \pi_n p_0 \\
 \pi_n(X, x) & \xrightarrow{\pi_n h} & \pi_n([\Delta^1, Y], h(x)) \\
 \parallel & & \downarrow \pi_n p_1 \\
 \pi_n(X, x) & \xrightarrow{\pi_n f_1} & \pi_n(Y, f_1(x))
 \end{array}$$

where  $p_0, p_1 : [\Delta^1, Y] \rightarrow Y$  are the morphisms induced by  $\delta_0^1, \delta_0^0 : \Delta^0 \rightarrow \Delta^1$  (respectively). But [theorem 1.4.27](#) implies that  $\pi_n p_0$  and  $\pi_n p_1$  are isomorphisms, so we are done.  $\blacksquare$

The homotopy groups of a Kan complex are a complete homotopy invariant. More precisely, we have the following analogue of a theorem of Whitehead [[1949](#)]:

**Theorem 1.4.29** (Whitehead). *Let  $X$  and  $Y$  be Kan complexes. For any morphism  $f : X \rightarrow Y$ , the following are equivalent:*

- (i)  $f : X \rightarrow Y$  admits an intrinsic homotopy inverse.
- (ii)  $f : X \rightarrow Y$  is an  $\infty$ -connected morphism of Kan complexes.
- (iii)  $f : X \rightarrow Y$  admits a factorisation of the form  $q \circ j$ , where  $j$  is an anodyne extension and  $q$  is a trivial Kan fibration.

*Proof.* (i)  $\Rightarrow$  (ii). By [corollary 1.3.14](#),  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is a bijection, and using [corollary 1.4.28](#), it is not hard to see that  $\pi_n f : \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$  is an isomorphism for all positive integers  $n$  and all vertices  $x$  of  $X$ .

(ii)  $\Rightarrow$  (iii). [Proposition 1.4.7](#) says we may factor  $f$  as  $p \circ j$ , where  $j$  is an anodyne extension and  $p$  is a Kan fibration; note that the domain of  $p$  is automatically a Kan complex. By [proposition 1.4.18](#), anodyne extensions of Kan complexes admit homotopy inverses, so  $i$  is an  $\infty$ -connected morphism of Kan complexes; hence, applying [proposition 1.4.26](#), we may deduce that  $p$  is  $\infty$ -connected morphism if (and only if)  $f : X \rightarrow Y$  is  $\infty$ -connected. But [theorem 1.4.27](#) says  $p$  is  $\infty$ -connected if and only if it is a trivial Kan fibration, so we are done.

(iii)  $\Rightarrow$  (i). Propositions 1.4.11 and 1.4.18 say that both  $p$  and  $i$  admits intrinsic homotopy inverses, so the same is true for  $f = p \circ i$ .  $\blacksquare$

**Corollary 1.4.30.** *Let  $i : X \rightarrow Y$  be a monomorphism. If  $X$  and  $Y$  are Kan complexes, then the following are equivalent:*

- (i)  $i : X \rightarrow Y$  is an anodyne extension.
- (ii)  $i : X \rightarrow Y$  is an  $\infty$ -connected morphism of Kan complexes.

*Proof.* (i)  $\Rightarrow$  (ii). Apply theorem 1.4.29.

(ii)  $\Rightarrow$  (i). If  $i$  is an  $\infty$ -connected morphism of Kan complexes, then  $i$  admits a factorisation of the form  $q \circ j$ , where  $j$  is an anodyne extension and  $q$  is a trivial Kan fibration. The right lifting property of  $q$  implies there is a morphism  $h$  such that  $p \circ h = \text{id}_Y$  and  $h \circ i = j$ ; in particular,  $i$  is a retract of  $j$ . Thus,  $i$  is an anodyne extension.  $\blacksquare$

**Lemma 1.4.31.** *Let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{sSet}$ , let  $L$  be a simplicial set, and let  $K \subseteq L$  and  $J \subseteq K$  be simplicial subsets. If  $Y$  is a Kan complex and  $f : X \rightarrow Y$  has the homotopical right lifting property with respect to both  $J \hookrightarrow K$  and  $K \hookrightarrow L$ , then  $f : X \rightarrow Y$  also has the homotopical right lifting property with respect to  $J \hookrightarrow L$ .*

*Proof.* See Lemma 3.4 in [Dugger and Isaksen, 2004].  $\square$

**Theorem 1.4.32.** *Let  $f : X \rightarrow Y$  be a morphism of Kan complexes. The following are equivalent:*

- (i)  $f : X \rightarrow Y$  is an  $\infty$ -connected morphism of Kan complexes.
- (ii)  $f : X \rightarrow Y$  has the homotopical right lifting property with respect to all monomorphisms between finite simplicial sets.
- (iii)  $f : X \rightarrow Y$  has the homotopical right lifting property with respect to all boundary inclusions  $\partial\Delta^n \hookrightarrow \Delta^n$ .

*Proof.* See Proposition 4.1 in [Dugger and Isaksen, 2004].  $\square$

## 1.5 The Kan–Quillen model structure

**Prerequisites.** §§ 1.3, 1.4 4.1, A.3.

In [1967], Quillen constructed an axiomatic framework for doing homotopy theory in abstract categories, which he called ‘closed model categories’, and showed that  $\mathbf{sSet}$  can be endowed with a model structure such that the resulting homotopy theory is equivalent in a strong sense to the homotopy theory of topological spaces.

The following characterisation of weak homotopy equivalences appears in [Quillen, 1967, Ch. II, §3]; we follow Joyal and Tierney [2008] in taking it as our definition. Recalling that  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  from proposition 1.2.4 is the functor sending a simplicial set  $X$  to the set  $\pi_0$  of its connected components,

**Definition 1.5.1.** A **weak homotopy equivalence of simplicial sets** is a morphism  $f : W \rightarrow Z$  such that, for every Kan complex  $K$ , the induced map

$$\pi_0[f, K] : \pi_0[Z, K] \rightarrow \pi_0[W, K]$$

is a bijection of sets.

**Proposition 1.5.2** (Formal Whitehead theorem).

- (i) *If a morphism in  $\mathbf{sSet}$  admits an intrinsic homotopy left inverse and an intrinsic homotopy right inverse, then it is a weak homotopy equivalence.*
- (ii) *A morphism in  $\mathbf{Kan}$  is a weak homotopy equivalence if and only if it admits an intrinsic homotopy inverse.*

*Proof.* (i). If  $f : X \rightarrow Y$  admits an intrinsic homotopy left inverse (resp. an intrinsic homotopy right inverse), then  $\pi_0[f, K] : \pi_0[Y, K] \rightarrow \pi_0[X, K]$  is injective (resp. surjective) for all simplicial sets  $K$ . In particular,  $f : X \rightarrow Y$  is a weak homotopy equivalence as soon as it has both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse.

(ii). Let  $f : X \rightarrow Y$  be a weak homotopy equivalence of Kan complexes. The definition says  $f : X \rightarrow Y$  is a weak homotopy equivalence if and only if  $\pi_0[f, K] : \pi_0[Y, K] \rightarrow \pi_0[X, K]$  is a bijection for all Kan complexes  $K$ , so (recalling remark 1.3.7) we may obtain an intrinsic homotopy left inverse for  $f : X \rightarrow Y$  by taking  $K = X$ , say  $g : Y \rightarrow X$ . By naturality, the following

diagram commutes:

$$\begin{array}{ccc}
 \pi_0[Y, Y] & \xrightarrow{\pi_0[f, Y]} & \pi_0[X, Y] \\
 \pi_0[Y, g] \downarrow & & \downarrow \pi_0[Y, g] \\
 \pi_0[Y, X] & \xrightarrow{\pi_0[f, X]} & \pi_0[X, X] \\
 \pi_0[Y, f] \downarrow & & \downarrow \pi_0[X, f] \\
 \pi_0[Y, Y] & \xrightarrow{\pi_0[f, Y]} & \pi_0[X, Y]
 \end{array}$$

Thus, by chasing the homotopy class of  $\text{id}_Y$ , we deduce that  $g : Y \rightarrow X$  is also an intrinsic homotopy right inverse for  $f : X \rightarrow Y$ , as required.  $\blacksquare$

**Lemma 1.5.3.** *Anodyne extensions are weak homotopy equivalences.*

*Proof.* If  $i : X \rightarrow Y$  is an anodyne extension, then  $[i, K] : [Y, K] \rightarrow [X, K]$  is a trivial Kan fibration for all Kan complexes  $K$ , by [corollary 1.4.16](#). Applying [lemma 1.3.22](#), we then deduce that  $i : X \rightarrow Y$  is a weak homotopy equivalence.  $\blacksquare$

**Definition 1.5.4.** A **weakly contractible simplicial set** is a simplicial set  $X$  for which the unique morphism  $X \rightarrow \Delta^0$  in  $\mathbf{sSet}$  is a weak homotopy equivalence.

**Proposition 1.5.5.** *Let  $X$  be a Kan complex. The following are equivalent:*

- (i)  $X$  is a weakly contractible simplicial set.
- (ii)  $X \rightarrow \Delta^0$  admits an intrinsic homotopy inverse.
- (iii)  $X \rightarrow \Delta^0$  is a trivial Kan fibration.

*Proof.* (i)  $\Leftrightarrow$  (ii). Apply [proposition 1.5.2](#).

(ii)  $\Leftrightarrow$  (iii). Use theorems [1.4.27](#) and [1.4.29](#).  $\blacksquare$

**REMARK 1.5.6.** Not all weak homotopy equivalences admit an intrinsic homotopy inverse. For instance, if  $X$  is the nerve of the following category,

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \dots$$

then every morphism  $\Delta^0 \rightarrow X$  is an anodyne extension (and in particular  $X$  is weakly contractible), but none of them admit an intrinsic homotopy right inverse.

**Proposition 1.5.7.**

- (i) For each simplicial set  $X$ , there exist a Kan complex  $RX$  and a weak homotopy equivalence  $i : X \rightarrow RX$ .
- (ii) For each simplicial set  $X$ , the functor  $\pi_0[X, -] : \mathbf{Kan} \rightarrow \mathbf{Set}$  factors through  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$  as a representable functor on  $\mathbf{H}$ .
- (iii) The functor  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$  extends to a functor  $\pi : \mathbf{sSet} \rightarrow \mathbf{H}$  that sends weak homotopy equivalences to isomorphisms, and this extension is unique up to unique isomorphism.

*Proof.* (i). Proposition 1.4.7 implies there is an anodyne extension  $i : X \rightarrow RX$  where  $RX$  is a Kan complex, and lemma 1.5.3 says anodyne extensions are weak homotopy equivalences.

(ii). Given  $i : X \rightarrow RX$  as in claim (i), the maps  $\pi_0[i, K] : \pi_0[RX, K] \rightarrow \pi_0[X, K]$  are bijections (natural in  $K$ ), so we may as well assume  $X$  is a Kan complex. We may then apply propositions 1.4.23 and 1.5.2 to deduce that the functor  $\pi_0[X, -] : \mathbf{Kan} \rightarrow \mathbf{Set}$  factors through  $\pi : \mathbf{Kan} \rightarrow \mathbf{H}$ , and remark 1.3.7 implies that the resulting functor  $\mathbf{H} \rightarrow \mathbf{Set}$  is isomorphic to  $\mathbf{H}(\pi X, -)$ .

(iii). Formally, what we seek is a functor  $F : \mathbf{sSet} \rightarrow \mathbf{H}$  such that, for all Kan complexes  $Y$  and  $K$ ,

$$\mathbf{H}(FY, \pi K) = \pi_0[Y, K]$$

and, for all weak homotopy equivalences  $f : X \rightarrow Y$  in  $\mathbf{sSet}$ , the induced hom-set map  $\mathbf{H}(Ff, \pi K) : \mathbf{H}(FY, \pi K) \rightarrow \mathbf{H}(FX, \pi K)$  is a bijection for all Kan complexes  $K$ . Clearly, for any such  $F$  and any simplicial set  $X$ , there must be bijections

$$\mathbf{H}(FX, \pi K) \cong \pi_0[X, K]$$

that are natural in  $K$ , but by claim (ii), this is representable as a functor  $\mathbf{H} \rightarrow \mathbf{Set}$  for each  $X$ , so we can certainly construct such a functor  $F$ , and it is unique up to unique isomorphism. ■

**Corollary 1.5.8.** *The inclusion  $\mathbf{H} \hookrightarrow \mathrm{Ho}_{\Delta^1} \mathbf{sSet}$  admits a left adjoint.* ■

**Proposition 1.5.9.**

- (i) A Kan fibration  $p : X \rightarrow Y$  is trivial if and only if it is a weak homotopy equivalence.

(ii) *A monomorphism  $i : Z \rightarrow W$  is an anodyne extension if and only if it is a weak homotopy equivalence.*

*Proof.* (i). See Proposition 3.4.1 in [Joyal and Tierney, 2008].

(ii). See Lemma 7 in [Quillen, 1967, Ch. II, §3] or Proposition 3.4.2 in [Joyal and Tierney, 2008].  $\square$

**Theorem 1.5.10.**  *$\mathbf{sSet}$ , regarded as a  $\mathbf{sSet}$ -enriched category via its cartesian closed structure, is a simplicial model category where*

- *the cofibrations are the monomorphisms in  $\mathbf{sSet}$ ,*
- *the fibrations are the Kan fibrations, and*
- *the weak equivalences are the weak homotopy equivalences.*

*This is the **Kan–Quillen model structure on simplicial sets.***

*Proof.* We know  $\mathbf{sSet}$  has limits and colimits for all small diagrams and is a cartesian closed category, so it satisfies axioms CM1 and SM0. Using the definition of weak homotopy equivalence given above, the class of weak homotopy equivalences has the 2-out-of-6 property by lemma A.4.14, hence axiom CM2 is satisfied. Proposition 1.4.7 plus theorem 4.1.12 then shows that the announced cofibrations, fibrations, and weak equivalences do indeed constitute a model structure on  $\mathbf{sSet}$ . Finally, we note that proposition 1.4.15 is precisely the condition required by axiom SM7.  $\blacksquare$

**Proposition 1.5.11.** *Let  $f : W \rightarrow Z$  be a weak homotopy equivalence of simplicial sets and let  $X$  be any simplicial set.*

- (i) *The morphism  $f \times \text{id}_X : W \times X \rightarrow Z \times X$  is a weak homotopy equivalence.*
- (ii) *If  $X$  is a Kan complex, then  $[f, X] : [Z, X] \rightarrow [W, X]$  is a weak homotopy equivalence.*
- (iii) *If  $W$  and  $Z$  are Kan complexes, then  $[X, f] : [X, W] \rightarrow [X, Z]$  is a weak homotopy equivalence.*

*Proof.* (i). We must show that, for all Kan complexes  $K$ , the induced map

$$\pi_0[f \times \text{id}_X, K] : \pi_0[Z \times X, K] \rightarrow \pi_0[W \times X, K]$$



is a bijection. However, we have a commutative diagram

$$\begin{array}{ccc} \pi_0[Z \times X, \mathbf{K}] & \xrightarrow{\pi_0[\text{id}_X \times f, \mathbf{K}]} & \pi_0[W \times X, \mathbf{K}] \\ \cong \downarrow & & \downarrow \cong \\ \pi_0[Z, [X, \mathbf{K}]] & \xrightarrow{\pi_0[f, [X, \mathbf{K}]]} & \pi_0[W, [X, \mathbf{K}]] \end{array}$$

and (by [corollary 1.4.16](#))  $[X, \mathbf{K}]$  is a Kan complex, so  $\pi_0[f, [X, \mathbf{K}]]$  is a bijection; hence,  $\pi_0[f \times \text{id}_X, \mathbf{K}]$  is indeed a bijection for all Kan complexes  $\mathbf{K}$ .

(ii). If  $X$  is a Kan complex, then [corollary 1.4.16](#) says that  $[-, X]$  is a right Quillen functor; but every simplicial set is cofibrant, so Ken Brown’s lemma ([4.3.6](#)) implies  $[-, X]$  preserves weak homotopy equivalences.

(iii). Similarly, for any simplicial set  $X$ ,  $[X, -]$  is a right Quillen functor, and so Ken Brown’s lemma implies  $[X, -]$  preserves weak homotopy equivalences between Kan complexes. ■

**Proposition 1.5.12.**

(i) *Equipping **Set** with the discrete model structure,<sup>[8]</sup> the adjunction*

$$\pi_0 \dashv \text{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$$

*is a Quillen adjunction.<sup>[9]</sup>*

(ii) *For every map  $f : X \rightarrow Y$ , the morphism  $\text{disc } f : \text{disc } X \rightarrow \text{disc } Y$  is a Kan fibration.*

(iii) *The functor  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  sends weak homotopy equivalences to bijections.*

*Proof.* (i). Since every map is a cofibration in the discrete model structure on **Set**, it is enough (by [proposition 4.3.2](#)) to show that  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  sends anodyne extensions in **sSet** to bijections; and by [proposition 1.4.12](#), it suffices to show that the maps  $\pi_0 \Lambda_k^n \rightarrow \pi_0 \Delta^n$  induced by the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$ . But this is an immediate consequence of the fact that each  $\Lambda_k^n$  and  $\Delta^n$  is connected.

[8] See [example 4.1.5](#).

[9] See [definition 4.3.1](#).

(ii). Every map is a fibration in the discrete model structure on **Set**, and  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{sSet}$  is a right Quillen functor, so each  $\text{disc } f : \text{disc } X \rightarrow \text{disc } Y$  is indeed a Kan fibration.

(iii). Every simplicial set is cofibrant, so this is a consequence of Ken Brown's lemma (4.3.6). ■

**Proposition 1.5.13.** *Let  $\mathcal{W}$  be a subcategory of **sSet** that satisfies these conditions:*

- *Every identity morphism in **sSet** is in  $\mathcal{W}$ .*
- *$\mathcal{W}$  has the 2-out-of-3 property in **sSet**.*
- *For every simplicial set  $X$ , the projection  $p_X : X \times \Delta^1 \rightarrow X$  is in  $\mathcal{W}$ .*

*Then:*

- (i) *Given a parallel pair  $f_0, f_1 : X \rightarrow Y$  in **sSet** and an intrinsic homotopy  $\alpha : f_0 \Rightarrow f_1$ , the morphism  $f_0$  is in  $\mathcal{W}$  if and only if  $f_1$  is in  $\mathcal{W}$ .*
- (ii) *If  $\mathcal{W}$  has the special 2-out-of-4 property, then every trivial Kan fibration is in  $\mathcal{W}$ .*
- (iii) *If  $\mathcal{W}$  is closed under retracts or has the 2-out-of-6 property in **sSet**, then every trivial Kan fibration is in  $\mathcal{W}$ .*

*Proof.* (i). This follows from [remark 1.3.2](#).

(ii). This is a special case of [proposition 5.3.34](#).

(iii). Apply [lemma A.4.17](#). ■

**Proposition 1.5.14.** *Let  $\mathcal{W}$  be a subcategory of **sSet** that satisfies these conditions:*

- (a) *The class of monomorphisms that are in  $\mathcal{W}$  is closed under pushouts, transfinite composition, and retracts.*
- (b)  *$\mathcal{W}$  has the 2-out-of-3 property in **sSet**, and for all simplicial sets  $X$ , the morphism  $\text{id} : X \rightarrow X$  is in  $\mathcal{W}$ .*
- (c) *For all natural numbers  $n$ , the unique morphism  $\Delta^n \rightarrow \Delta^0$  is in  $\mathcal{W}$ .*

Then every weak homotopy equivalence is in  $\mathcal{W}$ .

*Proof.* First, we should show that all the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  are in  $\mathcal{W}$ . We proceed by induction on  $n$ . For  $n = 1$ , observe that conditions (a) and (b) together imply that every isomorphism is in  $\mathcal{W}$ , and so we may use the 2-out-of-3 property to deduce that the horn inclusions  $\Lambda_0^1 \hookrightarrow \Delta^1$  and  $\Lambda_1^1 \hookrightarrow \Delta^1$  are in  $\mathcal{W}$ .

Now, suppose that the horn inclusions  $\Lambda_k^m \hookrightarrow \Delta^m$  are in  $\mathcal{W}$  for all  $m < n$ . It is not hard to see that the horn  $\Lambda_l^n$  can be constructed by adjoining  $m$  copies of  $\Delta^m$  along various horn inclusions (for  $0 < m < n$ ), so conditions (a) and (b) imply that the  $l$ -th vertex  $\Delta^0 \rightarrow \Lambda_l^n$  is in  $\mathcal{W}$ . Condition (c) says that the unique morphism  $\Delta^n \rightarrow \Delta^0$  is in  $\mathcal{W}$ , so we can then use the 2-out-of-3 property to deduce that the horn inclusion  $\Lambda_l^n \hookrightarrow \Delta^n$  is in  $\mathcal{W}$ .

Having shown that the horn inclusions are in  $\mathcal{W}$ , we can use Quillen’s small object argument (theorem 0.5.12) together with corollaries 0.5.7 and 0.5.13 and condition (a) to deduce that all anodyne extensions are in  $\mathcal{W}$ . Notice that, if  $p : X \rightarrow Y$  is a trivial Kan fibration, then there is a morphism  $s : Y \rightarrow X$  such that  $p \circ s = \text{id}_Y$ , and by 1.5.9,  $s : Y \rightarrow X$  is an anodyne extension. Hence, condition (b) implies that all trivial Kan fibrations are in  $\mathcal{W}$  as well. Finally, using the fact that every weak homotopy equivalence factors as an anodyne extension followed by a trivial Kan fibration, we conclude that every weak homotopy equivalence is in  $\mathcal{W}$ . ■

**Corollary 1.5.15.** *The subcategory of weak homotopy equivalences in  $\mathbf{sSet}$  is the smallest subcategory satisfying the conditions in the proposition.*

*Proof.* Proposition 1.5.9 says that the class of monomorphisms that are weak homotopy equivalences is precisely the class of anodyne extensions, which has the required closure properties by proposition A.3.14. Thus, the class of weak homotopy equivalences satisfies condition (a), and the remaining conditions are easily verified. ■

¶ 1.5.16. Let  $P^n$  be the partially ordered set of non-empty subsets of  $[n]$  and, for each monotone map  $f : [n] \rightarrow [m]$ , let  $f_* : P^n \rightarrow P^m$  be the map induced by taking images. Taking nerves, this defines a functor  $N(P^\bullet) : \mathbf{\Delta} \rightarrow \mathbf{sSet}$ . Note that there is a natural surjective monotone map  $\max : P^n \rightarrow [n]$ , each with a canonical (but not natural!) splitting, so we get a natural transformation  $N(\max) : N(P^\bullet) \Rightarrow \mathbf{\Delta}^\bullet$  whose components are split epimorphisms.

**Definition 1.5.17.** The **extension** of a simplicial set  $X$  is the simplicial set  $\text{Ex}(X)$  defined by the formula below:

$$\text{Ex}(X)_n = \mathbf{sSet}(\mathbf{N}(P^n), X)$$

The **canonical embedding** is the morphism  $i_X : X \rightarrow \text{Ex}(X)$  induced by  $\mathbf{N}(\max) : \mathbf{N}(P^\bullet) \Rightarrow \Delta^\bullet$ ; note that it is a split monomorphism in  $\mathbf{sSet}$ .

**REMARK 1.5.18.** Every simplex of  $\mathbf{N}(P^n)$  is uniquely determined by its vertices and  $P^n$  has only finitely many elements, so  $\mathbf{N}(P^n)$  is a finite simplicial set. In particular, each  $\text{Ex}(X)_n$  is a finite weighted limit of the diagram  $X : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ .

**¶ 1.5.19.** Let  $\text{Sd} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  be (the functor part of) a left Kan extension of  $\mathbf{N}(\max) : \Delta \rightarrow \mathbf{sSet}$  along  $\Delta^\bullet : \Delta \rightarrow \mathbf{sSet}$ . Using the formulas of [theorem A.5.15](#), we see there is a natural bijection of the form below:

$$\mathbf{sSet}(\text{Sd}(X), Y) \cong \mathbf{sSet}(X, \text{Ex}(Y))$$

In other words, we have the following adjunction:

$$\text{Sd} \dashv \text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$$

**Definition 1.5.20.** The **subdivision** of a simplicial set  $X$  is the simplicial set  $\text{Sd}(X)$  defined above.

**Lemma 1.5.21.** *Let  $X$  be a simplicial set. For each morphism  $z : \Lambda_k^n \rightarrow \text{Ex}(X)$ , there exists a morphism  $w : \Delta_k^n \rightarrow \text{Ex}(\text{Ex}(X))$  making the diagram below commute:*

$$\begin{array}{ccc} \Lambda_k^n & \xrightarrow{z} & \text{Ex}(X) \\ \downarrow & & \downarrow i_{\text{Ex}(X)} \\ \Delta_k^n & \xrightarrow{w} & \text{Ex}(\text{Ex}(X)) \end{array}$$

*Proof.* See Lemma 3.2 in [Kan, 1957], or Lemma 4.7 in [GJ, Ch. III]. □

**Lemma 1.5.22.** *The functor  $\text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves Kan fibrations. In particular, if  $X$  is a Kan complex, then so is  $\text{Ex}(X)$ .*

*Proof.* See Lemma 3.4 in [Kan, 1957], or Lemma 4.5 in [GJ, Ch. III], or Corollary 2.1.27 in [Cisinski, 2006]. □

**Lemma 1.5.23.** *For any simplicial set  $X$ , the canonical embedding  $i_X : X \rightarrow \text{Ex}(X)$  is a weak homotopy equivalence.*

*Proof.* See Theorem 4.6 in [GJ, Ch. III], or Proposition 2.1.30 in [Cisinski, 2006]; but see also Lemma 3.7 in [Kan, 1957] for the case where  $X$  is a Kan complex.  $\square$

**Corollary 1.5.24.** *The functor  $\text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves trivial Kan fibrations.*

*Proof.* Combine proposition 1.5.9 with lemmas 1.5.22 and 1.5.23.  $\blacksquare$

**Corollary 1.5.25.** *We have the following Quillen equivalence:*

$$\mathbf{Sd} \dashv \text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$$

*Proof.* Lemma 1.5.22 and corollary 1.5.24 say  $\text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  is a right Quillen functor, so (by proposition 4.3.2) the indicated adjunction is indeed a Quillen adjunction. Consider the derived adjunction:

$$\mathbf{LSd} \dashv \mathbf{REx} : \text{Ho } \mathbf{sSet} \rightarrow \text{Ho } \mathbf{sSet}$$

By proposition 1.5.9 and lemma 1.5.23,  $\text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  is a homotopical functor, so  $\text{Ho } \text{Ex} : \text{Ho } \mathbf{sSet} \rightarrow \text{Ho } \mathbf{sSet}$  is well defined and isomorphic to both  $\text{id}$  and  $\mathbf{REx}$ . Hence,  $\mathbf{LSd}$  is also isomorphic to  $\text{id}$ , and (recalling theorem 4.4.1) we may then apply theorem 4.3.12.  $\blacksquare$

$\P$  1.5.26. For each simplicial set  $X$ , we define  $\text{Ex}^\infty(X)$  to be the colimit of the diagram below:

$$X \xrightarrow{i_X} \text{Ex}(X) \xrightarrow{i_{\text{Ex}(X)}} \text{Ex}^2(X) \xrightarrow{i_{\text{Ex}^2(X)}} \text{Ex}^3(X) \longrightarrow \dots$$

The above defines a functor  $\text{Ex}^\infty : \mathbf{sSet} \rightarrow \mathbf{sSet}$  and a natural transformation  $i^\infty : \text{id}_{\mathbf{sSet}} \Rightarrow \text{Ex}^\infty$ .

**Theorem 1.5.27.**

- (i) *For all simplicial sets  $X$ , the morphism  $i_X^\infty : X \rightarrow \text{Ex}^\infty(X)$  is an anodyne extension.*
- (ii) *For all simplicial sets  $X$ , the simplicial set  $\text{Ex}^\infty(X)$  is a Kan complex.*
- (iii) *The functor  $\text{Ex}^\infty : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves Kan fibrations, trivial Kan fibrations, and finite limits.*

*Proof.* (i). Recalling [proposition 1.5.9](#) and [lemma 1.5.23](#), we see that the canonical embedding  $i_X : X \rightarrow \text{Ex}(X)$  is an anodyne extension for all simplicial sets  $X$ ; but [proposition A.3.14](#) implies that the class of anodyne extensions is closed under transfinite composition, and  $i_X^\infty : X \rightarrow \text{Ex}^\infty(X)$  is a transfinite composite of these canonical embeddings, so  $i_X^\infty$  is also an anodyne extension.

(ii). Since horns are finite simplicial sets, any horn  $\Lambda_k^n \rightarrow \text{Ex}^\infty(X)$  must factor as  $\Lambda_k^n \rightarrow \text{Ex}^m(X) \rightarrow \text{Ex}^\infty(X)$  for some  $m$ . We then apply [lemma 1.5.21](#) to deduce that  $\text{Ex}^\infty(X)$  is a Kan complex.

(iii). Similar reasoning applied to [lemma 1.5.22](#) (resp. [corollary 1.5.24](#)) shows that  $\text{Ex}^\infty : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves Kan fibrations (resp. trivial Kan fibrations). On the other hand, since  $\text{Ex} : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves finite limits and  $\text{Ex}^\infty$  is a filtered colimit of iterations of  $\text{Ex}$ , [corollary 0.2.26](#) implies  $\text{Ex}^\infty$  also preserves finite limits. ■

**Proposition 1.5.28.** *There exist a functor  $R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  and a natural transformation  $i : \text{id}_{\mathbf{sSet}} \Rightarrow R$  satisfying the following conditions:*

- *For all simplicial sets  $X$ ,  $RX$  is a Kan complex and  $i_X : X \rightarrow RX$  is an anodyne extension.*
- *$R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves Kan fibrations and trivial Kan fibrations.*
- *$R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  preserves finite limits.*

*Moreover, any such functor  $R$  preserves and reflects weak homotopy equivalences.*

*Proof.* By [theorem 1.5.27](#), we may take  $(R, i)$  to be  $(\text{Ex}^\infty, i^\infty)$ . Given any such  $(R, i)$ , consider the following commutative diagram in  $\mathbf{sSet}$ :

$$\begin{array}{ccc} X & \xrightarrow{i_X} & RX \\ f \downarrow & & \downarrow Rf \\ Y & \xrightarrow{i_Y} & RY \end{array}$$

Using [proposition 1.5.9](#) and the 2-out-of-3 property of weak homotopy equivalences, we see that  $f : X \rightarrow Y$  is a weak homotopy equivalence if and only if  $Rf : RX \rightarrow RY$  is a weak homotopy equivalence. ■

REMARK 1.5.29. We may construct a different functor satisfying the conditions of the above proposition by using an appropriate geometric realisation functor: see Proposition 2.4 and Proposition 10.10 in [GJ, Ch. I].

REMARK 1.5.30. Fixing a fibrant replacement functor  $R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  as in proposition 1.5.28, we have the following explicit construction of  $\mathbf{Ho sSet}$ :

- The objects are simplicial sets.
- For any two simplicial sets  $X$  and  $Y$ ,  $\mathbf{Ho sSet}(X, Y) = \pi_0[RX, RY]$ .
- Composition and identity morphisms are constructed as in  $\mathbf{H}$ .
- The localising functor  $\gamma : \mathbf{sSet} \rightarrow \mathbf{Ho sSet}$  inverting weak homotopy equivalences is the one sending  $f : X \rightarrow Y$  to the homotopy class of  $Rf : RX \rightarrow RY$ .

The **homotopy category of simplicial sets** is the category  $\mathbf{Ho sSet}$ .

**Definition 1.5.31.** Two simplicial sets have the **same weak homotopy type** if they are isomorphic in  $\mathbf{Ho sSet}$ .

REMARK 1.5.32. Freyd [1970] proved that  $\mathbf{H}$  is not a concrete category, i.e. that there does not exist a faithful functor  $\mathbf{H} \rightarrow \mathbf{Set}$ ; in particular,  $\mathbf{H}$  cannot be an accessible category. Nonetheless, the notion of weak homotopy type is stable under universe enlargement in the following sense:

- (i) The property of being a weak homotopy equivalence is universe-independent: indeed, it is clear that the property of being a trivial Kan fibration is universe-independent, so we may apply remark 0.5.18 to the (trivial cofibration, Kan fibration) factorisation system to test whether or not a morphism is a weak homotopy equivalence in a universe-independent way.
- (ii) Moreover, the property of being a Kan complex is universe-independent, and  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  is a left adjoint between locally presentable categories, so the hom-set  $\mathbf{H}(K, L)$  depends only on the choice of Kan complexes  $K$  and  $L$  and does not depend on the choice of universe. Similarly, whether or not  $K$  and  $L$  have the same homotopy type is universe-independent.
- (iii) Thus, for any two simplicial sets  $X$  and  $Y$ , the hom-set  $\mathbf{Ho sSet}(X, Y)$  is well-defined up to natural bijection independently of the choice of universe, and whether or not  $X$  and  $Y$  have the same weak homotopy type is also universe-independent.

## 1.6 Bisimplicial sets and cosimplicial simplicial sets

**Prerequisites.** §§ 1.1, 1.5, 4.3, 4.5, A.5, A.6.

**Definition 1.6.1.** A **bisimplicial set** is a simplicial object in  $\mathbf{sSet}$ , i.e. a functor  $\Delta^{\text{op}} \rightarrow \mathbf{sSet}$ , and a **morphism of bisimplicial sets** is a natural transformation of such functors. We write  $\mathbf{ssSet}$  for the **category of bisimplicial sets**.

**Definition 1.6.2.** Let  $X_{\bullet}$  be a bisimplicial set and let  $n$  be a natural number. The  **$n$ -th horizontal level** of  $X_{\bullet}$  is the simplicial set  $X_n$ , and the  **$n$ -th vertical level** of  $X_{\bullet}$  is the simplicial set  $(X_{\bullet})_n$ .

**Definition 1.6.3.** A **Reedy weak homotopy equivalence of bisimplicial sets** is a morphism in  $\mathbf{ssSet}$  that is a weak homotopy equivalence of simplicial sets in each horizontal level.

**Theorem 1.6.4.**  $\mathbf{ssSet}$  is a DHK model category where

- the cofibrations are the monomorphisms in  $\mathbf{ssSet}$ ,
- the fibrations are the Reedy fibrations, and
- the weak equivalences are the Reedy weak homotopy equivalences.

This is the **Reedy model structure on bisimplicial sets**.

*Proof.* Given theorem 4.5.20, it suffices to show that the Reedy cofibrations are precisely the monomorphisms in  $\mathbf{ssSet}$ : see Theorem 15.8.7 in [Hirschhorn, 2003]. □

**Corollary 1.6.5.** The Reedy model structure on  $\mathbf{ssSet}$  is the injective model structure on the functor category  $[\Delta^{\text{op}}, \mathbf{sSet}]$ . ■

**Definition 1.6.6.** The **realisation** of a bisimplicial set  $X_{\bullet}$  is the simplicial set  $|X_{\bullet}|$  defined by the following coend in  $\mathbf{sSet}$ :

$$|X_{\bullet}| = \int^{\Delta^{\text{op}} \times \Delta} \Delta^n \times X_n$$

**Lemma 1.6.7.** For each bisimplicial set  $X_{\bullet}$ , there is an isomorphism

$$|X_{\bullet}| \cong \text{diag } X$$

where  $\text{diag } X$  is the simplicial set defined by  $(\text{diag } X)_n = (X_n)_n$ , and this isomorphism is natural in  $X$ .



*Proof.* The Yoneda lemma for coends (proposition A.6.17) yields natural bijections of the form below:

$$\int^{[n]:\Delta} \Delta([m], [n]) \times (X_n)_m \cong (X_m)_m$$

Thus,  $|X_\bullet| \cong \text{diag } X$ . ■

**Corollary 1.6.8.**

- (i) If  $X_\bullet$  is a bisimplicial set whose horizontal levels are discrete,<sup>[10]</sup> then the realisation  $|X_\bullet|$  is naturally isomorphic to the simplicial set  $(X_\bullet)_0$ .
- (ii) If  $X_\bullet$  is a bisimplicial set whose vertical levels are discrete, then the realisation  $|X_\bullet|$  is naturally isomorphic to the simplicial set  $(X_0)_\bullet$ . ■

**Theorem 1.6.9.**

- (i) The functor  $|-| : \mathbf{ssSet} \rightarrow \mathbf{sSet}$  has left and right adjoints.
- (ii)  $|-|$  sends Reedy weak homotopy equivalences in  $\mathbf{ssSet}$  to weak homotopy equivalences in  $\mathbf{sSet}$ .
- (iii) Equipping  $\mathbf{ssSet}$  with the Reedy model structure and  $\mathbf{sSet}$  with the Kan–Quillen model structure,  $|-| : \mathbf{ssSet} \rightarrow \mathbf{sSet}$  is a left Quillen functor.

*Proof.* (i). Using the isomorphism  $\mathbf{ssSet} \cong [\Delta^{\text{op}} \times \Delta^{\text{op}}, \mathbf{Set}]$  and lemma 1.6.7, we may identify  $|-|$  as the functor  $\delta^*$  induced by the diagonal embedding  $\delta : \Delta \rightarrow \Delta \times \Delta$ , and corollary A.5.17 says  $\delta^*$  has left and right adjoints.

(ii). See Theorem 15.11.11 in [Hirschhorn, 2003], or Proposition 1.7 in [GJ].

(iii). From claims (i) and (ii) it follows that  $|-|$  is a left Quillen functor; alternatively, see Proposition 3.6 in [GJ, Ch. VII]. □

**Definition 1.6.10.** A **cosimplicial simplicial set** is a cosimplicial object in  $\mathbf{sSet}$ , i.e. a functor  $\Delta \rightarrow \mathbf{sSet}$ , and a morphism of cosimplicial simplicial sets is a natural transformation of such functors. We write  $\mathbf{csSet}$  for the **category of cosimplicial simplicial sets**.

**Definition 1.6.11.** Let  $X^\bullet$  be a cosimplicial simplicial set and let  $n$  be a natural number. The  **$n$ -th horizontal level** of  $X^\bullet$  is the simplicial set  $X^n$ , and the  **$n$ -th vertical level** of  $X^\bullet$  is the cosimplicial set  $(X^\bullet)_n$ .

[10] Recall definition 1.2.5.

**Definition 1.6.12.** A **Reedy weak homotopy equivalence of cosimplicial simplicial sets** is a morphism in  $\mathbf{csSet}$  that is a weak homotopy equivalence of simplicial sets in each horizontal level.

**Lemma 1.6.13.** *Let  $X^\bullet$  be a cosimplicial simplicial set. The limit  $\lim_{\leftarrow \Delta} X^\bullet$  in  $\mathbf{sSet}$  can be computed as the equaliser of the coface operators  $\delta^0, \delta^1 : X^0 \rightarrow X^1$ .*

*Proof.* This is a straightforward exercise.  $\diamond$

**Definition 1.6.14.** The **maximal augmentation** of a cosimplicial simplicial set  $X^\bullet$  is the limit  $\lim_{\leftarrow \Delta} X^\bullet$ .

**Theorem 1.6.15.**  $\mathbf{csSet}$  is a *DHK model category* where

- the cofibrations are the monomorphisms in  $\mathbf{csSet}$  that induce isomorphisms of maximal augmentations,
- the fibrations are the Reedy fibrations, and
- the weak equivalences are the Reedy weak homotopy equivalences.

This is the **Reedy model structure on cosimplicial simplicial sets**.

*Proof.* Given [theorem 4.5.20](#), it suffices to show that the Reedy cofibrations are precisely the announced ones: see Theorem 15.9.9 in [Hirschhorn, 2003].  $\blacksquare$

**Definition 1.6.16.** The **totalisation** of a cosimplicial simplicial set  $X^\bullet$  is the simplicial set  $\mathrm{Tot} X^\bullet$  defined by the following end in  $\mathbf{sSet}$ :

$$\mathrm{Tot} X^\bullet = \int_{[n]:\Delta} [\Delta^n, X^n]$$

**Lemma 1.6.17.** *Let  $Y^\bullet : \Delta \rightarrow \mathbf{sSet}$  be a cosimplicial simplicial set. Then there is a bijection*

$$\mathbf{sSet}(X, \mathrm{Tot} Y^\bullet) \cong \int_{[m]:\Delta} \mathbf{Set}(X_m, (Y^m)_m)$$

for each simplicial set  $X$ , and this bijection is natural in  $X$  and  $Y$ .

*Proof.* Using the Yoneda lemma for ends and the interchange law ([theorem A.6.16](#)), we have the following natural bijections:

$$\mathbf{sSet}\left(X, \int_{[n]:\Delta} [\Delta^n, Y^n]\right) \cong \int_{[n]:\Delta} \mathbf{sSet}(X, [\Delta^n, Y^n])$$

$$\begin{aligned}
 &\cong \int_{[n]:\Delta} \mathbf{sSet}(X \times \Delta^n, Y^n) \\
 &\cong \int_{[n]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(X_m \times \Delta([m], [n]), (Y^n)_m) \\
 &\cong \int_{[n]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(X_m, \mathbf{Set}(\Delta([m], [n]), (Y^n)_m)) \\
 &\cong \int_{[m]:\Delta} \mathbf{Set}\left(X_m, \int_{[n]:\Delta} \mathbf{Set}(\Delta([m], [n]), (Y^n)_m)\right) \\
 &\cong \int_{[m]:\Delta} \mathbf{Set}(X_m, (Y^m)_m) \quad \blacksquare
 \end{aligned}$$

**Theorem 1.6.18.**

- (i) *The functor  $\mathrm{Tot} : \mathbf{csSet} \rightarrow \mathbf{sSet}$  has a left adjoint.*
- (ii) *For each simplicial set  $X$  and each cosimplicial simplicial set  $Y^\bullet$ , the canonical comparison morphism  $\mathrm{Tot}[X, Y^\bullet] \rightarrow [X, \mathrm{Tot} Y^\bullet]$  is an isomorphism.*
- (iii) *Equipping  $\mathbf{csSet}$  with the Reedy model structure and  $\mathbf{sSet}$  with the Kan–Quillen model structure,  $\mathrm{Tot} : \mathbf{csSet} \rightarrow \mathbf{sSet}$  is a right Quillen functor.*

*Proof.* (i). It is straightforward to check that the functor sending a simplicial set  $X$  to the cosimplicial simplicial set  $\Delta^\bullet \times X$  is a left adjoint for  $\mathrm{Tot}$ ; see also [proposition A.6.14](#).

(ii). Since right adjoints preserve ends, we have the natural isomorphisms shown below:

$$\begin{aligned}
 [X, \mathrm{Tot} Y^\bullet] &= \left[ X, \int_{[n]:\Delta} [\Delta^n, Y^n] \right] \\
 &\cong \int_{[n]:\Delta} [X, [\Delta^n, Y^n]] \\
 &\cong \int_{[n]:\Delta} [\Delta^n, [X, Y^n]] \\
 &= \mathrm{Tot}[X, Y^\bullet]
 \end{aligned}$$

(iii). See Theorem 18.6.7 in [Hirschhorn, 2003]. □

## 1.7 Bar and cobar complexes

**Prerequisites.** §§ I.1, I.3, I.6, A.5, A.6.

**Definition 1.7.1.** Let  $\mathbb{C}$  be a small category.

The **bar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathbf{Set}$  weighted by  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  is the simplicial set  $B_\bullet(G, \mathbb{C}, F)$ , where

$$B_n(G, \mathbb{C}, F) = \coprod_{(c_0, \dots, c_n)} (Gc_n \times \mathbb{C}(c_{n-1}, c_n) \times \cdots \times \mathbb{C}(c_0, c_1) \times Fc_0)$$

with  $(c_0, \dots, c_n)$  ranging over  $(n+1)$ -tuples of objects in  $\mathbb{C}$ , face maps defined by the following formulae,

$$\begin{aligned} d_0^n(y, f_n, \dots, f_1, x) &= (y, f_n, \dots, f_2, F(f_1)(x)) \\ d_i^n(y, f_n, \dots, f_1, x) &= (y, f_n, \dots, f_{i+1} \circ f_i, \dots, f_1, x) \\ d_n^n(y, f_n, \dots, f_1, x) &= (G(f_n)(y), f_{n-1}, \dots, f_1, x) \end{aligned}$$

and degeneracy maps defined as below:

$$\begin{aligned} s_0^n(y, f_n, \dots, f_1, x) &= (y, f_n, \dots, f_1, \text{id}_{c_0}, x) \\ s_i^n(y, f_n, \dots, f_1, x) &= (y, f_n, \dots, f_{i+1}, \text{id}_{c_i}, f_i, \dots, f_1, x) \\ s_n^n(y, f_n, \dots, f_1, x) &= (y, \text{id}_{c_n}, f_n, \dots, f_1, x) \end{aligned}$$

The **cobar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathbf{Set}$  weighted by  $G : \mathbb{C} \rightarrow \mathbf{Set}$  is the cosimplicial set  $C^\bullet(G, \mathbb{C}, F)$ , where

$$C^n(G, \mathbb{C}, F) = \prod_{(c_0, \dots, c_n)} [Gc_n \times \mathbb{C}(c_n, c_{n-1}) \times \cdots \times \mathbb{C}(c_1, c_0), Fc_0]$$

with  $(c_0, \dots, c_n)$  ranging over  $(n+1)$ -tuples of objects in  $\mathbb{C}$ , coface maps defined by the following formulae,

$$\begin{aligned} \delta_n^0(x)_{(c_0, \dots, c_n)} &= \left( (y, f_n, \dots, f_1) \mapsto F(f_1) \left( x_{(c_1, \dots, c_n)}(y, f_n, \dots, f_2) \right) \right) \\ \delta_n^i(x)_{(c_0, \dots, c_n)} &= \left( (y, f_n, \dots, f_1) \mapsto x_{(\dots, c_i, \dots)}(y, f_n, \dots, f_i \circ f_{i+1}, \dots, f_1) \right) \\ \delta_n^n(x)_{(c_0, \dots, c_n)} &= \left( (y, f_n, \dots, f_1) \mapsto x_{(c_0, \dots, c_{n-1})}(G(f_n)(y), f_{n-1}, \dots, f_1) \right) \end{aligned}$$

and codegeneracy maps defined as below:

$$\sigma_n^0(x)_{(c_0, \dots, c_n)} = \left( (y, f_n, \dots, f_1) \mapsto x_{c_0, c_0, \dots, c_n}(y, f_n, \dots, f_1, \text{id}_{c_0}) \right)$$

$$\begin{aligned}\sigma_n^i(x)_{(c_0, \dots, c_n)} &= ((y, f_n, \dots, f_1) \mapsto x_{\dots c_i, c_i, \dots} (y, f_n, \dots, f_{i+1}, \text{id}_{c_i}, f_i, \dots, f_1)) \\ \sigma_n^n(x)_{(c_0, \dots, c_n)} &= ((y, f_n, \dots, f_1) \mapsto x_{c_0, \dots, c_n, c_n} (y, \text{id}_{c_n}, f_n, \dots, f_1))\end{aligned}$$

REMARK 1.7.2. It is clear that  $B_\bullet(G, \mathbb{C}, F)$  is covariantly functorial in both  $F$  and  $G$ , while  $C^\bullet(G, \mathbb{C}, F)$  is contravariantly functorial in  $G$  and covariantly functorial in  $F$ . One may also verify that there are bijections

$$\mathbf{Set}(B_n(G, \mathbb{C}, F), X) \cong C^n(G, \mathbb{C}^{\text{op}}, \mathbf{Set}(F, X))$$

that are natural in  $n$ ,  $F$ ,  $G$ , and  $X$ : this is one sense in which the bar complex and cobar complex are formally dual.

**Example 1.7.3.** The nerve  $N(\mathbb{C})$  of a small category  $\mathbb{C}$  is isomorphic to the bar complex  $B_\bullet(\Delta 1, \mathbb{C}, \Delta 1)$ , so there is a canonical morphism  $B_\bullet(G, \mathbb{C}, F) \rightarrow N(\mathbb{C})$  for any  $F : \mathbb{C} \rightarrow \mathbf{Set}$  and any  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ .

REMARK 1.7.4. More generally, the bar complex  $B_\bullet(G, \mathbb{C}, F)$  is isomorphic to the nerve of the following category:

- The objects are tuples  $(c, x, y)$ , where  $c$  is an object in  $\mathbb{C}$ ,  $x$  is an element of  $Fc$ , and  $y$  is an element of  $Gc$ .
- The morphisms  $f : (c, x, y) \rightarrow (c', x', y')$  are morphisms  $f : c \rightarrow c'$  in  $\mathbb{C}$  such that  $F(f)(x) = x'$  and  $G(f)(y') = y$ .
- Composition and identities are inherited from  $\mathbb{C}$ .

In particular,  $B_\bullet(\Delta 1, \mathbb{C}, h^c)$  may be identified with the nerve of the slice category  ${}^c/\mathbb{C}$ , and  $B_\bullet(h_c, \mathbb{C}, \Delta 1)$  with the nerve of the slice category  $\mathbb{C}/_c$ .

**Definition 1.7.5.** Let  $\mathbb{C}$  be a small category and let  $\mathcal{M}$  be a locally small category.

- A **bar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$  weighted by  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  is a simplicial object  $B_\bullet(G, \mathbb{C}, F)$  in  $\mathcal{M}$  with bijections

$$\mathcal{M}(B_n(G, \mathbb{C}, F), M) \cong C^n(G, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M))$$

that are natural in both  $n$  and  $M$ .

- A **cobar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$  weighted by  $G : \mathbb{C} \rightarrow \mathbf{Set}$  is a cosimplicial object  $C^\bullet(G, \mathbb{C}, F)$  in  $\mathcal{M}$  with bijections

$$\mathcal{M}(M, C^n(G, \mathbb{C}, F)) \cong C^n(G, \mathbb{C}, \mathcal{M}(M, F))$$

that are natural in both  $n$  and  $M$ .

REMARK 1.7.6. Of course, this definition agrees with the previous one (up to isomorphism) in the special case  $\mathcal{M} = \mathbf{Set}$ , and it is clear that a cobar complex in  $\mathcal{M}$  for a diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$  weighted by  $G : \mathbb{C} \rightarrow \mathbf{Set}$  becomes a bar complex in  $\mathcal{M}^{\text{op}}$  for  $F^{\text{op}} : \mathbb{C}^{\text{op}} \rightarrow \mathcal{M}^{\text{op}}$  weighted by the same  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , and vice versa.

REMARK 1.7.7. By general considerations about the representability of limits, we see that bar complexes exist for all small diagrams and weights if  $\mathcal{M}$  has co-products for small families of objects, while cobar complexes exist for all small diagrams and weights if  $\mathcal{M}$  has products for small families of objects.

**Lemma 1.7.8.** *Let  $\mathbb{C}$  be a small category. For each diagram  $F : \mathbb{C} \rightarrow \mathbf{Set}$  and each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , we have a bijection*

$$[\mathbb{C}, \mathbf{Set}](G, F) \cong \lim_{\Delta}^{\leftarrow} C^{\bullet}(G, \mathbb{C}, F)$$

that is natural in both  $F$  and  $G$ .

*Proof.* It is not hard to see that the (non-full) subcategory  $\{[0] \rightrightarrows [1]\}$  is coinital in  $\Delta$ , so it suffices to show that there is an equaliser diagram of the following form,

$$[\mathbb{C}, \mathbf{Set}](G, F) \longrightarrow C^0(G, \mathbb{C}, F) \begin{array}{c} \xrightarrow{\delta^0} \\ \xrightarrow{\delta^1} \end{array} C^1(G, \mathbb{C}, F)$$

However, if we take the map  $[\mathbb{C}, \mathbf{Set}](G, F) \rightarrow C^0(G, \mathbb{C}, F)$  to be the one sending a natural transformation  $\alpha : G \rightrightarrows F$  to its underlying family of maps  $(\alpha_c : Gc \rightarrow Fc \mid c \in \text{ob } \mathbb{C})$ , then it is clear that the diagram is indeed an equaliser. ■

**Proposition 1.7.9.** *Let  $\mathbb{C}$  be a small category and let  $\mathcal{M}$  be a locally small category.*

- *If  $B_{\bullet}(G, \mathbb{C}, F)$  is a bar complex in  $\mathcal{M}$ , then the colimit  $\lim_{\Delta^{\text{op}}} B_{\bullet}(G, \mathbb{C}, F)$  exists in  $\mathcal{M}$  if and only if the weighted colimit  $G \star_{\mathbb{C}} F$  exists in  $\mathcal{M}$ , and the two are isomorphic:*

$$G \star_{\mathbb{C}} F \cong \lim_{\Delta^{\text{op}}} B_{\bullet}(G, \mathbb{C}, F)$$

- *If  $C^{\bullet}(G, \mathbb{C}, F)$  is a cobar complex in  $\mathcal{M}$ , then the limit  $\lim_{\Delta} C^{\bullet}(G, \mathbb{C}, F)$  exists in  $\mathcal{M}$  if and only if the weighted limit  $\{G, F\}^{\mathbb{C}}$  exists in  $\mathcal{M}$ , and the*

two are isomorphic:

$$\{G, F\}^{\mathbb{C}} \cong \varprojlim_{\Delta} B_{\bullet}(G, \mathbb{C}, F)$$

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $M$  be any object in  $\mathcal{M}$ . Using lemma A.5.12, proposition A.5.13, and the above lemma, we obtain the following natural bijections:

$$\begin{aligned} \{G, \mathcal{M}(F, M)\}^{\mathbb{C}^{\text{op}}} &\cong [\mathbb{C}^{\text{op}}, \mathbf{Set}](G, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\Delta} C^{\bullet}(G, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\Delta} \mathcal{M}(B_{\bullet}(G, \mathbb{C}, F), M) \end{aligned}$$

It follows by the Yoneda lemma that  $G \star_{\mathbb{C}} F \cong \varinjlim_{\Delta^{\text{op}}} B_{\bullet}(G, \mathbb{C}, F)$ . ■

**Lemma 1.7.10.** *Let  $\mathbb{C}$  be a small category.*

- (i) *For each natural number  $n$  and each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , the functor  $C^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$  preserves limits, weighted limits, and ends.*
- (ii) *For each natural number  $n$  and each diagram  $F : \mathbb{C} \rightarrow \mathbf{Set}$ , the functor  $C^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\text{op}} \rightarrow \mathbf{Set}$  sends colimits to limits, weighted colimits to weighted limits, and coends to ends.*

*Proof.* Obvious. ◆

**Proposition 1.7.11.** *Let  $\mathbb{C}$  be a small category and let  $\mathcal{M}$  be a locally small category. If  $\mathcal{M}$  has coproducts for small families of objects, then:*

- (i) *For each natural number  $n$  and each weight  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ , the functor  $B_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \rightarrow \mathcal{M}$  preserves colimits, weighted colimits, and coends.*
- (ii) *For each natural number  $n$  and each diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$ , the functor  $B_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \rightarrow \mathcal{M}$  preserves colimits, weighted colimits, and coends.*

*Dually, if  $\mathcal{M}$  has products for small families of objects, then:*

- (i) *For each natural number  $n$  and each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , the functor  $C^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$  preserves limits, weighted limits, and ends.*

- (ii) For each natural number  $n$  and each diagram  $F : \mathbb{C} \rightarrow \mathbf{Set}$ , the functor  $C^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\text{op}} \rightarrow \mathbf{Set}$  sends colimits to limits, weighted colimits to weighted limits, and coends to ends.

*Proof.* We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

**Lemma 1.7.12.** *Let  $\mathbb{C}$  be a small category.*

- Let  $F : \mathbb{C} \rightarrow \mathbf{Set}$  be a diagram and let  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  be a weight. For all sets  $X$ , we have bijections

$$B_n(G \times X, \mathbb{C}, F) \cong X \times B_n(G, \mathbb{C}, F) \cong B_n(G, \mathbb{C}, X \times F)$$

that are natural in  $X$ ,  $F$ , and  $G$ .

- Let  $F : \mathbb{C} \rightarrow \mathbf{Set}$  be a diagram and let  $G : \mathbb{C} \rightarrow \mathbf{Set}$  be a weight. For all sets  $X$ , we have bijections

$$C^n(X \times G, \mathbb{C}, F) \cong [X, C^n(G, \mathbb{C}, F)] \cong C^n(G, \mathbb{C}, [X, F])$$

that are natural in  $X$ ,  $F$ , and  $G$ .

*Proof.* Obvious. ◆

**Proposition 1.7.13.** *Let  $\mathbb{C}$  be a small category and let  $\mathcal{M}$  be a locally small category. If  $\mathcal{M}$  has coproducts for small families of objects, then:*

- (i) Let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram, let  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  be a weight and let  $M$  be any object in  $\mathcal{M}$ . We then have bijections

$$\mathcal{M}(B_n(G, \mathbb{C}, F), M) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(B_n(h_{c'}, \mathbb{C}, h_c), \mathcal{M}(Gc' \odot Fc, M))$$

that are natural in  $n$ ,  $F$ ,  $G$ , and  $M$ .

- (ii) If  $\mathcal{M}$  is cotensored, then for each natural number  $n$  and each weight  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ , the functor  $B_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \rightarrow \mathcal{M}$  has a right adjoint, namely the functor that sends an object  $M$  to the diagram  $c \mapsto B_n(G, \mathbb{C}^{\text{op}}, h_c) \pitchfork M$ .
- (iii) For each natural number  $n$  and each diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$ , the functor  $B_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \rightarrow \mathcal{M}$  has a right adjoint, namely the functor that sends an object  $M$  to the weight  $c \mapsto C^n(h_c, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M))$ .



Dually, if  $\mathcal{M}$  has products for small families of objects, then:

- (i') Let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram, let  $G : \mathbb{C} \rightarrow \mathbf{Set}$  be a weight, and let  $M$  be an object in  $\mathcal{M}$ . We then have bijections

$$\mathcal{M}(M, C^n(G, \mathbb{C}, F)) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(\mathbf{B}_n(\hat{h}_{c'}, \mathbb{C}, \hat{h}^c), \mathcal{M}(M, Gc' \pitchfork Fc))$$

that are natural in  $n$ ,  $F$ ,  $G$ , and  $M$ .

- (ii') If  $\mathcal{M}$  is tensored, then for each natural number  $n$  and each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , the functor  $C^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \rightarrow \mathcal{M}$  has a left adjoint, namely the functor that sends an object  $M$  to the diagram  $c \mapsto \mathbf{B}_n(G, \mathbb{C}^{\text{op}}, \hat{h}_c) \odot M$ .
- (iii') For each natural number  $n$  and each diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$ , the functor  $C^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\text{op}} \rightarrow \mathcal{M}$  has a left adjoint, namely the functor that sends an object  $M$  to the weight  $c \mapsto C^n(\hat{h}^c, \mathbb{C}, \mathcal{M}(M, F))$ .

*Proof.* The two sets of claims are formally dual; we will prove the first version.

- (i). Using the interchange law for ends ([theorem A.6.16](#)), the Yoneda lemma for ends ([proposition A.6.17](#)), and [proposition I.7.II](#), we obtain the following natural bijections:

$$\begin{aligned} & \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(\mathbf{B}_n(\hat{h}_{c'}, \mathbb{C}, \hat{h}^c), \mathcal{M}(Gc' \odot Fc, M)) \\ & \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} C^n(\hat{h}_{c'}, \mathbb{C}^{\text{op}}, \mathbf{Set}(\hat{h}^c, \mathcal{M}(Gc' \odot Fc, M))) \\ & \cong \int_{c' : \mathbb{C}} \int_{c : \mathbb{C}} C^n(\hat{h}_{c'}, \mathbb{C}^{\text{op}}, \mathbf{Set}(\hat{h}^c, \mathcal{M}(Gc' \odot Fc, M))) \\ & \cong \int_{c' : \mathbb{C}} C^n\left(\hat{h}_{c'}, \mathbb{C}^{\text{op}}, \int_{c : \mathbb{C}} \mathbf{Set}(\hat{h}^c, \mathcal{M}(Gc' \odot Fc, M))\right) \\ & \cong \int_{c' : \mathbb{C}} C^n(\hat{h}_{c'}, \mathbb{C}^{\text{op}}, \mathcal{M}(Gc' \odot F, M)) \\ & \cong \int_{c' : \mathbb{C}} C^n(Gc' \times \hat{h}_{c'}, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ & \cong C^n\left(\int^{c' : \mathbb{C}} Gc' \times \hat{h}_{c'}, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)\right) \\ & \cong C^n(G, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \end{aligned}$$

$$\cong \mathcal{M}(\mathbf{B}_n(G, \mathbb{C}, F), M)$$

(ii). Similarly, we have the following natural bijections:

$$\begin{aligned} \mathcal{M}(\mathbf{B}_n(G, \mathbb{C}, F), M) &\cong \mathbf{C}^n(G, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ &\cong \mathbf{C}^n\left(G, \mathbb{C}^{\text{op}}, \int_{c:\mathbb{C}} \mathbf{Set}(h^c, \mathcal{M}(Fc, M))\right) \\ &\cong \int_{c:\mathbb{C}} \mathbf{C}^n(G, \mathbb{C}^{\text{op}}, \mathbf{Set}(h^c, \mathcal{M}(Fc, M))) \\ &\cong \int_{c:\mathbb{C}} \mathbf{Set}(\mathbf{B}_n(G, \mathbb{C}, h^c), \mathcal{M}(Fc, M)) \\ &\cong \int_{c:\mathbb{C}} \mathcal{M}(Fc, \mathbf{B}_n(G, \mathbb{C}, h^c) \pitchfork M) \end{aligned}$$

Now apply [remark A.6.5](#).

(iii). Along the same lines:

$$\begin{aligned} \mathcal{M}(\mathbf{B}_n(G, \mathbb{C}, F), M) &\cong \mathbf{C}^n(G, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ &\cong \mathbf{C}^n\left(\int_{c:\mathbb{C}} Gc \times h_c, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)\right) \\ &\cong \int_{c:\mathbb{C}} \mathbf{C}^n(Gc \times h_c, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ &\cong \int_{c:\mathbb{C}} \mathbf{C}^n(Gc \times h_c, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M)) \\ &\cong \int_{c:\mathbb{C}} \mathbf{Set}(Gc, \mathbf{C}^n(h_c, \mathbb{C}^{\text{op}}, \mathcal{M}(F, M))) \end{aligned}$$

Note that in the last step we are appealing to [lemma 1.7.12](#). ■

**REMARK 1.7.14.** The above proposition shows that bar complexes are a certain kind of weighted colimit, while cobar complexes are a certain kind of weighted limit.

**Definition 1.7.15.** Let  $\mathbb{C}$  be a small category, let  $\mathcal{A}$  be any category and let  $\mathcal{M}$  be a locally small category.

- Given  $\odot : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$ , a **bar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$  weighted by  $G : \mathbb{C}^{\text{op}} \rightarrow \mathcal{A}$  is a simplicial object  $B_{\bullet}(G, \mathbb{C}, F)$  equipped with bijections

$$\mathcal{M}(B_n(G, \mathbb{C}, F), M) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(B_n(h_{c'}, \mathbb{C}, h^c), \mathcal{M}(Gc' \odot Fc, M))$$

that are natural in both  $n$  and  $M$ .

- Given  $\pitchfork : \mathcal{A}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$ , a **cobar complex** for a diagram  $F : \mathbb{C} \rightarrow \mathcal{M}$  weighted by  $G : \mathbb{C} \rightarrow \mathcal{A}$  is a cosimplicial object  $C^{\bullet}(G, \mathbb{C}, F)$  equipped with bijections

$$\mathcal{M}(M, C^n(G, \mathbb{C}, F)) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(B_n(h_{c'}, \mathbb{C}, h^c), \mathcal{M}(M, Gc' \pitchfork Fc))$$

that are natural in both  $n$  and  $M$ .

**REMARK 1.7.16.** Although the definition given here is stated using an end, one can also state a version that only uses products. Thus these generalised bar (resp. cobar) complexes exist in a locally small category  $\mathcal{M}$  as soon as  $\mathcal{M}$  has coproducts (resp. products) for small families of objects.

**REMARK 1.7.17.** In the case where  $\mathcal{A} = \mathcal{M} = \mathbf{sSet}$ , we will almost always take  $A \odot M = A \times M$  and  $A \pitchfork M = [A, M]$ . With this choice, the formulae of [definition 1.7.1](#) (understood appropriately) can be applied verbatim.

**Theorem 1.7.18.** Let  $\mathbb{C}$  and  $\mathbb{D}$  be two small categories, let  $\mathcal{A}$  and  $\mathcal{M}$  be two locally small categories, and let  $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ ,  $\odot : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$ ,  $\pitchfork : \mathcal{A}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$ , and  $\underline{\mathcal{M}} : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{A}$  be functors. Suppose  $\mathcal{A}$  has coproducts for small families of objects, that there are bijections

$$\mathcal{M}(A \odot M, N) \cong \mathcal{A}(A, \underline{\mathcal{M}}(M, N)) \cong \mathcal{M}(M, A \pitchfork N)$$

that are natural in  $A$ ,  $M$ , and  $N$ , and that there are isomorphisms

$$\begin{aligned} (A \otimes B) \odot M &\cong A \odot (B \odot M) \\ (A \otimes B) \pitchfork M &\cong A \pitchfork (B \pitchfork M) \end{aligned}$$

that are natural in  $A$ ,  $B$ , and  $M$ .

## I. SIMPLICIAL SETS

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- Let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram, let  $G : \mathbb{D}^{\text{op}} \rightarrow \mathcal{A}$  be a weight, and let  $H : \mathbb{C}^{\text{op}} \times \mathbb{D} \rightarrow \mathcal{A}$  be a functor. If  $\mathcal{M}$  has coproducts for small families of objects, then there is an isomorphism

$$\mathbf{B}_m(\mathbf{B}_n(G, \mathbb{D}, H), \mathbb{C}, F) \cong \mathbf{B}_n(G, \mathbb{D}, \mathbf{B}_m(H, \mathbb{C}, F))$$

that is natural in  $m, n, F, G,$  and  $H$ .

- Let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram, let  $G : \mathbb{D} \rightarrow \mathcal{A}$  be a weight, and let  $H : \mathbb{D}^{\text{op}} \times \mathbb{C} \rightarrow \mathcal{A}$  be a functor. If  $\mathcal{M}$  has products for small families of objects, then there is an isomorphism

$$\mathbf{C}^m(\mathbf{B}_n(G, \mathbb{D}^{\text{op}}, H), \mathbb{C}, F) \cong \mathbf{C}^n(G, \mathbb{D}, \mathbf{C}^m(H, \mathbb{C}, F))$$

that is natural in  $m, n, F, G,$  and  $H$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $M$  be any object in  $\mathcal{M}$  and let  $K : \mathbb{D}^{\text{op}} \times \mathbb{C}^{\text{op}} \times \mathbb{D} \times \mathbb{C} \rightarrow \mathbf{Set}$  be the functor defined below:

$$K(d', c', d, c) = \mathcal{A}(Gd' \otimes H(c', d), \underline{\mathcal{M}}(Fc, M))$$

Notice that we have the following natural bijections:

$$\begin{aligned} K(d', c', d, c) &\cong \mathcal{M}((Gd' \otimes H(c', d)) \odot Fc, M) \\ &\cong \mathcal{M}(Gd' \odot (H(c', d) \odot Fc), M) \\ &\cong \mathcal{M}(H(c', d) \odot Fc, Gd' \pitchfork M) \end{aligned}$$

Now, using the definition of the generalised bar complex, we obtain the natural bijections shown below:

$$\begin{aligned} &\mathcal{M}(\mathbf{B}_m(\mathbf{B}_n(G, \mathbb{D}, H), \mathbb{C}, F), M) \\ &\cong \int_{(c', c)} \mathbf{Set}(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c), \mathcal{M}(\mathbf{B}_m(G, \mathbb{D}, H(c', -)) \odot Fc, M)) \\ &\cong \int_{(c', c)} \mathbf{Set}(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c), \mathcal{A}(\mathbf{B}_m(G, \mathbb{D}, H(c', -)), \underline{\mathcal{M}}(Fc, M))) \\ &\cong \int_{(c', c)} \mathbf{Set}\left(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c), \int_{(d', d)} \mathbf{Set}(\mathbf{B}_m(h_{d'}, \mathbb{D}, h^d), K(d', c', d, c))\right) \\ &\cong \int_{(c', c)} \int_{(d', d)} \mathbf{Set}(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c) \times \mathbf{B}_m(h_{d'}, \mathbb{D}, h^d), K(d', c', d, c)) \end{aligned}$$

On the other hand,

$$\begin{aligned}
 & \mathcal{M}(\mathbf{B}_n(G, \mathbb{D}, \mathbf{B}_m(H, \mathbb{C}, F)), M) \\
 & \cong \int_{(d', d)} \mathbf{Set}(\mathbf{B}_n(\mathfrak{h}_{d'}, \mathbb{D}, \mathfrak{h}^d), \mathcal{M}(Gd' \odot \mathbf{B}_m(H(-, d), \mathbb{C}, F), M)) \\
 & \cong \int_{(d', d)} \mathbf{Set}(\mathbf{B}_n(\mathfrak{h}_{d'}, \mathbb{D}, \mathfrak{h}^d), \mathcal{M}(\mathbf{B}_m(H(-, d), \mathbb{C}, F), Gd' \pitchfork M)) \\
 & \cong \int_{(d', d)} \mathbf{Set}\left(\mathbf{B}_n(\mathfrak{h}_{d'}, \mathbb{D}, \mathfrak{h}^d), \int_{(c', c)} \mathbf{Set}(\mathbf{B}_m(\mathfrak{h}_{c'}, \mathbb{C}, \mathfrak{h}^c), K(d', c', d, c))\right) \\
 & \cong \int_{(d', d)} \int_{(c', c)} \mathbf{Set}(\mathbf{B}_n(\mathfrak{h}_{d'}, \mathbb{D}, \mathfrak{h}^d) \times \mathbf{B}_m(\mathfrak{h}_{c'}, \mathbb{C}, \mathfrak{h}^c), K(d', c', d, c))
 \end{aligned}$$

and so, applying the interchange law for ends ([theorem A.6.16](#)), we obtain a natural bijection

$$\mathcal{M}(\mathbf{B}_m(\mathbf{B}_n(G, \mathbb{D}, H), \mathbb{C}, F), M) \cong \mathcal{M}(\mathbf{B}_n(G, \mathbb{D}, \mathbf{B}_m(H, \mathbb{C}, F)), M)$$

and the claim follows by the Yoneda lemma. ■

**Definition 1.7.19.** Let  $\mathbb{C}$  be a small category.

- Given  $\odot : \mathcal{A} \times \mathbf{sSet} \rightarrow \mathbf{sSet}$ , the **bar construction** for a diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  weighted by a functor  $G : \mathbb{C}^{\text{op}} \rightarrow \mathcal{A}$  is the following coend:

$$\mathbf{B}(G, \mathbb{C}, F) = \int^{[n]:\Delta} \Delta^n \times \mathbf{B}_n(G, \mathbb{C}, F)$$

In other words,  $\mathbf{B}(G, \mathbb{C}, F)$  is the realisation  $|\mathbf{B}_\bullet(G, \mathbb{C}, F)|$ .

- Given  $\pitchfork : \mathcal{A}^{\text{op}} \times \mathbf{sSet} \rightarrow \mathbf{sSet}$ , the **cobar construction** for a diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  weighted by a functor  $G : \mathbb{C} \rightarrow \mathcal{A}$  is the following end:

$$\mathbf{C}(G, \mathbb{C}, F) = \int_{[n]:\Delta} [\Delta^n, \mathbf{B}_n(G, \mathbb{C}, F)]$$

In other words,  $\mathbf{C}(G, \mathbb{C}, F)$  is the totalisation  $\text{Tot } \mathbf{C}^\bullet(G, \mathbb{C}, F)$ .

**Lemma 1.7.20.** *Let  $\mathbb{C}$  be a small category, let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram, and let  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{sSet}$  be a weight. We then have bijections*

$$(\mathbf{B}(G, \mathbb{C}, F))_n \cong \mathbf{B}_n(G_n, \mathbb{C}, F_n)$$

*that are natural in  $n$ .*

*Proof.* Apply lemma 1.6.7 to remark 1.7.17. ■

**Proposition 1.7.21.** *Let  $\mathbb{C}$  be a small category and let  $\mathcal{A}$  be any category.*

- *Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram, let  $G : \mathbb{C}^{\text{op}} \rightarrow \mathcal{A}$  be a weight, and let  $M$  be a simplicial set. Given  $\odot : \mathcal{A} \times \mathbf{sSet} \rightarrow \mathbf{sSet}$ , we have bijections*

$$\mathbf{sSet}(\mathbf{B}(G, \mathbb{C}, F), M) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{sSet}(\mathbf{B}_\bullet(h_{c'}, \mathbb{C}, h^c), [Gc' \odot Fc, M])$$

*that are natural in  $F$ ,  $G$ , and  $M$ .*

- *Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram, let  $G : \mathbb{C} \rightarrow \mathcal{A}$  be a weight, and let  $M$  be a simplicial set. Given  $\pitchfork : \mathcal{A}^{\text{op}} \times \mathbf{sSet} \rightarrow \mathbf{sSet}$ , we have bijections*

$$\mathbf{sSet}(M, \mathbf{C}(G, \mathbb{C}, F)) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{sSet}(\mathbf{B}_\bullet(h_{c'}, \mathbb{C}, h^c), [M, Gc' \pitchfork Fc])$$

*that are natural in  $F$ ,  $G$ , and  $M$ .*

*Proof.* We will prove the first claim; the second can be proved in a similar way. By definition, we have the natural bijection

$$\mathbf{sSet}(\mathbf{B}(G, \mathbb{C}, F), M) \cong \int_{[n] : \Delta} \mathbf{sSet}(\Delta^n \times \mathbf{B}_n(G, \mathbb{C}, F), M)$$

and furthermore, we also have the following:

$$\begin{aligned} \mathbf{sSet}(\Delta^n \times \mathbf{B}_n(G, \mathbb{C}, F), M) &\cong \mathbf{sSet}(\mathbf{B}_n(G, \mathbb{C}, F), [\Delta^n, M]) \\ &\cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c), \mathbf{sSet}(Gc' \odot Fc, [\Delta^n, M])) \\ &\cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{Set}(\mathbf{B}_n(h_{c'}, \mathbb{C}, h^c), \mathbf{sSet}(\Delta^n, [Gc' \odot Fc, M])) \\ &\cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{sSet}(\text{disc } \mathbf{B}_n(h_{c'}, \mathbb{C}, h^c) \times \Delta^n, [Gc' \odot Fc, M]) \end{aligned}$$

Thus, applying the interchange law for ends (theorem A.6.16) and corollary 1.6.8, we obtain

$$\mathbf{sSet}(\mathbf{B}(G, \mathbb{C}, F), M) \cong \int_{(c', c) : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{sSet}(\mathbf{B}_\bullet(h_{c'}, \mathbb{C}, h^c), [Gc' \odot Fc, M])$$

as required. ■

**Proposition 1.7.22.** *Let  $\mathbb{C}$  be a small category.*

- (i) *For each weight  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{sSet}$ , the functor  $B(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  has a right adjoint, namely the functor that sends a simplicial set  $M$  to the diagram  $c \mapsto [B(G, \mathbb{C}^{\text{op}}, h_c), M]$ .*
- (ii) *For each diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$ , the functor  $B(-, \mathbb{C}, F) : [\mathbb{C}^{\text{op}}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  has a right adjoint, namely the functor that sends an object  $M$  to the weight  $c \mapsto C(h_c, \mathbb{C}^{\text{op}}, [F, M])$ .*
- (iii) *For each simplicial set  $X$ , there are isomorphisms*

$$B(X \times G, \mathbb{C}, F) \cong X \times B(G, \mathbb{C}, F) \cong B(G, \mathbb{C}, X \times F)$$

*that are natural in  $X$ ,  $F$ , and  $G$ .*

*Dually:*

- (i') *For each weight  $G : \mathbb{C} \rightarrow \mathbf{sSet}$ , the functor  $C(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  has a left adjoint, namely the functor that sends a simplicial set  $M$  to the diagram  $c \mapsto B(G, \mathbb{C}^{\text{op}}, h_c) \times M$ .*
- (ii') *For each diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$ , the functor  $C(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{sSet}]^{\text{op}} \rightarrow \mathbf{sSet}$  has a right adjoint, namely the functor that sends an object  $M$  to the weight  $c \mapsto C(h^c, \mathbb{C}, [M, F])$ .*
- (iii') *For each simplicial set  $X$ , there are isomorphisms*

$$C(X \times G, \mathbb{C}, F) \cong [X, C(G, \mathbb{C}, F)] \cong C(G, \mathbb{C}, [X, F])$$

*that are natural in  $X$ ,  $F$ , and  $G$ .*

*Proof.* We will prove the first set of claims; the second can be proved in a similar way.

- (i). Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram. By definition, we have the following natural bijections:

$$\begin{aligned} [\mathbb{C}, \mathbf{sSet}](F, [B(G, \mathbb{C}^{\text{op}}, h_\bullet), M]) &\cong \int_{c:\mathbb{C}} \mathbf{sSet}(Fc, [B(G, \mathbb{C}^{\text{op}}, h_c), M]) \\ &\cong \int_{c:\mathbb{C}} \mathbf{sSet}(B(G, \mathbb{C}^{\text{op}}, h_c), [Fc, M]) \end{aligned}$$

Now, using [proposition 1.7.21](#), we also obtain these natural bijections:

$$\begin{aligned} \mathbf{sSet}(\mathbf{B}(G, \mathbb{C}^{\text{op}}, h_c), [Fc, M]) \\ &\cong \int_{(c'', c') : \mathbb{C} \times \mathbb{C}^{\text{op}}} \mathbf{sSet}(\mathbf{B}(h^{c''}, \mathbb{C}^{\text{op}}, h_{c'}), [Gc'' \times \mathbb{C}(c', c), [Fc, M]]) \\ &\cong \int_{(c'', c') : \mathbb{C} \times \mathbb{C}^{\text{op}}} \mathbf{sSet}(\mathbf{B}(h^{c''}, \mathbb{C}^{\text{op}}, h_{c'}), [Gc'' \times \mathbb{C}(c', c) \times Fc, M]) \end{aligned}$$

Applying the interchange law for ends ([theorem A.6.16](#)) and the Yoneda lemma for coends ([proposition A.6.17](#)), we deduce that

$$\begin{aligned} [\mathbb{C}, \mathbf{sSet}](F, [\mathbf{B}(G, \mathbb{C}^{\text{op}}, h_\bullet), M]) \\ &\cong \int_{(c'', c') : \mathbb{C}^{\text{op}} \times \mathbb{C}} \mathbf{sSet}(\mathbf{B}(h^{c''}, \mathbb{C}^{\text{op}}, h_{c'}), [Gc'' \times Fc', M]) \\ &\cong \mathbf{sSet}(\mathbf{B}(G, \mathbb{C}, F), M) \end{aligned}$$

naturally in  $F$  and  $M$ , as required.

(ii). The proof is similar to that of claim (i).

(iii). Apply lemmas [1.7.12](#) to [1.7.20](#). (For the dual claim, use [theorem 1.6.18](#) instead.) ■

**Theorem 1.7.23.** *Let  $\mathbb{C}$  and  $\mathbb{D}$  be two small categories.*

- *Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram, let  $G : \mathbb{D}^{\text{op}} \rightarrow \mathbf{sSet}$  be a weight, and let  $H : \mathbb{C}^{\text{op}} \times \mathbb{D} \rightarrow \mathbf{sSet}$  be a functor. There is then an isomorphism*

$$\mathbf{B}_m(\mathbf{B}_n(G, \mathbb{D}, H), \mathbb{C}, F) \cong \mathbf{B}_n(G, \mathbb{D}, \mathbf{B}_m(H, \mathbb{C}, F))$$

*that is natural in  $m, n, F, G,$  and  $H$ .*

- *Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a diagram, let  $G : \mathbb{D} \rightarrow \mathbf{sSet}$  be a weight, and let  $H : \mathbb{D}^{\text{op}} \times \mathbb{C} \rightarrow \mathbf{sSet}$  be a functor. There is then an isomorphism*

$$\mathbf{C}^m(\mathbf{B}_n(G, \mathbb{D}^{\text{op}}, H), \mathbb{C}, F) \cong \mathbf{C}^n(G, \mathbb{D}, \mathbf{C}^m(H, \mathbb{C}, F))$$

*that is natural in  $m, n, F, G,$  and  $H$ .*

*Proof.* The proof is essentially the same as that of [theorem 1.7.18](#). ■



**Proposition 1.7.24.** *Let  $\mathbb{C}$  be a small category.*

- For each diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  and each functor  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ , there is a morphism  $B(G, \mathbb{C}, F) \rightarrow G \star_{\mathbb{C}} F$ , and it is natural in both  $F$  and  $G$ .
- For each diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  and each functor  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , there is a morphism  $\{G, F\}^{\mathbb{C}} \rightarrow C(G, \mathbb{C}, F)$ , and it is natural in both  $F$  and  $G$ .

*Proof.* By [theorem A.6.13](#) and [proposition 1.7.9](#), we have the following natural isomorphisms:

$$\int_{[n]:\Delta} B_n(G, \mathbb{C}, F) \cong \Delta 1 \star_{\Delta^{\text{op}}} B_{\bullet}(G, \mathbb{C}, F) \cong \varinjlim_{\Delta^{\text{op}}} B_{\bullet}(G, \mathbb{C}, F) \cong G \star_{\mathbb{C}} F$$

$$\int_{[n]:\Delta} C^n(G, \mathbb{C}, F) \cong \{\Delta 1, C^{\bullet}(G, \mathbb{C}, F)\}^{\Delta} \cong \varprojlim_{\Delta} C^{\bullet}(G, \mathbb{C}, F) \cong \{G, F\}^{\mathbb{C}}$$

The claim then follows from the existence of a (unique) natural transformation  $\Delta^{\bullet} \Rightarrow \Delta 1$ . ■

**Definition 1.7.25.** Let  $\mathbb{C}$  be a small category, let  $\mathcal{M}$  be a locally small category, and let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram.

- The **bar resolution** of  $F$  is the diagram  $B_{\bullet}(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \rightarrow [\Delta^{\text{op}}, \mathcal{M}]$  defined by the following formula,

$$c \mapsto B_{\bullet}(h_c, \mathbb{C}, F)$$

where  $h_c : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$  is the representable functor  $\mathbb{C}(-, c)$ .

- The **cobar resolution** of  $F$  is the diagram  $C^{\bullet}(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \rightarrow [\Delta, \mathcal{M}]$  defined by the following formula,

$$c \mapsto C^{\bullet}(h^c, \mathbb{C}, F)$$

where  $h^c : \mathbb{C} \rightarrow \mathbf{Set}$  is the representable functor  $\mathbb{C}(c, -)$ .

**Lemma 1.7.26.** *Let  $\mathbb{C}$  be a small category and let  $F : \mathbb{C} \rightarrow \mathbf{Set}$  be a diagram.*

- (i) *There is an isomorphism*

$$F \cong \varprojlim_{\Delta} C^{\bullet}(\mathbb{C}, \mathbb{C}, F)$$

*and it is natural in  $F$ .*

(ii) For each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , there is an isomorphism

$$\{G, \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathbf{C}^\bullet(G, \mathbb{C}, F)$$

and it is natural in both  $F$  and  $G$ .

(iii) For each object  $c$  in  $\mathbb{C}$ , there exist maps  $\eta_c : Fc \rightarrow \mathbf{C}^0(\mathcal{h}^c, \mathbb{C}, F)$ ,  $\varepsilon_c : \mathbf{C}^0(\mathcal{h}^c, \mathbb{C}, F) \rightarrow Fc$ , and  $h_{n,c} : \mathbf{C}^{n+1}(\mathcal{h}^c, \mathbb{C}, F) \rightarrow \mathbf{C}^n(\mathcal{h}^c, \mathbb{C}, F)$  satisfying these identities:

$$\begin{aligned} \delta_1^1 \circ \eta_c &= \delta_1^0 \circ \eta_c \\ \varepsilon_c \circ \eta_c &= \text{id} \\ h_{0,c} \circ \delta_1^0 &= \eta_c \circ \varepsilon_c \\ h_{n,c} \circ \delta_{n+1}^i &= \delta_n^i \circ h_{n-1,c} && \text{if } 0 \leq i \leq n \\ h_{n,c} \circ \delta_{n+1}^{n+1} &= \text{id} \\ \sigma_n^i \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^i && \text{if } 0 \leq i \leq n \\ h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^{n+1} \end{aligned}$$

These maps are moreover natural in  $F$ , and  $\eta_c$  is also natural in  $c$ .

*Proof.* (i). By lemma 1.7.8, there are bijections

$$[\mathbb{C}, \mathbf{Set}](\mathcal{h}^c, F) \cong \lim_{\leftarrow \Delta} \mathbf{C}^\bullet(\mathcal{h}^c, \mathbb{C}, F)$$

that are natural in  $c$  and  $F$ , so the Yoneda lemma implies  $F \cong \lim_{\leftarrow \Delta} \circ \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$ , naturally in  $F$ .

(ii). Applying the Yoneda lemma for ends (proposition A.6.17), we obtain the following natural bijections:

$$\begin{aligned} &\int_{c:\mathbb{C}} [G(c), [\mathbb{C}(c, c_n) \times \mathbb{C}(c_n, c_{n-1}) \times \cdots \times \mathbb{C}(c_1, c_0), F(c_0)]] \\ &\cong \int_{c:\mathbb{C}} [\mathbb{C}(c, c_n), [G(c) \times \mathbb{C}(c_n, c_{n-1}) \times \cdots \times \mathbb{C}(c_1, c_0), F(c_0)]] \\ &\cong [G(c_n) \times \mathbb{C}(c_n, c_{n-1}) \times \cdots \times \mathbb{C}(c_1, c_0), F(c_0)] \end{aligned}$$

Theorem A.6.13 implies that there is a natural isomorphism

$$\{G, \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \int_{c:\mathbb{C}} [G, \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)]$$

and it is now clear that the claim holds.

(iii). Let  $\eta_c$ ,  $\varepsilon_c$ , and  $h_{n,c}$  be the maps defined below:

$$\begin{aligned}\eta_c(x)_{(c_0)} &= (y \mapsto F(y)(x)) \\ \varepsilon_c(x) &= x_{(c)}(\text{id}_c) \\ h_{n,c}(x)_{(c_0, \dots, c_n)} &= \left( (y, f_n, \dots, f_1) \mapsto x_{(c_0, \dots, c_n, c)}(\text{id}_c, y, f_n, \dots, f_1) \right)\end{aligned}$$

By construction, we have  $\varepsilon_c \circ \eta_c = \text{id}_{F_c}$ , and it is not hard to check that the other identities are satisfied. For naturality of  $\eta_c$  in  $c$ , observe that, given  $f : c \rightarrow c'$  in  $\mathbb{C}$ , we have

$$\begin{aligned}\eta_{c'}(F(f)(x))_{(c_0)} &= (y \mapsto F(y)(F(f)(x))) \\ &= (y \mapsto F(y \circ f)(x)) \\ &= (y \mapsto F(h^f(y))(x)) \\ &= C^0(h^f, \mathbb{C}, F)(\eta_c(x))_{(c_0)}\end{aligned}$$

and so  $\eta_{c'} \circ F(f) = C^0(h^f, \mathbb{C}, F) \circ \eta_c$ , as required. ■

**Proposition 1.7.27.** *Let  $\mathbb{C}$  be a small category, let  $\mathcal{M}$  be a locally small category, and let  $F : \mathbb{C} \rightarrow \mathcal{M}$  be a diagram. If the bar resolution  $B_\bullet(\mathbb{C}, \mathbb{C}, F)$  exists, then:*

(i) *There is an isomorphism*

$$F \cong \lim_{\Delta^{\text{op}}} \circ B_\bullet(\mathbb{C}, \mathbb{C}, F)$$

*and it is natural in  $F$ .*

(ii) *For each weight  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{Set}$ , there is an isomorphism*

$$G \star_{\mathbb{C}} B_\bullet(\mathbb{C}, \mathbb{C}, F) \cong B_\bullet(G, \mathbb{C}, F)$$

*and it is natural in both  $F$  and  $G$ .*

(iii) *For each object  $c$  in  $\mathbb{C}$ , there exist morphisms  $\eta_c : Fc \rightarrow B_0(h_c, \mathbb{C}, F)$ ,  $\varepsilon_c : B_0(h_c, \mathbb{C}, F) \rightarrow Fc$ , and  $h_c^n : B_n(h_c, \mathbb{C}, F) \rightarrow B_{n+1}(h_c, \mathbb{C}, F)$  satisfying these identities:*

$$\varepsilon_c \circ d_1^1 = \varepsilon_c \circ d_0^1$$

$$\begin{aligned}
\varepsilon_c \circ \eta_c &= \text{id} \\
d_0^1 \circ h_c^0 &= s \circ r \\
d_i^{n+1} \circ h_c^n &= h_c^{n-1} \circ d_i^n && \text{if } 0 \leq i \leq n \\
d_{n+1}^{n+1} \circ h_c^n &= \text{id} \\
h_c^{n+1} \circ s_i^n &= s_i^{n+1} \circ h_c^n && \text{if } 0 \leq i \leq n \\
h_c^{n+1} \circ h_c^n &= s_{n+1}^{n+1} \circ h_c^n
\end{aligned}$$

These morphisms are moreover natural in  $F$ , and  $\varepsilon_c$  is also natural in  $c$ .

Dually, if the cobar resolution  $\mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$  exists, then:

(i) There is an isomorphism

$$F \cong \varprojlim_{\Delta} \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in  $F$ .

(ii) For each weight  $G : \mathbb{C} \rightarrow \mathbf{Set}$ , there is an isomorphism

$$\{G, \mathbf{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathbf{C}^\bullet(G, \mathbb{C}, F)$$

and it is natural in both  $F$  and  $G$ .

(iii) For each object  $c$  in  $\mathbb{C}$ , there exist morphisms  $\eta_c : Fc \rightarrow \mathbf{C}^0(\mathcal{H}^c, \mathbb{C}, F)$ ,  $\varepsilon_c : \mathbf{C}^0(\mathcal{H}^c, \mathbb{C}, F) \rightarrow Fc$ , and  $h_{n,c} : \mathbf{C}^{n+1}(\mathcal{H}^c, \mathbb{C}, F) \rightarrow \mathbf{C}^n(\mathcal{H}^c, \mathbb{C}, F)$  satisfying these identities:

$$\begin{aligned}
\delta_1^1 \circ \eta_c &= \delta_1^0 \circ \eta_c \\
\varepsilon_c \circ \eta_c &= \text{id} \\
h_{0,c} \circ \delta_1^0 &= \eta_c \circ \varepsilon_c \\
h_{n,c} \circ \delta_{n+1}^i &= \delta_n^i \circ h_{n-1,c} && \text{if } 0 \leq i \leq n \\
h_{n,c} \circ \delta_{n+1}^{n+1} &= \text{id} \\
\sigma_n^i \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^i && \text{if } 0 \leq i \leq n \\
h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^{n+1}
\end{aligned}$$

These morphisms are moreover natural in  $F$ , and  $\eta_c$  is also natural in  $c$ .

*Proof.* We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

## 1.8 Homotopy limits and colimits

**Prerequisites.** §§ 1.2, 1.5, 1.6, 1.7, 3.3, 3.4.

**REMARK 1.8.1.** It is important to stress that there is an asymmetry between the theory of homotopy colimits and the theory of homotopy limits in  $\mathbf{sSet}$  because not all simplicial sets are fibrant. As such, it will often be necessary to restrict our attention to Kan complexes when working with homotopy limits.

**Proposition 1.8.2.** *Let  $\mathbb{C}$  be a small category. For any weight  $G : \mathbb{C} \rightarrow \mathbf{sSet}$ , if  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  is a diagram of Kan complexes, then:*

- (i) *The cobar complex  $C^\bullet(G, \mathbb{C}, F)$  is a Reedy-fibrant cosimplicial simplicial set, and each horizontal level is a Kan complex.*
- (ii) *The cobar construction  $C(G, \mathbb{C}, F)$  is a Kan complex.*

*Proof.* (i). Recalling [remark 1.7.17](#), we have the following formula:

$$C^n(G, \mathbb{C}, F) \cong \prod_{(c_0, \dots, c_n)} [Gc_n \times \mathbb{C}(c_n, c_{n-1}) \times \dots \times \mathbb{C}(c_1, c_0), Fc_0]$$

Since  $Fc_0$  is a Kan complex, [corollary 1.4.16](#) and [proposition 4.3.17](#) imply that  $C^n(G, \mathbb{C}, F)$  is also a Kan complex. The same results, plus the fact that the class of Kan fibrations is closed under pullbacks ([proposition A.3.14](#)), can then be used to show that the matching morphisms for  $C^\bullet(G, \mathbb{C}, F)$  are Kan fibrations. Thus, the cobar complex is Reedy-fibrant.

(ii). [Theorem 1.6.18](#) says Tot sends Reedy fibrations in  $[\mathbf{\Delta}, \mathbf{sSet}]$  to Kan fibrations in  $\mathbf{sSet}$ , so  $C(G, \mathbb{C}, F)$  is a Kan complex when  $C^\bullet(G, \mathbb{C}, F)$  is Reedy-fibrant. ■

**Corollary 1.8.3.** *Kan is closed in  $\mathbf{sSet}$  under cobar complexes and cobar constructions.* ■

**Definition 1.8.4.** Let  $C$  be a category and let  $F, F' : C \rightarrow \mathbf{sSet}$  be functors. A **natural weak homotopy equivalence**  $F \Rightarrow F'$  is a natural transformation whose components are weak homotopy equivalences of simplicial sets.

**Proposition 1.8.5.** *Let  $\mathbb{C}$  be a small category.*

- (i) *For each weight  $G : \mathbb{C}^{\text{op}} \rightarrow \mathbf{sSet}$ , if  $\varphi : F \Rightarrow F'$  is a natural weak homotopy equivalence of diagrams  $\mathbb{C} \rightarrow \mathbf{sSet}$ , then the induced morphism  $B(G, \mathbb{C}, \varphi) : B(G, \mathbb{C}, F) \rightarrow B(G, \mathbb{C}, F')$  is a weak homotopy equivalence of simplicial sets.*
- (ii) *For each diagram  $F : \mathbb{C} \rightarrow \mathbf{sSet}$ , if  $\psi : G \Rightarrow G'$  is a natural weak homotopy equivalence of weights  $\mathbb{C}^{\text{op}} \rightarrow \mathbf{sSet}$ , then the induced morphism  $B(\psi, \mathbb{C}, F) : B(G, \mathbb{C}, F) \rightarrow B(G', \mathbb{C}, F)$  is a weak homotopy equivalence of simplicial sets.*

*Dually:*

- (i') *For each weight  $G : \mathbb{C} \rightarrow \mathbf{sSet}$ , if  $\varphi : F \Rightarrow F'$  is a natural weak homotopy equivalence of diagrams  $\mathbb{C} \rightarrow \mathbf{Kan}$ , then the induced morphism  $C(G, \mathbb{C}, \varphi) : C(G, \mathbb{C}, F) \rightarrow C(G, \mathbb{C}, F')$  is a weak homotopy equivalence of Kan complexes.*
- (ii') *For each diagram  $F : \mathbb{C} \rightarrow \mathbf{Kan}$ , if  $\psi : G \Rightarrow G'$  is a natural weak homotopy equivalence of weights  $\mathbb{C}^{\text{op}} \rightarrow \mathbf{sSet}$ , then the induced morphism  $C(\psi, \mathbb{C}, F) : C(G, \mathbb{C}, F) \rightarrow C(G', \mathbb{C}, F)$  is a weak homotopy equivalence of Kan complexes.*

*Proof.* Use [proposition 1.5.11](#) and [corollary 4.3.18](#) to show that the corresponding morphisms of bar/cobar complexes is a Reedy weak homotopy equivalence, and then apply [theorems 1.6.9](#) and [1.6.18](#) to deduce that the induced morphism of bar/cobar constructions is a weak homotopy equivalence. ■

**Lemma 1.8.6.** *Let  $\mathbb{C}$  be a small category. For each object  $c$  in  $\mathbb{C}$ , the bar construction  $B(\Delta 1, \mathbb{C}, \mathfrak{h}^c)$  is a weakly contractible simplicial set.*

*Proof.* It is not hard to see that  $B(\Delta 1, \mathbb{C}, \mathfrak{h}^c)$  is naturally isomorphic to the nerve  $N({}^c/\mathbb{C})$ , and since  ${}^c/\mathbb{C}$  has an initial object, the unique functor  ${}^c/\mathbb{C} \rightarrow \mathbb{1}$  has a left adjoint. Using [proposition 1.2.1](#), we may obtain a backward contracting homotopy for  $N({}^c/\mathbb{C})$  onto  $\Delta^0$ , and thus, by [proposition 1.3.16](#),  $B(\Delta 1, \mathbb{C}, \mathfrak{h}^c)$  is a weakly contractible simplicial set. ■

**Proposition 1.8.7.** *Let  $F : \mathbb{C} \rightarrow \mathbf{sSet}$  be a small diagram.*

- (i) *There is a natural weak homotopy equivalence  $\bar{\varepsilon}_F : \mathbf{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$  that is natural in  $F$ .*
- (ii) *The induced morphism*

$$\lim_{\xrightarrow{\mathbb{C}}} \bar{\varepsilon}_{\mathbf{B}(\mathbb{C}, \mathbb{C}, F)} : \lim_{\xrightarrow{\mathbb{C}}} \mathbf{B}(\mathbb{C}, \mathbb{C}, \mathbf{B}(\mathbb{C}, \mathbb{C}, F)) \rightarrow \lim_{\xrightarrow{\mathbb{C}}} \mathbf{B}(\mathbb{C}, \mathbb{C}, F)$$

*is a weak homotopy equivalence of simplicial sets.*

- (iii) *The bar resolution functor  $\mathbf{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \rightarrow [\mathbb{C}, \mathbf{sSet}]$  preserves natural weak homotopy equivalences.*

*Dually:*

- (i') *There is a natural weak homotopy equivalence  $\bar{\eta}_F : F \Rightarrow \mathbf{C}(\mathbb{C}, \mathbb{C}, F)$  that is natural in  $F$ .*
- (ii') *If  $Fc$  is a Kan complex for each object  $c$  in  $\mathbb{C}$ , then the induced morphism*

$$\lim_{\xleftarrow{\mathbb{C}}} \bar{\eta}_{\mathbf{C}(\mathbb{C}, \mathbb{C}, F)} : \lim_{\xleftarrow{\mathbb{C}}} \mathbf{C}(\mathbb{C}, \mathbb{C}, F) \rightarrow \lim_{\xleftarrow{\mathbb{C}}} \mathbf{C}(\mathbb{C}, \mathbb{C}, \mathbf{C}(\mathbb{C}, \mathbb{C}, F))$$

*is a weak homotopy equivalence of Kan complexes.*

- (iii') *The cobar resolution functor  $\mathbf{C}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \rightarrow [\mathbb{C}, \mathbf{sSet}]$  preserves natural weak homotopy equivalences.*

*Proof.* We will prove the first set of claims; the second set can be proved in a similar way.

(i). [Proposition 1.3.16](#) applied to [proposition 1.7.27](#) implies that the components of the natural transformation  $\varepsilon_F : \mathbf{B}_{\bullet}(\mathbb{C}, \mathbb{C}, F) \Rightarrow \Delta \circ F$  are (halves of) simplicial homotopy equivalences. [Lemma 1.6.7](#) plus [corollary 2.3.9](#) then says that the induced morphisms  $\mathbf{B}(\mathcal{h}^c, \mathbb{C}, F) \rightarrow Fc$  are (halves of) simple homotopy equivalences, and so  $|\varepsilon_F| : \mathbf{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$  is a componentwise weak equivalence, by [proposition 1.5.2](#). It is clear that  $|\varepsilon_F|$  is natural in  $F$  and is the required natural weak homotopy equivalence  $\mathbf{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$ .

(ii). [Proposition 1.7.24](#) and [remark A.5.II](#) imply there is a natural isomorphism  $\lim_{\xrightarrow{\mathbb{C}}} \mathbf{B}(\mathbb{C}, \mathbb{C}, F) \cong \mathbf{B}(\Delta 1, \mathbb{C}, F)$ . [Lemma 1.8.6](#) implies the unique natural

## I. SIMPLICIAL SETS

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transformation  $B(\Delta 1, \mathbb{C}, \mathbb{C}) \Rightarrow \Delta 1$  is a natural weak homotopy equivalence, and therefore (by [proposition 1.8.5](#)) the induced morphism

$$B(B(\Delta 1, \mathbb{C}, \mathbb{C}), \mathbb{C}, F) \rightarrow B(\Delta 1, \mathbb{C}, F)$$

is a weak homotopy equivalence. Using the fact that the functor  $B(-, \mathbb{C}, F)$  preserves weighted  $\mathbf{sSet}$ -colimits, it can be shown that the following diagram commutes,

$$\begin{array}{ccc} B(\Delta 1, \mathbb{C}, B(\mathbb{C}, \mathbb{C}, F)) & \xrightarrow{\alpha_F} & B(B(\Delta 1, \mathbb{C}, \mathbb{C}), \mathbb{C}, F) \\ \lim_{\rightarrow \mathbb{C}} \bar{\varepsilon}_{B(\Delta 1, \mathbb{C}, F)} \downarrow & & \downarrow \\ B(\Delta 1, \mathbb{C}, F) & \xlongequal{\quad} & B(\Delta 1, \mathbb{C}, F) \end{array}$$

where  $\alpha_F$  is the natural isomorphism of [theorem 1.7.23](#), and thus we may deduce that  $\bar{\varepsilon}_{B(\Delta 1, \mathbb{C}, F)}$  is indeed a weak homotopy equivalence.

(iii). Consider a natural transformation  $\varphi : F \Rightarrow F'$  such that the components of  $\varphi$  are weak homotopy equivalences. Then, we have the following commutative diagram of natural transformations:

$$\begin{array}{ccc} B(\mathbb{C}, \mathbb{C}, F) & \xrightarrow{|\varepsilon_F|} & F \\ B(\mathbb{C}, \mathbb{C}, \varphi) \downarrow & & \downarrow \varphi \\ B(\mathbb{C}, \mathbb{C}, F') & \xrightarrow{|\varepsilon_{F'}|} & F' \end{array}$$

The 2-out-of-3 property for weak homotopy equivalences then implies that the components of  $B(\mathbb{C}, \mathbb{C}, \varphi)$  must also be weak homotopy equivalences. ■

**Definition 1.8.8.** Let  $\mathbb{C}$  be a small category.

- A **homotopy colimit functor** for diagrams  $\mathbb{C} \rightarrow \mathbf{sSet}$  is a homotopical left approximation for the functor  $\lim_{\rightarrow \mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$ .
- A **homotopy limit functor** for diagrams  $\mathbb{C} \rightarrow \mathbf{sSet}$  is a homotopical right approximation for the functor  $\lim_{\leftarrow \mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$ .

**Theorem 1.8.9.** Let  $\mathbb{C}$  be a small category and let  $R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  be a fibrant replacement functor for  $\mathbf{sSet}$ .



- (i) The functor  $\lim_{\rightarrow \mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  sends natural weak homotopy equivalences between diagrams of the form  $B(\mathbb{C}, \mathbb{C}, F)$  to weak homotopy equivalences in  $\mathbf{sSet}$ .
- (ii)  $B(\mathbb{C}, \mathbb{C}, -)$  is (the functor part of) a functorial left deformation retract for  $\lim_{\rightarrow \mathbb{C}}$ .
- (iii)  $B(\Delta 1, \mathbb{C}, -)$  is a homotopy colimit functor for diagrams  $\mathbb{C} \rightarrow \mathbf{sSet}$ .

*Dually:*

- (i') The functor  $\lim_{\leftarrow \mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  sends natural weak homotopy equivalences between diagrams of the form  $C(\mathbb{C}, \mathbb{C}, RF)$  to weak homotopy equivalences in  $\mathbf{sSet}$ .
- (ii')  $C(\mathbb{C}, \mathbb{C}, -)$  is (the functor part of) a functorial right deformation retract for  $\lim_{\leftarrow \mathbb{C}}$ .
- (iii')  $C(\Delta 1, \mathbb{C}, R-)$  is a homotopy limit functor for diagrams  $\mathbb{C} \rightarrow \mathbf{sSet}$ .

*Proof.* We will prove the first set of claims; the second set can be proved in a similar way.

(i) and (ii). Propositions 1.7.24, 1.8.5, and 1.8.7 together imply that  $B(\mathbb{C}, \mathbb{C}, -)$  and  $\bar{\varepsilon} : B(\mathbb{C}, \mathbb{C}, -) \Rightarrow \text{id}_{\mathbf{sSet}}$  satisfy the hypotheses of proposition 3.4.5 and so constitute a left deformation for the functor  $\lim_{\rightarrow \mathbb{C}}$ .

(iii). Apply theorem 3.4.10. ■



## SIMPLICIAL CATEGORIES

### 2.1 Basics

**Prerequisites.** §§0.2, 1.1, 1.2, A.2.

In this section, we use the explicit universe convention.

**Definition 2.1.1.** A **simplicial category**  $C_\bullet$  consists of the following data:

- For each natural number  $n$ , a category  $C_n$ .
- For each natural number  $n$  and  $0 \leq i \leq n$ , a functor  $d_i^n : C_n \rightarrow C_{n-1}$  and a functor  $s_i^n : C_n \rightarrow C_{n+1}$ .

These functors are moreover required to satisfy the simplicial identities. The **underlying category** of  $C_\bullet$  is the category  $C_0$ .

**REMARK 2.1.2.** In short, a simplicial category is a simplicial object in the meta-category of all categories. Thus, we may refer to the functors  $d_i^n$  and  $s_i^n$  as **face operators** and **degeneracy operators**, just as in the general case.

**Definition 2.1.3.** Given two simplicial categories  $C_\bullet$  and  $D_\bullet$ , a **simplicial functor**  $F_\bullet : C_\bullet \rightarrow D_\bullet$  consists of a functor  $F_n : C_n \rightarrow D_n$  for each natural number  $n$ , such that the functors  $F_n$  are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n F_n = F_{n-1} d_i^n \qquad s_i^n F_n = F_{n+1} s_i^n$$

**Definition 2.1.4.** Given two simplicial functors  $F_\bullet, F'_\bullet : C_\bullet \rightarrow D_\bullet$ , a **simplicial natural transformation**  $\varphi_\bullet : F_\bullet \Rightarrow F'_\bullet$  consists of a natural transformation

$\varphi_n : F_n \Rightarrow F'_n$  for each natural number  $n$ , such that the natural transformations  $\varphi_n$  are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n \varphi_n = \varphi_{n-1} d_i^n \qquad s_i^n \varphi_n = \varphi_{n+1} s_i^n$$

**Definition 2.1.5.** Let  $\mathbf{U}$  be a universe. A **U-small** (resp. **locally U-small**) **simplicial category** is a simplicial category  $\mathcal{C}_\bullet$  such that each  $\mathcal{C}_n$  is  $\mathbf{U}$ -small (resp. locally  $\mathbf{U}$ -small).

**Example 2.1.6.** If  $\mathcal{C}$  is a  $\mathbf{U}$ -small category, then we have a  $\mathbf{U}$ -small constant simplicial category  $\mathcal{C}_\bullet$ , where  $\mathcal{C}_n = \mathcal{C}$  for all  $n$ , with the trivial face and degeneracy operators.

**Definition 2.1.7.** The **bisimplicial nerve** of a simplicial category  $\mathcal{C}_\bullet$  is the bisimplicial set  $\mathbf{N}^{\text{ss}}(\mathcal{C}_\bullet)$  defined by the following formula:

$$(\mathbf{N}^{\text{ss}}(\mathcal{C}_\bullet))_m = \mathbf{N}(\mathcal{C}_m)_n$$

In other words, the  $m$ -simplices of the  $n$ -th level of  $\mathbf{N}^{\text{ss}}(\mathcal{C}_\bullet)$  are the composable strings of morphisms in  $\mathcal{C}_m$  of length  $n$ .

**Example 2.1.8.** Let  $\mathcal{C}$  be an ordinary category, and consider the simplicial category  $\mathcal{C}_\bullet$  defined by  $\mathcal{C}_n = [\mathbf{I}[n], \mathcal{C}]$ , where  $\mathbf{I}[n]$  denotes the groupoid obtained by freely inverting all the arrows in  $[n]$ . The bisimplicial nerve  $\mathbf{N}^{\text{ss}}(\mathcal{C}_\bullet)$  is then (isomorphic to) the **classifying diagram** of  $\mathcal{C}$ , in the sense of Rezk [2001].

**Proposition 2.1.9.** Let  $\mathbf{U}$  be a universe, let  $[\Delta^{\text{op}}, \mathbf{Cat}]$  be the category of  $\mathbf{U}$ -small simplicial categories, and let  $\mathbf{ssSet}$  be the category of bisimplicial sets.

- (i)  $[\Delta^{\text{op}}, \mathbf{Cat}]$  is a locally finitely presentable  $\mathbf{U}$ -category.
- (ii)  $\mathbf{N}^{\text{ss}} : [\Delta^{\text{op}}, \mathbf{Cat}] \rightarrow \mathbf{ssSet}$  is a fully faithful  $\aleph_0$ -accessible functor.
- (iii)  $\mathbf{N}^{\text{ss}}$  has a left adjoint.

*Proof.* (i). This is an instance of [proposition 0.2.43](#).

(ii). That  $\mathbf{N}^{\text{ss}} : [\Delta^{\text{op}}, \mathbf{Cat}] \rightarrow \mathbf{ssSet}$  is a fully faithful  $\aleph_0$ -accessible functor essentially follows from the fact that  $\mathbf{N} : \mathbf{Cat} \rightarrow \mathbf{sSet}$  is so: see [proposition 1.2.1](#) and the accessible adjoint functor theorem ([0.2.49](#)).

(iii). It is also clear that  $\mathbf{N}^{\text{ss}}$  preserves limits for  $\mathbf{U}$ -small diagrams, so we may apply the accessible adjoint functor theorem to construct a left adjoint for  $\mathbf{N}^{\text{ss}}$ . ■

**Definition 2.1.10.** A **simplicially enriched category**  $\underline{C}$  consists of the following data:

- A set of objects,  $\text{ob } C$ .
- A simplicial set of morphisms,  $\text{mor } \underline{C}$ .
- A pair of simplicial maps  $\text{dom}, \text{codom} : \text{mor } \underline{C} \rightarrow \text{disc ob } C$ .
- For each element  $C$  of  $\text{ob } C$ , a vertex  $\text{id}_C$  in  $\text{mor } \underline{C}$  such that  $\text{dom } \text{id}_C = C$  and  $\text{codom } \text{id}_C = C$ .
- A simplicial map  $\underline{C}^{[2]} \rightarrow \text{mor } \underline{C}$ , written as  $(\beta, \alpha) \mapsto \beta \circ \alpha$ , where  $\underline{C}^{[2]}$  is the simplicial set defined by the following pullback diagram:

$$\begin{array}{ccc} \underline{C}^{[2]} & \longrightarrow & \text{mor } \underline{C} \\ \downarrow & & \downarrow \text{codom} \\ \text{mor } \underline{C} & \xrightarrow{\text{dom}} & \text{disc ob } C \end{array}$$

These are moreover required to satisfy the following condition:

- For each natural number  $n$ , the given identities and binary operation induce a category with  $\text{ob } C$  for its object-set and  $(\text{mor } \underline{C})_n$  for its morphism-set.

As usual, we write  $\underline{C}(C, C')$  for the simplicial subset of  $\text{mor } \underline{C}$  consisting of those simplices  $\alpha$  such that  $\text{dom } \alpha = C$  and  $\text{codom } \alpha = C'$ .

The **underlying category** of a simplicial category  $\underline{C}$  is the category  $C$  obtained by taking  $C(C', C) = \underline{C}(C', C)_0$ , with the evident identity morphisms and induced composition. By **object** or **morphism** in  $\underline{C}$ , we shall always mean an object or morphism in the underlying category  $C$ .

**REMARK 2.1.11.** It is clear from the definition that a simplicially enriched category  $\underline{C}$  induces a simplicial category  $C_\bullet$ , but not every simplicial category arises in this fashion: simplicially enriched categories correspond to the simplicial categories  $C_\bullet$  where  $\text{ob } C_\bullet$  is a constant simplicial set.

**Definition 2.1.12.** Given two simplicially enriched categories  $\underline{C} \rightarrow \underline{D}$ , a **simplicially enriched functor**  $\underline{F} : \underline{C} \rightarrow \underline{D}$  consists of a map  $\text{ob } F : \text{ob } C \rightarrow \text{ob } D$  and a simplicial map  $\text{mor } F : \text{mor } \underline{C} \rightarrow \text{mor } \underline{D}$  that respect the structure of simplicially enriched categories in the obvious sense.

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REMARK 2.1.13. There is a natural bijection between simplicially enriched functors  $\underline{C}$  and  $\underline{D}$  and simplicial functors  $C_\bullet \rightarrow D_\bullet$ , where  $C_\bullet$  and  $D_\bullet$  are the simplicial categories associated with  $\underline{C}$  and  $\underline{D}$ .

Of course, just as in the simplicial case, a simplicially enriched functor  $\underline{F} : \underline{C} \rightarrow \underline{D}$  has a underlying functor  $F : C \rightarrow D$  between the underlying categories.

**Definition 2.1.14.** Given two simplicially enriched functors  $\underline{F}, \underline{F}' : \underline{C} \rightarrow \underline{D}$ , a **simplicially enriched natural transformation**  $\varphi : \underline{F} \Rightarrow \underline{F}'$  consists of a morphism  $\varphi_C : FC \rightarrow F'C$  in  $D$  for each object  $C$  in  $C$ , such that the following diagram commutes for all pairs  $(C, C')$ :

$$\begin{array}{ccc}
 & \underline{D}(FC, FC') & \\
 \text{mor } \underline{F} \nearrow & & \searrow \underline{D}(FC, \varphi_{C'}) \\
 \underline{C}(C, C') & & \underline{D}(FC, F'C') \\
 \text{mor } \underline{F}' \searrow & & \nearrow \underline{D}(\varphi_C, F'C') \\
 & \underline{D}(F'C, F'C') & 
 \end{array}$$

REMARK 2.1.15. It is not hard to see that any simplicially enriched natural transformation has an underlying natural transformation; but unlike simplicially enriched functors, being a simplicially enriched natural transformation merely a property, rather than an extra structure.

Less obviously, the bijection between simplicially enriched functors and simplicial functors also extends to a bijection between simplicially enriched natural transformations and simplicial natural transformations. In particular, to check whether a natural transformation is simplicially enriched, it is enough to check whether it is levelwise natural.

**Definition 2.1.16.** Let  $\mathbf{U}$  be a universe. A  **$\mathbf{U}$ -small simplicially enriched category** is a simplicially enriched category  $\underline{C}$  such that  $\text{ob } C$  is a  $\mathbf{U}$ -set and  $\text{mor } C$  is a simplicial  $\mathbf{U}$ -set. A **locally  $\mathbf{U}$ -small simplicially enriched category** is a simplicially enriched category  $\underline{C}$  such that  $\text{ob } C$  is a  $\mathbf{U}$ -class and, for each pair  $(C', C)$  of elements of  $\text{ob } C$ , the simplicial set  $\underline{C}(C', C)$  is a simplicial  $\mathbf{U}$ -set.

REMARK 2.1.17. If  $\mathbf{sSet}$  is the category of simplicial  $\mathbf{U}$ -sets, then a locally  $\mathbf{U}$ -small simplicially enriched category is essentially the same thing as a  $\mathbf{sSet}$ -enriched category whose object-set is a  $\mathbf{U}$ -class.

**Example 2.1.18.** Since  $\mathbf{sSet}$  is a cartesian closed category (theorem A.2.22), we may make it a simplicially enriched category by taking  $\underline{\mathbf{sSet}}(X, Y) = [X, Y]$ . This is a locally  $\mathbf{U}$ -small simplicially enriched category, since each  $[X, Y]$  is a simplicial  $\mathbf{U}$ -set.

**Definition 2.1.19.** A **discrete simplicially enriched category** is a simplicially enriched category  $\underline{\mathcal{C}}$  such that  $\text{mor } \underline{\mathcal{C}}$  is a constant simplicial set.

**Proposition 2.1.20.** *Let  $\mathbf{U}$  be a universe. If  $\mathcal{C}$  is a locally  $\mathbf{U}$ -small category, then there exists a locally  $\mathbf{U}$ -small discrete simplicially enriched category  $\underline{\mathcal{C}}$  whose underlying category is  $\mathcal{C}$  such that, for all simplicially enriched categories  $\underline{\mathcal{D}}$ , the map sending a simplicially enriched functor  $\underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$  to its underlying ordinary functor  $\mathcal{C} \rightarrow \mathcal{D}$  is a bijection.*

*Proof.* Obvious. ◆

**Definition 2.1.21.** Given two simplicially enriched functors  $\underline{F}, \underline{F}' : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$ , the **simplicial set of natural transformations**  $\underline{F} \Rightarrow \underline{F}'$  is the simplicial set  $\underline{\text{Nat}}(\underline{F}, \underline{F}')$  defined by the equaliser diagram shown below,

$$\underline{\text{Nat}}(\underline{F}, \underline{F}') \longrightarrow \prod_{C \in \text{ob } \underline{\mathcal{C}}} \underline{\mathcal{D}}(FC, F'C) \rightrightarrows \prod_{\substack{C \in \text{ob } \underline{\mathcal{C}} \\ C' \in \text{ob } \underline{\mathcal{C}}}} [\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C)]$$

where the two maps are defined in components by the following composites, respectively,

$$\underline{\mathcal{D}}(FC, F'C) \longrightarrow [\underline{\mathcal{D}}(F'C, F'C'), \underline{\mathcal{D}}(FC, F'C')] \xrightarrow{(\text{mor } \underline{F}')^*} [\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C)]$$

$$\underline{\mathcal{D}}(FC', F'C') \longrightarrow [\underline{\mathcal{D}}(FC, FC'), \underline{\mathcal{D}}(FC', F'C')] \xrightarrow{(\text{mor } \underline{F})^*} [\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C)]$$

with the first arrow in each diagram being the exponential transpose of the composition map.

**Proposition 2.1.22.** *Let  $\mathbf{U}$  be a universe.*

- (i) *If  $\underline{\mathcal{D}}$  and  $\underline{\mathcal{E}}$  are  $\mathbf{U}$ -small simplicially enriched categories, then there exist a  $\mathbf{U}$ -small simplicially enriched category  $\underline{\mathcal{D}} \times \underline{\mathcal{E}}$  and simplicially enriched functors  $p_1 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \rightarrow \underline{\mathcal{D}}$  and  $p_2 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \rightarrow \underline{\mathcal{E}}$  such that  $(p_1, p_2)$  induce*

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a bijection between simplicially enriched functors  $\langle \underline{F}, \underline{G} \rangle : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}} \times \underline{\mathcal{E}}$  and pairs  $(\underline{F}, \underline{G})$  of simplicially enriched functors, where  $\underline{F} : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{D}}$  and  $\underline{G} : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{E}}$ , where  $\underline{\mathcal{C}}$  varies over all simplicially enriched categories.

- (ii) If  $\underline{\mathcal{D}}$  is a  $\mathbf{U}$ -small simplicially enriched category and  $\underline{\mathcal{E}}$  is a locally  $\mathbf{U}$ -small simplicially enriched category, then there exist a locally  $\mathbf{U}$ -small simplicially enriched category  $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$  and a simplicially enriched functor  $\text{ev} : [\underline{\mathcal{D}}, \underline{\mathcal{E}}] \times \underline{\mathcal{D}} \rightarrow \underline{\mathcal{E}}$  such that  $\text{ev}$  induces a bijection between simplicially enriched functors  $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \rightarrow \underline{\mathcal{E}}$  and simplicially enriched functors  $\underline{\mathcal{C}} \rightarrow [\underline{\mathcal{D}}, \underline{\mathcal{E}}]$ , where  $\underline{\mathcal{C}}$  varies over all simplicially enriched categories.
- (iii) If  $\underline{\mathcal{D}}$  and  $\underline{\mathcal{E}}$  are both  $\mathbf{U}$ -small simplicially enriched categories, then  $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$  is also  $\mathbf{U}$ -small.

*Proof.* Omitted, but straightforward. ◆

**Proposition 2.1.23.** *Let  $\mathbf{U}$  be a universe, let  $\mathbf{SCat}$  be the category of  $\mathbf{U}$ -small simplicially enriched categories, and let  $[\Delta^{\text{op}}, \mathbf{Cat}]$  be the category of  $\mathbf{U}$ -small simplicial categories.*

- (i)  $\mathbf{SCat}$ , regarded as a full subcategory of  $[\Delta^{\text{op}}, \mathbf{Cat}]$ , is closed under limits and colimits for all  $\mathbf{U}$ -small diagrams.
- (ii)  $\mathbf{SCat}$  is a cartesian closed category.
- (iii) The inclusion  $\mathbf{SCat} \hookrightarrow [\Delta^{\text{op}}, \mathbf{Cat}]$  has a left adjoint, and  $\mathbf{SCat}$  is a locally finitely presentable  $\mathbf{U}$ -category.

*Proof.* (i). The functor  $[\Delta^{\text{op}}, \text{ob}] : [\Delta^{\text{op}}, \mathbf{Cat}] \rightarrow \mathbf{sSet}$  has a left adjoint and a right adjoint, so it follows that a limit or colimit for diagrams of simplicially enriched categories, computed as a simplicial category, will have object-space a discrete simplicial set and thus be isomorphic to a simplicially enriched category.

(ii). This is implied by [proposition 2.1.22](#).

(iii). It is not hard to directly construct a left adjoint for the inclusion  $\mathbf{SCat} \hookrightarrow [\Delta^{\text{op}}, \mathbf{Cat}]$ , and once this is done, we may apply the classification theorem for locally presentable categories ([0.2.39](#)) to deduce (from [proposition 2.1.9](#)) that  $\mathbf{SCat}$  is also locally finitely presentable. Alternatively, one may instead first show that  $\mathbf{SCat}$  is locally finitely presentable and then use the accessible adjoint functor theorem ([0.2.49](#)) to construct a left adjoint for the inclusion. ◆



**Definition 2.1.24.** Let  $\underline{C}$  be a simplicially enriched category.

- A **tensor product** of a simplicial set  $X$  and an object  $A$  in  $\underline{C}$  is an object  $X \odot A$  in  $\underline{C}$  equipped with a family of isomorphisms of simplicial sets of the form

$$\underline{C}(X \odot A, B) \cong [X, \underline{C}(A, B)]$$

that are simplicially natural as  $B$  varies in  $\underline{C}$ . We may also say that  $X \odot A$  is a **simplicial copower** of  $A$  by  $X$ .

- A **cotensor product** of a simplicial set  $X$  and an object  $B$  in  $\underline{C}$  is an object  $X \pitchfork B$  in  $\underline{C}$  equipped with a family of isomorphisms of simplicial sets of the form

$$\underline{C}(A, X \pitchfork B) \cong [X, \underline{C}(A, B)]$$

that are simplicially natural as  $A$  varies in  $\underline{C}$ . We may also say that  $X \pitchfork B$  is a **simplicial power** of  $B$  by  $X$ .

**Definition 2.1.25.** Let  $\mathbf{U}$  be a universe and let  $\underline{\mathbf{sSet}}$  be the simplicially enriched category of simplicial  $\mathbf{U}$ -sets.

- A **simplicially tensored  $\mathbf{U}$ -category** is a locally  $\mathbf{U}$ -small simplicially enriched category  $\underline{C}$  equipped with a simplicially enriched functor

$$\odot : \underline{\mathbf{sSet}} \times \underline{C} \rightarrow \underline{C}$$

and a family of isomorphisms in  $\mathbf{sSet}$  of the form

$$\underline{C}(X \odot A, B) \cong \underline{\mathbf{sSet}}(X, \underline{C}(A, B))$$

constituting a simplicially enriched natural transformation in  $A$ ,  $B$ , and  $X$ .

- A **simplicially cotensored  $\mathbf{U}$ -category** is a locally  $\mathbf{U}$ -small simplicially enriched category  $\underline{C}$  equipped with a simplicially enriched functor

$$\pitchfork : \underline{\mathbf{sSet}}^{\text{op}} \times \underline{C} \rightarrow \underline{C}$$

and a family of isomorphisms in  $\mathbf{sSet}$  of the form

$$\underline{\mathbf{sSet}}(X, \underline{C}(A, B)) \cong \underline{C}(A, X \pitchfork B)$$

constituting a simplicially enriched natural transformation in  $A$ ,  $B$ , and  $X$ .

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**REMARK 2.1.26.** The simplicially enriched functor  $\odot$  (resp.  $\pitchfork$ ) is unique up to unique natural isomorphism, so a locally  $\mathbf{U}$ -small simplicially enriched category is a simplicially tensored (resp. cotensored)  $\mathbf{U}$ -category in at most one way up to isomorphism.

**Theorem 2.1.27.** *Let  $\underline{C}$  be a locally  $\mathbf{U}$ -small simplicially enriched category. The following are equivalent:*

- (i)  $\underline{C}$  is a simplicially tensored  $\mathbf{U}$ -category.
- (ii)  $\underline{C}$  has tensor products  $X \odot A$  for all simplicial  $\mathbf{U}$ -sets  $X$  and all objects  $A$  in  $\underline{C}$ .
- (iii) There exist a functor  $\odot : \mathbf{sSet} \times \underline{C} \rightarrow \underline{C}$  and natural bijections of hom-sets

$$C(X \odot A, B) \cong \mathbf{sSet}(X, \underline{C}(A, B))$$

together with a natural isomorphism  $\eta : \text{id}_C \Rightarrow 1 \odot (-)$  and natural isomorphisms  $\mu_{X,Y} : X \odot (Y \odot (-)) \Rightarrow (X \times Y) \odot (-)$  satisfying the conditions in [remark B.2.2](#).

Dually, the following are equivalent:

- (i')  $\underline{C}$  is a simplicially cotensored  $\mathbf{U}$ -category.
- (ii')  $\underline{C}$  has cotensor products  $X \pitchfork B$  for all simplicial  $\mathbf{U}$ -sets  $X$  and all objects  $B$  in  $\underline{C}$ .
- (iii') There exist a functor  $\pitchfork : \mathbf{sSet}^{\text{op}} \times \underline{C} \rightarrow \underline{C}$  and natural bijections of hom-sets

$$\mathbf{sSet}(X, \underline{C}(A, B)) \cong C(A, X \pitchfork B)$$

together with a natural isomorphism  $\eta : \text{id}_C \Rightarrow 1 \pitchfork (-)$  and natural isomorphisms  $\mu_{X,Y} : X \pitchfork (Y \pitchfork (-)) \Rightarrow (X \times Y) \pitchfork (-)$  satisfying the conditions in [remark B.2.2](#).

*Proof.* See [???]. ■

**Proposition 2.1.28.** *Let  $C$  be a category and let  $\mathbb{S}^{[\bullet]} = (S^{[\bullet]}, \varepsilon^{[\bullet]}, \delta^{[\bullet]})$  be a cosimplicial object in the category of comonads on  $C$ . If  $\mathbb{S}^{[0]} = (\text{id}, \text{id}, \text{id})$ , then*

$C$  is the underlying ordinary category of a simplicially enriched category  $\underline{C}$  where the hom-spaces are given by the formula below,

$$\underline{C}(A, B) \cong C(S^{[\bullet]}A, B)$$

with composition in level  $n$  induced by the comultiplication  $\delta^{[n]} : S^{[n]} \Rightarrow S^{[n]}S^{[n]}$ .

*Proof.* Let  $C_n$  be the Kleisli category associated with the comonad  $S^{[n]}$ . Clearly, these fit together to form a simplicial category  $C_\bullet$  such that  $\text{ob } C_\bullet$  is a constant simplicial set; so by [remark 2.1.11](#), we have the required simplicially enriched category  $\underline{C}$ . ■

**Definition 2.1.29.** Let  $\underline{C}$  be a locally  $\mathbf{U}$ -small simplicially enriched category and let  $F : \mathbb{D} \rightarrow C$  be a diagram in  $C$ .

- A **conical colimit** for  $F$  in  $\underline{C}$  is an object  $A$  and a cocone  $\lambda : F \Rightarrow \Delta A$  such that, for all objects  $B$  in  $C$ , the hom-functor  $\underline{C}(-, B) : C^{\text{op}} \rightarrow \mathbf{sSet}$  sends  $\lambda$  to a limiting cone in  $\mathbf{sSet}$ .
- A **conical limit** for  $F$  in  $\underline{C}$  is an object  $B$  and a cone  $\lambda : \Delta B \Rightarrow F$  such that, for all objects  $A$  in  $C$ , the hom-functor  $\underline{C}(A, -) : C \rightarrow \mathbf{sSet}$  sends  $\lambda$  to a limiting cone in  $\mathbf{sSet}$ .

**REMARK 2.1.30.** Every conical colimit (resp. limit) for  $F$  in  $\underline{C}$  is a colimit (resp. limit) for  $F$  in the underlying category  $C$ , but the converse is not true in general.

**Proposition 2.1.31.** *Let  $\underline{C}$  be a locally  $\mathbf{U}$ -small simplicially enriched category and let  $F : \mathbb{D} \rightarrow C$  be a diagram in  $C$ . If  $\underline{C}$  has cotensor products with the standard simplices, then the following are equivalent for any cocone  $\lambda : F \Rightarrow \Delta A$ :*

- (i)  $\lambda$  is a conical colimit for  $F$  in the simplicially enriched category  $\underline{C}$ .
- (ii)  $\lambda$  is a colimit for  $F$  in the underlying category  $C$ .

*Dually, if  $\underline{C}$  has tensor products with the standard simplices, then the following are equivalent for any cone  $\lambda : \Delta B \Rightarrow F$ :*

- (i')  $\lambda$  is a conical limit for  $F$  in the simplicially enriched category  $\underline{C}$ .
- (ii')  $\lambda$  is a limit for  $F$  in the underlying category  $C$ .

*Proof.* (i)  $\Rightarrow$  (ii). Immediate.

(ii)  $\Rightarrow$  (i). It suffices to show that, for each natural number  $n$ , the canonical comparison map

$$\underline{C}\left(\lim_{\rightarrow \mathbb{D}} F, T\right)_n \rightarrow \lim_{\leftarrow \mathbb{D}} \underline{C}(F, T)_n$$

is a bijection; but by the Yoneda lemma,

$$\underline{C}(S, T)_n \cong \mathbf{sSet}(\Delta^n, \underline{C}(S, T))$$

and the definition of  $\Delta^n \pitchfork (-)$  implies there is a natural bijection of the form below,

$$\mathbf{sSet}(\Delta^n, \underline{C}(S, T)) \cong \underline{C}(S, \Delta^n \pitchfork T)$$

therefore the functor  $\underline{C}(-, T)_n : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  is representable.  $\blacksquare$

**Definition 2.1.32.** Let  $\underline{C}$  be a locally  $\mathbf{U}$ -small simplicially enriched category, let  $\underline{\mathbb{D}}$  be a  $\mathbf{U}$ -small simplicially enriched category, and let  $\underline{F} : \underline{\mathbb{D}} \rightarrow \underline{C}$  be a simplicially enriched functor.

- Given a simplicially enriched functor  $\underline{W} : \underline{\mathbb{D}}^{\text{op}} \rightarrow \mathbf{sSet}$ , a  **$\underline{W}$ -weighted colimit** for  $\underline{F}$  is an object  $\underline{W} \star_{\underline{\mathbb{D}}} \underline{F}$  equipped with a simplicially enriched natural isomorphism of the following form:

$$\underline{C}(\underline{W} \star_{\underline{\mathbb{D}}} \underline{F}, -) \cong [\underline{\mathbb{D}}^{\text{op}}, \mathbf{sSet}](\underline{W}, \underline{C}(\underline{F}, -))$$

- Given a simplicially enriched functor  $\underline{W} : \underline{\mathbb{D}} \rightarrow \mathbf{sSet}$ , a  **$\underline{W}$ -weighted limit** for  $\underline{F}$  is an object  $\{\underline{W}, \underline{F}\}^{\underline{\mathbb{D}}}$  equipped with a simplicially enriched natural isomorphism of the following form:

$$\underline{C}(-, \{\underline{W}, \underline{F}\}^{\underline{\mathbb{D}}}) \cong [\underline{\mathbb{D}}, \mathbf{sSet}](\underline{W}, \underline{C}(-, \underline{F}))$$

**REMARK 2.1.33.** When  $\underline{\mathbb{D}}$  is the free simplicial enrichment of an ordinary category  $\mathbb{D}$ , ordinary cocones (resp. cones) on diagrams  $F : \mathbb{D} \rightarrow \mathcal{C}$  are automatically simplicially enriched, and thus conical colimits (resp. limits) for  $F$  are the same thing as  $\Delta 1$ -weighted colimits (resp. limits) for  $\underline{F}$ , where  $\Delta 1$  denotes the constant functor with value 1 in  $\mathbf{sSet}$ .

**Definition 2.1.34.** Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \subseteq \mathbf{U}^+$ .

- A  **$\mathbf{U}$ -cocomplete simplicially enriched category** is a locally  $\mathbf{U}^+$ -small simplicially enriched category  $\underline{\mathcal{C}}$  such that, for all  $\mathbf{U}$ -small simplicially enriched diagrams  $F : \underline{\mathbb{D}} \rightarrow \underline{\mathcal{C}}$  and all  $\mathbf{U}$ -small weights  $W : \underline{\mathbb{D}}^{\text{op}} \rightarrow \mathbf{sSet}$ ,  $\underline{\mathcal{C}}$  has a  $W$ -weighted colimit for  $F$ .
- A  **$\mathbf{U}$ -complete simplicially enriched category** is a locally  $\mathbf{U}^+$ -small simplicially enriched category  $\underline{\mathcal{C}}$  such that, for all  $\mathbf{U}$ -small simplicially enriched diagrams  $F : \underline{\mathbb{D}} \rightarrow \underline{\mathcal{C}}$  and all  $\mathbf{U}$ -small weights  $W : \underline{\mathbb{D}} \rightarrow \mathbf{sSet}$ ,  $\underline{\mathcal{C}}$  has a  $W$ -weighted limit for  $F$ .

**Proposition 2.1.35.** Let  $\underline{\mathcal{C}}$  be a locally  $\mathbf{U}$ -small simplicially enriched category.

- $\underline{\mathcal{C}}$  is  $\mathbf{U}$ -cocomplete if and only if  $\underline{\mathcal{C}}$  is simplicially tensored and has conical colimits for all  $\mathbf{U}$ -small diagrams.
- $\underline{\mathcal{C}}$  is  $\mathbf{U}$ -complete if and only if  $\underline{\mathcal{C}}$  is simplicially cotensored and conical limits for all  $\mathbf{U}$ -small diagrams.
- $\underline{\mathcal{C}}$  is both  $\mathbf{U}$ -cocomplete and  $\mathbf{U}$ -complete if and only if  $\underline{\mathcal{C}}$  is both simplicially tensored and cotensored and the underlying category  $\mathcal{C}$  is  $\mathbf{U}$ -cocomplete and  $\mathbf{U}$ -complete.

*Proof.* See [???]. □

## 2.2 Homotopical aspects

**Prerequisites.** §§ 1.2, 1.3, 2.1, A.4.

**Definition 2.2.1.** Let  $\mathcal{V}$  be a category with finite products and let  $F : \mathbf{sSet} \rightarrow \mathcal{V}$  be a functor that preserves finite products. The  **$F$ -localisation** of a locally small simplicially enriched category  $\underline{\mathcal{C}}$  is the following  $\mathcal{V}$ -enriched category  $F[\underline{\mathcal{C}}]$ :

- The objects in  $F[\underline{\mathcal{C}}]$  are the objects in  $\underline{\mathcal{C}}$ .
- For each pair  $(X, Y)$  of objects in  $\underline{\mathcal{C}}$ , the hom-object  $F[\underline{\mathcal{C}}](X, Y)$  is the object  $F(\underline{\mathcal{C}}(X, Y))$ .
- Identities and composition in  $F[\underline{\mathcal{C}}]$  are inherited from  $\underline{\mathcal{C}}$  via  $F$ .

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**REMARK 2.2.2.** It is clear that  $F$ -localisation is 2-functorial and moreover preserves finite products of simplicially enriched categories; unlike localisation of relative categories,  $F$ -localisation may or may not have a universal property. Nonetheless, there is always a localising functor  $\mathcal{C} \rightarrow F[\underline{\mathcal{C}}]$  between the underlying categories.

**Definition 2.2.3.** Let  $\underline{\mathcal{C}}$  be a locally small simplicially enriched category. A parallel pair of morphisms  $g_0, g_1 : A \rightarrow B$  in  $\underline{\mathcal{C}}$  are  **$F$ -homotopic** if their images under the localising functor  $\mathcal{C} \rightarrow F[\underline{\mathcal{C}}]$  are equal, in which case we write  $g_0 \stackrel{F}{\sim} g_1$ .

**Example 2.2.4.** The notion of intrinsic homotopy in  $\mathbf{sSet}$  is obtained as the special case where  $F$  is the connected components functor  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$ .<sup>[1]</sup>

**Definition 2.2.5.** Let  $\underline{\mathcal{C}}$  be a locally small simplicially enriched category. A **weak  $F$ -homotopy equivalence** in  $\underline{\mathcal{C}}$  is a morphism in  $\underline{\mathcal{C}}$  whose image in  $F[\underline{\mathcal{C}}]$  is an isomorphism. An  **$F$ -homotopy equivalence** in  $\underline{\mathcal{C}}$  is a pair  $(f, g)$ , where  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are morphisms in  $\underline{\mathcal{C}}$  such that  $g \circ f \stackrel{F}{\sim} \text{id}_A$  and  $f \circ g \stackrel{F}{\sim} \text{id}_B$ . Two morphisms  $f : A \rightarrow B$  and  $g : B \rightarrow A$  are **mutual  $F$ -homotopy inverses** when  $(f, g)$  constitute an  $F$ -homotopy equivalence.

**REMARK 2.2.6.** By [lemma A.4.14](#), the class of weak  $F$ -homotopy equivalences in  $\underline{\mathcal{C}}$  automatically has the 2-out-of-6 property in  $\mathcal{C}$ .

**Lemma 2.2.7.** *Let  $\underline{\mathcal{C}}$  be a locally small simplicially enriched category, let  $\mathcal{V}$  be a cartesian closed category, and let  $F : \mathbf{sSet} \rightarrow \mathcal{V}$  be a functor that preserves finite products.*

- *If  $\underline{\mathcal{C}}$  is tensored over  $\mathbf{sSet}$ ,  $f : X \rightarrow Y$  is a weak  $F$ -homotopy equivalence in  $\mathbf{sSet}$ , and  $g : A \rightarrow B$  is a weak  $F$ -homotopy equivalence in  $\underline{\mathcal{C}}$ , then the morphism  $f \odot g : X \odot A \rightarrow Y \odot B$  is a weak  $F$ -homotopy equivalence in  $\underline{\mathcal{C}}$ .*
- *If  $\underline{\mathcal{C}}$  is cotensored over  $\mathbf{sSet}$ ,  $f : X \rightarrow Y$  is a weak  $F$ -homotopy equivalence in  $\mathbf{sSet}$ , and  $g : A \rightarrow B$  is a weak  $F$ -homotopy equivalence in  $\underline{\mathcal{C}}$ , then the morphism  $f \pitchfork g : Y \pitchfork A \rightarrow X \pitchfork B$  is a weak  $F$ -homotopy equivalence in  $\underline{\mathcal{C}}$ .*

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[1] Recall [proposition 1.2.4](#).

*Proof.* Since  $\odot$  (resp.  $\pitchfork$ ) is a simplicially enriched functor  $\mathbf{sSet} \times \underline{C} \rightarrow \underline{C}$  (resp.  $\mathbf{sSet}^{\text{op}} \times \underline{C} \rightarrow \underline{C}$ ), it induces a  $\mathcal{V}$ -enriched functor  $F[\mathbf{sSet}] \times F[\underline{C}] \rightarrow F[\underline{C}]$  (resp.  $F[\mathbf{sSet}]^{\text{op}} \times F[\underline{C}] \rightarrow F[\underline{C}]$ ) and so *a fortiori* must preserve weak  $F$ -homotopy equivalences.  $\blacksquare$

**Definition 2.2.8.** A **simplicial homotopy**  $\alpha : f_0 \Rightarrow f_1$  in a simplicially enriched category  $\underline{C}$  is an edge  $\alpha$  in  $\text{mor } \underline{C}$  such that  $d^0(\alpha) = f_1$  and  $d^1(\alpha) = f_0$ . For each morphism  $f : X \rightarrow Y$  in  $\underline{C}$ , we define  $\text{id}_f : f \Rightarrow f$  to be the simplicial homotopy  $s_0(f)$ .

**REMARK 2.2.9.** Because  $\text{ob } \underline{C}$  is a discrete set, we must have  $\text{dom } f_0 = \text{dom } f_1$  and  $\text{codom } f_0 = \text{codom } f_1$ .

**Lemma 2.2.10.** *Let  $\underline{C}$  be a locally small simplicially enriched category, and let  $\alpha : f_0 \Rightarrow f_1$  be an intrinsic homotopy of morphisms in  $\mathbf{sSet}$ .*

- *If  $\underline{C}$  is tensored over  $\mathbf{sSet}$ , then for any morphism  $g : A \rightarrow B$  in  $\underline{C}$ ,  $\alpha \odot \text{id}_g : f_0 \odot g \Rightarrow f_1 \odot g$  is a simplicial homotopy of morphisms in  $\underline{C}$ .*
- *If  $\underline{C}$  is cotensored over  $\mathbf{sSet}$ , then for any morphism  $g : A \rightarrow B$  in  $\underline{C}$ ,  $\alpha \pitchfork \text{id}_g : f_0 \pitchfork g \Rightarrow f_1 \pitchfork g$  is a simplicial homotopy of morphisms in  $\underline{C}$ .*

*Proof.* This is an immediate consequence of the fact that  $\odot$  (resp.  $\pitchfork$ ) is a simplicially enriched functor  $\mathbf{sSet} \times \underline{C} \rightarrow \underline{C}$  (resp.  $\mathbf{sSet}^{\text{op}} \times \underline{C} \rightarrow \underline{C}$ ).  $\blacksquare$

**Proposition 2.2.11.** *Let  $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$  be the connected components functor, let  $\pi : \mathbf{sSet} \rightarrow \mathbf{H}$  be the weak homotopy type functor,<sup>[2]</sup> and let  $\underline{C}$  be a locally small simplicially enriched category.*

- (i) *A morphism in  $\underline{C}$  is a weak  $\pi_0$ -homotopy equivalence if and only if it is a weak  $\pi$ -homotopy equivalence.*
- (ii) *The localising functor  $\gamma : \underline{C} \rightarrow \pi_0[\underline{C}]$  induces a bijection between simplicially enriched functors  $\underline{C} \rightarrow \underline{D}$  and ordinary functors  $\underline{C} \rightarrow \underline{D}$ , where  $\underline{D}$  is an ordinary category (regarded as a simplicially enriched category via [proposition 2.1.20](#)).*
- (iii) *If  $\underline{C}$  is moreover tensored or cotensored over  $\mathbf{sSet}$ , then  $\pi_0[\underline{C}]$  is the localisation of  $\underline{C}$  at the weak  $\pi$ -equivalences.*

[2] Recall [proposition 1.5.7](#).

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*Proof.* (i). The underlying category of the  $\mathbf{H}$ -enriched category  $\pi[\underline{C}]$  is naturally isomorphic to the category  $\pi_0[\underline{C}]$ , since  $\mathbf{H}(1, \pi X) \cong \pi_0 X$ , and the property of being an isomorphism in a  $\mathbf{H}$ -enriched category depends only on the underlying category.

(ii). By [proposition 1.2.4](#), a morphism from a simplicial set  $X$  to a discrete set  $Y$  must factor through  $\pi_0 X$  in a unique way, so a simplicially enriched functor  $\underline{C} \rightarrow \mathcal{D}$  must factor through  $\pi_0[\underline{C}]$ .

(iii). Simplicially tensored categories and simplicially cotensored categories are formally dual; we will prove the claim for case where  $\underline{C}$  is tensored over  $\mathbf{sSet}$ .

First, consider a simplicial homotopy  $\alpha : f_0 \Rightarrow f_1$  of morphisms  $A \rightarrow B$  in  $\underline{C}$ . Transposing across the tensor–hom adjunction yields  $H : \Delta^1 \odot A \rightarrow B$  making the diagram below commute:

$$\begin{array}{ccc}
 \Delta^0 \odot A & \xleftarrow{\eta_A} & A \\
 \delta^1 \odot \text{id}_A \downarrow & & \downarrow f_0 \\
 \Delta^1 \odot A & \xrightarrow{H} & B \\
 \delta^0 \odot \text{id}_A \uparrow & & \uparrow f_1 \\
 \Delta^0 \odot A & \xleftarrow{\eta_A} & A
 \end{array}$$

Using [lemma 2.2.7](#), it is not hard to see that  $\delta^0 \odot \text{id}_A$  and  $\delta^1 \odot \text{id}_A$  are  $\pi_0$ -homotopy equivalences in  $\underline{C}$  with common  $\pi_0$ -homotopy inverse  $\sigma^0 \odot \text{id}_A$ , so any functor that sends weak  $\pi_0$ -homotopy equivalences to isomorphisms must also identify  $f_0$  and  $f_1$ , and hence, must factor through  $\pi_0[\underline{C}]$ .  $\blacksquare$

**Proposition 2.2.12.** *Let  $\underline{C}$  be a simplicially enriched category.*

- (i) *The localising functor  $\underline{C} \rightarrow \pi_0[\underline{C}]$  is full and surjective on objects.*
- (ii) *A morphism in  $\underline{C}$  is a weak  $\pi$ -homotopy equivalence if and only if it has a  $\pi$ -homotopy inverse.*
- (iii) *Two objects in  $\underline{C}$  are isomorphic in  $\pi[\underline{C}]$  if and only if there is a  $\pi$ -homotopy equivalence between them in  $\underline{C}$ .*

*Proof.* Claim (i) is just the observation that the canonical map  $X_0 \rightarrow \pi_0 X$  is surjective, and the rest follows straightforwardly.  $\blacklozenge$



**Definition 2.2.13.** The **simplicial homotopy category** of a locally small simplicially enriched category  $\underline{C}$  is the  $\mathbf{H}$ -enriched category  $\pi[\underline{C}]$ , and a **simplicial homotopy equivalence** in  $\underline{C}$  is a  $\pi$ -homotopy equivalence.

REMARK 2.2.14. By remark 1.5.32, the notion of  $\pi$ -equivalence is stable under universe enlargement, as is the notion of simplicial homotopy category.

REMARK 2.2.15. It is sometimes convenient to consider other localisations; for example, if  $\pi_1 : \mathbf{sSet} \rightarrow \mathbf{Grpd}$  is the fundamental groupoid functor,<sup>[3]</sup> then the 2-category  $\pi_1[\underline{C}]$  has the following properties:

- (i) The underlying category of  $\pi_1[\underline{C}]$  is naturally isomorphic to the underlying category of  $\underline{C}$  itself.
- (ii) Given a parallel pair  $f_0, f_1 : A \rightarrow B$  in  $\underline{C}$ , there exists a 2-cell  $f_0 \Rightarrow f_1$  if and only if  $f_0$  and  $f_1$  are  $\pi$ -homotopic in  $\underline{C}$ .
- (iii) A morphism is a simplicial homotopy equivalence in  $\underline{C}$  if and only if it is an equivalence in the 2-category  $\pi_1[\underline{C}]$ .

However, if  $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$  is the fundamental category functor,<sup>[4]</sup> then the 2-category  $\tau_1[\underline{C}]$  in general only enjoys the first of the above properties.

**Definition 2.2.16.** A **Dwyer–Kan equivalence of simplicially enriched categories** is a simplicially enriched functor  $\underline{F} : \underline{C} \rightarrow \underline{D}$  such that the induced  $\mathbf{H}$ -enriched functor  $\pi[\underline{F}] : \pi[\underline{C}] \rightarrow \pi[\underline{D}]$  is fully faithful and essentially surjective on objects.

## 2.3 Simplicial and cosimplicial objects

**Prerequisites.** §§ 1.1, 2.1, A.6.

¶ 2.3.1. Recall that a **simplicial object** in a category is a diagram of shape  $\Delta^{\text{op}}$ , and dually, a **cosimplicial object** is a diagram of shape  $\Delta$ . Let us write  $\mathbf{sM}$  for the category of simplicial objects in  $\mathcal{M}$ , and  $\mathbf{cM}$  for the category of cosimplicial objects in  $\mathcal{M}$ .

[3] Recall proposition 1.2.7.

[4] Recall proposition 1.2.1.

**Proposition 2.3.2.** *Let  $\mathcal{M}$  be a locally small category. Then there is a functor  $\underline{\text{Hom}} : (\mathbf{sM})^{\text{op}} \times \mathbf{sM} \rightarrow \mathbf{sSet}$  equipped with bijections*

$$\mathbf{sSet}(\Delta^0, \underline{\text{Hom}}(A, B)) \cong \mathbf{sM}(A, B)$$

*that are natural in  $A_\bullet$  and  $B_\bullet$ , and this makes  $\mathbf{sM}$  into a locally small simplicially enriched category.*

*Proof.* Define  $\underline{\text{Hom}} : (\mathbf{sM})^{\text{op}} \times \mathbf{sM} \rightarrow \mathbf{sSet}$  by the following end formula:

$$\underline{\text{Hom}}(A, B)_n = \int_{[m]:\Delta} \mathbf{Set}(\Delta([m], [n]), \mathcal{M}(A_m, B_m))$$

This is clearly functorial. Concretely, an element  $f$  of  $\underline{\text{Hom}}(A, B)_n$  is the same thing as a  $\coprod_{[m]:\Delta} \Delta([m], [n])$ -indexed family of morphisms  $f_\varphi : A_m \rightarrow B_m$  in  $\mathcal{M}$ , such that for any two morphisms  $\varphi : [m] \rightarrow [n], \psi : [l] \rightarrow [m]$  in  $\Delta$ , the diagram in  $\mathcal{M}$  shown below commutes:

$$\begin{array}{ccc} A_m & \xrightarrow{f_\varphi} & B_m \\ \psi^* \downarrow & & \downarrow \psi^* \\ A_l & \xrightarrow{f_{\varphi \circ \psi}} & B_l \end{array}$$

Decomposing an element  $g$  of  $\underline{\text{Hom}}(B, C)_n$  the same way, we obtain the following commutative diagram in  $\mathcal{M}$ ,

$$\begin{array}{ccccc} A_m & \xrightarrow{f_\varphi} & B_m & \xrightarrow{g_\varphi} & C_m \\ \psi^* \downarrow & & \psi^* \downarrow & & \downarrow \psi^* \\ A_l & \xrightarrow{f_{\varphi \circ \psi}} & B_l & \xrightarrow{g_{\varphi \circ \psi}} & C_l \end{array}$$

and thus we have an element of  $\underline{\text{Hom}}(A, C)_n$ . This is certainly natural in  $n$ , and this is clearly the required associative composition with identity.

It remains to be shown that a morphism  $\Delta^0 \rightarrow \underline{\text{Hom}}(A, B)$  is the same thing as a morphism  $A_\bullet \rightarrow B_\bullet$ . Given a morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$ , we define an element  $f[n]$  of  $\underline{\text{Hom}}(A, B)_n$  for each object  $[n]$  in  $\Delta^{\text{op}}$  as follows: given  $\varphi : [m] \rightarrow [n]$ , we set  $(f[n])_\varphi = f_m$  and naturality of  $f_m$  makes the diagram in  $\mathcal{M}$  shown below commute for every morphism  $\psi : [l] \rightarrow [m]$  in  $\Delta$ :

$$\begin{array}{ccc} A_m & \xrightarrow{(f[n])_\varphi} & B_m \\ \psi^* \downarrow & & \downarrow \psi^* \\ A_l & \xrightarrow{(f[n])_{\varphi \circ \psi}} & B_l \end{array}$$

Thus, we have a morphism  $\Delta^0 \rightarrow \underline{\text{Hom}}(A, B)$ . Conversely, given a family of elements  $f[-]$  such that  $\theta^*(f[n]) = f[n']$  for all  $\theta : [n'] \rightarrow [n]$ , we discover

$$(f[n])_\varphi = (f[n])_{\varphi \circ \text{id}_{[m]}} = (\varphi^*(f[n]))_{\text{id}_{[m]}} = (f[m])_{\text{id}_{[m]}}$$

for all morphisms  $\varphi : [m] \rightarrow [n]$  in  $\Delta$ , so we get a morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$  by setting  $f_m = (f[m])_{\text{id}_{[m]}}$ . This establishes the natural bijection in the statement of the proposition.  $\blacksquare$

**Proposition 2.3.3.** *Let  $\mathcal{M}$  be a locally small category and let  $X$  be a finite simplicial set.*

- *If  $\mathcal{M}$  has finite colimits, then for all cosimplicial objects  $A^\bullet$  in  $\mathcal{M}$ , there exists an object  $X \star A$  in  $\mathcal{M}$  equipped with bijections*

$$\mathcal{M}(X \star A, B) \cong \mathbf{sSet}(X, \mathcal{M}(A^\bullet, B))$$

*that are natural in  $B$ .*

- *If  $\mathcal{M}$  has finite limits, then for all simplicial objects  $B_\bullet$  in  $\mathcal{M}$ , there exists an object  $\{X, B\}$  in  $\mathcal{M}$  equipped with bijections*

$$\mathcal{M}(A, \{X, B\}) \cong \mathbf{sSet}(X, \mathcal{M}(A, B_\bullet))$$

*that are natural in  $A$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

Applying the Yoneda lemma, we see that  $\Delta^n \star A$  must be (isomorphic to)  $A^n$ . It is not hard to see that, if  $X : \mathcal{J} \rightarrow \mathbf{sSet}$  is a diagram such that  $Xj \star A$  exists for all  $j$  in  $\mathcal{J}$ , then  $\left(\lim_{\rightarrow j \in \mathcal{J}} Xj\right) \star A$  must be (isomorphic to)  $\lim_{\rightarrow j \in \mathcal{J}} (Xj \star A)$  when the latter exists; thus, the class of simplicial sets  $X$  for which  $X \star A$  exists must be closed under finite colimits (because  $\mathcal{M}$  has colimits for finite diagrams). We may then use [proposition 1.1.16](#) to deduce that  $X \star A$  exists if  $X$  is a finite simplicial set.  $\blacksquare$

**REMARK 2.3.4.** The same is true for a general simplicial set  $X$  when  $\mathcal{M}$  has limits and colimits for all small diagrams: see [theorem A.6.13](#).

**Proposition 2.3.5.** *Let  $\mathcal{M}$  be a locally small category and let  $X$  be a finite simplicial set.*

- *If  $\mathcal{M}$  has finite colimits, then for all cosimplicial objects  $A^\bullet$  in  $\mathcal{M}$ , there exists a cosimplicial object  $(X \odot A)^\bullet$  in  $\mathcal{M}$  equipped with isomorphisms*

$$\underline{\mathbf{cM}}(X \odot A, B) \cong [X, \underline{\mathbf{cM}}(A, B)]$$

*that are simplicially natural in  $B^\bullet$ .*

- *If  $\mathcal{M}$  has finite limits, then for all simplicial objects  $B_\bullet$  in  $\mathcal{M}$ , there exists a simplicial object  $(X \pitchfork B)_\bullet$  in  $\mathcal{M}$  equipped with isomorphisms*

$$\underline{\mathbf{sM}}(A, X \pitchfork B) \cong [X, \underline{\mathbf{sM}}(A, B)]$$

*that are natural in  $A_\bullet$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

It is clear that  $\Delta^n \times X$  is a finite simplicial set for all  $n \geq 0$  when  $X$  is a finite simplicial set, so the objects  $(\Delta^n \times X) \star A$  exist in  $\mathcal{M}$  (by [proposition 2.3.3](#)). We then define  $(X \odot A)^\bullet$  by taking  $(X \odot A)^n = (\Delta^n \times X) \star A$ . Using the calculus of ends, we have the following natural bijections:

$$\begin{aligned} & \underline{\mathbf{cM}}(X \odot A, B)_n \\ &= \int_{[m]:\Delta} \mathbf{Set}(\Delta([m], [n]), \mathcal{M}((\Delta^m \times X) \star A, B^m)) \\ & \quad \text{by definition} \\ &\cong \int_{[m]:\Delta} \mathbf{Set}\left(\Delta([m], [n]), \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [m]) \times X_l, \mathcal{M}(A^l, B^m))\right) \\ & \quad \text{by definition and [remark A.6.5](#)} \\ &\cong \int_{[m]:\Delta} \int_{[l]:\Delta} \mathbf{Set}(\Delta([m], [n]), \mathbf{Set}(\Delta([l], [m]) \times X_l, \mathcal{M}(A^l, B^m))) \\ & \quad \text{by [proposition A.6.10](#)} \\ &\cong \int_{[m]:\Delta} \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [m]), \mathbf{Set}(\Delta([m], [n]) \times X_l, \mathcal{M}(A^l, B^m))) \\ & \quad \text{by exponential adjunction} \\ &\cong \int_{[l]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(\Delta([l], [m]), \mathbf{Set}(\Delta([m], [n]) \times X_l, \mathcal{M}(A^l, B^m))) \end{aligned}$$

$$\begin{aligned}
 & \text{by the interchange law (theorem A.6.16)} \\
 & \cong \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [n]) \times X_l, \mathcal{M}(A^l, B^l)) \\
 & \text{by the Yoneda lemma for ends (proposition A.6.17)}
 \end{aligned}$$

On the other hand:

$$\begin{aligned}
 & [X, \underline{\mathbf{cM}}(A, B)]_n \\
 & \cong \mathbf{sSet}(\Delta^n \times X, \underline{\mathbf{cM}}(A, B)) \\
 & \quad \text{by remark A.2.23} \\
 & \cong \int_{[m]:\Delta} \mathbf{Set}\left(\Delta([m], [n]) \times X_m, \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [m]), \mathcal{M}(A^l, B^l))\right) \\
 & \quad \text{by definition and remark A.6.5} \\
 & \cong \int_{[m]:\Delta} \int_{[l]:\Delta} \mathbf{Set}(\Delta([m], [n]) \times X_m, \mathbf{Set}(\Delta([l], [m]), \mathcal{M}(A^l, B^l))) \\
 & \quad \text{by proposition A.6.10} \\
 & \cong \int_{[m]:\Delta} \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [m]), \mathbf{Set}(\Delta([m], [n]) \times X_m, \mathcal{M}(A^l, B^l))) \\
 & \quad \text{by exponential adjunction} \\
 & \cong \int_{[l]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(\Delta([l], [m]), \mathbf{Set}(\Delta([m], [n]) \times X_m, \mathcal{M}(A^l, B^l))) \\
 & \quad \text{by the interchange law (theorem A.6.16)} \\
 & \cong \int_{[l]:\Delta} \mathbf{Set}(\Delta([l], [n]) \times X_l, \mathcal{M}(A^l, B^l)) \\
 & \quad \text{by the Yoneda lemma for ends (proposition A.6.17)}
 \end{aligned}$$

Thus, we have isomorphisms

$$\underline{\mathbf{cM}}(X \odot A, B) \cong [X, \underline{\mathbf{cM}}(A, B)]$$

that are natural in  $B^\bullet$ . Moreover,

$$\underline{\mathbf{cM}}(X \odot A, B) \times \underline{\mathbf{cM}}(B, C) \cong [X, \underline{\mathbf{cM}}(A, B) \times \underline{\mathbf{cM}}(B, C)]$$

and so a similar calculation may be used to verify *simplicial* naturality in  $B^\bullet$ . ■

**Definition 2.3.6.** Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category.

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- A **realisation** of a simplicial object  $A_\bullet$  in  $\underline{\mathcal{M}}$  is an object  $|A_\bullet|$  in  $\underline{\mathcal{M}}$  with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}(|A_\bullet|, -) \cong [\underline{\Delta}, \underline{\mathbf{sSet}}](\underline{\Delta}^\bullet, \underline{\mathcal{M}}(A_\bullet, -))$$

- A **totalisation** of a cosimplicial object  $B^\bullet$  in  $\underline{\mathcal{M}}$  is an object  $|B^\bullet|$  in  $\underline{\mathcal{M}}$  with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}(-, \text{Tot } B^\bullet) \cong [\underline{\Delta}, \underline{\mathbf{sSet}}](\underline{\Delta}^\bullet, \underline{\mathcal{M}}(-, B^\bullet))$$

**Proposition 2.3.7.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category.*

- *If  $\underline{\mathcal{M}}$  is cocomplete as a simplicially enriched category, then realisations exist for all simplicial objects in  $\underline{\mathcal{M}}$ .*
- *If  $\underline{\mathcal{M}}$  is complete as a simplicially enriched category, then totalisations exist for all cosimplicial objects in  $\underline{\mathcal{M}}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $A_\bullet$  be a simplicial object in  $\underline{\mathcal{M}}$ . If  $\underline{\mathcal{M}}$  is cocomplete, then there must exist an object  $\Delta^\bullet \star_{\Delta^{\text{op}}} A_\bullet$  and a simplicially enriched natural isomorphism of the form below,

$$\underline{\mathcal{M}}(\Delta^\bullet \star_{\Delta^{\text{op}}} A_\bullet, B) \cong [\underline{\Delta}, \underline{\mathbf{sSet}}](\underline{\Delta}^\bullet, \underline{\mathcal{M}}(A_\bullet, B))$$

so we may take  $|A_\bullet| = \Delta^\bullet \star_{\Delta^{\text{op}}} A_\bullet$ . ■

**Proposition 2.3.8.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category.*

- *Let  $X$  be a simplicial set and let  $A_\bullet$  be a simplicial object in  $\underline{\mathcal{M}}$ . If  $\underline{\mathcal{M}}$  is cocomplete and  $X \boxtimes A_\bullet$  is the simplicial object defined below,*

$$(X \boxtimes A_\bullet)_n = X_n \odot A_n$$

*then there is an isomorphism*

$$\text{Tot}(X \boxtimes A_\bullet) \cong X \odot \text{Tot } A_\bullet$$

*and it is natural in both  $X$  and  $A_\bullet$ .*

- Let  $X$  be a simplicial set and let  $B^\bullet$  be a cosimplicial object in  $\underline{\mathcal{M}}$ . If  $\mathcal{M}$  is complete and  $X \boxtimes G^\bullet$  is the cosimplicial object defined below,

$$(X \boxtimes B^\bullet)^n = X_n \boxtimes B^n$$

then there is an isomorphism

$$|X \boxtimes B^\bullet| \cong X \boxtimes |B^\bullet|$$

and it is natural in both  $X$  and  $B^\bullet$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Using the calculus of ends, we have the following natural bijections:

$$\begin{aligned}
 \mathcal{M}(X \odot |A_\bullet|, B) &\cong \mathbf{sSet}(X, \underline{\mathcal{M}}(|A_\bullet|, B)) \\
 &\text{by definition} \\
 &\cong \mathbf{sSet}\left(X, \int_{[n]:\Delta} [\Delta^n, \underline{\mathcal{M}}(A_n, B)]\right) \\
 &\text{by theorem A.6.13} \\
 &\cong \int_{[n]:\Delta} \mathbf{sSet}(X, [\Delta^n, \underline{\mathcal{M}}(A_n, B)]) \\
 &\text{by proposition A.6.10} \\
 &\cong \int_{[n]:\Delta} \mathbf{sSet}(X \times \Delta^n, \underline{\mathcal{M}}(A_n, B)) \\
 &\text{by exponential adjunction} \\
 &\cong \int_{[n]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(X_m \times \Delta([m], [n]), \underline{\mathcal{M}}(A_n, B)_m) \\
 &\text{by remark A.6.5} \\
 &\cong \int_{[n]:\Delta} \int_{[m]:\Delta} \mathbf{Set}(X_m, \mathbf{Set}(\Delta([m], [n]), \underline{\mathcal{M}}(A_n, B)_m)) \\
 &\text{by exponential adjunction} \\
 &\cong \int_{[m]:\Delta} \mathbf{Set}\left(X_m, \int_{[n]:\Delta} \mathbf{Set}(\Delta([m], [n]), \underline{\mathcal{M}}(A_n, B)_m)\right) \\
 &\text{by the interchange law (theorem A.6.16)} \\
 &\cong \int_{[m]:\Delta} \mathbf{Set}(X_m, \underline{\mathcal{M}}(A_m, B)_m)
 \end{aligned}$$

$$\begin{aligned}
 & \text{by the Yoneda lemma for ends (proposition A.6.17)} \\
 & \cong \int_{[m]:\Delta} \mathcal{M}(X_m \odot A_m, B)_m \\
 & \text{by definition} \\
 & \cong \int_{[m]:\Delta} \mathbf{sSet}(\Delta^m, \underline{\mathcal{M}}(X_m \odot A_m, B)) \\
 & \text{by the ordinary Yoneda lemma} \\
 & \cong \mathcal{M}(|X \square A_\bullet|, B)
 \end{aligned}$$

Applying the Yoneda lemma once more, we deduce that  $|X \square A_\bullet|$  is naturally isomorphic to  $X \odot |A_\bullet|$ .  $\blacksquare$

**Corollary 2.3.9.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category.*

- *Let  $f_\bullet, f'_\bullet : A_\bullet \rightarrow B_\bullet$  be a parallel pair of morphisms in  $[\Delta^{\text{op}}, \underline{\mathcal{M}}]$ . If  $\underline{\mathcal{M}}$  is cocomplete as a simplicially enriched category and there exists a morphism  $H : \Delta^1 \square A_\bullet \rightarrow B_\bullet$  making the following diagram commute,*

$$\begin{array}{ccc}
 \Delta^0 \square A_\bullet & \xleftarrow{\cong} & A_\bullet \\
 \delta^1 \square \text{id}_{A_\bullet} \downarrow & & \downarrow f_\bullet \\
 \Delta^1 \square A_\bullet & \xrightarrow{H} & B_\bullet \\
 \delta^0 \square \text{id}_{A_\bullet} \uparrow & & \uparrow f'_\bullet \\
 \Delta^0 \square A_\bullet & \xleftarrow{\cong} & A_\bullet
 \end{array}$$

*then there is a simplicial homotopy  $\alpha : |f| \Rightarrow |f'|$ .*

- *Let  $f^\bullet, f'^\bullet : A^\bullet \rightarrow B^\bullet$  be a parallel pair of morphisms in  $[\Delta, \underline{\mathcal{M}}]$ . If  $\underline{\mathcal{M}}$  is complete as a simplicially enriched category and there exists a morphism  $H : A^\bullet \rightarrow \Delta^1 \boxtimes B^\bullet$  making the following diagram commute,*

$$\begin{array}{ccc}
 B^\bullet & \xrightarrow{\cong} & \Delta^0 \boxtimes B^\bullet \\
 f^\bullet \uparrow & & \uparrow \delta^1 \boxtimes \text{id}_{B^\bullet} \\
 A^\bullet & \xrightarrow{H} & \Delta^1 \boxtimes B^\bullet \\
 f'^\bullet \downarrow & & \downarrow \delta^0 \boxtimes \text{id}_{B^\bullet} \\
 B^\bullet & \xrightarrow{\cong} & \Delta^0 \boxtimes B^\bullet
 \end{array}$$

*then there is a simplicial homotopy  $\alpha : |f| \Rightarrow |f'|$ .*



*Proof.* The Yoneda lemma implies there are natural bijections

$$\mathcal{M}(\Delta^1 \odot A, B) \cong \underline{\mathcal{M}}(A, B)_1 \cong \mathcal{M}(A, \Delta^1 \pitchfork B)$$

so the required simplicial homotopy is obtained by applying realisation to the displayed diagrams. ■

**Proposition 2.3.10.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category.*

- *If  $\underline{\mathcal{M}}$  is cocomplete and cotensored, then we have the following adjunction of ordinary categories:*

$$|-| \dashv \Delta^\bullet \pitchfork (-) : \mathcal{M} \rightarrow [\Delta^{\text{op}}, \mathcal{M}]$$

- *If  $\underline{\mathcal{M}}$  is complete and tensored, then we have the following adjunction of ordinary categories:*

$$\Delta^\bullet \odot (-) \dashv \text{Tot} : [\Delta, \mathcal{M}] \rightarrow \mathcal{M}$$

*Proof.* By definition, we have the following natural bijections:

$$\mathcal{M}(|A_\bullet|, B) \cong [\Delta, \mathbf{sSet}](\Delta^\bullet, \underline{\mathcal{M}}(A_\bullet, B)) \cong [\Delta^{\text{op}}, \mathcal{M}](A_\bullet, \Delta^\bullet \pitchfork B)$$

$$\mathcal{M}(A, \text{Tot } B^\bullet) \cong [\Delta, \mathbf{sSet}](\Delta^\bullet, \underline{\mathcal{M}}(A, B^\bullet)) \cong [\Delta^{\text{op}}, \mathcal{M}](\Delta^\bullet \odot A, B^\bullet) \quad \blacksquare$$

## 2.4 Homotopy-coherent diagrams

**Prerequisites.** §§ 1.1, 1.2, 2.1, 2.2, 6.1.

**Lemma 2.4.1.** *Let  $\mathbf{Cat}$  be the category of small categories and let  $\mathbf{Grph} = \mathbf{sSet}_{\leq 1}$  be the category of reflexive graphs.*

- The forgetful functor  $U : \mathbf{Cat} \rightarrow \mathbf{Grph}$  has a left adjoint, say  $F$ .*
- The forgetful functor  $U : \mathbf{Cat} \rightarrow \mathbf{Grph}$  preserves any colimiting cocone that  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  preserves; in particular it is an  $\aleph_0$ -accessible functor.*
- For any reflexive graph  $X$ , the unit morphism  $\eta_X : X \rightarrow UFX$  is a bijection on vertices.*

*Proof.* (i). We may factor  $U : \mathbf{Cat} \rightarrow \mathbf{Grph}$  as the nerve functor  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  followed by the brutal  $\mathbb{I}$ -truncation functor  $(-)_{\leq 1} : \mathbf{sSet} \rightarrow \mathbf{Grph}$ ; but each of these has a left adjoint, by propositions 1.2.1 and 1.2.II.

(ii). We deduce this claim from the above discussion by noting that  $(-)_{\leq 1}$  is *itself* a left adjoint; for accessibility, we appeal to the accessible adjoint functor theorem (0.2.49).

(iii). This follows straightforwardly from the explicit description of  $\tau_1$ . ■

**Definition 2.4.2.** With notation as in the lemma, the **standard resolution** of a small category  $\mathbb{C}$  is the small simplicial category  $\mathbf{S}(\mathbb{C})_\bullet$  defined by the following formulae:

$$\begin{aligned} \mathbf{S}(\mathbb{C})_n &= (FU)^{n+1}(\mathbb{C}) \\ d_i^n &= (FU)^{n-i+1} \varepsilon_{(FU)^{i+1}(\mathbb{C})} \\ s_i^n &= (FU)^{n-i} F \eta_{U(FU)^i(\mathbb{C})} \end{aligned}$$

Here,  $\eta$  and  $\varepsilon$  are the unit and counit of the adjunction  $F \dashv U : \mathbf{Cat} \rightarrow \mathbf{Grph}$ . The **standard augmentation** for a category  $\mathbb{C}$  is the unique simplicial functor  $(\varepsilon_{\mathbb{C}})_\bullet : \mathbf{S}(\mathbb{C})_\bullet \rightarrow \mathbb{C}$  given in degree 0 by the counit  $\varepsilon_{\mathbb{C}} : FU(\mathbb{C}) \rightarrow \mathbb{C}$ .

REMARK 2.4.3. The fact that the above formulae do satisfy the simplicial identities is an instance of the general construction of simplicial objects using a comonad. More subtly, the fact that the standard resolution of  $\mathbb{C}$  is stable under universe enlargement is an instance of the stability of accessible adjunctions.

REMARK 2.4.4. Although the standard resolution  $\mathbf{S}(\mathbb{C})_\bullet$  of a category  $\mathbb{C}$  is most naturally defined as a simplicial category, the fact that  $\text{ob } \mathbf{S}(\mathbb{C})_\bullet$  is a constant simplicial set enables us to view it as a simplicially enriched category  $\underline{\mathbf{S}}(\mathbb{C})$ , per remark 2.1.II.

**Proposition 2.4.5.** *For any small category  $\mathbb{C}$ , the standard augmentation  $\varepsilon_{\mathbb{C}} : \mathbf{S}(\mathbb{C}) \rightarrow \mathbb{C}$  is a Dwyer–Kan equivalence of simplicially enriched categories.*

*Proof.* Given any pair  $(A, B)$  of objects in  $\mathbb{C}$ ,  $\varepsilon_{\mathbb{C}} : \underline{\mathbf{S}}(\mathbb{C}) \rightarrow \mathbb{C}$  induces a hom-space morphism  $\mathbf{S}(\mathbb{C})_\bullet(A, B) \rightarrow \mathbb{C}(A, B)$  which admits a backward contracting homotopy (defined in degree 0 by the adjunction unit), so it is a weak homotopy equivalence by propositions 1.5.2 and 1.3.16. ■

**Corollary 2.4.6.** *The functor  $\pi_0[\underline{\mathbf{S}}(\underline{\mathcal{C}})] \rightarrow \underline{\mathcal{C}}$  induced by the standard augmentation is an isomorphism of categories. ■*

**Definition 2.4.7.** Let  $\mathcal{J}$  be an ordinary category. A **homotopy-coherent diagram** of shape  $\mathcal{J}$  in a simplicially enriched category  $\underline{\mathcal{C}}$  is a simplicially enriched functor  $\underline{\mathbf{S}}(\mathcal{J}) \rightarrow \underline{\mathcal{C}}$ .

REMARK 2.4.8. It is worth thinking about the data that comprise a homotopy-coherent diagram of shape  $\mathcal{J}$ : in degree 0, one must specify a morphism  $F(f)$  in  $\mathcal{C}$  for every non-trivial morphism  $f$  in  $\mathcal{J}$  (but this assignment need not be functorial!); in degree 1, for every composable string of non-trivial morphisms of positive length, such as  $f_3 \circ f_2 \circ f_1$ , one has a simplicial homotopy from the “free” composition to the “true” composition, e.g.

$$\mu_{f_3, f_2, f_1} : F(f_3) \circ F(f_2) \circ F(f_1) \Rightarrow F(f_3 \circ f_2 \circ f_1)$$

and so on in higher degrees. The phrase ‘homotopy-coherent’ alludes to the relations imposed by the higher simplices: for instance, for each composable triple  $(f_3, f_2, f_1)$  as above, one has a pair of 2-cells in  $\text{mor } \underline{\mathcal{C}}$  as in the diagram below:

$$\begin{array}{ccc}
 & F(f_3 \circ f_2) \circ F(f_1) & \\
 \mu_{f_3, f_2} \circ \text{id}_{f_1} \nearrow & \Downarrow & \searrow \mu_{f_3 \circ f_2, f_1} \\
 F(f_3) \circ F(f_2) \circ F(f_1) & \xrightarrow{\mu_{f_3, f_2, f_1}} & F(f_3 \circ f_2 \circ f_1) \\
 \searrow \text{id}_{f_3} \circ \mu_{f_2, f_1} & \Uparrow & \nearrow \mu_{f_3, f_2 \circ f_1} \\
 & F(f_3) \circ F(f_2 \circ f_1) & 
 \end{array}$$

In particular, if  $\underline{\mathcal{C}}$  is obtained from a 2-category  $\mathfrak{C}$  by applying the nerve functor  $N : \mathbf{Cat} \rightarrow \mathbf{sSet}$  to its hom-categories, a homotopy-coherent diagram of shape  $\mathcal{J}$  in  $\underline{\mathcal{C}}$  is the same thing as a normalised lax 2-functor  $\mathcal{J} \rightarrow \mathfrak{C}$ .

**Definition 2.4.9.** The **homotopy-coherent nerve** of a simplicially enriched category  $\underline{\mathcal{C}}$  is the simplicial set defined by the formula below,

$$N^{\text{hc}}(\underline{\mathcal{C}})_n = \{\text{simplicially enriched functors } \underline{\mathbf{S}}([n]) \rightarrow \underline{\mathcal{C}}\}$$

with face and degeneracy maps induced by the coface and codegeneracy maps in  $\Delta$ .

**Proposition 2.4.10.** *Let  $\mathbf{SCat}$  be the category of small simplicially enriched categories.*

- (i)  $N^{\text{hc}} : \mathbf{SCat} \rightarrow \mathbf{sSet}$  has a left adjoint, which is the unique (up to unique isomorphism) colimit-preserving functor  $\underline{\mathbf{C}} : \mathbf{sSet} \rightarrow \mathbf{SCat}$  such that  $\underline{\mathbf{C}}(\Delta^n) = \underline{\mathbf{S}}([n])$ .
- (ii)  $N^{\text{hc}} : \mathbf{SCat} \rightarrow \mathbf{sSet}$  and  $\underline{\mathbf{C}} : \mathbf{sSet} \rightarrow \mathbf{SCat}$  are both accessible functors.
- (iii) If  $\mathbb{C}$  is a small category regarded as a simplicially enriched category, then  $N^{\text{hc}}(\mathbb{C})$  is naturally isomorphic to  $\mathbf{N}(\mathbb{C})$ .

*Proof.* (i). Apply [theorem 1.1.11](#).

(ii). This is an instance of the accessible adjoint functor theorem ([0.2.49](#)).

(iii). This follows from [proposition 2.2.11](#) and [corollary 2.4.6](#). ■

**Definition 2.4.11.** Given a simplicial set  $X$ , the **associated simplicially enriched category** is the simplicially enriched category  $\underline{\mathbf{C}}(X)$  constructed above.

**REMARK 2.4.12.** The stability of accessible adjunctions under universe enlargement implies that the simplicially enriched category  $\underline{\mathbf{C}}(X)$  associated with a simplicial set  $X$  does not depend on the choice of universe.

**REMARK 2.4.13.** One way of getting a good grip on the hom-spaces of  $\underline{\mathbf{C}}(X)$  for a general simplicial set  $X$  is to use the formalism of necklaces introduced by Dugger and Spivak [[2011b](#)].

**Theorem 2.4.14** (Riehl).

- (i) For any simplicial set  $X$  and any pair  $(a, b)$  of vertices of  $X$ , the hom-space  $\underline{\mathbf{C}}(X)(a, b)$  is 3-coskeletal.
- (ii) For any category  $\mathbb{C}$  and any pair  $(A, B)$  of objects in  $\mathbb{C}$ , the hom-space  $\underline{\mathbf{C}}(\mathbf{N}(\mathbb{C}))(A, B)$  is 2-coskeletal.
- (iii) For any category  $\mathbb{C}$ , its associated simplicially enriched category  $\underline{\mathbf{C}}(\mathbf{N}(\mathbb{C}))$  is naturally isomorphic to the standard resolution  $\underline{\mathbf{S}}(\mathbb{C})$ .

*Proof.* See Theorems 4.1, 6.4, and 6.7 in [[Riehl, 2011c](#)]. □

**Corollary 2.4.15.** *For any simplicially enriched category  $\underline{\mathcal{C}}$  and any ordinary category  $\mathcal{J}$ , there is a bijection*

$$\begin{aligned} \{ \text{simplicial maps } \mathbf{N}(\mathcal{J}) \rightarrow \mathbf{N}^{\text{hc}}(\underline{\mathcal{C}}) \} \\ \cong \{ \text{homotopy-coherent diagrams of shape } \mathcal{J} \text{ in } \underline{\mathcal{C}} \} \end{aligned}$$

and it is natural in  $\mathcal{J}$  and in  $\underline{\mathcal{C}}$ . ■

**REMARK 2.4.16.** The above result can also be proven directly, and the uniqueness of representations for functors up to unique isomorphism then implies that  $\underline{\mathbf{C}}(\mathbf{N}(\mathcal{J}))$  must be isomorphic to  $\underline{\mathbf{S}}(\mathcal{J})$ .

**Definition 2.4.17.** A **fibrant simplicially enriched category** is a simplicially enriched category  $\underline{\mathcal{C}}$  such that the hom-spaces  $\underline{\mathcal{C}}(A, B)$  are Kan complexes for all pairs  $(A, B)$  of objects in  $\underline{\mathcal{C}}$ .

**Definition 2.4.18.** Let  $F$  and  $G$  be homotopy-coherent diagrams of shape  $\mathcal{J}$  in a simplicially enriched category  $\underline{\mathcal{C}}$ . A **homotopy-coherent natural transformation**  $F \Rightarrow G$  is a homotopy-coherent diagram of shape  $\mathcal{J} \times [1]$  such that the restriction along  $\underline{\mathbf{S}}(\text{id}_{\mathcal{J}} \times \delta^1)$  is  $F$  and the restriction along  $\underline{\mathbf{S}}(\text{id}_{\mathcal{J}} \times \delta^0)$  is  $G$ .

Unfortunately, it is in general not possible to compose homotopy-coherent natural transformations, and even when it is possible, the composite is usually only well-defined up to higher homotopy. Instead, in good situations, what we get is a quasicategory:

**Theorem 2.4.19.** *Let  $\mathcal{J}$  be a small category and let  $\underline{\mathcal{C}}$  be a small simplicially enriched category. Consider the following simplicial set:*

$$[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}} = [\mathbf{N}(\mathcal{J}), \mathbf{N}^{\text{hc}}(\underline{\mathcal{C}})]$$

- (i) *There is a natural identification of the vertices of  $[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}}$  as homotopy-coherent diagrams of shape  $\mathcal{J}$  in  $\underline{\mathcal{C}}$ , and similarly, there is a natural identification of the edges as homotopy-coherent natural transformations.*
- (ii) *If  $\underline{\mathcal{C}}$  is fibrant, then the homotopy-coherent nerve  $\mathbf{N}^{\text{hc}}(\underline{\mathcal{C}})$  is a small quasicategory.*
- (iii) *Under the same hypothesis,  $[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}}$  is a small quasicategory.*

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*Proof.* (i). Apply [corollary 2.4.15](#) to the explicit description of exponential objects in the category of simplicial  $\mathbf{U}$ -sets.

(ii). See Theorem 2.1 in [Cordier and Porter, 1986].

(iii). Use [corollary 6.2.13](#). □

Let us say that a locally small simplicially enriched category  $\underline{\mathcal{C}}$  admits **rectification for homotopy-coherent diagrams** if, for all small categories  $\mathcal{J}$ , we have a commutative diagram of functors of the form below,

$$\begin{array}{ccc} [\mathcal{J}, \underline{\mathcal{C}}] & \longrightarrow & \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}} \\ \downarrow & \nearrow \text{---} & \\ \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] & & \end{array}$$

where  $[\mathcal{J}, \underline{\mathcal{C}}] \rightarrow \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}}$  is the functor

$$[\mathcal{J}, \underline{\mathcal{C}}] \cong \tau_1 \mathbf{N}([\mathcal{J}, \underline{\mathcal{C}}]) \cong \tau_1[\mathbf{N}(\mathcal{J}), \mathbf{N}(\underline{\mathcal{C}})] \rightarrow \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}}$$

induced by the canonical morphism  $\mathbf{N}(\underline{\mathcal{C}}) \rightarrow \mathbf{N}^{\text{hc}}(\underline{\mathcal{C}})$ ,  $[\mathcal{J}, \underline{\mathcal{C}}] \rightarrow \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$  is the localising functor, and  $\pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] \rightarrow \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\text{hc}}$  is fully faithful and essentially surjective on objects. (Note that this functor is unique *if* it exists, because the localising functor  $[\mathcal{J}, \underline{\mathcal{C}}] \rightarrow \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$  is full and bijective on objects.)

**Theorem 2.4.20** (Cordier–Porter). *Let  $\underline{\mathcal{C}}$  be a locally small simplicially enriched category. Consider the following conditions:*

- (i)  $\underline{\mathcal{C}}$  is fibrant and complete as a simplicially enriched category.
- (ii)  $\underline{\mathcal{C}}$  is fibrant and cocomplete as a simplicially enriched category.
- (iii)  $\underline{\mathcal{C}}$  is the simplicially enriched category of Kan complexes.

*If  $\underline{\mathcal{C}}$  satisfies any one of the above conditions, then  $\underline{\mathcal{C}}$  admits rectification for homotopy-coherent diagrams.*

*Proof.* (i). See Theorem 4.7 in [Cordier and Porter, 1986].

(ii). This follows from claim (i) by duality.

(iii). See the remark following Corollary 2.3 in [Cordier and Porter, 1997]. □

## 2.5 Simplicial model categories

**Prerequisites.** §§ 1.5, 2.1, 4.1, 4.3, 4.7, 4.8.

**Definition 2.5.1.** Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category. A **simplicial model structure** on  $\underline{\mathcal{M}}$  is a model structure on the underlying model category  $\mathcal{M}$  that satisfies the following axiom:<sup>[5]</sup>

**SM7.** If  $i : Z \rightarrow W$  is a cofibration in  $\mathcal{M}$  and  $p : X \rightarrow Y$  is a fibration in  $\mathcal{M}$ , and the square in the diagram below is a pullback square in  $\mathbf{sSet}$ ,

$$\begin{array}{ccc}
 \underline{\mathcal{M}}(W, X) & \xrightarrow{\quad \underline{\mathcal{M}}(W, p) \quad} & \underline{\mathcal{M}}(W, Y) \\
 \downarrow \underline{\mathcal{M}}(i, X) & \dashrightarrow^{i^* \square p_*} & \downarrow \underline{\mathcal{M}}(i, Y) \\
 \underline{\mathcal{M}}(Z, X) \times_{\underline{\mathcal{M}}(Z, Y)} \underline{\mathcal{M}}(W, Y) & \longrightarrow & \underline{\mathcal{M}}(W, Y) \\
 \downarrow & & \downarrow \underline{\mathcal{M}}(i, Y) \\
 \underline{\mathcal{M}}(Z, X) & \xrightarrow{\quad \underline{\mathcal{M}}(Z, p) \quad} & \underline{\mathcal{M}}(Z, Y)
 \end{array}$$

then the unique morphism  $i^* \square p_*$  making the diagram commute is a Kan fibration; moreover, if either  $i : Z \rightarrow W$  or  $p : X \rightarrow Y$  is a weak equivalence, then  $i^* \square p_*$  is a trivial Kan fibration.

A **simplicial model category** is a locally small simplicially enriched category  $\underline{\mathcal{M}}$  that is equipped with a simplicial model structure and satisfies the additional axioms below:

**SM0.** For each finite simplicial set  $K$  and each object  $X$  in  $\mathcal{M}$ , the tensor product  $K \odot X$  and the cotensor product  $K \pitchfork X$  exist in  $\underline{\mathcal{M}}$ .

**CM1.**  $\mathcal{M}$  has finite limits and finite colimits.

A **simplicial derivable category** is a locally small simplicially enriched category  $\underline{\mathcal{M}}$  that is equipped with a simplicial model structure and satisfies the additional axioms below:

- If  $W$  is a cofibrant object in  $\mathcal{M}$ , then the functor  $\underline{\mathcal{M}}(W, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  preserves fibrant objects, fibrations, and trivial fibrations; and if  $X$  is a fibrant object in  $\mathcal{M}$ , then the functor  $\underline{\mathcal{M}}(-) : \mathcal{M}^{\text{op}} \rightarrow \mathbf{sSet}$  sends cofibrant objects (resp. cofibrations, trivial cofibrations) in  $\mathcal{M}$  to Kan complexes (resp. Kan fibrations, trivial Kan fibrations) in  $\mathbf{sSet}$ .

[5] This presentation is due to Quillen [1967].

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- The underlying ordinary category  $\mathcal{M}$  equipped with the given model structure is a derivable category.

REMARK 2.5.2. **Proposition 2.1.31** implies that limits and colimits in a simplicial model category are automatically conical (i.e. limits and colimits in the simplicially enriched sense).

REMARK 2.5.3. Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category equipped whose underlying ordinary category is equipped with a model structure. Then  $\underline{\mathcal{M}}$  is a simplicial model category if and only if  $\underline{\mathcal{M}}^{\text{op}}$  is a simplicial model category.

**Proposition 2.5.4.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category whose underlying ordinary category is equipped with a model structure. If  $\underline{\mathcal{M}}$  satisfies axioms SM0 and CM1, then the following are equivalent:*

- (i) *Axiom SM7 is satisfied.*
- (ii) *For all fibrations (resp. trivial fibrations)  $p : X \rightarrow Y$  in  $\mathcal{M}$ , if  $i : Z \rightarrow W$  is a boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  and the square in the diagram below is a pullback square in  $\mathcal{M}$ ,*

$$\begin{array}{ccc}
 W \pitchfork X & \xrightarrow{\text{id}_W \pitchfork p} & W \pitchfork Y \\
 \downarrow i \pitchfork p & \searrow & \downarrow i \pitchfork \text{id}_Y \\
 (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) & \longrightarrow & W \pitchfork Y \\
 \downarrow & & \downarrow i \pitchfork \text{id}_Y \\
 Z \pitchfork X & \xrightarrow{\text{id}_Z \pitchfork p} & Z \pitchfork Y
 \end{array}$$

$i \pitchfork \text{id}_X$  is the curved arrow from  $W \pitchfork X$  to  $Z \pitchfork X$ .

*then the unique morphism  $i \pitchfork p$  making the diagram commute is a fibration (resp. trivial fibration); and for all fibrations  $p : X \rightarrow Y$  in  $\mathcal{M}$ , if  $i : Z \rightarrow W$  is a horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ , then the morphism  $i \pitchfork p$  defined as above is a trivial fibration.*

- (iii) *For all cofibrations (resp. trivial cofibrations)  $j : X \rightarrow Y$  in  $\mathcal{M}$ , if  $i : Z \rightarrow W$  is a boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  and the square in the diagram*



below is a pushout square in  $\mathcal{M}$ ,

$$\begin{array}{ccc}
 Z \odot X & \xrightarrow{\text{id}_Z \odot j} & Z \odot Y \\
 i \odot \text{id}_X \downarrow & & \downarrow \\
 W \odot X & \longrightarrow & (W \odot X) \cup^{Z \odot X} (Z \odot Y) \\
 & \searrow & \downarrow i \sqcup j \\
 & & W \odot Y \\
 & \xrightarrow{\text{id}_W \odot j} & 
 \end{array}$$

then the unique morphism  $i \sqcup j$  making the diagram commute is a cofibration (resp. trivial cofibration); and for all cofibrations  $j : X \rightarrow Y$  in  $\mathcal{M}$ , if  $i : Z \rightarrow W$  is a horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ , then the morphism  $i \sqcup j$  defined as above is a trivial cofibration.

*Proof.* This is (essentially) a special case of [proposition 5.4.1](#). ■

**Proposition 2.5.5.** *Let  $\underline{\mathcal{M}}$  be a locally small simplicially enriched category with an initial object  $0$  and a terminal object  $1$  (in the simplicially enriched sense) and suppose  $\underline{\mathcal{M}}$  is equipped with a simplicial model structure.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$ , then the functor  $\underline{\mathcal{M}}(A, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  preserves weighted limits, fibrant objects, fibrations, and trivial fibrations.*
- *If  $B$  is a fibrant object in  $\mathcal{M}$ , then the functor  $\underline{\mathcal{M}}(-, B) : \mathcal{M}^{\text{op}} \rightarrow \mathbf{sSet}$  preserves weighted limits, fibrant objects, fibrations, and trivial fibrations.*

*In particular, every simplicial model category is a simplicial derivable category.*

*Proof.* The two claims are formally dual; we will prove the first version.

Essentially by definition, the functor  $\underline{\mathcal{M}}(A, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  preserves any weighted limits that exist in  $\underline{\mathcal{M}}$ . [Lemma 4.1.15](#) says the unique morphism  $0 \rightarrow A$  is a cofibration if and only if  $A$  is a cofibrant object in  $\mathcal{M}$ , so we may then apply axiom SM7 to deduce that  $\underline{\mathcal{M}}(A, -)$  preserves fibrant objects, fibrations, and trivial fibrations.

To conclude, we need only apply [remark 2.5.2](#) and [proposition 4.1.16](#). ■

**Lemma 2.5.6.** *Let  $\underline{\mathcal{M}}$  be a simplicial derivable category, let  $\mathcal{M}_c$  be the full subcategory of cofibrant objects in  $\mathcal{M}$ , and let  $\mathcal{M}_f$  be the full subcategory of fibrant objects in  $\mathcal{M}$ .*

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- If  $A$  is a cofibrant object in  $\mathcal{M}$ , then  $\underline{\mathcal{M}}(A, -) : \mathcal{M}_f \rightarrow \mathbf{sSet}$  is a homotopical functor.
- If  $B$  is a fibrant object in  $\mathcal{M}$ , then  $\underline{\mathcal{M}}(-, B) : \mathcal{M}_c^{\text{op}} \rightarrow \mathbf{sSet}$  is a homotopical functor.

In particular,  $\underline{\mathcal{M}}(-, -) : \mathcal{M}_c^{\text{op}} \times \mathcal{M}_f \rightarrow \mathbf{sSet}$  is a homotopical functor.

*Proof.* By definition,  $\underline{\mathcal{M}}(A, -)$  sends trivial fibrations in  $\mathcal{M}$  to trivial Kan fibrations when  $A$  is cofibrant, and  $\underline{\mathcal{M}}(-, B)$  sends trivial cofibrations in  $\mathcal{M}$  to trivial Kan fibrations when  $B$  is fibrant, we may apply [lemma 4.1.32](#). ■

**Theorem 2.5.7.** Let  $\underline{\mathcal{M}}$  be a simplicial derivable category, let  $(\mathcal{M}_c, Q, p)$  be a left Quillen deformation retract of  $\mathcal{M}$ , and let  $(\mathcal{M}_f, R, i)$  be a right Quillen deformation retract of  $\mathcal{M}$ .

- $(\mathcal{M}_c^{\text{op}} \times \mathcal{M}_f, Q \times R, (p, i))$  is a right deformation retract for the functor  $\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{sSet}$ .
- $\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{sSet}$  has a total right derived functor; furthermore, if  $(\mathcal{M}_c, Q, p)$  and  $(\mathcal{M}_f, R, i)$  are functorial deformation retracts, then  $\underline{\mathcal{M}}(-, -)$  also has a homotopical right approximation.

*Proof.* (i). This is [lemma 2.5.6](#).

(ii). Apply theorems [3.3.13](#) and [3.4.10](#). ■

**Definition 2.5.8.** Let  $\underline{\mathcal{M}}$  be a simplicial derivable category. A **derived hom-space functor** for  $\underline{\mathcal{M}}$  is a total right derived functor for the functor  $\underline{\mathcal{M}}(-, -) : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{sSet}$ . We write  $\mathbf{RHom}_{\underline{\mathcal{M}}} : (\text{Ho } \mathcal{M})^{\text{op}} \times \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathbf{sSet}$  for (the functor part of) a derived hom-space functor for  $\underline{\mathcal{M}}$ .

**Proposition 2.5.9.** Let  $\underline{\mathcal{M}}$  be a simplicial model category.

- If  $A$  is a cofibrant object in  $\mathcal{M}$ , then the cosimplicial object  $\Delta^\bullet \odot A$  is (the object part of) a left frame on  $A$ .
- If  $B$  is a fibrant object in  $\mathcal{M}$ , then the simplicial object  $\Delta^\bullet \pitchfork B$  is (the object part of) a right frame on  $B$ .

*Proof.* See Remark 5.2.10 in [Hovey, 1999]. □

**Corollary 2.5.10.** *Let  $\underline{\mathcal{M}}$  be a simplicial model category. If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $B$  is a fibrant object in  $\mathcal{M}$ , then:*

- *The hom-space  $\underline{\mathcal{M}}(A, B)$  is (the object part of) a left homotopy function complex from  $A$  to  $B$ .*
- *The hom-space  $\underline{\mathcal{M}}(A, B)$  is (the object part of) a right homotopy function complex from  $A$  to  $B$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

By [proposition 2.5.9](#), the cosimplicial object  $\tilde{A}^\bullet = \Delta^\bullet \odot A$  is (the object part of) a left frame on  $A$ ; but there is a natural isomorphism between the left hom-complex  $\text{Hom}_{\mathcal{M}}(\tilde{A}, B)$  and the hom-space  $\underline{\mathcal{M}}(A, B)$ , and  $B$  is fibrant by hypothesis, so we are done. ■

**REMARK 2.5.11.** In particular, the derived hom-spaces of the simplicial model category  $\underline{\mathcal{M}}$  agree with the derived hom-spaces of the underlying model category  $\mathcal{M}$ .

**Proposition 2.5.12.** *Let  $\underline{\mathcal{M}}$  be a simplicial model category.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$ , then  $(\Delta^1 \odot A, \delta^1 \odot \text{id}_A, \delta^0 \odot \text{id}_A, \sigma^0 \odot \text{id}_A)$  is a cylinder object for  $\Delta^0 \odot A$  (and hence, isomorphic to a cylinder object for  $A$ ).*
- *If  $B$  is a fibrant object in  $\mathcal{M}$ , then  $(\Delta^1 \pitchfork B, \delta^1 \pitchfork \text{id}_B, \delta^0 \pitchfork \text{id}_B, \sigma^0 \pitchfork \text{id}_B)$  is a path object for  $\Delta^0 \pitchfork B$  (and hence, isomorphic to a path object for  $B$ ).*

*Proof.* Apply [propositions 2.5.9](#) and [4.7.21](#); but see also [Lemma 3.5](#) in [GJ], or [Lemma 9.5.4](#) in [Hirschhorn, 2003]. ■

**Corollary 2.5.13.** *Let  $\underline{\mathcal{M}}$  be a simplicial model category. If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $B$  is a fibrant object in  $\mathcal{M}$ , then the canonical map*

$$\text{Ho } \mathcal{M}(A, B) \rightarrow \pi_0 \underline{\mathcal{M}}(A, B)$$

*is a natural bijection.*

*Proof.* [Proposition 2.5.12](#) says that  $(\Delta^1 \odot A, \delta^1 \odot \text{id}_A, \delta^0 \odot \text{id}_A, \sigma^0 \odot \text{id}_A)$  is a cylinder object for  $\Delta^0 \odot A$ , so if  $B$  is fibrant, we may apply [lemma 4.2.14](#) and [theorem 4.4.1](#) to deduce that the connected components of  $\underline{\mathcal{M}}(A, B)$  are in natural bijection with the homotopy classes of morphisms  $A \rightarrow B$ . ■

## 2.6 Simplicial localisation

**Prerequisites.** §§ 1.2, 1.5, 2.1, 2.2, 2.4, 3.1, A.4.

When one passes from a relative category to its homotopy category by freely inverting the weak equivalences, one loses much of the homotopical information. Dwyer and Kan [1980a,b,c] instead proposed a more sophisticated notion of localisation that produces a simplicial category retaining all the homotopical information, at least in the case of a simplicial model category.

**Definition 2.6.1.** The **standard resolution** of a small relative category  $C$  is the simplicial relative category  $\mathbf{S}(C)_\bullet$  where  $\text{und } \mathbf{S}(C)_\bullet = \mathbf{S}(\text{und } C)_\bullet$  and  $\text{weq } \mathbf{S}(C)_\bullet = \mathbf{S}(\text{weq } C)_\bullet$ . The **standard simplicial localisation** of  $C$  is the simplicial category  $\mathbf{Lo}(C)_\bullet$  obtained by applying  $\text{Ho}$  to  $\mathbf{S}(C)_\bullet$  degreewise, and the **simplicial localising functor** is the induced simplicial functor  $\mathbf{S}(C)_\bullet \rightarrow \mathbf{Lo}(C)_\bullet$ .

**REMARK 2.6.2.** As in [remark 2.4.4](#), the face and degeneracy operators of the simplicial category  $\mathbf{Lo}(C)_\bullet$  are trivial, so we may regard it as a simplicially enriched category  $\underline{\mathbf{Lo}}(C)$ .

**Proposition 2.6.3.** *Let  $C$  be a small relative category. The standard augmentation for  $C$  induces an isomorphism  $\pi_0[\underline{\mathbf{Lo}}(C)] \rightarrow \text{Ho } C$ .*

*Proof.* Let  $\mathcal{D}$  be an ordinary category and let  $F : C \rightarrow \mathcal{D}$  be a functor that sends weak equivalences in  $C$  to isomorphisms in  $\mathcal{D}$ . Then, composing with the standard augmentation  $(\varepsilon_C)_\bullet : \mathbf{S}(C)_\bullet \rightarrow C$  yields a simplicial functor  $\mathbf{S}(C)_\bullet \rightarrow \mathcal{D}$  that sends weak equivalences in each  $\mathbf{S}(C)_n$  to isomorphisms in  $\mathcal{D}$ , so the degreewise universal property of  $\mathbf{Lo}(C)_\bullet$  yields a unique simplicial functor  $\mathbf{Lo}(C)_\bullet \rightarrow \mathcal{D}$  making the diagram below commute (strictly),

$$\begin{array}{ccc} \mathbf{S}(C)_\bullet & \xrightarrow{(\varepsilon_C)_\bullet} & C \\ \downarrow & & \downarrow F \\ \mathbf{Lo}(C)_\bullet & \dashrightarrow & \mathcal{D} \end{array}$$

where  $\mathbf{S}(C)_\bullet \rightarrow \mathbf{Lo}(C)_\bullet$  is the simplicial localising functor.  $\mathcal{D}$  is an ordinary category, so [proposition 2.2.11](#) says the corresponding simplicially enriched functor  $\underline{\mathbf{Lo}}(C) \rightarrow \mathcal{D}$  factors through the  $\pi_0$ -localising functor  $\underline{\mathbf{Lo}}(C) \rightarrow \pi_0[\underline{\mathbf{Lo}}(C)]$  in a unique way. Thus,  $\pi_0[\underline{\mathbf{Lo}}(C)]$  has the universal property of  $\text{Ho } C$ , and the required isomorphism  $\pi_0[\underline{\mathbf{Lo}}(C)] \rightarrow \text{Ho } C$  is induced by the ordinary localising functor  $C \rightarrow \text{Ho } C$ . ■

**Proposition 2.6.4.** *Let  $\mathcal{C}$  be a small relative category. The following are equivalent for a morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$ :*

- (i) *The morphism  $f : X \rightarrow Y$  is a weak equivalence in  $\mathcal{C}$ .*
- (ii) *The morphism in  $\mathbf{Lo}(\mathcal{C})$  corresponding to  $f : X \rightarrow Y$  is an isomorphism.*
- (iii) *The morphism in  $\mathbf{Lo}(\mathcal{C})_0$  corresponding to  $f : X \rightarrow Y$  is an isomorphism.*

*Proof.* (i)  $\Rightarrow$  (ii). For each natural number  $n$ , the morphism in  $\mathbf{S}(\mathcal{C})_n$  corresponding to  $f : X \rightarrow Y$  is a weak equivalence (by definition), so its image in  $\mathbf{Lo}(\mathcal{C})_n$  is an isomorphism. Thus, the morphism corresponding to  $f$  in the simplicially enriched category  $\mathbf{Lo}(\mathcal{C})$  is an isomorphism.

(ii)  $\Rightarrow$  (iii). Immediate.

(iii)  $\Rightarrow$  (i). Since  $\mathbf{und} \mathbf{S}(\mathcal{C})_0$  and  $\mathbf{weq} \mathbf{S}(\mathcal{C})_0$  are free categories, the morphisms in  $\mathbf{Lo}(\mathcal{C})_0$  can be represented by reduced composable strings generated by morphisms in  $\mathbf{und} \mathcal{C}$  and the formal inverses of morphisms in  $\mathbf{weq} \mathcal{C}$ . Thus, a morphism in  $\mathcal{C}$  corresponds to an isomorphism in  $\mathbf{Lo}(\mathcal{C})_0$  if and only if it is a weak equivalence in  $\mathcal{C}$ . ■

**Definition 2.6.5.** Let  $X$  and  $Y$  be objects in a relative category  $\mathcal{C}$ . A **hammock** in  $\mathcal{C}$  from  $X$  to  $Y$  of width  $k$  and length  $n$  is a commutative diagram in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccccccccccc}
 X & \longrightarrow & Z_{0,1} & \longrightarrow & Z_{0,2} & \longrightarrow & \cdots & \longrightarrow & Z_{0,n-2} & \longrightarrow & Z_{0,n-1} & \longrightarrow & Y \\
 \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
 X & \longrightarrow & Z_{1,1} & \longrightarrow & Z_{1,2} & \longrightarrow & \cdots & \longrightarrow & Z_{1,n-2} & \longrightarrow & Z_{1,n-1} & \longrightarrow & Y \\
 \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
 \vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \vdots \\
 \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
 X & \longrightarrow & Z_{k-1,1} & \longrightarrow & Z_{k-1,2} & \longrightarrow & \cdots & \longrightarrow & Z_{k-1,n-2} & \longrightarrow & Z_{k-1,n-1} & \longrightarrow & Y \\
 \parallel & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \parallel \\
 X & \longrightarrow & Z_{k,1} & \longrightarrow & Z_{k,2} & \longrightarrow & \cdots & \longrightarrow & Z_{k,n-2} & \longrightarrow & Z_{k,n-1} & \longrightarrow & Y
 \end{array}$$

such that the following conditions are satisfied:

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- In each column, all horizontal arrows point in the same direction.
- All leftward-pointing arrows are weak equivalences.
- All vertical arrows are weak equivalences.

We allow both  $k$  and  $n$  to be zero; if  $n = 0$  then we must have  $X = Y$ .

A **reduced hammock** in  $C$  is a hammock with these additional properties:

- In each column, not every horizontal arrow is an identity morphism.
- Arrows in adjacent columns point in opposite directions.

¶ 2.6.6. It is clear that we can transform any hammock into a reduced hammock by iteratively omitting any column of identity morphisms and composing any adjacent columns where possible.

**Definition 2.6.7.** Let  $C$  be a small relative category. The **hammock localisation** of  $C$  is the simplicial category  $\underline{\mathbf{Lo}}^{\mathbf{H}}(C)$  defined below:

- The objects are those in  $C$ .
- For each pair  $(X, Y)$  of objects, the hom-space  $\underline{\mathbf{Lo}}^{\mathbf{H}}(C)(X, Y)$  is the simplicial set whose  $k$ -simplices are the reduced hammocks of width  $k$  (and any length), with face (resp. degeneracy) operators defined by omitting (resp. repeating) a row of objects and reducing the resulting hammock if necessary.
- Composition is defined by concatenation of hammocks (reducing as necessary), and identities are hammocks of zero length.

REMARK 2.6.8. The hom-space  $\underline{\mathbf{Lo}}^{\mathbf{H}}(C)(X, Y)$  can be constructed as a colimit as shown below,

$$\underline{\mathbf{Lo}}^{\mathbf{H}}(C)(X, Y) \cong \varinjlim_{\mathbf{T}^{\text{op}}} \mathbf{N}(C^*(X, Y))$$

where  $\mathbf{N} : \mathbf{Cat} \rightarrow \mathbf{sSet}$  is the nerve functor and  $C^*(X, Y) : \mathbf{T}^{\text{op}} \rightarrow \mathbf{Cat}$  is the functor described in [remark A.4.24](#).

REMARK 2.6.9. Unlike the standard simplicial localisation, the hammock localisation of a relative category  $C$  is equipped with a natural functor  $C \rightarrow \underline{\mathbf{Lo}}^{\mathbf{H}}(C)$  that is bijective on objects and faithful (but not necessarily full).

**Theorem 2.6.10.** *Let  $\mathcal{C}$  be a small relative category and let  $\underline{\mathcal{D}}$  be the following simplicially enriched category:*

- *The objects are those in  $\mathcal{C}$ .*
- *For each pair  $(X, Y)$  of objects, the hom-space  $\underline{\mathcal{D}}(X, Y)$  is given by*

$$\underline{\mathcal{D}}(X, Y)_n = \underline{\mathbf{Lo}}^{\mathbf{H}}(\mathbf{S}(\mathcal{C})_n)(X, Y)_n$$

*where  $\mathbf{S}(\mathcal{C})_{\bullet}$  is the standard resolution of  $\mathcal{C}$ .*

- *Composition and identities are inherited from  $\underline{\mathbf{Lo}}^{\mathbf{H}}(\mathbf{S}(\mathcal{C})_{\bullet})$ .*

*Let  $\underline{\mathbf{Lo}}(\mathcal{C})$  be the standard simplicial localisation of  $\mathcal{C}$  and let  $\underline{\mathbf{Lo}}^{\mathbf{H}}(\mathcal{C})$  be the hammock localisation of  $\mathcal{C}$ . Then:*

- (i) *The simplicially enriched functor  $\underline{\mathcal{D}} \rightarrow \underline{\mathbf{Lo}}^{\mathbf{H}}(\mathcal{C})$  induced by the standard augmentation  $\mathbf{S}(\mathcal{C}) \rightarrow \mathcal{C}$  is a Dwyer–Kan equivalence.*
- (ii) *The simplicially enriched functor  $\underline{\mathcal{D}} \rightarrow \underline{\mathbf{Lo}}(\mathcal{C})$  induced by the localising functors  $\mathbf{S}(\mathcal{C})_n \rightarrow \mathbf{Lo}(\mathcal{C})_n$  is a Dwyer–Kan equivalence.*

*Proof.* See Proposition 2.2 in [Dwyer and Kan, 1980b]. □

**Corollary 2.6.11.** *If  $f : X \rightarrow Y$  is a weak equivalence in a small relative category  $\mathcal{C}$ , then its image in  $\underline{\mathbf{Lo}}^{\mathbf{H}}(\mathcal{C})$  is a simplicial homotopy equivalence. ■*

**Definition 2.6.12.** A **Dwyer–Kan equivalence of relative categories** is a relative functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that the induced simplicially enriched functor  $\underline{\mathbf{Lo}}^{\mathbf{H}}(F) : \underline{\mathbf{Lo}}^{\mathbf{H}}(\mathcal{C}) \rightarrow \underline{\mathbf{Lo}}^{\mathbf{H}}(\mathcal{D})$  is a Dwyer–Kan equivalence of simplicially enriched categories.

**Definition 2.6.13.** Let  $\mathcal{C}_{\bullet}$  be a simplicial category. The **flattening** of  $\mathcal{C}_{\bullet}$  is the relative category  $\mathcal{C}^{\flat}$  defined below:

- The objects are pairs  $(n, X)$ , where  $n$  is a natural number and  $X$  is an object in  $\mathcal{C}_n$ .
- A morphism  $(n, X) \rightarrow (m, Y)$  is a pair  $(\varphi, f)$ , where  $\varphi : [m] \rightarrow [n]$  is a morphism in  $\mathbf{\Delta}$  and  $f : \varphi^* X \rightarrow Y$  is a morphism in  $\mathcal{C}_m$ ; a weak equivalence is any morphism of the form  $(\varphi, \text{id}) : (n, X) \rightarrow (m, \varphi^* X)$ .

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- Given morphisms  $(\varphi, f) : (n, X) \rightarrow (m, Y)$  and  $(\psi, g) : (m, Y) \rightarrow (l, Z)$ , their composite is  $(\varphi \circ \psi, g \circ \psi^* f) : (n, X) \rightarrow (l, Z)$ .

In other words,  $C^b$  is the Grothendieck construction applied to  $C_\bullet$  considered as a functor  $\mathbf{\Delta}^{\text{op}} \rightarrow \mathbf{Cat}$ .

**Theorem 2.6.14.** *Let  $\mathbf{RelCat}$  be the category of small relative categories and let  $\mathbf{SCat}$  be the category of small simplicially enriched categories.*

- (i) *There is a zigzag of natural Dwyer–Kan equivalences between  $\text{id}_{\mathbf{SCat}}$  and  $\underline{\mathbf{Lo}}^{\mathbf{H}}((-)^b)$ .*
- (ii) *There is a zigzag of natural Dwyer–Kan equivalences between  $\text{id}_{\mathbf{RelCat}}$  and  $\mathbf{Lo}^{\mathbf{H}}(-)^b$ .*
- (iii) *If we regard  $\mathbf{RelCat}$  as a homotopical category where the weak equivalences are the Dwyer–Kan equivalences of relative categories, then the functors*

$$\underline{\mathbf{Lo}}^{\mathbf{H}}(-) : \mathbf{RelCat} \rightarrow \mathbf{SCat} \quad (-)^b : \mathbf{SCat} \rightarrow \mathbf{RelCat}$$

*are a mutually quasi-inverse pair of homotopical equivalences.*

*Proof.* See paragraph 2.5 and Proposition 3.1 in [Barwick and Kan, 2012].  $\square$

**Theorem 2.6.15** (Relative Yoneda embedding). *Let  $C$  be a small relative category and let  $h_\bullet : C \rightarrow [C^{\text{op}}, \mathbf{sSet}]_h$  be the relative functor defined by the formula below:*

$$h_Y(X) = \underline{\mathbf{Lo}}^{\mathbf{H}}(C)(X, Y)$$

- (i) *For each pair  $(X, Y)$  of objects in  $C$ , the induced hom-space morphism*

$$\underline{\mathbf{Lo}}^{\mathbf{H}}(C)(X, Y) \rightarrow \underline{\mathbf{Lo}}^{\mathbf{H}}([C^{\text{op}}, \mathbf{sSet}]_h)(X, Y)$$

*is a weak homotopy equivalence of simplicial sets.*

- (ii) *Let  $\mathcal{D}$  be the full relative subcategory of  $[C^{\text{op}}, \mathbf{sSet}]_h$  spanned by the relative functors  $C^{\text{op}} \rightarrow \mathbf{sSet}$  that are naturally weakly equivalent to one in the image of  $h_\bullet$ . Then the functor  $h_\bullet : C \rightarrow \mathcal{D}$  is a Dwyer–Kan equivalence of relative categories.*

*Proof.* See paragraph 4.3 in [Barwick and Kan, 2011].  $\square$



## HOMOTOPICAL CATEGORIES

### 3.1 Basics

**Prerequisites.** §A.4.

**Definition 3.1.1.** A relative category  $C$  is a **category with weak equivalences** if it is semi-saturated and  $\text{weq } C$  has the 2-out-of-3 property, and it is a **homotopical category** if  $\text{weq } C$  has the 2-out-of-6 property. A **homotopical functor** is a relative functor between homotopical categories.

**REMARK 3.1.2.** If  $C$  is a relative category such that  $\text{weq } C$  has the 2-out-of-6 property, then every isomorphism in  $C$  is automatically a weak equivalence. Indeed, suppose  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are mutual inverses in  $C$ ; then the fact that  $g \circ f = \text{id}_X$  and  $f \circ g = \text{id}_Y$  are in  $\text{weq } C$  implies that  $f$  and  $g$  must also be in  $\text{weq } C$ . Recalling [lemma A.4.14](#), it follows that every homotopical category is a category with weak equivalences.

¶ 3.1.3. To simplify notation, we will usually not distinguish between  $\text{und } C$  and  $C$ . For example, when  $C$  and  $D$  are relative categories, then by ‘ordinary functor  $C \rightarrow D$ ’ we mean a functor  $\text{und } C \rightarrow \text{und } D$ .

**Example 3.1.4.** Any saturated relative category is automatically a homotopical category, by [corollary A.4.15](#). In particular, any minimal saturated relative category is a homotopical category. On the other hand, any maximal relative category is obviously a homotopical category.

**REMARK 3.1.5.** A relative category  $C$  is a category with weak equivalences or a homotopical category if and only if the opposite relative category  $C^{\text{op}}$  is.

**Lemma 3.1.6.** *Let  $A$  be an object in a homotopical category (resp. category with weak equivalences)  $C$ . Then the slice category  $C_{/A}$  is also a homotopical category (resp. category with weak equivalences) if we declare a morphism in  $C_{/A}$  to be a weak equivalence if and only if it is a weak equivalence in  $C$ .*

*Proof.* Use lemma A.4.14 on the projection functor  $C_{/A} \rightarrow C$ . ■

**Lemma 3.1.7.** *Any relative subcategory  $\mathcal{D}$  of a homotopical category (resp. category with weak equivalences)  $C$  is also a homotopical category (resp. category with weak equivalences).*

*Proof.* Use lemma A.4.14 on the inclusion  $\mathcal{D} \hookrightarrow C$ . ■

**Lemma 3.1.8.** *Let  $C$  be a relative category, let  $\mathcal{D}$  be a saturated homotopical category, and let  $F : C \rightarrow \mathcal{D}$  be a relative functor. If a morphism in  $C$  is a weak equivalence if and only if its image under  $F$  is a weak equivalence in  $\mathcal{D}$ , then  $C$  is also a saturated homotopical category.*

*Proof.* Consider the induced functor  $\text{Ho } F : \text{Ho } C \rightarrow \text{Ho } \mathcal{D}$ . Let  $f : X \rightarrow Y$  be a morphism in  $C$  such that  $f$  is an isomorphism in  $\text{Ho } C$ . Since  $\text{Ho } F$  is a functor,  $Ff$  must be an isomorphism in  $\text{Ho } \mathcal{D}$ ; but  $\mathcal{D}$  is saturated, so  $Ff$  is a weak equivalence in  $\mathcal{D}$ . We may therefore deduce that  $f$  is a weak equivalence in  $C$ . ■

**Corollary 3.1.9.** *Any relative subcategory of a saturated homotopical category is a saturated homotopical category.* ■

**Lemma 3.1.10.** *Let  $C$  and  $\mathcal{D}$  be two relative categories. If  $\mathcal{D}$  is a homotopical category (resp. category with weak equivalences), then the relative functor category  $[C, \mathcal{D}]_h$  is also a homotopical category (resp. category with weak equivalences).*

*Proof.* This is a straightforward check. ◆

**Lemma 3.1.11.** *Let  $C$  and  $\mathcal{D}$  be two relative categories. If  $\mathcal{D}$  is a saturated homotopical category, then the relative functor category  $[C, \mathcal{D}]_h$  is also a saturated homotopical category.*

*Proof.* For each object  $C$  in  $C$ , we have a homotopical functor  $C^* : [C, \mathcal{D}]_h \rightarrow \mathcal{D}$  that evaluates an object  $F$  in  $[C, \mathcal{D}]_h$  at  $C$ . Thus, we obtain a functor  $\text{Ho } C^* : \text{Ho } [C, \mathcal{D}]_h \rightarrow \text{Ho } \mathcal{D}$ .

Consider a morphism  $\varphi : F \Rightarrow F'$  in  $[C, D]_h$  such that  $\varphi$  is an isomorphism in  $\text{Ho } [C, D]_h$ . Since  $\text{Ho } C^*$  is a functor,  $(\text{Ho } C^*)(\varphi)$  must be an isomorphism in  $\text{Ho } C$ ; but  $C$  is a saturated homotopical category, so that implies the component  $\varphi_C$  is a weak equivalence in  $C$ . We therefore conclude that  $\varphi$  is a weak equivalence in  $[C, D]_h$ . ■

**Definition 3.1.12.** Two objects in a relative category are **weakly equivalent** if they can be connected by a *zigzag* of weak equivalences; we define  $X \stackrel{w}{\simeq} Y$  to mean that  $X$  and  $Y$  are weakly equivalent.

REMARK 3.1.13. If  $X$  and  $Y$  are weakly equivalent in a relative category  $C$ , then they are isomorphic in  $\text{Ho } C$ . The converse is certainly true if  $C$  is saturated, but is false if  $C$  is not semi-saturated.

**Definition 3.1.14.** A **homotopically replete subcategory** of a relative category  $C$  is a relative subcategory  $D$  with the following property:

- If  $D$  is an object in  $D$  and  $f : C \rightarrow D$  is a weak equivalence in  $C$ , then both  $C$  and  $f$  are in  $D$ .
- If  $D$  is an object in  $D$  and  $g : D \rightarrow C$  is a weak equivalence in  $C$ , then both  $C$  and  $g$  are in  $D$ .

REMARK 3.1.15. Any full relative subcategory  $D$  of a relative category  $C$  is homotopically replete if and only if it has the following property:

- If  $D$  is an object in  $D$  and  $C$  an object in  $C$  that is weakly equivalent to  $D$ , then  $C$  is in  $D$ .

**Definition 3.1.16.** A parallel pair of morphisms in a relative category  $C$  are **weakly homotopic** if they are equal in  $\text{Ho } C$ ; we define  $f \stackrel{w}{\sim} g$  to mean that  $f$  and  $g$  are weakly homotopic.

**Definition 3.1.17.** An **equivalence** in a relative category  $C$  is a pair  $(f, g)$ , where  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are morphisms in  $C$  such that  $g \circ f \stackrel{w}{\sim} \text{id}_X$  and  $f \circ g \stackrel{w}{\sim} \text{id}_Y$ . Two morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  in  $C$  are **mutual quasi-inverses** when  $(f, g)$  constitute an equivalence in  $C$ .

REMARK 3.1.18. It follows from the definitions that quasi-inverses are unique up to weak homotopy.

**Lemma 3.1.19.** *If the localising functor  $\gamma : C \rightarrow \text{Ho } C$  for a relative category  $C$  is full, then the following are equivalent for all morphisms  $f : X \rightarrow Y$  in  $C$ :*

- $f$  is a morphism in  $C$  and has a quasi-inverse.
- $\gamma f$  is an isomorphism in  $C$ .

*Proof.* Obvious. ◆

**REMARK 3.1.20.** Clearly, any isomorphism in any relative category has a quasi-inverse; but this implies that in a relative category that is *not* semi-saturated, a morphism that has a quasi-inverse need *not* be a weak equivalence. On other hand, if  $f$  is a morphism in a *saturated* homotopical category and  $f$  has a quasi-inverse, then  $f$  must be a weak equivalence.

**Definition 3.1.21.** A relative category  $C$  has the **Whitehead property** when the following are equivalent:

- $f$  is a weak equivalence in  $C$ .
- $f$  is a morphism in  $C$  and has a quasi-inverse.

**Theorem 3.1.22.** *Let  $C$  be a relative category. The following are equivalent:*

- (i)  $C$  has the Whitehead property.
- (ii) The localising functor  $\gamma : C \rightarrow \text{Ho } C$  is full, and  $C$  is a saturated homotopical category.

*Proof.* (i)  $\Rightarrow$  (ii). By [theorem A.4.26](#), every morphism  $\gamma X_0 \rightarrow \gamma X_n$  in  $\text{Ho } C$  is of the form

$$(\gamma f_n)^{-1} \circ \dots \circ \gamma h_2 \circ (\gamma f_1)^{-1} \circ \gamma h_1$$

for some morphisms  $h_1 : X_0 \rightarrow Y_1, f_1 : X_1 \rightarrow Y_1, h_2 : X_1 \rightarrow Y_2$ , etc. in  $C$ , where  $f_1, \dots, f_n$  are weak equivalences. By the Whitehead property, each  $f_i : X_i \rightarrow Y_i$  has a quasi-inverse in  $C$ , say  $g_i : Y_i \rightarrow X_i$ . Since  $\gamma g_i = (\gamma f_i)^{-1}$ , it follows that

$$(\gamma f_n)^{-1} \circ \dots \circ h_2 \circ (\gamma f_1)^{-1} \circ \gamma h_1 = \gamma(g_n \circ \dots \circ h_2 \circ g_1 \circ h_1)$$

and therefore  $\gamma : C \rightarrow \text{Ho } C$  is indeed full.

In particular, every morphism  $f : X \rightarrow Y$  in  $C$  such that  $\gamma f : \gamma X \rightarrow \gamma Y$  is an isomorphism in  $\text{Ho } C$  must have a quasi-inverse, and hence must be a weak

equivalence, in view of the Whitehead property. We therefore conclude that  $\mathcal{C}$  is a saturated homotopical category.

(ii)  $\Rightarrow$  (i). The converse follows from the definitions and [lemma 3.1.19](#). ■

**REMARK 3.1.23.** The Whitehead property is in general not inherited by slice categories or by functor categories. For example, if  $q \circ f = p$  and  $g$  is a quasi-inverse for  $f$ , it is only guaranteed that  $q \simeq p \circ g$ .

**Definition 3.1.24.** Let  $F, G : \mathcal{C} \rightarrow \mathcal{D}$  be two ordinary functors between relative categories. A **natural weak equivalence**  $\alpha : F \Rightarrow G$  is a natural transformation such that  $\alpha_C : FC \rightarrow GC$  is a weak equivalence in  $\mathcal{D}$  for all objects  $C$  in  $\mathcal{C}$ , and we say  $F$  and  $G$  are **naturally weakly equivalent** if they can be connected by a *zigzag* of natural weak equivalences.

**REMARK 3.1.25.** This is precisely the notion of weak equivalence in the relative functor category  $[\text{min und } \mathcal{C}, \mathcal{D}]_{\text{h}}$ . Although the definition above applies to all functors, if  $H : \mathcal{D} \rightarrow \mathcal{E}$  is an ordinary functor, then the natural transformation  $H\alpha : HF \Rightarrow HG$  is only guaranteed to be a natural weak equivalence if we assume  $H$  is a relative functor.

**Definition 3.1.26.** A **relative equivalence** is a relative functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  for which there exists a relative functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  such that  $GF$  is naturally weakly equivalent to  $\text{id}_{\mathcal{C}}$  and  $FG$  is naturally weakly equivalent to  $\text{id}_{\mathcal{D}}$ . Such a  $G$  is said to be a **relative inverse** of  $F$ . When  $\mathcal{C}$  and  $\mathcal{D}$  are homotopical categories, we may say **homotopical equivalence** and **homotopical inverse** instead of ‘relative equivalence’ and ‘relative inverse’.

**Proposition 3.1.27.** *If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a relative equivalence of relative categories with relative inverse  $G : \mathcal{D} \rightarrow \mathcal{C}$ , then  $\text{Ho } F : \text{Ho } \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$  is an equivalence of categories, with quasi-inverse  $\text{Ho } G : \text{Ho } \mathcal{D} \rightarrow \text{Ho } \mathcal{C}$ .* ■

**Definition 3.1.28.** An **adjoint relative equivalence** is an adjunction of the form below,

$$F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$$

where  $\mathcal{C}$  and  $\mathcal{D}$  are relative categories,  $F$  and  $G$  are relative functors, and both the adjunction unit and counit are natural weak equivalences. When  $\mathcal{C}$  and  $\mathcal{D}$  are homotopical categories, we may say **adjoint homotopical equivalence** instead of ‘adjoint relative equivalence’.

**Proposition 3.1.29.** *An adjoint relative equivalence of relative categories descends to an adjoint equivalence of homotopy categories.*

*Proof.* Use the 2-functoriality of  $\text{Ho} : \mathfrak{RelCat} \rightarrow \mathfrak{Cat}$  (corollary A.4.20). ■

**Definition 3.1.30.** A **homotopically contractible category** is a homotopical category  $C$  such that the unique (homotopical) functor  $C \rightarrow \mathbb{1}$  is a homotopical equivalence, where  $\mathbb{1}$  is the trivial category with only one object.

**Proposition 3.1.31.** *Let  $C$  be a homotopical category. The following are equivalent:*

- (i)  $C$  is homotopically contractible.
- (ii)  $C$  is inhabited, and for every object  $A$  in  $C$ , the constant functor  $\Delta A$  is naturally weakly equivalent to  $\text{id}_C$ .
- (iii) There exists an object  $A$  in  $C$  such that  $\Delta A$  and  $\text{id}_C$  are naturally weakly equivalent.

*Proof.* Obvious. (This is paragraph 37.6 in [DHKS].) ◆

## 3.2 Homotopical Kan extensions

**Prerequisites.** §§3.1, A.4.

**Definition 3.2.1.** Let  $C$  be a homotopical category. A **homotopically initial object** in  $C$  is an object  $A$  for which there exists a zigzag of natural transformations of the form

$$\Delta A \rightsquigarrow F \xrightarrow{\alpha} G \rightsquigarrow \text{id}_C$$

where  $\Delta A : C \rightarrow C$  is the constant functor with value  $A$ ,  $\alpha_A : FA \rightarrow GA$  is a weak equivalence in  $C$ , and the squiggles denote (possibly trivial) zigzags of natural weak equivalences. Dually, a **homotopically terminal object** in  $C$  is a homotopically initial object in  $C^{\text{op}}$ .

**Proposition 3.2.2.** *Let  $C$  be a homotopical category. If  $A$  is a homotopically initial (resp. homotopically terminal) object in  $C$ , then:*

- (i) *Any object in  $C$  weakly equivalent to  $A$  is also a homotopically initial (resp. homotopically terminal) object in  $C$ .*

- (ii)  $A$  is an initial (resp. terminal) object in  $\text{Ho } C$ .
- (iii) If  $C$  is a minimal homotopical category, then  $A$  is an initial (resp. terminal) object in  $C$  as well.

Conversely, any initial (resp. terminal) object in  $C$  is also homotopically initial (resp. homotopically terminal).

*Proof.* Obvious. (This is Proposition 38.3 in [DHKS].) ◆

**Proposition 3.2.3.** *If  $A$  is a homotopically initial object in a homotopical category  $C$ , then for any object  $Z$  in  $C$ , the zigzag category  $C^{(\mathbf{T})}(A, Z)$  is connected.*

*Proof.* By theorem A.4.26, there is a bijection between the connected components of  $C^{(\mathbf{T})}(A, Z)$  and the morphisms  $A \rightarrow Z$  in  $\text{Ho } C$ ; but we know  $A$  is an initial object in  $\text{Ho } C$ , so  $C^{(\mathbf{T})}(A, Z)$  has exactly one connected component. ■

**Lemma 3.2.4.** *Let  $H : C \rightarrow D$  be a relative functor and let  $F : C \rightarrow D$  be an ordinary functor. If  $\text{weq } D$  has the 2-out-of-3 property and  $F$  is naturally weakly equivalent to  $H$ , then  $F$  is also a relative functor.*

*Proof.* Apply the 2-out-of-3 property inductively. ◆

**Lemma 3.2.5.** *If  $A$  and  $A'$  be homotopically initial objects in a homotopical category  $C$ , then  $A \stackrel{\text{w}}{\cong} A'$ , and moreover every morphism  $A \rightarrow A'$  in  $C$  is a weak equivalence.*

*Proof.* This is paragraph 38.5 in [DHKS].

Suppose, as in the definition, that we have endofunctors  $F, F', G, G'$  on  $C$  and natural transformations  $\alpha : F \Rightarrow G, \alpha' : F' \Rightarrow G'$ , such that  $F \stackrel{\text{w}}{\cong} \Delta A, F' \stackrel{\text{w}}{\cong} \Delta A', G \stackrel{\text{w}}{\cong} \text{id}_C$ , and  $G' \stackrel{\text{w}}{\cong} \text{id}_C$ , and the morphisms  $\alpha_A : FA \rightarrow GA$  and  $\alpha'_{A'} : FA' \rightarrow GA'$  are both weak equivalences. Note that the previous lemma implies  $G$  and  $G'$  are both homotopical functors, while a similar argument shows that  $F$  and  $F'$  sends *all* morphisms to weak equivalences.

Let  $f : A \rightarrow A'$  be a morphism in  $C$ . By applying the 2-out-of-3 property repeatedly in the following diagram,

$$\begin{array}{ccccccc}
 A & \rightsquigarrow & FA & \xrightarrow{\alpha_A} & GA & \rightsquigarrow & A \\
 \parallel & & \downarrow Ff & & \downarrow Gf & & \downarrow f \\
 A & \rightsquigarrow & FA' & \xrightarrow{\alpha_{A'}} & GA' & \rightsquigarrow & A'
 \end{array}$$

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we see that  $f$  is a weak equivalence if and only if  $\alpha_{A'} : FA' \rightarrow GA'$  is a weak equivalence. Since  $\alpha'_{A'} : F'A' \rightarrow G'A'$  is a weak equivalence, and  $GA' \stackrel{w}{\simeq} A'$ , it follows that  $\alpha'_{GA'} : FGA' \rightarrow G'GA'$  is a weak equivalence, and since  $G$  is homotopical, so  $G\alpha'_{GA'} : GFGA' \rightarrow GG'GA'$  is also a weak equivalence. Similarly,  $\alpha_A : FA \rightarrow GA$  is a weak equivalence, and  $A \stackrel{w}{\simeq} FA' \stackrel{w}{\simeq} G'FA'$ , so  $\alpha_{G'FA'} : FG'FA' \rightarrow GG'FA'$  is a weak equivalence as well.

Now, by applying the 2-out-of-6 property to the diagram below,

$$\begin{array}{ccccc}
 FF'FA' & \xrightarrow{\alpha_{F'FA'}} & GF'FA' & \xrightarrow{GF'\alpha_{A'}} & GF'GA' \\
 F\alpha'_{FA'} \downarrow & & G\alpha'_{FA'} \downarrow & & \downarrow G\alpha'_{GA'} \\
 FG'FA' & \xrightarrow{\alpha_{G'FA'}} & GG'FA' & \xrightarrow{GG'\alpha_{A'}} & GG'GA'
 \end{array}$$

we may deduce that  $GG'\alpha_{A'} : GG'FA' \rightarrow GG'GA'$  is a weak equivalence, and hence that  $\alpha_{A'} : FA' \rightarrow GA'$  is a weak equivalence, as required.  $\blacksquare$

$\P$  3.2.6. We will say that an object in a homotopical category  $\mathcal{C}$  characterised by a homotopical universal property is **homotopically unique** if the full subcategory spanned by such objects inside the homotopical category of objects in  $\mathcal{C}$  equipped with the relevant additional structure.

**Proposition 3.2.7.** *Let  $\mathcal{C}$  be a homotopically contractible category.*

- (i) *Every morphism in  $\mathcal{C}$  is a weak equivalence.*
- (ii) *The unique functor  $\mathrm{Ho} \mathcal{C} \rightarrow \mathbb{1}$  is an equivalence of categories.*
- (iii) *If  $\mathcal{C}$  is a minimal homotopical category, then  $\mathcal{C} \rightarrow \mathbb{1}$  is also an equivalence of categories.*
- (iv) *The opposite homotopical category  $\mathcal{C}^{\mathrm{op}}$  and the homotopical functor category  $[\mathcal{D}, \mathcal{C}]_{\mathrm{h}}$  (for any homotopical category  $\mathcal{D}$ ) are also homotopically contractible.*
- (v) *Every object in  $\mathcal{C}$  is both homotopically initial and homotopically terminal.*

*Proof.* Obvious. (This is paragraph 37.6 in [DHKS].)  $\blacklozenge$

**Proposition 3.2.8.** *Let  $\mathcal{C}$  be a homotopical category. If  $\mathcal{D}$  is the full homotopical subcategory of  $\mathcal{C}$  spanned by the homotopically initial (or homotopically terminal) objects, then  $\mathcal{D}$  is homotopically contractible.*



*Proof.* This follows from lemma 3.2.5. ■

REMARK 3.2.9. Even if  $\mathcal{C}$  is a saturated homotopical category, an object that is initial in  $\text{Ho } \mathcal{C}$  need not be homotopically initial in  $\mathcal{C}$ . Indeed, let  $\mathcal{C}$  be the maximal homotopical category generated by a graph of the following form:



No object in  $\mathcal{C}$  is homotopically initial, because the length of the shortest zigzag connecting two objects cannot be bounded above; yet every object in  $\text{Ho } \mathcal{C}$  is initial. The same argument shows that  $\mathcal{C}$  is not homotopically contractible, but  $\text{Ho } \mathcal{C}$  is certainly contractible.

**Definition 3.2.10.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{C} \rightarrow \mathcal{E}$  be two ordinary functors between homotopical categories. A **homotopical left Kan extension** (resp. **homotopical right Kan extension**) of  $G$  along  $F$  is a homotopically initial (resp. homotopically terminal) object of the homotopical category  $(G \downarrow F^*)_{\text{h}}$  (resp.  $(F^* \downarrow G)_{\text{h}}$ ) described below:

- The objects are pairs  $(H, \alpha)$  where  $H$  is a homotopical functor  $\mathcal{D} \rightarrow \mathcal{E}$  and  $\alpha$  is a natural transformation of type  $G \Rightarrow HF$  (resp.  $HF \Rightarrow G$ ).
- The morphisms  $(H', \alpha') \rightarrow (H, \alpha)$  are those natural transformations  $\beta : H' \Rightarrow H$  such that  $\beta F \bullet \alpha' = \alpha$  (resp.  $\alpha \bullet \beta F = \alpha'$ ).
- The weak equivalences are the natural weak equivalences.

REMARK 3.2.11. Note that any homotopical Kan extension of  $F : \mathcal{C} \rightarrow \mathcal{D}$  along  $G : \mathcal{C} \rightarrow \mathcal{E}$  has, by definition, an underlying *homotopical* functor  $H : \mathcal{D} \rightarrow \mathcal{E}$ .

**Corollary 3.2.12.** *Homotopical Kan extensions are homotopically unique, any two homotopical left (resp. right) Kan extensions of  $G$  along  $F$  are naturally weakly equivalent.* ■

**Definition 3.2.13.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{C} \rightarrow \mathcal{E}$  be two ordinary functors between homotopical categories, and let  $L : \mathcal{E} \rightarrow \mathcal{F}$  be a homotopical functor. We say  $L$  **preserves** a homotopical left (resp. right) Kan extension  $(H, \alpha)$  of  $G$  along  $F$  if  $(LH, L\alpha)$  is a homotopical left (resp. right) Kan extension of  $LF$  along  $G$ . If a homotopical Kan extension is preserved by *all* homotopical functors, then it is said to be **absolute**.

### 3.3 Quillen–Verdier derived functors

**Prerequisites.** §§3.1, A.4, A.1, A.5

The fact that  $\text{Ho} : \mathfrak{RelCat} \rightarrow \mathfrak{Cat}$  is a 2-functor means that relative functors  $F : C \rightarrow D$  descend to functors  $\text{Ho } F : \text{Ho } C \rightarrow \text{Ho } D$  in a very well-behaved way. However, what can we say about ordinary (i.e. not necessarily relative) functors  $C \rightarrow D$ ?

In this section, we follow [DHKS, §§40–43]; however, we will use a weaker definition of ‘deformation retract’ and a stronger definition of ‘total derived functor’.

**Definition 3.3.1.** Let  $C$  and  $D$  be relative categories, and let  $\gamma_C : C \rightarrow \text{Ho } C$  and  $\gamma_D : D \rightarrow \text{Ho } D$  be the localising functors.

- A **total left derived functor** for an ordinary functor  $F : C \rightarrow D$  is an absolute right (!) Kan extension of  $\gamma_D F : C \rightarrow \text{Ho } D$  along  $\gamma_C : C \rightarrow \text{Ho } C$ .
- A **total right derived functor** for an ordinary functor  $G : D \rightarrow C$  is an absolute left (!) Kan extension of  $\gamma_C G : D \rightarrow \text{Ho } C$  along  $\gamma_D : D \rightarrow \text{Ho } D$ .

REMARK 3.3.2. The above definition is essentially due to Verdier [1963], but the formulation using Kan extensions is due to Quillen [1967, Ch. I, §4]. We deviate from convention by demanding that the Kan extensions be *absolute*; this is in order to make theorem 3.3.5 true.

REMARK 3.3.3. As with everything defined by a universal property, total derived functors are unique up to unique isomorphism *if* they exist.

**Definition 3.3.4.** Let  $C$  and  $D$  be relative categories and let  $F \dashv G : D \rightarrow C$  be an adjunction of ordinary categories. A **derived adjunction** for  $F \dashv G$  consists of

- a left derived functor  $(\mathbf{L}F, \alpha)$  for  $F$ ,
- a right derived functor  $(\mathbf{R}G, \beta)$  for  $G$ , and
- an adjunction  $\mathbf{L}F \dashv \mathbf{R}G : \text{Ho } D \rightarrow \text{Ho } C$  with unit  $\bar{\eta} : \text{id}_{\text{Ho } C} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$  and counit  $\bar{\epsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \text{id}_{\text{Ho } D}$ ,

such that  $(\alpha, \beta)$  constitute a conjugate pair of natural transformations. We refer to  $\bar{\eta}$  as the **derived unit** and  $\bar{\epsilon}$  as the **derived counit**.

**Theorem 3.3.5.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories and let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  be an ordinary adjunction. If  $(\mathbf{L}F, \alpha)$  is a total left derived functor for  $F$  and  $(\mathbf{R}G, \beta)$  is a total right derived functor for  $G$ , then there exist unique natural transformations  $\bar{\eta} : \text{id}_{\text{Ho } \mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$  and  $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \text{id}_{\text{Ho } \mathcal{D}}$  making  $\mathbf{L}F \dashv \mathbf{R}G : \text{Ho } \mathcal{D} \rightarrow \text{Ho } \mathcal{C}$  a derived adjunction for  $F \dashv G$  with derived unit  $\bar{\eta}$  and derived counit  $\bar{\varepsilon}$ .*

*Proof.* Let  $\eta$  and  $\varepsilon$  be the unit and counit of the adjunction  $F \dashv G$ . First, we prove that  $\bar{\eta}$  and  $\bar{\varepsilon}$  are unique if they exist. Indeed, if they exist, then  $(\alpha, \beta)$  is a conjugate pair of natural transformations, so we must have the equations shown below:

$$\beta F \cdot \gamma_C \eta = (\mathbf{R}G)\alpha \cdot \bar{\eta} \gamma_C \qquad \bar{\varepsilon} \gamma_D \cdot (\mathbf{L}F)\beta = \gamma_D \varepsilon \cdot \alpha G$$

However,  $((\mathbf{R}G)(\mathbf{L}F), (\mathbf{R}G)\alpha)$  is a left Kan extension of  $(\mathbf{R}G)\gamma_D F$  along  $\gamma_C$  and  $((\mathbf{L}F)(\mathbf{R}G), (\mathbf{L}F)\beta)$  is a right Kan extension of  $(\mathbf{L}F)\gamma_C G$  along  $\gamma_D$ , so  $\bar{\eta}$  and  $\bar{\varepsilon}$  are uniquely determined as natural transformations by these equations.

Next, we prove that the natural transformations  $\bar{\eta}$  and  $\bar{\varepsilon}$  defined above satisfy the left and right triangle identities. Using naturality and the defining equations for  $\bar{\eta}$  and  $\bar{\varepsilon}$ , we obtain the following:

$$\begin{aligned} \alpha \cdot (\bar{\varepsilon}(\mathbf{L}F) \cdot (\mathbf{L}F)\bar{\eta})\gamma_C &= \alpha \cdot \bar{\varepsilon}(\mathbf{L}F)\gamma_C \cdot (\mathbf{L}F)\bar{\eta}\gamma_C \\ &= \bar{\varepsilon}\gamma_D F \cdot (\mathbf{L}F)(\mathbf{R}G)\alpha \cdot (\mathbf{L}F)\bar{\eta}\gamma_C \\ &= \bar{\varepsilon}\gamma_D F \cdot (\mathbf{L}F)\beta F \cdot (\mathbf{L}F)\gamma_C \eta \\ &= \gamma_D \varepsilon F \cdot \alpha G F \cdot (\mathbf{L}F)\gamma_C \eta \\ &= \gamma_D \varepsilon F \cdot \gamma_D F \eta \cdot \alpha \\ &= \gamma_D(\varepsilon F \cdot F \eta) \cdot \alpha \end{aligned}$$

Since  $(\mathbf{L}F, \alpha)$  is a right Kan extension of  $F$  along  $\gamma_C$ , this implies that  $\bar{\eta}$  and  $\bar{\varepsilon}$  satisfy the left triangle identity if  $\eta$  and  $\varepsilon$  do. A formally dual calculation shows that the same is true for the right triangle identity. Thus, we have the required derived adjunction. ■

**Definition 3.3.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories. A **left deformation retract** for an ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a triple  $(\mathcal{C}^\circ, Q, \rho)$  where

- $\mathcal{C}^\circ$  is a full subcategory of  $\mathcal{C}$  with the induced relative subcategory structure,

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- $Q$  is a pair of maps  $\text{ob } C \rightarrow \text{ob } C$  and  $\text{mor } C \rightarrow \text{mor } C$  (but not necessarily functorial), and
- $p$  assigns to each object  $X$  in  $C$  a weak equivalence  $p_X : QX \rightarrow X$ ,

and these data are required to satisfy the following axioms:

**DR1.** For all objects  $X$  in  $C$ , the object  $QX$  is in  $C^\circ$ .

**DR2.** For all morphisms  $f : X \rightarrow Y$  in  $C$ , we have  $p_Y \circ Qf = f \circ p_X$ , i.e. the diagram in  $C$  shown below commutes,

$$\begin{array}{ccc} QX & \xrightarrow{p_X} & X \\ Qf \downarrow & & \downarrow f \\ QY & \xrightarrow{p_Y} & Y \end{array}$$

and if  $f$  is a weak equivalence in  $C$ , then so is  $Qf$ .

**DR3.** The inclusion  $C^\circ \hookrightarrow C$  induces a fully faithful functor  $\text{Ho } C^\circ \rightarrow \text{Ho } C$ .

**DR4.** The restriction  $F|_{C^\circ} : C^\circ \rightarrow D$  is a relative functor.

An ordinary functor  $F : C \rightarrow D$  is **left deformable** if there exists a left deformation retract for  $F$ . A **left deformation retract** of a relative category  $C$  is a left deformation retract for  $\text{id}_C$ .

Dually, a **right deformation retract** for an ordinary functor  $G : D \rightarrow C$  is a triple  $(D^\circ, R, i)$  where

- $D^\circ$  is a full subcategory of  $D$  with the induced relative subcategory structure,
- $R$  is a pair of maps  $\text{ob } D \rightarrow \text{ob } D$  and  $\text{mor } D \rightarrow \text{mor } D$  (but not necessarily functorial), and
- $i$  assigns to each object  $A$  in  $D$  a weak equivalence  $i_A : A \rightarrow RA$ ,

and these data are required to satisfy the following axioms:

**DR1.** For all objects  $A$  in  $D$ , the object  $RA$  is in  $D^\circ$ .

**DR2.** For all morphisms  $g : A \rightarrow B$  in  $\mathcal{D}$ , we have  $Rg \circ i_A = i_B \circ g$ , i.e. the diagram in  $\mathcal{D}$  shown below commutes,

$$\begin{array}{ccc} A & \xrightarrow{i_A} & RA \\ g \downarrow & & \downarrow Rg \\ B & \xrightarrow{i_B} & RB \end{array}$$

and if  $g$  is a weak equivalence in  $\mathcal{D}$ , then so is  $Rg$ .

**DR3.** The inclusion  $\mathcal{D}^\circ \hookrightarrow \mathcal{D}$  induces a fully faithful functor  $\mathrm{Ho} \mathcal{D}^\circ \rightarrow \mathrm{Ho} \mathcal{D}$ .

**DR4.** The restriction  $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \rightarrow \mathcal{C}$  is a relative functor.

An ordinary functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is **right deformable** if there exists a weak right deformation retract for  $G$ . A **right deformation retract** of a relative category  $\mathcal{D}$  is a right deformation retract for  $\mathrm{id}_{\mathcal{D}}$ .

**REMARK 3.3.7.** Every relative functor is both left deformable and right deformable, with trivial left and right deformation retracts.

**REMARK 3.3.8.** Given any weak left (resp. right) deformation retract  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  (resp.  $(\mathcal{D}^\circ, R, i)$ ), the canonical functor  $\mathrm{Ho} \mathcal{C}^\circ \rightarrow \mathrm{Ho} \mathcal{C}$  (resp.  $\mathrm{Ho} \mathcal{D}^\circ \rightarrow \mathrm{Ho} \mathcal{D}$ ) is not only fully faithful but also essentially surjective on objects, so the categories  $\mathrm{Ho} \mathcal{C}^\circ$  and  $\mathrm{Ho} \mathcal{C}$  (resp.  $\mathrm{Ho} \mathcal{D}^\circ$  and  $\mathrm{Ho} \mathcal{D}$ ) are equivalent.

**Proposition 3.3.9.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories, and let  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  be a left deformation retract for  $F : \mathcal{C} \rightarrow \mathcal{D}$ .*

- (i) *If  $\mathcal{Q}$  is functorial, then the composite  $F\mathcal{Q} : \mathcal{C} \rightarrow \mathcal{D}$  is a relative functor.*
- (ii) *If  $\mathcal{C}_F^\circ$  is the full subcategory of  $\mathcal{C}$  spanned by the objects  $X$  such that the morphism  $Fp_X : F\mathcal{Q}X \rightarrow FX$  is weak equivalence in  $\mathcal{D}$ , then  $\mathcal{C}^\circ \subseteq \mathcal{C}_F^\circ$ .*
- (iii) *If moreover  $\mathrm{weq} \mathcal{D}$  has the 2-out-of-3 property in  $\mathcal{D}$ , then  $(\mathcal{C}_F^\circ, \mathcal{Q}, p)$  is also a left deformation retract for  $F$ .*

*Dually, let  $(\mathcal{D}^\circ, R, i)$  be a right deformation retract for  $G : \mathcal{D} \rightarrow \mathcal{C}$ .*

- (i') *If  $\mathcal{Q}$  is functorial, then the composite  $GR : \mathcal{D} \rightarrow \mathcal{C}$  is a relative functor.*
- (ii') *If  $\mathcal{D}_G^\circ$  is the full subcategory of  $\mathcal{D}$  spanned by the objects  $A$  such that the morphism  $Gi_A : GA \rightarrow GRA$  is weak equivalence in  $\mathcal{C}$ , then  $\mathcal{D}^\circ \subseteq \mathcal{D}_G^\circ$ .*

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(iii') *If moreover  $\text{weq } C$  has the 2-out-of-3 property in  $C$ , then  $(\mathcal{D}_G^\circ, \mathbf{R}, i)$  is also a right deformation retract for  $F$ .*

*Proof.* (i). Immediate from the definitions.

(ii). Let  $\tilde{X}$  be an object in  $C^\circ$ . By definition,  $Q\tilde{X}$  is also an object in  $C^\circ$ , and  $F|_{C^\circ}$  is a relative functor, so  $Fp_{\tilde{X}} : FQ\tilde{X} \rightarrow F\tilde{X}$  is a weak equivalence in  $C$ .

(iii). Let  $X$  and  $Y$  be objects in  $C_F^\circ$  and let  $f : X \rightarrow Y$  be a weak equivalence in  $C$ . Consider the following commutative diagram in  $\mathcal{D}$ :

$$\begin{array}{ccc} FQX & \xrightarrow{Fp_X} & FX \\ FQf \downarrow & & \downarrow Ff \\ FQY & \xrightarrow{Fp_Y} & FY \end{array}$$

$FQf$  is a weak equivalence in  $\mathcal{D}$  by claim (i), and both  $Fp_X$  and  $Fp_Y$  are weak equivalences by the definition of  $C_F^\circ$ , so using the 2-out-of-3 property of  $\text{weq } \mathcal{D}$ , we may deduce that  $Ff$  is a weak equivalence in  $\mathcal{D}$  too. Thus,  $F|_{C_F^\circ}$  is a relative functor, as required for  $(C_F^\circ, Q, p)$  to be a left deformation retract for  $F$ .  $\blacksquare$

**Proposition 3.3.10.** *Let  $C$  and  $\mathcal{D}$  be relative categories, and let  $\gamma_C : C \rightarrow \text{Ho } C$  and  $\gamma_D : \mathcal{D} \rightarrow \text{Ho } \mathcal{D}$  be the respective localising functors.*

- *If  $(C^\circ, Q, p)$  is a weak left deformation retract for an ordinary functor  $F : C \rightarrow \mathcal{D}$ , then there exist a right Kan extension  $(\mathbf{L}F, \alpha)$  of  $\gamma_D F$  along  $\gamma_C$  such that  $(\mathbf{L}F)\gamma_C = \gamma_D FQ$  and  $\alpha = \gamma_D Fp$ . (In particular,  $\gamma_D FQ$  is functorial even if  $Q$  is not.)*
- *If  $(\mathcal{D}^\circ, \mathbf{R}, i)$  is a weak right deformation retract for an ordinary functor  $G : \mathcal{D} \rightarrow C$ , then there exist a left Kan extension  $(\mathbf{R}G, \beta)$  of  $\gamma_C G$  along  $\gamma_D$  such that  $(\mathbf{R}G)\gamma_D = \gamma_C GR$  and  $\beta = \gamma_C Gi$ . (In particular,  $\gamma_C GR$  is functorial even if  $\mathbf{R}$  is not.)*

*Proof.* The two claims are formally dual; we will prove the first version.

To simplify notation, we may assume without loss of generality that  $\mathcal{D}$  is a minimal saturated relative category and that  $\gamma_D = \text{id}_{\mathcal{D}}$ . Henceforth, we write  $\gamma$  instead of  $\gamma_C$ . First, observe that  $\gamma Q$  is functorial (even if  $Q$  is not) because each  $\gamma p_X : \gamma QX \rightarrow \gamma X$  is an isomorphism, so (using axioms DR1 and DR3) there is a unique functor  $\tilde{Q} : \text{Ho } C \rightarrow \text{Ho } C^\circ$  such that  $\tilde{Q}\gamma = \gamma Q$ . Let  $\gamma^\circ : C^\circ \rightarrow \text{Ho } C^\circ$

be the localising functor for  $C^\circ$ . Since  $F|_{C^\circ}$  is a relative functor (by axiom DR4), we must have  $F|_{C^\circ} = \tilde{F}\gamma^\circ$  for a unique functor  $\tilde{F} : \text{Ho } C^\circ \rightarrow D$ . We may then define  $\mathbf{L}F$  to be the functor  $\tilde{F}\tilde{Q}$ . We define  $\alpha : (\mathbf{L}F)\gamma \Rightarrow F$  by taking  $\alpha_X = Fp_X$ ; by axiom DR2, this is indeed a natural transformation.

It remains to be shown that  $(\mathbf{L}F, \alpha)$  is a right Kan extension of  $F : C \rightarrow D$  along  $\gamma : C \rightarrow \text{Ho } C$ . Let  $H : \text{Ho } C \rightarrow D$  be a functor and let  $\varphi : H\gamma \Rightarrow F$  be any natural transformation. By restricting along the inclusion  $C^\circ \rightarrow C$ , we obtain a natural transformation  $\varphi|_{C^\circ} : H|_{\text{Ho } C^\circ}\gamma^\circ \Rightarrow F|_{C^\circ}$ , so there is a unique natural transformation  $\tilde{\varphi} : H|_{\text{Ho } C^\circ} \Rightarrow \tilde{F}$  such that  $\tilde{\varphi}\gamma^\circ = \varphi|_{C^\circ}$  (by the 2-dimensional universal property of  $\text{Ho } C$ ). Since  $\gamma p$  is a natural isomorphism, there is then a unique natural transformation  $\bar{\varphi} : H \Rightarrow \mathbf{L}F$  such that  $\bar{\varphi}_{\gamma X} \circ H\gamma p_X = \tilde{\varphi}_{\gamma^\circ QX}$  for all objects  $X$  in  $C$ . We then have  $\alpha \bullet \bar{\varphi}\gamma = \varphi$ , and  $\bar{\varphi}$  is the unique such natural transformation because the canonical functor  $\text{Ho } C^\circ \rightarrow \text{Ho } C$  is essentially surjective on objects.  $\blacksquare$

**Definition 3.3.II.** Let  $B, C, D, \mathcal{E}$  be relative categories. Given a composable pair of ordinary functors  $F : C \rightarrow D$  and  $G : D \rightarrow \mathcal{E}$ , a **lax left deformation retract** for  $(G, F)$  consists of

- a left deformation retract  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  for  $F$ , and
- a left deformation retract  $(D^\circ, Q^{D^\circ}, p^{D^\circ})$  for  $G$ ,

such that  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  is also a left deformation retract for  $GF$  as well. A **strong left deformation retract** for  $(G, F)$  is a lax left deformation retract as above such that  $F$  sends objects in  $C^\circ$  to objects in  $D^\circ$ . We say a composable pair of functors is **laxly left deformable** (resp. **strongly left deformable**) if it admits a lax left deformation (resp. strong left deformation).

Dually, given a composable pair of ordinary functors  $F : C \rightarrow B$  and  $G : D \rightarrow C$ , an **oplax right deformation retract** for  $(F, G)$  consists of

- a right deformation retract  $(C^\circ, R^{C^\circ}, i^{C^\circ})$  for  $F$ , and
- a right deformation retract  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  for  $G$ ,

such that  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  is a right deformation retract for  $GF$  as well. A **strong right deformation retract** for  $(F, G)$  is an oplax right deformation retract as above such that  $G$  sends objects in  $D^\circ$  to objects in  $C^\circ$ . We say a composable pair of functors is **oplaxly right deformable** (resp. **strongly right deformable**) if it admits an oplax right deformation (resp. strong right deformation).

**Lemma 3.3.12.**

- Let  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  be a left deformation retract for  $F : C \rightarrow D$  and let  $(D^\circ, Q^{D^\circ}, p^{D^\circ})$  be a left deformation retract for  $G : D \rightarrow \mathcal{E}$ . If  $F$  maps objects in  $C^\circ$  to objects in  $D^\circ$ , then  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  is a left deformation retract for  $GF : C \rightarrow \mathcal{E}$ .

Dually:

- Let  $(C^\circ, R^{C^\circ}, i^{C^\circ})$  be a right deformation retract for  $F : C \rightarrow B$  and let  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  be a right deformation retract for  $G : D \rightarrow C$ . If  $G$  maps objects in  $D^\circ$  to objects in  $C^\circ$ , then  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  is a right deformation retract for  $FG : D \rightarrow B$ .

*Proof.* Our hypotheses imply that the restriction  $GF|_{C^\circ} : C^\circ \rightarrow \mathcal{E}$  is a relative functor, so  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  satisfies the conditions required to be a left deformation retract for  $GF : C \rightarrow \mathcal{E}$ . ■

**Theorem 3.3.13.** Let  $C, D$ , and  $\mathcal{E}$  be relative categories, and let  $\gamma_C : C \rightarrow \text{Ho } C$ ,  $\gamma_D : D \rightarrow \text{Ho } D$ , and  $\gamma_{\mathcal{E}} : \mathcal{E} \rightarrow \text{Ho } \mathcal{E}$  be the respective localising functors.

- (i) Let  $F : C \rightarrow D$  be an ordinary functor. If  $(C^\circ, Q, p)$  is any left deformation retract for  $F$ , then  $F$  has a total left derived functor  $(\mathbf{L}F, \alpha)$  such that  $(\mathbf{L}F)\gamma_C = \gamma_D FQ$  and  $\alpha = \gamma_D Fp$ .
- (ii) Let  $F, F' : C \rightarrow D$  be a parallel pair of ordinary functors. If  $(\mathbf{L}F, \alpha)$  and  $(\mathbf{L}F', \alpha')$  are total left derived functors for  $F$  and  $F'$  (respectively), then for any natural transformation  $\varphi : F \Rightarrow F'$ , there exists a unique natural transformation  $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$  such that  $\alpha' \bullet (\mathbf{L}\varphi)\gamma_C = \gamma_D \varphi \bullet \alpha$ .
- (iii) Moreover, if  $(C^\circ, Q, p)$  is a left deformation retract for both  $F$  and  $F'$ , then  $(\mathbf{L}\varphi)\gamma_C = \gamma_D \varphi Q$ .
- (iv) Let  $F : C \rightarrow D$  and  $G : D \rightarrow \mathcal{E}$  be ordinary functors between relative categories. If  $(\mathbf{L}F, \alpha^F)$ ,  $(\mathbf{L}G, \alpha^G)$ , and  $(\mathbf{L}(GF), \alpha^{GF})$  are total left derived functors for  $F$ ,  $G$ , and  $GF$  (respectively), then there is a unique natural transformation  $\mu_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$  such that  $\alpha^{GF} \bullet \mu_{G,F}\gamma_C = \alpha^G F \bullet (\mathbf{L}G)\alpha^F$ .
- (v) If  $(G, F)$  is moreover a strongly left deformable composable pair, then the canonical comparison  $\mu_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$  is an isomorphism.



Dually:

- (i') Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be an ordinary functor. If  $(\mathcal{D}^\circ, \mathbf{R}, i)$  is any right deformation retract for  $G$ , then  $G$  has a total right derived functor  $(\mathbf{R}G, \beta)$  such that  $(\mathbf{R}G)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}}GR$  and  $\beta = \gamma_{\mathcal{C}}Gi$ .
- (ii') Let  $G, G' : \mathcal{D} \rightarrow \mathcal{C}$  be a parallel pair of ordinary functors. If  $(\mathbf{R}G, \beta)$  and  $(\mathbf{R}G', \beta')$  are total right derived functors for  $G$  and  $G'$  (respectively), then for any natural transformation  $\psi : G' \Rightarrow G$ , there exists a unique natural transformation  $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$  such that  $(\mathbf{R}\psi)\gamma_{\mathcal{D}} \cdot \beta' = \beta \cdot \gamma_{\mathcal{C}}\psi$ .
- (iii') Moreover, if  $(\mathcal{D}^\circ, \mathbf{R}, i)$  is a right deformation retract for both  $G$  and  $G'$ , then  $(\mathbf{R}\psi)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}}\psi R$ .
- (iv') Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be ordinary functors between relative categories. If  $(\mathbf{R}F, \beta^F)$ ,  $(\mathbf{R}G, \beta^G)$ , and  $(\mathbf{R}(FG), \beta^{FG})$  are total right derived functors for  $F$ ,  $G$ , and  $FG$  (respectively), then there is a unique natural transformation  $\delta_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$  such that  $\delta_{F,G}\gamma_{\mathcal{D}} \cdot \beta^{FG} = (\mathbf{R}F)\beta^G \cdot \beta^F G$ .
- (v') If  $(F, G)$  is moreover a strongly right deformable composable pair, then the canonical comparison  $\delta_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$  is an isomorphism.

*Proof.* (i). By [proposition 3.3.10](#), the functor  $\gamma_{\mathcal{D}}F : \mathcal{C} \rightarrow \text{Ho } \mathcal{D}$  has a right Kan extension along  $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$ , say  $(\mathbf{L}F, \alpha)$ , characterised by the announced equations. We must verify that  $(\mathbf{L}F, \alpha)$  is an absolute right Kan extension, i.e. that  $(H(\mathbf{L}F), H\alpha)$  is a right Kan extension for any functor  $H : \text{Ho } \mathcal{D} \rightarrow \mathcal{E}$  whatsoever.

It is clear that  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is also a left deformation retract for  $H\gamma_{\mathcal{D}}F : \mathcal{C} \rightarrow \mathcal{E}$ , so the cited proposition yields a right Kan extension  $(L', \alpha')$  of  $H\gamma_{\mathcal{D}}F$  along  $\gamma_{\mathcal{C}}$ . There is then a unique natural transformation  $\varphi : H(\mathbf{L}F) \Rightarrow L'$  such that  $\alpha' \cdot \varphi\gamma_{\mathcal{C}} = H\alpha$ , i.e. the following diagram commutes for all objects  $X$  in  $\mathcal{C}$ :

$$\begin{array}{ccc} H(\mathbf{L}F)\gamma_{\mathcal{C}}X & \xrightarrow{\varphi\gamma_{\mathcal{C}}X} & L'\gamma_{\mathcal{C}}X \\ H\alpha_X \downarrow & & \downarrow \alpha'_X \\ H\gamma_{\mathcal{D}}FX & \xlongequal{\quad} & H\gamma_{\mathcal{D}}FX \end{array}$$

However, if  $\tilde{X}$  is in  $\mathcal{C}^\circ$ , then  $\alpha_{\tilde{X}}$  and  $\alpha'_{\tilde{X}}$  are isomorphisms, and so  $\varphi\gamma_{\mathcal{C}}X$  must be an isomorphism as well. Since the canonical functor  $\text{Ho } \mathcal{C}^\circ \rightarrow \text{Ho } \mathcal{C}$  is essentially

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surjective on objects,  $\varphi : H(\mathbf{L}F) \Rightarrow L'$  must be a natural isomorphism. In particular,  $(H(\mathbf{L}F), H\alpha)$  is indeed a right Kan extension.

(ii). Noting that  $\gamma_D \varphi \bullet \alpha$  is a natural transformation  $(\mathbf{L}F)\gamma_C \Rightarrow \gamma_D F'$ , the universal property of  $(\mathbf{L}F', \alpha')$  yields a unique natural transformation  $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$  such that  $\gamma_D \varphi \bullet \alpha = \alpha' \bullet (\mathbf{L}\varphi)\gamma_C$ , as required.

(iii). We must have

$$\gamma_D F p \bullet (\mathbf{L}\varphi)\gamma_C = \gamma_D \varphi \bullet \gamma_D F' p = \gamma_D F p \bullet \gamma_D \varphi Q$$

as required.

(iv). Since  $\alpha^G F \bullet (\mathbf{L}G)\alpha^F$  is a natural transformation  $(\mathbf{L}G)(\mathbf{L}F)\gamma_C \Rightarrow \gamma_D GF$ , the universal property of  $(\mathbf{L}(GF), \alpha^{GF})$  yields the required natural transformation  $\mu_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ .

(v). Let  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  and  $(D^\circ, Q^{D^\circ}, p^{D^\circ})$  constitute a strong left deformation retract for  $(G, F)$ , and let  $(\mathbf{L}F, \alpha^F)$ ,  $(\mathbf{L}G, \alpha^G)$ ,  $(\mathbf{L}(GF), \alpha^{GF})$  be the total left derived functors for  $F$  and  $G$ , respectively, as constructed in claim (i). Then,

$$\begin{aligned} \alpha^{GF} \bullet \mu_{G,F}\gamma_C &= \alpha^G F \bullet (\mathbf{L}G)\alpha^F \\ &= \gamma_\varepsilon G p^{D^\circ} F \bullet \gamma_\varepsilon G Q^{D^\circ} F p^{C^\circ} \\ &= \gamma_\varepsilon GF p^{C^\circ} \bullet \gamma_\varepsilon G p^{D^\circ} F Q^{C^\circ} \end{aligned}$$

so we must have  $\mu_{G,F}\gamma_C = \gamma_\varepsilon G p^{D^\circ} F Q^{C^\circ}$ ; but  $\gamma_\varepsilon G p^{D^\circ} F Q^{C^\circ}$  is a natural isomorphism because  $F$  sends objects in  $C^\circ$  to objects in  $D^\circ$  and  $G$  preserves weak equivalences in  $D^\circ$ , so we deduce that  $\mu_{G,F}$  is also a natural isomorphism (using the fact that  $\gamma_C : C \rightarrow \text{Ho } C$  is bijective on objects).  $\blacksquare$

**Corollary 3.3.14.** *Let  $C$  and  $D$  be relative categories.*

- *If  $F : C \rightarrow D$  is a relative functor, then  $(\text{Ho } F, \text{id})$  is a total left derived functor for  $F$ .*
- *If  $G : D \rightarrow C$  is a relative functor, then  $(\text{Ho } G, \text{id})$  is a total right derived functor for  $G$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

By [remark 3.3.7](#), the trivial right deformation retract is a right deformation retract for  $F : C \rightarrow D$ . Thus,  $\text{Ho } F : \text{Ho } C \rightarrow \text{Ho } D$  together with  $\text{id} : (\text{Ho } F)\gamma_C \Rightarrow \gamma_D F$  constitute a total left derived functor for  $F$ .  $\blacksquare$

**Proposition 3.3.15.** *Let  $\mathcal{C}$  be a relative category.*

- *If  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is a left deformation retract of  $\mathcal{C}$  and  $\mathcal{W}$  is a subcategory of  $\mathcal{C}$  such that  $\text{weq } \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \text{weq } \mathcal{C}$ , then the functor  $\mathcal{C}[\mathcal{W}^{-1}] \rightarrow \text{Ho } \mathcal{C}$  induced by the inclusion  $\mathcal{W} \hookrightarrow \text{weq } \mathcal{C}$  has a fully faithful left adjoint.*
- *If  $(\mathcal{C}^\circ, \mathcal{R}, i)$  is a right deformation retract of  $\mathcal{C}$  and  $\mathcal{W}$  is a subcategory of  $\mathcal{C}$  such that  $\text{weq } \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \text{weq } \mathcal{C}$ , then the functor  $\mathcal{C}[\mathcal{W}^{-1}] \rightarrow \text{Ho } \mathcal{C}$  induced by the inclusion  $\mathcal{W} \hookrightarrow \text{weq } \mathcal{C}$  has a fully faithful right adjoint.*

*Proof.* The two claims are formally dual; we will prove the first version.

Consider the localising functor  $\gamma_{\mathcal{W}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$ . Since  $\text{weq } \mathcal{C}^\circ \subseteq \mathcal{W}$ ,  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is a left deformation retract for  $\gamma_{\mathcal{W}}$ , so (by [theorem 3.3.13](#)) there exists an absolute right Kan extension  $(F, \alpha)$  of  $\gamma_{\mathcal{W}} : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$  along the localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$ . Since  $\gamma$  factors through  $\gamma_{\mathcal{W}}$ , say  $\gamma = G\gamma_{\mathcal{W}}$ , the 2-dimensional universal property of  $\mathcal{C}[\mathcal{W}^{-1}]$  yields a natural transformation  $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{C}[\mathcal{W}^{-1}]}$  such that  $\varepsilon\gamma_{\mathcal{W}} = \alpha$ ; similar arguments show that  $(F, \varepsilon)$  is an absolute right Kan extension of  $\text{id} : \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$  along  $G : \mathcal{C}[\mathcal{W}^{-1}] \rightarrow \text{Ho } \mathcal{C}$ , so  $F$  is a left adjoint for  $G$  with counit  $\varepsilon$ , by [proposition A.5.2.1](#).

It remains to be shown that  $F : \text{Ho } \mathcal{C} \rightarrow \mathcal{C}[\mathcal{W}^{-1}]$  is fully faithful. Consider the natural transformation  $G\varepsilon : GFG \Rightarrow G$ . The total derived functor theorem says  $\varepsilon\gamma_{\mathcal{W}} : FG\gamma_{\mathcal{W}} \Rightarrow \gamma_{\mathcal{W}}$  is given by  $\gamma_{\mathcal{W}}p$ , so  $G\varepsilon\gamma_{\mathcal{W}}$  is given by  $G\gamma_{\mathcal{W}}p$ , which is a natural isomorphism. Since  $\gamma_{\mathcal{W}}$  is bijective on objects, we deduce that  $G\varepsilon$  itself is a natural isomorphism. Thus,  $\eta G : G \Rightarrow GFG$  is a natural isomorphism (by the right triangle identity), and since  $G$  is bijective on objects, we may use [proposition A.1.3](#) to see that  $F$  is fully faithful. ■

**Definition 3.3.16.** The 2-category of small left deformation retracts is defined as follows:

- The objects are pairs  $(\mathcal{C}, \mathcal{C}^\circ, \mathcal{Q}^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ})$  where  $\mathcal{C}$  is a small relative category and  $(\mathcal{C}^\circ, \mathcal{Q}^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ})$  is a left deformation retract of  $\mathcal{C}$ .
- A 1-morphism  $F : (\mathcal{C}, \mathcal{C}^\circ, \mathcal{Q}^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}) \rightarrow (\mathcal{D}, \mathcal{D}^\circ, \mathcal{Q}^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ})$  is an ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , such that  $(\mathcal{C}^\circ, \mathcal{Q}^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ})$  is a left deformation retract for  $F$ , and  $F$  sends objects in  $\mathcal{C}^\circ$  to objects in  $\mathcal{D}^\circ$ .
- The 2-morphisms are ordinary natural transformations.

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- All compositions and identities are inherited from 2-category of small categories.

We write  $\mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f}$  for this 2-category, and we write  $\mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{F}\mathfrak{u}\mathfrak{n}$  for its hom-sets.

The **2-category of small right deformation retracts** is defined dually:

- The objects are pairs  $(D, D^\circ, R^{D^\circ}, i^{D^\circ})$  where  $D$  is a small relative category and  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  is a right deformation retract of  $D$ .
- A 1-morphism  $G : (D, D^\circ, R^{D^\circ}, i^{D^\circ}) \rightarrow (C, C^\circ, R^{C^\circ}, i^{C^\circ})$  is an ordinary functor  $G : D \rightarrow C$ , such that  $(D^\circ, R^{D^\circ}, i^{D^\circ})$  is a right deformation retract for  $G$ , and  $G$  sends objects in  $D^\circ$  to objects in  $C^\circ$ .
- The 2-morphisms are ordinary natural transformations.
- All compositions and identities are inherited from 2-category of small categories.

We write  $\mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f}$  for this 2-category, and we write  $\mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f}\mathfrak{F}\mathfrak{u}\mathfrak{n}$  for its hom-sets.

**REMARK 3.3.17.** The duality principle for deformation retracts can be formalised as follows: there is a 2-functor  $\mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f}^{\text{co}} \rightarrow \mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f}$  that sends  $(C, C^\circ, Q^{C^\circ}, p^{C^\circ})$  to its opposite  $(C^{\text{op}}, (C^\circ)^{\text{op}}, (Q^{C^\circ})^{\text{op}}, (p^{C^\circ})^{\text{op}})$ , and it has an evident strict inverse  $\mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f}^{\text{co}} \rightarrow \mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f}$ . Note that these two 2-functors reverse the direction of 2-morphisms but preserve the direction of 1-morphisms!

**Corollary 3.3.18.** *There are two pseudofunctors,  $\mathbf{L}$  and  $\mathbf{R}$ , where:*

- $\mathbf{L}$  is a pseudofunctor  $\mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f} \rightarrow \mathfrak{C}\mathfrak{a}\mathfrak{t}$  that sends an object  $(C, C^\circ, Q^{C^\circ}, p^{C^\circ})$  to the homotopy category  $\text{Ho } C$ , a 1-morphism  $F : (C, C^\circ, Q^{C^\circ}, p^{C^\circ}) \rightarrow (D, D^\circ, Q^{D^\circ}, p^{D^\circ})$  to its total left derived functor  $\mathbf{L}F : \text{Ho } C \rightarrow \text{Ho } D$ , and a 2-morphism  $\varphi : F \Rightarrow F'$  to the derived natural transformation  $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$ , and  $\mathbf{L}$  preserves identity 1-morphisms strictly.
- $\mathbf{R}$  is a pseudofunctor  $\mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f} \rightarrow \mathfrak{C}\mathfrak{a}\mathfrak{t}$  that sends an object  $(D, D^\circ, R^{D^\circ}, i^{D^\circ})$  to the homotopy category  $\text{Ho } C$ , a 1-morphism  $G : (D, D^\circ, R^{D^\circ}, i^{D^\circ}) \rightarrow (C, C^\circ, R^{C^\circ}, i^{C^\circ})$  to its total right derived functor  $\mathbf{R}G : \text{Ho } D \rightarrow \text{Ho } C$ , and a 2-morphism  $\psi : G' \Rightarrow G$  to the derived natural transformation  $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$ , and  $\mathbf{R}$  preserves identity 1-morphisms strictly.

- $\mathbf{L}$  and  $\mathbf{R}$  are compatible with the duality principle, in the sense that the following diagrams commute (strictly):

$$\begin{array}{ccc}
 \mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f}^{\text{co}} & \xrightarrow{\mathbf{L}^{\text{co}}} & \mathfrak{C}\mathfrak{a}\mathfrak{t}^{\text{co}} \\
 (-)^{\text{op}} \downarrow & & \downarrow (-)^{\text{op}} \\
 \mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f} & \xrightarrow{\mathbf{R}} & \mathfrak{C}\mathfrak{a}\mathfrak{t}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathfrak{R}\mathfrak{D}\mathfrak{e}\mathfrak{f}^{\text{co}} & \xrightarrow{\mathbf{R}^{\text{co}}} & \mathfrak{C}\mathfrak{a}\mathfrak{t}^{\text{co}} \\
 (-)^{\text{op}} \downarrow & & \downarrow (-)^{\text{op}} \\
 \mathfrak{L}\mathfrak{D}\mathfrak{e}\mathfrak{f} & \xrightarrow{\mathbf{L}} & \mathfrak{C}\mathfrak{a}\mathfrak{t}
 \end{array}$$

*Proof.* The main claims follow from [theorem 3.3.13](#); the only thing left to check is that the collection of 2-isomorphisms  $\boldsymbol{\mu}$  and  $\boldsymbol{\delta}$  satisfy the coherence laws for pseudofunctors; that is, we should show that the following diagrams commute:

$$\begin{array}{ccc}
 (\mathbf{L}H)(\mathbf{L}G)(\mathbf{L}F) & \xrightarrow{(\mathbf{L}H)\boldsymbol{\mu}_{G,F}} & (\mathbf{L}H)\mathbf{L}(GF) \\
 \boldsymbol{\mu}_{H,G}(\mathbf{L}F) \downarrow & & \downarrow \boldsymbol{\mu}_{H,GF} \\
 \mathbf{L}(HG)(\mathbf{L}F) & \xrightarrow{\boldsymbol{\mu}_{HG,F}} & \mathbf{L}(HGF)
 \end{array}$$
  

$$\begin{array}{ccc}
 \mathbf{R}(FGH) & \xrightarrow{\boldsymbol{\delta}_{F,GH}} & (\mathbf{R}F)\mathbf{R}(GH) \\
 \boldsymbol{\delta}_{FG,H} \downarrow & & \downarrow (\mathbf{R}F)\boldsymbol{\delta}_{G,H} \\
 \mathbf{R}(FG)(\mathbf{R}H) & \xrightarrow{\boldsymbol{\delta}_{F,G}(\mathbf{R}H)} & (\mathbf{R}F)(\mathbf{R}G)(\mathbf{R}H)
 \end{array}$$

However, using the explicit formulae for  $\boldsymbol{\mu}$  and  $\boldsymbol{\delta}$  in the proof of the theorem, it is easy to see that these diagrams do indeed commute.  $\blacksquare$

**Definition 3.3.19.** A **deformable adjunction** between two relative categories is an ordinary adjunction where the left adjoint is left deformable and the right adjoint is right deformable.

**Theorem 3.3.20.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories and let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  be an adjunction of ordinary categories, with unit  $\eta : \text{id}_{\mathcal{C}} \Rightarrow GF$  and counit  $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{D}}$ .*

- If  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  is a deformable adjunction, then it admits a derived adjunction.
- Let  $F' \dashv G' : \mathcal{D}' \rightarrow \mathcal{C}'$  be another adjunction, with unit  $\eta'$  and counit  $\varepsilon'$ , and let  $H : \mathcal{C}' \rightarrow \mathcal{C}$  and  $K : \mathcal{D}' \rightarrow \mathcal{D}$  be relative functors. If

- $(\mathcal{C}^\circ, Q, p)$  is a left deformation retract for  $F$ ,

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- $(C'^{\circ}, Q', p')$  is a left deformation retract for  $F'$ ,
- $H$  sends objects in  $C'^{\circ}$  to objects in  $C^{\circ}$ ,
- $(D^{\circ}, R, i)$  is a right deformation retract for  $G$ ,
- $(D'^{\circ}, R', i')$  is a right deformation retract for  $G'$ , and
- $K$  sends objects in  $D'^{\circ}$  to objects in  $D^{\circ}$ ,

then for any conjugate pair of natural transformations,

$$\varphi : FH \Rightarrow KF' \qquad \psi : HG' \Rightarrow GK$$

the derived natural transformations

$$\mathbf{L}\varphi : (\mathbf{L}F)(\mathbf{H}o H) \Rightarrow (\mathbf{H}o K)(\mathbf{L}F') \quad \mathbf{R}\psi : (\mathbf{H}o K)(\mathbf{R}G') \Rightarrow (\mathbf{R}G)(\mathbf{H}o K)$$

also constitute a conjugate pair.

- (iii) Let  $F' \dashv G' : D' \rightarrow D$  be another adjunction, with unit  $\eta'$  and counit  $\varepsilon'$ . If  $(F', F)$  is strongly left deformable and  $(G, G')$  is strongly right deformable, then the three derived adjunctions

$$\begin{aligned} \mathbf{L}F \dashv \mathbf{R}G &: \mathbf{H}o D \rightarrow \mathbf{H}o C \\ \mathbf{L}F' \dashv \mathbf{R}G' &: \mathbf{H}o D' \rightarrow \mathbf{H}o D \\ \mathbf{L}(F'F) \dashv \mathbf{R}(GG') &: \mathbf{H}o D' \rightarrow \mathbf{H}o C \end{aligned}$$

make  $(\boldsymbol{\mu}_{F',F}, \boldsymbol{\delta}_{G,G'})$  a conjugate pair of natural transformations, i.e.

$$\begin{aligned} (\boldsymbol{\delta}_{G,G'} \mathbf{L}(F'F)) \bullet \bar{\eta}'' &= ((\mathbf{R}G)(\mathbf{R}G') \boldsymbol{\mu}_{F',F}) \bullet (\mathbf{R}G) \bar{\eta}' (\mathbf{L}F) \bullet \bar{\eta} \\ \bar{\varepsilon}'' \bullet (\boldsymbol{\mu}_{F',F} \mathbf{R}(GG')) &= \bar{\varepsilon}' \bullet (\mathbf{L}F') \bar{\varepsilon}' (\mathbf{R}G') \bullet ((\mathbf{L}F)(\mathbf{L}F') \boldsymbol{\delta}_{G,G'}) \end{aligned}$$

where  $\bar{\eta}''$  and  $\bar{\varepsilon}''$  are the unit and counit for  $\mathbf{L}(F'F) \dashv \mathbf{R}(GG')$ .

*Proof.* (i). We appeal to theorems 3.3.5 and 3.3.13.

(ii). Recall the following characterisations of  $\mathbf{L}\varphi$  and  $\mathbf{R}\psi$ :

$$\begin{aligned} \gamma_D K F' p' \bullet (\mathbf{L}\varphi) \gamma_{C'} &= \gamma_D \varphi \bullet \gamma_D F p H \\ (\mathbf{R}\psi) \gamma_{D'} \bullet \gamma_C H G' i' &= \gamma_C G i K \bullet \gamma_C \psi \end{aligned}$$

We wish to show that these equations hold:

$$(1) \qquad \bar{\varepsilon}(\mathbf{H}o K) \bullet (\mathbf{L}F)(\mathbf{R}\psi) = (\mathbf{H}o K) \bar{\varepsilon}' \bullet (\mathbf{L}\varphi)(\mathbf{R}G')$$

$$(2) \quad (\mathbf{R}G)(\mathbf{L}\varphi) \bullet \bar{\eta}(\mathrm{Ho} H) = (\mathbf{R}\psi)(\mathbf{L}F') \bullet (\mathrm{Ho} H)\bar{\eta}'$$

By [proposition A.1.5](#), it suffices to show that equation (1) is satisfied, and since the canonical functor  $\mathrm{Ho} \mathcal{D}'^\circ \rightarrow \mathrm{Ho} \mathcal{D}'$  is essentially surjective on objects, equation (1) holds if and only if the following equation holds for all  $\hat{A}$  in  $\mathcal{D}'^\circ$ :

$$(3) \quad \bar{\varepsilon}_{\gamma_D K \hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'} \hat{A}} = (\mathrm{Ho} K)\bar{\varepsilon}'_{\gamma_{D'} \hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'} G' R' \hat{A}}$$

We observe that  $G' i'_{\hat{A}} : G' \hat{A} \rightarrow G' R' \hat{A}$  is a weak equivalence in  $C'$  (because  $(\mathcal{D}'^\circ, R', i')$  is a right deformation retract for  $G'$ ), so  $\gamma_C H G' i'_{\hat{A}}$  is invertible, and we must have

$$(\mathbf{R}\psi)_{\gamma_{D'} \hat{A}} = \gamma_C G i_{K \hat{A}} \circ \gamma_C \psi_{\hat{A}} \circ (\gamma_C H G' i'_{\hat{A}})^{-1}$$

and hence,

$$(\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'} \hat{A}} = \gamma_D F Q G i_{K \hat{A}} \circ \gamma_D F Q \psi_{\hat{A}} \circ (\gamma_D F Q H G' i'_{\hat{A}})^{-1}$$

therefore:

$$\begin{aligned} \bar{\varepsilon}_{\gamma_D K \hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{D'} \hat{A}} &= \gamma_D \varepsilon_{K \hat{A}} \circ \gamma_D F p_{G \hat{A}} \circ \gamma_D F Q \psi_{\hat{A}} \circ (\gamma_D F Q H G' i'_{\hat{A}})^{-1} \\ &= \gamma_D \varepsilon_{K \hat{A}} \circ \gamma_D F \psi_{\hat{A}} \circ \gamma_D F p_{H G' \hat{A}} \circ (\gamma_D F Q H G' i'_{\hat{A}})^{-1} \end{aligned}$$

On the other hand,

$$\bar{\varepsilon}'_{\gamma_{D'} \hat{A}} = \gamma_{D'} \varepsilon'_{\hat{A}} \circ \gamma_{D'} F' p'_{G' \hat{A}} \circ (\gamma_{D'} F' Q' G' i'_{\hat{A}})^{-1}$$

and so,

$$\begin{aligned} (\mathrm{Ho} K)\bar{\varepsilon}'_{\gamma_{D'} \hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'} G' R' \hat{A}} &= \gamma_D K \varepsilon'_{\hat{A}} \circ \gamma_D K F' p'_{G' \hat{A}} \circ (\gamma_D K F' Q' G' i'_{\hat{A}})^{-1} \circ (\mathbf{L}\varphi)_{\gamma_{C'} G' R' \hat{A}} \\ &= \gamma_D K \varepsilon'_{\hat{A}} \circ \gamma_D K F' p'_{G' \hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{C'} G' \hat{A}} \circ (\gamma_D F Q H G' i'_{\hat{A}})^{-1} \\ &= \gamma_D K \varepsilon'_{\hat{A}} \circ \gamma_D \varphi_{G' \hat{A}} \circ \gamma_D F p_{H G' \hat{A}} \circ (\gamma_D F Q H G' i'_{\hat{A}})^{-1} \end{aligned}$$

but  $\varepsilon_{K \hat{A}} \circ F \psi_{\hat{A}} = K \varepsilon'_{\hat{A}} \circ \varphi_{G' \hat{A}}$  by hypothesis, so equation (3) indeed holds.

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(iii). Suppose

- $(C^\circ, Q, p)$  is a left deformation retract for  $F$ ,
- $(C'^\circ, Q', p')$  is a left deformation retract for  $F'$ ,
- $F$  sends objects in  $C^\circ$  to objects in  $C'^\circ$ ,
- $(D^\circ, R, i)$  is a right deformation retract for  $G$ ,
- $(D'^\circ, R', i')$  is a right deformation retract for  $G'$ , and
- $G'$  sends objects in  $D'^\circ$  to objects in  $D^\circ$ ,

and recall that the comparison isomorphisms are characterised by the following equations:

$$\mu_{F',F}\gamma_C = \gamma_{D'}F'p'FQ \qquad \delta_{G,G'}\gamma_{D'} = \gamma_C GiG'R'$$

Thus,  $((\mathbf{R}G)(\mathbf{R}G') \circ \mu_{F',F}) \cdot (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \cdot \bar{\eta})\gamma_C$  expands to

$$\begin{aligned} & \gamma_C G R G' R' F' p' F Q \\ & \quad \cdot \gamma_C (G R G' i' F' Q' F Q \cdot G R \eta' Q' F Q) \cdot (\gamma_C G R p' F Q)^{-1} \\ & \qquad \qquad \qquad \cdot \gamma_C (G i F Q \cdot \eta Q) \cdot (\gamma_C p)^{-1} \end{aligned}$$

and a straightforward calculation then shows

$$\begin{aligned} & ((\delta_{G,G'}^{-1} \circ \mu_{F',F}) \cdot (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \cdot \bar{\eta})\gamma_C \\ & \quad = \gamma_C GiG'R'F'FQ \cdot \gamma_C (GG'i'F'FQ \cdot G\eta FQ \cdot \eta Q) \cdot (\gamma_C p)^{-1} \end{aligned}$$

but the RHS is precisely the definition of  $((\delta_{G,G'}\mathbf{L}(F'F)) \cdot \bar{\eta}'')\gamma_C$ . The dual calculation proves the other equation.  $\blacksquare$

**Corollary 3.3.21.** *Let  $C, C', D, D'$  be relative categories, let  $F \dashv G : D \rightarrow C$  and  $F' \dashv G' : D' \rightarrow C'$  be two adjunctions of ordinary categories, and let  $H : C' \rightarrow C$  and  $K : D' \rightarrow D$  be homotopical functors. Suppose we have a conjugate pair of natural transformations as in the diagrams below:*

$$(L) \quad \begin{array}{ccc} C' & \xrightarrow{H} & C \\ F' \downarrow & \not\cong_{\varphi} & \downarrow F \\ D' & \xrightarrow{K} & D \end{array} \qquad (R) \quad \begin{array}{ccc} D' & \xrightarrow{K} & D \\ G' \downarrow & \not\cong_{\psi} & \downarrow G \\ C' & \xrightarrow{H} & C \end{array}$$



Assume the following hypotheses:

- $(C^\circ, Q, p)$  is a left deformation retract for  $F$ .
- $(C'^\circ, Q', p')$  is a left deformation retract for  $F'$ .
- $H$  sends objects in  $C'^\circ$  to objects in  $C^\circ$ .
- $(D^\circ, R, i)$  is a right deformation retract for  $G$ .
- $(D'^\circ, R', i')$  is a right deformation retract for  $G'$ .
- $K$  sends objects in  $D'^\circ$  to objects in  $D^\circ$ .

Then, considering the derived natural transformations  $\mathbf{L}\varphi$  and  $\mathbf{R}\psi$ :

$$\begin{array}{ccc}
 \text{Ho } C' & \xrightarrow{\text{Ho } H} & \text{Ho } C \\
 \mathbf{L}F' \downarrow & \swarrow_{\mathbf{L}\varphi} & \downarrow \mathbf{L}F \\
 \text{Ho } D' & \xrightarrow{\text{Ho } K} & \text{Ho } D
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Ho } D' & \xrightarrow{\text{Ho } K} & \text{Ho } D \\
 \mathbf{R}G' \downarrow & \nearrow^{\mathbf{R}\psi} & \downarrow \mathbf{R}G \\
 \text{Ho } C' & \xrightarrow{\text{Ho } H} & \text{Ho } C
 \end{array}
 \tag{L'} \qquad \tag{R'}$$

- If diagram (R) satisfies the left Beck–Chevalley condition, then so does (R').
- If diagram (L) satisfies the right Beck–Chevalley condition, then so does (L').

*Proof.* The theorem says that  $\mathbf{L}\varphi$  and  $\mathbf{R}\psi$  constitute a conjugate pair of natural transformations, and by [theorem 3.3.13](#) it is clear that  $\mathbf{L}\varphi$  (resp.  $\mathbf{R}\psi$ ) is a natural isomorphism if  $\varphi$  (resp.  $\psi$ ) is a natural isomorphism.  $\blacksquare$

**Proposition 3.3.22.** *Let  $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$  be relative categories.*

- *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be functors and suppose  $(G, F)$  is laxly left deformable. If the canonical comparison  $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$  is a natural isomorphism and  $\mathcal{E}$  is a saturated homotopical category, then  $(G, F)$  is a left deformable composable pair.*

*Dually:*

- *Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be functors and suppose  $(F, G)$  is oplaxly right deformable. If the canonical comparison  $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$  is a natural isomorphism and  $\mathcal{C}$  is a saturated homotopical category, then  $(F, G)$  is a left deformable composable pair.*

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*Proof.* Let  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  and  $(D^\circ, Q^{D^\circ}, p^{D^\circ})$  constitute a lax left deformation retract for  $(G, F)$ . By [theorem 3.3.13](#), we may assume without loss of generality that  $(\mathbf{L}F)\gamma_C = \gamma_D FQ$ ,  $(\mathbf{L}G)\gamma_D = \gamma_C GQ^{D^\circ}$ , and  $\mu_{G,F}\gamma_C = \gamma_C Gp^{D^\circ} FQ^{C^\circ}$ . Our hypothesis says  $\mu_{G,F}$  is a natural isomorphism and  $\mathcal{E}$  is a saturated homotopical category, so the morphisms  $Gp_{FQ^{C^\circ}X}^{D^\circ} : GQ^{D^\circ} FQ^{C^\circ} X \Rightarrow GFQ^{C^\circ} X$  are weak equivalences, for all objects  $X$  in  $C$ .

Now, let  $\tilde{X}$  be an object in  $C^\circ$ . The following diagram commutes,

$$\begin{array}{ccc} GQ^{D^\circ} FQ^{C^\circ} \tilde{X} & \xrightarrow{Gp_{FQ^{C^\circ}\tilde{X}}^{D^\circ}} & GFQ^{C^\circ} \tilde{X} \\ GQ^{D^\circ} Fp_{F\tilde{X}}^{C^\circ} \downarrow & & \downarrow GFp_{F\tilde{X}}^{C^\circ} \\ GQ^{D^\circ} F\tilde{X} & \xrightarrow{Gp_{F\tilde{X}}^{D^\circ}} & GF\tilde{X} \end{array}$$

and since  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  is a left deformation retract for both  $F$  and  $GF$ , it follows that the downward-pointing arrows in the above diagrams are weak equivalences in  $\mathcal{E}$ ; so using the 2-out-of-3 property of  $\text{weq } \mathcal{E}$  and the fact that  $Gp_{FQ^{C^\circ}\tilde{X}}^{D^\circ}$  is a weak equivalence, we deduce that  $Gp_{F\tilde{X}}^{D^\circ}$  is a weak equivalence in  $\mathcal{E}$ . Thus, recalling [proposition 3.3.9](#), we obtain a left deformation retract  $(D_G^\circ, Q^{D^\circ}, p^{D^\circ})$  for  $G$  such that  $F$  sends every object in  $C^\circ$  to an object in  $D_G^\circ$ , and so  $(G, F)$  is indeed strongly left deformable.  $\blacksquare$

**Corollary 3.3.23.** *Let  $C$ ,  $D$ , and  $\mathcal{E}$  be relative categories, and let*

$$F_! \dashv F^* : D \rightarrow C \qquad G_! \dashv G^* : \mathcal{E} \rightarrow D$$

*be adjunctions of ordinary categories. If  $C$  and  $\mathcal{E}$  are saturated homotopical categories, then the following are equivalent:*

- (i)  $(G_!, F_!)$  is strongly left deformable and  $(F^*, G^*)$  is strongly right deformable.
- (ii)  $(G_!, F_!)$  is laxly left deformable and  $(F^*, G^*)$  is strongly right deformable.
- (iii)  $(G_!, F_!)$  is strongly left deformable and  $(F^*, G^*)$  is oplaxly right deformable.

*Proof.* [Theorem 3.3.20](#) says  $(\mu_{G_!, F_!}, \delta_{F^*, G^*})$  is a conjugate pair of natural transformations, and the pasting lemma ([A.I.II](#)) implies  $\mu_{G_!, F_!}$  is a natural isomorphism if and only if  $\delta_{F^*, G^*}$  is a natural isomorphism, so the equivalence of the three statements follows from the proposition above.  $\blacksquare$

**Proposition 3.3.24.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be two relative categories, let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  be an adjunction of ordinary categories with unit  $\eta$  and counit  $\varepsilon$ , let  $(\mathcal{C}^\circ, Q, p)$  be a left deformation retract for  $F$ , and let  $(\mathcal{D}^\circ, R, i)$  be a right deformation retract for  $G$ . Consider the following statements:*

- (i) *For all objects  $\tilde{X}$  in  $\mathcal{C}^\circ$  and all objects  $\hat{B}$  in  $\mathcal{D}^\circ$ , if  $F\tilde{X} \rightarrow \hat{B}$  is a weak equivalence in  $\mathcal{D}$ , then its right adjoint transpose  $\tilde{X} \rightarrow G\hat{B}$  is a weak equivalence in  $\mathcal{C}$ .*
- (ii) *For all objects  $X$  in  $\mathcal{C}$ , the morphism  $Gi_{FQX} \circ \eta_{QX} : QX \rightarrow GRFQX$  is a weak equivalence in  $\mathcal{C}$ .*
- (iii) *The derived unit  $\bar{\eta} : \text{id}_{\text{Ho}\mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$  is a natural isomorphism.*
- (i') *For all objects  $\tilde{X}$  in  $\mathcal{C}^\circ$  and all objects  $\hat{B}$  in  $\mathcal{D}^\circ$ , if  $\tilde{X} \rightarrow G\hat{B}$  is a weak equivalence in  $\mathcal{C}$ , then its left adjoint transpose  $F\tilde{X} \rightarrow \hat{B}$  is a weak equivalence in  $\mathcal{D}$ .*
- (ii') *For all objects  $B$  in  $\mathcal{D}$ , the morphism  $\varepsilon_{RB} \circ Fp_{GRB} : FQGRB \Rightarrow RB$  is a weak equivalence in  $\mathcal{D}$ .*
- (iii') *The derived counit  $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \text{id}_{\text{Ho}\mathcal{D}}$  is a natural isomorphism.*

*We have the implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii); if  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property, then (ii)  $\Rightarrow$  (i); and if  $\mathcal{C}$  is a saturated homotopical category, then (iii)  $\Rightarrow$  (ii). Dually, (i')  $\Rightarrow$  (ii')  $\Rightarrow$  (iii'); if  $\text{weq } \mathcal{D}$  has the 2-out-of-3 property, then (ii')  $\Rightarrow$  (i'); and if  $\mathcal{D}$  is a saturated homotopical category, then (iii')  $\Rightarrow$  (ii').*

*Proof.* (i)  $\Rightarrow$  (ii). We have a weak equivalence  $i_{FQX} : FQX \rightarrow RFQX$ , and  $QX$  is an object in  $\mathcal{C}^\circ$ , so by the hypothesis, its right adjoint transpose  $Gi_{FQX} \circ \eta_{QX}$  is also a weak equivalence.

(ii)  $\Rightarrow$  (iii). The derived unit is given by  $\bar{\eta}\gamma_C = \gamma_C(Gi_{FQ} \bullet \eta_Q) \circ (\gamma_C p)^{-1}$ , which is certainly a natural isomorphism if  $Gi_{FQX} \circ \eta_{QX}$  is a weak equivalence for all  $X$ .

(ii)  $\Rightarrow$  (i). Assume  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property. Given  $\tilde{X}$  in  $\mathcal{C}^\circ$ , the diagram below commutes,

$$\begin{array}{ccccc} Q\tilde{X} & \xrightarrow{\eta_{Q\tilde{X}}} & GFQ\tilde{X} & \xrightarrow{Gi_{FQ\tilde{X}}} & GRFQ\tilde{X} \\ p_{\tilde{X}} \downarrow & & & & \downarrow GRFp_{\tilde{X}} \\ \tilde{X} & \xrightarrow{\eta_{\tilde{X}}} & GF\tilde{X} & \xrightarrow{Gi_{F\tilde{X}}} & GRF\tilde{X} \end{array}$$

but the top row and the two vertical arrows are weak equivalences in  $\mathcal{C}$ , so the bottom row must be a weak equivalence as well, by the 2-out-of-3 property.

Let  $g : F\tilde{X} \rightarrow \hat{B}$  be a weak equivalence in  $\mathcal{D}$ , and let  $f = Gg \circ \eta_{\tilde{X}}$  be its right adjoint transpose in  $\mathcal{C}$ . We know  $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \rightarrow \mathcal{C}$  is a relative functor, so  $GRg : GRF\tilde{X} \rightarrow GR\hat{B}$  is a weak equivalence in  $\mathcal{C}$ ; but

$$Gi_{\hat{B}} \circ f = Gi_{\hat{B}} \circ Gg \circ \eta_{\tilde{X}} = GRg \circ (Gi_{F\tilde{X}} \circ \eta_{\tilde{X}})$$

and we know  $Gi_{\hat{B}} : G\hat{B} \rightarrow GR\hat{B}$  is a weak equivalence in  $\mathcal{C}$ , so by the 2-out-of-3 property again,  $f$  must be a weak equivalence in  $\mathcal{C}$ .

(iii)  $\Rightarrow$  (ii). Now assume  $\mathcal{C}$  is a saturated homotopical category. If  $\bar{\eta}$  is a natural isomorphism, then each  $\gamma_{\mathcal{C}}(Gi_{FQ} \circ \eta_Q)$  must also be a natural isomorphism, and so each  $Gi_{FQX} \circ \eta_{QX}$  is a weak equivalence, by the saturation hypothesis.  $\blacksquare$

**Corollary 3.3-25.** *With notation as above, suppose the **Quillen equivalence condition** is satisfied:*

- For all objects  $\tilde{X}$  in  $\mathcal{C}^\circ$  and all objects  $\hat{B}$  in  $\mathcal{D}^\circ$ , a morphism  $F\tilde{X} \rightarrow \hat{B}$  is a weak equivalence in  $\mathcal{D}$  if and only if its right adjoint transpose  $\tilde{X} \rightarrow G\hat{B}$  is a weak equivalence in  $\mathcal{C}$ .

*Then the derived adjunction is an adjoint equivalence of categories.*  $\blacksquare$

## 3.4 DHKS derived functors

**Prerequisites.** §§3.1, 3.2, 3.3.

Notice that in [theorem 3.3.13](#), we constructed derived functors by restricting to a relatively equivalent full subcategory on which the functor respects weak equivalences. This suggests that, by strengthening the definition of ‘deformation

retract', we may be able to construct derived functors without first passing to the homotopy category.

In this section we follow [DHKS, Ch. VII].

**Definition 3.4.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories. A **functorial left deformation retract** for an ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a triple  $(\mathcal{C}^\circ, Q, p)$  where

- $\mathcal{C}^\circ$  is a full subcategory of  $\mathcal{C}$  with the induced relative subcategory structure,
- $Q : \mathcal{C} \rightarrow \mathcal{C}$  is a relative functor, and
- $p : Q \Rightarrow \text{id}_{\mathcal{C}}$  is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction  $F|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \rightarrow \mathcal{D}$  is a relative functor.
- For all objects  $X$  in  $\mathcal{C}$ , the object  $QX$  is in  $\mathcal{C}^\circ$ .

An ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is **functorially left deformable** if there exists a functorial left deformation retract for  $F$ .

Dually, a **functorial right deformation retract** for an ordinary functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a triple  $(\mathcal{D}^\circ, R, i)$  where

- $\mathcal{D}^\circ$  is a full subcategory of  $\mathcal{D}$  with the induced relative subcategory structure,
- $R : \mathcal{D} \rightarrow \mathcal{D}$  is a relative functor, and
- $i : \text{id}_{\mathcal{D}} \Rightarrow R$  is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction  $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \rightarrow \mathcal{C}$  is a relative functor.
- For all objects  $A$  in  $\mathcal{D}$ , the object  $RA$  is in  $\mathcal{D}^\circ$ .

An ordinary functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  is **functorially right deformable** if there exists a functorial right deformation retract for  $G$ .

**REMARK 3.4.2.** Every relative functor is both functorially left deformable and functorially right deformable, with trivial functorial left and right deformation retracts.

REMARK 3.4.3. The definition above is the one found in [DHKS, §40] under the name ‘deformation retract’; they do not consider the non-functorial version.

**Lemma 3.4.4.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories.*

- *If  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is a functorial left deformation retract for an ordinary functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , then  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is also a left deformation retract for  $F$ .*
- *If  $(\mathcal{D}^\circ, \mathcal{R}, i)$  is a functorial right deformation retract for an ordinary functor  $G : \mathcal{D} \rightarrow \mathcal{C}$ , then  $(\mathcal{D}^\circ, \mathcal{R}, i)$  is also a right deformation retract for  $G$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

It is clear that axioms DR1, DR2, and DR4 are satisfied, so we need only check axiom DR3. For this, we simply observe that the inclusion  $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$  and the relative functor  $\mathcal{Q} : \mathcal{C} \rightarrow \mathcal{C}^\circ$  (together with the natural weak equivalence  $p : \mathcal{Q} \Rightarrow \text{id}_{\mathcal{C}}$ ) constitute a relative equivalence of relative categories; thus, [proposition 3.1.27](#) implies the canonical functor  $\text{Ho } \mathcal{C}^\circ \rightarrow \text{Ho } \mathcal{C}$  is fully faithful, as required. ■

**Proposition 3.4.5.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be relative categories.*

- *Let  $\mathcal{Q} : \mathcal{C} \rightarrow \mathcal{C}$  be a relative functor, let  $p : \mathcal{Q} \Rightarrow \text{id}_{\mathcal{C}}$  be a natural weak equivalence, and let  $\mathcal{C}^\circ$  be the full subcategory of  $\mathcal{C}$  spanned by the image of  $\mathcal{Q}$ . If  $\text{weq } \mathcal{D}$  has the 2-out-of-3 property in  $\mathcal{D}$  and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor such that  $F\mathcal{Q}$  is a relative functor and  $Fq\mathcal{Q} : F\mathcal{Q}\mathcal{Q} \Rightarrow F\mathcal{Q}$  is a natural weak equivalence, then  $(\mathcal{C}^\circ, \mathcal{Q}, p)$  is a functorial left deformation retract for  $F$ .*

*Dually:*

- *Let  $\mathcal{R} : \mathcal{D} \rightarrow \mathcal{D}$  be a relative functor, let  $i : \text{id}_{\mathcal{D}} \Rightarrow \mathcal{R}$  be a natural weak equivalence, and let  $\mathcal{D}^\circ$  be the full subcategory of  $\mathcal{D}$  spanned by the image of  $\mathcal{R}$ . If  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property in  $\mathcal{C}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a functor such that  $G\mathcal{R}$  is a relative functor and  $Gi\mathcal{R} : G\mathcal{R}\mathcal{R} \Rightarrow G\mathcal{R}$  is a natural weak equivalence, then  $(\mathcal{D}^\circ, \mathcal{R}, i)$  is a functorial right deformation retract for  $G$ .*

*Proof.* Let  $f : QX \rightarrow QY$  be a weak equivalence in  $C^\circ$ . By naturality, the following diagram commutes:

$$\begin{array}{ccc} FQX & \xrightarrow{FQf} & FQY \\ Fp_{QX} \downarrow & & \downarrow Fp_{QY} \\ FQX & \xrightarrow{Ff} & FQY \end{array}$$

We know  $FQf$ ,  $Fp_{QX}$ , and  $Fp_{QY}$  are weak equivalences in  $D$ , so using the 2-out-of-3 property of  $\text{weq } D$ , we deduce that  $Ff$  is also a weak equivalence in  $D$ . Thus  $F|_{C^\circ}$  is a relative functor, as required.  $\blacksquare$

**Definition 3.4.6.** Let  $C$  and  $D$  be homotopical categories. A **homotopical left approximation** for an ordinary functor  $F : C \rightarrow D$  is a homotopical right (!) Kan extension of  $F$  along  $\text{id}_C$ . Dually, a **homotopical right approximation** for an ordinary functor  $G : D \rightarrow C$  is a homotopical left (!) Kan extension of  $G$  along  $\text{id}_D$ .

**REMARK 3.4.7.** More explicitly, a homotopical left approximation for  $F : C \rightarrow D$  is a homotopically terminal object in the homotopical category  $([C, D]_h \downarrow F)_h$  described below:

- The objects are pairs  $(K, \alpha)$  where  $K$  is a homotopical functor  $C \rightarrow D$  and  $\alpha$  is a natural transformation of type  $K \Rightarrow F$ .
- The morphisms  $(K', \alpha') \rightarrow (K, \alpha)$  are those natural transformations  $\psi : K' \Rightarrow K$  such that  $\alpha \bullet \psi = \alpha'$ .
- The weak equivalences are the natural weak equivalences.

Dually, a homotopical right approximation for  $G : D \rightarrow C$  is a homotopically initial object in the homotopical category  $(F \downarrow [D, C]_h)_h$ . By [corollary 3.2.12](#), homotopical approximations are homotopically unique.

We have the following special case:

**Proposition 3.4.8.** *Let  $Q$  be a homotopical endofunctor on a homotopical category  $C$  and let  $p : Q \Rightarrow \text{id}_C$  be a natural transformation. The following are equivalent:*

- (i)  $(Q, p)$  is a homotopical left approximation for  $\text{id}_C$ .

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(ii)  $(C, Q, p)$  is a functorial left deformation retract for  $\text{id}_C$ .

Dually, let  $R$  be a homotopical endofunctor on a homotopical category  $\mathcal{D}$ , and let  $i : \text{id}_{\mathcal{D}} \Rightarrow R$  be a natural transformation. The following are equivalent:

(i')  $(R, i)$  is a homotopical right approximation for  $\text{id}_C$ .

(ii')  $(\mathcal{D}, R, i)$  is a functorial right deformation retract for  $\text{id}_{\mathcal{D}}$ .

*Proof.* (i)  $\Rightarrow$  (ii). If  $(Q, p)$  is a homotopical left approximation for  $\text{id}_C$ , then there must exist a commutative diagram of the form below,

$$\begin{array}{ccccccc}
 \text{id}_C & \longrightarrow & Q_1 & \longleftarrow & Q_2 & \longrightarrow & \cdots & \longleftarrow & Q \\
 \text{id} \downarrow & & p_1 \downarrow & & p_2 \downarrow & & & & \downarrow p \\
 \text{id}_C & \longequal{\quad} & \text{id}_C & \longequal{\quad} & \text{id}_C & \longequal{\quad} & \cdots & \longequal{\quad} & \text{id}_C
 \end{array}$$

where all the arrows in the top row are natural weak equivalences. Using 2-out-of-3 property, we deduce (by induction) that  $p_1, p_2, \dots, p$  are also natural weak equivalences; thus  $(C, Q, p)$  is indeed a functorial left deformation retract for  $\text{id}_C$ .

(ii)  $\Rightarrow$  (i). If  $(C, Q, p)$  is a functorial left deformation retract for  $\text{id}_C$ , then  $p : Q \Rightarrow \text{id}_C$  is a natural weak equivalence; but  $(\text{id}_C, \text{id}_{\text{id}_C})$  is a terminal object in  $([C, C]_{\text{h}} \downarrow \text{id}_C)_{\text{h}}$ , so by [proposition 3.2.2](#),  $(Q, p)$  must be a homotopically terminal object. ■

**Definition 3.4.9.** Let  $F, F' : C \rightarrow D$  be ordinary functors between homotopical categories, and let  $\varphi : F \Rightarrow F'$  be a natural transformation. We define the homotopical category  $([\min 2, [C, D]_{\text{h}}]_{\text{h}} \downarrow \varphi)_{\text{h}}$  as follows:

- The objects are tuples  $(H, H', \alpha, \alpha', \theta)$  where  $H$  and  $H'$  are homotopical functors  $C \rightarrow D$ ,  $\alpha$  and  $\alpha'$  are natural transformations of type  $H \Rightarrow F$  and  $H' \Rightarrow F'$  (respectively), and  $\theta : H \Rightarrow H'$  is a natural transformation such that  $\varphi \bullet \alpha = \alpha' \bullet \theta$ .
- The morphisms  $(H, H', \alpha, \alpha', \theta) \rightarrow (K, K', \beta, \beta', \chi)$  are pairs  $(\zeta, \zeta')$  of natural transformations, where  $\zeta : H \Rightarrow K$  and  $\zeta' : H' \Rightarrow K'$ , such that  $\chi \bullet \zeta = \zeta' \bullet \theta$ ,  $\beta \bullet \zeta = \alpha$ , and  $\beta' \bullet \zeta' = \alpha'$ .
- The weak equivalences are those  $(\zeta, \zeta')$  where both  $\zeta$  and  $\zeta'$  are natural weak equivalences.



A **homotopical left approximation** for  $\varphi$  is a homotopically terminal object  $(\mathbb{L}F, \mathbb{L}F', \delta, \delta', \mathbb{L}\varphi)$  in  $([\min 2, [C, \mathcal{D}]_h]_h \downarrow \varphi)_h$  such that  $(\mathbb{L}F, \delta)$  is a homotopical left approximation for  $F$  and  $(\mathbb{L}F', \delta')$  is a homotopical left approximation for  $F'$ .

Dually, let  $G, G' : \mathcal{D} \rightarrow \mathcal{C}$  be ordinary functors between homotopical categories, and let  $\psi : G' \Rightarrow G$  be a natural transformation. We define the homotopical category  $(\psi \downarrow [\min 2, [D, C]_h]_h)$  as follows:

- The objects are tuples  $(H, H', \alpha, \alpha', \theta)$  where  $H$  and  $H'$  are homotopical functors  $\mathcal{D} \rightarrow \mathcal{C}$ ,  $\alpha$  and  $\alpha'$  are natural transformations of type  $G \Rightarrow H$  and  $G' \Rightarrow H'$  (respectively), and  $\theta : H' \Rightarrow H$  is a natural transformation such that  $\alpha \bullet \psi = \theta \bullet \alpha'$ .
- The morphisms  $(K, K', \beta, \beta', \chi) \rightarrow (H, H', \alpha, \alpha', \theta)$  are pairs  $(\zeta, \zeta')$  of natural transformations, where  $\zeta : K \Rightarrow H$  and  $\zeta' : K' \Rightarrow H'$ , such that  $\zeta \bullet \chi = \theta \bullet \zeta'$ ,  $\zeta \bullet \beta = \alpha$ , and  $\zeta' \bullet \beta' = \alpha'$ .
- The weak equivalences are those  $(\zeta, \zeta')$  where both  $\zeta$  and  $\zeta'$  are natural weak equivalences.

A **homotopical right approximation** for  $\psi$  is a homotopically initial object  $(\mathbb{R}G, \mathbb{R}G', \delta, \delta', \mathbb{R}\psi)$  in  $(\psi \downarrow [\min 2, [D, C]_h]_h)$  such that  $(\mathbb{R}G, \delta)$  is a homotopical right approximation for  $G$  and  $(\mathbb{R}G', \delta')$  is a homotopical right approximation for  $G'$ .

**Theorem 3.4.10.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be homotopical categories.*

- (i) *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an ordinary functor. If  $(C^\circ, Q, p)$  is a functorial left deformation retract for  $F$ , then  $(FQ, Fp)$  is a homotopical absolute right Kan extension of  $F$  along  $\text{id}_{\mathcal{C}}$ .*
- (ii) *Let  $F, F' : \mathcal{C} \rightarrow \mathcal{D}$  be a parallel pair of ordinary functors. If  $(C^\circ, Q, p)$  is a functorial left deformation retract for both  $F$  and  $F'$ , then for any natural transformation  $\varphi : F \Rightarrow F'$ ,  $(FQ, F'Q, Fp, F'p, \varphi Q)$  is a homotopical left approximation for  $\varphi$ .*
- (iii) *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{E}$  be ordinary functors between homotopical categories. If  $(G, F)$  is strongly left deformable, then, for any homotopical left approximation  $(\mathbb{L}F, \delta^F)$  for  $F$  and any homotopical left approximation  $(\mathbb{L}G, \delta^G)$  for  $G$ ,  $(\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F)$  is a homotopical left approximation for  $GF$ .*

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Dually:

(i') Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be an ordinary functor. If  $(\mathcal{D}^\circ, \mathbf{R}, i)$  is a functorial right deformation retract for  $F$ , then  $(GR, Gi)$  is a homotopical absolute left Kan extension of  $G$  along  $\text{id}_{\mathcal{D}}$ .

(ii') Let  $G, G' : \mathcal{D} \rightarrow \mathcal{C}$  be a parallel pair of ordinary functors. If  $(\mathcal{D}^\circ, \mathbf{R}, i)$  is a functorial right deformation retract for both  $G$  and  $G'$ , then for any natural transformation  $\psi : G' \Rightarrow G$ ,  $(GR, G'R, Gi, G'i, \psi R)$  is a homotopical right approximation for  $\psi$ .

(iii') Let  $F : \mathcal{C} \rightarrow \mathcal{B}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be ordinary functors between homotopical categories. If  $(F, G)$  is strongly right deformable, then, for any homotopical right approximation  $((\mathbb{R}F), \delta^F)$  for  $F$  and any homotopical right approximation  $((\mathbb{R}G), \delta^G)$  for  $G$ ,  $((\mathbb{R}F)(\mathbb{R}G), \delta^F \circ \delta^G)$  is a homotopical right approximation for  $FG$ .

*Proof.* (i). Let  $H : \mathcal{D} \rightarrow \mathcal{E}$  and  $K : \mathcal{C} \rightarrow \mathcal{E}$  be any two homotopical functors, and let  $\alpha : K \Rightarrow HF$  be any natural transformation. Then, we have the following commutative diagram of natural transformations,

$$\begin{array}{ccccc}
 K & \xleftarrow{Kp} & KQ & \xrightarrow{\alpha Q} & HFQ \\
 & \searrow \alpha & \downarrow & \swarrow HFp & \\
 & & HF & & 
 \end{array}$$

and, for any other homotopical functor  $K' : \mathcal{C} \rightarrow \mathcal{E}$  and natural transformation  $\psi : K' \Rightarrow K$ , for  $\alpha' = \alpha \bullet \psi$ , the diagram

$$\begin{array}{ccccc}
 K' & \xleftarrow{K'p} & K'Q & \xrightarrow{\alpha' Q} & HFQ \\
 \psi \downarrow & & \downarrow \psi Q & & \parallel \\
 K & \xleftarrow{Kp} & KQ & \xrightarrow{\alpha Q} & HFQ \\
 & \searrow \alpha & \downarrow & \swarrow HFp & \\
 & & HF & & 
 \end{array}$$

also commutes; thus,  $(HFQ, HFp)$  is indeed a homotopically terminal object in  $([\mathcal{C}, \mathcal{E}]_{\text{h}} \downarrow HF)_{\text{h}}$ .

(ii). Suppose  $(H, H', \alpha, \alpha', \theta)$  is an object in  $([\min \mathcal{L}, [C, D]_{\text{h}}]_{\text{h}} \downarrow \varphi)_{\text{h}}$ . The diagram below commutes,

$$\begin{array}{ccccc}
 & & HQ & \xrightarrow{\theta Q} & H'Q \\
 & \swarrow H_p & \downarrow & & \swarrow H'_p \\
 H & \xrightarrow{\theta} & H' & & \\
 \downarrow \alpha & & \downarrow \alpha Q & & \downarrow \alpha' Q \\
 & & FQ & \xrightarrow{\varphi Q} & F'Q \\
 & \swarrow F_p & \downarrow & & \swarrow F'_p \\
 F & \xrightarrow{\varphi} & F' & & 
 \end{array}$$

and  $(H_p, H'_p)$  is a weak equivalence, so  $(FQ, F'Q, F_p, F'_p, \varphi Q)$  is indeed a homotopically terminal object in  $([\min \mathcal{L}, [C, D]_{\text{h}}]_{\text{h}} \downarrow \varphi)_{\text{h}}$ .

(iii). Let  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  and  $(D^\circ, Q^{D^\circ}, p^{D^\circ})$  be functorial left deformation retracts for  $F$  and  $G$  respectively, and suppose  $F$  maps objects in  $C^\circ$  to objects in  $D^\circ$ . To begin, observe that  $Gp^{D^\circ} FQ^{C^\circ} : GQ^{D^\circ} FQ^{C^\circ} \Rightarrow GFQ^{C^\circ}$  is a natural weak equivalence; and, as established above, both  $\delta^F Q^{C^\circ} : (\mathbb{L}F)Q^{C^\circ} \Rightarrow FQ^{C^\circ}$  and  $\delta^G Q^{D^\circ} : (\mathbb{L}G)Q^{D^\circ} \Rightarrow GQ^{D^\circ}$  are natural weak equivalences, so their horizontal composite  $(\delta^G Q^{D^\circ}) \circ (\delta^F Q^{C^\circ})$  is also a natural weak equivalence. We also know that  $(C^\circ, Q^{C^\circ}, p^{C^\circ})$  is a functorial left deformation retract for  $GF$ , so  $(GFQ^{C^\circ}, GFp^{C^\circ})$  is a homotopical left approximation for  $GF$ . Now, noting that the following diagram commutes,

$$\begin{array}{ccccc}
 (\mathbb{L}G)Q^{D^\circ} (\mathbb{L}F)Q^{C^\circ} & \xrightarrow{(\delta^G Q^{D^\circ}) \circ (\delta^F Q^{C^\circ})} & GQ^{D^\circ} FQ^{C^\circ} & \xrightarrow{Gp^{D^\circ} FQ^{C^\circ}} & GFQ^{C^\circ} \\
 \downarrow ((\mathbb{L}G)p^{D^\circ}) \circ ((\mathbb{L}F)p^{C^\circ}) & & \downarrow (Gp^{D^\circ}) \circ (Fp^{C^\circ}) & & \downarrow GFp^{C^\circ} \\
 (\mathbb{L}G)(\mathbb{L}F) & \xrightarrow{\delta^G \circ \delta^F} & GF & \xlongequal{\quad\quad\quad} & GF
 \end{array}$$

we conclude that  $(\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F$  and  $(GFQ^{C^\circ}, GFp^{C^\circ})$  are weakly equivalent in  $([C, \mathcal{E}]_{\text{h}} \downarrow GF)_{\text{h}}$ , and so  $(\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F$  is also a homotopical left approximation for  $GF$ , by [proposition 3.2.2](#).  $\blacksquare$

**REMARK 3.4.11.** Unlike the situation we had with total derived functors, the assignment  $F \mapsto FQ$  (resp.  $G \mapsto GR$ ) is not a lax (resp. oplax) 2-functor, because we do not have a natural transformation  $\text{id}_C \Rightarrow Q$  (resp.  $R \Rightarrow \text{id}_D$ ).

**Corollary 3.4.12.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be homotopical categories, and let  $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \mathrm{Ho} \mathcal{C}$  and  $\gamma_{\mathcal{D}} : \mathcal{D} \rightarrow \mathrm{Ho} \mathcal{D}$  be the respective localising functors.*

- *If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a left deformable functor and  $(\mathbb{L}F, \delta)$  is any homotopical left approximation for  $F$ , then  $(\mathrm{Ho}(\mathbb{L}F), \gamma_{\mathcal{D}}\delta)$  is a total left derived functor for  $F$ .*
- *If  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a right deformable functor and  $(\mathbb{R}G, \delta)$  is any homotopical right approximation for  $G$ , then  $(\mathrm{Ho}(\mathbb{R}G), \gamma_{\mathcal{C}}\delta)$  is a total right derived functor for  $G$ .*

*Proof.* Combine theorems 3.3.13 and 3.4.10. ■

## 3.5 Two-arrow calculi

**Prerequisites.** §§3.1, A.4.

**Definition 3.5.1.** Let  $\mathcal{C}$  be a relative category.

- We say  $\mathcal{C}$  admits a **calculus of spans** if, for any morphism  $f : X \rightarrow Y$  and any weak equivalence  $v : \tilde{Y} \rightarrow Y$  in  $\mathcal{C}$ , there exists a pullback square in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} \tilde{X} & \overset{f'}{\dashrightarrow} & \tilde{Y} \\ v' \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y \end{array}$$

where  $v' : \tilde{X} \rightarrow X$  is also a weak equivalence in  $\mathcal{C}$ .

- We say  $\mathcal{C}$  admits a **calculus of cospans** if, for any weak equivalence  $u : Y \rightarrow \hat{Y}$  and any morphism  $g : Y \rightarrow Z$  in  $\mathcal{C}$ , there exists a pushout square in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ u \downarrow & & \downarrow u' \\ \hat{Y} & \overset{g'}{\dashrightarrow} & \hat{Z} \end{array}$$

where  $u' : Z \rightarrow \hat{Z}$  is also a weak equivalence in  $\mathcal{C}$ .

We follow J. F. Jardine [2009] in using the following terminology:

**Definition 3.5.2.** Let  $\mathcal{C}$  be a relative category.

- A **cocycle**  $(f, v) : X \rightrightarrows Y$  in  $\mathcal{C}$  is a span of the form below,

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} Y$$

where  $v : \tilde{X} \rightarrow X$  is a weak equivalence in  $\mathcal{C}$  and  $f : \tilde{X} \rightarrow Y$  is any morphism. The **cocycle category**  $\mathcal{C}^{\rightsquigarrow}(X, Y)$  is the category whose objects are cocycles  $X \rightrightarrows Y$  in  $\mathcal{C}$  and whose morphisms are commutative diagrams of the following form,

$$\begin{array}{ccccc} X & \xleftarrow{v} & \tilde{X} & \xrightarrow{f} & Y \\ \parallel & & \downarrow & & \parallel \\ X & \xleftarrow{v'} & \tilde{X}' & \xrightarrow{f'} & Y \end{array}$$

with composition and identities inherited from  $\mathcal{C}$ .

- A **cycle**  $(u, f) : X \rightrightarrows Y$  in  $\mathcal{C}$  is a cospan of the form below,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

where  $u : Y \rightarrow \hat{Y}$  is a weak equivalence in  $\mathcal{C}$  and  $f : X \rightarrow \hat{Y}$  is any morphism. The **cycle category**  $\mathcal{C}^{\rightsquigarrow}(X, Y)$  is the category whose objects are cycles  $X \rightrightarrows Y$  in  $\mathcal{C}$  and whose morphisms are commutative diagrams of the following form,

$$\begin{array}{ccccc} X & \xrightarrow{f} & \hat{Y} & \xleftarrow{u} & Y \\ \parallel & & \downarrow & & \parallel \\ X & \xrightarrow{f'} & \hat{Y}' & \xleftarrow{u'} & Y \end{array}$$

with composition and identities inherited from  $\mathcal{C}$ .

**REMARK 3.5.3.** In many cases of interest,  $\mathcal{C}$  will be a relative category where  $\text{weq } \mathcal{C}$  does *not* have the 2-out-of-3 property; as such, we cannot assume that the underlying morphism of a morphism of cocycles or cycles is a weak equivalence.

¶ 3.5.4. Let  $\mathcal{C}$  be a relative category that admits a calculus of spans. Given a pair of cocycles in  $\mathcal{C}$ , say  $(f, v)$  and  $(g, v')$  as below,

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} Y \xleftarrow{v'} \tilde{Y} \xrightarrow{g} Z$$

a **composition** for the pair is a commutative diagram of the following form,

$$\begin{array}{ccccc}
 & & W & & \\
 & & \swarrow v'' & \searrow f'' & \\
 & \tilde{X} & & & \tilde{Y} \\
 & \swarrow v & & \searrow v' & \\
 X & & Y & & Z \\
 & \searrow f & & \swarrow f' & 
 \end{array}$$

where the diamond is a pullback square with  $v'' : W \rightarrow \tilde{X}$  a weak equivalence in  $C$ , and the **composite** is the cocycle  $(f' \circ f'', v \circ v'')$ . It is clear that compositions exist and are unique up to unique isomorphism (in the appropriate sense). Moreover, composition is associative and unital up to coherent natural isomorphism, so we get a **bicategory of cocycles** in  $C$ , which we denote by  $C^{\rightsquigarrow}$ , and we have an obvious pseudofunctor  $C \rightarrow C^{\rightsquigarrow}$  that sends a morphism  $f : X \rightarrow Y$  in  $C$  to the cocycle  $(f, \text{id}_X)$ .

Dually, if  $C$  is a relative category that admits a calculus of cospans, then we get a **bicategory of cycles** in  $C$ , which we denote by  $C^{\leftarrow\sim}$ , and we have an obvious pseudofunctor  $C \rightarrow C^{\leftarrow\sim}$  that sends a morphism  $f : X \rightarrow Y$  in  $C$  to the cycle  $(\text{id}_Y, f)$ .

**REMARK 3.5.5.** If  $C$  is a small relative category, then the category of cocycles or cycles between any two objects is a small category; but if  $C$  is merely locally small, then the category of cocycles or cycles may not even be essentially small.

**Theorem 3.5.6** (Fundamental theorem of calculi of spans and cospans). *Let  $C$  be a small relative category and let  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$  be the connected components functor.<sup>[1]</sup>*

- *If  $C$  admits a calculus of spans and  $\pi_0[C^{\rightsquigarrow}]$  is the category obtained by applying  $\pi_0$  to the hom-categories of the bicategory of cocycles, then the pseudofunctor  $C \rightarrow C^{\rightsquigarrow}$  induces an isomorphism  $\text{Ho } C \rightarrow \pi_0[C^{\rightsquigarrow}]$ .*
- *If  $C$  admits a calculus of cospans and  $\pi_0[C^{\leftarrow\sim}]$  is the category obtained by applying  $\pi_0$  to the hom-categories of the bicategory of cycles, then the pseudofunctor  $C \rightarrow C^{\leftarrow\sim}$  induces an isomorphism  $\text{Ho } C \rightarrow \pi_0[C^{\leftarrow\sim}]$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

[1] Recall [proposition A.2.15](#).

Let  $v : \tilde{X} \rightarrow X$  be a weak equivalence in  $\mathcal{C}$ . We must first show that the cocycle  $(v, \text{id}_{\tilde{X}}) : \tilde{X} \rightrightarrows X$  becomes an isomorphism in  $\pi_0[\mathcal{C}^{\sim\rightarrow}]$ . Consider the cocycle  $(\text{id}_{\tilde{X}}, v) : X \rightrightarrows \tilde{X}$ . The following diagram commutes,

$$\begin{array}{ccccc} X & \xleftarrow{v} & \tilde{X} & \xrightarrow{v} & X \\ \parallel & & \downarrow v & & \parallel \\ X & \xleftarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \end{array}$$

so  $(v, \text{id}_{\tilde{X}}) \circ (\text{id}_{\tilde{X}}, v) = (\text{id}_X, \text{id}_X)$  in  $\pi_0[\mathcal{C}^{\sim\rightarrow}]$ . On the other hand, given a pullback square in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} K & \xrightarrow{p_1} & \tilde{X} \\ p_0 \downarrow & & \downarrow v \\ \tilde{X} & \xrightarrow{v} & X \end{array}$$

where  $p_0 : K \rightarrow \tilde{X}$  is a weak equivalence, the universal property of  $K$  yields a unique morphism  $\Delta : X \rightarrow K$  making the diagram below commute:

$$\begin{array}{ccccc} X & \xleftarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \\ \parallel & & \Delta \downarrow & & \parallel \\ X & \xleftarrow{p_0} & K & \xrightarrow{p_1} & X \end{array}$$

Thus,  $(\text{id}_{\tilde{X}}, v) \circ (v, \text{id}_{\tilde{X}}) = (\text{id}_{\tilde{X}}, \text{id}_{\tilde{X}})$  in  $\pi_0[\mathcal{C}^{\sim\rightarrow}]$ . It now follows that every morphism  $X \rightarrow Y$  in  $\pi_0[\mathcal{C}^{\sim\rightarrow}]$  is of the form  $(f, \text{id}_{\tilde{X}}) \circ (v, \text{id}_{\tilde{X}})^{-1}$  for some weak equivalence  $v : \tilde{X} \rightarrow X$  in  $\mathcal{C}$  and some morphism  $f : \tilde{X} \rightarrow Y$ ; hence, the induced functor  $\text{Ho } \mathcal{C} \rightarrow \pi_0[\mathcal{C}^{\sim\rightarrow}]$  is a bijection on objects and full.

It remains to be shown that the functor  $\text{Ho } \mathcal{C} \rightarrow \pi_0[\mathcal{C}^{\sim\rightarrow}]$  is faithful. Suppose we have the following commutative diagram in  $\mathcal{C}$ ,

$$\begin{array}{ccccc} X & \xleftarrow{v} & \tilde{X} & \xrightarrow{f} & X \\ \parallel & & \downarrow h & & \parallel \\ X & \xleftarrow{v'} & \tilde{X}' & \xrightarrow{f'} & X \end{array}$$

where  $v : \tilde{X} \rightarrow X$  and  $v' : \tilde{X}' \rightarrow X$  are weak equivalences in  $\mathcal{C}$ . The 2-out-of-3 property of isomorphisms in  $\text{Ho } \mathcal{C}$  ensures  $h : \tilde{X} \rightarrow \tilde{X}'$  is an isomorphism in  $\text{Ho } \mathcal{C}$ , so:

$$f \circ v^{-1} = (f' \circ h) \circ (h \circ v')^{-1} = f' \circ v'^{-1}$$

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We may therefore define a functor  $\pi_0[C^{\rightsquigarrow}] \rightarrow \text{Ho } C$  that sends the connected component of a cocycle  $(f, v) : X \rightrightarrows Y$  in  $C$  to the morphism  $f \circ v^{-1}$  in  $\text{Ho } C$ ; and using the fact that localising functor  $C \rightarrow \text{Ho } C$  is an epimorphism in **Cat**, we see that this functor is a left inverse for the functor  $\text{Ho } C \rightarrow \pi_0[C^{\rightsquigarrow}]$  constructed in the previous paragraph. Thus  $\text{Ho } C \rightarrow \pi_0[C^{\rightsquigarrow}]$  is indeed an isomorphism. ■

**Proposition 3.5.7.** *Let  $C$  be a relative category in which  $\text{weq } C$  has the 2-out-of-3 property, and let  $X$  and  $Y$  be objects in  $C$ .*

- *If  $C$  admits a calculus of spans, then the cocycle category  $C^{\rightsquigarrow}(X, Y)$  (considered as a maximal relative category) also admits a calculus of spans.*
- *If  $C$  admits a calculus of cospans, then the cycle category  $C^{\rightsquigarrow\sim}(X, Y)$  (considered as a maximal relative category) also admits a calculus of cospans.*

*Proof.* The two claims are formally dual; we will prove the first version.

Since  $\text{weq } C$  has the 2-out-of-3 property in  $C$ , the underlying morphisms of morphisms of cocycles must be weak equivalences in  $C$ . It follows that pullbacks in  $C^{\rightsquigarrow}(X, Y)$  exist and can be constructed componentwise in  $C$ . ■

**Corollary 3.5.8.** *Let  $C$  be a relative category in which  $\text{weq } C$  has the 2-out-of-3 property.*

- *Let  $(f, v)$  and  $(f', v')$  be two cocycles  $X \rightrightarrows Y$  in  $C$ . If  $C$  admits a calculus of spans, then  $(f, v)$  and  $(f', v')$  are in the same connected component of  $C^{\rightsquigarrow}(X, Y)$  if and only if there exists a commutative diagram in  $C$  of the following form,*

$$\begin{array}{ccccc}
 X & \xleftarrow{v} & \bullet & \xrightarrow{f} & Y \\
 \parallel & & \uparrow w_1 & & \parallel \\
 X & \xleftarrow{w_3} & \bullet & \xrightarrow{f_3} & Y \\
 \parallel & & \downarrow w_2 & & \parallel \\
 X & \xleftarrow{v'} & \bullet & \xrightarrow{f'} & Y
 \end{array}$$

where  $w_1, w_2, w_3$  are weak equivalences in  $C$ .

- *Let  $(u, g)$  and  $(u', g')$  be two cycles  $X \rightrightarrows Y$  in  $C$ . If  $C$  admits a calculus of cospans, then  $(u, g)$  and  $(u', g')$  are in the same connected component*



of  $C^{\rightarrow\sim}(X, Y)$  if and only if there exists a commutative diagram in  $C$  of the following form,

$$\begin{array}{ccccc}
 X & \xrightarrow{g} & \bullet & \xleftarrow{u} & Y \\
 \parallel & & \downarrow w_1 & & \parallel \\
 X & \xrightarrow{g_3} & \bullet & \xleftarrow{w_3} & Y \\
 \parallel & & \uparrow w_2 & & \parallel \\
 X & \xrightarrow{g'} & \bullet & \xleftarrow{u'} & Y
 \end{array}$$

where  $w_1, w_2, w_3$  are weak equivalences in  $C$ .

*Proof.* Combine the fundamental theorem of calculi of spans and cospans (3.5.6) with the previous proposition. ■

The following definition is due to Gabriel and Zisman [GZ].

**Definition 3.5.9.** Let  $C$  be a relative category. We say  $C$  admits a **calculus of right fractions** if the following axioms are satisfied:

- (Right Ore condition). Given any morphism  $f : X \rightarrow Y$  in  $C$  and any weak equivalence  $v : X \rightarrow \tilde{X}$ , there exists a commutative diagram of the form below,

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{f'} & \tilde{Y} \\
 \downarrow v' & & \downarrow v \\
 X & \xrightarrow{f} & Y
 \end{array}$$

where  $v' : \tilde{X} \rightarrow X$  is also a weak equivalence in  $C$ .

- (Right cancellability). Given any parallel pair  $f_0, f_1 : X \rightarrow Y$  in  $C$ , if  $t : Y \rightarrow T$  is a weak equivalence in  $C$  such that  $t \circ f_0 = t \circ f_1$ , then there exists a weak equivalence  $s : S \rightarrow X$  such that  $f_0 \circ s = f_1 \circ s$ .

Dually, we say  $C$  admits a **calculus of left fractions** if the following axioms are satisfied:

- (Left Ore condition). Given any weak equivalence  $u : Y \rightarrow \hat{Y}$  and any morphism  $g : Y \rightarrow Z$  in  $C$ , there exists a commutative diagram of the

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form below,

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ u \downarrow & & \downarrow u' \\ \hat{Y} & \xrightarrow[g']{} & \hat{Z} \end{array}$$

where  $u' : Z \rightarrow \hat{Z}$  is also a weak equivalence in  $\mathcal{C}$ .

- (Left cancellability). Given any parallel pair  $g_0, g_1 : Y \rightarrow Z$  in  $\mathcal{C}$ , if  $s : S \rightarrow Y$  is a weak equivalence in  $\mathcal{C}$  such that  $g_0 \circ s = g_1 \circ s$ , then there exists a weak equivalence  $t : Z \rightarrow T$  such that  $t \circ g_0 = t \circ g_1$ .

**REMARK 3.5.10.** Although we cannot compose cocycles (resp. cycles) using pull-backs (resp. pushouts) and form a bicategory of cocycles (resp. cycles) in a relative category  $\mathcal{C}$  with a calculus of right fractions (resp. calculus of left fractions), the axioms are still enough to give a well-defined category  $\pi_0[\mathcal{C}^{\rightsquigarrow}]$  (resp.  $\pi_0[\mathcal{C}^{\leftarrow}]$ ).

**Lemma 3.5.II.** *Let  $Y$  be an object in a relative category  $\mathcal{C}$ .*

- *Let  $(\mathcal{C}_{/Y})_{\text{w}}$  be the full subcategory of the slice category  $\mathcal{C}_{/Y}$  spanned by the objects  $v : \tilde{Y} \rightarrow Y$  where  $v$  is a weak equivalence in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a calculus of right fractions, then  $(\mathcal{C}_{/Y})_{\text{w}}^{\text{op}}$  is a filtered category.<sup>[2]</sup>*
- *Let  $({}^Y/\mathcal{C})_{\text{w}}$  be the full subcategory of the slice category  ${}^Y/\mathcal{C}$  spanned by the objects  $u : Y \rightarrow \hat{Y}$  where  $u$  is a weak equivalence in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a calculus of left fractions, then  $({}^Y/\mathcal{C})_{\text{w}}$  is a filtered category.*

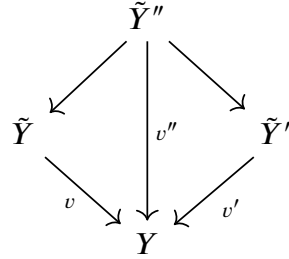
*Proof.* The two claims are formally dual; we will prove the first version.

To begin, we observe that  $\text{id} : Y \rightarrow Y$  is an object in  $(\mathcal{C}_{/Y})_{\text{w}}$ , so  $(\mathcal{C}_{/Y})_{\text{w}}$  is indeed an inhabited category. Now suppose we have two objects in  $(\mathcal{C}_{/Y})_{\text{w}}$ , say  $v' : \tilde{Y}' \rightarrow Y$  and  $v : \tilde{Y} \rightarrow Y$ . Then the right Ore condition ensures there is a

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[2] Recall [definition 0.2.1](#).

commutative diagram in  $\mathcal{C}$  of the form below,



where  $v'' : \tilde{Y}'' \rightarrow Y$  is a weak equivalence in  $\mathcal{C}$ . Finally, suppose we have a parallel pair of morphisms in  $(\mathcal{C}/Y)_w$ , say  $f_0, f_1 : \tilde{Y} \rightarrow \tilde{Y}'$  such that  $v' \circ f_0 = v' \circ f_1 = v$ . The right cancellability condition then yields a weak equivalence  $s : \tilde{S} \rightarrow \tilde{Y}$  such that  $f_0 \circ s = f_1 \circ s$ . This completes the proof that  $(\mathcal{C}/Y)_w$  is a cofiltered category. ■

**Theorem 3.5.12** (Fundamental theorem of calculi of fractions). *Let  $\mathcal{C}$  be a relative category.*

- *Let  $Y$  and  $Z$  be objects in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a calculus of right fractions, then the hom-ensemble maps*

$$\begin{aligned}
 \mathcal{C}(\tilde{Y}, Z) &\rightarrow \text{Ho } \mathcal{C}(Y, Z) \\
 f &\mapsto f \circ v^{-1}
 \end{aligned}$$

*defined by each weak equivalence  $v : \tilde{Y} \rightarrow Y$  in  $\mathcal{C}$  constitute a colimiting cocone over the evident filtered diagram of shape  $(\mathcal{C}/Y)_w^{\text{op}}$ .*

- *Let  $X$  and  $Y$  be objects in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a calculus of left fractions, then the hom-ensemble maps*

$$\begin{aligned}
 \mathcal{C}(X, \hat{Y}) &\rightarrow \text{Ho } \mathcal{C}(X, Y) \\
 g &\mapsto u^{-1} \circ g
 \end{aligned}$$

*defined by each weak equivalence  $u : Y \rightarrow \hat{Y}$  in  $\mathcal{C}$  constitute a colimiting cocone over the evident filtered diagram of shape  $({}^Y/\mathcal{C})_w$ .*

*Proof.* See Proposition 2.4 in [GZ, Ch. I]. □

**Proposition 3.5.13.** *Let  $\mathcal{C}$  be a relative category. Let  $(f, v)$  and  $(f', v')$  be two cocycles  $Y \rightrightarrows Z$  in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a calculus of right fractions, then the following are equivalent:*

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- (i) The cocycles  $(f, v)$  and  $(f', v')$  are in the same connected component of  $C^{\rightsquigarrow}(Y, Z)$ .
- (ii) We have  $f \circ v^{-1} = f' \circ v'^{-1}$  in  $\text{Ho } C$ .
- (iii) There exists a commutative diagram in  $C$  of the form below,

$$\begin{array}{ccccc}
 Y & \xleftarrow{v} & \bullet & \xrightarrow{f} & Z \\
 \parallel & & \uparrow & & \parallel \\
 Y & \xleftarrow{w_3} & \bullet & \longrightarrow & Z \\
 \parallel & & \downarrow & & \parallel \\
 Y & \xleftarrow{v'} & \bullet & \xrightarrow{f'} & Z
 \end{array}$$

where  $w_3$  is a weak equivalence in  $C$ .

Dually, let  $(u, g)$  and  $(u', g')$  be two cocycles  $X \rightarrow Y$  in  $C$ . If  $C$  admits a calculus of left fractions, then the following are equivalent:

- (i') The cycles  $(u, g)$  and  $(u', g')$  are in the same connected component of  $C^{\rightsquigarrow}(X, Y)$ .
- (ii') We have  $u^{-1} \circ g = u'^{-1} \circ g'$  in  $\text{Ho } C$ .
- (iii') There exists a commutative diagram in  $C$  of the form below,

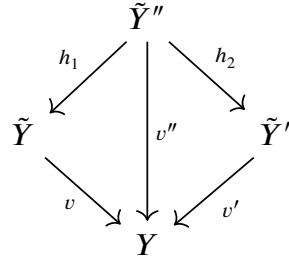
$$\begin{array}{ccccc}
 X & \xrightarrow{g} & \bullet & \xleftarrow{u} & Y \\
 \parallel & & \downarrow & & \parallel \\
 X & \longrightarrow & \bullet & \xleftarrow{w_3} & Y \\
 \parallel & & \uparrow & & \parallel \\
 X & \xrightarrow{g'} & \bullet & \xleftarrow{u'} & Y
 \end{array}$$

where  $w_3$  is a weak equivalence in  $C$ .

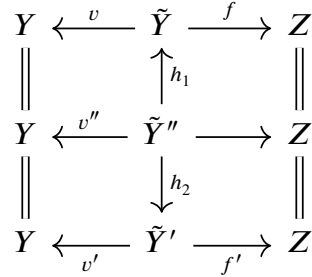
*Proof.* (i)  $\Rightarrow$  (ii). It is clear that any two cocycles in the same connected component of  $C^{\rightsquigarrow}(Y, Z)$  must represent the same morphism  $Y \rightarrow Z$  in  $\text{Ho } C$ .

(ii)  $\Rightarrow$  (iii). Suppose  $(f, v)$  and  $(f', v')$  represent the same morphism in  $\text{Ho } C$ . Using the explicit description of filtered colimits of ensembles, we deduce that

there is a commutative diagram in  $\mathcal{C}$  of the form below,



where  $v''$  is a weak equivalence in  $\mathcal{C}$  and  $f \circ h_1 = f' \circ h_2$ . Thus, the following diagram commutes, as required:



(iii)  $\Rightarrow$  (i). Immediate. ■

**Proposition 3.5.14.** *Let  $\mathcal{C}$  be a homotopical category. If  $\mathcal{C}$  admits*

- *a calculus of spans, or*
- *a calculus of cospans, or*
- *a calculus of right fractions, or*
- *a calculus of left fractions*

*then  $\mathcal{C}$  is a saturated homotopical category.*

*Proof.* The four cases are similar; we will assume that  $\mathcal{C}$  admits a calculus of spans.

Suppose  $f : X \rightarrow Y$  is a morphism that is invertible in  $\text{Ho } \mathcal{C}$ . Then there exists a cocycle  $(g, v) : Y \rightrightarrows X$  in  $\mathcal{C}$  such that  $g \circ v^{-1}$  is a two-sided inverse for

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$f$  in  $\text{Ho } C$ . Construct a commutative diagram in  $C$  of the form below,

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{f'} & \tilde{Y} \\ v' \downarrow & & \downarrow v \\ X & \xrightarrow{f} & Y \end{array}$$

where  $v' : \tilde{X} \rightarrow X$  is a weak equivalence in  $C$ . The fundamental theorem of calculi of spans (3.5.6) implies that  $(f \circ g, v) = (\text{id}_Y, \text{id}_Y)$  and  $(g \circ f', v') = (\text{id}_X, \text{id}_X)$  in  $\pi_0[C^{\rightsquigarrow}]$ , so by corollary 3.5.8, we must have commutative diagrams of the form below:

$$\begin{array}{ccc} Y & \xleftarrow{v} & \tilde{Y} & \xrightarrow{f \circ g} & Y \\ \parallel & & \uparrow & & \parallel \\ Y & \xleftarrow{\quad} & \bullet & \xrightarrow{\quad} & Y \\ \parallel & & \downarrow & & \parallel \\ Y & \xleftarrow{\text{id}} & Y & \xrightarrow{\text{id}} & Y \end{array} \qquad \begin{array}{ccc} X & \xleftarrow{v'} & \tilde{X} & \xrightarrow{g \circ f'} & X \\ \parallel & & \uparrow & & \parallel \\ X & \xleftarrow{\quad} & \bullet & \xrightarrow{\quad} & X \\ \parallel & & \downarrow & & \parallel \\ X & \xleftarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \end{array}$$

Thus, by repeatedly using the 2-out-of-3 property of  $\text{weq } C$  in  $C$ , we see that  $f \circ g$  and  $g \circ f'$  are weak equivalences in  $C$ , and by using the 2-out-of-6 property, we deduce that  $f$  (as well as  $g$  and  $f'$ ) is indeed a weak equivalence in  $C$ . ■

One advantage of calculi of fractions over calculi of spans and cospans is the following:

**Proposition 3.5.15.** *Let  $C$  be a relative category and let  $\gamma : C \rightarrow \text{Ho } C$  be the localising functor.*

- *If  $C$  admits a calculus of right fractions, then  $\gamma : C \rightarrow \text{Ho } C$  preserves limits for any finite diagram in  $C$ .*
- *If  $C$  admits a calculus of left fractions, then  $\gamma : C \rightarrow \text{Ho } C$  preserves colimits for any finite diagram in  $C$ .*

*Proof.* Apply theorems 0.2.12 and 3.5.12. ■

**Definition 3.5.16.** Let  $C$  be a relative category.

- A **colocal object** (or **right-closed object**) in  $\mathcal{C}$  is an object  $X$  in  $\mathcal{C}$  such that the hom-ensemble map

$$\mathcal{C}(X, v) : \mathcal{C}(X, \tilde{Y}) \rightarrow \mathcal{C}(X, Y)$$

is a bijection for all weak equivalences  $v : \tilde{Y} \rightarrow Y$  in  $\mathcal{C}$ .

- A **local object** (or **left-closed object**) in  $\mathcal{C}$  is an object  $Y$  in  $\mathcal{C}$  such that the hom-ensemble map

$$\mathcal{C}(u, Y) : \mathcal{C}(\hat{X}, Y) \rightarrow \mathcal{C}(X, Y)$$

is a bijection for all weak equivalences  $u : X \rightarrow \hat{X}$  in  $\mathcal{C}$ .

**Proposition 3.5.17.** *Let  $\mathcal{C}$  be a relative category. If  $\mathcal{C}$  admits a calculus of right fractions, then the following are equivalent for an object  $X$  in  $\mathcal{C}$ :*

- (i)  $X$  is a colocal object in  $\mathcal{C}$ .
- (ii) For all weak equivalences  $v : \tilde{Y} \rightarrow Y$  in  $\mathcal{C}$ , the hom-ensemble map

$$\mathcal{C}(X, v) : \mathcal{C}(X, \tilde{Y}) \rightarrow \mathcal{C}(X, Y)$$

is a surjection.

- (iii) The map  $\mathcal{C}(X, Y) \rightarrow \text{Ho } \mathcal{C}(\gamma X, \gamma Y)$  induced by the localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  is a bijection.

*Dually, if  $\mathcal{C}$  admits a calculus of left fractions, then the following are equivalent for an object  $Y$  in  $\mathcal{C}$ :*

- (i')  $Y$  is a local object in  $\mathcal{C}$ .
- (ii') For all weak equivalences  $u : X \rightarrow \hat{X}$  in  $\mathcal{C}$ , the hom-ensemble map

$$\mathcal{C}(u, Y) : \mathcal{C}(\hat{X}, Y) \rightarrow \mathcal{C}(X, Y)$$

is a surjection.

- (iii') The map  $\mathcal{C}(X, Y) \rightarrow \text{Ho } \mathcal{C}(\gamma X, \gamma Y)$  induced by the localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  is a bijection.

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*Proof.* (i)  $\Rightarrow$  (ii). Obvious.

(ii)  $\Rightarrow$  (iii). The fundamental theorem of calculi of fractions (3.5.12) says that there is a natural bijection

$$\lim_{v: (C/X)_w^{\text{op}}} C(\text{dom } v, Y) \cong \text{Ho } C(\gamma X, \gamma Y)$$

where  $v$  varies over the weak equivalences in  $C$  with codomain  $X$  (considered as a full subcategory of the slice category  $C_{/X}$ ). Note that each weak equivalence  $v : \tilde{X} \rightarrow X$  is a split epimorphism, so  $\text{Ho } C(X, Y)$  is a filtered colimit for a diagram of injective maps. In particular, the map  $C(X, Y) \rightarrow \text{Ho } C(\gamma X, \gamma Y)$  is injective. On the other hand, if  $i : X \rightarrow \tilde{X}$  is a section of a weak equivalence  $v : \tilde{X} \rightarrow X$ , then  $\gamma(v)^{-1} = \gamma(i)$ . Thus, the map  $C(X, Y) \rightarrow \text{Ho } C(\gamma X, \gamma Y)$  is also surjective.

(iii)  $\Rightarrow$  (i). Let  $v : \tilde{Y} \rightarrow Y$  be any weak equivalence in  $C$ . The hom-ensemble bijection in the hypothesis is natural, so we have the following commutative diagram:

$$\begin{array}{ccc} C(X, \tilde{Y}) & \xrightarrow{\gamma} & \text{Ho } C(\gamma X, \gamma \tilde{Y}) \\ c(X, v) \downarrow & & \downarrow \text{Ho } C(\gamma(v)) \\ C(X, Y) & \xrightarrow{\gamma} & \text{Ho } C(\gamma X, \gamma Y) \end{array}$$

Since  $\gamma(v) : \gamma \tilde{Y} \rightarrow \gamma Y$  is an isomorphism in  $\text{Ho } C$ , the map  $C(X, v)$  must be a bijection. Thus,  $X$  is a colocal object in  $C$ . ■

¶ 3.5.18. Given a functor  $F : C \rightarrow D$ , an  **$F$ -isomorphism** is a morphism in  $C$  that  $F$  sends to an isomorphism in  $D$ . Note that  $C$ , together with the class of  $F$ -isomorphisms, is then a saturated homotopical category by lemma 3.1.8.

**Proposition 3.5.19.** *Let  $C$  be a relative category. Consider the following statements:*

- (i) *The localising functor  $\gamma : C \rightarrow \text{Ho } C$  has a left adjoint.*
- (ii) *The localising functor  $\gamma : C \rightarrow \text{Ho } C$  has a fully faithful left adjoint.*
- (iii) *For each object  $X$  in  $C$ , there exists a colocal object  $\tilde{X}$  and a  $\gamma$ -isomorphism  $p : \tilde{X} \rightarrow X$ .*



We always have the implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii), and if  $\mathcal{C}$  admits a calculus of right fractions, then (iii)  $\Rightarrow$  (i) as well.

Dually:

(i') The localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  has a right adjoint.

(ii') The localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  has a fully faithful right adjoint.

(iii') For each object  $Y$  in  $\mathcal{C}$ , there exists a local object  $\hat{Y}$  and a  $\gamma$ -isomorphism  $i : Y \rightarrow \hat{Y}$ .

We always have the implications (i')  $\Rightarrow$  (ii')  $\Rightarrow$  (iii'), and if  $\mathcal{C}$  admits a calculus of left fractions, then (iii')  $\Rightarrow$  (i') as well.

*Proof.* (i)  $\Rightarrow$  (ii). This is [proposition A.4.2I](#).

(ii)  $\Rightarrow$  (iii). Let  $L : \text{Ho } \mathcal{C} \rightarrow \mathcal{C}$  be a left adjoint for  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$ . We then have the following natural bijection:

$$\mathcal{C}(L\gamma X, Y) \cong \text{Ho } \mathcal{C}(\gamma X, \gamma Y)$$

Since  $\gamma v : \gamma \tilde{Y} \rightarrow \gamma Y$  is an isomorphism for any weak equivalence  $v : \tilde{Y} \rightarrow Y$  in  $\mathcal{C}$ , it follows that  $L\gamma X$  is a colocal object in  $\mathcal{C}$ .

Now, consider the adjunction counit component  $\varepsilon_X : L\gamma X \rightarrow X$ . [Proposition A.1.3](#) says the adjunction unit  $\eta : \text{id}_{\text{Ho } \mathcal{C}} \Rightarrow \gamma L$  is a natural isomorphism, so the right triangle identity implies  $\gamma \varepsilon_X : \gamma L\gamma X \rightarrow \gamma X$  is an isomorphism, i.e.  $\varepsilon_X$  is a  $\gamma$ -isomorphism, as required.

(iii)  $\Rightarrow$  (i). Suppose  $\mathcal{C}$  admits a calculus of right fractions. [Proposition 3.5.17](#) says the localising functor  $\gamma : \mathcal{C} \rightarrow \text{Ho } \mathcal{C}$  induces a natural map

$$\mathcal{C}(\tilde{X}, Y) \rightarrow \text{Ho } \mathcal{C}(\gamma \tilde{X}, \gamma Y)$$

that is a bijection whenever  $\tilde{X}$  is a colocal object, so if  $p : \tilde{X} \rightarrow X$  is a  $\gamma$ -isomorphism, we obtain a bijection

$$\mathcal{C}(\tilde{X}, Y) \cong \text{Ho } \mathcal{C}(\gamma X, \gamma Y)$$

that is natural in  $Y$ . Since  $\gamma$  is bijective on objects, this implies  $\gamma$  has a left adjoint. ■

**Theorem 3.5.20** (Reflective localisations). *Let  $U : \mathcal{D} \rightarrow \mathcal{C}$  be a fully faithful functor. If  $U$  has a left adjoint, say  $F : \mathcal{C} \rightarrow \mathcal{D}$ , then:*

- (i) *Let  $\mathcal{U}$  be the smallest subcategory of  $\mathcal{C}$  that contains all identity morphisms and the components of the adjunction unit  $\eta : \text{id}_{\mathcal{C}} \Rightarrow UF$ . Then  $(\mathcal{C}, \mathcal{U})$  admits a calculus of left fractions.*
- (ii) *Any localisation of  $\mathcal{C}$  at  $\mathcal{U}$  is also a localisation of  $\mathcal{C}$  at  $F$ -isomorphisms.*
- (iii) *The canonical functor  $\bar{F} : \mathcal{C}[\mathcal{U}^{-1}] \rightarrow \mathcal{D}$  induced by  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful and essentially surjective on objects.*

*Dually, if  $U$  has a right adjoint, say  $H : \mathcal{C} \rightarrow \mathcal{D}$ , then:*

- (i') *Let  $\mathcal{V}$  be the smallest subcategory of  $\mathcal{C}$  that contains all identity morphisms and the components of the adjunction counit  $\varepsilon : UH \Rightarrow \text{id}_{\mathcal{C}}$ . Then  $(\mathcal{C}, \mathcal{V})$  admits a calculus of right fractions.*
- (ii') *Any localisation of  $\mathcal{C}$  at  $\mathcal{V}$  is also a localisation of  $\mathcal{C}$  at  $H$ -isomorphisms.*
- (iii') *The canonical functor  $\bar{H} : \mathcal{C}[\mathcal{V}^{-1}] \rightarrow \mathcal{D}$  induced by  $H : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful and essentially surjective on objects.*

*Proof.* (i). The naturality of  $\eta$  ensures that  $(\mathcal{C}, \mathcal{U})$  satisfies the left Ore condition. Suppose  $f_0, f_1 : UFX \rightarrow Y$  are morphisms in  $\mathcal{C}$  such that  $f_0 \circ \eta_X = f_1 \circ \eta_X$ . By [proposition A.1.3](#), the adjunction counit  $\varepsilon : FU \Rightarrow \text{id}_{\mathcal{D}}$  is a natural isomorphism, so the triangle identities imply that  $\eta UF = FU\eta$ . But  $\eta_Y \circ f_0 = UF f_0 \circ \eta_{UFX}$  and  $\eta_Y \circ f_1 = UF f_1 \circ \eta_{UFX}$ , so we may deduce that  $\eta_Y \circ f_0 = \eta_Y \circ f_1$ . Thus  $(\mathcal{C}, \mathcal{U})$  is left cancellable.

(ii). Let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . By naturality of  $\eta$ , the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UFX \\ f \downarrow & & \downarrow UFf \\ Y & \xrightarrow{\eta_Y} & UFY \end{array}$$

Thus, any functor that sends the components of  $\eta$  to isomorphisms must also make  $F$ -isomorphisms invertible. On the other hand,  $F\eta$  is a natural isomorphism because  $\varepsilon$  is, so any functor that makes  $F$ -isomorphisms invertible must also send the components of  $\eta$  to isomorphisms.

(iii). Since  $\varepsilon : FU \Rightarrow \text{id}_{\mathcal{D}}$  is a natural isomorphism, the functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is essentially surjective on objects, and so  $\bar{F} : \mathcal{C}[\mathcal{V}^{-1}] \rightarrow \mathcal{D}$  must also be essentially surjective on objects.

It remains to be shown that  $\bar{F}$  is a fully faithful functor. Let  $Y$  be an object in  $\mathcal{C}$ , and let  $f : X \rightarrow X'$  be an  $F$ -isomorphism. Since  $F \dashv U$ , we have the following commutative diagram:

$$\begin{array}{ccc} D(FX', FY) & \xrightarrow{\cong} & C(X', UFY) \\ D(Ff, FY) \downarrow & & \downarrow C(f, UFY) \\ D(FX, FY) & \xrightarrow{\cong} & C(X, UFY) \end{array}$$

We then see that  $UFY$  is a local object in  $\mathcal{C}$  (with respect to  $F$ -isomorphisms). Since  $\eta_Y : Y \rightarrow UFY$  is an  $F$ -isomorphism, we may then apply [proposition 3.5.19](#) to deduce that the localising functor  $\gamma : \mathcal{C} \rightarrow \mathcal{C}[\mathcal{V}^{-1}]$  has a fully faithful right adjoint that sends each object  $\gamma Y$  to  $UFY$ . Thus  $\bar{F}$  is indeed fully faithful. ■

## 3.6 Three-arrow calculi

**Prerequisites.** §§3.1, A.4.

In this section, we follow [\[DHKS, §36\]](#) and [\[Thomas, 2011\]](#).

**Definition 3.6.1.** Let  $\mathcal{C}$  be a relative category, let  $\mathcal{W} = \text{weq } \mathcal{C}$  be the subcategory of weak equivalences in  $\mathcal{C}$ , and let  $\mathcal{U}$  and  $\mathcal{V}$  be subcategories of  $\mathcal{W}$ . We say  $\mathcal{C}$  **admits a three-arrow calculus** for  $\mathcal{C}$  with respect to  $(\mathcal{U}, \mathcal{V})$  if the following conditions are satisfied:

**A1.** For each weak equivalence  $w$  in  $\mathcal{C}$ , there exist  $u$  in  $\mathcal{U}$  and  $v$  in  $\mathcal{V}$  such that  $w = v \circ u$ .

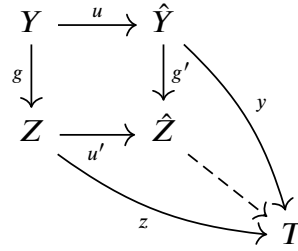
**A2.** Given a diagram of the form  $\hat{Y} \xleftarrow{u} Y \xrightarrow{g} Z$  in  $\mathcal{C}$  with  $u$  in  $\mathcal{U}$ , there exists a diagram of the form  $\hat{Y} \xrightarrow{g'} \hat{Z} \xleftarrow{u'} Z$  such that

- $g' \circ u = u' \circ g$ ,
- $u'$  is in  $\mathcal{U}$ , and

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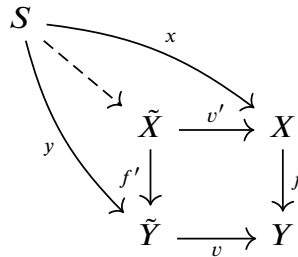
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- given any diagram of the form  $\hat{Y} \xrightarrow{y} T \xleftarrow{z} Z$  such that  $y \circ u = z \circ g$ , there exists a (not necessarily unique) morphism  $T \rightarrow \hat{Z}$  making the diagram below commute:



**A3.** Given a diagram of the form  $X \xrightarrow{f} Y \xleftarrow{v} \tilde{Y}$  in  $\mathcal{C}$  with  $v$  in  $\mathcal{V}$ , there exists a diagram of the form  $X \xleftarrow{v'} \tilde{X} \xrightarrow{f'} \tilde{Y}$  such that

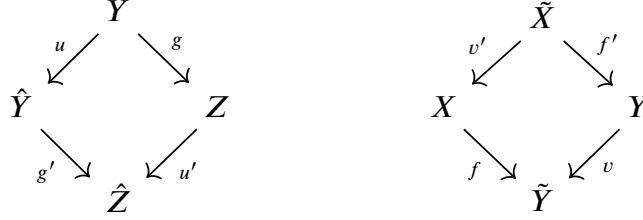
- $f \circ v' = v \circ g'$ ,
- $v'$  is in  $\mathcal{V}$ , and
- given any diagram of the form  $X \xleftarrow{x} S \xrightarrow{y} Y$  such that  $f \circ x = v \circ y$ , there exists a (not necessarily unique) morphism  $S \rightarrow \tilde{X}$  making the diagram below commute:



A **uni-fractionable category** is a relative category  $\mathcal{C}$  together with a pair of subcategories  $(\mathcal{U}, \mathcal{V})$  such that  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property in  $\mathcal{C}$  and  $\mathcal{C}$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ .

**REMARK 3.6.2.** Note that axiom A1 implies that  $\text{ob } \mathcal{U} = \text{ob } \mathcal{V} = \text{ob } \mathcal{C}$ ; in particular, every identity morphism in  $\mathcal{C}$  is also in  $\mathcal{U}$  and  $\mathcal{V}$ .

REMARK 3.6.3. Consider diagrams of the following forms,



where  $u, u'$  are in  $\mathcal{U}$  and  $v, v'$  are in  $\mathcal{V}$ . Under the assumption that  $\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{C}$ , the morphism  $g$  is in  $\mathcal{W}$  if and only if  $g'$  is in  $\mathcal{W}$ , and the morphism  $f$  is in  $\mathcal{W}$  if and only if  $f'$  is in  $\mathcal{W}$ .

**Definition 3.6.4.** Let  $\mathcal{C}$  be a relative category, let  $\mathcal{W} = \text{weq } \mathcal{C}$  be the subcategory of weak equivalences in  $\mathcal{C}$ , and let  $\mathcal{U}$  and  $\mathcal{V}$  be subcategories of  $\mathcal{W}$ . A **functorial three-arrow calculus** for  $\mathcal{C}$  with respect to  $(\mathcal{U}, \mathcal{V})$  consists of the following data:

- FA1.** A functorial factorisation system on  $\mathcal{W}$  with left class contained in  $\text{mor } \mathcal{U}$  and right class contained in  $\text{mor } \mathcal{V}$ .
- FA2.** A functor from the full subcategory of  $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$  spanned by those diagrams of the form  $\hat{Y} \xleftarrow{u} Y \xrightarrow{g} Z$ , where  $u$  is in  $\mathcal{U}$ , to the category  $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$ , such that each diagram  $\hat{Y} \xleftarrow{u} Y \xrightarrow{g} Z$  is sent to a diagram of the form  $\hat{Y} \xrightarrow{g'} \hat{Z} \xleftarrow{u'} Z$ , where  $g' \circ u = u' \circ g$ ,  $u'$  is in  $\mathcal{U}$ , and  $u'$  is an isomorphism if  $u$  is.
- FA3.** A functor from the full subcategory of  $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$  spanned by those diagrams of the form  $X \xrightarrow{f} Y \xleftarrow{v} \tilde{Y}$ , where  $v$  is in  $\mathcal{V}$ , to the category  $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$ , such that each diagram  $X \xrightarrow{f} Y \xleftarrow{v} \tilde{Y}$  is sent to a diagram of the form  $X \xleftarrow{v'} \tilde{X} \xrightarrow{f'} \tilde{Y}$ , where  $f \circ v' = v \circ g'$ ,  $v'$  is in  $\mathcal{V}$ , and  $v'$  is an isomorphism if  $v$  is.

If such data exist, then we say  $\mathcal{C}$  **admits a functorial three-arrow calculus** with respect to  $(\mathcal{U}, \mathcal{V})$ .

REMARK 3.6.5. If  $\text{mor } \mathcal{U}$  is closed under pushout in  $\mathcal{C}$ , then we may take pushouts to construct datum FA2; similarly, if  $\text{mor } \mathcal{V}$  is closed under pullback in  $\mathcal{C}$ , then we may take pullbacks to construct datum FA3.

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REMARK 3.6.6. A relative category  $\mathcal{C}$  admits a (functorial) three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$  if and only if the opposite relative category  $\mathcal{C}^{\text{op}}$  admits a (functorial) three-arrow calculus with respect to  $(\mathcal{V}, \mathcal{U})$ .

**Proposition 3.6.7.** *Let  $\mathcal{C}$  be a relative category and let  $\mathcal{U}$  and  $\mathcal{V}$  be subcategories of  $\mathcal{W} = \text{weq } \mathcal{C}$  (itself considered as a subcategory of  $\mathcal{C}$ ). If  $\mathcal{C}$  admits a functorial three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ , then  $\mathcal{C}$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ .*

*Proof.* Obviously, having datum FA1 implies axiom A1 is satisfied. Now suppose we have a commutative square of the form below in  $\mathcal{C}$ ,

$$\begin{array}{ccc} Y & \xrightarrow{u} & \hat{Y} \\ g \downarrow & & \downarrow y \\ Z & \xrightarrow{z} & T \end{array}$$

where  $u$  is in  $\mathcal{U}$ . The datum FA2 then gives us the following commutative diagram,

$$\begin{array}{ccccc} Y & \xrightarrow{u} & \hat{Y} & & \\ & \searrow & \downarrow y & & \\ & & T & \xrightarrow{\quad} & T \\ g \downarrow & & \downarrow g' & & \downarrow w \\ Z & \xrightarrow{\quad} & \hat{Z} & \xrightarrow{\quad} & T \\ & \searrow z & \downarrow u' & \searrow & \\ & & T & \xrightarrow{w} & \hat{T} \end{array}$$

and  $w : T \rightarrow \hat{T}$  is an isomorphism, thus, there exists a morphism  $\hat{Z} \rightarrow T$  making the diagram below commute:

$$\begin{array}{ccc} Y & \xrightarrow{u} & \hat{Y} \\ g \downarrow & & \downarrow g' \\ Z & \xrightarrow{u'} & \hat{Z} \\ & \searrow z & \downarrow y \\ & & T \end{array}$$

This shows that axiom A2 is satisfied, and the dual argument proves axiom A3. ■

**Proposition 3.6.8.** *Let  $\mathcal{A}$  and  $\mathcal{C}$  be relative categories. If  $\mathcal{C}$  admits a functorial three-arrow calculus, and either*

- *weq  $\mathcal{C}$  has the 2-out-of-3 property in  $\mathcal{C}$ , or*
- *$\mathcal{A}$  is a minimal relative category,*

*then the relative functor category  $[\mathcal{A}, \mathcal{C}]_{\text{h}}$  admits a functorial three-arrow calculus constructed componentwise from  $\mathcal{C}$ .*

*Proof.* Let  $(\mathcal{U}, \mathcal{V})$  be a functorial three-arrow calculus for  $\mathcal{C}$ . It is clear that, when  $\mathcal{A}$  is a minimal relative category, all the data constituting a three-arrow calculus for  $\mathcal{C}$  may be lifted componentwise to define a three-arrow calculus for  $[\mathcal{A}, \mathcal{C}]_{\text{h}}$ .

In general, we must check that  $[\mathcal{A}, \mathcal{C}]_{\text{h}}$  is closed under the various componentwise constructions. However, if  $f : A \rightarrow B$  is a weak equivalence in  $\mathcal{A}$  and  $\theta : X \Rightarrow Y$  is a natural weak equivalence of relative functors  $X, Y : \mathcal{A} \rightarrow \mathcal{M}$ , and  $\psi \bullet \varphi$  is the componentwise  $(\mathcal{U}, \mathcal{V})$ -factorisation of  $\theta$ , then the diagram below commutes,

$$\begin{array}{ccccc} XA & \xrightarrow{\varphi_A} & ZA & \xrightarrow{\psi_A} & YA \\ x f \downarrow & & z f \downarrow & & \downarrow Y f \\ XB & \xrightarrow{\varphi_B} & ZB & \xrightarrow{\psi_B} & YB \end{array}$$

and so by the 2-out-of-3 property of weq  $\mathcal{C}$ ,  $Zf$  is also a weak equivalence in  $\mathcal{C}$ , thus  $Z : \mathcal{A} \rightarrow \mathcal{M}$  is a relative functor. Similarly, one uses the 2-out-of-3 property of weq  $\mathcal{C}$  to ensure that the componentwise constructions satisfy the conditions to be data FA2 and FA3 for a functorial three-arrow calculus. ■

**Theorem 3.6.9** (Fundamental theorem of three-arrow calculi). *Let  $\mathcal{C}$  be a relative category such that weq  $\mathcal{C}$  has the 2-out-of-3 property in  $\mathcal{C}$ . If  $\mathcal{C}$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ , then:*

- (i) *Every morphism in  $\text{Ho } \mathcal{C}$  can be represented by a zigzag in  $\mathcal{C}$  of the form below,*

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

*where  $u$  is in  $\mathcal{U}$  and  $v$  is in  $\mathcal{V}$ .*

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(ii) Two such zigzags represent the same morphism in  $\text{Ho } C$  if and only if there exists a commutative diagram in  $C$  of the form

$$\begin{array}{ccccccc}
 X & \xleftarrow{v} & \tilde{X} & \xrightarrow{f} & \hat{Y} & \xleftarrow{u} & Y \\
 \parallel & & \uparrow v_1 & & \uparrow v_2 & & \parallel \\
 X & \xleftarrow{v_3} & \bullet & \xrightarrow{f_3} & \bullet & \xleftarrow{u_3} & Y \\
 \parallel & & \downarrow w_1 & & \downarrow w_2 & & \parallel \\
 X & \xleftarrow{v_4} & \bullet & \xrightarrow{f_4} & \bullet & \xleftarrow{u_4} & Y \\
 \parallel & & \uparrow u_1 & & \uparrow u_2 & & \parallel \\
 X & \xleftarrow{v'} & \tilde{X}' & \xrightarrow{f'} & \hat{Y}' & \xleftarrow{u'} & Y
 \end{array}$$

where  $u_1, u_2, u_3, u_4$  are in  $\mathcal{U}$ ,  $v_1, v_2, v_3, v_4$  are in  $\mathcal{V}$ , and  $w_1, w_2$  are weak equivalences in  $C$ .

*Proof.* For the functorial case, see paragraph 36.3 in [DHKS]; for the general case, see Lemma 4.9 and Theorem 5.13 in [Thomas, 2011].  $\square$

**Proposition 3.6.10.** *If  $C$  is a homotopical category that admits a three-arrow calculus, then  $C$  is a saturated homotopical category.*

*Proof.* Suppose  $C$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ . Let  $f : X \rightarrow Y$  be a morphism in  $C$  whose image in  $\text{Ho } C$  is an isomorphism, with inverse represented by the following zigzag,

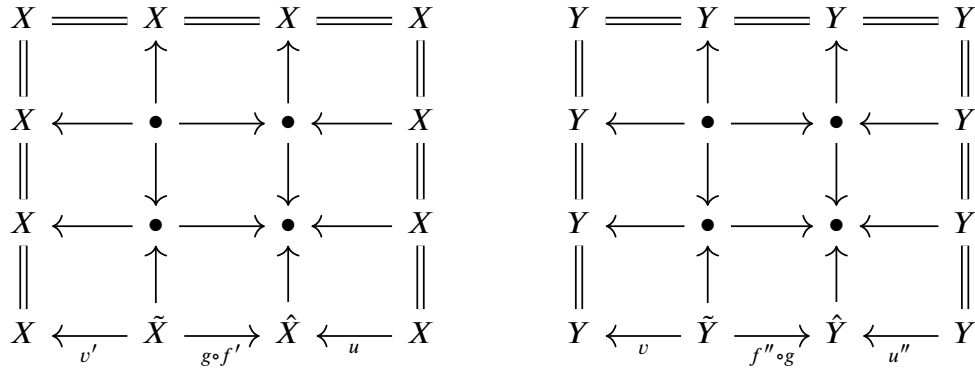
$$Y \xleftarrow{v} \tilde{Y} \xrightarrow{g} \hat{X} \xleftarrow{u} X$$

where  $u$  is in  $\mathcal{U}$  and  $v$  is in  $\mathcal{V}$ . Then, by axioms A2 and A3, there exist  $v'$  in  $\mathcal{V}$ ,  $f'$  in  $C$ ,  $u''$  in  $\mathcal{U}$ , and  $f''$  in  $C$  such that the diagrams below commute,

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{f'} & \tilde{Y} \\
 v' \downarrow & & \downarrow v \\
 X & \xrightarrow{f} & Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 X & \xrightarrow{f} & Y \\
 u \downarrow & & \downarrow u'' \\
 \hat{X} & \xrightarrow{f''} & \hat{Y}
 \end{array}$$

and by theorem 3.6.9, we have commutative diagrams in  $C$  of the following form,





where all leftward- and upward-pointing arrows are weak equivalences in  $\mathcal{C}$ . We may then deduce that *every* arrow appearing in the above diagrams are in  $\text{weq } \mathcal{C}$  by iteratively applying the 2-out-of-3 property of  $\text{weq } \mathcal{C}$ . In particular,  $g \circ f'$  and  $f'' \circ g$  are weak equivalences in  $\mathcal{C}$ , so the 2-out-of-6 property of  $\text{weq } \mathcal{C}$  implies that  $f', f'', g$  are all in  $\text{weq } \mathcal{C}$ . We then conclude that  $f$  is in  $\text{weq } \mathcal{C}$ , by using the 2-out-of-3 property again. ■

### 3.7 Categories of fibrant objects

**Prerequisites.** §§3.1, 3.5, A.4.

One particularly common kind of relative category with a calculus of spans is obtained by taking the full subcategory of fibrant objects in a model category. We can study these categories axiomatically following Brown [1973]:

**Definition 3.7.1.** A **category of fibrant objects** is a locally small category  $\mathcal{E}$  with finite products and equipped with a pair  $(\mathcal{W}, \mathcal{F})$  of subclasses of  $\text{mor } \mathcal{E}$  satisfying these axioms:

- (A)  $(\mathcal{E}, \mathcal{W})$  is a category with weak equivalences, i.e. every isomorphism is in  $\mathcal{W}$  and  $\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{E}$ .
- (B) Every isomorphism is in  $\mathcal{F}$ , and  $\mathcal{F}$  is closed under composition.
- (C) Pullbacks along morphisms in  $\mathcal{F}$  exist, and the pullback of a morphism that is in  $\mathcal{F}$  (resp.  $\mathcal{W} \cap \mathcal{F}$ ) is also a morphism that is in  $\mathcal{F}$  (resp.  $\mathcal{W} \cap \mathcal{F}$ ).

(D) For each object  $X$  in  $\mathcal{E}$ , there is a commutative diagram of the form below,

$$\begin{array}{ccc} X & \xrightarrow{i} & \text{Path}(X) \\ & \searrow \Delta & \downarrow p \\ & & X \times X \end{array}$$

where  $\Delta : X \rightarrow X \times X$  is the diagonal morphism,  $i : X \rightarrow \text{Path}(X)$  is in  $\mathcal{W}$ , and  $\text{Path}(X) \rightarrow X \times X$  is in  $\mathcal{F}$ .

(E) For any object  $X$  in  $\mathcal{E}$ , the unique morphism  $X \rightarrow 1$  is in  $\mathcal{F}$ .

In a category of fibrant objects as above,

- a **weak equivalence** is a morphism in  $\mathcal{W}$ ,
- a **fibration** is a morphism in  $\mathcal{F}$ , and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in  $\mathcal{W} \cap \mathcal{F}$ ; and

**Definition 3.7.2.** Let  $X$  be an object in a category of fibrant objects  $\mathcal{E}$ . A **path object** for  $X$  is a quadruple  $(\text{Path}(X), i, p_0, p_1)$ , where  $\text{Path}(X)$  is an object in  $\mathcal{E}$ ,  $i : X \rightarrow \text{Path}(X)$  is a weak equivalence, and  $p_0$  and  $p_1$  are retractions of  $i$  such that the morphism  $\langle p_0, p_1 \rangle : \text{Path}(X) \rightarrow X \times X$  is a fibration.

**REMARK 3.7.3.** Axiom D is precisely the statement that path objects exist in a category of fibrant objects.

**Lemma 3.7.4.** *Let  $X$  be an object in a category of fibrant objects  $\mathcal{E}$  and let  $(\text{Path}(X), i, p_0, p_1)$  be a path object for  $X$ . Then  $p_0, p_1 : \text{Path}(X) \rightarrow X$  are trivial fibrations.*

*Proof.* Axioms C and E imply the two projections  $X \times X \rightarrow X$  are fibrations, so axiom B implies  $p_0, p_1 : \text{Path}(X) \rightarrow X$  must be fibrations. By definition, we have  $p_0 \circ i = p_1 \circ i = \text{id}_X$ , so axiom A implies  $p_0$  and  $p_1$  are both weak equivalences, as required. ■

**Lemma 3.7.5** (Factorisation lemma). *Let  $f : X \rightarrow Y$  be a morphism in a category of fibrant objects  $\mathcal{E}$ . Then there exist a fibration  $v : E_f \rightarrow Y$  and a weak equivalence  $u : X \rightarrow E_f$  in  $\mathcal{E}$  such that  $f = v \circ u$  and  $u$  is a section of a trivial fibration.*

*Proof.* Let  $(\text{Path}(Y), i, p_0, p_1)$  be a path object for  $Y$  in  $\mathcal{E}$ . Form a pullback diagram in  $\mathcal{E}$  of the form below:

$$\begin{array}{ccc} E_f & \xrightarrow{g} & \text{Path}(Y) \\ q \downarrow & & \downarrow p_0 \\ X & \xrightarrow{f} & Y \end{array}$$

Note that  $p_0 : \text{Path}(Y) \rightarrow Y$  is a trivial fibration (by lemma 3.7.4), so this makes sense, and  $q : E_f \rightarrow X$  is also a trivial fibration (by axiom C). Let  $u : X \rightarrow E_f$  be the unique morphism such that  $q \circ u = \text{id}_X$  and  $g \circ u = i \circ f$ , and let  $v = p_1 \circ g$ . Then  $u$  is a section of a trivial fibration, hence is a weak equivalence by axiom A, and  $v$  is a fibration because we have the following commutative diagram,

$$\begin{array}{ccc} E_f & \xrightarrow{g} & \text{Path}(Y) \\ \downarrow \langle q, p_1 \circ g \rangle & & \downarrow \langle p_0, p_1 \rangle \\ v \left( X \times Y \right. & \xrightarrow{f \times \text{id}_Y} & \left. Y \times Y \right) \\ \downarrow & & \\ Y & & \end{array}$$

where the square in the diagram is a pullback square and  $X \times Y \rightarrow Y$  is the product projection (thus, a fibration by axioms C and E). ■

**Proposition 3.7.6.** *Let  $A$  be an object in a category of fibrant objects  $\mathcal{E}$  and let  $(\mathcal{E}_{/A})_f$  be the full subcategory of the slice category  $\mathcal{E}_{/A}$  spanned by the fibrations over  $A$ . Then  $(\mathcal{E}_{/A})_f$  is a category of fibrant objects where a morphism  $(\mathcal{E}_{/A})_f$  is a weak equivalence (resp. fibration) if and only if the underlying morphism in  $\mathcal{E}$  is a weak equivalence (resp. fibration).*

*Proof.* Since  $\mathcal{E}$  has pullbacks along fibrations,  $(\mathcal{E}_{/A})_f$  has finite products (and the inclusion  $(\mathcal{E}_{/A})_f \hookrightarrow \mathcal{E}_{/A}$  preserves them). It is clear that  $(\mathcal{E}_{/A})_f$  with the given weak equivalences and fibrations satisfies axioms A, B, C, and E; and for axiom D, we may apply the factorisation lemma (3.7.5). ■

**Proposition 3.7.7.** *Let  $\mathcal{E}$  be a category of fibrant objects and let  $\mathbb{D}$  be a small category. If  $\mathcal{E}$  admits functorial (weak equivalence, fibration)-factorisations, then the functor category  $[\mathbb{D}, \mathcal{E}]$  is a category of fibrant objects where a natural transformation is a weak equivalence (resp. fibration) if and only if its components are weak equivalences (resp. fibrations) in  $\mathcal{E}$ .*

*Proof.* Since  $\mathcal{E}$  has finite products,  $[\mathbb{D}, \mathcal{E}]$  also has finite products, and they are computed componentwise. It is clear that  $[\mathbb{D}, \mathcal{E}]$  with the given weak equivalences and fibrations satisfies axioms A, B, C, and E; and for axiom D, we may use the functorial (weak equivalence, fibration)-factorisation system on  $\mathcal{E}$ . ■

**Lemma 3.7.8.** *Let  $\mathcal{E}$  be a category of fibrant objects and let  $\mathcal{D}$  be a category of weak equivalences. If  $F : \mathcal{E} \rightarrow \mathcal{D}$  is a functor that sends trivial fibrations in  $\mathcal{E}$  to weak equivalences in  $\mathcal{D}$ , then  $F$  also sends weak equivalences in  $\mathcal{E}$  to weak equivalences in  $\mathcal{D}$ .*

*Proof.* Let  $f$  be a weak equivalence in  $\mathcal{E}$ . By the factorisation lemma (3.7.5), there exist morphisms  $u$  and  $v$  in  $\mathcal{E}$  such that  $f = v \circ u$ ,  $u$  is a section of a trivial fibration, and  $v$  is a trivial fibration. Since weak equivalences have the 2-out-of-3 property in  $\mathcal{D}$ ,  $Fu$  is a weak equivalence in  $\mathcal{D}$ ; hence,  $Ff$  is a weak equivalence in  $\mathcal{D}$ , as required. ■

**Corollary 3.7.9.** *Let  $\mathcal{E}$  be a category of fibrant objects and let  $\mathcal{W}$  be a subclass of  $\text{mor } \mathcal{E}$  that has the 2-out-of-3 property.*

- (i) *If every trivial fibration in  $\mathcal{E}$  is in  $\mathcal{W}$ , then every weak equivalence in  $\mathcal{E}$  is also in  $\mathcal{W}$ .*
- (ii) *In particular, the class of weak equivalences in  $\mathcal{E}$  is the smallest class of morphisms in  $\mathcal{E}$  that has the 2-out-of-3 property and contains the trivial fibrations.* ■

**Definition 3.7.10.** The **homotopy category** of a category of fibrant objects  $\mathcal{E}$  is the category  $\text{Ho } \mathcal{E}$  obtained by freely inverting the weak equivalences in  $\mathcal{E}$ , as in definition A.4.9.

**Proposition 3.7.11.** *Let  $\mathcal{E}$  be a category of fibrant objects and let  $\mathcal{V}$  be the class of trivial fibrations in  $\mathcal{E}$ .*

- (i)  *$\mathcal{E}$ , considered as a relative category with trivial fibrations as weak equivalences, admits a calculus of spans.*
- (ii) *The canonical comparison functor  $\mathcal{E}[\mathcal{V}^{-1}] \rightarrow \text{Ho } \mathcal{E}$  is an isomorphism of categories.*

(iii) Every morphism  $X \rightarrow Y$  in  $\text{Ho } \mathcal{E}$  can be represented by a cocycle in  $\mathcal{E}$  of the form below:

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} Y$$

*Proof.* (i). This is just axiom C for a category of fibrant objects.

(ii). Lemma 3.7.8 implies every weak equivalence in  $\mathcal{E}$  becomes invertible in  $\mathcal{E}[\mathcal{V}^{-1}]$ , so it has the same universal property as  $\text{Ho } \mathcal{E}$ ; thus the canonical comparison functor  $\mathcal{E}[\mathcal{V}^{-1}] \rightarrow \text{Ho } \mathcal{E}$  must be an isomorphism of categories.

(iii). This is a consequence of the fundamental theorem of calculi of spans (3.5.6). ■

**Proposition 3.7.12.** *Let  $f : X \rightarrow Y$  be a morphism in a category of fibrant objects  $\mathcal{E}$ .*

- (i) *The pullback functor  $f^* : \mathcal{E}_{/Y} \rightarrow \mathcal{E}_{/X}$  restricts to a functor  $f^* : (\mathcal{E}_{/Y})_{\text{f}} \rightarrow (\mathcal{E}_{/X})_{\text{f}}$ .*
- (ii) *The pullback functor  $f^* : (\mathcal{E}_{/Y})_{\text{f}} \rightarrow (\mathcal{E}_{/X})_{\text{f}}$  preserves weak equivalences and fibrations.*

*Proof.* (i). This is just axiom C for a category of fibrant objects.

(ii). Recalling the pullback pasting lemma, this is a consequence of axiom C and lemma 3.7.8. ■

**Lemma 3.7.13.** *Let  $\mathcal{E}$  be a category of fibrant objects and let  $s : X \rightarrow Y$  be a section of a trivial fibration in  $\mathcal{E}$ . Given a pullback diagram in  $\mathcal{E}$  of the form below,*

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ s' \downarrow & & \downarrow s \\ Y' & \xrightarrow{g} & Y \end{array}$$

*where  $g : Y' \rightarrow Y$  is a fibration in  $\mathcal{E}$ , the morphism  $s' : X' \rightarrow Y'$  is a weak equivalence in  $\mathcal{E}$ .*

### III. HOMOTOPICAL CATEGORIES

*Proof.* Let  $p : Y \rightarrow X$  be a trivial fibration in  $\mathcal{E}$  such that  $p \circ s = \text{id}_X$ . Form the following pullback diagram in  $\mathcal{E}$ :

$$\begin{array}{ccc} Z & \xrightarrow{q} & Y \\ r \downarrow & & \downarrow p \\ Y' & \xrightarrow{p \circ g} & X \end{array}$$

By axiom C,  $r : Z \rightarrow Y'$  is a trivial fibration and  $q : Z \rightarrow Y$  is a fibration. There is a unique morphism  $t : Y' \rightarrow Z$  such that  $r \circ t = \text{id}_{Y'}$ , and  $q \circ t = g$ ; note that  $t$  is then a weak equivalence by axiom A. Now, consider the commutative diagram in  $\mathcal{E}$  shown below:

$$\begin{array}{ccc} Y' & \xrightarrow{p \circ g} & X \\ \downarrow t & & \downarrow s \\ Z & \xrightarrow{q} & Y \\ \downarrow r & & \downarrow p \\ Y' & \xrightarrow{p \circ g} & X \end{array} \quad \begin{array}{c} \text{id} \\ \left. \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\} \\ \text{id} \end{array}$$

The lower square and the outer rectangle are pullback diagrams in  $\mathcal{E}$ , so the pullback pasting lemma implies the upper square is also a pullback diagram in  $\mathcal{E}$ . Similarly, the following diagram in  $\mathcal{E}$  commutes,

$$\begin{array}{ccccc} X' & \xrightarrow{\quad} & Y' & \xrightarrow{p \circ g} & X \\ s' \downarrow & & \downarrow t & & \downarrow s \\ Y' & \xrightarrow{t} & Z & \xrightarrow{q} & Y \end{array} \quad \begin{array}{c} f \\ \left. \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \end{array} \right\} \\ g \end{array}$$

and both the right square and the outer rectangle are pullback diagrams in  $\mathcal{E}$ , so the left square is a pullback diagram in  $\mathcal{E}$  as well. But [proposition 3.7.12](#) says that  $s^* : (\mathcal{E}_{/Y})_f \rightarrow (\mathcal{E}_{/X})_f$  preserves weak equivalences, so  $s' : X' \rightarrow Y'$  is indeed a weak equivalence, as required. ■

**Proposition 3.7.14.** *In a category of fibrant objects, the pullback of a weak equivalence along a fibration is again a weak equivalence.*

*Proof.* Axiom C says that the pullback of a trivial fibration is a trivial fibration, so the factorisation lemma ([3.7.5](#)) implies it is enough to prove that the pullback

of a section of a trivial fibration is a weak equivalence; but that is precisely the statement of [lemma 3.7.13](#). ■





## MODEL CATEGORIES

In [1967], Quillen introduced the notion of a ‘closed model category’ (but we shall say simply ‘model category’) for homotopy theory, so as to formalise the similarities between the homotopy theory of spaces and homological algebra. The idea was that, to do homotopy theory, one only really needs to know which morphisms are cofibrations, which are weak equivalences, and which are fibrations.

### 4.1 Basics

**Prerequisites.** §§3.1, 3.5, 3.6, A.3, A.4.

**Definition 4.1.1.** A **model structure** on a category  $\mathcal{M}$  is a triple  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  of subensembles of mor  $\mathcal{M}$  satisfying the following axioms:<sup>[1]</sup>

- CM2.**  $\mathcal{W}$  has the 2-out-of-3 property.
- CM3.**  $\mathcal{C}$ ,  $\mathcal{W}$ , and  $\mathcal{F}$  are closed under retracts.
- CM4.** Given a commutative diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} Z & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ W & \longrightarrow & Y \end{array}$$

where  $i$  is in  $\mathcal{C}$  and  $p$  is in  $\mathcal{F}$ , if at least one of  $i$  or  $p$  is also in  $\mathcal{W}$ , then there exists a morphism  $W \rightarrow X$  making both of the evident triangles commute.

[1] This presentation is due to Quillen [1969].

**CM5.** Any morphism  $f$  in  $\mathcal{M}$  may be factored in two ways:

- $f = p \circ i$  where  $i$  is in  $\mathcal{C} \cap \mathcal{W}$  and  $p$  is in  $\mathcal{F}$ , and
- $f = q \circ j$ , where  $j$  is in  $\mathcal{C}$  and  $q$  is in  $\mathcal{W} \cap \mathcal{F}$ .

Given a model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  on a category  $\mathcal{M}$ ,

- a **weak equivalence** is a morphism in  $\mathcal{W}$ ,
- a **cofibration** is a morphism in  $\mathcal{C}$ ,
- a **fibration** is a morphism in  $\mathcal{F}$ ,
- a **trivial cofibration** (or **acyclic cofibration**) is a morphism in  $\mathcal{C} \cap \mathcal{W}$ , and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in  $\mathcal{W} \cap \mathcal{F}$ ;
- a **cofibrant object** is an object  $W$  that is projective with respect to the class of trivial fibrations, i.e. for every trivial fibration  $p : X \rightarrow Y$  and every morphism  $w : W \rightarrow Y$ , there exists a morphism  $W \rightarrow X$  making the diagram below commute:

$$\begin{array}{ccc} & & X \\ & \nearrow & \downarrow p \\ W & \xrightarrow{w} & Y \end{array}$$

- a **fibrant object** is an object  $X$  that is injective with respect to the class of trivial cofibrations, i.e. for every trivial cofibration  $i : Z \rightarrow W$  and every morphism  $z : Z \rightarrow X$ , there exists a morphism  $W \rightarrow X$  making the diagram below commute:

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ \downarrow i & & \nearrow \\ W & & \end{array}$$

- a **cofibrant–fibrant object** is an object that is both cofibrant and fibrant.

A **model category** is a locally small category  $\mathcal{M}$  that is equipped with a model structure and satisfies the additional axiom below:

**CM1.**  $\mathcal{M}$  has finite limits and finite colimits.

A **derivable category** is a locally small category  $\mathcal{M}$  that is equipped with a model structure and satisfies the additional axioms below:

**DC0.** For each object  $X$  in  $\mathcal{M}$ , there exist

- a trivial cofibration  $X \rightarrow \hat{X}$  where  $\hat{X}$  is a fibrant object in  $\mathcal{M}$ , and
- a trivial fibration  $\tilde{X} \rightarrow X$  where  $\tilde{X}$  is a cofibrant object in  $\mathcal{M}$ .

**DC1.**  $\mathcal{M}$  has pushouts along morphisms in  $C \cap \mathcal{W}$ , and pullbacks along morphisms in  $\mathcal{W} \cap \mathcal{F}$ ; i.e. given diagrams in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ i \downarrow & & \\ W & & \end{array} \qquad \begin{array}{ccc} & & X \\ & & \downarrow p \\ Y' & \longrightarrow & Y \end{array}$$

if  $i : Z \rightarrow W$  is in  $C \cap \mathcal{W}$ , then the diagram on the left can be completed to a pushout square; and if  $p : X \rightarrow Y$  is in  $\mathcal{W} \cap \mathcal{F}$ , then the diagram on the right can be completed to a pullback square.

**REMARK 4.1.2.** Our definition of ‘cofibrant object’ (resp. ‘fibrant object’) is necessarily non-standard, because we do not always have initial objects (resp. terminal objects). Nonetheless, in a model category, our definitions agree with the standard ones: see [lemma 4.1.15](#).

**Definition 4.1.3.** A **DHK model category** is a model category satisfying the following variants of CM1 and CM5:

**CM1\*.**  $\mathcal{M}$  is complete and cocomplete.

**CM5\*.** The  $(C \cap \mathcal{W}, \mathcal{F})$  and  $(C, \mathcal{W} \cap \mathcal{F})$ -factorisations can be chosen *functorially* in the sense of [definition A.3.24](#).

**REMARK 4.1.4.** Hovey [1999] and Hirschhorn [2003] attribute the stronger definition of ‘model category’ to Dwyer, Hirschhorn and Kan [DHK], hence the name ‘DHK model category’; of course, this is the definition used in the cited works, as well as in [DHKS]. Note also that the definition in [Hovey, 1999] includes the functorial factorisations as a *structure* instead of a property. On the other hand, [DS] and [GJ] use Quillen’s 1969 definition essentially verbatim.

**Example 4.1.5.** Let  $\mathcal{M}$  be any category. The **discrete model structure** on  $\mathcal{M}$  is defined by the following data:

- The weak equivalences are the isomorphisms.
- Every morphism is both a cofibration and a fibration.

It is straightforward to directly verify that the axioms are satisfied in this case. Notice that if  $\mathcal{M}$  is complete and cocomplete, then the discrete model structure even makes  $\mathcal{M}$  into a DHK model category.

**Example 4.1.6.** The **mono–epi model structure** on **Set** is defined by the following data:

- Every morphism is a weak equivalence.
- The cofibrations are the injective maps.
- The fibrations are the surjective maps.

The key observation is that **Set** admits a mono–epi weak factorisation system;<sup>[2]</sup> in fact, we can even choose the mono–epi factorisations functorially: for example, given a map  $f : X \rightarrow Y$ , we may take the cograph factorisation  $X \rightarrow X \amalg Y \rightarrow Y$ , where  $X \rightarrow X \amalg Y$  is the coproduct insertion and  $X \amalg Y \rightarrow Y$  is the map  $(f, \text{id}_Y)$ .

**REMARK 4.1.7.** Let  $\mathcal{M}$  be a category. Then,  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  is a model structure on  $\mathcal{M}$  if and only if  $(\mathcal{F}^{\text{op}}, \mathcal{W}^{\text{op}}, \mathcal{C}^{\text{op}})$  is a model structure on  $\mathcal{M}^{\text{op}}$ .

**Lemma 4.1.8.** *Let  $\mathcal{M}$  be a category equipped with a model structure.*

- *If  $i : Z \rightarrow W$  is a cofibration in  $\mathcal{M}$  and  $Z$  is a cofibrant object, then  $W$  is also a cofibrant object.*
- *If  $p : X \rightarrow Y$  is a fibration in  $\mathcal{M}$  and  $Y$  is a fibrant object, then  $X$  is also a fibrant object.*

*Proof.* The two claims are formally dual; we will prove the first version.

---

[2] — not to be confused with the epi–mono *orthogonal* factorisation system!

Let  $p : X \rightarrow Y$  be a trivial fibration in  $\mathcal{M}$  and let  $w : W \rightarrow Y$  be any morphism in  $\mathcal{M}$ . Since  $Z$  is cofibrant, there exists a morphism  $z : Z \rightarrow X$  such that the diagram below commutes,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ i \downarrow & & \downarrow p \\ W & \xrightarrow{w} & Y \end{array}$$

and since  $i : Z \rightarrow W$  is a cofibration, axiom CM4 gives a morphism  $s : W \rightarrow X$  such that  $p \circ s = w$ . Thus  $W$  is also cofibrant. ■

**Lemma 4.1.9.** *In a category equipped with a model structure:*

- *Every trivial fibration with cofibrant codomain is a split epimorphism.*
- *Every trivial cofibration with fibrant domain is a split monomorphism.*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $p : X \rightarrow Y$  be a trivial fibration, and suppose  $Y$  is cofibrant. Consider the following diagram in  $\mathcal{M}$ :

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

By definition, there exists a morphism  $s : Y \rightarrow X$  such that  $p \circ s = \text{id}_Y$ . This shows that  $p : X \rightarrow Y$  is a split epimorphism. ■

**Lemma 4.1.10.** *Let  $\mathcal{M}$  be a category equipped with a model structure. The following are equivalent for a morphism  $f$  in  $\mathcal{M}$ :*

- (i)  *$f$  is a weak equivalence in  $\mathcal{M}$ .*
- (ii) *For any factorisation  $f = p \circ j$  in  $\mathcal{M}$  where  $p$  is a fibration and  $j$  is a trivial cofibration,  $p$  must be a trivial fibration.*
- (iii) *There exist a trivial cofibration  $j$  and a trivial fibration  $q$  such that  $f = q \circ j$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Use axiom CM2.

(ii)  $\Rightarrow$  (iii). Use axiom CM5.

(iii)  $\Rightarrow$  (i). Use axiom CM2 again. ■

**Lemma 4.1.11.** *Let  $\mathcal{M}$  be a category with a pair of weak factorisation systems  $(\mathcal{C}', \mathcal{F})$  and  $(\mathcal{C}, \mathcal{F}')$ . Assume  $\mathcal{W}$  is a subensemble of  $\text{mor } \mathcal{C}$  satisfying the following condition:*

$$\mathcal{W} \subseteq \{q \circ j \mid j \in \mathcal{C}', q \in \mathcal{F}'\}$$

(i)  $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$ .

(ii) If  $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$ , then  $\mathcal{F}' \subseteq \mathcal{F}$  and  $\mathcal{C} \cap \mathcal{W} = \mathcal{C}'$ .

*Dually:*

(i')  $\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{F}'$ .

(ii') If  $\mathcal{F}' \subseteq \mathcal{W} \cap \mathcal{F}$ , then  $\mathcal{C}' \subseteq \mathcal{C}$  and  $\mathcal{W} \cap \mathcal{F} = \mathcal{F}'$ .

*In particular, assuming  $\mathcal{C}' \cup \mathcal{F}' \subseteq \mathcal{W}$ , we have  $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$  if and only if  $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ .*

*Proof.* (i). Suppose  $i : X \rightarrow Z$  is in  $\mathcal{C} \cap \mathcal{W}$ ; then there must be  $j : X \rightarrow Y$  in  $\mathcal{C}'$  and  $q : Y \rightarrow Z$  in  $\mathcal{F}'$  such that  $i = q \circ j$ , and so we have the commutative diagram shown below:

$$\begin{array}{ccc} X & \xrightarrow{j} & Y \\ i \downarrow & & \downarrow q \\ Z & \xrightarrow{\text{id}} & Z \end{array}$$

Since  $i \square q$ ,  $i$  must be a retract of  $j$ ; hence, by [proposition A.3.14](#),  $i$  is in  $\mathcal{C}'$ , and therefore  $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$ .

(ii). If we know  $\mathcal{C}' \subseteq \mathcal{C}$ , then  $\mathcal{F}' \subseteq \mathcal{F}$  by [proposition A.3.3](#), and  $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$ , so from claim (i) it follows that  $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$ . ■

**Theorem 4.1.12.** *Let  $\mathcal{M}$  be a category and let  $\mathcal{C}, \mathcal{W}, \mathcal{F}$  be subclasses of  $\text{mor } \mathcal{M}$ . Assuming  $\mathcal{M}$  has either pushouts along morphisms in  $\mathcal{C} \cap \mathcal{W}$  or pullbacks along morphisms in  $\mathcal{W} \cap \mathcal{F}$ , the following are equivalent:*

(i)  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  is a model structure for  $\mathcal{M}$ .

(ii)  $\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{M}$ , and both  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  and  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  are weak factorisation systems for  $\mathcal{M}$ .

*Proof.* (i)  $\Rightarrow$  (ii). Axiom CM5 says that  $(C \cap \mathcal{W}, \mathcal{F})$ - and  $(C, \mathcal{W} \cap \mathcal{F})$ -factorisations exist, and axiom CM4 says we have the following inclusions:

$$\begin{array}{ll} C \subseteq \square(\mathcal{W} \cap \mathcal{F}) & \mathcal{W} \cap \mathcal{F} \subseteq C^\square \\ \mathcal{F} \subseteq (C \cap \mathcal{W})^\square & C \cap \mathcal{W} \subseteq \square\mathcal{F} \end{array}$$

Axiom CM3 implies each one of  $C, \mathcal{F}, C \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F}$  is closed under retracts, so we may apply [proposition A.3.16](#) to deduce that both  $(C, \mathcal{W} \cap \mathcal{F})$  and  $(C \cap \mathcal{W}, \mathcal{F})$  are indeed weak factorisation systems.

(ii)  $\Rightarrow$  (i). We may deduce from [proposition A.3.14](#) that  $C$  and  $\mathcal{F}$  are closed under retracts, and it remains to be shown that  $\mathcal{W}$  is closed under retracts. The two cases are formally dual; we will assume that  $\mathcal{M}$  has pushouts along morphisms in  $C \cap \mathcal{W}$ .

Let  $w : X \rightarrow Z$  be a morphism in  $\mathcal{W}$ , and consider a commutative diagram of the form below:

$$\begin{array}{ccccc} & & \text{id} & & \\ & & \curvearrowright & & \\ X' & \xrightarrow{s_X} & X & \xrightarrow{r_X} & X' \\ w' \downarrow & & w \downarrow & & \downarrow w' \\ Z' & \xrightarrow{s_Z} & Z & \xrightarrow{r_Z} & Z' \\ & & \curvearrowleft & & \\ & & \text{id} & & \end{array}$$

Choose a  $(C \cap \mathcal{W}, \mathcal{F})$  factorisation for  $w'$ , say  $w' = p' \circ j'$ , with  $j' : X' \rightarrow Y'$  in  $C \cap \mathcal{W}$  and  $p' : Y' \rightarrow Z'$  in  $\mathcal{F}$ . Construct the following commutative diagram,

$$\begin{array}{ccccc} X' & \xrightarrow{s_X} & X & \xrightarrow{r_X} & X' \\ j' \downarrow & & u \downarrow & & \downarrow j' \\ Y' & \xrightarrow{s_Y} & Y & \xrightarrow{r_Y} & Y' \\ p' \downarrow & & v \downarrow & & \downarrow p' \\ Z' & \xrightarrow{s_Z} & Z & \xrightarrow{r_Z} & Z' \end{array}$$

where the top left square is a pushout square,  $v \circ u = w$ , and  $r_Y \circ s_Y = \text{id}_Y$ . Since  $C \cap \mathcal{W}$  is closed under pushouts,  $u$  is also in  $C \cap \mathcal{W}$ , and by the 2-out-of-3

property,  $v$  is in  $\mathcal{W}$ . Thus,  $p'$  is in  $\mathcal{F}$  and is a retract of  $v$ :

$$\begin{array}{ccccc}
 & & \text{id} & & \\
 & \curvearrowright & & \curvearrowleft & \\
 Y' & \xrightarrow{s_Y} & Y & \xrightarrow{r_Y} & Y' \\
 p' \downarrow & & v \downarrow & & \downarrow p' \\
 Z' & \xrightarrow{s_Z} & Z & \xrightarrow{r_Z} & Z' \\
 & \curvearrowleft & & \curvearrowright & \\
 & & \text{id} & & 
 \end{array}$$

Using the 2-out-of-3 property again, choose a  $(\mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F})$ -factorisation of  $v$ , say  $v = q \circ j$ . Since  $j \square p'$ , there exists a morphism  $r$  such that  $r \circ j = r_Y$  and  $p' \circ r = r_Z \circ q$ . Putting  $s = j \circ s_Y$ , we obtain  $r \circ s = r_Y \circ s_Y = \text{id}_Y$ ; thus  $p'$  is a retract of  $q$  and must therefore be in  $\mathcal{F} \cap \mathcal{W}$ . Hence,  $w' = p' \circ j'$  is in  $\mathcal{W}$ . ■

**Corollary 4.1.13.** *Let  $\mathcal{M}$  be a derivable category.*

- *Pushouts of trivial cofibrations along any morphism in  $\mathcal{M}$  exist, and any such is a trivial cofibration.*
- *Pullbacks of trivial fibrations along any morphism in  $\mathcal{M}$  exist, and any such is a trivial fibration.*

*Proof.* Apply [proposition A.3.14](#). ■

**REMARK 4.1.14.** May and Ponto [2012, Ch. 14] define ‘model category’ to mean a complete and cocomplete locally small category  $\mathcal{M}$  equipped with a triple of classes  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  satisfying condition (ii) of the above proposition; if the two weak factorisation systems can be extended to a pair of functorial factorisation systems, then this is a DHK model category.

**Lemma 4.1.15.** *Let  $\mathcal{M}$  be a category equipped with a model structure. If  $\mathcal{M}$  has an initial object  $0$ , then the following are equivalent for any object  $W$  in  $\mathcal{M}$ :*

- (i)  *$W$  is a cofibrant object in  $\mathcal{M}$ .*
- (ii) *The unique morphism  $0 \rightarrow W$  has the left lifting property with respect to all trivial fibrations in  $\mathcal{M}$ .*
- (iii) *The unique morphism  $0 \rightarrow W$  is a cofibration.*



Dually, if  $\mathcal{M}$  has a terminal object  $1$ , then the following are equivalent for any object  $X$  in  $\mathcal{M}$ :

- (i')  $X$  is a fibrant object in  $\mathcal{M}$ .
- (ii') The unique morphism  $X \rightarrow 1$  has the right lifting property with respect to all trivial fibrations in  $\mathcal{M}$ .
- (iii') The unique morphism  $X \rightarrow 1$  is a fibration.

*Proof.* (i)  $\Leftrightarrow$  (ii). Obvious.

(ii)  $\Leftrightarrow$  (iii). By [theorem 4.1.12](#), any morphism that has the left lifting property with respect to all trivial fibrations must be a cofibration. ■

**Proposition 4.1.16.** *Let  $\mathcal{M}$  be a category equipped with a model structure. If  $\mathcal{M}$  satisfies axiom DC1 and has both an initial object and a terminal object, then  $\mathcal{M}$  is a derivable category. In particular, any model category is a derivable category.*

*Proof.* Use axiom CM5 to factorise the unique morphisms  $0 \rightarrow X$  and  $X \rightarrow 1$ , and then apply [lemma 4.1.15](#) to deduce that axiom DC0 is satisfied. ■

**Lemma 4.1.17.** *Let  $\mathcal{M}$  be a category equipped with a model structure and let  $A$  be an object in  $\mathcal{M}$ .*

- (i) *The slice category  $\mathcal{M}_{/A}$  (resp.  ${}^A/\mathcal{M}$ ) admits a **slice model structure**, where a morphism in  $\mathcal{M}_{/A}$  (resp.  ${}^A/\mathcal{M}$ ) is a weak equivalence, cofibration, or fibration if it is so in  $\mathcal{M}$ .*
- (ii) *The slice category  $\mathcal{M}_{/A}$  (resp.  ${}^A/\mathcal{M}$ ), equipped with the slice model structure, is a derivable category if  $\mathcal{M}$  is a derivable category.*
- (iii) *The slice category  $\mathcal{M}_{/A}$  (resp.  ${}^A/\mathcal{M}$ ), equipped with the slice model structure, is a model category if  $\mathcal{M}$  is a model category.*

*Proof.* The two halves of each claim are formally dual; we will prove the versions for  $\mathcal{M}_{/A}$ .

- (i). Use [lemmas 3.1.6](#) and [A.3.13](#).

(ii).  $\mathcal{M}_{/A}$  always has a terminal object, so axiom CM5 and lemma 4.1.15 imply one half of axiom DC0 in  $\mathcal{M}_{/A}$ ; for the other half, we may use axiom DC0 in  $\mathcal{M}$  directly.

It is well known that the projection functor  $\mathcal{M}_{/A} \rightarrow \mathcal{M}$  preserves and reflects pullbacks and pushouts, so pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in  $\mathcal{M}_{/A}$  if pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in  $\mathcal{M}$ . Thus  $\mathcal{M}_{/A}$  satisfies axiom DC1 if  $\mathcal{M}$  does.

(iii). The argument above also shows that  $\mathcal{M}_{/A}$  has finite limits and colimits if  $\mathcal{M}$  does. ■

**Lemma 4.1.18.** *Let  $(\mathcal{M}_i \mid i \in I)$  be a sequence of categories equipped with model structures.*

- (i) *The product category  $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$  admits a **product model structure**, where a morphism in  $\mathcal{M}$  is a weak equivalence, cofibration, or fibration if each component is so.*
- (ii)  *$\mathcal{M}$ , equipped with the product model structure, is a derivable category if each  $\mathcal{M}_i$  is a derivable category.*
- (iii)  *$\mathcal{M}$ , equipped with the product model structure, is a model category if each  $\mathcal{M}_i$  is a model category.*

*Proof.* Everything can be checked componentwise. ◆

**Lemma 4.1.19.** *Let  $\mathcal{M}$  be a category equipped with a model structure.*

- *Suppose we have a commutative diagram in  $\mathcal{M}$  of the form below,*

$$\begin{array}{ccc}
 & Z & \\
 i' \swarrow & & \searrow i \\
 W' & \xrightarrow{g} & W
 \end{array}$$

*where  $i : Z \rightarrow W$  and  $i' : Z \rightarrow W'$  are cofibrations and  $g : W' \rightarrow W$  is a weak equivalence. If a fibration  $p : X \rightarrow Y$  has the right lifting property with respect to  $i' : Z \rightarrow W'$ , then  $p : X \rightarrow Y$  also has the right lifting property with respect to  $i : Z \rightarrow W$ .*

- Suppose we have a commutative diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ & \searrow p & \swarrow p' \\ & & Y \end{array}$$

where  $p : X \rightarrow Y$  and  $p' : X' \rightarrow Y$  are fibrations and  $f : X \rightarrow X'$  is a weak equivalence. If a cofibration  $i : Z \rightarrow W$  has the left lifting property with respect to  $p' : X' \rightarrow Y$ , then  $i : Z \rightarrow W$  also has the left lifting property with respect to  $p : X \rightarrow Y$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Consider the following lifting problem in  $\mathcal{M}$ :

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ i \downarrow & & \downarrow p \\ W & \xrightarrow{w} & Y \end{array}$$

Suppose  $p : X \rightarrow Y$  is a fibration that has the right lifting property with respect to  $i' : Z \rightarrow W'$ . Then there must exist a morphism  $h' : W' \rightarrow X$  such that the diagram shown below commutes:

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ i' \downarrow & \nearrow h' & \downarrow p \\ W' & \xrightarrow{w \circ g} & Y \end{array}$$

Using lemma 4.1.10, choose a trivial cofibration  $j : W' \rightarrow W''$  and a trivial fibration  $q : W'' \rightarrow W$  such that  $g = q \circ j$ . Since  $p : X \rightarrow Y$  is a fibration, there is a morphism  $h'' : W'' \rightarrow X$  making the following diagram commute:

$$\begin{array}{ccc} W' & \xrightarrow{h'} & X \\ j \downarrow & \nearrow h'' & \downarrow p \\ W'' & \xrightarrow{w \circ q} & Y \end{array}$$

On the other hand,  $q : W'' \rightarrow W$  is a trivial fibration, so there is a morphism  $s : W \rightarrow W''$  such that the diagram shown below commutes:

$$\begin{array}{ccc} Z & \xrightarrow{j \circ i'} & W'' \\ i \downarrow & \nearrow s & \downarrow q \\ W & \xrightarrow{\text{id}} & W \end{array}$$

Let  $h = h'' \circ s$ . Then,

$$h \circ i = h'' \circ s \circ i = h'' \circ j \circ i' = h' \circ i' = z$$

and since  $q \circ s = \text{id}$ ,

$$p \circ h = p \circ h'' \circ s = w \circ q \circ s = w$$

so  $h : W \rightarrow X$  is indeed a solution to the lifting problem. ■

**Definition 4.1.20.** Let  $X$  be an object in a category  $\mathcal{M}$  equipped with a model structure.

- A **cofibrant replacement** for  $X$  is a pair  $(\tilde{X}, p)$  where  $\tilde{X}$  is a cofibrant object in  $\mathcal{M}$  and  $p$  is a weak equivalence  $\tilde{X} \rightarrow X$ .
- A **fibrant replacement** for  $X$  is a pair  $(\hat{X}, i)$  where  $\hat{X}$  is a fibrant object in  $\mathcal{M}$  and  $i$  is a weak equivalence  $X \rightarrow \hat{X}$ .
- A **fibrant cofibrant replacement** for  $X$  is a cofibrant replacement  $(\tilde{X}, p)$  where  $p : \tilde{X} \rightarrow X$  is a trivial fibration.
- A **cofibrant fibrant replacement** for  $X$  is a fibrant replacement  $(\hat{X}, i)$  where  $i : X \rightarrow \hat{X}$  is a trivial cofibration.

**Definition 4.1.21.** Let  $\mathcal{M}$  be a category equipped with a model structure.

- A **cofibrant replacement functor** for  $\mathcal{M}$  is a pair  $(Q, p)$ , where  $Q$  is an endofunctor on  $\mathcal{M}$  and  $p$  is a natural transformation  $Q \Rightarrow \text{id}_{\mathcal{M}}$  such that, for every object  $X$  in  $\mathcal{M}$ ,  $(QX, p_X)$  is a cofibrant replacement for  $X$ .
- A **fibrant replacement functor** for  $\mathcal{M}$  is a pair  $(R, i)$ , where  $R$  is an endofunctor on  $\mathcal{M}$  and  $i$  is a natural transformation  $\text{id}_{\mathcal{M}} \Rightarrow R$  such that, for every object  $X$  in  $\mathcal{M}$ ,  $(RX, i_X)$  is a fibrant replacement for  $X$ .
- A **fibrant cofibrant replacement functor** for  $\mathcal{M}$  is a pair  $(Q, p)$ , where  $Q$  is an endofunctor on  $\mathcal{M}$  and  $p$  is a natural transformation  $Q \Rightarrow \text{id}_{\mathcal{M}}$  such that, for every object  $X$  in  $\mathcal{M}$ ,  $(QX, p_X)$  is a fibrant cofibrant replacement for  $X$ .

- A **cofibrant fibrant replacement functor** for  $\mathcal{M}$  is a pair  $(R, i)$ , where  $R$  is an endofunctor on  $\mathcal{M}$  and  $i$  is a natural transformation  $\text{id}_{\mathcal{M}} \Rightarrow R$  such that, for every object  $X$  in  $\mathcal{M}$ ,  $(RX, i_X)$  is a cofibrant fibrant replacement for  $X$ .

REMARK 4.1.22. Note that a fibrant cofibrant replacement for  $X$  is precisely a cofibrant replacement for  $X$  that is fibrant as an object in  $\mathcal{M}_{/X}$ , and a cofibrant fibrant replacement for  $X$  is precisely a fibrant replacement for  $X$  that is cofibrant as an object in  ${}^X/\mathcal{M}$ .

Moreover, if  $X$  is fibrant and  $(\tilde{X}, p)$  is a fibrant cofibrant replacement for  $X$ , then  $\tilde{X}$  is both fibrant and cofibrant in  $\mathcal{M}$ , and if  $X$  is cofibrant and  $(\hat{X}, i)$  is a cofibrant fibrant replacement for  $X$ , then  $\hat{X}$  is both cofibrant and fibrant in  $\mathcal{M}$ .

**Proposition 4.1.23.**

- (i) *Any object in a derivable category has both a fibrant cofibrant replacement and a cofibrant fibrant replacement.*
- (ii) *Any DHK model category has both a fibrant cofibrant replacement functor and a cofibrant fibrant replacement functor.*

*Proof.* (i). This is axiom DC0.

(ii). Use axiom CM5\* to factorise the unique natural transformations  $\Delta 0 \Rightarrow \text{id}_{\mathcal{M}}$  and  $\text{id}_{\mathcal{M}} \Rightarrow \Delta 1$ , and then apply [lemma 4.1.15](#). ■

It should go without saying that any two cofibrant or fibrant replacements for a fixed object are weakly equivalent; however, more is true:

**Lemma 4.1.24.** *Let  $X$  be an object in a derivable category  $\mathcal{M}$ .*

- *Any two cofibrant replacements for  $X$  are weakly equivalent as objects in the slice model category  $\mathcal{M}_{/X}$ .*
- *Any two fibrant replacements for  $X$  are weakly equivalent as objects in the slice model category  ${}^X/\mathcal{M}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $(\tilde{X}, p)$  be a fibrant cofibrant replacement for  $X$ ; such exist, by [proposition 4.1.23](#). Let  $(\tilde{X}', p')$  be any cofibrant replacement for  $X$ . Then,  $p : \tilde{X} \rightarrow X$  is a trivial fibration, so there exists a morphism  $f : \tilde{X}' \rightarrow \tilde{X}$  such that  $p \circ f = p'$ . The 2-out-of-3 property of weak equivalences implies any such  $f : \tilde{X}' \rightarrow \tilde{X}$  is

a weak equivalence, so we may deduce that every cofibrant replacement for  $X$  is weakly equivalent to  $(\tilde{X}, p)$  as objects in  $\mathcal{M}_{/X}$ . ■

In the presence of functorial cofibrant and fibrant replacements, we can say something stronger still:

**Proposition 4.1.25.** *Let  $X$  be an object in a derivable category  $\mathcal{M}$ .*

- *If  $\mathcal{M}$  has a cofibrant replacement functor, then the full subcategory of the slice category  $\mathcal{M}_{/X}$  spanned by the cofibrant replacements for  $X$  is homotopically contractible.*
- *If  $\mathcal{M}$  has a fibrant replacement functor, then the full subcategory of the slice category  ${}^X\mathcal{M}$  spanned by the fibrant replacements for  $X$  is homotopically contractible.*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $(Q, p)$  be a cofibrant replacement functor for  $\mathcal{M}$ . Then, for each cofibrant replacement  $(\tilde{X}, q)$  for  $X$ , we have the following commutative diagram in  $\mathcal{M}$ :

$$\begin{array}{ccccc}
 \tilde{X} & \xleftarrow{p_{\tilde{X}}} & Q\tilde{X} & \xrightarrow{Qq} & QX \\
 & \searrow q & \downarrow & \swarrow p_X & \\
 & & X & & 
 \end{array}$$

Thus, the constant functor at  $(QX, p_X)$  is naturally weakly equivalent to the identity functor of the category of cofibrant replacements for  $X$ , and we may then apply [proposition 3.1.31](#) to deduce that it is homotopically contractible. ■

**REMARK 4.1.26.** In other words, cofibrant replacements (resp. fibrant replacements) are homotopically unique in a model category with functorial cofibrant replacements (resp. functorial fibrant replacements).

**Proposition 4.1.27.** *Let  $\mathcal{M}$  be a category with a model structure, let  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  be the model structure on  $\mathcal{M}$ , and let  $\mathcal{N}$  be a full subcategory of  $\mathcal{M}$ .*

- (i) *If  $\mathcal{N}$  is homotopically replete in  $\mathcal{M}$ , then the data*

$$(\mathcal{C} \cap \text{mor } \mathcal{N}, \mathcal{W} \cap \text{mor } \mathcal{N}, \mathcal{F} \cap \text{mor } \mathcal{N})$$

*constitute a model structure on  $\mathcal{N}$ .*

- (ii) If  $W$  is a cofibrant object in  $\mathcal{M}$  and is in  $\mathcal{N}$ , then  $W$  is a projective object in  $\mathcal{N}$  with respect to  $\mathcal{W} \cap \mathcal{F} \cap \text{mor } \mathcal{N}$ ; dually, if  $X$  is a fibrant object in  $\mathcal{M}$  and is in  $\mathcal{N}$ , then  $X$  is an injective object in  $\mathcal{N}$  with respect to  $\mathcal{C} \cap \mathcal{W} \cap \text{mor } \mathcal{N}$ .
- (iii) If  $\mathcal{M}$  is a derivable category, then so is  $\mathcal{N}$  when equipped with the above model structure.

*Proof.* (i). [Lemma 3.1.7](#) implies that axiom CM2 is satisfied. Since  $\mathcal{N}$  is a full subcategory of  $\mathcal{M}$ , the data  $(\mathcal{C} \cap \text{mor } \mathcal{N}, \mathcal{W} \cap \text{mor } \mathcal{N}, \mathcal{F} \cap \text{mor } \mathcal{N})$  satisfy axioms CM3 and CM4 because  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  do. Finally, for axiom CM5, we appeal to the hypothesis that  $\mathcal{N}$  is homotopically replete.

(ii). This follows from the assumption that  $\mathcal{N}$  is a full subcategory of  $\mathcal{M}$ .

(iii). It remains to be shown that pushouts along trivial cofibrations and pullbacks along trivial fibrations exist in  $\mathcal{N}$ . For this, simply apply [corollary 4.1.13](#) to the hypothesis that  $\mathcal{N}$  is homotopically replete and full.  $\blacksquare$

**Definition 4.1.28.** The **Quillen homotopy category** (or, more simply, **homotopy category**) of a derivable category  $\mathcal{M}$  is the category  $\text{Ho } \mathcal{M}$  obtained by freely inverting the weak equivalences in  $\mathcal{M}$ , as in [definition A.4.9](#).

**Definition 4.1.29.** A **saturated derivable category** is a derivable category that is saturated as a category with weak equivalences.

**Theorem 4.1.30.** Let  $\mathcal{M}$  be a derivable category and let  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  be the localising functor.

- (i) Let  $\mathcal{U}$  and  $\mathcal{V}$  be the classes of trivial cofibrations and trivial fibrations in  $\mathcal{M}$ , respectively. Then  $\mathcal{M}$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ , which is functorial if  $\mathcal{M}$  satisfies axiom CM5\*.
- (ii) Let  $X$  and  $Y$  be objects in  $\mathcal{M}$ , let  $v : X \rightarrow \tilde{X}$  and  $v' : X \rightarrow \tilde{X}'$  be trivial fibrations, let  $u : Y \rightarrow \hat{Y}$  and  $u' : Y \rightarrow \hat{Y}'$  be trivial cofibrations, and let  $f : \tilde{X} \rightarrow \hat{Y}$  and  $f' : \tilde{X}' \rightarrow \hat{Y}'$  be morphisms in  $\mathcal{M}$ . Then,

$$\gamma(u)^{-1} \circ \gamma(f) \circ \gamma(v)^{-1} = \gamma(u')^{-1} \circ \gamma(f') \circ \gamma(v')^{-1}$$

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if and only if there exists a commutative diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccccccc}
 X & \xleftarrow{v} & \tilde{X} & \xrightarrow{f} & \hat{Y} & \xleftarrow{u} & Y \\
 \parallel & & \uparrow v_1 & & \uparrow v_2 & & \parallel \\
 X & \xleftarrow{v_3} & \bullet & \xrightarrow{f_3} & \bullet & \xleftarrow{u_3} & Y \\
 \parallel & & \downarrow w_1 & & \downarrow w_2 & & \parallel \\
 X & \xleftarrow{v_4} & \bullet & \xrightarrow{f_4} & \bullet & \xleftarrow{u_4} & Y \\
 \parallel & & \uparrow u_1 & & \uparrow u_2 & & \parallel \\
 X & \xleftarrow{v'} & \tilde{X}' & \xrightarrow{f'} & \hat{Y}' & \xleftarrow{u'} & Y
 \end{array}$$

where  $u_1, u_2, u_3, u_4$  are trivial cofibrations,  $v_1, v_2, v_3, v_4$  are trivial fibrations, and  $w_1, w_2$  are weak equivalences. In any such diagram,  $u_2$  is a split monomorphism if  $\tilde{X}$  is cofibrant, and  $v_1$  is a split epimorphism if  $\hat{Y}'$  is fibrant.

- (iii)  $\mathcal{M}$  is a saturated derivable category if and only if the weak equivalences in  $\mathcal{M}$  have the 2-out-of-6 property.
- (iv) If  $X$  is a cofibrant object in  $\mathcal{M}$  and  $Y$  is a fibrant object in  $\mathcal{M}$ , then the hom-set map  $\mathcal{M}(X, Y) \rightarrow \text{Ho } \mathcal{M}(\gamma X, \gamma Y)$  is surjective.
- (v)  $\text{Ho } \mathcal{M}$  is a locally small category.

*Proof.* (i). Axioms CM2 and CM5 imply axiom A1 is satisfied, and axioms A2 and A3 follow from the above claims; that we get a functorial three-arrow calculus under axiom CM5\* is an obvious consequence of the universal property of pushouts and pullbacks.

(ii). This is a special case of the fundamental theorem of three-arrow calculi (3.6.9), plus lemma 4.1.9.

(iii). Apply proposition 3.6.10 and lemma A.4.14.

(iv). Consider a zigzag of the following form in  $\mathcal{M}$ ,

$$X \xleftarrow{v} X' \xrightarrow{f'} Y' \xleftarrow{u} Y$$

where  $u : Y \rightarrow Y'$  is a trivial cofibration and  $v : X' \rightarrow X$  is a trivial fibration. Let  $\bar{f} = \gamma(u)^{-1} \circ \gamma(f') \circ \gamma(v)^{-1}$  be the corresponding morphism in  $\text{Ho } \mathcal{M}$ ; note



that the fundamental theorem of three-arrow calculi says that every morphism  $\gamma X \rightarrow \gamma Y$  in  $\text{Ho } \mathcal{M}$  is of this form. Suppose  $X$  is cofibrant and  $Y$  is fibrant. Then [lemma 4.1.9](#) says  $u$  is a split monomorphism and  $v$  is a split epimorphism, so choose  $r : Y' \rightarrow Y$  and  $s : X \rightarrow X'$  such that  $r \circ u = \text{id}_Y$  and  $v \circ s = \text{id}_X$ . Since  $\gamma(u)$  and  $\gamma(v)$  are isomorphisms in  $\text{Ho } \mathcal{M}$ , we must have  $\gamma(u)^{-1} = \gamma(r)$  and  $\gamma(v)^{-1} = \gamma(s)$ . Hence, taking  $f = r \circ f' \circ s$ , we have  $\bar{f} = \gamma(f)$ , as required.

(v). By [proposition 4.1.23](#), every object in  $\mathcal{M}$  is weakly equivalent to both a cofibrant object and a fibrant object, so we may deduce that  $\text{Ho } \mathcal{M}$  is locally small from claim (iii). ■

**Corollary 4.1.31.** *Let  $\mathcal{M}$  be a derivable category. For any two objects  $X$  and  $Y$  in  $\mathcal{M}$ , every morphism  $X \rightarrow Y$  in  $\text{Ho } \mathcal{M}$  can be represented by a zigzag of the following form,*

$$X \xleftarrow{p} \tilde{X} \longrightarrow \hat{Y} \xleftarrow{i} Y$$

where  $(\tilde{X}, p)$  is any cofibrant replacement for  $X$  and  $(\hat{Y}, i)$  is any fibrant replacement for  $Y$ . ■

**Lemma 4.1.32.** *Let  $\mathcal{M}$  be a derivable category and let  $\mathcal{C}$  be a relative category where  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property and the special 2-out-of-4 property.*

- *Let  $\mathcal{M}_c$  be the full subcategory of cofibrant objects in  $\mathcal{M}$ . If a functor  $F : \mathcal{M}_c \rightarrow \mathcal{C}$  sends trivial cofibrations in  $\mathcal{M}_c$  to weak equivalences in  $\mathcal{C}$ , then  $F$  preserves all weak equivalences.*
- *Let  $\mathcal{M}_f$  be the full subcategory of fibrant objects in  $\mathcal{M}$ . If a functor  $G : \mathcal{M}_f \rightarrow \mathcal{C}$  sends trivial fibrations in  $\mathcal{M}_f$  to weak equivalences in  $\mathcal{C}$ , then  $G$  preserves all weak equivalences.*

*Proof.* The two claims are formally dual; we will prove the first version.

Axioms CM2 and CM5 imply that every weak equivalence in  $\mathcal{M}$  can be factored as a trivial cofibration followed by a trivial fibration, so it is enough to show that  $F$  sends trivial fibrations in  $\mathcal{M}_c$  to weak equivalences in  $\mathcal{C}$ . Let  $p : X \rightarrow Y$  be a trivial fibration in  $\mathcal{M}_c$ .  $Y$  is cofibrant, so [lemma 4.1.9](#) says  $p : X \rightarrow Y$  has a section  $s : Y \rightarrow X$ .

Let  $e = s \circ p$ . Since  $p : X \rightarrow Y$  is a trivial fibration, we may form a pullback square in  $\mathcal{M}$  of the following form:

$$\begin{array}{ccc} K & \xrightarrow{k_1} & X \\ k_0 \downarrow & & \downarrow p \\ X & \xrightarrow{p} & Y \end{array}$$

There is then a unique morphism  $\Delta : X \rightarrow K$  such that  $k_0 \circ \Delta = k_1 \circ \Delta = \text{id}_X$ . Since  $k_0 : K \rightarrow X$  is a trivial fibration (by [corollary 4.1.13](#)),  $\Delta : X \rightarrow K$  is a weak equivalence in  $\mathcal{M}$  and therefore factorises as  $q \circ j$  for some trivial cofibration  $j : X \rightarrow \tilde{K}$  and some trivial fibration  $q : \tilde{K} \rightarrow K$ ; note that  $\tilde{K}$  is a cofibrant object. There is also a unique morphism  $t : X \rightarrow K$  such that  $k_0 \circ t = \text{id}_X$  and  $k_1 \circ t = e$ ; and  $X$  is a cofibrant object, so there exists a morphism  $h : X \rightarrow \tilde{K}$  such that  $q \circ h = t$ . Taking  $q_0 = k_0 \circ q$  and  $q_1 = k_1 \circ q$ , we obtain the following commutative diagram in  $\mathcal{M}_c$ :

$$\begin{array}{ccccc} X & \xleftarrow{\text{id}} & X & \xrightarrow{e} & X \\ \parallel & & \downarrow h & & \parallel \\ X & \xleftarrow{q_0} & \tilde{K} & \xrightarrow{q_1} & X \\ \parallel & & \uparrow j & & \parallel \\ X & \xleftarrow{\text{id}} & X & \xrightarrow{\text{id}} & X \end{array}$$

Consider the image of the above diagram in  $\mathcal{C}$ . By hypothesis,  $Fj : FX \rightarrow F\tilde{K}$  is a weak equivalence in  $\mathcal{C}$ , and by repeatedly applying the 2-out-of-3 property of  $\text{weq } \mathcal{C}$ , we may deduce that  $Fe : FX \rightarrow FX$  is a weak equivalence in  $\mathcal{C}$  as well. But  $\text{weq } \mathcal{C}$  has the special 2-out-of-4 property, and  $Fe = Fs \circ Fp$ , so we may conclude that  $Fp : FX \rightarrow FY$  is a weak equivalence in  $\mathcal{C}$ , as required.  $\blacksquare$

**Proposition 4.1.33.** *Let  $\mathcal{M}$  be a derivable category. Let  $\mathcal{M}_c$  be the full subcategory of cofibrant objects in  $\mathcal{M}$ .*

- (i)  $\mathcal{M}_c$ , considered as a relative category with trivial cofibrations as weak equivalences, admits a calculus of cospans.
- (ii) The localisation of  $\mathcal{M}_c$  with respect to trivial cofibrations is isomorphic to the localisation of  $\mathcal{M}_c$  with respect to all weak equivalences.

(iii) Every morphism  $X \rightarrow Y$  in  $\text{Ho } \mathcal{M}_c$  can be represented by a cycle in  $\mathcal{M}_c$  of the form below,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y$$

where  $(\hat{Y}, i)$  is any cofibrant fibrant replacement for  $Y$ .

Dually, let  $\mathcal{M}_f$  be the full subcategory of fibrant objects in  $\mathcal{M}$ .

(i')  $\mathcal{M}_f$ , considered as a relative category with trivial fibrations as weak equivalences, admits a calculus of spans.

(ii') The localisation of  $\mathcal{M}_f$  with respect to trivial fibrations is isomorphic to the localisation of  $\mathcal{M}_c$  with respect to all weak equivalences.

(iii') Every morphism  $X \rightarrow Y$  in  $\text{Ho } \mathcal{M}_f$  can be represented by a cocycle in  $\mathcal{M}_f$  of the form below,

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} Y$$

where  $(\tilde{X}, p)$  is any fibrant cofibrant replacement for  $X$ .

*Proof.* (i). This is an immediate consequence of [corollary 4.1.13](#).

(ii). Suppose  $F : \mathcal{M}_c \rightarrow \mathcal{C}$  is a functor that sends trivial cofibrations in  $\mathcal{M}_c$  to isomorphisms in  $\mathcal{C}$ . It is clear that isomorphisms have the special 2-out-of-4 property, so we may apply [lemma 4.1.32](#) to deduce that  $F$  sends weak equivalences in  $\mathcal{M}_c$  to isomorphisms in  $\mathcal{C}$  as well. Hence, any localisation of  $\mathcal{M}_c$  with respect to trivial cofibrations must also be a localisation of  $\mathcal{M}_c$  with respect to weak equivalences.

(iii). The fundamental theorem of calculi of cospans ([3.5.6](#)) says every morphism  $X \rightarrow Y$  in  $\text{Ho } \mathcal{M}_c$  can be represented by a cycle in  $\mathcal{M}_c$  of the form below,

$$X \xrightarrow{g} Y' \xleftarrow{u} Y$$

where  $u : Y \rightarrow Y'$  is a trivial cofibration, and that two such cycles represent the same morphism if and only if they are in the same connected component of the cycle category  $\mathcal{M}_c^{\rightarrow\sim}(X, Y)$ . Let  $(\hat{Y}, i)$  be any cofibrant fibrant replacement for  $Y$ . Since  $u : Y \rightarrow Y'$  is a trivial cofibration and  $\hat{Y}$  is fibrant, axiom CM4 yields

a morphism  $h : Y' \rightarrow \hat{Y}$  such that  $h \circ u = i$ . Taking  $f = h \circ g$ , we have the following commutative diagram in  $\mathcal{M}_c$ :

$$\begin{array}{ccccc} X & \xrightarrow{g} & Y' & \xleftarrow{u} & Y \\ \parallel & & \downarrow h & & \parallel \\ X & \xrightarrow{f} & \hat{Y} & \xleftarrow{i} & Y \end{array}$$

Thus, the cycles  $(u, g)$  and  $(i, f)$  represent the same morphism in  $\text{Ho } \mathcal{M}_c$ . ■

**Proposition 4.1.34.** *Let  $\mathcal{M}$  be a derivable category.*

- *Let  $\mathcal{M}_c$  be the full subcategory of cofibrant objects in  $\mathcal{M}$ . The canonical functor  $\text{Ho } \mathcal{M}_c \rightarrow \text{Ho } \mathcal{M}$  induced by the inclusion  $\mathcal{M}_c \hookrightarrow \mathcal{M}$  is fully faithful and essentially surjective on objects.*
- *Let  $\mathcal{M}_f$  be the full subcategory of fibrant objects in  $\mathcal{M}$ . The canonical functor  $\text{Ho } \mathcal{M}_f \rightarrow \text{Ho } \mathcal{M}$  induced by the inclusion  $\mathcal{M}_f \hookrightarrow \mathcal{M}$  is fully faithful and essentially surjective on objects.*

*Proof.* The two claims are formally dual; we will prove the first version.

It is clear that [proposition 4.1.23](#) implies the functor  $\text{Ho } \mathcal{M}_c \rightarrow \text{Ho } \mathcal{M}$  is essentially surjective on objects; it remains to be shown that the functor is fully faithful. Consider the full subcategory  $\mathcal{M}_{cf}$  spanned by the cofibrant–fibrant objects in  $\mathcal{M}$ . By restricting the localising functors, we obtain the following commutative diagram,

$$\begin{array}{ccc} \mathcal{M}_{cf} & \xlongequal{\quad} & \mathcal{M}_{cf} \\ \downarrow & & \downarrow \\ \text{Ho } \mathcal{M}_c & \longrightarrow & \text{Ho } \mathcal{M} \end{array}$$

where  $\mathcal{M}_{cf} \rightarrow \text{Ho } \mathcal{M}_c$  and  $\mathcal{M}_{cf} \rightarrow \text{Ho } \mathcal{M}$  are essentially surjective on objects. [Theorem 4.1.30](#) implies  $\mathcal{M}_{cf} \rightarrow \text{Ho } \mathcal{M}$  is a full functor, so  $\text{Ho } \mathcal{M}_c \rightarrow \text{Ho } \mathcal{M}$  must also be full.

Now, consider a parallel pair of morphisms in  $\text{Ho } \mathcal{M}_c$ . [Proposition 4.1.33](#) says they can be represented by cycles of the following form,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y \qquad X \xrightarrow{f'} \hat{Y} \xleftarrow{i} Y$$

where  $(\hat{Y}, i)$  is any cofibrant fibrant replacement for  $Y$ . Suppose the two morphisms are equal in  $\text{Ho } \mathcal{M}$ . Then, there must be a commutative diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccccccc}
 X & \xleftarrow{\text{id}} & X & \xrightarrow{f} & \hat{Y} & \xleftarrow{i} & Y \\
 \parallel & & \uparrow v_1 & & \uparrow v_2 & & \parallel \\
 X & \xleftarrow{v_3} & \bullet & \xrightarrow{f_3} & \bullet & \xleftarrow{u_3} & Y \\
 \parallel & & \downarrow w_1 & & \downarrow w_2 & & \parallel \\
 X & \xleftarrow{v_4} & \bullet & \xrightarrow{f_4} & \bullet & \xleftarrow{u_4} & Y \\
 \parallel & & \uparrow u_1 & & \uparrow u_2 & & \parallel \\
 X & \xleftarrow{\text{id}} & X & \xrightarrow{f'} & \hat{Y} & \xleftarrow{i} & Y
 \end{array}$$

where  $u_1, u_2, u_3, u_4$  are trivial cofibrations,  $v_1, v_2, v_3, v_4$  are trivial fibrations, and  $w_1, w_2$  are weak equivalences. Since  $X$  is cofibrant, there exists a morphism  $s$  in  $\mathcal{M}$  such that  $v_1 \circ s = \text{id}_X$ , so (using lemma 4.1.8) we obtain the following commutative diagram in  $\mathcal{M}_c$ :

$$\begin{array}{ccccccc}
 X & \xleftarrow{\text{id}} & X & \xrightarrow{f} & \hat{Y} & \xleftarrow{i} & Y \\
 \parallel & & \uparrow \text{id} & & \uparrow v_2 & & \parallel \\
 X & \xleftarrow{\text{id}} & X & \xrightarrow{f_3 \circ s} & \bullet & \xleftarrow{u_3} & Y \\
 \parallel & & \downarrow w_1 \circ s & & \downarrow w_2 & & \parallel \\
 X & \xleftarrow{v_4} & \bullet & \xrightarrow{f_4} & \bullet & \xleftarrow{u_4} & Y \\
 \parallel & & \uparrow u_1 & & \uparrow u_2 & & \parallel \\
 X & \xleftarrow{\text{id}} & X & \xrightarrow{f'} & \hat{Y} & \xleftarrow{i} & Y
 \end{array}$$

Noting that axiom CM2 implies  $w_1 \circ s$  is a weak equivalence in  $\mathcal{M}_c$ , we may then deduce that the two zigzags also represent the same morphism in  $\text{Ho } \mathcal{M}_c$ . Thus, the functor  $\text{Ho } \mathcal{M}_c \rightarrow \text{Ho } \mathcal{M}$  is indeed faithful. ■

## 4.2 Left and right homotopy

Prerequisites. §4.1.

**Definition 4.2.1.** Let  $X$  be an object in a model category  $\mathcal{M}$ .

- A **cylinder object** for  $X$  is a quadruple  $(\text{Cyl}(X), i_0, i_1, p)$ , where  $\text{Cyl}(X)$  is an object in  $\mathcal{M}$ ,  $p : \text{Cyl}(X) \rightarrow X$  is a weak equivalence, and  $i_0$  and  $i_1$  are sections of  $p$  such that the morphism  $(i_0, i_1) : X + X \rightarrow \text{Cyl}(X)$  is a cofibration.
- A **path object** for  $X$  is a quadruple  $(\text{Path}(X), i, p_0, p_1)$ , where  $\text{Path}(X)$  is an object in  $\mathcal{M}$ ,  $i : X \rightarrow \text{Path}(X)$  is a weak equivalence, and  $p_0$  and  $p_1$  are retractions of  $i$  such that the morphism  $\langle p_0, p_1 \rangle : \text{Path}(X) \rightarrow X \times X$  is a fibration.

**REMARK 4.2.2.** Let  $(\text{Cyl}(X), i_0, i_1, p)$  be a cylinder object for  $X$ . By definition,  $p \circ i_0 = p \circ i_1 = \text{id}_X$ , and  $p$  is a weak equivalence, so by the 2-out-of-3 property,  $i_0$  and  $i_1$  must also be weak equivalences  $X \rightarrow \text{Cyl}(X)$ .

Dually, if  $(\text{Path}(X), i, p_0, p_1)$  is a path object for  $X$ , then  $p_0$  and  $p_1$  must be weak equivalences  $\text{Path}(X) \rightarrow X$ .

**Proposition 4.2.3.** Let  $X$  be an object in a model category  $\mathcal{M}$ .

- There exists a cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  for  $X$ , where the morphism  $p : \text{Cyl}(X) \rightarrow X$  is a trivial fibration.
- There exists a path object  $(\text{Path}(X), i, p_0, p_1)$  for  $X$ , where the morphism  $i : X \rightarrow \text{Path}(X)$  is a trivial cofibration.

*Proof.* Use axioms CM1 and CM5. ■

**Definition 4.2.4.** Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category  $\mathcal{M}$ , let  $(\text{Cyl}(X), i_0, i_1, p)$  be a cylinder object for  $X$ , and let  $(\text{Path}(Y), i, p_0, p_1)$  be a path object for  $Y$ .

- A **left homotopy** from  $f_0$  to  $f_1$  with respect to  $(\text{Cyl}(X), i_0, i_1, p)$  is a morphism  $H : \text{Cyl}(X) \rightarrow Y$  such that  $H \circ i_0 = f_0$  and  $H \circ i_1 = f_1$ .
- A **right homotopy** from  $f_0$  to  $f_1$  with respect to  $(\text{Path}(Y), i, p_0, p_1)$  is a morphism  $H : X \rightarrow \text{Path}(Y)$  such that  $p_0 \circ H = f_0$  and  $p_1 \circ H = f_1$ .
- We say  $f_0$  and  $f_1$  are **left homotopic** if there exists a left homotopy from  $f_0$  to  $f_1$  with respect to some cylinder object for  $X$ .

- We say  $f_0$  and  $f_1$  are **right homotopic** if there exists a right homotopy from  $f_0$  to  $f_1$  with respect to some path object for  $Y$ .

REMARK 4.2.5. If  $f_0$  and  $f_1$  are either left homotopic or right homotopic, then they must represent the same morphism in  $\text{Ho } \mathcal{M}$ . For definiteness, let us write  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  for the localising functor, and suppose  $H : \text{Cyl}(X) \rightarrow Y$  is a left homotopy from  $f_0$  to  $f_1$ . Since  $i_0$  and  $i_1$  are both sections of the weak equivalence  $p : \text{Cyl}(X) \rightarrow X$ , we must have  $\gamma i_0 = (\gamma p)^{-1} = \gamma i_1$ ; but  $f_0 = H \circ i_0$  and  $f_1 = H \circ i_1$ , so indeed  $\gamma f_0 = \gamma f_1$ . This is one of the reasons for calling  $\text{Ho } \mathcal{M}$  the *homotopy category* of  $\mathcal{M}$ .

However, it is not quite true that  $\gamma f_0 = \gamma f_1$  if and only if  $f_0$  and  $f_1$  are either left homotopic or right homotopic; this only happens in special cases. In general, being left/right homotopic fails to even be an equivalence relation.

**Definition 4.2.6.** Let  $f : X \rightarrow Y$  be a morphism in a model category  $\mathcal{M}$ .

- A **left homotopy left inverse** for  $f$  is a morphism  $g : Y \rightarrow X$  in  $\mathcal{M}$  such that  $g \circ f$  and  $\text{id}_X$  are left homotopic.
- A **right homotopy right inverse** for  $f$  is a morphism  $h : Y \rightarrow X$  in  $\mathcal{M}$  such that  $f \circ h$  and  $\text{id}_Y$  are right homotopic.
- A **right homotopy left inverse** for  $f$  is a morphism  $g : Y \rightarrow X$  in  $\mathcal{M}$  such that  $g \circ f$  and  $\text{id}_X$  are right homotopic.
- A **left homotopy right inverse** for  $f$  is a morphism  $h : Y \rightarrow X$  in  $\mathcal{M}$  such that  $f \circ h$  and  $\text{id}_Y$  are left homotopic.

A **homotopy equivalence** in  $\mathcal{M}$  is a pair  $(f, g)$  such that  $g$  (resp.  $f$ ) is both a left homotopy left inverse and a right homotopy right inverse for  $f$  (resp.  $g$ ). Two morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  in  $\mathcal{M}$  are **mutual homotopy inverses** when  $(f, g)$  constitute a homotopy equivalence in  $\mathcal{M}$ .

REMARK 4.2.7. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be morphisms in a model category.

- $g$  is a left homotopy left inverse for  $f$  if and only if  $f$  is a left homotopy right inverse for  $g$ .
- $g$  is a right homotopy left inverse for  $f$  if and only if  $f$  is a right homotopy left inverse for  $g$ .

However, note that the dual of ‘left homotopy left inverse’ is ‘right homotopy right inverse’, and the dual of ‘right homotopy left inverse’ is ‘left homotopy right inverse’!

**Lemma 4.2.8.** *Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category, and suppose  $f_0$  and  $f_1$  are either left or right homotopic. Then,  $f_0$  is a weak equivalence if and only if  $f_1$  is a weak equivalence.*

*Proof.* Assume  $f_0$  and  $f_1$  are left homotopic; the other case is formally dual. So, there exist a cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  for  $X$  and a morphism  $H : \text{Cyl}(X) \rightarrow Y$  such that  $H \circ i_0 = f_0$  and  $H \circ i_1 = f_1$ . Suppose  $f_0$  is a weak equivalence. By [remark 4.2.2](#),  $i_0$  is a weak equivalence, so the 2-out-of-3 property implies  $H$  is also a weak equivalence; but  $i_1$  is a weak equivalence as well, so  $f_1$  must be a weak equivalence too. A symmetrical argument proves that  $f_0$  is a weak equivalence if  $f_1$  is. ■

**Lemma 4.2.9.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be morphisms in a model category  $\mathcal{M}$ .*

- (i) *If  $g \circ f$  is either left or right homotopic to  $\text{id}_X$ , and  $f \circ g$  is either left or right homotopic to  $\text{id}_Y$ , then  $(f, g)$  is an equivalence in  $\mathcal{M}$  (in the sense of [definition 3.1.17](#)).*
- (ii) *If there exist morphisms  $g, h : Y \rightarrow X$  such that  $g \circ f$  is either left or right homotopic to  $\text{id}_X$  and  $f \circ h$  is either left or right homotopic to  $\text{id}_Y$ , then (the image of)  $f$  is an isomorphism in  $\text{Ho } \mathcal{M}$ .*

*Proof.* Obvious, given [remark 4.2.5](#). ◆

**Lemma 4.2.10.** *Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category  $\mathcal{M}$ .*

- (i) *Given any cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  for  $X$ ,  $f_0 \circ p : \text{Cyl}(X) \rightarrow Y$  is a left homotopy from  $f_0$  to itself.*
- (ii) *If  $H : \text{Cyl}(X) \rightarrow Y$  is a left homotopy from  $f_0$  to  $f_1$  with respect to a cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  for  $X$ , then the same  $H$  is a left homotopy from  $f_1$  to  $f_0$  for the cylinder object  $(\text{Cyl}(X), i_1, i_0, p)$ .*



Dually:

- (i') Given any path object  $(\text{Path}(Y), i, p_0, p_1)$  for  $Y$ ,  $i \circ f_0 : X \rightarrow \text{Path}(Y)$  is a right homotopy from  $f_0$  to itself.
- (ii') If  $H : X \rightarrow \text{Path}(Y)$  is a right homotopy from  $f_0$  to  $f_1$  with respect to a path object  $(\text{Path}(Y), i, p_0, p_1)$  for  $Y$ , then the same  $H$  is a right homotopy from  $f_1$  to  $f_0$  for the path object  $(\text{Path}(Y), i, p_1, p_0)$ .

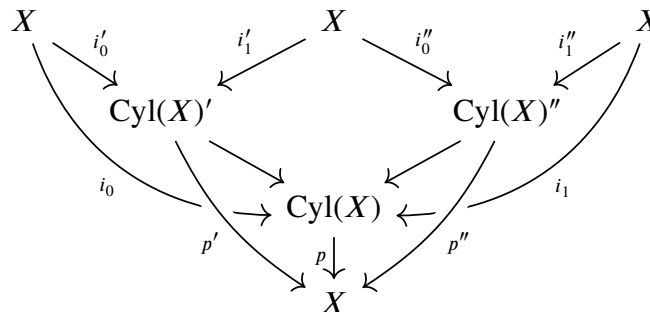
*Proof.* Obvious. ◆

**Lemma 4.2.II.** Let  $\mathcal{M}$  be a model category.

- If  $(\text{Cyl}(X), i_0, i_1, p)$  is a cylinder object for a cofibrant object in  $\mathcal{M}$ , then the insertions  $i_0, i_1 : X \rightarrow \text{Cyl}(X)$  are trivial cofibrations, and  $\text{Cyl}(X)$  is a cofibrant object in  $\mathcal{M}$ .
- If  $(\text{Path}(Y), i, p_0, p_1)$  is a path object for a fibrant object in  $\mathcal{M}$ , then the projections  $p_0, p_1 : \text{Path}(Y) \rightarrow Y$  are trivial fibrations, and  $\text{Path}(Y)$  is a fibrant object in  $\mathcal{M}$ .

*Proof.* See Lemmas 1.5 and 1.7 in [GJ], or Lemma 7.3.6 in [Hirschhorn, 2003]. □

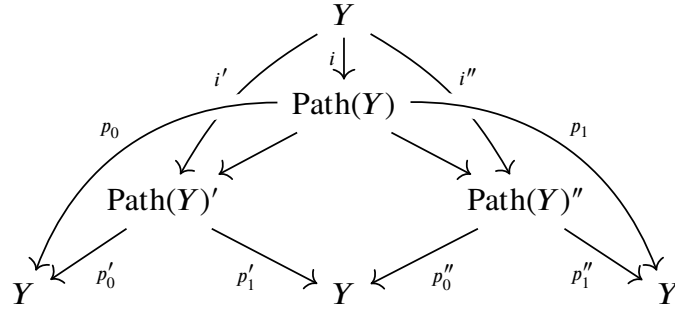
**Lemma 4.2.I2.** Let  $X$  be a cofibrant object in a model category  $\mathcal{M}$ . Given two cylinder objects for  $X$ , say  $(\text{Cyl}(X)', i'_0, i'_1, p')$  and  $(\text{Cyl}(X)'', i''_0, i''_1, p'')$ , there exists a third cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  such that the diagram below commutes,



and the diamond is a pushout diagram.

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Dually, if  $Y$  is a fibrant object in  $\mathcal{M}$ , and we have two path objects for  $Y$ , say  $(\text{Path}(Y)', i', p'_0, p'_1)$  and  $(\text{Path}(Y)'', i'', p''_0, p''_1)$ , then there exists a third path object  $(\text{Path}(Y), i, p_0, p_1)$  such that the diagram below commutes,



and the diamond is a pullback diagram.

*Proof.* See Lemmas 1.5 and 1.7 in [GJ, Ch. II], or Lemma 7.4.2 in [Hirschhorn, 2003].  $\square$

**Corollary 4.2.13.** Let  $f_0, f_1, f_2 : X \rightarrow Y$  be three parallel morphisms in a model category  $\mathcal{M}$ .

- (i) If  $f_0$  and  $f_1$  are left homotopic, and  $f_1$  and  $f_2$  are left homotopic, then  $f_0$  and  $f_2$  are also left homotopic.
- (ii) If  $f_0$  and  $f_1$  are right homotopic, and  $f_1$  and  $f_2$  are right homotopic, then  $f_0$  and  $f_2$  are also right homotopic.  $\blacksquare$

**Lemma 4.2.14.** Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category  $\mathcal{M}$ .

- (i) If  $X$  is cofibrant, and  $f_0$  and  $f_1$  are left homotopic, given any path object  $(\text{Path}(Y), i, p_0, p_1)$  for  $Y$ , there is a right homotopy  $H : X \rightarrow \text{Path}(Y)$  from  $f_0$  to  $f_1$ .
- (ii) If  $Y$  is fibrant, and  $f_0$  and  $f_1$  are right homotopic, given any cylinder object  $(\text{Cyl}(X), i_0, i_1, p)$  for  $X$ , there is a left homotopy  $H : \text{Cyl}(X) \rightarrow Y$  from  $f_0$  to  $f_1$ .

*Proof.* See Proposition 1.8 in [GJ, Ch. II], or Proposition 7.4.7 in [Hirschhorn, 2003].  $\square$

**Proposition 4.2.15.** Let  $X$  and  $Y$  be objects in a model category  $\mathcal{M}$ .

- (i) If  $X$  is cofibrant, then being left homotopic is an equivalence relation on the hom-set  $\mathcal{M}(X, Y)$ .
- (ii) If  $Y$  is fibrant, then being right homotopic is an equivalence relation on the hom-set  $\mathcal{M}(X, Y)$ .
- (iii) If  $X$  is cofibrant and  $Y$  is fibrant, then these two equivalence relations on  $\mathcal{M}(X, Y)$  coincide.

*Proof.* Use the preceding lemmas. ■

**Lemma 4.2.16.** *Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category  $\mathcal{M}$ .*

- (i) If  $f_0$  and  $f_1$  are right homotopic and  $g : W \rightarrow X$  is any morphism in  $\mathcal{M}$ , then  $f_0 \circ g$  and  $f_1 \circ g$  are also right homotopic.
- (ii) If  $f_0$  and  $f_1$  are left homotopic and  $g : Y \rightarrow Z$  is any morphism in  $\mathcal{M}$ , then  $g \circ f_0$  and  $g \circ f_1$  are also left homotopic.

*Proof.* Obvious. ◆

**Corollary 4.2.17.** *Let  $\mathcal{M}$  be a model category, and let  $\mathcal{M}_{\text{cf}}$  be the full subcategory spanned by the cofibrant–fibrant objects. Then the equivalence relation induced by homotopy is a congruence on  $\mathcal{M}_{\text{cf}}$ ; in particular, there exist a locally small category  $\mathcal{M}_{\text{h}}$  and a full functor  $\mathcal{M}_{\text{cf}} \rightarrow \mathcal{M}'$  with these properties:*

- The objects of  $\mathcal{M}_{\text{h}}$  are those of  $\mathcal{M}_{\text{cf}}$ .
- The hom-set  $\mathcal{M}_{\text{h}}(X, Y)$  is  $\mathcal{M}(X, Y)$  modulo homotopy.
- The functor  $\mathcal{M}_{\text{cf}} \rightarrow \mathcal{M}_{\text{h}}$  sends each morphism in  $\mathcal{M}_{\text{cf}}$  to its homotopy class. ■

The next result is a version of Whitehead’s theorem; however, this is a purely formal consequence of the model category axioms and has no real content, unlike the original theorem.

**Proposition 4.2.18.** *Let  $X$  and  $Y$  be cofibrant–fibrant objects in a model category  $\mathcal{M}$ . If  $f : X \rightarrow Y$  is a weak equivalence, then  $f$  has a homotopy inverse in  $\mathcal{M}$ .*

*Proof.* See Theorem 1.10 in [GJ, Ch. II], or Theorem 7.5.10 in [Hirschhorn, 2003].  $\square$

**Lemma 4.2.19.** *Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in a model category  $\mathcal{M}$ .*

- *If  $g : W \rightarrow X$  is a morphism with a right homotopy right inverse in  $\mathcal{M}$ , then  $f_0 \circ g$  and  $f_1 \circ g$  are right homotopic if and only if  $f_0$  and  $f_1$  are right homotopic.*
- *If  $g : Y \rightarrow Z$  is a morphism with a left homotopy left inverse in  $\mathcal{M}$ , then  $g \circ f_0$  and  $g \circ f_1$  are left homotopic if and only if  $f_0$  and  $f_1$  are left homotopic.*

*Proof.* This follows immediately from the definitions and lemma 4.2.16.  $\blacksquare$

**Corollary 4.2.20.** *Let  $W, X, Y, Z$  be cofibrant–fibrant objects in a model category  $\mathcal{M}$ , and let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms.*

- *If  $g : W \rightarrow X$  is a weak equivalence such that  $f_0 \circ g$  and  $f_1 \circ g$  are homotopic, then  $f_0$  and  $f_1$  are homotopic.*
- *If  $g : Y \rightarrow Z$  is a weak equivalence such that  $g \circ f_0$  and  $g \circ f_1$  are homotopic, then  $f_0$  and  $f_1$  are homotopic.*

*Proof.* Apply proposition 4.2.18 in conjunction with the above lemma.  $\blacksquare$

## 4.3 Quillen functors

**Prerequisites.** §§3.1, 3.3, 3.4, 4.1, A.5.

**Definition 4.3.1.**

- A **left Quillen functor** is a functor between derivable categories that has a right adjoint and preserves cofibrations and trivial cofibrations.
- A **right Quillen functor** is a functor between derivable categories that has a left adjoint and preserves fibrations and trivial fibrations.
- A **Quillen adjunction** is an adjunction

$$F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$$

where  $F$  is a left Quillen functor and  $G$  is a right Quillen functor.

- A **Quillen equivalence** is a Quillen adjunction as above satisfying this additional condition:
  - Given a cofibrant object  $A$  in  $\mathcal{N}$  and fibrant object  $Y$  in  $\mathcal{M}$ , a morphism  $FA \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$  if and only if its right adjoint transpose  $A \rightarrow GY$  is a weak equivalence in  $\mathcal{N}$ .

**Proposition 4.3.2.** *Let  $F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$  be an adjunction between categories with model structures. The following are equivalent:*

- (i)  $F$  preserves cofibrations and trivial cofibrations.
- (ii)  $G$  preserves fibrations and trivial fibrations.
- (iii)  $F$  preserves cofibrations and  $G$  preserves fibrations.
- (iv)  $F$  preserves trivial cofibrations and  $G$  preserves trivial fibrations.
- (v) (Assuming  $\mathcal{M}$  and  $\mathcal{N}$  are derivable categories.)  $F \dashv G$  is a Quillen adjunction.

*Proof.* Use proposition A.3.22. ■

**REMARK 4.3.3.** A functor between categories with model structures that preserves both trivial cofibrations and trivial fibrations must also preserve weak equivalences, since axioms CM2 and CM5 together imply that a morphism is a weak equivalence if and only if it is of the form  $p \circ i$  where  $i$  is a trivial cofibration and  $p$  is a trivial fibration. In particular, a functor that is both left and right Quillen must be homotopical.

**Proposition 4.3.4.** *Let  $F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$  be a Quillen adjunction.*

- $F$  sends cofibrant objects in  $\mathcal{N}$  to cofibrant objects in  $\mathcal{M}$ .
- $G$  sends fibrant objects in  $\mathcal{M}$  to fibrant objects in  $\mathcal{N}$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $B$  be a cofibrant object in  $\mathcal{N}$  and let  $p : X \rightarrow Y$  be a trivial fibration in  $\mathcal{M}$ . Since  $F \dashv G$ , we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{M}(FB, X) & \xrightarrow{\cong} & \mathcal{N}(B, GX) \\
 \mathcal{M}(FB, p) \downarrow & & \downarrow \mathcal{M}(B, Gp) \\
 \mathcal{M}(FB, Y) & \xrightarrow[\cong]{} & \mathcal{N}(B, GY)
 \end{array}$$

By hypothesis,  $Gp : GX \rightarrow GY$  is a trivial fibration in  $\mathcal{N}$ , so the hom-set map  $\mathcal{N}(B, Gp)$  is a surjection. It follows that  $\mathcal{M}(FB, p)$  is also a surjection, and thus  $FB$  is a cofibrant object in  $\mathcal{M}$ . ■

**Proposition 4.3.5.**

- (i) *The composite of two Quillen adjunctions is also a Quillen adjunction.*
- (ii) *The composite of two Quillen equivalences is also a Quillen equivalence.*

*Proof.* Obvious. ◆

**Lemma 4.3.6** (Kenneth S. Brown). *Let  $\mathcal{M}$  be a model category and let  $\mathcal{C}$  be a category with weak equivalences.*

- *Let  $\mathcal{M}_c$  be the full subcategory of cofibrant objects in  $\mathcal{M}$ . If  $F : \mathcal{M}_c \rightarrow \mathcal{C}$  sends trivial cofibrations in  $\mathcal{M}_c$  to weak equivalences in  $\mathcal{C}$ , then  $F$  also sends weak equivalences in  $\mathcal{M}_c$  to weak equivalences in  $\mathcal{C}$ .*
- *Let  $\mathcal{M}_f$  be the full subcategory of fibrant objects in  $\mathcal{M}$ . If  $F : \mathcal{M}_f \rightarrow \mathcal{C}$  sends trivial fibrations in  $\mathcal{M}_f$  to weak equivalences in  $\mathcal{C}$ , then  $F$  also sends weak equivalences in  $\mathcal{M}_f$  to weak equivalences in  $\mathcal{C}$ .*

*Proof.* See Lemma 9.9 in [DS], Lemma 7.7.1 in [Hirschhorn, 2003], or Lemma 14.5 in [DHKS]. □

The usual proof of the Ken Brown's lemma uses binary coproducts (or binary products, as the case may be), so it cannot be used in the case where the domain is merely a derivable category. Nonetheless, we have already proved something very similar, namely lemma 4.1.32.

**Proposition 4.3.7** (Dugger). *Let  $F \dashv G$  be an adjunction between DHK model categories. The following are equivalent:*

- (i)  *$F \dashv G$  is a Quillen adjunction.*
- (ii)  *$F$  preserves cofibrations between cofibrant objects and all trivial cofibrations.*
- (iii)  *$G$  preserves fibrations between fibrant objects and all trivial fibrations.*

*Proof.* See Proposition 8.5.4 in [Hirschhorn, 2003], or Corollary A.2 in [Dugger, 2001b]. □

**Definition 4.3.8.** Let  $\mathcal{M}$  be a derivable category.

- A **left Quillen deformation retract** (resp. **functorial left Quillen deformation retract**) of  $\mathcal{M}$  is a left deformation retract of  $\mathcal{M}$  of the form  $(\mathcal{M}_c, Q, p)$  where  $\mathcal{M}_c$  is the full subcategory of cofibrant objects in  $\mathcal{M}$ .
- A **right Quillen deformation retract** (resp. **functorial right Quillen deformation retract**) of  $\mathcal{M}$  is a right deformation retract of  $\mathcal{M}$  of the form  $(\mathcal{M}_f, R, i)$  where  $\mathcal{M}_f$  is the full subcategory of fibrant objects in  $\mathcal{M}$ .

**Lemma 4.3.9.** Let  $\mathcal{M}$  be a derivable category.

- *Left Quillen deformation retracts of  $\mathcal{M}$  exist.*
- *Right Quillen deformation retracts of  $\mathcal{M}$  exist.*

*Proof.* The two claims are formally dual; we will prove the first version.

For each object  $X$  in  $\mathcal{M}$ , choose a fibrant cofibrant replacement  $(QX, p_X)$ ; such exist by [proposition 4.1.23](#). Then, for each morphism  $f : X \rightarrow Y$  in  $\mathcal{M}$ , there exists a morphism  $Qf : QX \rightarrow QY$  making the diagram commute,

$$\begin{array}{ccc} QX & \xrightarrow{p_X} & X \\ Qf \downarrow & & \downarrow f \\ QY & \xrightarrow{p_Y} & Y \end{array}$$

because  $p_Y : QY \rightarrow Y$  is a trivial fibration and  $QX$  is cofibrant; note that axiom CM2 implies  $Qf$  is a weak equivalence if (and only if!)  $f$  is. Thus, axioms DR1–2 are satisfied. For axiom DR3, we refer to [proposition 4.1.34](#). Finally, we simply need to observe that axiom DR4 is trivial. ■

**Lemma 4.3.10.** Let  $\mathcal{M}$  be a derivable category.

- $(\mathcal{M}_c, Q, p)$  is a functorial left Quillen deformation for  $\mathcal{M}$  if and only if  $(Q, p)$  is a cofibrant replacement functor for  $\mathcal{M}$ .
- $(\mathcal{M}_f, R, i)$  is a functorial left Quillen deformation for  $\mathcal{M}$  if and only if  $(R, i)$  is a cofibrant replacement functor for  $\mathcal{M}$ .

*Proof.* Obvious. ◆

**Theorem 4.3.II.** *Let  $\mathcal{M}$  be a derivable category, let  $\mathcal{C}$  be a relative category, and let  $\gamma_{\mathcal{M}} : \mathcal{M} \rightarrow \mathbf{Ho} \mathcal{M}$  and  $\gamma_{\mathcal{C}} : \mathcal{C} \rightarrow \mathbf{Ho} \mathcal{C}$  be the respective localising functors. Suppose  $\mathbf{weq} \mathcal{C}$  has the 2-out-of-3 property and the special 2-out-of-4 property. If  $F : \mathcal{M} \rightarrow \mathcal{C}$  is a functor that sends trivial cofibrations in  $\mathcal{M}$  to weak equivalences in  $\mathcal{C}$ , then:*

- (i) *Any left Quillen deformation retract of  $\mathcal{M}$  is a left deformation retract for  $F$ ; in particular, a total left derived functor for  $F$  exists.*
- (ii) *If  $\mathcal{M}$  has a cofibrant replacement functor, then  $F$  is functorially left deformable and has a homotopical left approximation.*
- (iii) *If  $(\mathbf{L}F, \alpha)$  is any total left derived functor for  $F$ , then the extension counit component  $\alpha_X : (\mathbf{L}F)\gamma_{\mathcal{M}}X \rightarrow \gamma_{\mathcal{C}}FX$  is an isomorphism for all cofibrant objects  $X$  in  $\mathcal{M}$ .*

*Dually, if  $F : \mathcal{M} \rightarrow \mathcal{C}$  is a functor that sends trivial fibrations in  $\mathcal{M}$  to weak equivalences in  $\mathcal{C}$ , then:*

- (i') *Any right Quillen deformation retract of  $\mathcal{M}$  is a right deformation retract for  $F$ ; in particular, a total right derived functor for  $F$  exists.*
- (ii') *If  $\mathcal{M}$  has a fibrant replacement functor, then  $F$  is functorially right deformable and has a homotopical right approximation.*
- (iii') *If  $(\mathbf{R}F, \beta)$  is any total right derived functor for  $F$ , then the extension counit component  $\beta_X : (\mathbf{R}F)\gamma_{\mathcal{M}}X \rightarrow \gamma_{\mathcal{C}}FX$  is an isomorphism for all fibrant objects  $X$  in  $\mathcal{M}$ .*

*Proof.* (i). Let  $(\mathcal{M}_c, Q, p)$  be a left Quillen deformation retract of  $\mathcal{M}$ . Then  $F$  sends weak equivalences in  $\mathcal{M}_c$  to weak equivalences in  $\mathcal{C}$  by lemma 4.1.32, so  $(\mathcal{M}_c, Q, p)$  is indeed a left deformation retract for  $F$ . We may then apply theorem 3.3.13 to obtain a total left derived functor.

(ii). By the same argument, if  $(Q, p)$  is a cofibrant replacement functor for  $\mathcal{M}$ , then  $(\mathcal{M}_c, Q, p)$  is a functorial left deformation retract for  $F$ . We then appeal to theorem 3.4.10.

(iii). The extension counit has the required property because, for all cofibrant objects  $X$  in  $\mathcal{M}$ , the morphism  $Fp_X : FQX \rightarrow FX$  is a weak equivalence in  $\mathcal{C}$ ; but this is precisely the component of the extension counit at  $X$ . ■



**Theorem 4.3.12.** *Let  $F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$  be a Quillen adjunction.*

- (i) *Any left Quillen deformation retract of  $\mathcal{N}$  is a left deformation retract for  $F$ ; dually, any right Quillen deformation retract of  $\mathcal{M}$  is a right deformation retract for  $G$ .*
- (ii)  *$F \dashv G$  is a deformable adjunction; in particular, a derived adjunction exists.*
- (iii) *If  $F \dashv G$  is a Quillen equivalence, then the derived adjunction*

$$\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho} \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{N}$$

*is an adjoint equivalence of categories; and if  $\mathcal{M}$  and  $\mathcal{N}$  are saturated derivable categories, then the converse is true.*

*Proof.* (i). Since weak equivalences in derivable categories are closed under retracts (by axiom CM3), we may use [theorem 4.3.11](#).

(ii). That  $F \dashv G$  is a derivable adjunction follows immediately; then apply [theorem 3.3.20](#) for the existence of the derived adjunction.

(iii). This is a special case of [proposition 3.3.24](#). ■

**Proposition 4.3.13.** *Let  $\mathcal{L}$ ,  $\mathcal{M}$ , and  $\mathcal{N}$  be derivable categories.*

- *If  $F : \mathcal{N} \rightarrow \mathcal{M}$  and  $G : \mathcal{M} \rightarrow \mathcal{L}$  are left Quillen functors, then the composite  $(\mathbf{L}G)(\mathbf{L}F)$  is (the functor part of) a total left derived functor for  $GF$ .*
- *If  $F : \mathcal{N} \rightarrow \mathcal{P}$  and  $G : \mathcal{M} \rightarrow \mathcal{N}$  are right Quillen functors, then the composite  $(\mathbf{R}F)(\mathbf{R}G)$  is (the functor part of) a total right derived functor for  $FG$ .*

*Assuming  $\mathcal{M}$ ,  $\mathcal{N}$ , and  $\mathcal{L}$  have fibrant and cofibrant replacement functors:*

- *If  $F : \mathcal{N} \rightarrow \mathcal{M}$  and  $G : \mathcal{M} \rightarrow \mathcal{L}$  are left Quillen functors, then the composite  $(\mathbb{L}G)(\mathbb{L}F)$  is (the functor part of) a homotopical left approximation for  $GF$ .*
- *If  $F : \mathcal{N} \rightarrow \mathcal{P}$  and  $G : \mathcal{M} \rightarrow \mathcal{N}$  are right Quillen functors, then the composite  $(\mathbb{R}F)(\mathbb{R}G)$  is (the functor part of) a homotopical right approximation for  $FG$ .*

*Proof.* Use theorems 3.3.13, 3.4.10, and 4.3.11 with proposition 4.3.5. ■

**Definition 4.3.14.** Let  $\mathbb{A}$  be a small category and let  $\mathcal{M}$  be a category equipped with a model structure.

- The **injective model structure** on the functor category  $[\mathbb{A}, \mathcal{M}]$  is a model structure such that a morphism in  $[\mathbb{A}, \mathcal{M}]$  is a cofibration (resp. weak equivalence) if and only if all its components are cofibrations (resp. weak equivalences) in  $\mathcal{M}$ .
- The **projective model structure** on the functor category  $[\mathbb{A}, \mathcal{M}]$  is a model structure such that a morphism in  $[\mathbb{A}, \mathcal{M}]$  is a fibration (resp. weak equivalence) if and only if all its components are fibrations (resp. weak equivalences) in  $\mathcal{M}$ .

REMARK 4.3.15. The injective (resp. projective) model structure on  $[\mathbb{A}, \mathcal{M}]$  is unique if it exists, by theorem 4.1.12.

**Proposition 4.3.16.** Let  $\mathcal{M}$  be a derivable category, let  $\mathbb{A}$  be a small category, and let  $\Delta : \mathcal{M} \rightarrow [\mathbb{A}, \mathcal{M}]$  be the functor that sends an object  $X$  in  $\mathcal{M}$  to the constant functor  $\Delta X : \mathbb{A} \rightarrow \mathcal{M}$  with value  $X$ .

- If  $\mathcal{M}$  has colimits for diagrams of shape  $\mathbb{A}$ , then  $\Delta : \mathcal{M} \rightarrow [\mathbb{A}, \mathcal{M}]$  is a right Quillen functor with respect to the projective model structure on  $[\mathbb{A}, \mathcal{M}]$  when it exists.
- If  $\mathcal{M}$  has limits for diagrams of shape  $\mathbb{A}$ , then  $\Delta : \mathcal{M} \rightarrow [\mathbb{A}, \mathcal{M}]$  is a left Quillen functor with respect to the injective model structure on  $[\mathbb{A}, \mathcal{M}]$  when it exists.

*Proof.*  $\Delta$  certainly preserves fibrations (resp. cofibrations) and weak equivalences with respect to the projective (resp. injective) model structure, so by proposition 4.3.2,  $\lim_{\rightarrow \mathbb{A}} \dashv \Delta$  (resp.  $\Delta \dashv \lim_{\leftarrow \mathbb{A}}$ ) is a Quillen adjunction.<sup>[3]</sup> ■

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[3] Recall proposition 0.1.12.

**Proposition 4.3.17.** *Let  $\mathcal{M}$  be a category and let  $I$  be a set.*

- (i) *The functor category  $[I, \mathcal{M}]$  admits a model structure that is simultaneously an injective model structure and a projective model structure.*
- (ii) *If  $\mathcal{M}$  is a derivable category (resp. saturated derivable category, model category), then  $[I, \mathcal{M}]$  equipped with the above model structure is a derivable category (resp. saturated derivable category, model category).*
- (iii) *If  $\mathcal{M}$  is a derivable category and has products and coproducts for families of objects indexed by  $I$ , then  $\Delta : \mathcal{M} \rightarrow [I, \mathcal{M}]$  is both a left Quillen functor and a right Quillen functor.*
- (iv) *If  $\mathcal{M}$  is a model category, then the canonical exponential comparison functor  $\text{Ho}[I, \mathcal{M}] \rightarrow [I, \text{Ho } \mathcal{M}]$  is an isomorphism of categories.*

*Proof.* (i). If we declare the cofibrations (resp. weak equivalences, fibrations) in  $[I, \mathcal{M}]$  to be precisely the morphisms that are cofibrations (resp. weak equivalences, fibrations) componentwise, then the axioms CM2–5 may be verified componentwise as well.

(ii). Axioms DC0, DC1, and CM1 can be verified componentwise. If  $\mathcal{M}$  is saturated, then we can use [lemma 3.1.11](#) to deduce that  $[I, \mathcal{M}]$  is also saturated.

(iii). Apply [proposition 4.3.16](#).

(iv). Use [theorem 4.4.1](#) and the fact that the congruence of homotopy is componentwise in  $[I, \mathcal{M}]$ . ■

**Corollary 4.3.18.** *Let  $\mathcal{M}$  be a saturated derivable category and let  $I$  be a set.*

- *If  $\mathcal{M}$  has products for families of objects indexed by  $I$ , then the product of an  $I$ -indexed family of weak equivalences between fibrant objects is also a weak equivalence between fibrant objects.*
- *If  $\mathcal{M}$  has coproducts for families of objects indexed by  $I$ , then the coproduct of an  $I$ -indexed family of weak equivalences between cofibrant objects is also a weak equivalence between cofibrant objects.*

*Proof.* Apply [lemma 4.1.32](#) to the previous proposition. ■

**Proposition 4.3.19.** *Let  $\mathcal{M}$  be a derivable category and let  $\mathbb{A}$  be a small category.*

- *If  $\mathcal{M}$  has coproducts for families of size  $\leq |\text{mor } \mathbb{A}|$ , then the evaluation functors  $[\mathbb{A}, \mathcal{M}] \rightarrow \mathcal{M}$  are right Quillen functors with respect to the injective model structure on  $[\mathbb{A}, \mathcal{M}]$  (if it exists).*
- *If  $\mathcal{M}$  has products for families of size  $\leq |\text{mor } \mathbb{A}|$ , then the evaluation functors  $[\mathbb{A}, \mathcal{M}] \rightarrow \mathcal{M}$  are left Quillen functors with respect to the projective model structure on  $[\mathbb{A}, \mathcal{M}]$  (if it exists).*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $A$  be an object in  $\mathbb{A}$  and let  $A^* : [\mathbb{A}, \mathcal{M}] \rightarrow \mathcal{M}$  be the functor  $F \mapsto FA$ . It is not hard to check that  $A^*$  has a left adjoint  $A_! : \mathcal{M} \rightarrow [\mathbb{A}, \mathcal{M}]$ , namely the functor  $X \mapsto \mathbb{A}(A, -) \odot X$ . Since the class of cofibrations and the class of trivial cofibrations are both closed under coproducts, we see that  $A_! : \mathcal{M} \rightarrow [\mathbb{A}, \mathcal{M}]$  is a left Quillen functor with respect to the injective model structure. Thus, by [proposition 4.3.2](#),  $A^* : [\mathbb{A}, \mathcal{M}] \rightarrow \mathcal{M}$  is a right Quillen functor. ■

**Corollary 4.3.20.** *Let  $\mathcal{M}$  be a derivable category and let  $\mathbb{A}$  be a small category. Suppose the injective and projective model structures on  $[\mathbb{A}, \mathcal{M}]$  both exist. If  $\mathcal{M}$  has both coproducts and products for families of size  $\leq |\text{mor } \mathbb{A}|$ , then:*

- *Every fibration (resp. trivial fibration) in the injective model structure on  $[\mathbb{A}, \mathcal{M}]$  is a fibration (resp. trivial fibration) in the projective model structure.*
- *Every cofibration (resp. trivial cofibration) in the projective model structure on  $[\mathbb{A}, \mathcal{M}]$  is a cofibration (resp. trivial cofibration) in the injective model structure.*
- *The trivial adjunction*

$$\text{id} \dashv \text{id} : [\mathbb{A}, \mathcal{M}] \rightarrow [\mathbb{A}, \mathcal{M}]$$

*is a Quillen equivalence between the injective and projective model structures.* ■

**Definition 4.3.21.**

- A **left proper model category** is a model category  $\mathcal{M}$  with the following property: given a pushout diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} Z & \xrightarrow{g} & Z' \\ i \downarrow & & \downarrow i' \\ W & \xrightarrow{f} & W' \end{array}$$

if  $i : Z \rightarrow W$  is a cofibration and  $g : Z \rightarrow Z'$  is a weak equivalence, then  $f : W \rightarrow W'$  is also a weak equivalence.

- A **right proper model category** is a model category  $\mathcal{M}$  with the following property: given a pullback diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

if  $p : X \rightarrow Y$  is a fibration and  $g : Y' \rightarrow Y$  is a weak equivalence, then  $f : X' \rightarrow X$  is also a weak equivalence.

- A **proper model category** is a model category that is both left proper and right proper.

**Proposition 4.3.22** (Reedy). *Let  $\mathcal{M}$  be a model category.*

- *If all objects in  $\mathcal{M}$  are cofibrant, then  $\mathcal{M}$  is a left proper model category.*
- *If all objects in  $\mathcal{M}$  are fibrant, then  $\mathcal{M}$  is a right proper model category.*

*Proof.* See Proposition 13.1.2 in [Hirschhorn, 2003], or use proposition 3.7.14. □

**Proposition 4.3.23.** *Let  $g : Y' \rightarrow Y$  be a morphism in a model category  $\mathcal{M}$ .*

- (i) *There is a Quillen adjunction of the form below,*

$$\Sigma_g \dashv g^* : \mathcal{M}_Y \rightarrow \mathcal{M}_{Y'}$$

where  $\Sigma_g$  is the dependent sum functor.<sup>[4]</sup>

[4] See definition A.2.16.

- (ii) If  $\Sigma_g \dashv g^*$  is a Quillen equivalence, then  $g : Y' \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$ .
- (iii) If  $\mathcal{M}$  is a right proper model category and  $g : Y' \rightarrow Y$  is a weak equivalence, then  $\Sigma_g \dashv g^*$  is a Quillen equivalence.

*Proof.* (i). By definition, the functor  $\Sigma_g : \mathcal{M}_{/Y} \rightarrow \mathcal{M}_{/Y'}$  preserves cofibrations and trivial cofibrations, so recalling [lemma A.2.17](#) and [proposition 4.3.2](#), we have the announced Quillen adjunction.

(ii). Let  $(\tilde{Y}', q)$  be a cofibrant replacement for  $Y'$ . Clearly,  $\text{id}_{Y'}$  is isomorphic to  $g^*(\text{id}_Y)$ , and  $q : \tilde{Y}' \rightarrow Y'$  is a weak equivalence in  $\mathcal{M}_{/Y'}$ , so the Quillen equivalence condition implies that its adjoint transpose is a weak equivalence in  $\mathcal{M}_{/Y}$ , i.e. that  $g \circ q : \tilde{Y}' \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$ . We then use axiom CM2 to deduce that  $g : Y' \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$ .

(iii). Consider a fibrant object in  $\mathcal{M}_{/Y}$ , i.e. a fibration  $p : X \rightarrow Y$  in  $\mathcal{M}$ . Then we have a pullback square in  $\mathcal{M}$ :

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ g^*(p) \downarrow & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

Since  $\mathcal{M}$  is a right proper model category,  $f : X' \rightarrow X$  is a weak equivalence. Axiom CM2 then implies that a morphism  $l : g^*(p) \rightarrow p$  in  $\mathcal{M}_{/Y}$  is a weak equivalence if and only if its adjoint transpose  $\Sigma_g l \rightarrow p$  in  $\mathcal{M}_{/Y}$  is a weak equivalence. Thus,  $\Sigma_g \dashv g^*$  is indeed a Quillen equivalence.  $\blacksquare$

## 4.4 The homotopy category

**Prerequisites.** §§[4.1](#), [4.2](#), [4.3](#), [A.4](#).

**Theorem 4.4.1.** *Let  $\mathcal{M}$  be a model category and let  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  be the localising functor.*

- (i) *Ho  $\mathcal{M}$  is equivalent to the locally small category  $\mathcal{M}_h$  defined in [corollary 4.2.17](#), and  $\mathcal{M}$  is a saturated homotopical category.*

- (ii) If  $X$  and  $Y$  are cofibrant–fibrant objects in  $\mathcal{M}$ , then the hom-set map  $\mathcal{M}(X, Y) \rightarrow \text{Ho } \mathcal{M}(X, Y)$  induced by  $\gamma$  is surjective; and moreover for any parallel pair  $f_0, f_1 : X \rightarrow Y$  in  $\mathcal{M}$ , we have  $\gamma f_0 = \gamma f_1$  if and only if  $f_0$  and  $f_1$  are homotopic.

*Proof.* (i). This is Theorem 1.11 in [GJ, Ch. II], or Proposition 5.8 in [DS].

(ii). Implied by claim (i). □

**Corollary 4.4.2.** *Let  $f : X \rightarrow Y$  be a morphism in a model category  $\mathcal{M}$ . If  $f$  has a quasi-inverse in  $\mathcal{M}$  (in the sense of definition 3.1.17), then  $f$  is a weak equivalence in  $\mathcal{M}$ .*

*Proof.* If  $f$  has a quasi-inverse in  $\mathcal{M}$ , then (the image of)  $f$  is an isomorphism in  $\text{Ho } \mathcal{M}$ ; but  $\mathcal{M}$  is a saturated homotopical category, so  $f$  must be a weak equivalence in  $\mathcal{M}$ . ■

**Corollary 4.4.3.** *Let  $\mathcal{M}$  be a model category and let  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  be the localising functor.*

- (i) For any parallel pair  $f_0, f_1 : X \rightarrow Y$  in  $\mathcal{M}$ , if  $X$  is cofibrant and  $Y$  is fibrant, we have  $\gamma f_0 = \gamma f_1$  if and only if  $f_0$  and  $f_1$  are homotopic.
- (ii) The full subcategory  $\mathcal{M}_{\text{cf}}$  of cofibrant–fibrant objects in  $\mathcal{M}$  has the Whitehead property (in the sense of definition 3.1.21).

*Proof.* (i). As noted in remark 4.2.5, if  $f_0, f_1 : X \rightarrow Y$  are homotopic, then we must have  $\gamma f_0 = \gamma f_1$ . Conversely, suppose  $\gamma f_0 = \gamma f_1$  with  $X$  cofibrant and  $Y$  fibrant. Let  $(RX, i')$  be a cofibrant fibrant replacement for  $X$  and  $(QY, p')$  be a fibrant cofibrant replacement for  $Y$ . Then, there exists morphisms  $f'_0, f'_1 : RX \rightarrow QY$  such that  $f_0 = p' \circ f'_0 \circ i'$  and  $f_1 = p' \circ f'_1 \circ i'$ . Since  $i' : X \rightarrow RX$  and  $p' : QY \rightarrow Y$  are weak equivalences, we must have  $\gamma f'_0 = \gamma f'_1$  in  $\text{Ho } \mathcal{M}$ . The theorem then implies  $f'_0$  and  $f'_1$  are homotopic; thus  $f_0$  and  $f_1$  are also homotopic, by lemmas 4.2.14 and 4.2.16.

(ii). Apply theorem 3.1.22 in conjunction with lemma 4.2.9 and the above corollary. ■

**Corollary 4.4.4.** *Let  $f : X \rightarrow Y$  be a morphism between two cofibrant objects in a derivable category  $\mathcal{M}$ . If  $\mathcal{M}$  is a saturated homotopical category, then the following are equivalent:*

- (i) The morphism  $f : X \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$ .
- (ii) The hom-set map  $\text{Ho } \mathcal{M}(f, Z) : \text{Ho } \mathcal{M}(Y, Z) \rightarrow \text{Ho } \mathcal{M}(X, Z)$  is a bijection for all cofibrant–fibrant objects  $Z$  in  $\mathcal{M}$ .
- (iii) The hom-set map  $\mathcal{M}_h(f, Z) : \mathcal{M}_h(Y, Z) \rightarrow \mathcal{M}_h(X, Z)$  is a bijection for all cofibrant–fibrant objects  $Z$  in  $\mathcal{M}$ , where  $\mathcal{M}_h(Y, Z)$  (resp.  $\mathcal{M}_h(X, Z)$ ) denotes the set of all morphisms  $Y \rightarrow Z$  (resp.  $X \rightarrow Z$ ) in  $\mathcal{M}$  modulo homotopy.

*Proof.* (i)  $\Rightarrow$  (ii). Every weak equivalence in  $\mathcal{M}$  becomes an isomorphism in  $\text{Ho } \mathcal{M}$ , so in particular  $\text{Ho } \mathcal{M}(f, Z) : \text{Ho } \mathcal{M}(Y, Z) \rightarrow \text{Ho } \mathcal{M}(X, Z)$  must be a bijection.

(ii)  $\Leftrightarrow$  (iii). The previous corollary implies that the vertical arrows in the following commutative diagram are bijections,

$$\begin{array}{ccc} \mathcal{M}_h(Y, Z) & \xrightarrow{\mathcal{M}_h(f, Z)} & \mathcal{M}_h(X, Z) \\ \downarrow & & \downarrow \\ \text{Ho } \mathcal{M}(Y, Z) & \xrightarrow{\text{Ho } \mathcal{M}(f, Z)} & \text{Ho } \mathcal{M}(X, Z) \end{array}$$

and so  $\mathcal{M}_h(f, Z)$  is a bijection if and only if  $\text{Ho } \mathcal{M}(f, Z)$  is a bijection.

(ii)  $\Rightarrow$  (i). Suppose  $(\hat{X}, i_X)$  is a cofibrant fibrant replacement for  $X$  and  $(\hat{Y}, i_Y)$  is a cofibrant fibrant replacement for  $Y$ . Then, (by axiom CM4) there exists a morphism  $\hat{f} : \hat{X} \rightarrow \hat{Y}$  making the diagram below commute,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i_X \downarrow & & \downarrow i_Y \\ \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \end{array}$$

and by the 2-out-of-3 property,  $f$  is a weak equivalence if and only if  $\hat{f}$  is a weak equivalence. On the other hand, the following diagram also commutes,

$$\begin{array}{ccc} \text{Ho } \mathcal{M}(\hat{Y}, Z) & \xrightarrow{\text{Ho } \mathcal{M}(\hat{f}, Z)} & \text{Ho } \mathcal{M}(\hat{X}, Z) \\ \text{Ho } \mathcal{M}(i_Y, Z) \downarrow & & \downarrow \text{Ho } \mathcal{M}(i_X, Z) \\ \text{Ho } \mathcal{M}(Y, Z) & \xrightarrow{\text{Ho } \mathcal{M}(f, Z)} & \text{Ho } \mathcal{M}(X, Z) \end{array}$$



and so  $\text{Ho } \mathcal{M}(f, Z)$  is a bijection if and only if  $\text{Ho } \mathcal{M}(\hat{f}, Z)$  is a bijection; but  $\hat{X}$  and  $\hat{Y}$  are both cofibrant–fibrant objects, so if  $\text{Ho } \mathcal{M}(f, Z)$  is a bijection for all cofibrant–fibrant objects  $Z$ , then  $\hat{f}$  must be a weak equivalence (because  $\mathcal{M}$  is a saturated homotopical category). ■

**Proposition 4.4.5.** *Let  $\mathcal{M}$  be a derivable category, let  $\mathcal{U}$  be the class of trivial cofibrations in  $\mathcal{M}$ , and let  $\mathcal{V}$  be the class of trivial fibrations in  $\mathcal{M}$ .*

- *Let  $(\mathcal{M}_c, Q, p)$  be a left Quillen deformation retract of  $\mathcal{M}$  and let  $\mathcal{M}[\mathcal{U}^{-1}]$  be the localisation of  $\mathcal{M}$  with respect to the trivial cofibrations. Then the inclusion  $\mathcal{M}_c \hookrightarrow \mathcal{M}$  induces a fully faithful functor  $\text{Ho } \mathcal{M}_c \rightarrow \mathcal{M}[\mathcal{U}^{-1}]$ , and  $(Q, p)$  induces a right adjoint for that functor.*
- *Let  $(\mathcal{M}_f, R, i)$  be a right Quillen deformation retract of  $\mathcal{M}$  and let  $\mathcal{M}[\mathcal{V}^{-1}]$  be the localisation of  $\mathcal{M}$  with respect to the trivial fibrations. Then the inclusion  $\mathcal{M}_f \hookrightarrow \mathcal{M}$  induces a fully faithful functor  $\text{Ho } \mathcal{M}_f \rightarrow \mathcal{M}[\mathcal{V}^{-1}]$ , and  $(R, i)$  induces a left adjoint for that functor.*

*Proof.* The two claims are formally dual; we will prove the first version.

By [corollary 4.1.13](#),  $(\mathcal{M}, \mathcal{U})$  admits a calculus of cospans, so we may use the fundamental theorem of calculi of cospans ([3.5.6](#)) and [lemma 4.1.8](#) to deduce that the canonical functor  $\text{Ho } \mathcal{M}_c \rightarrow \mathcal{M}[\mathcal{U}^{-1}]$  is indeed fully faithful.

On the other hand, [lemma 4.1.32](#) says the localising functor  $\mathcal{M} \rightarrow \mathcal{M}[\mathcal{U}^{-1}]$  sends weak equivalences in  $\mathcal{M}_c$  to isomorphisms in  $\mathcal{M}[\mathcal{U}^{-1}]$ , so we may apply [proposition 3.3.15](#) to deduce that the canonical functor  $\mathcal{M}[\mathcal{U}^{-1}] \rightarrow \text{Ho } \mathcal{M}$  has a fully faithful left adjoint defined by  $Q$ . On the other hand, [proposition 4.1.34](#) implies that the canonical functor  $\text{Ho } \mathcal{M}_c \rightarrow \text{Ho } \mathcal{M}$  is fully faithful with a *right* adjoint defined by  $Q$ . Thus, we have the following hom-set bijections,

$$\text{Ho } \mathcal{M}_c(X, QY) \cong \text{Ho } \mathcal{M}(X, Y) \cong \mathcal{M}[\mathcal{U}^{-1}](QX, Y)$$

and these bijections are moreover natural in  $X$ . On the other hand, since  $X$  is a cofibrant object, the morphism  $p_X : QX \rightarrow X$  is invertible in  $\mathcal{M}[\mathcal{U}^{-1}]$ ; and  $p$  defines a natural transformation of functors  $\text{Ho } \mathcal{M}_c \rightarrow \mathcal{M}[\mathcal{U}^{-1}]$ , so we have obtain from  $(Q, p)$  a right adjoint for the functor  $\text{Ho } \mathcal{M}_c \rightarrow \mathcal{M}[\mathcal{U}^{-1}]$  induced by the inclusion  $\mathcal{M}_c \hookrightarrow \mathcal{M}$ , as required. ■

The following proposition extends a result of Joyal [[2010](#)].

**Proposition 4.4.6.** *Let  $\mathcal{M}$  and  $\mathcal{M}'$  be two saturated derivable categories with the same underlying category and let  $\mathcal{M}_f$  and  $\mathcal{M}'_f$  be the full subcategories of fibrant objects in  $\mathcal{M}$  and  $\mathcal{M}'$ , respectively. Consider the following statements:*

- (i) *Every weak equivalence in  $\mathcal{M}$  is a weak equivalence in  $\mathcal{M}'$ .*
- (ii)  *$\mathcal{M}'_f$  is a full relative subcategory of  $\mathcal{M}_f$ .*
- (iii) *Every fibrant object in  $\mathcal{M}'$  is a fibrant object in  $\mathcal{M}$ .*

*If  $\mathcal{M}$  and  $\mathcal{M}'$  have the same cofibrations, then (i)  $\Rightarrow$  (ii); we always have (ii)  $\Rightarrow$  (iii); and if  $\mathcal{M}'$  is a saturated derivable category with the same cofibrations as  $\mathcal{M}$ , then (iii)  $\Rightarrow$  (i).*

*Proof.* (i)  $\Rightarrow$  (ii). Every trivial cofibration in  $\mathcal{M}$  is a trivial cofibration in  $\mathcal{M}'$ , so every fibrant object in  $\mathcal{M}'$  is a fibrant object in  $\mathcal{M}$ . Since  $\mathcal{M}$  and  $\mathcal{M}'$  have the same cofibrations, they also have the same trivial fibrations (by [theorem 4.1.12](#)), so [lemma 4.1.32](#) implies every weak equivalence in  $\mathcal{M}'_f$  is also a weak equivalence in  $\mathcal{M}_f$ . But we assumed every weak equivalence in  $\mathcal{M}$  is a weak equivalence in  $\mathcal{M}'_f$ , so this implies that a morphism in  $\mathcal{M}'_f$  is a weak equivalence if and only if it is a weak equivalence in  $\mathcal{M}_f$ , as required.

(ii)  $\Rightarrow$  (iii). Immediate.

(iii)  $\Rightarrow$  (i). Since  $\mathcal{M}$  and  $\mathcal{M}'$  have the same cofibrations,  $\mathcal{M}'_f$  is a full relative subcategory of  $\mathcal{M}_f$ , and [proposition 4.4.5](#) implies that the induced functor  $\text{Ho } \mathcal{M}'_f \rightarrow \text{Ho } \mathcal{M}_f$  has a left adjoint, say  $L : \text{Ho } \mathcal{M}_f \rightarrow \text{Ho } \mathcal{M}'_f$ .

Let  $(R, i)$  and  $(R', i')$  be right Quillen deformations for  $\mathcal{M}$  and  $\mathcal{M}'$ , respectively. Axiom CM2 implies a morphism  $f : X \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$  (resp. in  $\mathcal{M}'$ ) if and only if  $Rf : RX \rightarrow RY$  is a weak equivalence in  $\mathcal{M}$  (resp.  $R'f : R'X \rightarrow R'Y$  is a weak equivalence in  $\mathcal{M}'$ ). The uniqueness of left adjoints implies  $LR \cong R'$  as functors  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho } \mathcal{M}'_f$ , where  $\mathcal{V}$  is the class of trivial fibrations in  $\mathcal{M}$  (or  $\mathcal{M}'$ ), so if  $\mathcal{M}'$  is a saturated derivable category, it follows that every weak equivalence in  $\mathcal{M}$  is also a weak equivalence in  $\mathcal{M}'$ . ■

**Theorem 4.4.7** (Determination principle). *The model structure on a derivable category is uniquely determined by any one of the following sets of data:*

- (i) *The cofibrations and the weak equivalences.*

- (ii) *The cofibrations and the trivial cofibrations.*
- (iii) *The cofibrations and the fibrant objects.*
- (iv) *The cofibrations and the fibrations.*
- (v) *The trivial cofibrations and the trivial fibrations.*
- (i') *The fibrations and the weak equivalences.*
- (ii') *The fibrations and the trivial fibrations.*
- (iii') *The fibrations and the cofibrant objects.*

*Proof.* (i) and (ii). By [theorem 4.1.12](#), the fibrations are precisely the morphisms with the right lifting property with respect to every trivial cofibration.

(iii). Apply [proposition 4.4.6](#) and reduce to case (i).

(iv). The trivial cofibrations are precisely the morphisms with the left lifting property with respect to all fibrations, and the trivial fibrations are precisely the morphisms with the right lifting property with respect to all cofibrations, so this reduces to case (v).

(v). Axioms CM2 and CM5 imply that every weak equivalence is of the form  $p \circ i$  where  $i$  is a trivial cofibration and  $p$  is a trivial fibration. Thus, the trivial cofibrations and the trivial fibrations together determine the weak equivalences. On the other hand, the trivial cofibrations determine the fibrations, and the trivial fibrations determine the cofibrations, thus the entire model structure is determined. ■

The following observation is due to Rezk [\[2012\]](#).

**Proposition 4.4.8.** *Let  $\mathcal{M}$  be a model category. The following are equivalent:*

- (i)  *$\mathcal{M}$  is a right proper model category.*
- (ii) *For all weak equivalences  $g : Y' \rightarrow Y$  in  $\mathcal{M}$ , the adjunction*

$$\Sigma_g \dashv g^* : \mathcal{M}_{/Y'} \rightarrow \mathcal{M}_{/Y}$$

*is a Quillen equivalence.*

(iii) For all weak equivalences  $g : Y' \rightarrow Y$  in  $\mathcal{M}$ , the derived adjunction

$$\mathbf{L}\Sigma_g \dashv \mathbf{R}g^* : \mathrm{Ho} \mathcal{M}_{/Y'} \rightarrow \mathrm{Ho} \mathcal{M}_{/Y}$$

is an adjoint equivalence.

*Proof.* (i)  $\Rightarrow$  (ii). See [proposition 4.3.23](#).

(ii)  $\Rightarrow$  (i). Consider a pullback diagram in  $\mathcal{M}$  of the form below,

$$\begin{array}{ccc} X' & \xrightarrow{f} & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{g} & Y \end{array}$$

where  $p : X \rightarrow Y$  is a fibration and  $g : Y' \rightarrow Y$  is a weak equivalence. Then  $p : X \rightarrow Y$  is a fibrant object in  $\mathcal{M}_{/Y}$ , and  $p' : X' \rightarrow Y'$  is isomorphic to  $g^*(p)$ . Let  $(\tilde{X}', q)$  be a cofibrant replacement for  $X'$ . The morphism  $q : p' \circ q \rightarrow p'$  is a weak equivalence in  $\mathcal{M}_{/Y'}$ , and since  $\Sigma_g \dashv g^* : \mathcal{M}_{/Y} \rightarrow \mathcal{M}_{/Y'}$  is a Quillen equivalence, its adjoint transpose is a weak equivalence in  $\mathcal{M}_{/Y}$ , i.e. the morphism  $f \circ q : \tilde{X}' \rightarrow X$  is a weak equivalence in  $\mathcal{M}$ . Axiom CM2 then implies  $f : X' \rightarrow X$  itself is a weak equivalence in  $\mathcal{M}$ , as required.

(ii)  $\Leftrightarrow$  (iii). Since model categories are saturated homotopical categories (by [theorem 4.4.1](#)), we may apply [proposition 3.3.24](#).  $\blacksquare$

## 4.5 Reedy diagrams

**Prerequisites.** §§4.1, 4.3

**Definition 4.5.1.** A **direct category** is a category  $C$  equipped with a function  $\mathrm{deg} : \mathrm{ob} C \rightarrow \mathbb{N}$  such that, if  $f : A \rightarrow B$  is a morphism in  $C$ , then  $\mathrm{deg} A \leq \mathrm{deg} B$ , with equality if and only if  $f = \mathrm{id}_A = \mathrm{id}_B$ . Dually, an **inverse category** is a category  $C$  equipped with a function  $\mathrm{deg} : \mathrm{ob} C \rightarrow \mathbb{N}$  such that, if  $f : A \rightarrow B$  is a morphism in  $C$ , then  $\mathrm{deg} A \geq \mathrm{deg} B$ , with equality if and only if  $f = \mathrm{id}_A = \mathrm{id}_B$ .

**REMARK 4.5.2.** The degree function for a direct or inverse category is not determined by the underlying category: for example, if  $\mathrm{deg}$  is a degree function for  $C$ , then so is  $A \mapsto 1 + \mathrm{deg} A$ . However, the partial order induced by  $\mathrm{deg}$

is determined by the underlying category of a direct (resp. inverse) category:  $\deg A \leq \deg B$  if and only if there exists a morphism  $A \rightarrow B$  (resp.  $B \rightarrow A$ ) in  $C$ ; note that this relation is indeed antisymmetric because the only morphisms that do not change the degree are identity morphisms.

**Definition 4.5.3.** A **Reedy category** is a category  $C$  equipped with two subcategories, the **direct subcategory**  $C_{\rightarrow}$  and the **inverse subcategory**  $C_{\leftarrow}$ , such that the following conditions are satisfied:

- $\text{ob } C = \text{ob } C_{\rightarrow} = \text{ob } C_{\leftarrow}$ .
- There exists a function  $\deg : \text{ob } C \rightarrow \mathbb{N}$  such that  $(C_{\rightarrow}, \deg)$  is a direct category and  $(C_{\leftarrow}, \deg)$  is an inverse category.
- Every morphism in  $C$  admits a unique factorisation of the form  $s \circ d$ , where  $d$  is in  $C_{\leftarrow}$  and  $s$  is in  $C_{\rightarrow}$ .

A **Reedy diagram** in a category  $\mathcal{M}$  is a functor  $C \rightarrow \mathcal{M}$ , where  $C$  is a Reedy category.

**REMARK 4.5.4.** Any direct (resp. inverse) category is a Reedy category in a trivial way: take the whole category as the direct (resp. inverse) subcategory, and take  $\text{disc ob } C$  as the inverse (resp. direct) subcategory.

**Example 4.5.5.** The simplex category  $\Delta$  is a Reedy category, where the direct subcategory consists of all injective maps, and the inverse subcategory consists of all surjective maps; note that the unique factorisation condition is implied by [theorem I.1.4](#).

**REMARK 4.5.6.** The opposite of any Reedy category is automatically a Reedy category, after exchanging the direct and inverse subcategories.

**Definition 4.5.7.** Let  $A$  be an object in a Reedy category  $C$ .

- The **latching category** of  $C$  at  $A$ , denoted by  $\partial(C_{\rightarrow} \downarrow A)$ , is the largest full subcategory of the slice category  $(C_{\rightarrow} \downarrow A)$  that does *not* contain the object  $\text{id}_A : A \rightarrow A$ .
- The **matching category** of  $C$  at  $A$ , denoted by  $\partial(A \downarrow C_{\leftarrow})$ , is the largest full subcategory of the slice category  $(A \downarrow C_{\leftarrow})$  that does *not* contain the object  $\text{id}_A : A \rightarrow A$ .

REMARK 4.5.8. If  $\mathcal{C}$  is a Reedy category whose direct (resp. inverse) subcategory is discrete, then all its latching (resp. matching) categories are empty.

**Definition 4.5.9.** Let  $\mathcal{M}$  be a category with limits and colimits for all finite (resp. small) diagrams, and let  $X : \mathcal{C} \rightarrow \mathcal{M}$  be a finite (resp. small) Reedy diagram.

- The **latching object** of  $X$  at  $A$ , denoted by  $L_A(X)$ , is the colimit of the diagram  $\partial(\mathcal{C}_{\rightarrow} \downarrow A) \rightarrow \mathcal{M}$  obtained by composing  $X : \mathcal{C} \rightarrow \mathcal{M}$  and the projection  $\partial(\mathcal{C}_{\rightarrow} \downarrow A) \rightarrow \mathcal{C}$ .
- The **matching object** of  $X$  at  $A$ , denoted by  $M_A(X)$ , is the limit of the diagram  $\partial(A \downarrow \mathcal{C}_{\leftarrow}) \rightarrow \mathcal{M}$  obtained by composing  $X : \mathcal{C} \rightarrow \mathcal{M}$  and the projection  $\partial(A \downarrow \mathcal{C}_{\leftarrow}) \rightarrow \mathcal{C}$ .
- The **latching morphism** of  $X$  at  $A$  is the morphism  $L_A(X) \rightarrow XA$  induced by the inclusion  $\partial(\mathcal{C}_{\rightarrow} \downarrow A) \hookrightarrow (\mathcal{C}_{\rightarrow} \downarrow A)$ .
- The **matching morphism** of  $X$  at  $A$  is the morphism  $XA \rightarrow M_A(X)$  induced by the inclusion  $\partial(A \downarrow \mathcal{C}_{\leftarrow}) \hookrightarrow (A \downarrow \mathcal{C}_{\leftarrow})$ .

REMARK 4.5.10. The latching object  $L_A(X)$  is functorial in  $A$  (as  $A$  varies in the direct subcategory), and the matching object  $M_A(X)$  is functorial in  $A$  (as  $A$  varies in the inverse subcategory). Of course, it goes without saying that  $L_A(X)$  and  $M_A(X)$  are both functorial in  $X$  (as  $X$  varies in  $[\mathcal{C}, \mathcal{M}]$ ).

**Definition 4.5.11.** Let  $\mathcal{M}$  be a category with limits and colimits for all finite (resp. small) diagrams, and let  $\varphi : X \Rightarrow Y$  be a natural transformation between two finite (resp. small) Reedy diagrams  $X, Y : \mathcal{C} \rightarrow \mathcal{M}$ .

- The **relative latching morphism**  $XA \cup^{L_A(X)} L_A(Y) \rightarrow YA$  is the unique morphism in  $\mathcal{M}$  making the diagram below commute,

$$\begin{array}{ccc}
 L_A(X) & \xrightarrow{L_A(\varphi)} & L_A(Y) \\
 \downarrow & & \downarrow \\
 XA & \longrightarrow & XA \cup^{L_A(X)} L_A(Y) \\
 & \searrow \varphi_A & \dashrightarrow \\
 & & YA
 \end{array}$$

where the arrows  $L_A(X) \rightarrow XA$  and  $L_A(Y) \rightarrow YA$  are the latching morphisms and the square is a pushout square.

- The **relative matching morphism**  $X A \rightarrow M_A(X) \times_{M_A(Y)} Y A$  is the unique morphism in  $\mathcal{M}$  making the diagram below commute,

$$\begin{array}{ccc}
 X A & \xrightarrow{\varphi_A} & Y A \\
 \downarrow & \dashrightarrow & \downarrow \\
 M_A(X) \times_{M_A(Y)} Y A & \xrightarrow{\quad} & Y A \\
 \downarrow & & \downarrow \\
 M_A(X) & \xrightarrow{M_A(\varphi)} & M_A(Y)
 \end{array}$$

where the arrows  $X A \rightarrow M_A(X)$  and  $Y A \rightarrow M_A(Y)$  are the latching morphisms and the square is a pullback square.

REMARK 4.5.12. If the direct subcategory of  $\mathbb{C}$  is discrete, then  $L_A(X)$  is an initial object in  $\mathcal{M}$  for all  $A$  and  $X$ , so the relative latching morphism of a natural transformation  $\varphi : X \Rightarrow Y$  at any object  $A$  in  $\mathbb{C}$  is (isomorphic to)  $\varphi_A : X A \rightarrow Y A$  itself.

Dually, if the inverse subcategory of  $\mathbb{C}$  is discrete, then  $M_A(X)$  is a terminal object in  $\mathcal{M}$  for all  $A$  and  $X$ , so the relative matching morphism of a natural transformation  $\varphi : X \Rightarrow Y$  at any object  $A$  in  $\mathbb{C}$  is (isomorphic to)  $\varphi_A : X A \rightarrow Y A$  itself.

**Definition 4.5.13.** Let  $\mathcal{M}$  be a model category, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, and assume  $\mathcal{M}$  has limits and colimits for all finite (resp. small) diagrams.

- A **Reedy weak equivalence** in  $[\mathbb{C}, \mathcal{M}]$  is a natural transformation such that all its components are weak equivalences in  $\mathcal{M}$ .
- A **Reedy cofibration** in  $[\mathbb{C}, \mathcal{M}]$  is a natural transformation such that all its relative latching morphisms are cofibrations in  $\mathcal{M}$ .
- A **Reedy fibration** in  $[\mathbb{C}, \mathcal{M}]$  is a natural transformation such that all its relative matching morphisms are fibrations in  $\mathcal{M}$ .

**Proposition 4.5.14.** *With notation as in the definition:*

- *If  $\varphi : X \Rightarrow Y$  is a Reedy cofibration (resp. trivial Reedy cofibration) in  $[\mathbb{C}, \mathcal{M}]$ , then, for each object  $A$  in  $\mathbb{C}$ , the morphisms  $L_A(\varphi) : L_A(X) \rightarrow L_A(Y)$  and  $\varphi_A : X A \rightarrow Y A$  are cofibrations (resp. trivial cofibrations).*

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- If  $\varphi : X \Rightarrow Y$  is a Reedy fibration (resp. trivial Reedy fibration) in  $[\mathbb{C}, \mathcal{M}]$ , then, for each object  $A$  in  $\mathbb{C}$ , the morphisms  $M_A(\varphi) : M_A(X) \rightarrow M_A(Y)$  and  $\varphi_A : XA \rightarrow YA$  are fibrations (resp. trivial fibrations).

*Proof.* See Propositions 15.3.11 and 15.3.14 in [Hirschhorn, 2003].

□ TODO: Check if this requires functorial factorisation. Surely not!

**Proposition 4.5.15.** *With notation as in the definition:*

- A Reedy cofibration in  $[\mathbb{C}, \mathcal{M}]$  is a Reedy weak equivalence if and only if all its relative latching morphisms are trivial cofibrations in  $\mathcal{M}$ .
- A Reedy fibration in  $[\mathbb{C}, \mathcal{M}]$  is a Reedy weak equivalence if and only if all its relative matching morphisms are trivial fibrations in  $\mathcal{M}$ .

*Proof.* See Theorem 15.3.15 in [Hirschhorn, 2003].

□ TODO: Check if this requires functorial factorisation. Surely not!

**Proposition 4.5.16.** *Let  $\mathcal{M}$  be a model category, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, and assume  $\mathcal{M}$  has limits and colimits for all finite (resp. small) diagrams.*

- If the injective model structure on  $[\mathbb{C}, \mathcal{M}]$  exists, then the trivial adjunction

$$\text{id} \dashv \text{id} : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$$

is a Quillen equivalence between the injective model structure and the Reedy model structure.

- If the projective model structure on  $[\mathbb{C}, \mathcal{M}]$  exists, then the trivial adjunction

$$\text{id} \dashv \text{id} : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$$

is a Quillen equivalence between the the Reedy model structure and the projective model structure.

*Proof.* This is an immediate consequence of [proposition 4.5.14](#). ■

**Proposition 4.5.17.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, let*

$$F \dashv G : \mathcal{N} \rightarrow \mathcal{M}$$



be a Quillen adjunction, and assume  $\mathcal{M}$  and  $\mathcal{N}$  have limits and colimits for all finite (resp. small) diagrams. Then the induced adjunction

$$[\mathbb{C}, F] \dashv [\mathbb{C}, G] : [\mathbb{C}, \mathcal{N}] \rightarrow [\mathbb{C}, \mathcal{M}]$$

is a Quillen adjunction, where  $[\mathbb{C}, \mathcal{M}]$  and  $[\mathbb{C}, \mathcal{N}]$  are equipped with their Reedy model structures.

*Proof.* Since  $F : \mathcal{M} \rightarrow \mathcal{N}$  preserves colimits and cofibrations, it follows that  $[\mathbb{C}, F] : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{N}]$  preserves Reedy cofibrations; dually,  $[\mathbb{C}, G] : [\mathbb{C}, \mathcal{N}] \rightarrow [\mathbb{C}, \mathcal{M}]$  preserves Reedy fibrations. Thus, by [proposition 4.3.2](#), we have a Quillen adjunction. ■

**Lemma 4.5.18.** *Let  $\mathcal{M}$  be a model category, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, and assume  $\mathcal{M}$  has limits and colimits for all finite (resp. small) diagrams.*

- *A diagram  $X : \mathbb{C} \rightarrow \mathcal{M}$  is Reedy cofibrant if and only if every latching morphism of  $X$  is a cofibration in  $\mathcal{M}$ .*
- *A diagram  $X : \mathbb{C} \rightarrow \mathcal{M}$  is Reedy fibrant if and only if every matching morphism of  $X$  is a fibration in  $\mathcal{M}$ .*

*Proof.* Let  $0$  be an initial object in  $\mathcal{M}$  and let  $1$  be a terminal object in  $\mathcal{M}$ . It is a standard fact that  $\Delta 0$  is an initial object in  $[\mathbb{C}, \mathcal{M}]$  and  $\Delta 1$  is a terminal object in  $[\mathbb{C}, \mathcal{M}]$ , so the claims follow from the observation that the latching morphism  $L_A(\Delta 0) \rightarrow 0$  and the matching morphism  $1 \rightarrow M_A(\Delta 1)$  are isomorphisms for all objects  $A$  in  $\mathbb{C}$ . ■

**Lemma 4.5.19.** *With notation as in the previous lemma:*

- *If  $X : \mathbb{C} \rightarrow \mathcal{M}$  is a Reedy cofibrant diagram, then, for every object  $A$  in  $\mathbb{C}$ , the object  $X A$  and the latching object  $L_A(X)$  are cofibrant objects in  $\mathcal{M}$ .*
- *If  $X : \mathbb{C} \rightarrow \mathcal{M}$  is a Reedy fibrant diagram, then, for every object  $A$  in  $\mathbb{C}$ , the object  $X A$  and the matching object  $M_A(X)$  are fibrant objects in  $\mathcal{M}$ .*

*Proof.* Apply [proposition 4.5.14](#). ■

**Theorem 4.5.20.** *With notation as in the definition, the announced weak equivalences, cofibrations, and fibrations constitute a model structure on  $[\mathbb{C}, \mathcal{M}]$ , called the **Reedy model structure**; moreover, if  $\mathcal{M}$  is a DHK model category, then so is  $[\mathbb{C}, \mathcal{M}]$  when equipped with the Reedy model structure.*

*Proof.* See Theorem 5.2.5 in [Hovey, 1999], or Theorem 15.3.4 in [Hirschhorn, 2003]. □

**Corollary 4.5.21.** *Let  $\mathcal{M}$  be a model category, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, and assume  $\mathcal{M}$  has limits and colimits for all finite (resp. small) diagrams.*

- *If the direct subcategory of  $\mathbb{C}$  is discrete, then the Reedy model structure on  $[\mathbb{C}, \mathcal{M}]$  is the injective model structure.*
- *If the inverse subcategory of  $\mathbb{C}$  is discrete, then the Reedy model structure on  $[\mathbb{C}, \mathcal{M}]$  is the projective model structure.*

*Proof.* This follows from the theorem and [remark 4.5.12](#). ■

**Definition 4.5.22.** Let  $C$  be a Reedy category.

- $C$  has **cofibrant constants** if, for every object  $A$  in  $\mathbb{C}$ , the latching category  $\partial(C_{\rightarrow} \downarrow A)$  has at most one connected component.
- $C$  has **fibrant constants** if, for every object  $A$  in  $\mathbb{C}$ , the matching category  $\partial(A \downarrow C_{\leftarrow})$  has at most one connected component.

**Example 4.5.23.** Let  $C$  be a Reedy category.

- If the direct subcategory of  $C$  is discrete, then  $C$  has cofibrant constants. (In fact, every latching category is empty.)
- If the inverse subcategory of  $C$  is discrete, then  $C$  has fibrant constants. (In fact, every matching category is empty.)

**Proposition 4.5.24.** *Let  $\mathcal{M}$  be a model category, let  $\mathbb{C}$  be a finite (resp. small) Reedy category, and assume  $\mathcal{M}$  has limits and colimits for all finite (resp. small) diagrams.*

- *If  $\mathbb{C}$  has cofibrant constants, then the functor  $\Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  is a left Quillen functor.*

- If  $\mathbb{C}$  has fibrant constants, then the functor  $\Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  is a right Quillen functor.

*Proof.* The two claims are formally dual; we will prove the second version.

If the matching category  $\partial(A \downarrow \mathbb{C}_{\leftarrow})$  is empty, then the matching object of  $\Delta X$  at  $A$  is a terminal object in  $\mathcal{M}$ , so the relative matching morphism of  $\Delta f$  at  $A$  is isomorphic to  $f : X \rightarrow Y$  in this case.

On the other hand, if the matching category  $\partial(A \downarrow \mathbb{C}_{\leftarrow})$  of  $\mathbb{C}$  has only one connected component, then the matching morphism  $X \rightarrow M_A(\Delta X)$  must be an isomorphism, so the relative matching morphism of  $\Delta f$  at  $A$  is an isomorphism, hence a (trivial) fibration in particular.

We now conclude that, for any fibration  $f : X \rightarrow Y$  in  $\mathcal{M}$ , every relative matching morphism of  $\Delta f : \Delta X \rightarrow \Delta Y$  is a fibration. Clearly, the functor  $\Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  preserves weak equivalences, so this completes the proof that  $\Delta$  is a right Quillen functor.  $\blacksquare$

**Theorem 4.5.25** (Hirschhorn). *Let  $\mathbb{C}$  be a small Reedy category.*

- (i)  $\mathbb{C}$  has cofibrant constants.
- (ii)  $\Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  is a left Quillen functor for all DHK model categories  $\mathcal{M}$ .
- (iii) For every cofibrant object  $X$  in any DHK model category  $\mathcal{M}$ , the constant diagram  $\Delta X : \mathbb{C} \rightarrow \mathcal{M}$  is Reedy cofibrant.

*Dually, the following are equivalent:*

- (i')  $\mathbb{C}$  has fibrant constants.
- (ii')  $\Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  is a right Quillen functor for all DHK model categories  $\mathcal{M}$ .
- (iii') For every fibrant object  $X$  in any DHK model category  $\mathcal{M}$ , the constant diagram  $\Delta X : \mathbb{C} \rightarrow \mathcal{M}$  is Reedy fibrant.

*Proof.* (i)  $\Rightarrow$  (ii). This is the content of the earlier proposition.

(ii)  $\Rightarrow$  (iii). Left Quillen functors preserve cofibrant objects.

(iii)  $\Rightarrow$  (i). Take  $\mathcal{M}$  to be **Set** equipped with the mono–epi model structure,<sup>[5]</sup> and consider the constant diagram  $\Delta 1$ . Since  $1$  is a cofibrant object in  $\mathcal{M}$ ,  $\Delta 1$  must be a Reedy cofibrant object in  $[\mathbb{C}, \mathcal{M}]$ . It is not hard to see that the latching object  $L_A(\Delta 1)$  is the set of connected components of the latching category  $\partial(\mathbb{C}_\rightarrow \downarrow A)$ , so by lemma 4.5.18,  $\partial(\mathbb{C}_\rightarrow \downarrow A)$  has at most one connected component. ■

**Corollary 4.5.26.** *Let  $\mathcal{M}$  be a DHK model category and let  $\mathbb{C}$  be a small Reedy category.*

- *If  $\mathbb{C}$  has fibrant constants, then the adjunction  $\lim_{\rightarrow \mathbb{C}} \dashv \Delta : \mathcal{M} \rightarrow [\mathbb{C}, \mathcal{M}]$  is deformable.*
- *If  $\mathbb{C}$  has cofibrant constants, then the adjunction  $\Delta \dashv \lim_{\leftarrow \mathbb{C}} : [\mathbb{C}, \mathcal{M}] \rightarrow \mathcal{M}$  is deformable.*

*Proof.* Apply theorem 4.3.12 to the above result. ■

For the remainder of this section, we follow [Barwick, 2007] and discuss the functoriality of the Reedy model structure.

**Definition 4.5.27.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be Reedy categories. A **morphism of Reedy categories**  $\mathcal{C} \rightarrow \mathcal{D}$  is a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  that sends every morphism in  $\mathcal{C}_\rightarrow$  to  $\mathcal{D}_\rightarrow$  and every morphism in  $\mathcal{C}_\leftarrow$  to  $\mathcal{D}_\leftarrow$ , or equivalently, a commutative diagram of functors of the form below:

$$\begin{array}{ccc}
 \mathcal{C}_\rightarrow & \xrightarrow{F_\rightarrow} & \mathcal{D}_\rightarrow \\
 \downarrow & & \downarrow \\
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 \uparrow & & \uparrow \\
 \mathcal{C}_\leftarrow & \xrightarrow{F_\leftarrow} & \mathcal{D}_\leftarrow
 \end{array}$$

**Lemma 4.5.28.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a morphism of Reedy categories. If  $\mathcal{D}$  is any object in  $\mathcal{D}$ , then:*

- *There is a unique Reedy category structure on the comma category  $(F \downarrow \mathcal{D})$  making the projection  $(F \downarrow \mathcal{D}) \rightarrow \mathcal{C}$  a morphism of Reedy categories.*

---

[5] See example 4.1.6.

- *There is a unique Reedy category structure on the comma category  $(D \downarrow F)$  making the projection  $(D \downarrow F) \rightarrow C$  a morphism of Reedy categories.*

*Proof.* Obvious. ◆

While it is true that any functor  $F : \mathbb{C} \rightarrow \mathbb{D}$  induces a homotopical functor  $F^* : [\mathbb{D}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$ , even if  $F$  is a morphism of Reedy categories,  $F^*$  need not be either a left Quillen functor or a right Quillen functor. Instead, we must consider the following:

**Definition 4.5.29.** Let  $C$  and  $D$  be Reedy categories.

- A **left fibration of Reedy categories** is a morphism  $F : C \rightarrow D$  such that, for any object  $D$  in  $D$ , the comma category  $(F \downarrow D)$  has fibrant constants.
- A **right fibration of Reedy categories** is a morphism  $F : C \rightarrow D$  such that, for any object  $D$  in  $D$ , the comma category  $(D \downarrow F)$  has cofibrant constants.

**REMARK 4.5.30.** A Reedy category  $C$  has fibrant (resp. cofibrant) constants if and only if the unique morphism  $C \rightarrow \mathbb{1}$  is a left (resp. right) fibration.

**REMARK 4.5.31.** A morphism  $F : C \rightarrow D$  of Reedy categories is a left (resp. right) fibration if and only if  $F^{\text{op}} : C^{\text{op}} \rightarrow D^{\text{op}}$  is a right (resp. left) fibration.

**Theorem 4.5.32** (Barwick). *Let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a morphism between small Reedy categories. The following are equivalent:*

- (i) *The morphism  $F : \mathbb{C} \rightarrow \mathbb{D}$  is a left fibration of Reedy categories.*
- (ii) *For every object  $D$  in  $\mathbb{D}$  and every object  $(C, h)$  in  $(F \downarrow D)$ , the matching category  $\partial((C, h) \downarrow (F \downarrow D)_{\leftarrow})$  has at most one connected component.*
- (iii) *The functor  $F^* : [\mathbb{D}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$  is a right Quillen functor for all DHK model categories  $\mathcal{M}$ .*

*Dually, the following are equivalent:*

- (i') *The morphism  $F : \mathbb{C} \rightarrow \mathbb{D}$  is a right fibration of Reedy categories.*
- (ii') *For every object  $D$  in  $\mathbb{D}$  and every object  $(C, h)$  in  $(D \downarrow F)$ , the latching category  $\partial((D \downarrow F) \downarrow (C, h)_{\rightarrow})$  has at most one connected component.*

(iii') *The functor  $F^* : [\mathbb{D}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$  is a left Quillen functor for all DHK model categories  $\mathcal{M}$ .*

*Proof.* See Lemma 2.6 and Theorem 2.7 in [Barwick, 2007]. ■

## 4.6 Virtual cofibrancy and fibrancy

**Prerequisites.** §§ 1.1, 3.1, 3.3, 4.1, 4.5, A.1, A.5.

In this section, we follow [DHKS, §23]. As usual, for each natural number  $n$ , let  $[n]$  denote the category  $\{0 \rightarrow \dots \rightarrow n\}$  corresponding to the finite ordinal  $\{0, \dots, n\}$ , and let  $\mathbf{\Delta}$  be the category whose objects are the  $[n]$  and whose morphisms are functors.

**Definition 4.6.1.** The **category of simplices** of a (small) category  $\mathbb{C}$  is the category  $\mathbf{\Delta}(\mathbb{C})$  defined below:

- The objects are functors  $[n] \rightarrow \mathbb{C}$ .
- The morphisms  $(f : [m] \rightarrow \mathbb{C}) \rightarrow (g : [n] \rightarrow \mathbb{C})$  are functors  $[m] \rightarrow [n]$  making the evident triangle commute (strictly).
- Composition and identities are the obvious ones.

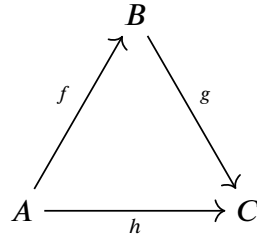
We write  $\pi_{\mathbf{\Delta}} : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbf{\Delta}$  for the evident projection functor that sends an object  $[n] \rightarrow \mathbb{C}$  in  $\mathbf{\Delta}(\mathbb{C})$  to the object  $[n]$  in  $\mathbf{\Delta}$ .

¶ 4.6.2. To elucidate the above definition, it is helpful to introduce some notation for the objects in  $\mathbf{\Delta}(\mathbb{C})$ . It is not hard to see that a functor  $f : [n] \rightarrow \mathbb{C}$  is the same thing as a string of  $n$  composable morphisms in  $\mathbb{C}$ , e.g.

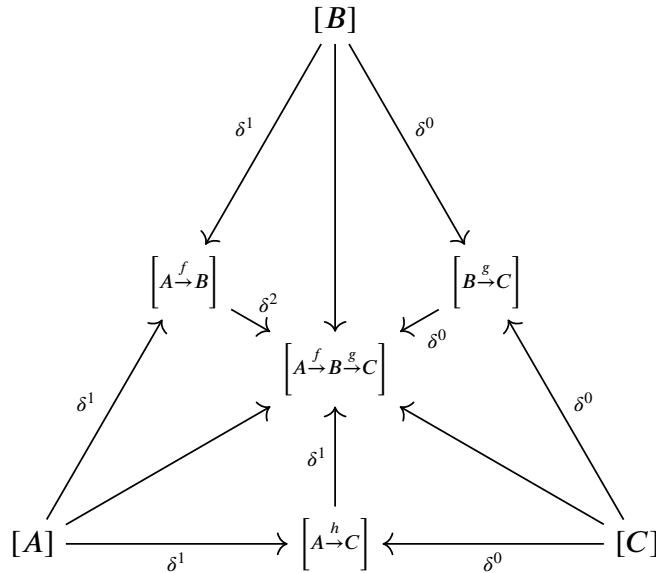
$$A_0 \xrightarrow{f_1} A_1 \longrightarrow \dots \longrightarrow A_{n-1} \xrightarrow{f_n} A_n$$

so let us write  $\left[ A_0 \xrightarrow{f_1} A_1 \cdots A_{n-1} \xrightarrow{f_n} A_n \right]$  for the corresponding object in  $\mathbf{\Delta}(\mathbb{C})$ . Since the projection  $\pi_{\mathbf{\Delta}} : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbf{\Delta}$  is faithful, we may borrow the notation of § 1.1 and write e.g.  $\delta^1 : [A_0] \rightarrow \left[ A_0 \xrightarrow{f_1} A_1 \right]$  for the unique morphism whose image under  $\pi_{\mathbf{\Delta}}$  is  $\delta^1 : [0] \rightarrow [1]$ .

Observe that, given a commutative triangle in  $\mathbb{C}$  of the form below,



we obtain the following commutative diagram in  $\mathbf{\Delta}(\mathbb{C})$ :



Similar phenomena occur for longer strings of composable morphisms. Thus, one may think of  $\mathbf{\Delta}(\mathbb{C})$  as being a kind of barycentric subdivision of  $\mathbb{C}$ ; notice also that the Mac Lane subdivision category  $\mathbb{C}^{\S}$  occurs as a subcategory of  $\mathbf{\Delta}(\mathbb{C})$ .

REMARK 4.6.3. There is an obvious natural isomorphism  $\mathbf{\Delta}(\mathbb{C}) \cong \mathbf{\Delta}(\mathbb{C}^{\text{op}})$  such that the following diagram of functors commutes,

$$\begin{array}{ccc}
 \mathbf{\Delta}(\mathbb{C}) & \xrightarrow{\cong} & \mathbf{\Delta}(\mathbb{C}^{\text{op}}) \\
 \pi_{\Delta} \downarrow & & \downarrow \pi_{\Delta} \\
 \mathbf{\Delta} & \xrightarrow{(-)^{\text{op}}} & \mathbf{\Delta}
 \end{array}$$

but in general there is no isomorphism between  $\mathbf{\Delta}(\mathbb{C})$  and  $\mathbf{\Delta}(\mathbb{C})^{\text{op}}$ .

**Proposition 4.6.4.** *Let  $X$  be a simplicial set, and let  $\Delta^\bullet : \Delta \rightarrow \mathbf{sSet}$  be the inclusion of the standard simplices.*

- (i) *The comma category  $(\Delta^\bullet \downarrow X)$  is a Reedy category, where the direct subcategory consists of all face operators and the inverse subcategory consists of all degeneracy operators.*
- (ii) *Moreover,  $(\Delta^\bullet \downarrow X)$  has fibrant constants.*

*Proof.* (i). The evident projection  $(\Delta^\bullet \downarrow X) \rightarrow \Delta$  is a discrete right fibration, so the Reedy category structure on  $\Delta$  induces one on  $(\Delta^\bullet \downarrow X)$ .

(ii). See Proposition 15.10.4 in [Hirschhorn, 2003]. □

**Corollary 4.6.5.** *The category  $\Delta(\mathbb{C})$  of simplices of a (small) category  $\mathbb{C}$  admits a Reedy category structure with fibrant constants.*

*Proof.* It is not hard to see that the category  $\Delta(\mathbb{C})$  as defined above is isomorphic to the comma category  $(\Delta^\bullet \downarrow N(\mathbb{C}))$ , where  $N(\mathbb{C})$  is the nerve of  $\mathbb{C}$ . ■

**Corollary 4.6.6.** *If  $\mathcal{M}$  is a DHK model category and  $\mathbb{C}$  is a small category, then:*

- *The functor  $\lim_{\rightarrow \Delta(\mathbb{C})} : [\Delta(\mathbb{C}), \mathcal{M}] \rightarrow \mathcal{M}$  sends Reedy weak equivalences between Reedy-cofibrant diagrams to weak equivalences between cofibrant objects.*
- *The functor  $\lim_{\leftarrow \Delta(\mathbb{C})^{\text{op}}} : [\Delta(\mathbb{C})^{\text{op}}, \mathcal{M}] \rightarrow \mathcal{M}$  sends Reedy weak equivalences between Reedy-fibrant diagrams to weak equivalences between fibrant objects.*

*Proof.* Apply Ken Brown's lemma (4.3.6) and corollary 4.5.26. ■

**Lemma 4.6.7.** *Let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between (small) categories.*

- (i)  *$\Delta(F) : \Delta(\mathbb{C}) \rightarrow \Delta(\mathbb{D})$  is a left fibration of Reedy categories.*
- (ii)  *$\Delta(F) : \Delta(\mathbb{C}) \rightarrow \Delta(\mathbb{D})$  is a right fibration of Reedy categories.*

*Proof.* (i). Let  $[D_0 \cdots D_n]$  be an object in  $\Delta(\mathbb{D})$ , let  $([C_0 \cdots C_m], h)$  be an object in the comma category  $(\Delta(F) \downarrow [D_0 \cdots D_n])$ . We will show that the matching category

$$\partial((([C_0 \cdots C_m], h) \downarrow (\Delta(F) \downarrow [D_0 \cdots D_n]))_{\leftarrow})$$



has at most one connected component.

First, note that the objects of this matching category are pairs  $(k, l)$ , where  $k$  is in  $\mathbf{\Delta}(\mathbb{C})_{\leftarrow}$ ,  $k \neq \text{id}_{[C_0 \dots C_m]}$ ,  $l$  is in  $\mathbf{\Delta}(\mathbb{D})$ , and  $h = l \circ \mathbf{\Delta}(F)k$ . Let  $(\sigma, \delta)$  be the codegeneracy–coface factorisation of  $\pi_{\Delta} h$  in  $\mathbf{\Delta}$ .

- If  $\sigma = \text{id}_{[m]}$ , then the matching category must be empty.
- If  $\sigma \neq \text{id}_{[m]}$ , then we may lift  $(\sigma, \delta)$  along the respective  $\pi_{\Delta}$  projections to obtain a terminal object in the matching category, so the matching category is connected *a fortiori*.

Thus, by [theorem 4.5.32](#),  $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbf{\Delta}(\mathbb{D})$  is a left fibration of Reedy categories.

(ii). A similar argument shows that  $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbf{\Delta}(\mathbb{D})$  is a right fibration of Reedy categories. ■

**Corollary 4.6.8.** *Let  $\mathcal{M}$  be a DHK model category and let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between small categories.*

- (i) *The functor  $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \rightarrow [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$  is a right Quillen functor.*
- (ii) *The functor  $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \rightarrow [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$  is a left Quillen functor.*

*Proof.* Apply [theorem 4.5.32](#). ■

**Definition 4.6.9.** Let  $\mathbb{C}$  be a (small) category and let  $\mathbf{\Delta}(\mathbb{C})$  be its category of simplices.

- The **left projection functor**  $\pi_L : \mathbf{\Delta}(\mathbb{C})^{\text{op}} \rightarrow \mathbb{C}$  is the functor defined by evaluating objects  $f : [n] \rightarrow \mathbb{C}$  in  $\mathbf{\Delta}(\mathbb{C})$  at the initial object in  $[n]$ .
- The **right projection functor**  $\pi_R : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbb{C}$  is the functor defined by evaluating objects  $f : [n] \rightarrow \mathbb{C}$  in  $\mathbf{\Delta}(\mathbb{C})$  at the terminal object in  $[n]$ .
- A **strong left equivalence** in  $\mathbf{\Delta}(\mathbb{C})$  is a morphism such that the underlying map in  $\mathbf{\Delta}$  preserves the initial object.
- A **strong right equivalence** in  $\mathbf{\Delta}(\mathbb{C})$  is a morphism such that the underlying map in  $\mathbf{\Delta}$  preserves the terminal object.

- The class of **weak left equivalences** in  $\mathbf{\Delta}(\mathbb{C})$  is the smallest subcategory that has the 2-out-of-6 property and contains all the strong left equivalences.
- The class of **weak right equivalences** in  $\mathbf{\Delta}(\mathbb{C})$  is the smallest subcategory that has the 2-out-of-6 property and contains all the strong right equivalences.

We write  $\mathbf{\Delta}(\mathbb{C})_L$  for the category of simplices of  $\mathbb{C}$  regarded as a relative category with weak equivalences the strong left equivalences, and we write  $\mathbf{\Delta}(\mathbb{C})_R$  for the category of simplices of  $\mathbb{C}$  regarded as a relative category with weak equivalences the strong right equivalences.

REMARK 4.6.10. The strong left (resp. right) equivalences in  $\mathbf{\Delta}(\mathbb{C})$  are closed under composition, and the left (resp. right) projection to  $\mathbb{C}$  sends strong left (resp. right) equivalences to identity morphisms, so if we regard  $\mathbf{\Delta}(\mathbb{C})$  as a relative category with weak equivalences the strong left (resp. right) equivalences, then the left (resp. right) projection functor becomes a relative functor.

Unfortunately, the subcategory of strong left (resp. right) equivalences in  $\mathbf{\Delta}(\mathbb{C})$  does not generally have the 2-out-of-6 property, or even the 2-out-of-3 property; one may rectify this by instead considering the class of weak left (resp. right) equivalences. An example of a weak left equivalence that is not a strong left equivalence is the morphism  $\delta^0 : \left[ A \xrightarrow{\text{id}} A \right] \rightarrow [A]$ : this is a weak left equivalence because  $\sigma^0 : [A] \rightarrow \left[ A \xrightarrow{\text{id}} A \right]$  is a strong left equivalence and  $\delta^0 \circ \sigma^0 = \text{id}_{[A]}$ , but  $\delta^0$  is not a strong left equivalence because the underlying cosimplicial operator in  $\mathbf{\Delta}$  sends 0 in [0] to 1 in [1].

REMARK 4.6.11. It is not hard to see that  $\mathbf{\Delta}(-)$  is a functor  $\mathbf{Cat} \rightarrow \mathbf{Cat}$  and that  $\pi_L$  (resp.  $\pi_R$ ) defines a natural transformation  $\mathbf{\Delta}(-)^{\text{op}} \Rightarrow \text{id}_{\mathbf{Cat}}$  (resp.  $\mathbf{\Delta}(-) \Rightarrow \text{id}_{\mathbf{Cat}}$ ).

**Lemma 4.6.12.** *Let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor, let  $\pi_L : \mathbf{\Delta}(\mathbb{C})^{\text{op}} \rightarrow \mathbb{C}$  be the left projection functor, and let  $\pi_R : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathbb{C}$  be the right projection functor. Then, for any object  $D$  in  $\mathbb{D}$ :*

- *The canonical comparison functor  $\mathbf{\Delta}((D \downarrow F))^{\text{op}} \rightarrow (D \downarrow F\pi_L)$  is an isomorphism.*
- *The canonical comparison functor  $\mathbf{\Delta}((F \downarrow D)) \rightarrow (F\pi_R \downarrow D)$  is an isomorphism.*

*Proof.* The two claims are formally dual; we will prove the first version.

As always, the comma category  $(D \downarrow F)$  fits into a comma square,

$$\begin{array}{ccc} (D \downarrow F) & \xrightarrow{P} & \mathbb{C} \\ \downarrow & \nearrow & \downarrow F \\ \mathbb{1} & \xrightarrow{D} & \mathbb{D} \end{array}$$

and the following diagram of functors commutes,

$$\begin{array}{ccc} \Delta((D \downarrow F))^{\text{op}} & \xrightarrow{\Delta(P)^{\text{op}}} & \Delta(\mathbb{C})^{\text{op}} \\ \pi_L \downarrow & & \downarrow \pi_L \\ (D \downarrow F) & \xrightarrow{P} & \mathbb{C} \end{array}$$

so the universal property of  $(D \downarrow F)$  gives us a canonical comparison functor  $\Delta((D \downarrow F))^{\text{op}} \rightarrow (D \downarrow F\pi_L)$ , as claimed. It is not hard to check that the second diagram is a pullback square, so the pasting lemma for comma squares implies that the comparison functor is an isomorphism. ■

**Proposition 4.6.13.** *Let  $\mathcal{M}$  be a DHK model category and let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between small categories.*

- *The functor  $\text{Ran}_{F\pi_L} : [\Delta(\mathbb{C})^{\text{op}}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  sends Reedy weak equivalences between Reedy-fibrant diagrams to componentwise weak equivalences between componentwise fibrant diagrams.*
- *The functor  $\text{Lan}_{F\pi_R} : [\Delta(\mathbb{C}), \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  sends Reedy weak equivalences between Reedy-cofibrant diagrams to componentwise weak equivalences between componentwise cofibrant diagrams.*

*Proof.* The two claims are formally dual; we will prove the second version.

Using the formula for  $\text{Lan}_{F\pi_R}$  given by [theorem A.5.15](#), we see that, for each object  $D$  in  $\mathbb{D}$ , the functor  $(\text{Lan}_{F\pi_R} -)(D) : [\Delta(\mathbb{C}), \mathcal{M}] \rightarrow \mathcal{M}$  is naturally isomorphic to the functor  $\varinjlim : [(F\pi_R \downarrow D), \mathcal{M}] \rightarrow \mathcal{M}$ ; but by [lemma 4.6.12](#), there is a canonical isomorphism  $(F\pi_R \downarrow D) \cong \Delta((F \downarrow D))$ , so  $(\text{Lan}_{F\pi_R} -)(D)$  is in turn naturally isomorphic to  $\varinjlim : [\Delta((F \downarrow D)), \mathcal{M}] \rightarrow \mathcal{M}$ . The claim now follows from [corollary 4.6.6](#). ■

**Theorem 4.6.14.** *Let  $\mathcal{M}$  be a DHK model category and let  $\mathbb{C}$  be a small category.*

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\pi_L^* \dashv \text{Ran}_{\pi_L} : [\mathbf{\Delta}(\mathbb{C})_L^{\text{op}}, \mathcal{M}]_h \rightarrow [\mathbb{C}, \mathcal{M}]$$

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\text{Lan}_{\pi_R} \dashv \pi_R^* : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbf{\Delta}(\mathbb{C})_R, \mathcal{M}]_h$$

*Proof.* See Proposition 23.2 in [DHKS]. □

**Definition 4.6.15.** Let  $\mathcal{M}$  be a DHK model category and let  $\mathbb{C}$  be a small category.

- A **virtually cofibrant diagram**  $X : \mathbb{C} \rightarrow \mathcal{M}$  is one for which there exists a Reedy-cofibrant diagram  $\tilde{X} : \mathbf{\Delta}(\mathbb{C}) \rightarrow \mathcal{M}$  such that  $\tilde{X}$  is in  $[\mathbf{\Delta}(\mathbb{C})_R, \mathcal{M}]_h$  and  $X \cong \text{Lan}_{\pi_R} \tilde{X}$ .
- A **virtually fibrant diagram**  $Y : \mathbb{C} \rightarrow \mathcal{M}$  is one for which there exists a Reedy-fibrant diagram  $\hat{Y} : \mathbf{\Delta}(\mathbb{C})^{\text{op}} \rightarrow \mathcal{M}$  such that  $\hat{Y}$  is in  $[\mathbf{\Delta}(\mathbb{C})_L^{\text{op}}, \mathcal{M}]_h$  and  $Y \cong \text{Ran}_{\pi_L} \hat{Y}$ .

We write  $[\mathbb{C}, \mathcal{M}]_{\text{vc}}$  for the full subcategory of  $[\mathbb{C}, \mathcal{M}]$  spanned by the virtually cofibrant diagrams, and we write  $[\mathbb{C}, \mathcal{M}]_{\text{vf}}$  for the full subcategory of  $[\mathbb{C}, \mathcal{M}]$  spanned by the virtually fibrant diagrams.

**Theorem 4.6.16.** *Let  $\mathcal{M}$  be a DHK model category and let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between small categories.*

- (i) *The functor  $\text{Lan}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  sends virtually cofibrant diagrams to componentwise cofibrant diagrams and preserves componentwise weak equivalences between such diagrams.*
- (ii) *If  $\text{Lan}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \rightarrow [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}]$  moreover restricts to a functor  $[\mathbf{\Delta}(\mathbb{C})_R, \mathcal{M}]_h \rightarrow [\mathbf{\Delta}(\mathbb{D})_R, \mathcal{M}]_h$ , then  $\text{Lan}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  preserves virtually cofibrant diagrams.*

(iii) If  $(Q, p)$  is a cofibrant replacement functor for  $[\Delta(\mathbb{C}), \mathcal{M}]$ , then

$$([\mathbb{C}, \mathcal{M}]_{\text{vc}}, \text{Lan}_{\pi_R} \circ Q \circ \pi_R^*, \varepsilon \bullet (\text{Lan}_{\pi_R} \circ p \circ \pi_R^*))$$

is a functorial left deformation retract for  $\text{Lan}_F$ , where  $\varepsilon$  is the counit of the adjunction  $\text{Lan}_{\pi_R} \dashv \pi_R^*$ .

(iv) The adjunction shown below is deformable:

$$\text{Lan}_F \dashv F^* : [\mathbb{D}, \mathcal{M}] \rightarrow [\mathbb{C}, \mathcal{M}]$$

(v) Given another functor  $G : \mathbb{D} \rightarrow \mathbb{E}$  between small categories,  $(\text{Lan}_G, \text{Lan}_F)$  is strongly left deformable.

Dually:

(i') The functor  $\text{Ran}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  sends virtually fibrant diagrams to componentwise fibrant diagrams and preserves componentwise weak equivalences between such diagrams.

(ii') If  $\text{Ran}_{\Delta(F)} : [\Delta(\mathbb{C})^{\text{op}}, \mathcal{M}] \rightarrow [\Delta(\mathbb{D})^{\text{op}}, \mathcal{M}]$  moreover restricts to a functor  $[\Delta(\mathbb{C})_{\text{L}}^{\text{op}}, \mathcal{M}]_{\text{h}} \rightarrow [\Delta(\mathbb{D})_{\text{L}}^{\text{op}}, \mathcal{M}]_{\text{h}}$ , then  $\text{Lan}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  preserves virtually cofibrant diagrams.

(iii') If  $(R, i)$  is a fibrant replacement functor for  $[\Delta(\mathbb{C})^{\text{op}}, \mathcal{M}]$ , then

$$([\mathbb{C}, \mathcal{M}]_{\text{vf}}, \text{Ran}_{\pi_L} \circ R \circ \pi_L^*, (\text{Ran}_{\pi_L} \circ i \circ \pi_L^*) \bullet \eta)$$

is a functorial right deformation retract for  $\text{Ran}_F$ , where  $\eta$  is the unit of the adjunction  $\pi_L^* \dashv \text{Ran}_{\pi_L}$ .

(iv') The adjunction shown below is deformable:

$$F^* \dashv \text{Ran}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$$

(v') Given another functor  $G : \mathbb{D} \rightarrow \mathbb{E}$  between small categories,  $(\text{Ran}_G, \text{Ran}_F)$  is strongly right deformable.

*Proof.* (i). Let  $\tilde{X}$  be a Reedy-cofibrant diagram  $\mathbb{C} \rightarrow \mathcal{M}$  that is in  $[\Delta(\mathbb{C})_{\text{R}}, \mathcal{M}]_{\text{h}}$  and let  $X = \text{Lan}_{\pi_R} \tilde{X}$ . There is a canonical isomorphism  $\text{Lan}_{F\pi_R} \tilde{X} \cong \text{Lan}_F \circ \text{Lan}_{\pi_R} \tilde{X}$ , so [proposition 4.6.13](#) implies  $\text{Lan}_F X$  is a componentwise cofibrant diagram  $\mathbb{D} \rightarrow \mathcal{M}$ .

Let  $\tilde{Y}$  be another Reedy-cofibrant diagram  $\mathbb{C} \rightarrow \mathcal{M}$  that is in  $[\Delta(\mathbb{C})_{\mathbb{R}}, \mathcal{M}]_{\text{h}}$ , let  $Y = \text{Lan}_{\pi_{\mathbb{R}}} \tilde{Y}$ , and let  $\varphi : X \Rightarrow Y$  be a componentwise weak equivalence. [Proposition 3.3.24](#) applied to [theorem 4.6.14](#) implies the adjunction unit components  $\tilde{X} \rightarrow \pi_{\mathbb{R}}^* X$  and  $\tilde{Y} \rightarrow \pi_{\mathbb{R}}^* Y$  are Reedy weak equivalences. Using axiom CM2 and CM5, factor  $\tilde{Y} \rightarrow \pi_{\mathbb{R}}^* Y$  as a trivial cofibration  $\theta : \tilde{Y} \rightarrow \tilde{Z}$  followed by a trivial fibration  $\tilde{Z} \rightarrow \pi_{\mathbb{R}}^* Y$ ; then by axiom CM4 there exists a natural transformation  $\psi : \tilde{X} \Rightarrow \tilde{Z}$  making the diagram in  $[\Delta(\mathbb{C}), \mathcal{M}]$  shown below commute:

$$\begin{array}{ccccc} \tilde{X} & \overset{\psi}{\dashrightarrow} & \tilde{Z} & \xleftarrow{\theta} & \tilde{Y} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_{\mathbb{R}}^*(X) & \xrightarrow{\pi_{\mathbb{R}}^*(\varphi)} & \pi_{\mathbb{R}}^*(Y) & \equiv & \pi_{\mathbb{R}}^*(Y) \end{array}$$

Since  $\pi_{\mathbb{R}}^*(\varphi)$  is a Reedy weak equivalence, it follows from axiom CM2 that  $\psi$  is also a Reedy weak equivalence. Transposing across the adjunction  $\text{Lan}_{\pi_{\mathbb{R}}} \dashv \pi_{\mathbb{R}}^*$ , we obtain a commutative diagram in  $[\mathbb{C}, \mathcal{M}]$ ,

$$\begin{array}{ccccc} \text{Lan}_{\pi_{\mathbb{R}}} \tilde{X} & \xrightarrow{\text{Lan}_{\pi_{\mathbb{R}}} \psi} & \text{Lan}_{\pi_{\mathbb{R}}} \tilde{Z} & \xleftarrow{\text{Lan}_{\pi_{\mathbb{R}}} \theta} & \text{Lan}_{\pi_{\mathbb{R}}} \tilde{Y} \\ \parallel & & \downarrow & & \parallel \\ X & \xrightarrow{\varphi} & Y & \equiv & Y \end{array}$$

to which we may then apply  $\text{Lan}_F$ , yielding the following commutative diagram in  $[\mathbb{D}, \mathcal{M}]$ :

$$\begin{array}{ccccc} \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{X} & \xrightarrow{\text{Lan}_{F\pi_{\mathbb{R}}} \psi} & \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{Z} & \xleftarrow{\text{Lan}_{F\pi_{\mathbb{R}}} \theta} & \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{Y} \\ \cong \downarrow & & \downarrow & & \downarrow \cong \\ \text{Lan}_F X & \xrightarrow{\text{Lan}_F \varphi} & \text{Lan}_F Y & \equiv & \text{Lan}_F Y \end{array}$$

Now,  $\text{Lan}_{F\pi_{\mathbb{R}}} \psi : \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{X} \rightarrow \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{Z}$  and  $\text{Lan}_{F\pi_{\mathbb{R}}} \theta : \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{Y} \rightarrow \text{Lan}_{F\pi_{\mathbb{R}}} \tilde{Z}$  are componentwise weak equivalences between componentwise cofibrant diagrams, by [proposition 4.6.13](#), so we deduce that  $\text{Lan}_F \varphi$  is also a componentwise weak equivalence between componentwise cofibrant diagrams as claimed, using the 2-out-of-3 property of weak equivalences in  $\mathcal{M}$ .

(ii). The following diagram of functors is commutative,

$$\begin{array}{ccc} \Delta(\mathbb{C}) & \xrightarrow{\Delta(F)} & \Delta(\mathbb{D}) \\ \pi_{\mathbb{R}} \downarrow & & \downarrow \pi_{\mathbb{R}} \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D} \end{array}$$

so there is a canonical natural isomorphism  $\text{Lan}_F \circ \text{Lan}_{\pi_R} \cong \text{Lan}_{\pi_R} \circ \text{Lan}_{\Delta(F)}$ . [Corollary 4.6.8](#) implies  $\text{Lan}_{\Delta(F)} : [\Delta(\mathbb{C}), \mathcal{M}] \rightarrow [\Delta(\mathbb{D}), \mathcal{M}]$  preserves Reedy-cofibrant diagrams, so it follows from the hypothesis that the functor  $\text{Lan}_F : [\mathbb{C}, \mathcal{M}] \rightarrow [\mathbb{D}, \mathcal{M}]$  preserves virtually cofibrant diagrams.

(iii). Having proved claim (i), it is now enough to show that the natural transformation  $\varepsilon \cdot (\text{Lan}_{\pi_R} \circ p \circ \pi_R^*) : \text{Lan}_{\pi_R} \circ Q \circ \pi_R^* \Rightarrow \text{id}_{[\mathbb{C}, \mathcal{M}]}$  is a natural weak equivalence; but this is also a consequence of [proposition 3.3.24](#) applied to [theorem 4.6.14](#).

(iv). The functor  $F^*$  is a homotopical functor, hence trivially right deformable, and claim (iii) implies  $\text{Lan}_F$  is left deformable.

(v). Since  $F^*$  and  $G^*$  are both homotopical functors,  $(F^*, G^*)$  is strongly right deformable, and we may deduce from claim (i) that  $(\text{Lan}_G, \text{Lan}_F)$  is laxly left deformable. Thus, by [lemma 3.1.11](#), [theorem 4.4.1](#), and [corollary 3.3.23](#), the composable pair  $(\text{Lan}_G, \text{Lan}_F)$  is strongly left deformable.  $\blacksquare$

**Lemma 4.6.17.** *Let  $\mathcal{M}$  be a DHK model category, let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between small categories, and let  $D$  be an object in  $\mathbb{D}$ .*

- Given the following comma square,

$$\begin{array}{ccc} \Delta((F \downarrow D)) & \longrightarrow & \mathbb{1} \\ \Delta(P) \downarrow & \nearrow & \downarrow D \\ \Delta(\mathbb{C}) & \xrightarrow{F\pi_R} & \mathbb{D} \end{array}$$

the derived left Beck–Chevalley transformation

$$\left( \mathbf{L} \lim_{\rightarrow \Delta((F \downarrow D))} \right) \circ (\text{Ho } \Delta(P)^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{L} \text{Lan}_{F\pi_R})$$

is a natural isomorphism.

Dually:

- Given the following comma square,

$$\begin{array}{ccc} \Delta((D \downarrow F))^{\text{op}} & \xrightarrow{\Delta(P)^{\text{op}}} & \Delta(\mathbb{C})^{\text{op}} \\ \downarrow & \nearrow & \downarrow F\pi_L \\ \mathbb{1} & \xrightarrow{D} & \mathbb{D} \end{array}$$

the derived right Beck–Chevalley transformation

$$\left( \mathbf{R} \lim_{\leftarrow \Delta((D \downarrow F))^{\text{op}}} \right) \circ (\text{Ho } \Delta(P)^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{R} \text{Ran}_{F\pi_L})$$

is a natural isomorphism.

*Proof.* Lemma 4.6.7 says  $\Delta(P) : \Delta((F \downarrow D)) \rightarrow \Delta(\mathbb{C})$  is a right fibration of Reedy categories, so by theorem 4.5.32,  $\Delta(P)^* : [\Delta((F \downarrow D)), \mathcal{M}] \rightarrow [\Delta(\mathbb{C}), \mathcal{M}]$  preserves Reedy-cofibrant diagrams. Proposition 7.1.19 implies that the left Beck–Chevalley transformation  $\lim_{\rightarrow (F\pi_R \downarrow D)} (-\Delta(P)) \Rightarrow (\text{Lan}_{F\pi_R} -)(D)$  is a natural isomorphism, hence by corollary 3.3.21, so too is its derived natural transformation.  $\blacksquare$

**Proposition 4.6.18.** *Let  $\mathcal{M}$  be a DHK model category, let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be a functor between small categories, and let  $D$  be an object in  $\mathbb{D}$ .*

- Given the following comma square,

$$\begin{array}{ccc} (F \downarrow D) & \longrightarrow & \mathbb{1} \\ P \downarrow & \nearrow & \downarrow D \\ \mathbb{C} & \xrightarrow{F} & \mathbb{D} \end{array}$$

the derived left Beck–Chevalley transformation

$$\left( \mathbf{L} \lim_{\rightarrow (F \downarrow D)} \right) \circ (\text{Ho } P^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{L} \text{Lan}_F)$$

is a natural isomorphism.

Dually:

- Given the following comma square,

$$\begin{array}{ccc} (D \downarrow F) & \xrightarrow{P} & \mathbb{C} \\ \downarrow & \nearrow & \downarrow F \\ \mathbb{1} & \xrightarrow{D} & \mathbb{D} \end{array}$$

the derived right Beck–Chevalley transformation

$$\left( \mathbf{R} \lim_{\leftarrow (D \downarrow F)} \right) \circ (\text{Ho } P^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{R} \text{Ran}_F)$$

is a natural isomorphism.



*Proof.* Consider the following diagram, where the 2-cells are the respective left Beck–Chevalley transformations:

$$\begin{array}{ccc}
 [\mathbf{\Delta}(\mathbb{C})_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}} & \xrightarrow{\Delta(P)^*} & [\mathbf{\Delta}((F \downarrow D))_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}} \\
 \downarrow & & \downarrow \\
 [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] & \xrightarrow{\Delta(P)^*} & [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \\
 \text{Lan}_{\pi_{\mathbf{R}}} \downarrow & \swarrow & \downarrow \text{Lan}_{\pi_{\mathbf{R}}} \\
 [\mathbb{C}, \mathcal{M}] & \xrightarrow{P^*} & [(F \downarrow D), \mathcal{M}] \\
 \text{Lan}_F \downarrow & \swarrow & \downarrow \lim \\
 [\mathbb{D}, \mathcal{M}] & \xrightarrow{D^*} & \mathcal{M}
 \end{array}$$

The pasting lemma (A.I.11) implies that left Beck–Chevalley transformations can be pasted together, and the preceding lemma says the derived left Beck–Chevalley transformation

$$\left( \mathbf{L} \lim_{\mathbf{\Delta}((F \downarrow D))} \right) \circ (\text{Ho } \mathbf{\Delta}(P)^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{L} \text{Lan}_{F\pi_{\mathbf{R}}})$$

is a natural isomorphism; but theorem 4.6.14 says that the adjunctions

$$\begin{aligned}
 \text{Lan}_{\pi_{\mathbf{R}}} \dashv \pi_{\mathbf{R}}^* : [\mathbb{C}, \mathcal{M}] &\rightarrow [\mathbf{\Delta}(\mathbb{C})_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}} \\
 \text{Lan}_{\pi_{\mathbf{R}}} \dashv \pi_{\mathbf{R}}^* : [(F \downarrow D), \mathcal{M}] &\rightarrow [\mathbf{\Delta}((F \downarrow D))_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}}
 \end{aligned}$$

satisfy the Quillen equivalence condition, so the commutative diagram shown below automatically satisfies the derived left Beck–Chevalley condition,

$$\begin{array}{ccc}
 [\mathbb{C}, \mathcal{M}] & \xrightarrow{P^*} & [(F \downarrow D), \mathcal{M}] \\
 \pi_{\mathbf{R}}^* \downarrow & & \downarrow \pi_{\mathbf{R}}^* \\
 [\mathbf{\Delta}(\mathbb{C})_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}} & \xrightarrow{\Delta(P)^*} & [\mathbf{\Delta}((F \downarrow D))_{\mathbf{R}}, \mathcal{M}]_{\mathbf{h}}
 \end{array}$$

and therefore, by cancelling natural isomorphisms, we conclude that the derived left Beck–Chevalley transformation

$$\left( \mathbf{L} \lim_{(F \downarrow D)} \right) \circ (\text{Ho } P^*) \Rightarrow (\text{Ho } D^*) \circ (\mathbf{L} \text{Lan}_F)$$

is a natural isomorphism, as claimed. ■

## 4.7 Framings and resolutions

**Prerequisites.** §§ 1.1, 1.2, 1.5, 2.3, 4.1, 4.2, 4.3, 4.5.

In homological algebra, one studies objects in categories without homotopical structure by embedding them in one that does, in such a way that objects in the original category become weakly equivalent to their presentations. The notion of ‘resolution’ in the sense of Dwyer and Kan [1980c] was invented for similar reasons: though not every model category has a simplicial enrichment, we can still replace objects with homotopically better-behaved simplicial (or co-simplicial) ones. It is also useful to simultaneously discuss the closely related notion of ‘framing’ introduced by Dwyer, Hirschhorn and Kan [DHK].

In this section, we follow [Hirschhorn, 2003, Ch. 16].

¶ 4.7.1. Since  $\mathbf{\Delta}$  is a Reedy category, theorem 4.5.20 says these categories admit model structures in which the weak equivalences are degreewise, at least when  $\mathcal{M}$  has enough limits and colimits. In fact, finite limits and colimits are enough, because the latching and matching categories of  $\mathbf{\Delta}$  at any object are always finite. In this section,  $\mathbf{s}\mathcal{M}$  and  $\mathbf{c}\mathcal{M}$  will always be equipped with the Reedy model structure.

**Lemma 4.7.2.** *Let  $\mathcal{M}$  be a model category.*

- A morphism  $A^\bullet \rightarrow B^\bullet$  in  $\mathbf{c}\mathcal{M}$  is a Reedy cofibration (resp. trivial Reedy cofibration) if and only if, for all  $n \geq 0$ , the morphism

$$(\Delta^n \star A) \cup^{\partial\Delta^n \star A} (\partial\Delta^n \star B) \rightarrow \Delta^n \star B$$

induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  is a cofibration (resp. trivial cofibration) in  $\mathcal{M}$ .

- A morphism  $A_\bullet \rightarrow B_\bullet$  in  $\mathbf{s}\mathcal{M}$  is a Reedy fibration (resp. trivial Reedy fibration) if and only if, for all  $n \geq 0$ , the morphism

$$\{\Delta^n, A\} \rightarrow \{\partial\Delta^n, A\} \times_{\{\partial\Delta^n, B\}} \{\Delta^n, B\}$$

induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  is a fibration (resp. trivial fibration) in  $\mathcal{M}$ .

*Proof.* It is not hard to check that the indicated morphisms are (isomorphic to) the relevant relative matching or latching morphisms; for the trivial fibration/cofibration case, apply proposition 4.5.15.  $\blacklozenge$

**Corollary 4.7.3.** *Let  $\mathcal{M}$  be a model category.*

- A cosimplicial object  $B^\bullet$  in  $\mathcal{M}$  is Reedy-cofibrant if and only if, for all  $n \geq 0$ , the morphism

$$\partial\Delta^n \star B \rightarrow \Delta^n \star B$$

induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  is a cofibration in  $\mathcal{M}$ .

- A simplicial object  $A_\bullet$  in  $\mathcal{M}$  is Reedy-fibrant if and only if, for all  $n \geq 0$ , the morphism

$$\{\Delta^n, A\} \rightarrow \{\partial\Delta^n, A\}$$

induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  is a fibration in  $\mathcal{M}$ . ■

**Proposition 4.7.4.** *Let  $\mathcal{M}$  be a model category.*

- A cosimplicial object  $B^\bullet$  in  $\mathcal{M}$  is Reedy-cofibrant if and only if, for all monomorphisms  $i : Z \rightarrow W$  between finite simplicial sets, the morphism

$$i \star \text{id}_B : Z \star B \rightarrow W \star B$$

induced by  $i$  is a cofibration in  $\mathcal{M}$ .

- A simplicial object  $A_\bullet$  in  $\mathcal{M}$  is Reedy-fibrant if and only if, for all monomorphisms  $i : Z \rightarrow W$  between finite simplicial sets, the morphism

$$\{i, A\} : \{W, A\} \rightarrow \{Z, A\}$$

induced by  $i$  is a fibration in  $\mathcal{M}$ .

*Proof.* The two claims are formally dual; we will prove the first version.

It is not hard to see that the class of monomorphisms between finite simplicial sets is the smallest class of morphisms between finite simplicial sets that contains the boundary inclusions  $\partial\Delta^n \hookrightarrow \Delta^n$  and is closed under composition, pushouts, and retracts: simply take an appropriate cellular decomposition. The class of cofibrations in  $\mathcal{M}$  is closed under composition, pushouts, and retracts by [proposition A.3.14](#), so we may then apply [corollary 4.7.3](#) to deduce that  $i \star A : Z \star A \rightarrow W \star A$  is a cofibration in  $\mathcal{M}$  for any monomorphism  $i : Z \rightarrow W$  between finite simplicial sets and any Reedy-cofibrant cosimplicial object  $A^\bullet$ . Conversely, any such cosimplicial object must be Reedy-cofibrant. ■

**Lemma 4.7.5.** *Let  $\mathcal{M}$  be a model category. There exist adjunctions of the form below,*

$$\begin{aligned} \mathrm{sk}^0 \dashv (-)^0 \dashv \mathrm{cosk}^0 : \mathcal{M} &\rightarrow \mathbf{cM} \\ \mathrm{sk}_0 \dashv (-)_0 \dashv \mathrm{cosk}_0 : \mathcal{M} &\rightarrow \mathbf{sM} \end{aligned}$$

where  $(-)^0 : \mathbf{cM} \rightarrow \mathcal{M}$  is the functor that sends a cosimplicial object  $A^\bullet$  to the object  $A^0$ , and dually,  $(-)_0 : \mathbf{sM} \rightarrow \mathcal{M}$  is the functor that sends a simplicial object  $A_\bullet$  to the object  $A_0$ .

*Proof.* It is straightforward to verify that the following formulae work,

$$\begin{aligned} \mathrm{sk}_0(A)_n &= A & \mathrm{cosk}_0(A)_n &= [n] \pitchfork A \\ \mathrm{sk}^0(A)^n &= [n] \odot A & \mathrm{cosk}^0(A)^n &= A \end{aligned}$$

where  $[n] \pitchfork A$  is the  $(n + 1)$ -fold power of  $A$ , and  $[n] \odot A$  is the  $(n + 1)$ -fold copower of  $A$ . ■

**Definition 4.7.6.** Let  $A$  be an object in a model category  $\mathcal{M}$ .

- A **cosimplicial resolution** of  $A$  is a Reedy-cofibrant replacement in  $\mathbf{cM}$  for the cosimplicial object  $\mathrm{cosk}^0(A)$ .
- A **simplicial resolution** of  $A$  is a Reedy-fibrant replacement in  $\mathbf{sM}$  for the simplicial object  $\mathrm{sk}_0(A)$ .

REMARK 4.7.7. [Proposition 4.7.4](#) implies that the the above definition is equivalent to the original definition of ‘resolution’ given by Dwyer and Kan [[1980c](#)].

**Proposition 4.7.8.** *Let  $\mathcal{M}$  be a model category.*

- (i) *Every object in  $\mathcal{M}$  has both a cosimplicial resolution and a simplicial resolution.*
- (ii) *If  $\mathcal{M}$  is a DHK model category, then cosimplicial resolutions and simplicial resolutions can both be chosen functorially.*

*Proof.* This follows from [proposition 4.1.23](#) and [theorem 4.5.20](#). ■

**Proposition 4.7.9.** *Let  $A$  be an object in a DHK model category  $\mathcal{M}$ .*

- *The full subcategory of the slice category  $\mathbf{c}\mathcal{M}_{/\mathrm{cosk}_0(A)}$  spanned by the co-simplicial resolutions of  $A$  is homotopically contractible.<sup>[6]</sup>*
- *The full subcategory of the slice category  $\mathrm{sk}_0(A)/\mathbf{s}\mathcal{M}$  spanned by the simplicial resolutions of  $A$  is homotopically contractible.*

*Proof.* This follows from [proposition 4.1.25](#) and [theorem 4.5.20](#). ■

**Lemma 4.7.10.** *Let  $\mathcal{M}$  be a model category.*

- $\mathrm{cosk}_0 : \mathcal{M} \rightarrow \mathbf{s}\mathcal{M}$  is a right Quillen functor.
- $\mathrm{sk}^0 : \mathcal{M} \rightarrow \mathbf{c}\mathcal{M}$  is a left Quillen functor.

*Proof.* The claims are formally dual; we will prove the first version.

By [proposition 4.3.2](#), it is enough to show that  $(-)_0 : \mathbf{s}\mathcal{M} \rightarrow \mathcal{M}$  preserves cofibrations and trivial cofibrations. However, the latching category at  $[0]$  is empty, so if  $f : A_\bullet \rightarrow B_\bullet$  is a Reedy cofibration, then  $f_0 : A_0 \rightarrow B_0$  must be a cofibration in  $\mathcal{M}$ . Since  $(-)_0$  preserves weak equivalences, it follows that  $(-)_0$  preserves trivial cofibrations. ■

**Lemma 4.7.11.** *Let  $\mathcal{M}$  be a model category.*

- *There is a unique natural transformation  $\Delta : \mathrm{sk}_0 \Rightarrow \mathrm{cosk}_0$  such that  $\varepsilon_A \circ (\Delta_A)_0 \circ \eta_A = \mathrm{id}_A$  for all objects  $A$  in  $\mathcal{M}$ , where  $\eta_A : A \rightarrow \mathrm{sk}_0(A)_0$  and  $\varepsilon_A : \mathrm{cosk}_0(A)_0 \rightarrow A$  are the components of the unit and counit of the respective adjunctions.*
- *There is a unique natural transformation  $\nabla : \mathrm{sk}^0 \Rightarrow \mathrm{cosk}^0$  such that  $\varepsilon_A \circ (\nabla_A)_0 \circ \eta_A = \mathrm{id}_A$  for all objects  $A$  in  $\mathcal{M}$ , where  $\eta_A : A \rightarrow \mathrm{sk}^0(A)^0$  and  $\varepsilon_A : \mathrm{cosk}^0(A)^0 \rightarrow A$  are the components of the unit and counit of the respective adjunctions.*

*Proof.* The two claims are formally dual; we will prove the first version.

It is not hard to check that  $\eta_A$  is an isomorphism, so  $\varepsilon_A \circ (\Delta_A)_0$  is uniquely determined. The universal property of  $\mathrm{cosk}_0(A)$  implies  $\Delta_A : \mathrm{sk}_0(A) \rightarrow \mathrm{cosk}_0(A)$  is determined by its adjoint transpose  $\varepsilon_A \circ (\Delta_A)_0 : \mathrm{sk}_0(A)_0 \rightarrow A$ , so  $\Delta_A$  is also uniquely determined. ■

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[6] See [definition 3.1.30](#).

**Definition 4.7.12.** Let  $A$  be an object in a model category  $\mathcal{M}$ .

- A **cosimplicial frame** on  $A$  is a pair  $(\tilde{A}^\bullet, p^\bullet)$ , where  $\tilde{A}^\bullet$  is a cosimplicial object in  $\mathcal{M}$ ,  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  is a Reedy weak equivalence with  $p^0 : \tilde{A}^0 \rightarrow \text{cosk}^0(A)^0$  an isomorphism, and  $\tilde{A}^\bullet$  is Reedy-cofibrant if  $A$  is cofibrant.
- A **simplicial frame** on  $A$  is a pair  $(\hat{A}_\bullet, i_\bullet)$ , where  $\hat{A}_\bullet$  is a simplicial object in  $\mathcal{M}$ ,  $i_\bullet : \text{sk}_0(A) \rightarrow \hat{A}_\bullet$  is a Reedy weak equivalence with  $i_0 : \text{sk}_0(A)_0 \rightarrow \hat{A}_0$  an isomorphism, and  $\hat{A}_\bullet$  is Reedy-fibrant if  $A$  is fibrant.
- A **left frame** on  $A$  is a tuple  $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ , where  $\tilde{A}^\bullet$  is a cosimplicial object in  $\mathcal{M}$ ,  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  is a Reedy weak equivalence with  $p^0 : \tilde{A}^0 \rightarrow \text{cosk}^0(A)^0$  an isomorphism,  $i^\bullet$  is a Reedy cofibration, and  $p^\bullet \circ i^\bullet = \nabla_A$ .
- A **right frame** on  $A$  is a tuple  $(\hat{A}_\bullet, i_\bullet, p_\bullet)$ , where  $\hat{A}_\bullet$  is a simplicial object in  $\mathcal{M}$ ,  $i_\bullet : \text{sk}_0(A) \rightarrow \hat{A}_\bullet$  is a Reedy weak equivalence with  $i_0 : \text{sk}_0(A)_0 \rightarrow \hat{A}_0$  an isomorphism,  $p_\bullet$  is a Reedy fibration, and  $p_\bullet \circ i_\bullet = \Delta_A$ .

**Proposition 4.7.13.** Let  $A$  be an object in a model category  $\mathcal{M}$ .

- (i) If  $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$  is a left frame on  $A$ , then  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ .
- (ii) If  $A$  is cofibrant, then every cosimplicial frame on  $A$  is a cosimplicial resolution of  $A$ .
- (iii) If  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial resolution of  $A$ , then  $\tilde{A}^\bullet$  is (the underlying cosimplicial object of) a cosimplicial frame on  $\tilde{A}^0$ , and  $(\tilde{A}^0, p^0)$  is (isomorphic to) a cofibrant replacement for  $A$ .

*Dually:*

- (i') If  $(\hat{A}_\bullet, i_\bullet, p_\bullet)$  is a right frame on  $A$ , then  $(\hat{A}_\bullet, i_\bullet)$  is a simplicial frame on  $A$ .
- (ii') If  $A$  is fibrant, then every simplicial frame on  $A$  is a simplicial resolution of  $A$ .
- (iii') If  $(\hat{A}_\bullet, i_\bullet)$  is a simplicial resolution of  $A$ , then  $\hat{A}_\bullet$  is (the underlying simplicial object of) a simplicial frame on  $\hat{A}_0$ , and  $(\hat{A}_0, i_0)$  is (isomorphic to) a fibrant replacement for  $A$ .

*Proof.* (i). Suppose  $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$  is a left frame on  $A$ . [Lemma 4.7.10](#) implies that  $\text{cosk}^0(A)$  is Reedy-cofibrant when  $A$  is cofibrant, so  $\tilde{A}^\bullet$  is Reedy-cofibrant when  $A$  is cofibrant. Thus  $(\tilde{A}^\bullet, p^\bullet)$  is indeed a cosimplicial frame on  $A$ .

(ii). If  $A$  is cofibrant and  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ , then  $\tilde{A}^\bullet$  is Reedy-cofibrant, and hence  $(\tilde{A}^\bullet, p^\bullet)$  is a Reedy-cofibrant replacement for  $\text{cosk}^0(A)$ .

(iii). Let  $q^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(\tilde{A}^0)$  be the component of the adjunction unit at  $\tilde{A}^\bullet$ . Since  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in  $\mathcal{M}$  implies  $q^\bullet$  is also a Reedy weak equivalence. Now,  $\tilde{A}^\bullet$  is Reedy-cofibrant by definition, it follows that  $(\tilde{A}^\bullet, q^\bullet)$  is a cosimplicial frame on  $\tilde{A}^0$ .

Finally, we note that [proposition 4.7.4](#) implies that  $\tilde{A}^0$  is a cofibrant object in  $\mathcal{M}$ , and  $p^0 : \tilde{A}^0 \rightarrow \text{cosk}^0(A)^0$  is a weak equivalence by definition, so  $(\tilde{A}^0, p^0)$  is (isomorphic to) a cofibrant replacement for  $A$ . ■

**REMARK 4.7.14.** The notions of ‘left frame’ and ‘right frame’ are originally due to Hovey [1999, §5.2], but he calls them ‘cosimplicial frame’ and ‘simplicial frame’ and does not give a name to the weaker notion. It is explained in loc. cit. that a left (resp. right) frame on  $A$  is a cosimplicial (resp. simplicial) frame that is almost Reedy-cofibrant (resp. Reedy-fibrant), in the sense that all but one of its latching (resp. matching) morphisms are cofibrations (resp. fibrations). One consequence of this is given in [proposition 4.7.21](#).

**Definition 4.7.15.** Let  $\mathcal{M}$  be a model category.

- A **left framing** for  $\mathcal{M}$  is a tuple  $(Q^\bullet, i^\bullet, p^\bullet)$ , where  $Q^\bullet : \mathcal{M} \rightarrow \mathbf{cM}$  is a functor,  $i^\bullet : \text{sk}^0 \Rightarrow Q^\bullet$  and  $p^\bullet : Q^\bullet \Rightarrow \text{cosk}^0$  are natural transformations, and  $(Q^\bullet A, (i_A)^\bullet, (p_A)^\bullet)$  is a left frame for all *cofibrant* objects  $A$  in  $\mathcal{M}$ .
- A **right framing** for  $\mathcal{M}$  is a tuple  $(R_\bullet, i_\bullet, p_\bullet)$ , where  $R_\bullet : \mathcal{M} \rightarrow \mathbf{sM}$  is a functor,  $i_\bullet : \text{sk}_0 \Rightarrow R_\bullet$  and  $p_\bullet : R_\bullet \Rightarrow \text{cosk}_0$  are natural transformations, and  $(R_\bullet A, (i_A)_\bullet, (p_A)_\bullet)$  is a right frame for all *fibrant* objects  $A$  in  $\mathcal{M}$ .

A **framed model category** is a model category equipped with a left framing and a right framing.

**Theorem 4.7.16.** *Let  $\mathcal{M}$  be a model category.*

- (i) *On each object  $A$  in  $\mathcal{M}$ , there exist a left frame  $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$  and a right frame  $(\hat{A}_\bullet, i_\bullet, p_\bullet)$  such that  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  is a trivial Reedy fibration and  $i_\bullet : \text{sk}_0(A) \rightarrow \hat{A}_\bullet$  is a trivial Reedy cofibration.*
- (ii) *If  $\mathcal{M}$  satisfies axiom CM5\*, then the left and right frames in (i) can be chosen functorially; in particular, left and right framings for  $\mathcal{M}$  exist.*

*Proof.* See Theorem 5.2.8 in [Hovey, 1999]. □

**Theorem 4.7.17.** *Let  $A$  be an object in a DHK model category  $\mathcal{M}$ .*

- *The nerve of the full subcategory of the slice category  $\mathbf{cM}_{/\text{cosk}^0(A)}$  spanned by the cosimplicial frames on  $A$  is weakly contractible.*
- *The nerve of the full subcategory of the slice category  $\text{sk}^0(A)/\mathbf{sA}$  spanned by the simplicial frames on  $A$  is weakly contractible.*

*Proof.* See Theorem 16.6.18 in [Hirschhorn, 2003]. □

**Proposition 4.7.18.** *Let  $\mathcal{M}$  be a model category.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ , then the morphism*

$$\Lambda_k^n \star \tilde{A} \rightarrow \Delta^n \star \tilde{A}$$

*induced by any horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  is a trivial cofibration in  $\mathcal{M}$ .*

- *If  $B$  is a fibrant object in  $\mathcal{M}$  and  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on  $B$ , then the morphism*

$$\{\Delta^n, \hat{B}\} \rightarrow \{\Lambda_k^n, \hat{B}\}$$

*induced by any horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  is a trivial fibration in  $\mathcal{M}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

First, note that [proposition 4.7.4](#) implies that  $\Lambda_k^n \star \tilde{A} \rightarrow \Delta^n \star \tilde{A}$  is a cofibration in  $\mathcal{M}$ . Since  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in  $\mathcal{M}$  implies that the morphism  $\Delta^n \star \tilde{A} \rightarrow \Delta^0 \star \tilde{A}$  is a weak equivalence in  $\mathcal{M}$  for all  $n \geq 0$ . It is clear that  $- \star \tilde{A}$  preserves finite colimits of finite simplicial sets, so we may then apply [Proposition 3.6.8](#) in [Hovey, 1999]. □



**Corollary 4.7.19.** *Let  $\mathcal{M}$  be a model category and let  $i : Z \rightarrow W$  be an anodyne extension between finite simplicial sets.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ , then the morphism*

$$i \star \text{id}_{\tilde{A}} : Z \star \tilde{A} \rightarrow W \star \tilde{A}$$

*induced by  $i : Z \rightarrow W$  is a trivial cofibration in  $\mathcal{M}$ .*

- *If  $B$  is a fibrant object in  $\mathcal{M}$  and  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on  $B$ , then the morphism*

$$\{i, \hat{B}\} : \{W, \hat{B}\} \rightarrow \{Z, \hat{B}\}$$

*induced by  $i : Z \rightarrow W$  is a trivial fibration in  $\mathcal{M}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

By [proposition 1.4.12](#), the class of anodyne extensions between finite simplicial sets is generated by the boundary inclusions under composition, pushouts, and retracts. We already know that  $- \star \tilde{A}$  sends horn inclusions to trivial cofibrations in  $\mathcal{M}$ , and it is clear that  $- \star \tilde{A}$  preserves composition, pushouts, and retracts, so [theorem 4.1.12](#) and [proposition A.3.14](#) imply that  $i \star \text{id}_{\tilde{A}}$  is a trivial cofibration in  $\mathcal{M}$ . ■

Cosimplicial frames and left frames (resp. simplicial frames and right frames) should be regarded as higher cylinder objects (resp. higher path objects). We can make this precise in two different ways:

**Proposition 4.7.20.** *Let  $\mathcal{M}$  be a model category.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ , then  $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$  is a cylinder object for  $\tilde{A}^0$  (and hence, isomorphic to a cylinder object for  $A$ ).*
- *If  $B$  is a fibrant object in  $\mathcal{M}$  and  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on  $B$ , then  $(\hat{B}_1, d_1, d_0, s_0)$  is a path object for  $\hat{B}_0$  (and hence, isomorphic to a path object for  $B$ ).*

*Proof.* The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism  $(\delta^1, \delta^0) : \tilde{A}^0 + \tilde{A}^0 \rightarrow \tilde{A}^1$  is isomorphic to the morphism  $\partial\Delta^1 \star \tilde{A} \rightarrow \Delta^1 \star \tilde{A}$  induced by  $\partial\Delta^1 \hookrightarrow \Delta^1$ , and the latter is a

cofibration by [proposition 4.7.4](#). On the other hand, the morphism  $\sigma^0 : \tilde{A}^1 \rightarrow \tilde{A}^0$  is a retraction for  $\delta^1 : \tilde{A}^0 \rightarrow \tilde{A}^1$ , and [proposition 4.7.18](#) implies the latter is (isomorphic to) a trivial cofibration; thus, by the 2-out-of-3 property of weak equivalences,  $\sigma^0 : \tilde{A}^1 \rightarrow \tilde{A}^0$  must be a weak equivalence.  $\blacksquare$

**Proposition 4.7.21.** *Let  $\mathcal{M}$  be a model category.*

- *If  $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$  is a left frame on an object in  $\mathcal{M}$ , then  $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$  is a cylinder object for  $\tilde{A}^0$ .*
- *If  $(\tilde{B}_\bullet, i_\bullet, p_\bullet)$  is a right frame on an object in  $\mathcal{M}$ , then  $(\tilde{B}_1, d_1, d_0, s_0)$  is a path object for  $\tilde{B}_0$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism  $(\delta^1, \delta^0) : \tilde{A}^0 + \tilde{A}^0 \rightarrow \tilde{A}^1$  is isomorphic to the relative latching morphism at [1] for  $i^\bullet : \text{sk}^0(A) \rightarrow \tilde{A}^\bullet$ , and the latter is a Reedy cofibration, so  $(\delta^1, \delta^0)$  is a cofibration in  $\mathcal{M}$ . On the other hand, we have the following commutative diagram,

$$\begin{array}{ccc} \tilde{A}^1 & \xrightarrow{p^1} & \text{cosk}^0(A)^1 \\ \sigma^0 \downarrow & & \downarrow \sigma^0 \\ \tilde{A}^0 & \xrightarrow{p^0} & \text{cosk}^0(A)^0 \end{array}$$

where  $p^0$  and  $p^1$  are weak equivalences in  $\mathcal{M}$ . Since  $\sigma^0 : \text{cosk}^0(A)^1 \rightarrow \text{cosk}^0(A)^0$  is an isomorphism (and so a weak equivalence *a fortiori*), the 2-out-of-3 property of weak equivalences implies  $\sigma^0 : \tilde{A}^1 \rightarrow \tilde{A}^0$  is also a weak equivalence.  $\blacksquare$

**Proposition 4.7.22.** *Let  $\mathcal{M}$  be a model category and let  $X$  be a finite simplicial set.*

- *If  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on a cofibrant object  $A$  in  $\mathcal{M}$ , then the cosimplicial object  $(X \odot \tilde{A})^\bullet$  is (the object part of) a cosimplicial frame on  $X \star \tilde{A}$ .*
- *If  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on a fibrant object  $B$  in  $\mathcal{M}$ , then the simplicial object  $(X \pitchfork \hat{B})_\bullet$  is (the object part of) a simplicial frame on  $\{X, \hat{B}\}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

To show that the cosimplicial object  $(X \odot \tilde{A})^\bullet$  is indeed (the object part of) a cosimplicial frame on  $X \star \tilde{A}$ , it suffices to verify that  $(X \odot \tilde{A})^\bullet$  is Reedy-cofibrant and all its codegeneracy operators are weak equivalences: the latter condition ensures that the counit component  $(X \odot \tilde{A})^\bullet \rightarrow \text{cosk}^0((X \odot \tilde{A})^0)$  is a Reedy weak equivalence, and we know that  $(X \odot \tilde{A})^0 \cong X \star \tilde{A}$ . By definition, we have the following natural bijections:

$$\begin{aligned} \mathcal{M}(Z \star (X \odot \tilde{A}), B) &\cong \mathbf{sSet}(Z, \mathcal{M}((X \odot \tilde{A})^\bullet, B)) \\ &\cong \mathbf{sSet}(Z, [X, \mathcal{M}(\tilde{A}^\bullet, B)]) \\ &\cong \mathbf{sSet}(Z \times X, \mathcal{M}(\tilde{A}^\bullet, B)) \\ &\cong \mathcal{M}((Z \times X) \star \tilde{A}, B) \end{aligned}$$

Since  $i \times \text{id}_X : Z \times X \rightarrow W \times X$  is a monomorphism between finite simplicial sets when  $i : Z \rightarrow W$  is, we may then use [proposition 4.7.4](#) to deduce that  $(X \odot \tilde{A})^\bullet$  is indeed Reedy-cofibrant.

It remains to be shown that the codegeneracy operators of  $(X \odot \tilde{A})^\bullet$  are weak equivalences. The cosimplicial identities and axiom CM2 implies it is enough to show that each coface operator  $\delta_n^i : (X \odot \tilde{A})^{n-1} \rightarrow (X \odot \tilde{A})^n$  is a weak equivalence. Since the unique morphism  $\Delta^n \rightarrow \Delta^0$  is a (weak) homotopy equivalence, we can use [proposition 1.5.11](#) and the 2-out-of-3 property of weak homotopy equivalences to deduce that, for each  $\delta_n^i : \Delta^{n-1} \rightarrow \Delta^n$ , the induced morphism  $\delta_n^i \times \text{id}_X : \Delta^0 \times X \rightarrow \Delta^n \times X$  is a weak homotopy equivalence. [Proposition 1.5.9](#) then says that  $\delta_n^i \times \text{id}_X$  is an anodyne extension, so by [corollary 4.7.19](#), the induced morphism  $(\Delta^{n-1} \times X) \star \tilde{A} \rightarrow (\Delta^n \times X) \star \tilde{A}$  is a trivial cofibration in  $\mathcal{M}$ . Thus, every coface operator  $(X \odot \tilde{A})^0 \rightarrow (X \odot \tilde{A})^n$  is a weak equivalence in  $\mathcal{M}$ . ■

## 4.8 Derived hom-spaces

**Prerequisites.** §§ [1.1](#), [1.5](#), [3.1](#), [3.3](#), [4.1](#), [4.3](#), [4.5](#), [4.7](#), [A.4](#).

Given a cofibrant object  $A$  and a fibrant object  $B$  in a model category  $\mathcal{M}$ , there ought to be a space of morphisms  $A \rightarrow B$ , at least well-defined up to weak equivalence, such that the set of connected components is in natural bijection with the hom-set  $\text{Ho } \mathcal{M}(A, B)$ , while homotopy classes of paths correspond to homotopy classes of homotopies of morphisms  $A \rightarrow B$ , and so on. For this, we will use the notion of ‘frame’ introduced in the previous section.

**Definition 4.8.1.** Let  $\mathcal{M}$  be a model category.

- A **cosimplicial frame** in  $\mathcal{M}$  is a cosimplicial object  $\tilde{A}^\bullet$  in  $\mathcal{M}$  for which there exist an object  $A$  and a morphism  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  such that  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ .
- A **simplicial frame** in  $\mathcal{M}$  is a simplicial object  $\hat{B}_\bullet$  in  $\mathcal{M}$  for which there exist an object  $B$  and a morphism  $i_\bullet : \text{sk}_0(B) \rightarrow \hat{B}_\bullet$  such that  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on  $B$ .
- A **cosimplicial resolution** in  $\mathcal{M}$  is a cosimplicial object  $\tilde{A}^\bullet$  in  $\mathcal{M}$  for which there exist an object  $A$  and a morphism  $p^\bullet : \tilde{A}^\bullet \rightarrow \text{cosk}^0(A)$  such that  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial resolution on  $A$ .
- A **simplicial resolution** in  $\mathcal{M}$  is a simplicial object  $\hat{B}_\bullet$  in  $\mathcal{M}$  for which there exist an object  $B$  and a morphism  $i_\bullet : \text{sk}_0(B) \rightarrow \hat{B}_\bullet$  such that  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial resolution on  $B$ .
- A **weakly constant cosimplicial object** in  $\mathcal{M}$  is a cosimplicial object in  $\mathcal{M}$  such that every coface and codegeneracy operator is a weak equivalence in  $\mathcal{M}$ .
- A **weakly constant simplicial object** in  $\mathcal{M}$  is a simplicial object in  $\mathcal{M}$  such that every face and degeneracy operator is a weak equivalence in  $\mathcal{M}$ .

**Proposition 4.8.2.** Let  $\mathcal{M}$  be a model category. Let  $A^\bullet$  be a cosimplicial object in  $\mathcal{M}$  and let  $p^\bullet : A^\bullet \rightarrow \text{cosk}^0(A^0)$  be the component of the adjunction unit at  $A^\bullet$ .

- (i)  $A^\bullet$  is a cosimplicial frame in  $\mathcal{M}$  if and only if  $(A^\bullet, p^\bullet)$  is a cosimplicial frame on  $A^0$ .
- (ii)  $A^\bullet$  is a cosimplicial resolution in  $\mathcal{M}$  if and only if  $A^\bullet$  is a cosimplicial frame in  $\mathcal{M}$  and  $A^0$  is cofibrant.
- (iii)  $A^\bullet$  is a cosimplicial resolution in  $\mathcal{M}$  if and only if  $(A^\bullet, p^\bullet)$  is a cosimplicial resolution of  $A^0$ .
- (iv)  $A^\bullet$  is a weakly constant cosimplicial object in  $\mathcal{M}$  if and only if the morphism  $p^\bullet : A^\bullet \rightarrow \text{cosk}^0(A^0)$  is a Reedy weak equivalence.

(v)  $A^\bullet$  is a cosimplicial resolution in  $\mathcal{M}$  if and only if  $A^\bullet$  is Reedy-cofibrant and weakly constant.

Dually, let  $B_\bullet$  be a simplicial object in  $\mathcal{M}$  and let  $i_\bullet : \text{sk}_0(B_0) \rightarrow B$  be the component of the adjunction counit at  $B_\bullet$ .

(i')  $B_\bullet$  is a simplicial frame in  $\mathcal{M}$  if and only if  $(B_\bullet, i_\bullet)$  is a simplicial frame on  $B_0$ .

(ii')  $B_\bullet$  is a simplicial resolution in  $\mathcal{M}$  if and only if  $B_\bullet$  is a simplicial frame in  $\mathcal{M}$  and  $B_0$  is fibrant.

(iii')  $B_\bullet$  is a simplicial resolution in  $\mathcal{M}$  if and only if  $(B_\bullet, i_\bullet)$  is a simplicial resolution of  $B_0$ .

(iv')  $B_\bullet$  is a weakly constant simplicial object in  $\mathcal{M}$  if and only if the morphism  $i_\bullet : \text{sk}_0(B_0) \rightarrow B_\bullet$  is a Reedy weak equivalence.

(v')  $B_\bullet$  is a simplicial resolution in  $\mathcal{M}$  if and only if  $B_\bullet$  is Reedy-fibrant and weakly constant.

*Proof.* These are straightforward consequences of the definitions and [proposition 4.7.13](#). ◆

**Lemma 4.8.3.** *Let  $\mathcal{M}$  be a model category.*

- Any cosimplicial resolution in  $\mathcal{M}$  is a degreewise cofibrant cosimplicial object in  $\mathcal{M}$ .
- Any simplicial resolution in  $\mathcal{M}$  is a degreewise fibrant simplicial object in  $\mathcal{M}$ .

*Proof.* This is a corollary of [proposition 4.7.4](#). ■

**Lemma 4.8.4.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories and let*

$$F \dashv G : \mathcal{N} \rightarrow \mathcal{M}$$

*be a Quillen adjunction.*

- The induced functor  $\mathbf{c}F : \mathbf{c}\mathcal{M} \rightarrow \mathbf{c}\mathcal{N}$  sends cosimplicial resolutions in  $\mathcal{M}$  to cosimplicial resolutions in  $\mathcal{N}$ .

- The induced functor  $\mathbf{s}G : \mathbf{s}\mathcal{N} \rightarrow \mathbf{s}\mathcal{M}$  sends simplicial resolutions in  $\mathcal{N}$  to simplicial resolutions in  $\mathcal{M}$ .

*Proof.* The two claims are formally dual; we will prove the first version.

By propositions 4.3.4 and 4.5.17,  $\mathbf{c}F$  sends Reedy cofibrant cosimplicial objects in  $\mathcal{M}$  to Reedy cofibrant cosimplicial objects in  $\mathcal{N}$ . Lemma 4.3.6 implies that  $\mathbf{c}F$  preserves weakly constancy for degreewise cofibrant cosimplicial objects; but lemma 4.8.3 says cosimplicial resolutions are degreewise cofibrant, so we are done. ■

**Definition 4.8.5.** Let  $\mathcal{M}$  be a model category.

- The **category of weakly constant cosimplicial objects** in  $\mathcal{M}$  is the full subcategory  $\mathbf{c}_w\mathcal{M}$  of  $\mathbf{c}\mathcal{M}$  spanned by the weakly constant cosimplicial objects. A **weak equivalence** (resp. **cofibration**, **fibration**) in  $\mathbf{c}_w\mathcal{M}$  is a Reedy weak equivalence (resp. cofibration, fibration).
- The **category of weakly constant simplicial objects** in  $\mathcal{M}$  is the full subcategory  $\mathbf{s}_w\mathcal{M}$  of  $\mathbf{s}\mathcal{M}$  spanned by the weakly constant simplicial objects. A **weak equivalence** (resp. **cofibration**, **fibration**) in  $\mathbf{s}_w\mathcal{M}$  is a Reedy weak equivalence (resp. cofibration, fibration).

REMARK 4.8.6. Any cosimplicial (resp. simplicial) object that is weakly equivalent to a weakly constant cosimplicial (resp. simplicial) object must itself be a weakly constant cosimplicial (resp. simplicial) object. Thus,  $\mathbf{c}_w\mathcal{M}$  (resp.  $\mathbf{s}_w\mathcal{M}$ ) together with the inherited notions of ‘weak equivalence’, ‘cofibration’, and ‘fibration’ is a derivable category (by proposition 4.1.27).

**Proposition 4.8.7.** Let  $\mathcal{M}$  be a model category.

- The following adjunction is an adjoint homotopical equivalence of homotopical categories:

$$(-)^0 \dashv \text{cosk}^0 : \mathcal{M} \rightarrow \mathbf{c}_w\mathcal{M}$$

In particular, we have an adjoint equivalence of homotopy categories:

$$\text{Ho}(-)^0 \dashv \text{Ho cosk}^0 : \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathbf{c}_w\mathcal{M}$$

- The following adjunction is an adjoint homotopical equivalence of homotopical categories:

$$\mathrm{sk}_0 \dashv (-)_0 : \mathbf{s}_w \mathcal{M} \rightarrow \mathcal{M}$$

In particular, we have an adjoint equivalence of homotopy categories:

$$\mathrm{Ho} \mathrm{sk}_0 \dashv \mathrm{Ho} (-)_0 : \mathrm{Ho} \mathbf{s}_w \mathcal{M} \rightarrow \mathrm{Ho} \mathcal{M}$$

*Proof.* The two claims are formally dual; we will prove the first version.

First of all, we note that  $\mathrm{cosk}^0(A)$  is a weakly constant cosimplicial object in  $\mathcal{M}$  for every object  $A$  in  $\mathcal{M}$ , so the adjunction in lemma 4.7.5 restricts to an adjunction between  $\mathcal{M}$  and  $\mathbf{c}_w \mathcal{M}$ . It is clear that the adjunction counit is a natural isomorphism, and proposition 4.8.2 says that the adjunction unit is a natural weak equivalence, so we indeed have an adjoint homotopical equivalence of homotopical categories. Finally, we apply proposition 3.1.29 to prove the claim about homotopy categories. ■

**Corollary 4.8.8.** *Let  $\mathcal{M}$  be a model category.*

- A morphism  $f^\bullet : A^\bullet \rightarrow B^\bullet$  in  $\mathbf{c}_w \mathcal{M}$  is a Reedy weak equivalence if and only if the component  $f^0 : A^0 \rightarrow B^0$  is a weak equivalence in  $\mathcal{M}$ .
- A morphism  $f_\bullet : A_\bullet \rightarrow B_\bullet$  in  $\mathbf{s}_w \mathcal{M}$  is a Reedy weak equivalence if and only if the component  $f_0 : A_0 \rightarrow B_0$  is a weak equivalences in  $\mathcal{M}$ . ■

**Definition 4.8.9.** Let  $\mathcal{M}$  be a model category.

- Let  $A^\bullet$  be a cosimplicial object in  $\mathcal{M}$  and let  $B$  be an object in  $\mathcal{M}$ . The **left hom-complex**  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is the simplicial set defined by the formula below:

$$(\mathrm{Hom}_{\mathcal{M}}(A, B))_n = \mathcal{M}(A^n, B)$$

- Let  $A$  be an object in  $\mathcal{M}$  and let  $B_\bullet$  be a simplicial object in  $\mathcal{M}$ . The **right hom-complex**  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is the simplicial set defined by the formula below:

$$(\mathrm{Hom}_{\mathcal{M}}(A, B))_m = \mathcal{M}(A, B_m)$$

- Let  $A^\bullet$  be a cosimplicial object in  $\mathcal{M}$  and let  $B_\bullet$  be a simplicial object in  $\mathcal{M}$ . The **total hom-complex**  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is the simplicial set defined by the formula below:

$$(\mathrm{Hom}_{\mathcal{M}}(A, B))_k = \mathcal{M}(A_k, B_k)$$

REMARK 4.8.10. Let  $\mathcal{M}$  be a model category.

- For each pair  $(A, B)$  of objects in  $\mathcal{M}$ , we have the following natural isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A), B) &\cong \mathrm{disc} \mathcal{M}(A, B) \\ \mathrm{Hom}_{\mathcal{M}}(A, \mathrm{sk}_0(B)) &\cong \mathrm{disc} \mathcal{M}(A, B)\end{aligned}$$

- For each cosimplicial object  $A^\bullet$  in  $\mathcal{M}$  and each object  $B$  in  $\mathcal{M}$ , we have the following natural isomorphism:

$$\mathrm{Hom}_{\mathcal{M}}(A, \mathrm{sk}_0(B)) \cong \mathrm{Hom}_{\mathcal{M}}(A, B)$$

- For each object  $A$  in  $\mathcal{M}$  and each simplicial object  $B_\bullet$  in  $\mathcal{M}$ , we have the following natural isomorphism:

$$\mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A), B) \cong \mathrm{Hom}_{\mathcal{M}}(A, B)$$

This justifies our use of the same notation for left, right, and total hom-complexes.

REMARK 4.8.11. Let  $\mathcal{M}$  be a model category.

- For each finite simplicial set  $Z$ , each cosimplicial object  $A^\bullet$  in  $\mathcal{M}$ , and each object  $B$  in  $\mathcal{M}$ , we have the following natural bijections:

$$\mathbf{sSet}(Z, \mathrm{Hom}_{\mathcal{M}}(A, B)) \cong \mathcal{M}(Z \star A, B) \cong \mathbf{cM}(A, Z \pitchfork B)$$

- For each finite simplicial set  $Z$ , each object  $A$  in  $\mathcal{M}$ , and each simplicial object  $B_\bullet$  in  $\mathcal{M}$ , we have the following natural bijections:

$$\mathbf{sSet}(Z, \mathrm{Hom}_{\mathcal{M}}(A, B)) \cong \mathbf{sM}(Z \odot A, B) \cong \mathcal{M}(A, \{Z, B\})$$

**Lemma 4.8.12.** *Let  $\mathcal{M}$  be a model category.*

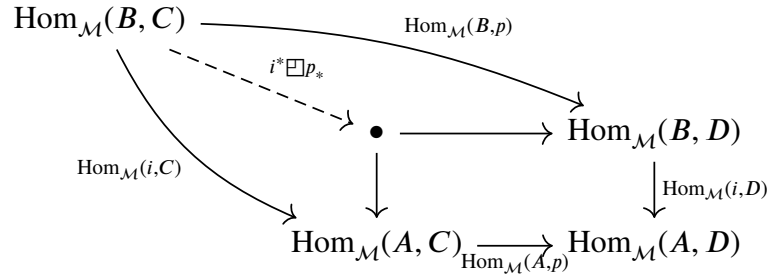
- *If  $i^\bullet : A^\bullet \rightarrow B^\bullet$  is a Reedy cofibration (resp. trivial Reedy cofibration) in  $\mathbf{cM}$ ,  $p : C \rightarrow D$  is a trivial fibration (resp. fibration) in  $\mathcal{M}$ , and the square in the diagram below is a pullback square in  $\mathbf{sSet}$ ,*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{M}}(B, C) & \xrightarrow{\mathrm{Hom}_{\mathcal{M}}(B, p)} & \mathrm{Hom}_{\mathcal{M}}(B, D) \\ \downarrow \mathrm{Hom}_{\mathcal{M}}(i, C) & \dashrightarrow^{i^\bullet \square p_\bullet} & \downarrow \mathrm{Hom}_{\mathcal{M}}(i, D) \\ \mathrm{Hom}_{\mathcal{M}}(A, C) & \xrightarrow{\mathrm{Hom}_{\mathcal{M}}(A, p)} & \mathrm{Hom}_{\mathcal{M}}(A, D) \end{array}$$



then the unique morphism  $i^* \square p_*$  making the diagram commute is a trivial Kan fibration.

- If  $i : A \rightarrow B$  is a cofibration (resp. trivial cofibration) in  $\mathcal{M}$ ,  $p_\bullet : C_\bullet \rightarrow D_\bullet$  is a trivial Reedy fibration (resp. Reedy fibration) in  $\mathbf{sM}$ , and the square in the diagram below is a pullback square in  $\mathbf{sSet}$ ,



then the unique morphism  $i^* \square p_*$  making the diagram commute is a trivial Kan fibration.

*Proof.* Recalling lemma 4.7.2 and remark 4.8.11, this is (essentially) a special case of proposition 5.4.1. ■

**Corollary 4.8.13.** Let  $\mathcal{M}$  be a model category.

- A morphism  $A^\bullet \rightarrow B^\bullet$  in  $\mathcal{M}$  is a Reedy cofibration if and only if the induced morphism

$$(W \star A) \cup^{W \star A} (Z \star B) \rightarrow W \star B$$

is a cofibration in  $\mathcal{M}$  for all monomorphisms  $Z \rightarrow W$  between finite simplicial sets.

- A morphism  $A_\bullet \rightarrow B_\bullet$  in  $\mathcal{M}$  is a Reedy fibration if and only if the induced morphism

$$\{W, A\} \rightarrow \{Z, A\} \times_{\{Z, B\}} \{W, B\}$$

is a fibration in  $\mathcal{M}$  for all monomorphisms  $Z \rightarrow W$  between finite simplicial sets.

*Proof.* Since the boundary inclusions  $\partial \Delta^n \hookrightarrow \Delta^n$  are monomorphisms, one direction of the implication is lemma 4.7.2, and the converse remains to be shown. By proposition 1.4.7, every monomorphism in  $\mathbf{sSet}$  has the left lifting property with respect to every trivial Kan fibration, so given lemma 4.8.12, this is again a special case of proposition 5.4.1. ■

**Lemma 4.8.14.** *Let  $\mathcal{M}$  be a model category.*

- *Given a Reedy cofibration  $A^\bullet \rightarrow B^\bullet$  between cosimplicial resolutions in  $\mathcal{M}$ , the morphism*

$$(\Delta^n \star A) \cup^{\Lambda_k^n \star A} (\Lambda_k^n \star B) \rightarrow \Delta^n \star B$$

*induced by any horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  is a trivial cofibration in  $\mathcal{M}$ .*

- *Given a Reedy fibration  $A_\bullet \rightarrow B_\bullet$  between simplicial resolutions in  $\mathcal{M}$ , the morphism*

$$\{\Delta^n, A\} \rightarrow \{\Lambda_k^n, A\} \times_{\{\Lambda_k^n, B\}} \{\Delta^n, B\}$$

*induced by any horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  is a trivial fibration in  $\mathcal{M}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

Consider the following commutative square in  $\mathcal{M}$ :

$$\begin{array}{ccc} \Lambda_k^n \star A & \longrightarrow & \Lambda_k^n \star B \\ \downarrow & & \downarrow \\ \Delta^n \star A & \longrightarrow & \Delta^n \star B \end{array}$$

Since  $A^\bullet$  and  $B^\bullet$  are cosimplicial resolutions, we may apply [proposition 4.7.18](#) to deduce that the vertical arrows are trivial cofibrations in  $\mathcal{M}$ . Thus, by axiom CM2 and [corollary 4.1.13](#), the morphism  $(\Delta^n \star A) \cup^{\Lambda_k^n \star A} (\Lambda_k^n \star B) \rightarrow \Delta^n \star B$  is a weak equivalence in  $\mathcal{M}$ . It remains to be shown that the morphism is a cofibration in  $\mathcal{M}$ ; but that is a special case of [corollary 4.8.13](#), so we are done.  $\blacksquare$

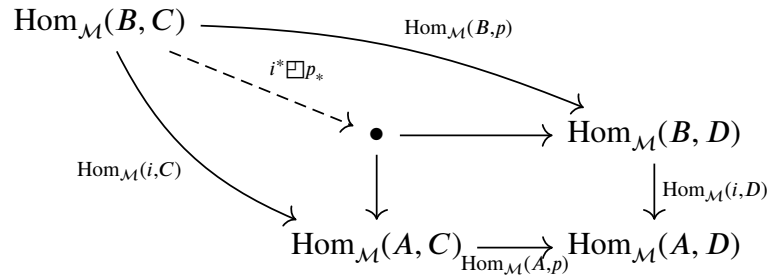
**Corollary 4.8.15.** *Let  $\mathcal{M}$  be a model category.*

- *If  $i^\bullet : A^\bullet \rightarrow B^\bullet$  is a Reedy cofibration between cosimplicial resolutions in  $\mathcal{M}$ ,  $p : C \rightarrow D$  is a fibration in  $\mathcal{M}$ , and the square in the diagram below is a pullback square in  $\mathbf{sSet}$ ,*

$$\begin{array}{ccc} \text{Hom}_{\mathcal{M}}(B, C) & \xrightarrow{\text{Hom}_{\mathcal{M}}(B, p)} & \text{Hom}_{\mathcal{M}}(B, D) \\ \downarrow \text{Hom}_{\mathcal{M}}(i, C) & \dashrightarrow^{i^\bullet \square p_\bullet} & \downarrow \text{Hom}_{\mathcal{M}}(i, D) \\ \text{Hom}_{\mathcal{M}}(A, C) & \xrightarrow{\text{Hom}_{\mathcal{M}}(A, p)} & \text{Hom}_{\mathcal{M}}(A, D) \end{array}$$

then the unique morphism  $i^* \square p_*$  making the diagram commute is a Kan fibration.

- If  $i : A \rightarrow B$  is a cofibration in  $\mathcal{M}$ ,  $p_\bullet : C_\bullet \rightarrow D_\bullet$  is a Reedy fibration between simplicial resolutions in  $\mathcal{M}$ , and the square in the diagram below is a pullback square in  $\mathbf{sSet}$ ,



then the unique morphism  $i^* \square p_*$  making the diagram commute is a Kan fibration.

*Proof.* Recalling lemma 4.8.14 and remark 4.8.11, this is (essentially) a special case of proposition 5.4.1. ■

**Lemma 4.8.16.** *Let  $\mathcal{M}$  be a model category.*

- If  $i^\bullet : A^\bullet \rightarrow B^\bullet$  is a Reedy cofibration between cosimplicial resolutions in  $\mathcal{M}$  and  $p : C \rightarrow D$  is a fibration in  $\mathcal{M}$  such that the induced morphism

$$\mathrm{Hom}_{\mathcal{M}}(B, C) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, C) \times_{\mathrm{Hom}_{\mathcal{M}}(A, D)} \mathrm{Hom}_{\mathcal{M}}(B, D)$$

is a weak homotopy equivalence, then  $p : C \rightarrow D$  has the right lifting property with respect to each  $i^n : A^n \rightarrow B^n$ .

- If  $i : A \rightarrow B$  is a cofibration in  $\mathcal{M}$  and  $p_\bullet : C_\bullet \rightarrow D_\bullet$  is a Reedy fibration between simplicial resolutions in  $\mathcal{M}$  such that the induced morphism

$$\mathrm{Hom}_{\mathcal{M}}(B, C) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, C) \times_{\mathrm{Hom}_{\mathcal{M}}(A, D)} \mathrm{Hom}_{\mathcal{M}}(B, D)$$

is a weak homotopy equivalence, then  $i : A \rightarrow B$  has the left lifting property with respect to each  $p_n : C_n \rightarrow D_n$ .

*Proof.* The two claims are formally dual; we will prove the first version.

**Corollary 4.8.15** says the indicated morphism of simplicial sets is a Kan fibration, hence, is a trivial Kan fibration. Since every simplicial set is cofibrant, the morphism is a (split) epimorphism; thus, for each natural number  $n$ , the map

$$\mathcal{M}(B^n, C) \rightarrow \mathcal{M}(A^n, C) \times_{\mathcal{M}(A^n, D)} \mathcal{M}(B^n, D)$$

is a surjection. We may then apply **lemma A.3.2** to deduce that  $p : C \rightarrow D$  has the right lifting property with respect to each  $i^n : A^n \rightarrow B^n$ . ■

**Proposition 4.8.17.** *Let  $\mathcal{M}$  be a model category.*

- *If  $A$  is a cofibrant object in  $\mathcal{M}$  and  $(\tilde{A}^\bullet, p^\bullet)$  is a cosimplicial frame on  $A$ , then the functor  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  preserves limits, fibrations, and trivial fibrations, and for all fibrant objects  $B$ , there is a natural bijection between  $\pi_0 \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, B)$  and  $\mathrm{Ho} \mathcal{M}(A, B)$ .*
- *If  $B$  is a fibrant object in  $\mathcal{M}$  and  $(\hat{B}_\bullet, i_\bullet)$  is a simplicial frame on  $B$ , then the functor  $\mathrm{Hom}_{\mathcal{M}}(-, \hat{B}) : \mathcal{M}^{\mathrm{op}} \rightarrow \mathbf{sSet}$  preserves limits, fibrations, and trivial fibrations, and for all cofibrant objects  $A$ , there is a natural bijection between  $\pi_0 \mathrm{Hom}_{\mathcal{M}}(A, \hat{B})$  and  $\mathrm{Ho} \mathcal{M}(A, B)$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

It is well-known that each  $\mathcal{M}(\tilde{A}^n, -) : \mathcal{M} \rightarrow \mathbf{Set}$  preserves limits, so by **lemma I.1.9**,  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  preserves limits as well. The fact that  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, -)$  preserves fibrations (resp. trivial fibrations) is a special case of **corollary 4.8.15** (resp. **lemma 4.8.12**).

Recalling that  $p^0 : \tilde{A}^0 \rightarrow \mathrm{cosk}^0(A)^0$  is an isomorphism, we get a natural morphism  $\mathrm{disc} \mathcal{M}(A, B) \rightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, B)$  for all objects  $B$  in  $\mathcal{M}$ , and it is a bijection on vertices. **Proposition 4.7.20** says that  $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$  is a cylinder object for  $\tilde{A}^0$ , so if  $B$  is fibrant, we may apply **lemma 4.2.14** and **theorem 4.4.1** to deduce that the connected components of  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, B)$  are in natural bijection with the homotopy classes of morphisms  $A \rightarrow B$ . ■

**Corollary 4.8.18.** *Let  $\mathcal{M}$  be a model category. If  $A^\bullet$  is a cosimplicial resolution in  $\mathcal{M}$ , then:*

- (i) *For each fibrant object  $B$  in  $\mathcal{M}$ , the left hom-complex  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is a Kan complex.*

- (ii) *The left hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathcal{M} \rightarrow \mathbf{sSet}$  sends weak equivalences between fibrant objects in  $\mathcal{M}$  to homotopy equivalences of Kan complexes.*
- (iii) *The total hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathbf{sM} \rightarrow \mathbf{sSet}$  sends Reedy weak equivalences between degreewise fibrant simplicial objects in  $\mathcal{M}$  to weak homotopy equivalences in  $\mathbf{sSet}$ .*

*Dually, if  $B_{\bullet}$  is a simplicial resolution in  $\mathcal{M}$ , then:*

- (i') *For each cofibrant object  $A$  in  $\mathcal{M}$ , the right hom-complex  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is a Kan complex.*
- (ii') *The right hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(-, B) : \mathcal{M}^{\mathrm{op}} \rightarrow \mathbf{sSet}$  sends weak equivalences between cofibrant objects in  $\mathcal{M}$  to homotopy equivalences of Kan complexes.*
- (iii') *The total hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(-, B) : (\mathbf{cM})^{\mathrm{op}} \rightarrow \mathbf{sSet}$  sends Reedy weak equivalences between degreewise cofibrant cosimplicial objects in  $\mathcal{M}$  to weak homotopy equivalences.*

*Proof.* (i) and (ii). Apply [proposition 4.8.2](#) and Ken Brown's lemma ([4.3.6](#)) to the previous proposition.

(iii). First, consider the functor  $\mathcal{M}(A^{\bullet}, -) : \mathbf{sM} \rightarrow \mathbf{ssSet}$  that sends an object  $B_{\bullet}$  in  $\mathbf{sM}$  to the bisimplicial set defined by the formula below:

$$(\mathcal{M}(A^{\bullet}, B_{\bullet}))_m = \mathrm{Hom}_{\mathcal{M}}(A, B_m)$$

Thus, by claim (ii),  $\mathcal{M}(A^{\bullet}, -)$  sends Reedy weak equivalences between degreewise fibrant simplicial objects in  $\mathcal{M}$  to Reedy weak equivalences in  $\mathbf{ssSet}$ . We may then use [lemma 1.6.7](#) and [theorem 1.6.9](#), we may deduce that the total hom-complex functor has the required property. ■

**Corollary 4.8.19.** *Let  $\mathcal{M}$  be a model category, let  $\mathbf{c}_r\mathcal{M}$  be the full subcategory of  $\mathbf{cM}$  spanned by the cosimplicial resolutions, and let  $\mathbf{s}_r\mathcal{M}$  be the full subcategory of  $\mathbf{sM}$  spanned by the simplicial resolutions. Then, the total hom-complex functor  $\mathrm{Hom}_{\mathcal{M}} : (\mathbf{c}_r\mathcal{M})^{\mathrm{op}} \times \mathbf{s}_r\mathcal{M} \rightarrow \mathbf{sSet}$  is a homotopical functor.*

*Proof.* Apply [lemma 4.8.3](#) to [corollary 4.8.18](#). ■

**Proposition 4.8.20.** *Let  $\mathcal{M}$  be a model category.*

- *If  $A^\bullet$  is a degreewise cofibrant weakly constant cosimplicial object in  $\mathcal{M}$ , then the functor  $\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathbf{sM} \rightarrow \mathbf{sSet}$  preserves weak equivalences between simplicial resolutions.*
- *If  $B_\bullet$  is a degreewise fibrant weakly constant simplicial object in  $\mathcal{M}$ , then the functor  $\mathrm{Hom}_{\mathcal{M}}(-, B) : (\mathbf{cM})^{\mathrm{op}} \rightarrow \mathbf{sSet}$  preserves weak equivalences between cosimplicial resolutions.*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $A^\bullet$  be a degreewise cofibrant weakly constant cosimplicial object in  $\mathcal{M}$  and let  $p^\bullet : A^\bullet \rightarrow \mathrm{cosk}^0(A^0)$  be the component of the adjunction unit. By [proposition 4.8.2](#),  $p^\bullet$  is a Reedy weak equivalence. Let  $f_\bullet : B_\bullet \rightarrow C_\bullet$  be a weak equivalence between simplicial resolutions. We then have the following commutative diagram in  $\mathbf{sSet}$ :

$$\begin{array}{ccc}
 \mathrm{Hom}_{\mathcal{M}}(A, B) & \xrightarrow{\mathrm{Hom}_{\mathcal{M}}(p, B)} & \mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A^0), B) \\
 \mathrm{Hom}_{\mathcal{M}}(A, f) \downarrow & & \downarrow \mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A^0), f) \\
 \mathrm{Hom}_{\mathcal{M}}(A, C) & \xrightarrow{\mathrm{Hom}_{\mathcal{M}}(p, C)} & \mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A^0), C)
 \end{array}$$

[Corollary 4.8.18](#) says that  $\mathrm{Hom}_{\mathcal{M}}(p, B)$  and  $\mathrm{Hom}_{\mathcal{M}}(p, C)$  are weak homotopy equivalences; but recalling [lemma 4.8.3](#), we may then use [remark 4.8.10](#) to deduce that  $\mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A^0), f)$  is a weak homotopy equivalence. Finally, we apply the 2-out-of-3 property of weak homotopy equivalences to conclude that  $\mathrm{Hom}_{\mathcal{M}}(A, f)$  itself is a weak homotopy equivalence. ■

**Proposition 4.8.21.** *Let  $\mathcal{M}$  be a model category. If  $A^\bullet$  is a cosimplicial resolution in  $\mathcal{M}$  and  $B_\bullet$  is a simplicial resolution in  $\mathcal{M}$ , then there is a natural diagram of weak homotopy equivalences in  $\mathbf{sSet}$  of the form below,*

$$\mathrm{Hom}_{\mathcal{M}}(A, B_0) \longrightarrow \mathrm{Hom}_{\mathcal{M}}(A, B) \longleftarrow \mathrm{Hom}_{\mathcal{M}}(A^0, B)$$

where  $\mathrm{Hom}_{\mathcal{M}}(A, B_0)$  is the left hom-complex,  $\mathrm{Hom}_{\mathcal{M}}(A^0, B)$  is the right hom-complex,  $\mathrm{Hom}_{\mathcal{M}}(A, B)$  is the total hom-complex, the rightward arrow is the morphism induced by the adjunction counit component  $i_\bullet : \mathrm{sk}_0(B_0) \rightarrow B_\bullet$ , and the leftward arrow is the morphism induced by the adjunction unit component  $p^\bullet : A^\bullet \rightarrow \mathrm{cosk}^0(A^0)$ .

*Proof.* The two halves of the claim are formally dual; we will show that there is a natural weak homotopy equivalence  $\mathrm{Hom}_{\mathcal{M}}(A, B_0) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, B)$ .

By [proposition 4.8.2](#),  $B_0$  is a fibrant object in  $\mathcal{M}$  and  $i_{\bullet} : \mathrm{sk}_0(B_0) \rightarrow B_{\bullet}$  is a Reedy weak equivalence. [Lemma 4.8.3](#) says that each  $B_m$  is a fibrant object in  $\mathcal{M}$ , so  $i_{\bullet}$  is moreover a Reedy weak equivalence between degreewise fibrant objects. Thus,  $\mathrm{Hom}_{\mathcal{M}}(A, \mathrm{sk}_0(B_0)) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, B)$  is a weak homotopy equivalence, by [corollary 4.8.18](#). Since the total hom-complex  $\mathrm{Hom}_{\mathcal{M}}(A, \mathrm{sk}_0(B_0))$  is naturally isomorphic to the left hom-complex  $\mathrm{Hom}_{\mathcal{M}}(A, B_0)$  (by [remark 4.8.10](#)), this is the required natural weak homotopy equivalence. ■

**Definition 4.8.22.** Let  $A$  and  $B$  be objects in a model category  $\mathcal{M}$ .

- A **left homotopy function complex** from  $A$  to  $B$  consists of the data  $(\tilde{A}^{\bullet}, p^{\bullet}, \hat{B}, i, \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B}))$ , where  $(\tilde{A}^{\bullet}, p^{\bullet})$  is a cosimplicial resolution of  $A$ ,  $(\hat{B}, i)$  is a fibrant replacement for  $B$ , and  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B})$  is the left hom-complex.
- A **right homotopy function complex** from  $A$  to  $B$  consists of the data  $(\tilde{A}, p, \hat{B}_{\bullet}, i_{\bullet}, \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B}))$ , where  $(\tilde{A}, p)$  is a cofibrant replacement for  $A$ ,  $(\hat{B}_{\bullet}, i_{\bullet})$  is a simplicial resolution of  $B$ , and  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B})$  is the right hom-complex.
- A **two-sided homotopy function complex** from  $A$  to  $B$  consists of the data  $(\tilde{A}^{\bullet}, p^{\bullet}, \hat{B}_{\bullet}, i_{\bullet}, \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B}))$ , where  $(\tilde{A}^{\bullet}, p^{\bullet})$  is a cosimplicial resolution of  $A$ ,  $(\hat{B}_{\bullet}, i_{\bullet})$  is a simplicial resolution of  $B$ , and  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B})$  is the total hom-complex.

We will often abuse notation and say  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B})$  is a (left, right, or two-sided) homotopy function complex from  $A$  to  $B$ , omitting mention of the other data.

Note that the weak homotopy type of  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, \hat{B})$  depends only on the isomorphism class of  $A$  and  $B$  in  $\mathrm{Ho} \mathcal{M}$ , by [corollary 4.8.18](#).

**Proposition 4.8.23.** *Let  $f : A \rightarrow B$  be a morphism in a model category  $\mathcal{M}$ .*

- *Let  $(\hat{A}, i_A)$  and  $(\hat{B}, i_B)$  be fibrant replacements for  $A$  and  $B$ , respectively, and let  $\hat{f} : \hat{A} \rightarrow \hat{B}$  be any morphism in  $\mathcal{M}$  making the diagram below commute:*

$$\begin{array}{ccc} A & \xrightarrow{i_A} & \hat{A} \\ f \downarrow & & \downarrow \hat{f} \\ B & \xrightarrow{i_B} & \hat{B} \end{array}$$

*The morphism  $f : A \rightarrow B$  is a weak equivalence in  $\mathcal{M}$  if and only if the induced morphism of left homotopy function complexes*

$$\mathrm{Hom}_{\mathcal{M}}(C, \hat{f}) : \mathrm{Hom}_{\mathcal{M}}(C, \hat{A}) \rightarrow \mathrm{Hom}_{\mathcal{M}}(C, \hat{B})$$

*is a weak homotopy equivalence for all cosimplicial resolutions  $C^\bullet$ .*

- *Let  $(\tilde{A}, p_A)$  and  $(\tilde{B}, p_B)$  be cofibrant replacements for  $A$  and  $B$ , respectively, and let  $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$  be any morphism in  $\mathcal{M}$  making the diagram below commute:*

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{p_A} & A \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{B} & \xrightarrow{p_B} & B \end{array}$$

*The morphism  $f : A \rightarrow B$  is a weak equivalence in  $\mathcal{M}$  if and only if the induced morphism of right homotopy function complexes*

$$\mathrm{Hom}_{\mathcal{M}}(\tilde{f}, C) : \mathrm{Hom}_{\mathcal{M}}(\tilde{B}, C) \rightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, C)$$

*is a weak homotopy equivalence for all simplicial resolutions  $C_\bullet$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

First, suppose  $f : A \rightarrow B$  is a weak equivalence in  $\mathcal{M}$ . Then, by axiom CM2,  $\hat{f} : \hat{A} \rightarrow \hat{B}$  is also a weak equivalence, so we may use [corollary 4.8.18](#) to deduce that  $\mathrm{Hom}_{\mathcal{M}}(C, \hat{f})$  is a weak homotopy equivalence for all cosimplicial resolutions  $C^\bullet$ .

Conversely, suppose  $\mathrm{Hom}_{\mathcal{M}}(C, \hat{f})$  is a weak homotopy equivalence for all cosimplicial resolutions  $C^\bullet$ . [Proposition 4.8.17](#) then implies that the hom-set map

$$\mathrm{Ho} \mathcal{M}(C^0, \hat{f}) : \mathrm{Ho} \mathcal{M}(C^0, \hat{A}) \rightarrow \mathrm{Ho} \mathcal{M}(C^0, \hat{B})$$



is a bijection for all cosimplicial resolutions  $C^\bullet$ ; but [theorem 4.7.16](#) implies every cofibrant object in  $\mathcal{M}$  occurs as  $C^0$  for some cosimplicial resolution  $C^\bullet$ , so using [proposition 4.1.23](#), we deduce that  $\hat{f} : \hat{A} \rightarrow \hat{B}$  is an isomorphism in  $\text{Ho } \mathcal{M}$ . [Theorem 4.4.1](#) then implies  $\hat{f} : \hat{A} \rightarrow \hat{B}$  must be a weak equivalence in  $\mathcal{M}$ , and therefore (using axiom CM2)  $f : A \rightarrow B$  itself is a weak equivalence in  $\mathcal{M}$ . ■

**Definition 4.8.24.** Let  $\mathcal{M}$  be a model category.

- A **derived left hom-space functor** for an object  $B$  in  $\mathcal{M}$  is a functor  $\mathbf{RHom}_{\mathcal{M}}(-, B) : (\text{Ho } \mathcal{M})^{\text{op}} \rightarrow \text{Ho sSet}$  equipped with natural isomorphisms

$$\mathbf{RHom}_{\mathcal{M}}(A^0, B) \cong \text{Hom}_{\mathcal{M}}(A, \hat{B})$$

in  $\text{Ho sSet}$ , where  $A^\bullet$  varies over the cosimplicial resolutions in  $\mathcal{M}$ ,  $(\hat{B}, i)$  varies over the fibrant replacements for  $B$ , and  $\text{Hom}_{\mathcal{M}}(A, \hat{B})$  is the left hom-complex.

- A **derived right hom-space functor** for an object  $A$  in  $\mathcal{M}$  is a functor  $\mathbf{RHom}_{\mathcal{M}}(A, -) : \text{Ho } \mathcal{M} \rightarrow \text{Ho sSet}$  equipped with natural isomorphisms

$$\mathbf{RHom}_{\mathcal{M}}(A, B_0) \cong \text{Hom}_{\mathcal{M}}(\tilde{A}, B)$$

in  $\text{Ho sSet}$ , where  $(\tilde{A}, p)$  varies over the cofibrant replacements for  $A$ ,  $B_\bullet$  varies over the simplicial resolutions in  $\mathcal{M}$ , and  $\text{Hom}_{\mathcal{M}}(\tilde{A}, B)$  is the right hom-complex.

- A **derived hom-space functor** for  $\mathcal{M}$  is a functor  $\mathbf{RHom}_{\mathcal{M}} : (\text{Ho } \mathcal{M})^{\text{op}} \times \text{Ho } \mathcal{M} \rightarrow \text{Ho sSet}$  equipped with natural isomorphisms

$$\mathbf{RHom}_{\mathcal{M}}(A^0, B_0) \cong \text{Hom}_{\mathcal{M}}(A, B)$$

in  $\text{Ho sSet}$ , where  $A^\bullet$  varies over the cosimplicial resolutions in  $\mathcal{M}$ ,  $B_\bullet$  varies over the simplicial resolutions in  $\mathcal{M}$ , and  $\text{Hom}_{\mathcal{M}}(A, B)$  is the total hom-complex.

We will often refer to the object  $\mathbf{RHom}_{\mathcal{M}}(A, B)$  as a **derived hom-space**, omitting mention of the other data.

The name ‘derived hom-space’ is justified by the following theorem.

**Theorem 4.8.25.** *Let  $\mathcal{M}$  be a model category, let  $(\mathbf{c}_r\mathcal{M}, \mathcal{Q}^\bullet, p^\bullet)$  be a left Quillen deformation retract of  $\mathbf{c}_w\mathcal{M}$ , and let  $(\mathbf{s}_r\mathcal{M}, \mathcal{R}_\bullet, i_\bullet)$  be a right Quillen deformation retract of  $\mathbf{s}_w\mathcal{M}$ .*

- (i)  *$((\mathbf{c}_r\mathcal{M})^{\text{op}} \times \mathbf{s}_r\mathcal{M}, \mathcal{Q}^\bullet \times \mathcal{R}_\bullet, (p^\bullet, i_\bullet))$  is a right deformation retract for the total hom-complex functor  $\text{Hom}_{\mathcal{M}} : (\mathbf{c}_w\mathcal{M})^{\text{op}} \times \mathbf{s}_w\mathcal{M} \rightarrow \mathbf{sSet}$ .*
- (ii)  *$\text{Hom}_{\mathcal{M}} : (\mathbf{c}_w\mathcal{M})^{\text{op}} \times \mathbf{s}_w\mathcal{M} \rightarrow \mathbf{sSet}$  has a total right derived functor; furthermore, if  $(\mathbf{c}_r\mathcal{M}, \mathcal{Q}^\bullet, p^\bullet)$  and  $(\mathbf{s}_r\mathcal{M}, \mathcal{R}_\bullet, i_\bullet)$  are functorial deformation retracts, then  $\text{Hom}_{\mathcal{M}}$  also has a homotopical right approximation.*
- (iii) *The functor  $\mathbf{RHom}_{\mathcal{M}}(\text{cosk}^0(-), \text{sk}_0(-)) : (\text{Ho } \mathcal{M})^{\text{op}} \times \text{Ho } \mathcal{M} \rightarrow \text{Ho } \mathbf{sSet}$  is a derived hom-space functor for  $\mathcal{M}$ .*

*Proof.* (i). Recall that [proposition 4.8.2](#) says every cofibrant weakly constant cosimplicial object is a cosimplicial resolution, and every fibrant weakly constant simplicial object is a simplicial resolution. Thus, a cofibrant replacement for  $\text{cosk}^0(A)$  is a cosimplicial resolution of  $A$ , and a fibrant replacement for  $\text{sk}_0(B)$  is a simplicial resolution of  $B$ . The claim then follows from [corollary 4.8.19](#).

(ii). Apply theorems [3.3.13](#) and [3.4.10](#).

(iii). This follows from claims (i) and (ii). ■

**Theorem 4.8.26.** *Let  $\mathcal{M}$  be a model category. If  $B$  is a fibrant object in  $\mathcal{M}$ , then:*

- (i) *The left hom-complex functor  $\text{Hom}_{\mathcal{M}}(-, B) : (\mathbf{c}_w\mathcal{M})^{\text{op}} \rightarrow \mathbf{sSet}$  sends trivial cofibrations in  $\mathbf{c}_w\mathcal{M}$  to weak homotopy equivalences in  $\mathbf{sSet}$ .*
- (ii) *The left hom-complex functor  $\text{Hom}_{\mathcal{M}}(-, B) : (\mathbf{c}_w\mathcal{M})^{\text{op}} \rightarrow \mathbf{sSet}$  admits a total right derived functor.*
- (iii) *The functor  $\mathbf{RHom}_{\mathcal{M}}(\text{cosk}^0(-), B) : (\text{Ho } \mathcal{M})^{\text{op}} \rightarrow \text{Ho } \mathbf{sSet}$  is a derived left hom-space functor.*

Dually, if  $A$  is a cofibrant object in  $\mathcal{M}$ , then:

- (i') The right hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathbf{s}_w\mathcal{M} \rightarrow \mathbf{sSet}$  sends trivial fibrations in  $\mathbf{s}_w\mathcal{M}$  to weak homotopy equivalences in  $\mathbf{sSet}$ .
- (ii') The right hom-complex functor  $\mathrm{Hom}_{\mathcal{M}}(A, -) : \mathbf{s}_w\mathcal{M} \rightarrow \mathbf{sSet}$  admits a total right derived functor.
- (iii') The functor  $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, \mathrm{sk}_0(-)) : \mathrm{Ho}\mathcal{M} \rightarrow \mathrm{Ho}\mathbf{sSet}$  is a derived right hom-space functor.

*Proof.* (i). Let  $f^\bullet : A^\bullet \rightarrow C^\bullet$  be a trivial cofibration in  $\mathbf{c}_w\mathcal{M}$ , and choose a simplicial frame  $(\hat{B}_\bullet, i_\bullet)$  on  $B$ . We then have a morphism of bisimplicial sets

$$\mathcal{M}(f^\bullet, \hat{B}_\bullet) : \mathcal{M}(C^\bullet, \hat{B}_\bullet) \rightarrow \mathcal{M}(A^\bullet, \hat{B}_\bullet)$$

and since each  $f^n : A^n \rightarrow C^n$  is a trivial cofibration in  $\mathcal{M}$  (by [proposition 4.5.14](#)), [proposition 4.8.17](#) says that the components

$$\mathcal{M}(f^n, \hat{B}_\bullet) : \mathcal{M}(C^n, \hat{B}_\bullet) \rightarrow \mathcal{M}(A^n, \hat{B}_\bullet)$$

are trivial Kan fibrations, hence weak homotopy equivalences *a fortiori*. Thus, applying [lemma 1.6.7](#) and [theorem 1.6.9](#), we deduce that the morphism

$$\mathrm{Hom}_{\mathcal{M}}(f, \hat{B}) : \mathrm{Hom}_{\mathcal{M}}(C, \hat{B}) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, \hat{B})$$

is a weak homotopy equivalence. Finally, using [proposition 4.8.21](#) and the 2-out-of-3 property of weak homotopy equivalences, we conclude that the morphism  $\mathrm{Hom}_{\mathcal{M}}(f, B) : \mathrm{Hom}_{\mathcal{M}}(C, B) \rightarrow \mathrm{Hom}_{\mathcal{M}}(A, B)$  is indeed a weak homotopy equivalence.

(ii). [Remark 4.8.6](#) says  $\mathbf{c}_w\mathcal{M}$  is a derivable category, so [theorem 4.3.11](#) yields the required total right derived functor.

(iii). The total derived functor theorem implies that  $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(A), B)$  is naturally isomorphic to the weak homotopy type of  $\mathrm{Hom}_{\mathcal{M}}(\tilde{A}, B)$  for any cofibrant replacement  $(\tilde{A}, p^\bullet)$  for  $\mathrm{cosk}^0(A)$ , and [proposition 4.8.2](#) says any such  $(\tilde{A}, p^\bullet)$  is a cosimplicial resolution of  $A$ , so  $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(\mathrm{cosk}^0(-), B)$  is indeed a derived left hom-space functor. ■

**Theorem 4.8.27.** *Let  $\mathcal{M}$  be a model category, let  $\mathbf{c}_r\mathcal{M}$  be the full subcategory of  $\mathbf{c}\mathcal{M}$  spanned by the cosimplicial resolutions, and let  $\mathbf{s}_r\mathcal{M}$  be the full subcategory of  $\mathbf{s}\mathcal{M}$  spanned by the simplicial resolutions. If  $\mathbf{h}_\bullet : \mathcal{M} \rightarrow [(\mathbf{c}_r\mathcal{M})^{\text{op}}, \mathbf{sSet}]_h$  is the functor defined by*

$$\mathbf{h}_B(A) = \text{Hom}_{\mathcal{M}}(A, B)$$

where  $\text{Hom}_{\mathcal{M}}(A, B)$  is the left hom-complex, then:

- (i)  $\mathbf{h}_\bullet$  sends fibrations (resp. trivial fibrations) in  $\mathcal{M}$  to componentwise Kan fibrations (resp. componentwise trivial Kan fibrations).
- (ii)  $\mathbf{h}_\bullet$  admits a total right derived functor.
- (iii) For each cosimplicial resolution  $A^\bullet$  in  $\mathcal{M}$  and each object  $B$  in  $\mathcal{M}$ ,  $\mathbf{Rh}_B(A)$  is a derived hom-space  $\mathbf{RHom}_{\mathcal{M}}(A^0, B)$ .

Dually, if  $\mathbf{h}^\bullet : \mathcal{M}^{\text{op}} \rightarrow [\mathbf{s}_r\mathcal{M}, \mathbf{sSet}]_h$  is the functor defined by

$$\mathbf{h}^A(B) = \text{Hom}_{\mathcal{M}}(A, B)$$

where  $\text{Hom}_{\mathcal{M}}(A, B)$  is the right hom-complex, then:

- (i')  $\mathbf{h}^\bullet$  sends cofibrations (resp. trivial cofibrations) in  $\mathcal{M}$  to componentwise Kan fibrations (resp. componentwise trivial Kan fibrations).
- (ii')  $\mathbf{h}^\bullet$  admits a total right derived functor.
- (iii') For each object  $A$  in  $\mathcal{M}$  and each simplicial resolution  $B_\bullet$  in  $\mathcal{M}$ ,  $\mathbf{Rh}^A(B)$  is a derived hom-space  $\mathbf{RHom}_{\mathcal{M}}(A, B_0)$ .

*Proof.* (i). This is [proposition 4.8.17](#); note that [corollary 4.8.18](#) implies that  $\mathbf{h}_B : (\mathbf{c}_r\mathcal{M})^{\text{op}} \rightarrow \mathbf{sSet}$  is indeed a homotopical functor.

(ii). Since the weak equivalences in  $[(\mathbf{c}_r\mathcal{M})^{\text{op}}, \mathbf{sSet}]_h$  are componentwise (by definition), we may apply [theorem 4.3.II](#).

(iii). The total derived functor theorem implies that  $\mathbf{Rh}_B(A)$  is isomorphic to the weak homotopy type of the left hom-complex  $\text{Hom}_{\mathcal{M}}(A, \hat{B})$ , where  $(\hat{B}, i)$  is any fibrant replacement for  $B$ , so  $\mathbf{Rh}_B(A)$  is a derived hom-space  $\mathbf{RHom}_{\mathcal{M}}(A^0, B)$ . ■

**Theorem 4.8.28.** *Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories, let*

$$F \dashv G : \mathcal{N} \rightarrow \mathcal{M}$$

*be a Quillen adjunction, and let*

$$\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho} \mathcal{N} \rightarrow \mathrm{Ho} \mathcal{M}$$

*be the derived adjunction. Then there are natural isomorphisms*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{N}}((\mathbf{L}F)A, B) \cong \mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, (\mathbf{R}G)B)$$

*in  $\mathrm{Ho} \mathbf{sSet}$ , where  $A$  varies in  $\mathrm{Ho} \mathcal{M}$  and  $B$  varies in  $\mathrm{Ho} \mathcal{N}$ .*

*Proof.* Let  $\tilde{A}$  be a cosimplicial resolution in  $\mathcal{M}$  and let  $B$  be a fibrant object in  $\mathcal{N}$ . Since  $F \dashv G$  is an adjunction, we have the following natural isomorphism of left hom-complexes;

$$\mathrm{Hom}_{\mathcal{N}}(F\tilde{A}, B) \cong \mathrm{Hom}_{\mathcal{M}}(\tilde{A}, GB)$$

moreover, by [proposition 4.3.4](#) and [lemma 4.8.4](#), both simplicial sets are (part of) left homotopy function complexes. [Theorem 4.3.11](#) and [proposition 4.8.7](#) then imply we have the required natural isomorphism in  $\mathrm{Ho} \mathbf{sSet}$ . ■



## TOPICS IN MODEL CATEGORIES

### 5.1 Combinatorial model categories

**Prerequisites.** §§0.2, 0.3, 0.5, 4.1, A.3.

**Definition 5.1.1.** A **cofibrantly generated model category** is a complete and cocomplete model category  $\mathcal{M}$  such that there exist a set  $\mathcal{I}$  of cofibrations and a set  $\mathcal{I}'$  of trivial cofibrations satisfying these conditions:

- $(\mathcal{I}, \mathcal{M})$  admits the small object argument, and  $\text{cof}_{\mathcal{M}} \mathcal{I}$  is the class of all cofibrations in  $\mathcal{M}$ .
- $(\mathcal{I}', \mathcal{M})$  admits the small object argument, and  $\text{cof}_{\mathcal{M}} \mathcal{I}'$  is the class of all trivial cofibrations in  $\mathcal{M}$ .

**REMARK 5.1.2.** By Quillen's small object argument (0.5.12), any cofibrantly generated model category satisfies axiom CM5\* and thus is a DHK model category.

**Theorem 5.1.3** (Kan's recognition principle). *Let  $\mathcal{M}$  be a complete and cocomplete locally small category, let  $\mathcal{W}$  be a subcategory of  $\mathcal{M}$  containing all the objects, and let  $\mathcal{I}$  and  $\mathcal{I}'$  be subsets of  $\text{mor } \mathcal{M}$ . Assume the following hypotheses:*

- $\mathcal{W}$  is closed under retracts and has the 2-out-of-3 property in  $\mathcal{M}$ .
- $(\mathcal{I}, \mathcal{M})$  and  $(\mathcal{I}', \mathcal{M})$  both admit the small object argument.
- $\text{inj}^{\mathcal{M}} \mathcal{I} \subseteq \mathcal{W} \cap \text{inj}^{\mathcal{M}} \mathcal{I}'$ .
- $\text{cof}_{\mathcal{M}} \mathcal{I}' \subseteq \mathcal{W} \cap \text{cof}_{\mathcal{M}} \mathcal{I}$ .

If, in addition, either

- $\text{inj}^{\mathcal{M}} \mathcal{I} = \mathcal{W} \cap \text{inj}^{\mathcal{M}} \mathcal{I}'$ , or
- $\text{cof}_{\mathcal{M}} \mathcal{I}' = \mathcal{W} \cap \text{cof}_{\mathcal{M}} \mathcal{I}$ .

then there exists a unique model structure on  $\mathcal{M}$  such that  $\text{cof}_{\mathcal{M}} \mathcal{I}$  is the class of cofibrations,  $\text{cof}_{\mathcal{M}} \mathcal{I}'$  is the class of trivial cofibrations, and  $\mathcal{W}$  is the class of weak equivalences.

*Proof.* See Theorem 11.3.1 in [Hirschhorn, 2003]. □

**Theorem 5.1.4** (Kan's lifting theorem). *Let  $\mathcal{M}$  be a complete and cocomplete locally small category, let  $\mathcal{N}$  be a cofibrantly generated model category. Assume the following hypotheses:*

- $F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$  is an adjunction of categories.
- $\mathcal{J}$  is a generating set of cofibrations in  $\mathcal{N}$ .
- $\mathcal{J}'$  is a generating set of trivial cofibrations in  $\mathcal{N}$ .
- $(\mathcal{I}, \mathcal{M})$  and  $(\mathcal{I}', \mathcal{M})$  admit the small object argument, where  $\mathcal{I}$  and  $\mathcal{I}'$  are the following sets:

$$\begin{aligned} \mathcal{I} &= \{Ff \mid f \in \mathcal{J}\} \\ \mathcal{I}' &= \{Ff \mid f \in \mathcal{J}'\} \end{aligned}$$

- $G$  sends relative  $\mathcal{I}'$ -cell complexes in  $\mathcal{M}$  to weak equivalences in  $\mathcal{N}$ .

Then:

- (i) *There is a unique model structure on  $\mathcal{M}$  with  $\text{cof}_{\mathcal{M}} \mathcal{I}$  as the class of cofibrations and  $\text{cof}_{\mathcal{M}} \mathcal{I}'$  as the class of trivial cofibrations.*
- (ii) *A morphism  $g : A \rightarrow B$  in  $\mathcal{M}$  is a weak equivalence in this model structure if and only if  $Gg : GA \rightarrow GB$  is a weak equivalence in  $\mathcal{N}$ .*
- (iii)  *$F \dashv G : \mathcal{M} \rightarrow \mathcal{N}$  is a Quillen adjunction with respect to this model structure.*

*Proof.* See Theorem 11.3.2 in [Hirschhorn, 2003]. □



**Theorem 5.1.5** (Existence of cofibrantly generated projective model structures). *Let  $\mathcal{M}$  be a cofibrantly generated model category. If  $\mathbb{A}$  is a small category, then the projective model structure on  $[\mathbb{A}, \mathcal{M}]$  exists and is cofibrantly generated.*

*Proof.* See Theorem 11.6.1 in [Hirschhorn, 2003]. □

The following is due to Smith [1998].

**Definition 5.1.6.** A **combinatorial model category** is a cofibrantly generated model category that is also a locally presentable category.

**REMARK 5.1.7.** Since locally presentable categories are automatically complete and cocomplete,<sup>[1]</sup> in light of [remark 0.5.9](#), to show that a locally presentable model category  $\mathcal{M}$  is a combinatorial model category, it is enough to verify that there exist sets  $\mathcal{I}$  and  $\mathcal{I}'$  such that  $\text{cof}_{\mathcal{M}} \mathcal{I}$  is the class of all cofibrations in  $\mathcal{M}$  and  $\text{cof}_{\mathcal{M}} \mathcal{I}'$  is the class of all trivial cofibrations in  $\mathcal{M}$ .

**Theorem 5.1.8** (Smith’s recognition principle). *Let  $\mathcal{M}$  be a locally presentable category, let  $\mathcal{W}$  be a subcategory of  $\mathcal{M}$  containing all the objects, and let  $\mathcal{I}$  be a subset of  $\text{mor } \mathcal{M}$ . Assume the following hypotheses:*

- $\mathcal{W}$  is closed under retracts and has the 2-out-of-3 property in  $\mathcal{M}$ .
- Every  $\mathcal{I}$ -injective morphism in  $\mathcal{M}$  is in  $\mathcal{W}$ .
- The class  $\text{cof}_{\mathcal{M}} \mathcal{I} \cap \mathcal{W}$  is closed under pushouts and transfinite composition.
- $\mathcal{W}$ , considered as a full subcategory of  $[\mathbb{2}, \mathcal{M}]$ , is an accessible subcategory of  $[\mathbb{2}, \mathcal{M}]$ .

*Then there exists a unique model structure on  $\mathcal{M}$  such that  $\text{cof}_{\mathcal{M}} \mathcal{I}$  is the class of cofibrations and  $\mathcal{W}$  is the class of weak equivalences, and this makes  $\mathcal{M}$  a combinatorial model category.*

*Proof.* See Theorem 1.7 in [Beke, 2000]. □

**Definition 5.1.9.** A **cellular combinatorial model category** is a combinatorial model category  $\mathcal{M}$  such that every cofibration is an effective monomorphism.

[1] See [theorem 0.2.39](#).

TODO: Replace this with the solution set condition.

REMARK 5.1.10. A cellular combinatorial model category is the same thing as a combinatorial model category that is also a cellular model category in the sense of Hirschhorn [2003, Ch. 12].

**Theorem 5.1.11** (Existence of combinatorial injective model structures). *Let  $\mathcal{M}$  be a combinatorial model category and let  $\mathbb{A}$  be a small category.*

- (i) *The injective model structure on  $[\mathbb{A}, \mathcal{M}]$  exists and is combinatorial.*
- (ii) *If  $\mathcal{M}$  is cellular, then so is  $[\mathbb{A}, \mathcal{M}]$  (with the injective model structure).*

*Proof.* (i). This theorem is due to Lurie; see [HTT, Proposition A.2.8.2].

(ii). This follows from the fact that limits and colimits in  $[\mathbb{A}, \mathcal{M}]$  are computed componentwise. □

**Definition 5.1.12.** Let  $\kappa$  and  $\lambda$  be regular cardinals. A **strongly  $(\kappa, \lambda)$ -combinatorial model category** is a combinatorial model category  $\mathcal{M}$  that satisfies these axioms:

- $\mathcal{M}$  is a locally  $\kappa$ -presentable category, and  $\kappa \triangleleft \lambda$ .
- $\mathbf{K}_\lambda(\mathcal{M})$  is closed under finite limits in  $\mathcal{M}$ .
- Each hom-set in  $\mathbf{K}_\kappa(\mathcal{M})$  is  $\lambda$ -small.
- There exist  $\lambda$ -small sets of morphisms in  $\mathbf{K}_\kappa(\mathcal{M})$  that cofibrantly generate the model structure of  $\mathcal{M}$ .

**Proposition 5.1.13.** *For any combinatorial model category  $\mathcal{M}$ , there exist regular cardinals  $\kappa$  and  $\lambda$  making  $\mathcal{M}$  into a strongly  $(\kappa, \lambda)$ -combinatorial model category.*

*Proof.* Apply [proposition 0.2.34](#), [lemma 0.2.37](#), and [remark 0.3.4](#). ■

**Proposition 5.1.14.** *Let  $\mathcal{M}$  be a strongly  $(\kappa, \lambda)$ -combinatorial model category.*

- (i) *There exist (trivial cofibration, fibration)- and (cofibration, trivial fibration)-factorisation functors that are  $\kappa$ -accessible and strongly  $\lambda$ -accessible.*
- (ii) *Let  $\mathcal{F}$  (resp.  $\mathcal{F}'$ ) be the full subcategory of  $[\mathbb{2}, \mathcal{M}]$  spanned by the fibrations (resp. trivial fibrations). Then  $\mathcal{F}$  and  $\mathcal{F}'$  are closed under colimits for small  $\kappa$ -filtered diagrams in  $[\mathbb{2}, \mathcal{M}]$ .*

*Proof.* (i). Since the weak factorisation systems on  $\mathcal{M}$  are cofibrantly generated by  $\lambda$ -small sets of morphisms in  $\mathbf{K}_\kappa(\mathcal{M})$  and  $\mathbf{K}_\kappa(\mathcal{M})$  is locally  $\lambda$ -small, we may apply the small object argument of either Quillen (theorem 0.5.12 and corollary 0.5.14) or Garner (proposition 0.5.23 and theorem 0.5.24) to obtain the required functorial weak factorisation systems.

(ii). This is corollary 0.5.27. ■

**Theorem 5.1.15.** *Let  $(L', R)$  and  $(L, R')$  be functorial weak factorisation systems on a locally presentable category  $\mathcal{M}$  and let  $\mathcal{F}$  and  $\mathcal{F}'$  be the full subcategories of  $[2, \mathcal{M}]$  spanned by the morphisms in the right class of the weak factorisation systems induced by  $(L', R)$  and  $(L, R')$ , respectively. Suppose  $\kappa$  and  $\lambda$  are regular cardinals satisfying the following hypotheses:*

- $\mathcal{M}$  is a locally  $\kappa$ -presentable category, and  $\kappa \triangleleft \lambda$ .
- $\mathcal{F}$  and  $\mathcal{F}'$  are closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ .
- $R, R' : [2, \mathcal{M}] \rightarrow [2, \mathcal{M}]$  preserve colimits for small  $\kappa$ -filtered diagrams and are strongly  $\lambda$ -accessible functors.

*Let  $\mathcal{C}'$  be the full subcategory of  $[2, \mathcal{M}]$  spanned by the morphisms in the left class of the weak factorisation system induced by  $(L', R)$  and let  $\mathcal{W}$  be the preimage of  $\mathcal{F}'$  under the functor  $R : [2, \mathcal{M}] \rightarrow [2, \mathcal{M}]$ . Then:*

- (i) *The functorial weak factorisation systems  $(L', R)$  and  $(L, R')$  restrict to functorial weak factorisation systems on  $\mathbf{K}_\lambda(\mathcal{M})$ .*
- (ii) *The inclusions  $\mathcal{F} \hookrightarrow [2, \mathcal{M}]$  and  $\mathcal{F}' \hookrightarrow [2, \mathcal{M}]$  are strongly  $\lambda$ -accessible functors.*
- (iii)  *$\mathcal{W}$  is closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ , and the inclusion  $\mathcal{W} \hookrightarrow [2, \mathcal{M}]$  is a strongly  $\lambda$ -accessible functor.*
- (iv)  *$\mathcal{C}' \subseteq \mathcal{W}$  if and only if the same holds in  $\mathbf{K}_\lambda(\mathcal{M})$ .*
- (v)  *$\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$  if and only if the same holds in  $\mathbf{K}_\lambda(\mathcal{M})$ .*
- (vi)  *$\mathcal{W}$  (regarded as a class of morphisms in  $\mathcal{M}$ ) has the 2-out-of-3 property in  $\mathcal{M}$  if and only if the same is true in  $\mathbf{K}_\lambda(\mathcal{M})$ .*

(vii) *The weak factorisation systems induced by  $(L', R)$  and  $(L, R')$  underlie a model structure on  $\mathcal{M}$  if and only if the restrictions to  $\mathbf{K}_\lambda(\mathcal{M})$  underlie a model structure on  $\mathbf{K}_\lambda(\mathcal{M})$ .*

*Proof.* (i). It is clear that we can restrict  $(L', R)$  and  $(L, R')$  to obtain functorial factorisation systems on  $\mathbf{K}_\lambda(\mathcal{M})$ , and these are functorial *weak* factorisation systems by [theorem A.3.31](#).

(ii). Since  $R, R' : [2, \mathcal{M}] \rightarrow [2, \mathcal{M}]$  are strongly  $\lambda$ -accessible, we may use [proposition 0.5.28](#) to deduce that the inclusions  $\mathcal{F} \hookrightarrow [2, \mathcal{M}]$  and  $\mathcal{F}' \hookrightarrow [2, \mathcal{M}]$  are strongly  $\lambda$ -accessible.

(iii). Since  $\mathcal{F}'$  is a replete subcategory of  $[2, \mathcal{M}]$ , we may use [proposition 0.3.30](#) to deduce that  $\mathcal{W}$  is closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$  and that the inclusion  $\mathcal{W} \hookrightarrow [2, \mathcal{M}]$  is a strongly  $\lambda$ -accessible functor.

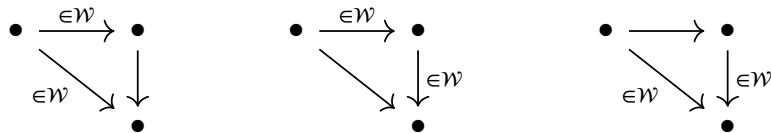
(iv). The endofunctor  $L' : [2, \mathcal{M}] \rightarrow [2, \mathcal{M}]$  is strongly  $\lambda$ -accessible, and  $\mathcal{W}$  is closed under colimits for small  $\lambda$ -filtered diagrams, so (recalling [propositions 0.2.43](#) and [0.2.46](#)) if  $L'$  sends the subcategory  $[2, \mathbf{K}_\lambda(\mathcal{M})]$  to  $\mathcal{W}$ , then the entirety of the image of  $L'$  must be contained in  $\mathcal{W}$ . [Proposition A.3.33](#) implies every object in  $\mathcal{C}'$  is a retract of an object in the image of  $L'$ , and claim (iii) implies  $\mathcal{W}$  is closed under retracts, so we may deduce that  $\mathcal{C}' \subseteq \mathcal{W}$  if and only if  $\mathcal{C}' \cap [2, \mathbf{K}_\lambda(\mathcal{M})] \subseteq \mathcal{W} \cap [2, \mathbf{K}_\lambda(\mathcal{M})]$ .

(v). Claims (ii) and (iii) and [proposition 0.3.30](#) imply the inclusion  $\mathcal{W} \cap \mathcal{F} \hookrightarrow [2, \mathcal{M}]$  is strongly  $\lambda$ -accessible; but by [propositions 0.2.46](#) and [0.3.29](#),

$$\mathbf{K}_\lambda(\mathcal{F}') = \mathcal{F}' \cap [2, \mathbf{K}_\lambda(\mathcal{M})] \quad \mathbf{K}_\lambda(\mathcal{W} \cap \mathcal{F}) = (\mathcal{W} \cap \mathcal{F}) \cap [2, \mathbf{K}_\lambda(\mathcal{M})]$$

so  $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$  if and only if  $\mathcal{F}' \cap [2, \mathbf{K}_\lambda(\mathcal{M})] = (\mathcal{W} \cap \mathcal{F}) \cap [2, \mathbf{K}_\lambda(\mathcal{M})]$ .

(vi). Consider the three full subcategories  $\Lambda_i^2(\mathcal{W})$  (where  $i \in \{0, 1, 2\}$ ) of  $[3, \mathcal{M}]$  spanned (respectively) by the diagrams of the form below:



By [proposition 0.3.15](#), each inclusion  $\Lambda_i^2(\mathcal{W}) \hookrightarrow [3, \mathcal{M}]$  is the pullback of a strongly  $\lambda$ -accessible inclusion of a full subcategory of  $[2, \mathcal{M}]^{\times 3}$  along the evident projection functor  $[3, \mathcal{M}] \rightarrow [2, \mathcal{M}]^{\times 3}$ ; thus, each inclusion  $\Lambda_i^2(\mathcal{W}) \hookrightarrow [3, \mathcal{M}]$

is a strongly  $\lambda$ -accessible functor. We may then use [proposition 0.3.29](#) as above to prove the claim.

(vii). Apply lemmas [4.1.10](#) and [4.1.11](#) and [theorem 4.1.12](#). ■

**Corollary 5.1.16.** *Let  $\mathcal{M}$  be a strongly  $(\kappa, \lambda)$ -combinatorial model category. Then the full subcategory  $\mathcal{W}$  of  $[2, \mathcal{M}]$  spanned by the weak equivalences is closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ , and the inclusion  $\mathcal{W} \hookrightarrow [2, \mathcal{M}]$  is a strongly  $\lambda$ -accessible functor.*

*Proof.* Combine [proposition 5.1.14](#) and [theorem 5.1.15](#). ■

**Definition 5.1.17.** Let  $\kappa$  and  $\lambda$  be regular cardinals. A  $(\kappa, \lambda)$ -compact model category is a model category  $\mathcal{M}$  that satisfies these axioms:

- $\mathcal{M}$  is a  $(\kappa, \lambda)$ -compactly generated category, and  $\kappa \triangleleft \lambda$ .
- $\mathcal{M}$  has limits for finite diagrams and colimits for  $\lambda$ -small diagrams.
- Each hom-set in  $\mathbf{K}_\kappa^\lambda(\mathcal{M})$  is  $\lambda$ -small.
- There exist  $\lambda$ -small sets of morphisms in  $\mathbf{K}_\kappa^\lambda(\mathcal{M})$  that cofibrantly generate the model structure of  $\mathcal{M}$ .

**Proposition 5.1.18.** *If  $\mathcal{M}$  is a strongly  $(\kappa, \lambda)$ -combinatorial model category, then  $\mathbf{K}_\lambda(\mathcal{M})$  is a  $(\kappa, \lambda)$ -compact model category (with the weak equivalences, cofibrations, and fibrations inherited from  $\mathcal{M}$ ).*

*Proof.* By [proposition 0.3.7](#),  $\mathbf{K}_\lambda(\mathcal{M})$  is a  $(\kappa, \lambda)$ -compactly generated category, and [lemma 0.2.17](#) implies it is closed under colimits for  $\lambda$ -small diagrams in  $\mathcal{M}$ . Now, choose a pair of functorial factorisation systems as in [proposition 5.1.14](#), and recall that [theorem A.3.31](#) says a morphism is in the left (resp. right) class of a functorial weak factorisation system if and only if it is a retract of the left (resp. right) half of its functorial factorisation. Since we chose factorisation functors that are strongly  $\lambda$ -accessible, it follows that the weak factorisation systems on  $\mathcal{M}$  restricts to weak factorisation systems on  $\mathbf{K}_\lambda(\mathcal{M})$ . It is then clear that  $\mathbf{K}_\lambda(\mathcal{M})$  inherits a model structure from  $\mathcal{M}$ , and [lemma 0.5.30](#) implies the model structure on  $\mathbf{K}_\lambda(\mathcal{M})$  can be cofibrantly generated by  $\lambda$ -small sets of morphisms in  $\mathbf{K}_\kappa(\mathcal{M})$ . The remaining axioms for a  $\lambda$ -compact model category are easily verified. ■

**Proposition 5.1.19.** *Let  $\mathcal{K}$  be a  $(\kappa, \lambda)$ -compact model category and let  $\mathcal{M}$  be the free  $\lambda$ -ind-completion  $\mathbf{Ind}^\lambda(\mathcal{K})$ . Then there is a unique way of making  $\mathcal{M}$  into a strongly  $(\kappa, \lambda)$ -combinatorial model category such that the canonical embedding  $\mathcal{K} \rightarrow \mathcal{M}$  preserves and reflects the model structure.*

*Proof.* We will regard  $\mathcal{K}$  as a full subcategory of  $\mathcal{M}$  via the canonical embedding  $\mathcal{K} \rightarrow \mathcal{M}$ . Let  $\mathcal{I}$  (resp.  $\mathcal{I}'$ ) be a  $\lambda$ -small set of morphisms in  $\mathbf{K}_\kappa^\lambda(\mathcal{K})$  that generate the cofibrations (resp. trivial cofibrations) in  $\mathcal{K}$ . Let  $(L', R)$  and  $(L, R')$  be functorial weak factorisation systems cofibrantly generated by  $\mathcal{I}'$  and  $\mathcal{I}$  respectively; by [corollary 0.5.14](#), we may assume  $R, R' : [2, \mathcal{M}] \rightarrow [2, \mathcal{M}]$  preserve colimits for small  $\kappa$ -filtered diagrams and are strongly  $\lambda$ -accessible functors.

Let  $\mathcal{F}$  and  $\mathcal{F}'$  be the full subcategories of  $[2, \mathcal{M}]$  spanned by the right class of the weak factorisation systems induced by  $(L', R)$  and  $(L, R')$ , respectively. It is not hard to see that any morphism in  $\mathcal{K}$  is an object in  $\mathcal{F}$  (resp.  $\mathcal{F}'$ ) if and only if it is a fibration (resp. trivial fibration) in  $\mathcal{K}$ . [Corollary 0.5.27](#) says  $\mathcal{F}$  and  $\mathcal{F}'$  are closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ , so we may now apply [theorem 5.1.15](#) to deduce that  $\mathcal{F}$  and  $\mathcal{F}'$  induce a model structure on  $\mathcal{M}$ . It is clear that  $\mathcal{M}$  equipped with this model structure is then a strongly  $(\kappa, \lambda)$ -combinatorial model category in a way that is compatible with the canonical embedding  $\mathcal{K} \rightarrow \mathcal{M}$ .

Finally, to see that the above construction is the unique way of making  $\mathcal{M}$  into a strongly  $(\kappa, \lambda)$ -combinatorial model category satisfying the given conditions, we simply have to observe that the model structure of a strongly  $(\kappa, \lambda)$ -combinatorial model category is necessarily cofibrantly generated by the cofibrations and trivial cofibrations in (a small skeleton of)  $\mathbf{K}_\kappa(\mathcal{M})$  (independently of the choice of  $\mathcal{I}$  and  $\mathcal{I}'$ ). ■

**REMARK 5.1.20.** Let  $\mathbf{U}$  and  $\mathbf{U}^+$  be universes, with  $\mathbf{U} \in \mathbf{U}^+$ , let  $\mathcal{M}$  be a strongly  $(\kappa, \lambda)$ -combinatorial model  $\mathbf{U}$ -category, and let  $\mathcal{M} \hookrightarrow \mathcal{M}^+$  be a  $(\kappa, \mathbf{U}, \mathbf{U}^+)$ -extension. By combining [propositions 5.1.18](#) and [5.1.19](#), we may deduce that there is a unique way of making  $\mathcal{M}^+$  into a strongly  $(\kappa, \lambda)$ -combinatorial model  $\mathbf{U}^+$ -category such that the embedding  $\mathcal{M} \hookrightarrow \mathcal{M}^+$  preserves and reflects the model structure. In other words, combinatorial model categories are stable under universe enlargement.

## 5.2 Algebraic model categories

**Prerequisites.** §§0.2, 0.3, 0.5, 4.I, 5.I, A.3.

Though model categories equipped with functorial factorisations are better-behaved than general model categories, one can often extract a bit more structure by using Garner’s small object argument (theorem 0.5.24). This leads to the notion of ‘algebraic model structure’, due to Riehl [2011a,b].

**Definition 5.2.1.** Let  $\mathcal{M}$  be a category. An **algebraic model structure** on  $\mathcal{M}$  consists of a pair of algebraic factorisation systems  $(\mathbf{L}', \mathbf{R})$  and  $(\mathbf{L}, \mathbf{R}')$  on  $\mathcal{M}$  and a morphism  $(\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R}')$  satisfying the following condition:

- There exists a model structure on  $\mathcal{M}$  such that the cofibrations are the left class of the weak factorisation system induced by  $(\mathbf{L}, \mathbf{R}')$  and the fibrations are the right class of the weak factorisation system induced by  $(\mathbf{L}', \mathbf{R})$ .

An **algebraic model category** is a category with limits and colimits for all finite diagrams and equipped with an algebraic model structure.

The following lemma, originally part of Theorem 3.8 in [Riehl, 2011b], is useful in the construction of algebraic model structures:

**Lemma 5.2.2.** *Let  $\mathcal{M}$  be a category with a model structure, let  $(\mathbf{L}, \mathbf{R}')$  be an algebraic factorisation system on  $\mathcal{M}$ , and suppose  $\mathcal{I}'$  is a generating set of trivial cofibrations in  $\mathcal{M}$ . If the left class of the weak factorisation system induced by  $(\mathbf{L}, \mathbf{R}')$  is the class of cofibrations, then there exists a subset  $\tilde{\mathcal{I}}'$  with the following properties:*

- $\tilde{\mathcal{I}}'$  has at most as many elements as  $\mathcal{I}'$ .
- The weak factorisation system on  $\mathcal{M}$  cofibrantly generated by  $\mathcal{I}'$  coincides with the one cofibrantly generated by  $\tilde{\mathcal{I}}'$ .
- Each morphism in  $\tilde{\mathcal{I}}'$  can be equipped with an  $\mathbf{L}$ -coalgebra structure.

*In particular, if  $(\mathbf{L}', \mathbf{R})$  is a free algebraic factorisation system cofibrantly generated by  $\tilde{\mathcal{I}}'$ , then there must exist a morphism  $(\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R}')$ .*

*Proof.* Let  $\tilde{\mathcal{I}}' = \{Le \mid e \in \mathcal{I}'\}$ . Since  $\mathbf{L}$  is a comonad, every morphism in  $\tilde{\mathcal{I}}'$  admits an  $\mathbf{L}$ -coalgebra structure. Consider the following commutative diagram

in  $\mathcal{M}$ :

$$\begin{array}{ccc} Z & \xrightarrow{Le} & W' \\ e \downarrow & & \downarrow R'e \\ W & \xrightarrow{\text{id}} & W \end{array}$$

Since  $e : Z \rightarrow W$  is a trivial cofibration and  $Re : W' \rightarrow W$  is a trivial fibration, there exists a morphism  $i : W \rightarrow W'$  filling in the diagram. Hence, every morphism in  $\mathcal{I}'$  is a retract of one in  $\tilde{\mathcal{I}}'$ , so by propositions A.3.3 and A.3.14, we have  $\tilde{\mathcal{I}}'^{\square} \subseteq \mathcal{I}'^{\square}$ . On the other hand, axiom CM2 implies  $Le : Z \rightarrow W'$  is a trivial cofibration, and so  $\tilde{\mathcal{I}}' \subseteq \square(\mathcal{I}'^{\square})$ . Thus, we have  $\mathcal{I}'^{\square} \subseteq \tilde{\mathcal{I}}'^{\square}$  as well. ■

**Proposition 5.2.3.** *Let  $\mathcal{M}$  be a combinatorial model category and let  $\mathcal{I}$  be a set of generating cofibrations in  $\mathcal{M}$ .*

- (i)  $\mathcal{I}$  cofibrantly generates an algebraically free algebraic factorisation system  $(\mathbf{L}, \mathbf{R}')$  on  $\mathcal{M}$ .
- (ii) There exists a set  $\tilde{\mathcal{I}}'$  of generating trivial cofibrations in  $\mathcal{M}$  such that  $\tilde{\mathcal{I}}'$  cofibrantly generates an algebraically free algebraic factorisation system  $(\mathbf{L}', \mathbf{R})$  on  $\mathcal{M}$  with a morphism  $\theta : (\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R}')$ .

*In particular,  $\mathcal{M}$  is the underlying model category of an algebraic model category.*

*Proof.* (i). Apply Garner's small object argument (theorem 0.5.24).

(ii). Use lemma 5.2.2. ■

**Definition 5.2.4.** Let  $\kappa$  and  $\lambda$  be regular cardinals. A **strongly  $(\kappa, \lambda)$ -algebraic model category** is an algebraic model category  $\mathcal{M}$  that satisfies these axioms:

- $\mathcal{M}$  is a locally  $\kappa$ -presentable category, and  $\kappa \triangleleft \lambda$ .
- $\mathbf{K}_{\lambda}(\mathcal{M})$  is closed under finite limits in  $\mathcal{M}$ .
- The underlying endofunctors of the two given algebraic factorisation systems on  $\mathcal{M}$  preserve colimits for small  $\kappa$ -filtered diagrams and are strongly  $\lambda$ -accessible functors.
- The full subcategory  $\mathcal{F}$  (resp.  $\mathcal{F}'$ ) of  $[2, \mathcal{M}]$  spanned by the fibrations (resp. trivial fibrations) in  $\mathcal{M}$  is closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ .



**Proposition 5.2.5.** *If  $\mathcal{M}$  is a strongly  $(\kappa, \lambda)$ -combinatorial model category, then there exist algebraic factorisation systems making  $\mathcal{M}$  a strongly  $(\kappa, \lambda)$ -algebraic model category.*

*Proof.* Let  $\mathcal{I}$  (resp.  $\mathcal{I}'$ ) be a  $\lambda$ -small set of morphisms in  $\mathbf{K}_\kappa(\mathcal{M})$  that generate the cofibrations (resp. trivial cofibrations) in  $\mathcal{M}$ . Replacing  $\mathcal{I}$  with  $\mathcal{I} \cup \mathcal{I}'$  if necessary, we may assume  $\mathcal{I}' \subseteq \mathcal{I}$ . Garner's small object argument (0.5.24) says that algebraically free algebraic factorisation systems cofibrantly generated by  $\mathcal{I}$  and  $\mathcal{I}'$  exist and are free, and since  $\mathcal{I}' \subseteq \mathcal{I}$ , the universal property of free algebraic factorisation systems ensures we have the required morphism of algebraic factorisation systems. Lemma 0.3.36 and proposition 0.5.23 then say that the underlying endofunctors of the algebraic factorisation systems preserve colimits for small  $\kappa$ -filtered diagrams and are strongly  $\lambda$ -accessible. Finally, by corollary 0.5.27, the two full subcategories of  $[2, \mathcal{M}]$  spanned by the fibrations and trivial fibrations are closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ . ■

**Proposition 5.2.6.** *Let  $\mathcal{M}$  be a strongly  $(\kappa, \lambda)$ -algebraic model category.*

- (i) *The algebraic model structure on  $\mathcal{M}$  restricts to an algebraic model structure on  $\mathbf{K}_\lambda(\mathcal{M})$ .*
- (ii) *The inclusions  $\mathcal{F} \hookrightarrow [2, \mathcal{M}]$  and  $\mathcal{F}' \hookrightarrow [2, \mathcal{M}]$  are strongly  $\lambda$ -accessible functors.*
- (iii)  *$\mathcal{W}$  is closed under colimits for small  $\kappa$ -filtered diagrams in  $[2, \mathcal{M}]$ , and the inclusion  $\mathcal{W} \hookrightarrow [2, \mathcal{M}]$  is a strongly  $\lambda$ -accessible functor.*

*Proof.* (i). By definition, the underlying endofunctors of the given algebraic factorisation systems are strongly  $\lambda$ -accessible and so send morphisms in  $\mathbf{K}_\lambda(\mathcal{M})$  back to  $\mathbf{K}_\lambda(\mathcal{M})$ . Thus, we obtain algebraic factorisation systems on  $\mathbf{K}_\lambda(\mathcal{M})$ , and it is clear that the given morphism of algebraic factorisation systems on  $\mathcal{M}$  restricts to a morphism of algebraic factorisation systems on  $\mathbf{K}_\lambda(\mathcal{M})$ . Since  $\mathbf{K}_\lambda(\mathcal{M})$  is a full subcategory of  $\mathcal{M}$ , it follows that the restricted data define an algebraic model structure on  $\mathbf{K}_\lambda(\mathcal{M})$ .

(ii) and (iii). Apply theorem 5.1.15. ■

### 5.3 Cisinski model categories

**Prerequisites.** §0.5, 3.1, 3.5, 4.1, 5.1, A.3, A.4.

In this section we follow [Cisinski, 2002] and [Cisinski, 2006, Ch. 1].

**Definition 5.3.1.** A **Cisinski model category** is a combinatorial model category whose underlying category is a Grothendieck topos and whose cofibrations are the monomorphisms.

**REMARK 5.3.2.** Grothendieck toposes are always locally presentable categories, so we may replace ‘combinatorial’ with ‘cofibrantly generated’ in the above definition. Since every monomorphism in a Grothendieck topos is an effective monomorphism, Cisinski model categories are cellular model categories.

**Example 5.3.3.** The Kan–Quillen model structure on **sSet** makes it into a Cisinski model category.

**REMARK 5.3.4.** In any topos, the unique morphism  $0 \rightarrow X$  is always a monomorphism; thus, in a Cisinski model category, every object is cofibrant.

**Proposition 5.3.5.** Let  $\mathcal{M}$  be a Grothendieck topos and let  $\mathcal{M}_f$  be a class of objects in  $\mathcal{M}$ . There is at most one model structure on  $\mathcal{M}$  making it a Cisinski model category with  $\mathcal{M}_f$  as the class of fibrant objects.

*Proof.* This is a special case of proposition 4.4.6. ■

**Proposition 5.3.6.** Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be cofibrations in a Cisinski model category  $\mathcal{M}$ . Suppose the square in the diagram below is a pushout square in  $\mathcal{M}$ :

$$\begin{array}{ccc}
 X \times Z & \xrightarrow{\text{id}_X \times g} & X \times W \\
 \downarrow f \times \text{id}_Z & & \downarrow \\
 Y \times Z & \xrightarrow{\quad} & (X \times W) \cup^{X \times Z} (Y \times Z) \\
 & \searrow \text{id}_Y \times g & \downarrow f \times \text{id}_W \\
 & & Y \times W
 \end{array}$$

(Note: A dashed arrow labeled  $f \boxminus g$  also points from  $(X \times W) \cup^{X \times Z} (Y \times Z)$  to  $Y \times W$ .)

- (i) The unique morphism  $f \boxminus g$  making the diagram commute is a cofibration.
- (ii) Assuming the class of trivial cofibrations in  $\mathcal{M}$  is closed under binary products, if either  $f$  or  $g$  is a trivial cofibration, then  $f \boxminus g$  is a trivial cofibration.

*Proof.* (i). The claim is certainly true when  $\mathcal{M}$  is a presheaf topos, and since the associated sheaf functor preserves colimits and finite limits, the claim holds for all sheaf toposes as well.

(ii). The two cases are symmetrical; we will assume  $f : X \rightarrow Y$  is a trivial cofibration. Clearly,  $f \times \text{id}_Z : X \times Z \rightarrow Y \times Z$  and  $f \times \text{id}_W : X \times W \rightarrow Y \times W$  are monomorphisms, so the hypothesis implies they are trivial cofibrations. The class of trivial cofibrations is closed under pushouts (by [proposition A.3.14](#)), so the morphism  $X \times W \rightarrow (X \times W) \cup^{X \times Z} (Y \times Z)$  is also a trivial cofibration. The 2-out-of-3 property of weak equivalences then implies  $f \sqsupseteq g$  must be a weak equivalence as well; hence, by claim (i), it is a trivial cofibration. ■

¶ 5.3.7. We will now see how to build Cisinski model structures. Throughout this section,  $\mathcal{M}$  will be a Grothendieck topos, say  $\mathcal{M} = \mathbf{Sh}(\mathbb{C}, J)$  for a small category  $\mathbb{C}$  equipped with a Grothendieck topology  $J$ .

**Definition 5.3.8.** A **Cisinski cylinder functor** for  $\mathcal{M}$  is a quadruple  $(I, \iota^0, \iota^1, \rho)$  where  $I : \mathcal{M} \rightarrow \mathcal{M}$  is a functor,  $\iota^0, \iota^1 : \text{id}_{\mathcal{M}} \Rightarrow I$  and  $\rho : I \Rightarrow \text{id}_{\mathcal{M}}$  are natural transformations, such that:

- $\rho \circ \iota^0 = \rho \circ \iota^1 = \text{id}_{\text{id}_{\mathcal{M}}}$ .
- The induced morphism  $\iota_X = ((\iota_X^0, \iota_X^1)) : X \amalg X \rightarrow IX$  is a monomorphism for every object  $X$  in  $\mathcal{M}$ .

We will often abuse notation and simply say that  $I$  is a cylinder functor, with the natural transformations  $\iota^0, \iota^1$ , and  $\rho$  understood.

REMARK 5.3.9. By symmetry,  $(I, \iota^0, \iota^1, \rho)$  is a Cisinski cylinder functor if and only if  $(I, \iota^1, \iota^0, \rho)$  is a Cisinski cylinder functor.

**Definition 5.3.10.** Let  $(I, \iota^0, \iota^1, \rho)$  be a Cisinski cylinder functor for  $\mathcal{M}$ , and let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathcal{M}$ . An  **$I$ -homotopy** in  $\mathcal{M}$  from  $f_0$  to  $f_1$  is a morphism  $H : IX \rightarrow Y$  such that  $H \circ \iota_X^0 = f_0$  and  $H \circ \iota_X^1 = f_1$ . We say  $f_0$  and  $f_1$  are  **$I$ -homotopic** if there is a zigzag of  $I$ -homotopies connecting  $f_0$  to  $f_1$ .

**Proposition 5.3.11.** Let  $(I, \iota^0, \iota^1, \rho)$  be a Cisinski cylinder functor for  $\mathcal{M}$ , and let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathcal{M}$ .

- (i) For any morphism  $g : Y \rightarrow Z$  in  $\mathcal{M}$ , if  $f_0$  and  $f_1$  are  $I$ -homotopic, then so are  $g \circ f_0$  and  $g \circ f_1$ .
- (ii) For any morphism  $g : W \rightarrow X$  in  $\mathcal{M}$ , if  $f_0$  and  $f_1$  are  $I$ -homotopic, then so are  $f_0 \circ g$  and  $f_1 \circ g$ .

*Proof.* Obvious. ◆

**Definition 5.3.12.** Let  $(I, l^0, l^1, \rho)$  be a Cisinski cylinder functor for  $\mathcal{M}$ . The  **$I$ -homotopy category** of  $\mathcal{M}$  is the category  $\text{Ho}_I \mathcal{M}$  defined below:

- The objects of  $\text{Ho}_I \mathcal{M}$  are those of  $\mathcal{M}$ .
- The hom-set  $\text{Ho}_I \mathcal{M}(X, Y)$  is  $\mathcal{M}(X, Y)$  modulo  $I$ -homotopy.
- Composition and identities are inherited from  $\mathcal{M}$ .

**Proposition 5.3.13.** Let  $(I, l^0, l^1, \rho)$  be a Cisinski cylinder functor and let  $\gamma : \mathcal{M} \rightarrow \text{Ho}_I \mathcal{M}$  be the functor that sends a morphism in  $\mathcal{M}$  to its  $I$ -homotopy class.

- (i) The functor  $\gamma : \mathcal{M} \rightarrow \text{Ho}_I \mathcal{M}$  is full.
- (ii) Let  $\mathcal{H}$  be the class of morphisms in  $\mathcal{M}$  that  $\gamma$  sends to isomorphisms. If  $\gamma\rho : \gamma I \Rightarrow \gamma$  is a natural isomorphism, then  $\gamma : \mathcal{M} \rightarrow \text{Ho}_I \mathcal{M}$  exhibits  $\text{Ho}_I \mathcal{M}$  as a localisation of  $\mathcal{M}$  at  $\mathcal{H}$ .

*Proof.* (i). Obvious.

(ii). Consider any functor  $F : \mathcal{M} \rightarrow \mathcal{C}$  such that  $F\rho : FI \Rightarrow F$  is a natural isomorphism. Then, we have  $Fl^0 = Fl^1$ , so  $F$  factors through  $\gamma : \mathcal{M} \rightarrow \text{Ho}_I \mathcal{M}$  in a unique way. In particular, if  $\gamma\rho : \gamma I \Rightarrow \gamma$  itself is a natural isomorphism, then  $\text{Ho}_I \mathcal{M}$  has the universal property of a localisation of  $\mathcal{M}$  at  $\mathcal{H}$ . ■

**Definition 5.3.14.** A **Cisinski trivial fibration** in  $\mathcal{M}$  is a morphism that has the right lifting property with respect to all monomorphisms.

**Proposition 5.3.15.** Let  $p : X \rightarrow Y$  be a Cisinski trivial fibration in  $\mathcal{M}$ .

- (i) There exists a morphism  $s : Y \rightarrow X$  such that  $p \circ s = \text{id}_Y$ .
- (ii) For any such  $s : Y \rightarrow X$  and any Cisinski cylinder functor  $(I, l^0, l^1, \rho)$  for  $\mathcal{M}$ , there exists an  $I$ -homotopy from  $\text{id}_X$  to  $s \circ p$ .

(iii) *The morphism  $p : X \rightarrow Y$  becomes an isomorphism in  $\text{Ho}_I \mathcal{M}$ .*

*Proof.* (i). The unique morphism  $0 \rightarrow Y$  is a monomorphism in any topos, so the right lifting property of  $p : X \rightarrow Y$  guarantees the existence of a section.

(ii). Consider the following commutative diagram in  $\mathcal{M}$ :

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(\text{id}_X, s \circ p)} & X \\ \iota_X \downarrow & & \downarrow p \\ IX & \xrightarrow{p \circ \rho_X} & Y \end{array}$$

By definition,  $\iota_X : X \amalg X \rightarrow IX$  is a monomorphism, so the right lifting property of  $p : X \rightarrow Y$  yields a morphism  $H : IX \rightarrow X$  such that  $H \circ \iota_X = (\text{id}_X, s \circ p)$  and  $p \circ H = p \circ \rho_X$ ; in particular,  $H$  is an  $I$ -homotopy from  $\text{id}_X$  to  $s \circ p$ .

(iii). Clearly, the morphisms  $p : X \rightarrow Y$  and  $s : Y \rightarrow X$  become mutual inverses in  $\text{Ho}_I \mathcal{M}$ . ■

¶ 5.3.16. Let  $\Omega$  be a subobject classifier for  $\mathcal{M}$  and let  $\top, \perp : 1 \rightarrow \Omega$  be the morphisms classifying the top and bottom subobjects of  $1$ , respectively. Then the following diagram is a pullback square by definition,

$$\begin{array}{ccc} 0 & \longrightarrow & 1 \\ \downarrow & & \downarrow \top \\ 1 & \xrightarrow{\perp} & \Omega \end{array}$$

so the induced morphism  $(\top, \perp) : 1 \amalg 1 \rightarrow \Omega$  is a monomorphism. Since monomorphisms are stable under pullback, the following definition is legitimate:

**Definition 5.3.17.** The **Lawvere cylinder functor** for  $\mathcal{M}$  is the cylinder functor  $(I, \iota^0, \iota^1, \rho)$  defined below:

- $I : \mathcal{M} \rightarrow \mathcal{M}$  is the functor  $\Omega \times -$ .
- The morphism  $\iota_X^0 : X \rightarrow \Omega \times X$  corresponds to  $\top \times \text{id}_X$ .
- The morphism  $\iota_X^1 : X \rightarrow \Omega \times X$  corresponds to  $\perp \times \text{id}_X$ .
- The morphism  $\rho_X : \Omega \times X \rightarrow X$  is the product projection.

**Proposition 5.3.18.** *Let  $X$  be any object in  $\mathcal{M}$  and let  $\Omega$  be the subobject classifier for  $\mathcal{M}$ .*

- (i) *The product projection  $p_X : \Omega \times X \rightarrow X$  is a Cisinski trivial fibration.*
- (ii) *For any Cisinski cylinder functor  $(I, \iota^0, \iota^1, \rho)$ , there exists a commutative diagram of the following form:*

$$\begin{array}{ccc}
 X \amalg X & \xrightarrow{(\top, \perp) \times \text{id}_X} & \Omega \times X \\
 \iota_X \downarrow & \nearrow \text{dashed arrow} & \downarrow p_X \\
 IX & \xrightarrow{\rho_X} & X
 \end{array}$$

*Proof.* (i). Since the class of Cisinski trivial fibrations is closed under pullbacks (by proposition A.3.14), it suffices to show that the morphism  $p_1 : \Omega \times 1 \rightarrow 1$  is a trivial fibration. However,  $\Omega$  is canonically an injective object in  $\mathcal{M}$  (with respect to the class of monomorphisms), i.e. the unique morphism  $\Omega \rightarrow 1$  has the right lifting property with respect to all monomorphisms, so  $p_1$  is indeed a Cisinski trivial fibration.

(ii). This follows from claim (i) and the requirement that  $\iota_X : X \amalg X \rightarrow IX$  be a monomorphism. ■

REMARK 5.3.19. Thus, any pair of morphisms that are homotopic with respect to the Lawvere cylinder functor must also be  $I$ -homotopic for any Cisinski cylinder functor  $(I, \iota^0, \iota^1, \rho)$ .

**Proposition 5.3.20.** *Let  $(I, \iota^0, \iota^1, \rho)$  be a cylinder functor for  $\mathcal{M}$  and let  $\mathcal{V}$  be the class of Cisinski trivial fibrations in  $\mathcal{M}$ .*

- (i) *There is a canonical identity-on-objects functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  compatible with the localising functors, and it is a full functor.*
- (ii) *If the natural morphism  $\rho_X : IX \rightarrow X$  is a Cisinski trivial fibration for all objects  $X$  in  $\mathcal{M}$ , then the canonical functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  is an isomorphism of categories.*
- (iii) *If  $(I, \iota^0, \iota^1, \rho)$  is the Lawvere cylinder functor for  $\mathcal{M}$ , then the canonical functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  is an isomorphism of categories.*

*Proof.* (i). Recall that Cisinski trivial fibrations are  $I$ -homotopy equivalences (by [proposition 5.3.15](#)), so there is indeed a canonical identity-on-objects functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  compatible with the localising functors. Since the localising functor  $\mathcal{M} \rightarrow \text{Ho}_I \mathcal{M}$  is full, the functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  must also be full.

(ii). The hypothesis implies any two  $I$ -homotopic morphisms in  $\mathcal{M}$  are equal as morphisms in  $\mathcal{M}[\mathcal{V}^{-1}]$ , so the canonical functor  $\mathcal{M}[\mathcal{V}^{-1}] \rightarrow \text{Ho}_I \mathcal{M}$  is indeed fully faithful and bijective on objects, as required.

(iii). [Proposition 5.3.18](#) says that the natural morphism  $\rho_X : IX \rightarrow X$  is a Cisinski trivial fibration for all objects  $X$  in  $\mathcal{M}$  when  $(I, i^0, i^1, \rho)$  is the Lawvere cylinder for  $\mathcal{M}$ . ■

**Definition 5.3.21.** An **elementary Cisinski homotopy structure** on  $\mathcal{M}$  is a Cisinski cylinder functor  $(I, i^0, i^1, \rho)$  satisfying these axioms:

**DH1.** The functor  $I : \mathcal{M} \rightarrow \mathcal{M}$  preserves monomorphisms and colimits for all small diagrams.

**DH2.** For all monomorphisms  $g : Z \rightarrow W$  in  $\mathcal{M}$ , the following diagrams are pullback squares:

$$\begin{array}{ccc} Z & \xrightarrow{g} & W \\ i_Z^0 \downarrow & & \downarrow i_W^0 \\ IZ & \xrightarrow{I_g} & IW \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{g} & W \\ i_Z^1 \downarrow & & \downarrow i_W^1 \\ IZ & \xrightarrow{I_g} & IW \end{array}$$

**Proposition 5.3.22.** *The Lawvere cylinder functor is an elementary Cisinski homotopy structure.*

*Proof.* The functor  $A \times -$  always preserves monomorphisms, and toposes are cartesian closed, so for any object  $A$  in  $\mathcal{M}$ , the functor  $A \times -$  preserves colimits. Thus the Lawvere cylinder functor satisfies axiom DH1. It is clear that axiom DH2 is also satisfied. ■

**Definition 5.3.23.** Let  $(I, i^0, i^1, \rho)$  be an elementary Cisinski homotopy structure on  $\mathcal{M}$ . A **class of  $I$ -anodyne extensions** is a class  $\mathcal{A}$  of morphisms in  $\mathcal{M}$  satisfying these axioms:

**An0.** There exists a subset  $\Lambda \subseteq \mathcal{A}$  such that the members of  $\Lambda$  are monomorphisms in  $\mathcal{M}$  and  $\mathcal{A} = \square(\Lambda^{\square})$ . We say  $\Lambda$  is a **generating set** for  $\mathcal{A}$ .

**An1.** If  $g : Z \rightarrow W$  is a monomorphism in  $\mathcal{M}$  and  $e \in \{0, 1\}$ , then given a commutative diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{i_Z^e} & IZ \\
 g \downarrow & & \downarrow \\
 W & \longrightarrow & V_e(g) \\
 & \searrow & \downarrow j_e \\
 & & IW \\
 & \nearrow i_W^e & \\
 & & 
 \end{array}$$

(Note: A curved arrow labeled  $Ig$  also points from  $IZ$  to  $IW$ .)

where the top-left square is a pushout square,  $j_e : V_e(g) \rightarrow IW$  is in  $\mathcal{A}$ .

**An2.** If  $g : Z \rightarrow W$  is in  $\mathcal{A}$ , then given a commutative diagram

$$\begin{array}{ccc}
 Z \amalg Z & \xrightarrow{i_Z} & IZ \\
 g \downarrow & & \downarrow \\
 W \amalg W & \longrightarrow & U(g) \\
 & \searrow & \downarrow j \\
 & & IW \\
 & \nearrow i_W & \\
 & & 
 \end{array}$$

(Note: A curved arrow labeled  $Ig$  also points from  $IZ$  to  $IW$ .)

where the top-left square is a pushout square,  $j : U(g) \rightarrow IW$  is in  $\mathcal{A}$ .

**REMARK 5.3.24.** Since  $I$  preserves colimits for all small diagrams,  $I0$  must be an initial object in  $\mathcal{M}$ . Thus, by taking  $Z = 0$ , we see that the morphisms  $i_W^0, i_W^1 : W \rightarrow IW$  are always in any class of  $I$ -anodyne extensions.

**Proposition 5.3.25.** *Let  $(I, i^0, i^1, \rho)$  be an elementary Cisinski homotopy structure on  $\mathcal{M}$ , let  $\mathcal{A}$  be a class of  $I$ -anodyne extensions, and let  $\Lambda$  be a generating set for  $\mathcal{A}$ .*

- (i) *There exists a functorial factorisation system on  $\mathcal{M}$  with  $\mathcal{A}$  as its left class.*
- (ii)  *$\mathcal{A}$  is the smallest class of morphisms containing  $\Lambda$  that is closed under pushouts, transfinite composition, and retracts.*
- (iii) *Every morphism that is in  $\mathcal{A}$  is a monomorphism.*



*Proof.* (i). Apply Quillen's small object argument (theorem 0.5.12).

(ii). This is corollary 0.5.13.

(iii). The class of monomorphisms in a Grothendieck topos is closed under push-outs, transfinite composition, and retracts because the class of injections in **Set** is closed under the same operations. Since  $\Lambda$  is a collection of monomorphisms, so too is  $\mathcal{A}$ . ■

**Definition 5.3.26.** A **Cisinski homotopy structure** on  $\mathcal{M}$  is an elementary Cisinski homotopy structure on  $\mathcal{M}$  together with a class of anodyne extensions.

**Definition 5.3.27.** Let  $\mathcal{A}$  be the class of anodyne extensions of a Cisinski homotopy structure on  $\mathcal{M}$ . An  **$\mathcal{A}$ -fibrant object** in  $\mathcal{M}$  is an object  $X$  such that the unique morphism  $X \rightarrow 1$  has the right lifting property with respect to  $\mathcal{A}$ .

**Definition 5.3.28.** Let  $(I, \mathcal{A})$  be a Cisinski homotopy structure on  $\mathcal{M}$ . A **weak equivalence** with respect to  $(I, \mathcal{A})$  is a morphism  $f : W \rightarrow Z$  in  $\mathcal{M}$  such that, for every  $\mathcal{A}$ -fibrant object  $X$ , the induced map

$$\mathrm{Ho}_I \mathcal{M}(f, X) : \mathrm{Ho}_I \mathcal{M}(Z, X) \rightarrow \mathrm{Ho}_I \mathcal{M}(W, X)$$

is a bijection of sets.

**Proposition 5.3.29.**  $\mathcal{M}$  together with the class of weak equivalences with respect to a Cisinski homotopy structure  $(I, \mathcal{A})$  constitute a saturated homotopical category.

*Proof.* Obvious. ◆

**Proposition 5.3.30.** Let  $(I, \mathcal{A})$  be a Cisinski homotopy structure on  $\mathcal{M}$ . Then every morphism in  $\mathcal{A}$  is a weak equivalence with respect to  $(I, \mathcal{A})$ .

*Proof.* See Proposition 2.23 in [Cisinski, 2002]. □

**Corollary 5.3.31.** Let  $\mathcal{W}$  be the class of weak equivalences with respect to  $(I, \mathcal{A})$  and let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathcal{M}$ . If  $f_0$  and  $f_1$  are  $I$ -homotopic, then  $f_0$  and  $f_1$  become equal in  $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$ .

*Proof.* It suffices to verify the case where there is an  $I$ -homotopy  $H : IX \rightarrow Y$  from  $f_0$  to  $f_1$ . By [remark 5.3.24](#), the morphisms  $i_X^0, i_X^1 : X \rightarrow IX$  are anodyne extensions, and so are invertible in  $\text{Ho}(\mathcal{M}, \mathcal{W})$ . We have  $\rho_X \circ i_X^0 = \rho_X \circ i_X^1 = \text{id}_X$  by definition, so  $i_X^0$  and  $i_X^1$  must be equal in  $\text{Ho}(\mathcal{M}, \mathcal{W})$ ; but  $H \circ i_X^0 = f_0$  and  $H \circ i_X^1 = f_1$ , so  $f_0$  and  $f_1$  must be equal in  $\text{Ho}(\mathcal{M}, \mathcal{W})$ . ■

**Theorem 5.3.32.** *Let  $(I, \mathcal{A})$  be a Cisinski homotopy structure on  $\mathcal{M}$ . Then  $\mathcal{M}$  is a combinatorial model category where*

- *the cofibrations are the monomorphisms in  $\mathcal{M}$ ,*
- *the weak equivalences are the weak equivalences with respect to  $(I, \mathcal{A})$ , and*
- *the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.*

*This is the **Cisinski model structure** on  $\mathcal{M}$  defined by  $(I, \mathcal{A})$ .*

*Proof.* See Théorème 2.13 in [Cisinski, 2002]. □

**Definition 5.3.33.** An  $\mathcal{M}$ -localiser is a class  $\mathcal{W}$  of morphisms in  $\mathcal{M}$  satisfying the following axioms:

- L1.**  $\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{M}$ .
- L2.** Every Cisinski trivial fibration is in  $\mathcal{W}$ .
- L3.** The class of monomorphisms that are in  $\mathcal{W}$  is closed under pushout and transfinite composition.

A **generating set** for  $\mathcal{W}$  is a set  $S$  such that  $\mathcal{W}$  is the smallest  $\mathcal{M}$ -localiser containing  $S$ . An **accessible  $\mathcal{M}$ -localiser** is an  $\mathcal{M}$ -localiser that admits a generating set.

**Proposition 5.3.34.** *Let  $\mathcal{W}$  be a class of morphisms in  $\mathcal{M}$  satisfying the following axioms:*

- FS1.** *For any object  $X$  in  $\mathcal{M}$ , the morphism  $\text{id} : X \rightarrow X$  is in  $\mathcal{W}$ .*
- FS2.**  *$\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{M}$ .*
- FS3.**  *$\mathcal{W}$  has the special 2-out-of-4 property in  $\mathcal{M}$ .*

Then the following are equivalent:

- (i) Every Cisinski trivial fibration is in  $\mathcal{W}$ .
- (ii) Let  $(I, i^0, i^1, \rho)$  be the Lawvere cylinder functor for  $\mathcal{M}$ . For all objects  $X$  in  $\mathcal{M}$ , the morphism  $\rho_X : IX \rightarrow X$  is in  $\mathcal{W}$ .
- (iii) There exists a Cisinski cylinder functor  $(I, i^0, i^1, \rho)$  for  $\mathcal{M}$  such that the morphism  $\rho_X : IX \rightarrow X$  is in  $\mathcal{W}$  for all objects  $X$  in  $\mathcal{M}$ .

*Proof.* (i)  $\Rightarrow$  (ii). This was shown in [proposition 5.3.18](#).

(ii)  $\Rightarrow$  (iii). Immediate.

(iii)  $\Rightarrow$  (i). Let  $p : X \rightarrow Y$  be a Cisinski trivial fibration in  $\mathcal{M}$ . [Proposition 5.3.15](#) then says that there exists a morphism  $s : Y \rightarrow X$  and an  $I$ -homotopy from  $\text{id}_X$  to  $s \circ p$ , i.e. a morphism  $H : IX \rightarrow X$  such that  $H \circ i_X^0 = \text{id}_X$  and  $H \circ i_X^1 = s \circ p$ . Since  $\rho_X : IX \rightarrow X$  is in  $\mathcal{W}$  and  $\rho_X \circ i_X^0 = \rho_X \circ i_X^1 = \text{id}_X$ , axioms FS1 and FS2 imply that  $i_X^0, i_X^1 : X \rightarrow IX$  are in  $\mathcal{W}$ , and so  $H : IX \rightarrow X$  is also in  $\mathcal{W}$ , and hence  $s \circ p : X \rightarrow X$  is in  $\mathcal{W}$  as well. We may now use axiom FS3 to deduce that  $p : X \rightarrow Y$  is in  $\mathcal{W}$ . ■

**Proposition 5.3.35.** *Let  $(I, \mathcal{A})$  be a Cisinski homotopy structure on  $\mathcal{M}$ . Then the class of weak equivalences with respect to  $(I, \mathcal{A})$  is an accessible  $\mathcal{M}$ -localiser.*

*Proof.* See Proposition 3.8 in [[Cisinski, 2002](#)]. □

**Theorem 5.3.36.** *Let  $\mathcal{W}$  be any accessible  $\mathcal{M}$ -localiser. Then  $\mathcal{M}$  is a combinatorial model category where*

- the cofibrations are the monomorphisms in  $\mathcal{M}$ ,
- the weak equivalences are the morphisms that are in  $\mathcal{W}$ , and
- the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.

*This is the **Cisinski model structure** on  $\mathcal{M}$  associated with  $\mathcal{W}$ .*

*Proof.* See Théorème 3.9 in [[Cisinski, 2002](#)]. □

**Corollary 5.3.37.** *If  $\mathcal{W}$  is any  $\mathcal{M}$ -localiser (not necessarily accessible), then  $\mathcal{W}$  is closed under retracts.*

*Proof.* See Corollaire 3.10 in [[Cisinski, 2002](#)]. □

## 5.4 Monoidal model categories

**Prerequisites.** §§4.1, 4.2, 4.3, B.1, B.2.

**Proposition 5.4.1.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories with pullbacks, let  $\mathcal{E}$  be a category with pushouts, and let  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$ ,  $\mathcal{J} \subseteq \text{mor } \mathcal{D}$  and  $\mathcal{K} \subseteq \text{mor } \mathcal{E}$  be subensembles. Suppose we have the following functors*

$$\begin{aligned} \otimes &: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \\ \pitchfork &: \mathcal{D}^{\text{op}} \times \mathcal{E} \rightarrow \mathcal{C} \\ \circ - &: \mathcal{E} \times \mathcal{C}^{\text{op}} \rightarrow \mathcal{D} \end{aligned}$$

and natural bijections:

$$\begin{aligned} \mathcal{E}(\mathcal{C} \otimes \mathcal{D}, E) &\cong \mathcal{C}(\mathcal{C}, \mathcal{D} \pitchfork E) \\ \mathcal{E}(\mathcal{C} \otimes \mathcal{D}, E) &\cong \mathcal{D}(\mathcal{D}, E \circ - \mathcal{C}) \\ \mathcal{C}(\mathcal{C}, \mathcal{D} \pitchfork E) &\cong \mathcal{D}(\mathcal{D}, E \circ - \mathcal{C}) \end{aligned}$$

Then the following are equivalent:

- (i) If  $f : \mathcal{C} \rightarrow \mathcal{C}'$  is in  $\mathcal{I}$ ,  $g : \mathcal{D} \rightarrow \mathcal{D}'$  is in  $\mathcal{J}$ , and the square in the diagram below is a pushout square in  $\mathcal{E}$ ,

$$\begin{array}{ccc} \mathcal{C} \otimes \mathcal{D} & \xrightarrow{\text{id}_{\mathcal{C}} \otimes g} & \mathcal{C} \otimes \mathcal{D}' \\ f \otimes \text{id}_{\mathcal{D}} \downarrow & & \downarrow \\ \mathcal{C}' \otimes \mathcal{D} & \xrightarrow{\quad} & (\mathcal{C} \otimes \mathcal{D}') \cup^{\mathcal{C} \otimes \mathcal{D}} (\mathcal{C}' \otimes \mathcal{D}) \\ & & \downarrow f \pitchfork g \\ & & \mathcal{C}' \otimes \mathcal{D}' \\ \text{id}_{\mathcal{C}'} \otimes g \nearrow & & \nearrow \end{array}$$

then the unique morphism  $f \pitchfork g$  making the diagram commute is in  $\mathcal{K}$ .

- (ii) If  $g : \mathcal{D} \rightarrow \mathcal{D}'$  is in  $\mathcal{J}$ ,  $h : \mathcal{E} \rightarrow \mathcal{E}'$  is in  $\mathcal{K}$ , and the square in the diagram below is a pullback square in  $\mathcal{C}$ ,

$$\begin{array}{ccc} \mathcal{D}' \pitchfork \mathcal{E} & \xrightarrow{g \pitchfork \text{id}_{\mathcal{E}}} & \mathcal{D} \pitchfork \mathcal{E} \\ g \pitchfork h \dashrightarrow & & \downarrow \text{id}_{\mathcal{D}} \pitchfork h \\ (\mathcal{D}' \pitchfork \mathcal{E}') \times_{\mathcal{D} \pitchfork \mathcal{E}'} (\mathcal{D} \pitchfork \mathcal{E}) & \xrightarrow{\quad} & \mathcal{D} \pitchfork \mathcal{E} \\ \text{id}_{\mathcal{D}'} \pitchfork h \searrow & & \downarrow \text{id}_{\mathcal{D}} \pitchfork h \\ \mathcal{D}' \pitchfork \mathcal{E}' & \xrightarrow{g \pitchfork \text{id}_{\mathcal{E}'}} & \mathcal{D} \pitchfork \mathcal{E}' \end{array}$$

then the unique morphism  $g \boxplus h$  making the diagram commute is in  $\mathcal{I}^\square$ .

(iii) If  $h : E \rightarrow E'$  is in  $\mathcal{K}$ ,  $f : C \rightarrow C'$  is in  $\mathcal{I}$  and the square in the diagram below is a pullback square in  $\mathcal{D}$ ,

$$\begin{array}{ccc}
 E \circ C' & \xrightarrow{\quad \text{id}_{E \circ C'} \quad} & E \circ C \\
 \downarrow h \circ \text{id}_C & \searrow h \boxplus f & \downarrow h \circ \text{id}_C \\
 (E' \circ C') \times_{E' \circ C} (E \circ C) & \xrightarrow{\quad} & E \circ C \\
 \downarrow & & \downarrow \\
 E' \circ C' & \xrightarrow{\quad \text{id}_{E' \circ C'} \quad} & E' \circ C
 \end{array}$$

then the unique morphism  $h \boxplus f$  making the diagram commute is in  $\mathcal{J}^\square$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $f : C \rightarrow C'$  be in  $\mathcal{I}$ , let  $g : D \rightarrow D'$  be in  $\mathcal{J}$ , let  $h : E \rightarrow E'$  be in  $\mathcal{K}$ , and suppose we have a commutative diagram of the following form:

$$\begin{array}{ccc}
 C & \xrightarrow{\quad} & D' \pitchfork E \\
 f \downarrow & \nearrow & \downarrow g \boxplus h \\
 C' & \xrightarrow{\quad} & (D' \pitchfork E') \times_{D' \pitchfork E'} (D \pitchfork E)
 \end{array}$$

By the universal property of pullbacks, this corresponds to a commutative diagram in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccccc}
 C & \xrightarrow{\quad} & D' \pitchfork E & \xrightarrow{\quad \text{id}_{D'} \pitchfork h \quad} & D' \pitchfork E' \\
 f \downarrow & \nearrow & \downarrow g \pitchfork \text{id}_E & \searrow & \downarrow g \pitchfork \text{id}_{E'} \\
 C' & \xrightarrow{\quad} & D \pitchfork E & \xrightarrow{\quad \text{id}_D \pitchfork h \quad} & D \pitchfork E'
 \end{array}$$

and, by adjoint transposition, to a commutative diagram in  $\mathcal{E}$  of the form

$$\begin{array}{ccccc}
 C \otimes D & \xrightarrow{\quad \text{id}_C \otimes g \quad} & C \otimes D' & \xrightarrow{\quad} & E \\
 f \otimes \text{id}_D \downarrow & & \downarrow f \otimes \text{id}_{D'} & \searrow & \downarrow h \\
 C' \otimes D & \xrightarrow{\quad} & C' \otimes D' & \xrightarrow{\quad} & E'
 \end{array}$$

whence, by the universal property of pushouts, commutative diagram in  $\mathcal{E}$  of the following form:

$$\begin{array}{ccc} (C \otimes D') \cup^{C \otimes D} (C' \otimes D) & \longrightarrow & E \\ f \sqcup g \downarrow & \nearrow & \downarrow h \\ C' \otimes D' & \longrightarrow & E' \end{array}$$

But  $(f \sqcup g) \sqsupseteq h$ , so we conclude that  $f \sqsupseteq (g \sqcup h)$ .

(ii)  $\Rightarrow$  (iii), (i)  $\Rightarrow$  (ii). A similar argument works. ■

**Definition 5.4.2.** Let  $\mathcal{C}$ ,  $\mathcal{D}$ , and  $\mathcal{E}$  be three model categories. A **Quillen adjunction of two variables** consists of three functors  $\otimes, \pitchfork, \circ$  with natural bijections as in the proposition satisfying the following (equivalent) axioms:

- (a) If  $h : E \rightarrow E'$  is a fibration in  $\mathcal{E}$  and  $f : C \rightarrow C'$  is a cofibration in  $\mathcal{C}$ , then the morphism  $h \sqcup f : E \circ C' \rightarrow (E' \circ C') \times_{E' \circ C} (E \circ C)$  is a fibration in  $\mathcal{D}$ , which is a weak equivalence if either  $h$  or  $f$  is.
- (b) If  $f : C \rightarrow C'$  is a cofibration in  $\mathcal{C}$  and  $g : D \rightarrow D'$  is a cofibration in  $\mathcal{D}$ , then the morphism  $f \sqcup g : C \otimes D \rightarrow (C \otimes D') \cup^{C \otimes D} (C' \otimes D)$  is a cofibration in  $\mathcal{E}$ , which is a weak equivalence if either  $f$  or  $g$  is.
- (c) If  $g : D \rightarrow D'$  is a cofibration in  $\mathcal{C}$  and  $h : E \rightarrow E'$  is a fibration in  $\mathcal{D}$ , then the morphism  $g \sqcup h : D' \pitchfork E \rightarrow (D' \pitchfork E') \times_{D' \pitchfork E} (D \pitchfork E)$  is a fibration in  $\mathcal{C}$ , which is a weak equivalence if either  $g$  or  $h$  is.

**Proposition 5.4.3.** Let  $(\otimes, \pitchfork, \circ)$  be a Quillen adjunction of two variables as above.

- (i) For each cofibrant object  $C$  in  $\mathcal{C}$ , the adjunction

$$C \otimes (-) \dashv (-) \circ C : \mathcal{E} \rightarrow \mathcal{D}$$

is a Quillen adjunction.

- (ii) For each cofibrant object  $D$  in  $\mathcal{D}$ , the adjunction

$$(-) \otimes D \dashv D \pitchfork (-) : \mathcal{E} \rightarrow \mathcal{C}$$

is a Quillen adjunction.

(iii) For each fibrant object  $E$  in  $\mathcal{E}$ , the adjunction

$$E \circ (-) \dashv (-) \pitchfork E : \mathcal{D}^{\text{op}} \rightarrow \mathcal{C}$$

is a Quillen adjunction.

*Proof.* Immediate from the definitions. ◆

**Corollary 5.4.4.**

- (i) For each object  $C$  in  $\mathcal{C}$ ,  $C \otimes (-)$  preserves weak equivalences between cofibrant objects, and  $(-) \otimes C$  preserves weak equivalences between fibrant objects.
- (ii) For each object  $D$  in  $\mathcal{D}$ ,  $(-) \otimes D$  preserves weak equivalences between cofibrant objects, and  $D \pitchfork (-)$  preserves weak equivalences between fibrant objects.
- (iii) For each object  $E$  in  $\mathcal{E}$ ,  $E \circ (-)$  sends weak equivalences between cofibrant objects in  $\mathcal{C}$  to weak equivalences between fibrant objects in  $\mathcal{D}$ , and  $(-) \pitchfork E$  sends weak equivalences between cofibrant objects in  $\mathcal{D}$  to weak equivalences between fibrant objects in  $\mathcal{D}$ .

*Proof.* Apply Ken Brown's lemma (4.3.6). ■

**Lemma 5.4.5.** Let  $\mathcal{V}$  be a monoidal category, let  $\mathcal{M}$  be a model category with fibrant and cofibrant replacement functors, and let  $p : \tilde{I} \rightarrow I$  be a morphism in  $\mathcal{V}$ , where  $I$  is the monoidal unit of  $\mathcal{V}$ .

If  $\mathcal{M}$  has a left  $\mathcal{V}$ -action  $\otimes$  and right adjoint right  $\mathcal{V}^{\text{op}}$ -action  $\circ$  such that the adjunction

$$\tilde{I} \otimes (-) \dashv (-) \circ \tilde{I} : \mathcal{M} \rightarrow \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

- (i) For all cofibrant objects  $X$  in  $\mathcal{M}$ ,  $p \otimes \text{id}_X : \tilde{I} \otimes X \rightarrow I \otimes X$  is a weak equivalence.
- (ii) For all fibrant objects  $Y$  in  $\mathcal{M}$ ,  $\text{id}_Y \circ p : Y \circ I \rightarrow Y \circ \tilde{I}$  is a weak equivalence.

If  $\mathcal{M}$  has a right  $\mathcal{V}$ -action  $\otimes$  and a right adjoint left  $\mathcal{V}^{\text{op}}$ -action  $\circ$  such that the adjunction

$$(-) \otimes \tilde{I} \dashv \tilde{I} \circ (-) : \mathcal{M} \rightarrow \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

- (i') For all cofibrant objects  $X$  in  $\mathcal{M}$ ,  $\text{id}_X \otimes p : X \otimes \tilde{I} \rightarrow X \otimes I$  is a weak equivalence.
- (ii') For all fibrant objects  $Y$  in  $\mathcal{M}$ ,  $p \multimap \text{id}_Y : I \multimap Y \rightarrow \tilde{I} \multimap Y$  is a weak equivalence.

*Proof.* Since  $\eta_X : X \rightarrow I \otimes X$  is a natural isomorphism, the adjunction

$$I \otimes (-) \dashv (-) \multimap I : \mathcal{M} \rightarrow \mathcal{M}$$

is an adjoint equivalence of categories, and *a fortiori* a Quillen equivalence, and the natural transformations  $p \otimes (-)$  and  $(-) \multimap p$  constitute a conjugate pair. [Theorem 3.3.20](#) says that the derived natural transformations for  $p \otimes (-)$  and  $(-) \multimap p$  constitute a conjugate pair of natural transformations between the derived adjunctions. Applying [proposition 3.3.24](#) to [theorem 4.3.12](#), we deduce that the following are equivalent:

- For all cofibrant objects  $X$ ,  $p \otimes \text{id}_X$  is a weak equivalence.
- The left derived natural transformation for  $p \otimes (-)$  is a natural isomorphism.
- The right derived natural transformation for  $(-) \otimes p$  is a natural isomorphism.
- For all fibrant objects  $Y$ ,  $\text{id}_Y \multimap p$  is a weak equivalence. ■

The following definition is due to Hovey [[1999](#), §4.2]:

**Definition 5.4.6.** A **monoidal model category** is a biclosed monoidal category  $\mathcal{M}$  equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The right  $\mathcal{M}$ -hom system  $(\otimes, \multimap, \multimap)$ , where  $\multimap$  (resp.  $\multimap$ ) is the right (resp. left) internal hom functor of  $\mathcal{M}$ , is a Quillen adjunction of two variables.
- **Unit axiom.** For each cofibrant replacement  $(\tilde{I}, p)$  of the monoidal unit  $I$  and each cofibrant object  $X$  in  $\mathcal{M}$ , the morphisms  $p \otimes \text{id}_X : \tilde{I} \otimes X \rightarrow I \otimes X$  and  $\text{id}_X \otimes p : X \otimes \tilde{I} \rightarrow X \otimes I$  are weak equivalences in  $\mathcal{M}$ .

**Lemma 5.4.7.** *Let  $\mathcal{M}$  be a biclosed monoidal category equipped with a model structure satisfying the pushout–product axiom, and let  $X$  be any object in  $\mathcal{M}$ . The following are equivalent:*



- (i) *There exists a cofibrant replacement  $(\tilde{I}, p)$  of the monoidal unit  $I$  such that  $p \otimes \text{id}_X$  and  $\text{id}_X \otimes p$  are weak equivalences in  $\mathcal{M}$ .*
- (ii) *There exists a fibrant cofibrant replacement  $(QI, q)$  of the monoidal unit  $I$  such that  $q \otimes \text{id}_X$  and  $\text{id}_X \otimes q$  are weak equivalences in  $\mathcal{M}$ .*
- (iii) *For any cofibrant replacement  $(\tilde{I}, p)$  of the monoidal unit  $I$ , both  $p \otimes \text{id}_X$  and  $\text{id}_X \otimes p$  are weak equivalences in  $\mathcal{M}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Let  $(QI, q)$  be a fibrant cofibrant replacement of  $I$ ; such exists by [proposition 4.1.23](#). Since  $\tilde{I}$  is cofibrant, axiom CM5 implies there is a morphism  $w : \tilde{I} \rightarrow QI$  such that  $q \circ w = p$ , and the 2-out-of-3 property implies  $w$  is a weak equivalence. [Corollary 5.4.4](#) says  $w \otimes \text{id}_X$  and  $\text{id}_X \otimes w$  are weak equivalences, thus by the 2-out-of-3 property again  $q \otimes \text{id}_X$  and  $\text{id}_X \otimes q$  must be weak equivalences.

(ii)  $\Rightarrow$  (iii). A similar argument works.

(iii)  $\Rightarrow$  (i). Obvious, given the existence of cofibrant replacements. ■

**Corollary 5.4.8.** *Let  $\mathcal{M}$  be a biclosed monoidal category equipped with a model structure. If the monoidal unit  $I$  is a cofibrant object in  $\mathcal{M}$ , then the following are equivalent:*

- (i)  *$\mathcal{M}$  is a monoidal model category.*
- (ii)  *$\mathcal{M}$  satisfies the pushout–product axiom.* ■

TODO: State the version without the assumption that the unit is cofibrant.

**Proposition 5.4.9.** *Let  $\mathcal{M}$  be a monoidal model category, let  $I$  be the monoidal unit, and let  $\dashv : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$  be the right internal hom functor. If  $I$  is a cofibrant object and  $(J, i_0, i_1, p)$  is a cylinder object for  $I$ , then  $(J \dashv X, i, p_0, p_1)$  is a path object for all fibrant  $X$ , where  $i : X \rightarrow [J, X]$  is the morphism induced by  $p : J \rightarrow I$ , and  $p_0, p_1 : [J, X] \rightarrow X$  are (respectively) the morphisms induced by  $i_0, i_1 : I \rightarrow J$ .*

*Proof.* Since  $I$  is a cofibrant object,  $I+I$  is cofibrant (by [proposition A.3.14](#)), and hence  $J$  itself is cofibrant. [Corollary 5.4.4](#) says the functor  $(-)\dashv X : \mathcal{M}^{\text{op}} \rightarrow \mathcal{M}$  sends weak equivalences between cofibrant objects in  $\mathcal{M}$  to weak equivalences between fibrant objects in  $\mathcal{M}$  when  $X$  is fibrant, so it follows that the morphism  $i : X \rightarrow [J, X]$  is a weak equivalence. Similarly, since the morphism  $I+I \rightarrow J$

induced by  $i_0$  and  $i_1$  is a cofibration, the morphism  $[J, X] \rightarrow X \times X$  induced by  $p_0$  and  $p_1$  is a fibration, so  $([J, X], i, p_0, p_1)$  is indeed a path object for  $X$ . ■

The following definition can be found in [Rezk, 2010, §2] and [Simpson, 2012, §7.7].

**Definition 5.4.10.** A **cartesian model category** is a cartesian closed category  $\mathcal{M}$  equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The left  $\mathcal{M}$ -hom system  $(\times, [-, -], [-, -])$  is a Quillen adjunction of two variables.
- **Cofibrant unit axiom.** Every terminal object in  $\mathcal{M}$  is cofibrant.

**Example 5.4.11.** The Kan–Quillen model structure on  $\mathbf{sSet}$  makes it a cartesian model category:  $\mathbf{sSet}$  is a cartesian closed combinatorial model category (*a fortiori* a DHK model category), all simplicial sets are cofibrant, and the pushout–product axiom is just [proposition 1.4.15](#).

**Proposition 5.4.12.** *Let  $\mathcal{M}$  be a Cisinski model category.<sup>[2]</sup> The following are equivalent:*

- (i)  $\mathcal{M}$  is a cartesian model category.
- (ii) The class of weak equivalences in  $\mathcal{M}$  is closed under binary products.
- (iii) The class of trivial cofibrations in  $\mathcal{M}$  is closed under binary products.

*Proof.* (i)  $\Rightarrow$  (ii). Since all objects in  $\mathcal{M}$  are cofibrant, [corollary 5.4.4](#) implies that, for any object  $Y$  in  $\mathcal{M}$ , the functor  $(-) \times Y : \mathcal{M} \rightarrow \mathcal{M}$  preserves weak equivalences. Thus, the class of weak equivalences in  $\mathcal{M}$  is closed under binary products.

(ii)  $\Rightarrow$  (iii). The class of monomorphisms is always closed under binary products, so the class of trivial cofibrations (i.e. monic weak equivalences) is closed under binary products if the class of weak equivalences is.

(iii)  $\Rightarrow$  (i). This is the content of [proposition 5.3.6](#). ■

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[2] See [definition 5.3.1](#).

**Theorem 5.4.13.** *If  $\mathcal{M}$  is a monoidal model category, then there is an induced monoidal biclosed structure on  $\mathrm{Ho} \mathcal{M}$  where the monoidal product is the left derived functor of the monoidal product in  $\mathcal{M}$  and the coherence data is inherited from  $\mathcal{M}$ .*

*Proof.* See Theorem 4.3.2 in [Hovey, 1999]. □

**Proposition 5.4.14.** *Let  $\mathcal{M}$  be a cartesian model category and let  $\mathcal{M}_f$  be the full subcategory of fibrant objects.*

- (i)  $\mathcal{M}_f$  is closed under products of small families of objects in  $\mathcal{M}$ , and  $[X, Y]$  is fibrant if  $X$  is cofibrant and  $Y$  is fibrant.
- (ii) The localising functor  $\gamma : \mathcal{M}_f \rightarrow \mathrm{Ho} \mathcal{M}$  preserves products of small families of objects; in particular,  $\mathrm{Ho} \mathcal{M}$  has products for all small families of objects.
- (iii)  $\mathrm{Ho} \mathcal{M}$  is a cartesian closed category, and  $\gamma[X, Y]$  is naturally isomorphic to  $[\gamma X, \gamma Y]$  when  $X$  is cofibrant and  $Y$  is fibrant.
- (iv) Let  $\Gamma : \mathcal{M} \rightarrow \mathbf{Set}$  be the functor  $\mathcal{M}(1, -)$  and let  $\tau_0 : \mathcal{M} \rightarrow \mathbf{Set}$  be the functor  $\mathrm{Ho} \mathcal{M}(\gamma 1, \gamma -)$ . The functor  $\tau_0$  preserves small products in  $\mathcal{M}_f$ , and the component  $\chi_Y : \Gamma Y \Rightarrow \tau_0 Y$  of the natural transformation  $\chi : \Gamma \Rightarrow \tau_0$  induced by the functor  $\gamma$  is surjective for all fibrant objects  $Y$  in  $\mathcal{M}$ .

*Proof.* (i). That  $\mathcal{M}_f$  is closed in  $\mathcal{M}$  under small products is a straightforward consequence of [proposition A.3.14](#), and pushout–product axiom for cartesian model structures implies the other half of the claim.

(ii). [Proposition 4.3.17](#) says  $\mathrm{Ho}[I, \mathcal{M}] \rightarrow [I, \mathrm{Ho} \mathcal{M}]$  is an equivalence of categories for all sets  $I$ , so products in  $\mathrm{Ho} \mathcal{M}$  coincide with homotopy products. Homotopy products in  $\mathcal{M}_f$  coincide with ordinary products, hence the localising functor  $\gamma : \mathcal{M}_f \rightarrow \mathrm{Ho} \mathcal{M}$  preserves small products. Since every object in  $\mathcal{M}$  is weakly equivalent to one in  $\mathcal{M}_f$ , it follows that  $\mathrm{Ho} \mathcal{M}$  has products for all small families of objects.

(iii). Apply [theorem 5.4.13](#).

(iv). As a representable functor,  $\mathrm{Ho} \mathcal{M}(\gamma 1, -) : \mathrm{Ho} \mathcal{M} \rightarrow \mathbf{Set}$  preserves small products, and by claim (ii),  $\gamma : \mathcal{M}_f \rightarrow \mathrm{Ho} \mathcal{M}$  preserves small products, so

$\tau_0 : \mathcal{M}_f \rightarrow \mathbf{Set}$  indeed preserves small products. [Theorem 4.1.30](#) says that the localising functor induces hom-set maps  $\mathcal{M}(X, Y) \rightarrow \mathrm{Ho} \mathcal{M}(\gamma X, \gamma Y)$  that are surjective when  $X$  is cofibrant and  $Y$  is fibrant; since  $1$  is cofibrant by hypothesis, it follows that the map  $\chi_Y : \Gamma Y \rightarrow \tau_0 Y$  is surjective for all cofibrant objects  $Y$ . ■

Under stronger hypotheses, the homotopy category of a cartesian model category admits a description à la Hurewicz:

**Proposition 5.4.15.** *Let  $\mathcal{M}$  be a cartesian model category, let  $\mathcal{M}_f$  be the full subcategory of fibrant objects, and let  $\mathrm{Ho} \mathcal{M}_f$  be the localisation of  $\mathcal{M}_f$  at the weak equivalences. If all fibrant objects in  $\mathcal{M}$  are cofibrant, then:*

- (i)  $\mathcal{M}_f$  is a cartesian closed category.
- (ii) The natural transformation  $\chi : \Gamma \Rightarrow \tau_0$  induces a functor  $\mathcal{M}_f \rightarrow \tau_0[\underline{\mathcal{M}}_f]$  that is a bijection on objects, full, and preserves small products and exponential objects.
- (iii) Let  $f_0, f_1 : X \rightarrow Y$  be a parallel pair of morphisms in  $\mathcal{M}_f$ . Then  $f_0$  and  $f_1$  are (right) homotopic if and only if they are sent to the same morphism in  $\tau_0[\underline{\mathcal{M}}_f]$ .
- (iv) The canonical functor  $\mathrm{Ho} \mathcal{M}_f \rightarrow \tau_0[\underline{\mathcal{M}}_f]$  is an isomorphism of categories.

*Proof.* (i). Since all fibrant objects are cofibrant, the exponential object  $[X, Y]$  is fibrant for all  $X$  and  $Y$  in  $\mathcal{M}_f$ ; and by [proposition 5.4.14](#),  $\mathcal{M}_f$  is closed under products of small objects in  $\mathcal{M}$ , so it follows that  $\mathcal{M}_f$  is a cartesian closed category.

(ii). This is a straightforward consequence of the fact that  $\tau_0 : \mathcal{M}_f \rightarrow \mathbf{Set}$  preserves small products, that we have a natural bijection  $\Gamma[X, Y] \cong \mathcal{M}(X, Y)$  for all objects  $X$  and  $Y$ , and that  $\chi_Z : \Gamma Z \rightarrow \tau_0 Z$  is a surjection for all fibrant objects  $Z$ .

(iii). Suppose  $f_0, f_1 : X \rightarrow Y$  are related by a right homotopy, i.e. there exists a path object  $(P, i, p_0, p_1)$  for  $Y$  and a morphism  $f : X \rightarrow P$  such that  $p_0 \circ f = f_0$  and  $p_1 \circ f = f_1$ . Since  $p_0, p_1 : P \rightarrow Y$  are retractions of the weak equivalence  $i : Y \rightarrow P$ , the two maps  $\tau_0[X, P] \rightarrow \tau_0[X, Y]$  induced by  $p_0$  and  $p_1$  must be

equal. In particular,  $\chi_{[X,Y]} : \Gamma[X, Y] \rightarrow \tau_0[X, Y]$  must map  $f_0$  and  $f_1$  to the same element.

Conversely, if  $f_0$  and  $f_1$  are sent to the same morphism in  $\tau_0[\underline{\mathcal{M}}_f]$ , then there must exist a cylinder object  $(J, i_0, i_1, p)$  for 1 and a morphism  $\bar{h} : J \rightarrow [X, Y]$  such that  $h \circ i_0$  (resp.  $h \circ i_1$ ) is the exponential transpose of  $f_0$  (resp.  $f_1$ ). Taking exponential transposes again and using the fact that  $[J, Y]$  is a path object for  $Y$ , we deduce that  $f_0$  and  $f_1$  are right homotopic.

(iv). The formal Whitehead theorem implies that weak equivalences in  $\mathcal{M}_f$  are mapped to isomorphisms in  $\tau_0[\underline{\mathcal{M}}_f]$ , so the functor  $\mathcal{M} \rightarrow \tau_0[\underline{\mathcal{M}}_f]$  induces a functor  $\text{Ho } \mathcal{M}_f \rightarrow \tau_0[\underline{\mathcal{M}}_f]$ . A standard argument then shows that it is an isomorphism: see e.g. [theorem 4.4.I](#). ■

**Proposition 5.4.16.** *Let  $\mathcal{M}$  be a cartesian model category. If all objects in  $\mathcal{M}$  are cofibrant, then:*

- (i) *The functors  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  and  $\tau_0 : \mathcal{M} \rightarrow \mathbf{Set}$  both preserve finite products.*
- (ii) *A morphism  $f : X \rightarrow Y$  in  $\mathcal{M}$  is a weak equivalence if and only if the induced maps*

$$\tau_0[f, Z] : \tau_0[Y, Z] \rightarrow \tau_0[X, Z]$$

*are bijections for all fibrant objects  $Z$  in  $\mathcal{M}$ .*

- (iii) *The inclusion  $\mathcal{M}_f \hookrightarrow \mathcal{M}$  induces a fully faithful functor  $\tau_0[\underline{\mathcal{M}}_f] \rightarrow \tau_0[\underline{\mathcal{M}}]$  with a left adjoint.*

*Proof.* (i). It suffices to show that  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  preserves finite products; that  $\tau_0 : \mathcal{M} \rightarrow \mathbf{Set}$  preserves finite products will follow automatically. It is not hard to check that  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  preserves terminal objects for all model categories  $\mathcal{M}$ , and we will now show that  $\gamma$  preserves binary products.

The pushout–product axiom implies that, for all cofibrant objects  $Y$ , the functor  $- \times Y : \mathcal{M} \rightarrow \mathcal{M}$  is a left Quillen functor. Since we are assuming that all objects are cofibrant, [corollary 5.4.4](#) implies that  $- \times Y$  preserves weak equivalences. We may then deduce that  $- \times - : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{M}$  preserves all weak equivalences, and hence that it is a homotopical left approximation for itself. Thus, the localising functor  $\gamma : \mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  indeed preserves binary products.

(ii). If  $f : X \rightarrow Y$  is a weak equivalence, then  $[f, Z] : [Y, Z] \rightarrow [X, Z]$  is a weak equivalence for all fibrant objects  $Z$ , and hence  $\tau_0[f, Z]$  must be a bijection. Conversely, suppose  $\tau_0[f, Z]$  is a bijection for all fibrant objects  $Z$ . Let  $R : \mathcal{M} \rightarrow \mathcal{M}$  be a fibrant replacement functor for  $\mathcal{M}$ . Then, the morphism  $Rf : RX \rightarrow RY$  also induces bijections  $\tau_0[Rf, Z]$  for all fibrant objects  $Z$ , and since  $RX$  and  $RY$  are in  $\mathcal{M}_f$ , the Yoneda lemma implies that  $Rf : RX \rightarrow RY$  is sent to an isomorphism in  $\tau_0[\underline{\mathcal{M}}_f]$ , and hence must be a weak equivalence in  $\mathcal{M}_f$ . The 2-out-of-3 property of weak equivalences then implies  $f : X \rightarrow Y$  is a weak equivalence in  $\mathcal{M}$ .

(iii). It is clear that the induced functor  $\tau_0[\underline{\mathcal{M}}_f] \rightarrow \tau_0[\underline{\mathcal{M}}]$  is indeed fully faithful, and it is not hard to check that a fibrant replacement functor provides the required left adjoint  $\tau_0[\underline{\mathcal{M}}] \rightarrow \tau_0[\underline{\mathcal{M}}_f]$ . ■

**Definition 5.4.17.** An **isocofibration** is a functor that is injective on objects. An **isofibration** is a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that, for every object  $C$  in  $\mathcal{C}$  and every isomorphism  $f : FC \rightarrow D$  in  $\mathcal{D}$ , there exists an isomorphism  $\tilde{f} : C \rightarrow \tilde{D}$  in  $\mathcal{C}$  such that  $F\tilde{f} = f$ .

**Proposition 5.4.18.** *Let **Cat** be the category of small categories. The following data constitute a model structure on **Cat**:*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isocofibrations.*
- *The fibrations are the isofibrations.*

*Moreover, the factorisations for axiom CM5 may be chosen functorially, so that **Cat** becomes a DHK model category. This model structure is called the **canonical model structure on Cat**.*

*Proof.* It is not hard to show that **Cat** has limits and colimits for all small diagrams, so axiom CM1\* is satisfied. It is also clear that the announced class of weak equivalences has the 2-out-of-3 property, so by [theorem 4.1.12](#), it is enough to show that we have a pair of compatible weak factorisation systems.

Let  $I : \mathbb{A} \rightarrow \mathbb{B}$  be an isofibration and  $P : \mathbb{C} \rightarrow \mathbb{D}$  be an isofibration, and suppose we have a commutative diagram of the following form:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{F} & \mathbb{C} \\ I \downarrow & & \downarrow P \\ \mathbb{B} & \xrightarrow{G} & \mathbb{D} \end{array}$$

First, suppose  $P$  is a weak equivalence. Then,  $P$  must be surjective on objects, so we may define a map  $H : \text{ob } \mathbb{B} \rightarrow \text{ob } \mathbb{C}$  by taking  $HB = FA$  if  $B = IA$  for some  $A$ , and if  $B$  is not in the image of  $A$ , define  $HB$  to be any object in  $\mathbb{C}$  such that  $PHB = GB$ ; there is then a unique way of extending  $H$  to a functor  $\mathbb{B} \rightarrow \mathbb{C}$  making the evident diagram commute.

Next, instead suppose  $I$  is a weak equivalence. Then,  $I$  may be regarded as the inclusion of a full subcategory that is essentially surjective on objects. For each object  $B$  in  $\mathbb{B}$  that is not in the image of  $I$ , fix an object  $A$  in  $\mathbb{A}$  and an isomorphism  $IA \xrightarrow{\cong} B$ . Since  $P$  is an isofibration, for each such  $B$  we may also choose an object  $C$  in  $\mathbb{C}$  and an isomorphism  $FA \xrightarrow{\cong} C$  whose image under  $P$  is  $GIA \xrightarrow{\cong} GB$ . There is then a unique functor  $H : \mathbb{B} \rightarrow \mathbb{C}$  that makes the evident diagram commute and sends  $B$  to the chosen  $C$  and  $IA \xrightarrow{\cong} B$  to  $FA \xrightarrow{\cong} C$ .

It remains to be shown that every functor can be factorised in the required manner. Let  $F : \mathbb{C} \rightarrow \mathbb{D}$  be any functor. Consider the iso-comma category  $(F \wr \mathbb{D})$ :

- The objects are triples  $(C, D, \alpha)$ , where  $C$  is an object in  $\mathbb{C}$ ,  $D$  is an object in  $\mathbb{D}$ , and  $\alpha : FC \rightarrow D$  is an *isomorphism* in  $\mathbb{D}$ .
- The morphisms  $(C, D, \alpha) \rightarrow (C', D', \alpha')$  is a morphism  $f : C \rightarrow C'$  in  $\mathbb{C}$  together with a morphism  $g : D \rightarrow D'$  in  $\mathbb{D}$  such that  $g \circ \alpha = \alpha' \circ Ff$ .<sup>[3]</sup>
- Composition and identities are inherited from  $\mathbb{C}$  and  $\mathbb{D}$ .

There is an evident isofibration  $I : \mathbb{C} \rightarrow (F \wr \mathbb{D})$  sending an object  $C$  in  $\mathbb{C}$  to the object  $(C, FC, \text{id}_{FC})$ , and it is easy to see that  $I$  is a weak equivalence. On the other hand, the projection  $P : (F \wr \mathbb{D}) \rightarrow \mathbb{D}$  is an isofibration by construction, and obviously  $F = PI$ . Thus, we have factored  $F$  as a trivial isofibration

[3] However, because  $\alpha$  and  $\alpha'$  are isomorphisms,  $f$  freely and uniquely determines  $g$ .

followed by an isofibration, and it is clear that this construction is functorial in  $F$ .

Now, consider instead the category  $\mathbf{M}(F)$  defined below:

- $\text{ob } \mathbf{M}(F) = \text{ob } \mathbb{C} \amalg \text{ob } \mathbb{D}$ .
- If  $C$  and  $C'$  are objects in  $\mathbb{C}$ , while  $D$  and  $D'$  are objects in  $\mathbb{D}$ , then:

$$\text{Hom}(C, C') = \mathbb{D}(FC, FC')$$

$$\text{Hom}(C, D') = \mathbb{D}(FC, D')$$

$$\text{Hom}(D, C') = \mathbb{D}(D, FC')$$

$$\text{Hom}(D, D') = \mathbb{D}(D, D')$$

- Composition and identities are inherited from  $\mathbb{D}$ .

There is an evident isocofibration  $I : \mathbb{C} \rightarrow \mathbf{M}(F)$  that sends an object  $C$  in  $\mathbb{C}$  to the corresponding object in  $\mathbf{M}(F)$  and sends a morphism  $f : C \rightarrow C'$  in  $\mathbb{C}$  to the morphism in  $\mathbf{M}(F)$  corresponding to  $Ff : FC \rightarrow FC'$  in  $\mathbb{D}$ . On the other hand, there is an evident projection  $P : \mathbf{M}(F) \rightarrow \mathbb{D}$  that is fully faithful and surjective on objects, i.e.  $P$  is a trivial isofibration. Of course,  $F = PI$ , so this is a factorisation of  $F$  as an isocofibration followed by a trivial isofibration, and it is clear that this construction is functorial in  $F$ . ■

**Theorem 5.4.19.** *Let  $\mathbf{Cat}$  be considered as a model category via the canonical model structure.*

- (i) *Every object in  $\mathbf{Cat}$  is both cofibrant and fibrant.*
- (ii)  *$\mathbf{Cat}$  is a combinatorial model category.*
- (iii)  *$\mathbf{Cat}$  is a cartesian model category.*

*Proof.* (i). The unique functor  $\emptyset \rightarrow \mathbb{C}$  is vacuously an isocofibration, and the unique functor  $\mathbb{C} \rightarrow \mathbb{1}$  is certainly an isofibration.

(ii).  $\mathbf{Cat}$  is a locally finitely presentable category,<sup>[4]</sup> and it remains to be shown that the canonical model structure is a cofibrantly generated model structure.

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[4] — because e.g.  $\mathbf{Cat}$  is the category of models for a finite limit sketch; see Proposition 1.51 in [LPAC] or Proposition 5.6.4 in [Borceux, 1994b].



By the very definition of isofibration, the set  $\{\{0\} \hookrightarrow \mathbf{I2}\}$  is a generating set of trivial isofibrations, where  $\mathbf{I2}$  is the groupoid containing only a pair of non-trivial isomorphisms. It is also straightforward to see that a functor is ...

- ... surjective on objects if and only if it has the right lifting property with respect to the unique functor  $\emptyset \rightarrow \mathbf{1}$ ;
- ... full if and only if it has the right lifting property with respect to the inclusion  $\text{disc } 2 \rightarrow \mathbf{2}$ ; and
- ... faithful if and only if it has the right lifting property with respect to the surjective functor  $\mathbb{E} \rightarrow \mathbf{2}$ , where  $\mathbb{E}$  is the category with a parallel pair of non-trivial morphisms.

However, a functor is a trivial isofibration if and only if it is fully faithful and surjective on objects, so  $\{\emptyset \rightarrow \mathbf{1}, \text{disc } 2 \rightarrow \mathbf{2}, \mathbb{E} \rightarrow \mathbf{2}\}$  is a set of generating isofibrations.

(iii). Let  $F : \mathbb{C} \rightarrow \mathbb{C}'$  and  $G : \mathbb{D} \rightarrow \mathbb{D}'$  be isofibrations, and consider the functor  $F \boxtimes G$  defined by the diagram below:

$$\begin{array}{ccc}
 \mathbb{C} \times \mathbb{D} & \xrightarrow{\text{id}_{\mathbb{C}} \times G} & \mathbb{C} \times \mathbb{D}' \\
 F \times \text{id}_{\mathbb{D}} \downarrow & & \downarrow \\
 \mathbb{C}' \otimes \mathbb{D} & \longrightarrow & (\mathbb{C} \times \mathbb{D}') \cup^{\mathbb{C} \times \mathbb{D}} (\mathbb{C}' \times \mathbb{D}) \\
 & \searrow & \downarrow F \times \text{id}_{\mathbb{D}'} \\
 & & \mathbb{C}' \times \mathbb{D}' \\
 & \xrightarrow{\text{id}_{\mathbb{C}'} \times G} & \\
 & \nearrow F \boxtimes G & 
 \end{array}$$

The functor  $\text{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$  has both left and right adjoints, so it is easy to see that  $F \boxtimes G$  is an isofibration. Moreover, if  $F : \mathbb{C} \rightarrow \mathbb{C}'$  is a trivial isofibration, one may directly verify that  $F \times \text{id}_{\mathbb{D}} : \mathbb{C} \times \mathbb{D} \rightarrow \mathbb{C}' \times \mathbb{D}$  and  $F \times \text{id}_{\mathbb{D}'} : \mathbb{C} \times \mathbb{D}' \rightarrow \mathbb{C}' \times \mathbb{D}'$  are trivial isofibrations; but trivial isofibrations are closed under pushout, so applying the 2-out-of-3 property of weak equivalences, we conclude that  $F \boxtimes G$  is a trivial isofibration if  $F$  is. The symmetrical argument shows that  $F \boxtimes G$  is a trivial isofibration if  $G$  is.

Having shown that  $\mathbf{Cat}$  satisfies the pushout-product axiom, we must now verify that  $\mathbf{Cat}$  is cartesian closed and has a cofibrant unit; but the former is a very well-known fact, and the latter follows from claim (i). ■

**Theorem 5.4.20.** *Let  $\mathbf{Grpd}$  be the category of small groupoids.*

(i) *The following data constitute a model structure on  $\mathbf{Grpd}$ :*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isofibrations.*
- *The fibrations are the isofibrations.*

*This model structure is called the **canonical model structure on  $\mathbf{Grpd}$** .*

(ii) *Every object in  $\mathbf{Grpd}$  is both cofibrant and fibrant.*

(iii)  *$\mathbf{Grpd}$  is a combinatorial model category.*

(iv)  *$\mathbf{Grpd}$  is a cartesian model category.*

(v) *The inclusion  $\text{und} : \mathbf{Grpd} \rightarrow \mathbf{Cat}$  preserves and reflects weak equivalences, isofibrations, and isofibrations; moreover, it is both a left Quillen functor and a right Quillen functor.*

*Proof.* (i). The proof of [proposition 5.4.18](#) goes through for  $\mathbf{Grpd}$  without modifications.

(ii) – (iv). These can be proven in essentially the same way as [proposition 5.4.18](#), though one should note that the generating isofibrations and generating trivial isofibrations for  $\mathbf{Grpd}$  are different.

(v). It is clear that  $\text{und} : \mathbf{Grpd} \rightarrow \mathbf{Cat}$  has the announced preservation and reflection properties. One may check that  $\text{und}$  has a left adjoint  $\mathbf{I} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  and a right adjoint  $\text{iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$ , so  $\text{und}$  is both a left Quillen functor and a right Quillen functor. ■

## 5.5 Bousfield localisation

**Prerequisites.** §§4.1, 4.3, 4.8, 5.1.

**Definition 5.5.1.** Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .

- An  **$S$ -local object** in  $\mathcal{M}$  is a fibrant object  $X$  in  $\mathcal{M}$  such that, for every morphism  $g : Z \rightarrow W$  that is in  $S$ , the induced morphism

$$\mathbf{RHom}_{\mathcal{M}}(g, X) : \mathbf{RHom}_{\mathcal{M}}(W, X) \rightarrow \mathbf{RHom}_{\mathcal{M}}(Z, X)$$

is an isomorphism in  $\mathbf{Ho sSet}$ .

- An  **$S$ -colocal object** in  $\mathcal{M}$  is a cofibrant object  $W$  in  $\mathcal{M}$  such that, for every morphism  $f : X \rightarrow Y$  that is in  $S$ , the induced morphism

$$\mathbf{RHom}_{\mathcal{M}}(W, f) : \mathbf{RHom}_{\mathcal{M}}(W, X) \rightarrow \mathbf{RHom}_{\mathcal{M}}(W, Y)$$

is an isomorphism in  $\mathbf{Ho sSet}$ .

**Lemma 5.5.2.** Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .

- If  $f : X \rightarrow Y$  is a weak equivalence between fibrant objects in  $\mathcal{M}$ , then  $X$  is an  $S$ -local object in  $\mathcal{M}$  if and only if  $Y$  is  $S$ -local.
- If  $g : Z \rightarrow W$  is a weak equivalence between cofibrant objects in  $\mathcal{M}$ , then  $Z$  is an  $S$ -colocal object in  $\mathcal{M}$  if and only if  $W$  is  $S$ -colocal.

*Proof.* Immediate, because  $\mathbf{RHom}_{\mathcal{M}}$  is a functor  $(\mathbf{Ho } \mathcal{M})^{\text{op}} \times \mathbf{Ho } \mathcal{M} \rightarrow \mathbf{Ho sSet}$ . ■

**Proposition 5.5.3.** Let  $\mathcal{M}$  and  $\mathcal{N}$  be model categories and let

$$F \dashv G : \mathcal{N} \rightarrow \mathcal{M}$$

be a Quillen adjunction.

- If  $S$  is a class of morphisms in  $\mathcal{M}$  and  $F : \mathcal{M} \rightarrow \mathcal{N}$  sends every morphism that is in  $S$  to a weak equivalence in  $\mathcal{N}$ , then  $G : \mathcal{N} \rightarrow \mathcal{M}$  sends every fibrant object in  $\mathcal{N}$  to an  $S$ -local object in  $\mathcal{M}$ .
- If  $\mathcal{T}$  is a class of morphisms in  $\mathcal{N}$  and  $G : \mathcal{N} \rightarrow \mathcal{M}$  sends every morphism that is in  $\mathcal{T}$  to a weak equivalence in  $\mathcal{M}$ , then  $F : \mathcal{M} \rightarrow \mathcal{N}$  sends every cofibrant object in  $\mathcal{M}$  to a  $\mathcal{T}$ -colocal object in  $\mathcal{N}$ .

*Proof.* The two claims are formally dual; we will prove the first version.

By [proposition 4.3.4](#),  $G$  sends fibrant objects in  $\mathcal{N}$  to fibrant objects in  $\mathcal{M}$ , and by [theorem 4.8.28](#), we have natural isomorphism in  $\text{Ho sSet}$  of the form below:

$$\mathbf{RHom}_{\mathcal{N}}((\mathbf{L}F)Z, B) \cong \mathbf{RHom}_{\mathcal{M}}(Z, (\mathbf{R}G)B)$$

[Proposition 4.8.23](#) then implies  $(\mathbf{R}G)B$  is an  $S$ -local object when  $B$  is a fibrant object in  $\mathcal{N}$ . ■

**Definition 5.5.4.** Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .

- An  **$S$ -local equivalence** in  $\mathcal{M}$  is a morphism  $g : Z \rightarrow W$  in  $\mathcal{M}$  such that the induced morphism

$$\mathbf{RHom}_{\mathcal{M}}(g, X) : \mathbf{RHom}_{\mathcal{M}}(W, X) \rightarrow \mathbf{RHom}_{\mathcal{M}}(Z, X)$$

is an isomorphism in  $\text{Ho sSet}$  for all  $S$ -local objects  $X$  in  $\mathcal{M}$ .

- An  **$S$ -colocal equivalence** in  $\mathcal{M}$  is a morphism  $f : X \rightarrow Y$  in  $\mathcal{M}$  such that the induced morphism

$$\mathbf{RHom}_{\mathcal{M}}(W, f) : \mathbf{RHom}_{\mathcal{M}}(W, X) \rightarrow \mathbf{RHom}_{\mathcal{M}}(W, Y)$$

is an isomorphism in  $\text{Ho sSet}$  for all  $S$ -colocal objects  $W$  in  $\mathcal{M}$ .

**REMARK 5.5.5.** For any given model category  $\mathcal{M}$  and any class  $S$  of morphisms in  $\mathcal{M}$ , the class of  $S$ -local equivalences (resp.  $S$ -colocal equivalences) is saturated (by [lemma 3.1.8](#)). Note that every morphism that is in  $S$  is automatically an  $S$ -local equivalence (resp.  $S$ -colocal equivalence), as is every weak equivalence in  $\mathcal{M}$ .

**Lemma 5.5.6.** *Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .*

- *A morphism between  $S$ -local objects in  $\mathcal{M}$  is an  $S$ -local equivalence if and only if it is a weak equivalence in  $\mathcal{M}$ .*
- *A morphism between  $S$ -colocal objects in  $\mathcal{M}$  is an  $S$ -colocal equivalence if and only if it is a weak equivalence in  $\mathcal{M}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

It is clear that every weak equivalence in  $\mathcal{M}$  is an  $S$ -local equivalence. Conversely, suppose  $g : Z \rightarrow W$  is an  $S$ -local equivalence between  $S$ -local objects in  $\mathcal{M}$ . [Proposition 4.8.17](#) then implies that  $g : Z \rightarrow W$  induces bijections

$$\mathrm{Ho} \mathcal{M}(g, X) : \mathrm{Ho} \mathcal{M}(W, X) \rightarrow \mathrm{Ho} \mathcal{M}(Z, X)$$

for all  $S$ -local objects  $X$ , so  $g : Z \rightarrow W$  must be an isomorphism in  $\mathrm{Ho} \mathcal{M}$ . Since  $\mathcal{M}$  is a saturated homotopical category (by [theorem 4.4.1](#)), we may then deduce that  $g : Z \rightarrow W$  is a weak equivalence in  $\mathcal{M}$ .  $\blacksquare$

**Definition 5.5.7.** Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$  and let  $X$  be an object in  $\mathcal{M}$ .

- An  **$S$ -local replacement** for  $X$  is a pair  $(\hat{X}, i)$  where  $\hat{X}$  is an  $S$ -local object in  $\mathcal{M}$  and  $i : X \rightarrow \hat{X}$  is an  $S$ -local equivalence.
- An  **$S$ -colocal replacement** for  $X$  in  $\mathcal{M}$  is a pair  $(\tilde{X}, p)$  where  $\tilde{X}$  is an  $S$ -colocal object in  $\mathcal{M}$  and  $p : \tilde{X} \rightarrow X$  is an  $S$ -colocal equivalence.
- A **cofibrant  $S$ -local replacement** for  $X$  in  $\mathcal{M}$  is an  $S$ -local replacement  $(\hat{X}, i)$  where  $i : X \rightarrow \hat{X}$  is cofibration in  $\mathcal{M}$  (and also an  $S$ -local equivalence).
- A **fibrant  $S$ -colocal replacement** for  $X$  in  $\mathcal{M}$  is an  $S$ -colocal replacement  $(\tilde{X}, p)$  where  $p : \tilde{X} \rightarrow X$  is a fibration in  $\mathcal{M}$  (and also an  $S$ -local equivalence).

**Proposition 5.5.8.** *Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .*

- *If every cofibrant–fibrant object in  $\mathcal{M}$  admits an  $S$ -local replacement, then the full subcategory of  $\mathrm{Ho} \mathcal{M}$  spanned by the  $S$ -local objects is a reflective subcategory of  $\mathrm{Ho} \mathcal{M}$ .*
- *If every cofibrant–fibrant object in  $\mathcal{M}$  admits an  $S$ -colocal replacement, then the full subcategory of  $\mathrm{Ho} \mathcal{M}$  spanned by the  $S$ -colocal objects is a coreflective subcategory of  $\mathrm{Ho} \mathcal{M}$ .*

*Proof.* Let  $\mathcal{L}$  be the full subcategory of  $\mathrm{Ho} \mathcal{M}$  spanned by the  $S$ -local objects. If  $g : Z \rightarrow W$  is an  $S$ -local equivalence between  $S$ -local objects in  $\mathcal{M}$ , then [proposition 4.8.17](#) implies that  $g : Z \rightarrow W$  induces bijections

$$\mathrm{Ho} \mathcal{M}(g, X) : \mathrm{Ho} \mathcal{M}(W, X) \rightarrow \mathrm{Ho} \mathcal{M}(Z, X)$$

for all  $S$ -local objects  $X$ . Thus, for any  $S$ -local replacement  $(\hat{Z}, i)$ , we have natural bijections

$$\mathrm{Ho} \mathcal{M}(\hat{Z}, X) \cong \mathrm{Ho} \mathcal{M}(Z, X)$$

where  $X$  varies among the  $S$ -local objects in  $\mathrm{Ho} \mathcal{M}$ . [Proposition 4.1.23](#) implies every object in  $\mathcal{M}$  is weakly equivalent to a cofibrant–fibrant object, so this implies that the inclusion  $\mathcal{L} \hookrightarrow \mathrm{Ho} \mathcal{M}$  has a left adjoint, as required.  $\blacksquare$

**Lemma 5.5.9.** *Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .*

- *If  $\mathcal{M}$  is a left proper model category, then every  $S$ -local object is injective with respect to the class of cofibrations in  $\mathcal{M}$  that are also  $S$ -local equivalences.*
- *If  $\mathcal{M}$  is a right proper model category, then every  $S$ -colocal object is projective with respect to the class of fibrations in  $\mathcal{M}$  that are also  $S$ -colocal equivalences.*

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $i : Z \rightarrow W$  be a morphism in  $\mathcal{M}$  that is a cofibration and also an  $S$ -local equivalence. Using axiom CM5 and [theorem 4.5.20](#), we may choose cosimplicial resolutions  $(\tilde{Z}^\bullet, p_Z^\bullet)$  and  $(\tilde{W}^\bullet, p_W^\bullet)$  for  $Z$  and  $W$  (respectively) and a Reedy cofibration  $\tilde{i}^\bullet : \tilde{Z}^\bullet \rightarrow \tilde{W}^\bullet$  such that the following diagram in  $\mathcal{M}$  commutes:

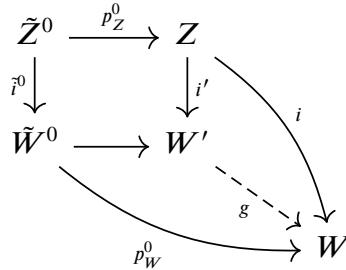
$$\begin{array}{ccc} \tilde{Z}^0 & \xrightarrow{p_Z^0} & Z \\ \tilde{i}^0 \downarrow & & \downarrow i \\ \tilde{W}^0 & \xrightarrow{p_W^0} & W \end{array}$$

Let  $X$  be an  $S$ -local object. We wish to show that the unique morphism  $X \rightarrow 1$  has the right lifting property with respect to  $i : Z \rightarrow W$ . Since  $i : Z \rightarrow W$  is an  $S$ -local equivalence, the induced morphism of left homotopy function complexes

$$\mathrm{Hom}_{\mathcal{M}}(\tilde{i}, X) : \mathrm{Hom}_{\mathcal{M}}(\tilde{W}, X) \rightarrow \mathrm{Hom}_{\mathcal{M}}(\tilde{Z}, X)$$

is a homotopy equivalence of Kan complexes; but since  $\tilde{i}^\bullet : \tilde{Z}^\bullet \rightarrow \tilde{W}^\bullet$  is a Reedy cofibration between cosimplicial resolutions in  $\mathcal{M}$  and  $X$  is a fibrant object in  $\mathcal{M}$ , [lemma 4.8.16](#) says the unique morphism  $X \rightarrow 1$  has the right lifting property with respect to  $\tilde{i}^0 : \tilde{Z}^0 \rightarrow \tilde{W}^0$ .

Now, suppose the square in the diagram below is a pushout diagram in  $\mathcal{M}$ :



Since  $\mathcal{M}$  is a left proper model category, axiom CM2 implies  $g : W' \rightarrow W$  is a weak equivalence in  $\mathcal{M}$ . Proposition A.3.14 says the unique morphism  $X \rightarrow 1$  also has the right lifting property with respect to  $i' : Z \rightarrow W'$ , so we may use lemma 4.1.19 to deduce that  $X \rightarrow 1$  has the right lifting property with respect to  $i : Z \rightarrow W$ . ■

**Definition 5.5.10.** Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .

- The **left Bousfield localisation** of  $\mathcal{M}$  with respect to  $S$  is a model category  $\mathbf{L}_S\mathcal{M}$  whose underlying category and cofibrations are the same as  $\mathcal{M}$  and whose weak equivalences are the  $S$ -local equivalences.
- The **right Bousfield localisation** of  $\mathcal{M}$  with respect to  $S$  is a model category  $\mathbf{R}_S\mathcal{M}$  whose underlying category and fibrations are the same as  $\mathcal{M}$  and whose weak equivalences are the  $S$ -colocal equivalences.

REMARK 5.5.11. The left (resp. right) Bousfield localisation of  $\mathcal{M}$  with respect to  $S$  is unique *if it exists*, by theorem 4.1.12. Note that the theorem also implies that the trivial fibrations in  $\mathbf{L}_S\mathcal{M}$  (resp. trivial cofibrations in  $\mathbf{R}_S\mathcal{M}$ ) are the same as the trivial fibrations (resp. trivial cofibrations) in  $\mathcal{M}$ .

**Proposition 5.5.12.** *Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .*

- *If the left Bousfield localisation  $\mathbf{L}_S\mathcal{M}$  exists, then the trivial adjunction*

$$\text{id} \dashv \text{id} : \mathbf{L}_S\mathcal{M} \rightarrow \mathcal{M}$$

*is a Quillen adjunction, and the right derived functor  $\text{Ho } \mathbf{L}_S\mathcal{M} \rightarrow \text{Ho } \mathcal{M}$  is fully faithful.*

- If the right Bousfield localisation  $\mathbf{R}_S\mathcal{M}$  exists, then the trivial adjunction

$$\mathrm{id} \dashv \mathrm{id} : \mathcal{M} \rightarrow \mathbf{R}_S\mathcal{M}$$

is a Quillen adjunction, and the left derived functor  $\mathrm{Ho} \mathbf{R}_S\mathcal{M} \rightarrow \mathrm{Ho} \mathcal{M}$  is fully faithful.

*Proof.* The two claims are formally dual; we will prove the first version.

By definition,  $\mathrm{id} : \mathcal{M} \rightarrow \mathbf{L}_S\mathcal{M}$  preserves cofibrations, and since every weak equivalence in  $\mathcal{M}$  is an  $S$ -local equivalence,  $\mathrm{id} : \mathcal{M} \rightarrow \mathbf{L}_S\mathcal{M}$  also preserves trivial cofibrations. [Proposition 4.3.2](#) then says the trivial adjunction is indeed a Quillen adjunction. Recalling [proposition A.1.3](#), we use [theorem 4.3.11](#) and [proposition 3.3.24](#) to deduce that the right derived functor is fully faithful. ■

**Proposition 5.5.13.** *Let  $S$  be a class of morphisms in a model category  $\mathcal{M}$ .*

- If the left Bousfield localisation  $\mathbf{L}_S\mathcal{M}$  exists, then fibrant objects in  $\mathbf{L}_S\mathcal{M}$  are  $S$ -local objects in  $\mathcal{M}$ ; in addition, if  $\mathcal{M}$  is left proper, then  $S$ -local objects in  $\mathcal{M}$  are fibrant objects in  $\mathbf{L}_S\mathcal{M}$ .
- If the right Bousfield localisation  $\mathbf{R}_S\mathcal{M}$  exists, then cofibrant objects in  $\mathbf{R}_S\mathcal{M}$  are  $S$ -colocal objects in  $\mathcal{M}$ ; in addition, if  $\mathcal{M}$  is right proper, then  $S$ -colocal objects in  $\mathcal{M}$  are cofibrant objects in  $\mathbf{R}_S\mathcal{M}$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Consider the Quillen adjunction of [proposition 5.5.12](#):

$$\mathrm{id} \dashv \mathrm{id} : \mathbf{L}_S\mathcal{M} \rightarrow \mathcal{M}$$

Since every weak equivalence in  $\mathcal{M}$  is an  $S$ -local equivalence, [proposition 5.5.3](#) says every fibrant object in  $\mathbf{L}_S\mathcal{M}$  is an  $S$ -local object in  $\mathcal{M}$ . Conversely, if  $\mathcal{M}$  is left proper, then [lemma 5.5.9](#) says every  $S$ -local object in  $\mathcal{M}$  is a fibrant object in  $\mathbf{L}_S\mathcal{M}$ . ■

**Theorem 5.5.14** (Existence of left Bousfield localisations). *Let  $\mathcal{M}$  be a left proper cellular model category and let  $S$  be a set of morphisms in  $\mathcal{M}$ .*

- (i) *The left Bousfield localisation  $\mathbf{L}_S\mathcal{M}$  exists.*
- (ii)  *$\mathbf{L}_S\mathcal{M}$  is a left proper cellular model category.*



(iii) *If  $\mathcal{M}$  is the underlying model category of a simplicial model category  $\underline{\mathcal{M}}$ ,<sup>[5]</sup> then the model structure on  $\mathbf{L}_S\mathcal{M}$  defines a simplicial model structure on  $\underline{\mathcal{M}}$  as well.*

*Proof.* See Theorem 4.1.1 in [Hirschhorn, 2003].

□

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[5] See §2.5.



## QUASICATEGORIES

Quasicategories were first defined by Boardman and Vogt [BV] as simplicial classes that satisfy the “restricted Kan condition”. The modern name is due to Joyal [2002], who worked out much of the basic theory.

As the word itself suggests, a quasicategory is a structure that is like a category. More precisely, it is a model for an  $(\infty, 1)$ -category, i.e. a weak higher category with  $n$ -morphisms for all  $n \geq 0$ , such that every  $n$ -morphism with  $n > 1$  is (weakly) invertible; alternatively, one may think of quasicategories as being homotopy-coherent categories, i.e. a structure which is like a category but only up to a specified, coherent system of homotopies.

### 6.1 Basics

**Prerequisites.** §§0.1, 1.1, 1.2, A.2.

In this section we use the explicit universe convention.

**Definition 6.1.1.** An **inner horn** is a simplicial subset of the form  $\Lambda_k^n \subseteq \Delta^n$ , where  $n \geq 2$  and  $0 < k < n$ , where  $\Lambda_k^n$  is the union of the faces of  $\Delta^n$  that include the  $k$ -th vertex. (See also [definition 1.3.19.](#))

**Definition 6.1.2.** A **quasicategory** is a simplicial set  $X$  such that the unique morphism  $X \rightarrow 1$  has the right lifting property with respect to all inner horn inclusions.

¶ 6.1.3. Quasicategories are also called  **$\infty$ -categories** (by e.g. Lurie [HTT]) or **weak Kan complexes** (by e.g. Cordier and Porter [1986]). We will usually use *bold upright* calligraphic letters to denote quasicategories, e.g.  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ . As

with ‘category’, the word ‘quasicategory’ always means a quasicategory that is not necessarily small, even when we are using the implicit universe convention.

**Proposition 6.1.4.** *Let  $\mathcal{C}$  be a category and let  $\mathbf{N}(\mathcal{C})$  be its nerve. For  $n \geq 2$  and  $0 < k < n$ , the unique morphism  $\mathbf{N}(\mathcal{C}) \rightarrow 1$  is right orthogonal to the inner horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ ; in particular,  $\mathbf{N}(\mathcal{C})$  is a quasicategory.*

*Proof.* This is a straightforward exercise using induction on  $n$ . ◇

¶ 6.1.5. We will often refer to vertices of a quasicategory as **objects**, and edges as **morphisms**. The **domain** of a morphism  $f$  in a quasicategory is the object  $d_1(f)$ , and the **codomain** of  $f$  is the object  $d_0(f)$ . An **identity morphism** is a degenerate edge; we define  $f : x \rightarrow y$  to mean that  $x$  is the domain of  $f$  and  $y$  is the codomain of  $f$ . The **identity morphism** of an object  $x$  in a quasicategory is the degenerate edge  $s_0(x)$ , which we also denote by  $\mathrm{id}_x$ . Note that all this terminology is compatible with the identification of categories  $\mathcal{C}$  with their nerves  $\mathbf{N}(\mathcal{C})$ .

Also observe that there is an automorphism  $(-)^{\mathrm{op}} : \mathbf{\Delta} \rightarrow \mathbf{\Delta}$  that sends coface maps  $\delta_n^i : [n-1] \rightarrow [n]$  to  $\delta_n^{n-i} : [n-1] \rightarrow [n]$  and codegeneracy maps  $\sigma_n^i : [n] \rightarrow [n+1]$  to  $\sigma_n^{n-i} : [n] \rightarrow [n+1]$  for all  $n \geq 0$  and  $0 \leq i \leq n$ . This in turn induces an automorphism on the category of simplicial sets.

**Definition 6.1.6.** The **opposite** of a simplicial set  $X$  is the simplicial set  $X^{\mathrm{op}}$  obtained by composing  $X : \mathbf{\Delta}^{\mathrm{op}} \rightarrow \mathbf{Set}$  with  $(-)^{\mathrm{op}} : \mathbf{\Delta} \rightarrow \mathbf{\Delta}$ .

REMARK 6.1.7. It is not hard to check that a simplicial set  $X$  is a quasicategory if and only if the simplicial set  $X^{\mathrm{op}}$  is a quasicategory.

**Definition 6.1.8.** Let  $f_0$  and  $f_1$  be a parallel pair of morphisms in a quasicategory.

- We say  $f_0$  and  $f_1$  are **left homotopic** if there exists a 2-simplex  $\alpha$  such that  $d_1(\alpha) = f_0$ ,  $d_0(\alpha) = f_1$ , and  $d_2(\alpha) = s_0(d_0(f_0))$ .
- We say  $f_0$  and  $f_1$  are **right homotopic** if there exists a 2-simplex  $\alpha$  such that  $d_2(\alpha) = f_0$ ,  $d_1(\alpha) = f_1$ , and  $d_0(\alpha) = s_0(d_0(f_0))$ .
- We say  $f_0$  and  $f_1$  are **homotopic** if they are both left and right homotopic, and we write  $f_0 \sim f_1$  in this case.

Obviously, two edges are left homotopic in a quasicategory  $\mathcal{C}$  if and only if they are right homotopic in  $\mathcal{C}^{\text{op}}$ . In fact:

**Lemma 6.1.9.** *Let  $f_0$  and  $f_1$  be a parallel pair of morphisms in a quasicategory  $\mathcal{C}$ . The following are equivalent:*

- (i)  $f_0$  and  $f_1$  are left homotopic.
- (ii)  $f_0$  and  $f_1$  are right homotopic.
- (iii)  $f_0$  and  $f_1$  are homotopic.

*Proof.* (i)  $\Leftrightarrow$  (ii). By duality, it suffices to show that (i)  $\Rightarrow$  (ii). Let  $\alpha$  be a 2-simplex of  $\mathcal{C}$  such that  $d_1(\alpha) = f$ ,  $d_0(\alpha) = f'$ , and  $d_2(\alpha) = s_0(d_0(f))$ . Using the right lifting property of  $\mathcal{C} \rightarrow 1$  with respect to the inner horn inclusion  $\Lambda_1^3 \rightarrow \mathcal{C}$ , it is straightforward to obtain a 3-simplex  $\xi$  such that  $d_2(\xi) = \alpha$ ,  $d_3(\xi) = s_0(f_1)$ , and  $d_0(\xi) = s_1(f_1)$ ; thus the 2-simplex  $d_1(\xi)$  is the required witness for the claim that  $f_0$  and  $f_1$  are right homotopic.

(i) and (ii)  $\Leftrightarrow$  (iii). This is by definition. ■

**Lemma 6.1.10.** *Let  $\mathcal{C}$  be a quasicategory. The relation of homotopy is an equivalence relation on the set of edges of  $\mathcal{C}$ .*

*Proof.* See Proposition 1.2.3.5 in [HTT], or Lemma 4.11 in [BV]. □

**Definition 6.1.11.** The **homotopy category** of a quasicategory  $\mathcal{C}$  is the category  $\text{Ho } \mathcal{C}$  defined below:

- The objects are the objects in  $\mathcal{C}$ .
- A morphism  $x \rightarrow y$  is a homotopy class of morphisms  $f : x \rightarrow y$  in  $\mathcal{C}$ .
- The identity morphism  $x \rightarrow x$  is the homotopy class of the morphism  $\text{id}_x$ .
- Composition is induced by the existence of fillers for the inner horn  $\Lambda_1^2$ : if  $\alpha$  is a 2-simplex of  $\mathcal{C}$ , then we have  $d_0(\alpha) \circ d_2(\alpha) = d_1(\alpha)$ .

**Lemma 6.1.12.** *The above construction is indeed a category.*

*Proof.* See Proposition 1.2.3.8 in [HTT]. □

**Definition 6.1.13.** Let  $\mathbf{U}$  be a universe. A  **$\mathbf{U}$ -small quasicategory** is a quasicategory whose underlying simplicial set is  $\mathbf{U}$ -small.

**Proposition 6.1.14.** Let  $\mathbf{U}$  be a universe, let  $\mathbf{sSet}$  be the category of simplicial  $\mathbf{U}$ -sets, and let  $\mathbf{Cat}$  be the category of  $\mathbf{U}$ -small categories.

- (i) The functor  $\mathbf{N} : \mathbf{Cat} \rightarrow \mathbf{sSet}$  that sends a  $\mathbf{U}$ -small category  $\mathcal{C}$  to its nerve  $\mathbf{N}(\mathcal{C})$  has a left adjoint  $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$  that sends a simplicial  $\mathbf{U}$ -set  $X$  to its fundamental category  $\tau_1 X$ .
- (ii) The functor  $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$  preserves finite products.
- (iii) For each quasicategory  $\mathcal{C}$ , there is a canonical isomorphism  $\tau_1 \mathcal{C} \cong \mathbf{Ho} \mathcal{C}$ .

*Proof.* Claims (i) and (ii) were previously proven in [proposition 1.2.1](#), and claim (iii) is essentially a consequence of the fact that  $\tau_1 X$  can be presented explicitly in terms of generators and relations as in [remark 1.2.3](#).  $\square$

$\P$  6.1.15. Henceforth, we will regard all ordinary categories as quasicategories by implicitly identifying a category  $\mathcal{C}$  with its nerve  $\mathbf{N}(\mathcal{C})$ . Continuing the terminological conventions in paragraph [6.1.5](#), we now define functors and natural transformations in the context of quasicategories.

**Definition 6.1.16.** A **functor** between quasicategories is a morphism of simplicial sets whose domain and codomain are quasicategories.

Recall that [theorem A.2.22](#) implies that the category of simplicial  $\mathbf{U}$ -sets is cartesian closed for all universes  $\mathbf{U}$ . For brevity, we will identify morphisms  $X \rightarrow Y$  with vertices of the exponential object  $[X, Y]$ ; thus, a functor  $\mathcal{C} \rightarrow \mathcal{D}$  will be both a morphism between simplicial sets and an vertex in  $[\mathcal{C}, \mathcal{D}]$ .

**Definition 6.1.17.** Let  $f_0, f_1 : \mathcal{C} \rightarrow \mathcal{D}$  be functors between quasicategories.

- A **natural transformation**  $\alpha : f_0 \Rightarrow f_1$  is an edge  $\alpha : f_0 \rightarrow f_1$  in the exponential object  $[\mathcal{C}, \mathcal{D}]$ .
- Two natural transformations are **homotopic** if they are *isomorphic* in the fundamental category  $\tau_1[\mathcal{C}, \mathcal{D}]$ .

**REMARK 6.1.18.** It is a fact that the exponential object  $[X, Y]$  is a quasicategory when  $Y$  is quasicategory: see [corollary 6.2.12](#). Thus the fundamental category  $\tau_1[\mathcal{C}, \mathcal{D}]$  can be computed using the homotopy category construction.

**Definition 6.1.19.** Let  $\mathcal{C}$  be a quasicategory. An **equivalence** in  $\mathcal{C}$  is a morphism  $f$  whose homotopy class is invertible in  $\mathrm{Ho} \mathcal{C}$ , and a **quasi-inverse** for  $f$  is a morphism in  $\mathcal{C}$  whose homotopy class is an inverse for (the homotopy class of)  $f$  in  $\mathrm{Ho} \mathcal{C}$ .

One of the requirements for a model of the theory of  $(\infty, 1)$ -categories is that the groupoid-like instances should be models of  $\infty$ -groupoids. If by ‘ $\infty$ -groupoid’ one means a (weak) homotopy type of Kan complexes, then the following result is relevant:

**Proposition 6.1.20.** *Let  $\mathcal{C}$  be a quasicategory. The following are equivalent:*

- (i)  $\mathcal{C}$  (as a simplicial set) is a Kan complex.
- (ii) Every morphism in  $\mathcal{C}$  is an equivalence.
- (iii)  $\mathrm{Ho} \mathcal{C}$  is a groupoid.

*Proof.* See Corollary 1.5 in [Joyal, 2002]. □

There is also a homotopy-coherent notion of equivalence. Let  $\mathbf{I}\mathbb{2}$  be the groupoid obtained by freely inverting the arrows in the category  $\mathbb{2}$  freely generated by a morphism  $0 \rightarrow 1$ .

**Definition 6.1.21.** A **homotopy-coherent equivalence** in a quasicategory  $\mathcal{C}$  is a functor  $\mathbf{I}[1] \rightarrow \mathcal{C}$ .

**REMARK 6.1.22.** More explicitly, a homotopy-coherent equivalence in  $\mathcal{C}$  consists of the following data:

- A pair of objects in  $\mathcal{C}$ , say  $x$  and  $y$ .
- A pair of morphisms in  $\mathcal{C}$ , say  $f : x \rightarrow y$  and  $g : y \rightarrow x$ .
- A pair of 2-simplices, say  $\alpha$  and  $\beta$ , witnessing the fact that  $\mathrm{id}_x \sim g \circ f$  and  $f \circ g \sim \mathrm{id}_y$ .
- A pair of 3-simplices witnessing the fact that  $\alpha$  and  $\beta$  satisfy (versions of) the triangle identities for adjunctions.
- etc.

That is, for each natural number  $n$ , we have a pair of  $(n + 1)$ -simplices witnessing a coherence axiom for the given pair of  $n$ -simplices. Note that the data for  $n \leq 2$  already determine a mutually quasi-inverse pair of equivalences in  $\mathcal{C}$ ; we will refer to  $f : x \rightarrow y$  as the **underlying morphism** of the homotopy-coherent equivalence.

When  $\mathcal{C}$  is an ordinary category, the 2-simplices are unique *if* they exist, and given the 2-simplices, the required  $n$ -simplices exist and are unique for  $n \geq 2$ .

In other words, every isomorphism in an ordinary category can be equipped with the structure of a homotopy-coherent equivalence. It turns out the same is true for quasicategories:

**Proposition 6.1.23.** *Let  $\mathcal{C}$  be a quasicategory. If  $f$  is an equivalence in  $\mathcal{C}$ , then there is a homotopy-coherent equivalence whose underlying morphism is  $f$ .*

*Proof.* See Corollary 1.6 in [Joyal, 2002], or Theorem 4.14 in [TQA]. □

**Definition 6.1.24.** Let  $\mathbf{U}$  be a universe. The **homotopy 2-category of  $\mathbf{U}$ -small quasicategories** is the following 2-category  $\mathfrak{Qcat}$ :

- The objects are  $\mathbf{U}$ -small quasicategories.
- The category of morphisms  $\mathcal{C} \rightarrow \mathcal{D}$  is the fundamental category  $\tau_1[\mathcal{C}, \mathcal{D}]$ , which we also denote by  $\mathbf{QFun}(\mathcal{C}, \mathcal{D})$ .
- Composition and identity morphisms are induced by  $\tau_1$  from the cartesian closed structure of  $\mathbf{sSet}$ .

The construction of the 2-category  $\mathfrak{Qcat}$  enables us to apply definitions from formal category theory to the context of quasicategories.

**Definition 6.1.25.** Let  $f_0, f_1 : \mathcal{C} \rightarrow \mathcal{D}$  be functors between quasicategories. A **natural equivalence** is a natural transformation  $\alpha : f_0 \Rightarrow f_1$  whose image in the fundamental category  $\tau_1[\mathcal{C}, \mathcal{D}]$  is an isomorphism.

As with natural transformations of functors between ordinary categories, natural transformations of functors between quasicategories have components. It is a non-trivial fact that a natural transformation is a natural equivalence if and only if its components are equivalences:



**Theorem 6.1.26.** *Let  $f_0, f_1 : \mathcal{C} \rightarrow \mathcal{D}$  be functors between quasicategories and let  $\alpha : f_0 \Rightarrow f_1$  be a natural transformation. Then  $\alpha$  is a natural equivalence if and only if, for every object  $c$  in  $\mathcal{C}$ , the morphism  $\alpha_c : f_0(x) \rightarrow f_1(x)$  is an equivalence in  $\mathcal{D}$ .*

*Proof.* See Theorem 5.14 in [TQA], or Lemma 2.3.8 in [Riehl and Verity, 2013a] □

**Definition 6.1.27.** An **equivalence of quasicategories** is an equivalence in the 2-category  $\mathcal{Q}\mathbf{cat}$ , i.e. a tuple  $(f, g, \eta, \varepsilon)$  where:

- $f : \mathcal{C} \rightarrow \mathcal{D}$  and  $g : \mathcal{D} \rightarrow \mathcal{C}$  are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow g \circ f$  and  $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathcal{D}}$  are natural equivalences.

We will often abuse notation and say that  $f$  is an equivalence of quasicategories, omitting mention of the other data.

**Definition 6.1.28.** An **adjunction of quasicategories** is an adjunction in the 2-category  $\mathcal{Q}\mathbf{cat}$ , i.e. a tuple  $(f, g, \eta, \varepsilon)$  where:

- $f : \mathcal{C} \rightarrow \mathcal{D}$  and  $g : \mathcal{D} \rightarrow \mathcal{C}$  are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow g \circ f$  and  $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathcal{D}}$  are natural transformations.
- The triangle identities are satisfied:

$$\begin{aligned} (\varepsilon \circ \mathrm{id}_f) \bullet (\mathrm{id}_f \circ \eta) &= \mathrm{id}_f && \text{in } \mathbf{QFun}(\mathcal{C}, \mathcal{D}) \\ (\mathrm{id}_g \circ \varepsilon) \bullet (\eta \circ \mathrm{id}_g) &= \mathrm{id}_g && \text{in } \mathbf{QFun}(\mathcal{D}, \mathcal{C}) \end{aligned}$$

**REMARK 6.1.29.** There also exist homotopy-coherent versions of the above definitions, but it is a theorem of Riehl and Verity [2013b] that every adjunction of quasicategories can be extended to a homotopy-coherent adjunction.

**Lemma 6.1.30.** *Let  $\mathbf{U}$  be a universe, let  $\mathbf{Cat}$  be the category of  $\mathbf{U}$ -small categories, and let  $\mathbf{Qcat}$  be the category of  $\mathbf{U}$ -small quasicategories. The functor  $\mathrm{Ho} : \mathbf{Qcat} \rightarrow \mathbf{Cat}$  is isomorphic to (the underlying functor of) a representable 2-functor  $\mathcal{Q}\mathbf{cat} \rightarrow \mathcal{C}\mathbf{at}$ .*

*Proof.* This is an immediate consequence of the natural isomorphism  $[1, -] \cong \mathrm{id}_{\mathcal{Q}\mathbf{cat}}$  and the fact that  $\mathbf{QFun}(1, -)$  is a 2-functor  $\mathcal{Q}\mathbf{cat} \rightarrow \mathcal{C}\mathbf{at}$ . ■

## 6.2 The Joyal model structure

**Prerequisites.** §§0.5, 1.5, 4.I, 5.4, 6.I, A.2, A.4.

Just as there is a model structure on  $\mathbf{Cat}$  whose homotopy category is the category of small categories modulo natural isomorphism of functors, there is a model structure on  $\mathbf{sSet}$ , due to Joyal [TQ1], whose homotopy category is the category of small quasicategories modulo natural equivalence of functors.

¶ 6.2.I. Throughout this section,  $\tau_0$  denotes the functor  $\mathbf{sSet} \rightarrow \mathbf{Set}$  that sends a simplicial set  $X$  to the set of isomorphism classes of objects in the fundamental category  $\tau_1 X$ . Note that it can be factored as  $\pi_0 \circ \text{iso} \circ \tau_1$ , where  $\text{iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  is the *right* adjoint of the inclusion  $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$ .

**Definition 6.2.2.** A **weak categorical equivalence** is a morphism  $f : W \rightarrow Z$  of simplicial sets such that the induced map

$$\tau_0[f, \mathcal{K}] : \tau_0[Z, \mathcal{K}] \rightarrow \tau_0[W, \mathcal{K}]$$

is a bijection for all small quasicategories  $\mathcal{K}$ .

**Lemma 6.2.3.** *Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between small quasicategories. The following are equivalent:*

- (i)  $f$  is (part of) an equivalence of quasicategories.
- (ii)  $f$  is a weak categorical equivalence.
- (iii) For all small quasicategories  $\mathcal{K}$ , the induced map

$$\tau_0[\mathcal{K}, f] : \tau_0[\mathcal{K}, \mathcal{C}] \rightarrow \tau_0[\mathcal{K}, \mathcal{D}]$$

is a bijection.

*Proof.* (i)  $\Rightarrow$  (ii). It is not hard to see that  $f : \mathcal{C} \rightarrow \mathcal{D}$  is (part of) an equivalence of quasicategories if and only if the induced functor

$$\tau_1[f, \mathcal{K}] : \tau_1[\mathcal{D}, \mathcal{K}] \rightarrow \tau_1[\mathcal{C}, \mathcal{K}]$$

is (part of) an equivalence of categories for all small quasicategories  $\mathcal{K}$ . The functor  $\pi_0 \circ \text{iso} : \mathbf{Cat} \rightarrow \mathbf{Set}$  sends equivalences to bijections, so we may deduce that  $f : \mathcal{C} \rightarrow \mathcal{D}$  is a weak categorical equivalence.

(i)  $\Rightarrow$  (iii). The proof is similar to that of (i)  $\Rightarrow$  (ii).

(ii)  $\Rightarrow$  (i), (iii)  $\Rightarrow$  (i). These are straightforward exercises in chasing identity morphisms. ■

**Proposition 6.2.4.** *sSet with the class of weak categorical equivalences constitute a saturated relative category; in particular, the class of weak categorical equivalences has the 2-out-of-6 property.*

*Proof.* The collection of functors  $\tau_0[-, \mathcal{K}] : \mathbf{sSet} \rightarrow \mathbf{Set}$ , as  $\mathcal{K}$  varies over the small quasicategories, jointly reflect isomorphisms as weak categorical equivalences, so the class of weak categorical equivalences must be saturated. For the 2-out-of-6 property, see [corollary A.4.15](#). ■

**Definition 6.2.5.** An **inner fibration** of simplicial sets is a morphism  $f : X \rightarrow Y$  with the right lifting property with respect to the inner horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  for all  $n \geq 2$  and  $0 < i < n$ .

**REMARK 6.2.6.** It is clear that a simplicial set  $X$  is a quasicategory if and only if the unique morphism  $X \rightarrow 1$  is an inner fibration. Unfortunately, these are *not* the fibrations in the Joyal model structure.

**Definition 6.2.7.** An **inner anodyne extension** is a morphism of simplicial sets with the left lifting property with respect to all inner fibrations.

**Proposition 6.2.8.** *There exist an  $\aleph_0$ -accessible functor  $M : [2, \mathbf{sSet}] \rightarrow \mathbf{sSet}$  and two natural transformations  $i : \text{dom} \Rightarrow M$  and  $p : M \Rightarrow \text{codom}$  such that, for all objects  $f$  in  $[2, \mathbf{sSet}]$ :*

- $f = p_f \circ i_f$ .
- $i_f$  is a relative  $\mathcal{I}'$ -cell complex, where  $\mathcal{I}'$  is the set of inner horn inclusions.
- $p_f$  is an inner fibration of simplicial sets.

*Proof.* Using [proposition 0.2.45](#), it is not hard to see that the inner horn inclusions are  $\aleph_0$ -compact as objects in  $[2, \mathbf{sSet}]$ . We then apply [corollary 0.5.14](#). ■

**Corollary 6.2.9.** *There exist an  $\aleph_0$ -accessible functor  $R : \mathbf{sSet} \rightarrow \mathbf{sSet}$  and a natural transformation  $i : \text{id}_{\mathbf{sSet}} \Rightarrow R$  such that, for all objects  $X$  in  $\mathbf{sSet}$ :*

- $RX$  is a small quasicategory.
- $i_X : X \rightarrow RX$  is an inner anodyne extension. ■

**Corollary 6.2.10.** *Inner anodyne extensions are monomorphisms.*

*Proof.* The class of monomorphisms in  $\mathbf{sSet}$  is closed under coproducts, pushouts, transfinite composition, and retracts, so we may apply [corollary 0.5.13](#). ■

**Theorem 6.2.11.** *Let  $i : Z \rightarrow W$  be a monomorphism in  $\mathbf{sSet}$  and let  $p : X \rightarrow Y$  be an inner fibration. Suppose we have a commutative diagram*

$$\begin{array}{ccc}
 [W, X] & \xrightarrow{[i, X]} & [Z, X] \\
 \downarrow [W, p] & \searrow q & \downarrow [Z, p] \\
 & L(i, p) & \longrightarrow [Z, X] \\
 & \downarrow & \\
 [W, Y] & \xrightarrow{[i, Y]} & [Z, Y]
 \end{array}$$

where the square in the lower right is a pullback square.

- (i) *The unique morphism  $q : [W, X] \rightarrow L(i, p)$  making the diagram commute is an inner fibration.*
- (ii) *If  $i : Z \rightarrow W$  is an inner anodyne extension, then  $q : [W, X] \rightarrow L(i, p)$  is a trivial Kan fibration.*
- (iii) *If  $p : Z \rightarrow W$  is a trivial Kan fibration, then so is  $q : [W, X] \rightarrow L(i, p)$ .*

*Proof.* (i) and (ii). See Theorem 2.18 in [\[TQA\]](#), or Propositions 2.5 and 2.6 in [\[Dugger and Spivak, 2011a\]](#).

(iii). This is a special case of [proposition 1.4.15](#). □

**Corollary 6.2.12.**

- (i) *If  $p : X \rightarrow Y$  is an inner fibration, then for all simplicial sets  $W$ , the morphism  $[W, p] : [W, X] \rightarrow [W, Y]$  is also an inner fibration.*
- (ii) *If  $i : Z \rightarrow W$  is a monomorphism (resp. inner anodyne extension) and  $\mathcal{K}$  is a small quasicategory, then the morphism  $[i, \mathcal{K}] : [W, \mathcal{K}] \rightarrow [Z, \mathcal{K}]$  is an inner fibration (resp. trivial Kan fibration).*
- (iii) *If  $W$  is any simplicial set and  $\mathcal{K}$  is a small quasicategory, then  $[W, \mathcal{K}]$  is also a small quasicategory.*

*Proof.* The proof is similar to that of [corollary 1.4.16](#). ■

**Corollary 6.2.13.** *Qcat is an exponential ideal of sSet; in particular, Qcat is a cartesian closed category.* ■

**Proposition 6.2.14.** *Let  $f : W \rightarrow Z$  be a morphism in sSet. The following are equivalent:*

(i) *For all small quasicategories  $\mathcal{K}$ , the induced functor*

$$[f, \mathcal{K}] : [Z, \mathcal{K}] \rightarrow [W, \mathcal{K}]$$

*is (part of) an equivalence of quasicategories.*

(ii) *For all small quasicategories  $\mathcal{K}$ , the induced functor*

$$\mathrm{Ho} [f, \mathcal{K}] : \mathrm{Ho} [Z, \mathcal{K}] \rightarrow \mathrm{Ho} [W, \mathcal{K}]$$

*is (part of) an equivalence of categories.*

(iii) *The morphism  $f : W \rightarrow Z$  is a weak categorical equivalence.*

*Proof.* (i)  $\Rightarrow$  (ii). This is a corollary of [lemma 6.1.30](#).

(ii)  $\Rightarrow$  (iii). Any equivalence of categories must induce a bijection on isomorphism classes of objects.

(iii)  $\Rightarrow$  (i). Suppose  $f : W \rightarrow Z$  is a weak categorical equivalence, i.e. that the induced map

$$\tau_0 [f, \mathcal{K}] : \tau_0 [Z, \mathcal{K}] \rightarrow \tau_0 [W, \mathcal{K}]$$

is a bijection of sets for all small quasicategories  $\mathcal{K}$ . Then, for all simplicial sets  $X$  and all small quasicategories  $\mathcal{K}$ , the induced map

$$\tau_0 [f, [X, \mathcal{K}]] : \tau_0 [Z, [X, \mathcal{K}]] \rightarrow \tau_0 [W, [X, \mathcal{K}]]$$

is a bijection, because  $[X, \mathcal{K}]$  is a quasicategory by [corollary 6.2.13](#). [Proposition A.2.11](#) then implies that the induced map

$$\tau_0 [X, [f, \mathcal{K}]] : \tau_0 [X, [Z, \mathcal{K}]] \rightarrow \tau_0 [X, [W, \mathcal{K}]]$$

is a bijection for all simplicial sets  $X$  and all small quasicategories  $\mathcal{K}$ . Thus, by [lemma 6.2.3](#), the induced functor  $[f, \mathcal{K}] : [Z, \mathcal{K}] \rightarrow [W, \mathcal{K}]$  is an equivalence of quasicategories. ■

**Proposition 6.2.15.** *The class of weak categorical equivalences is closed under binary products.*

*Proof.* Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be weak categorical equivalences. Since  $f \times g = (\text{id}_Y \times g) \circ (f \times \text{id}_Z)$ , it suffices (by symmetry) to show that  $f \times \text{id}_Z : X \times Z \rightarrow Y \times Z$  is a weak categorical equivalence, i.e. that the induced map

$$\tau_0[f \times \text{id}_Z, \mathcal{K}] : \tau_0[Y \times Z, \mathcal{K}] \rightarrow \tau_0[X \times Z, \mathcal{K}]$$

is a bijection for all small quasicategories  $\mathcal{K}$ . By [proposition A.2.11](#), it is the same to show that

$$\tau_0[f, [Z, \mathcal{K}]] : \tau_0[Y, [Z, \mathcal{K}]] \rightarrow \tau_0[X, [Z, \mathcal{K}]]$$

is a bijection for all small quasicategories  $\mathcal{K}$ ; but [corollary 6.2.13](#) says that the exponential object  $[Z, \mathcal{K}]$  is a small quasicategory and  $f$  is a weak categorical equivalence, so the maps are indeed bijections. ■

**Proposition 6.2.16.** *Trivial Kan fibrations are weak categorical equivalences.*

*Proof.* See Proposition 1.22 in [\[TQA\]](#). □

**Corollary 6.2.17.** *Inner anodyne extensions are weak categorical equivalences.*

*Proof.* Let  $f : Z \rightarrow W$  be an inner anodyne extension. By [corollary 6.2.12](#), the morphism  $[f, \mathcal{K}] : [W, \mathcal{K}] \rightarrow [Z, \mathcal{K}]$  is a trivial Kan fibration for all small quasicategories  $\mathcal{K}$ ; hence, by [propositions 6.2.14](#) and [6.2.16](#),  $f : Z \rightarrow W$  is a weak categorical equivalence. ■

**REMARK 6.2.18.** It is *a priori* not clear whether the notion of weak categorical equivalence is stable under universe enlargement, but in fact it is. First, notice that the notion of weak categorical equivalence *between quasicategories* is stable under universe enlargement, by [lemma 6.2.3](#). Given any morphism  $f : X \rightarrow Y$  in  $\mathbf{sSet}$ , we may apply the functor  $R$  of [corollary 6.2.9](#) to get a commutative diagram of the form below,

$$\begin{array}{ccc} X & \xrightarrow{i_X} & RX \\ f \downarrow & & \downarrow Rf \\ Y & \xrightarrow{i_Y} & RY \end{array}$$

and [proposition 6.2.4](#) implies that the class of weak categorical equivalences has the 2-out-of-3 property, so  $f : X \rightarrow Y$  is a weak categorical equivalence if and only if  $Rf : RX \rightarrow RY$  is an equivalence of quasicategories. Since  $R$  and  $i$  are stable under universe enlargement, it follows that the property of  $f$  being a weak categorical equivalence is also stable.

**Definition 6.2.19.** An **isofibration** of quasicategories is a functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  with the following properties:

- $f$  (as a morphism of simplicial sets) is an inner fibration.
- $f$  has the right lifting property with respect to the inclusion  $\{0\} \hookrightarrow \mathbf{I2}$ .

**Proposition 6.2.20.** *Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between small quasicategories.*

- (i) *If  $f$  (as a morphism of simplicial sets) has the right lifting property with respect to all monomorphisms in  $\mathbf{sSet}$ , then  $f$  is an isofibration.*
- (ii) *If  $\mathcal{D}$  is an ordinary category, then  $f$  is an inner fibration.*
- (iii) *Assuming  $f$  (as a morphism of simplicial sets) is an inner fibration,  $f : \mathcal{C} \rightarrow \mathcal{D}$  is an isofibration if and only if  $\mathrm{Ho} f : \mathrm{Ho} \mathcal{C} \rightarrow \mathrm{Ho} \mathcal{D}$  is an isofibration.*

*Proof.* (i). This is an immediate consequence of the fact that isofibrations are morphisms that have the right lifting property with respect to certain monomorphisms in  $\mathbf{sSet}$ .

(ii). See Proposition 2.2 in [\[TQA\]](#).

(iii). See Proposition 4.5 in [\[TQA\]](#). □

**Theorem 6.2.21** (Joyal). *Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between small quasicategories. The following are equivalent:*

- (i)  *$f$  is an isofibration of quasicategories.*
- (ii)  *$f$  (as a morphism of simplicial sets) has the right lifting property with respect to all monic weak categorical equivalences in  $\mathbf{sSet}$ .*

*Proof.* See Theorem 6.11 in [\[TQA\]](#), or combine Proposition 2.2.5.8 and Corollary 2.4.6.5 in [\[HTT\]](#). □

**Theorem 6.2.22** (Joyal). *The following data constitute a cofibrantly generated model structure on  $\mathbf{sSet}$ :*

- *The weak equivalences are the weak categorical equivalences.*
- *The cofibrations are the monomorphisms.*
- *The fibrations are the morphisms that have the right lifting property with respect to monomorphisms that are weak categorical equivalences.*

*This model structure is called the **Joyal model structure for quasicategories**, and the fibrant objects are the quasicategories.*

*Proof.* See Theorem 6.12 in [TQA], or combine Proposition 2.2.5.8 with Theorems 2.2.5.1 and 2.4.6.1 in [HTT], or Theorem 2.13 in [Dugger and Spivak, 2011a]. □

REMARK 6.2.23. Joyal's determination principle (proposition 4.4.6) implies the Joyal model structure is stable under universe enlargement. Indeed, the claim is obvious for the class of cofibrations, the class of fibrant objects, and lemma 6.2.3 implies that the class of weak equivalences between fibrant objects is stable under universe enlargement; but this is enough data to uniquely determine a model structure.

**Proposition 6.2.24.** *The Joyal model structure for quasicategories is cartesian.*

*Proof.* The Joyal model structure for quasicategories is a Cisinski model structure, so we may apply proposition 5.4.12 to proposition 6.2.15 to deduce the claim. ■

**Proposition 6.2.25.** *Let  $\mathbf{Cat}$  be the category of small categories, let  $\mathbf{sSet}$  be the category of small simplicial sets, let  $\mathbf{Qcat}$  be the full subcategory spanned by the small quasicategories, and let  $\mathbf{Ho Qcat}$  be the localisation of  $\mathbf{Qcat}$  at the weak categorical equivalences.*

(i) *The adjunction*

$$\tau_1 \dashv \mathbf{N} : \mathbf{Cat} \rightarrow \mathbf{sSet}$$

*is a Quillen adjunction with respect to the canonical model structure on  $\mathbf{Cat}$  and the Joyal model structure on  $\mathbf{sSet}$ .*



(ii) *The functors  $\tau_1$  and  $N$  preserve weak equivalences, and the induced adjunction*

$$\mathrm{Ho} \tau_1 \dashv \mathrm{Ho} N : \mathrm{Ho} \mathbf{Cat} \rightarrow \mathrm{Ho} \mathbf{Qcat}$$

*exhibits  $\mathrm{Ho} \mathbf{Cat}$  as a reflective exponential ideal of  $\mathrm{Ho} \mathbf{Qcat}$ .*

*Proof.* (i). See Proposition 6.14 in [TQA].

(ii). Apply [theorem 5.4.19](#), Ken Brown's lemma ([4.3.6](#)), propositions [5.4.14](#) and [A.2.13](#), and the 2-functoriality of  $\mathrm{Ho}$  ([corollary A.4.20](#)). ■



## DERIVATORS

### 7.1 Basics

**Prerequisites.** §§3.1, 3.6, A.1, A.5.

The notion of derivator has a somewhat complicated history; the name and the original idea are due to Grothendieck [1983, 1991], but Heller [1988] studied essentially the same thing independently. The distinguishing characteristic of the theory of derivators is its agnosticism: a derivator is a way of studying homotopy-coherent diagrams and their limits/colimits without using any particular model for homotopical algebra.

In this section, we use the explicit universe convention, all 2-categories and 2-functors will be strict unless otherwise stated, and for simplicity, we say ‘coproduct’, ‘product’, ‘pullback’, etc. instead of ‘2-coproduct’, ‘2-product’, ‘2-pullback’ etc., i.e. we tacitly assume that these have the relevant 2-dimensional universal property in addition to the usual 1-dimensional universal property.

**Definition 7.1.1.** A **derivator domain** is 2-category  $\mathfrak{K}$  satisfying these axioms:

**D0.**  $\mathfrak{K}$  has an initial object  $0$ , a terminal object  $1$ , and tensors with the category  $\mathbb{2} = \{0 \rightarrow 1\}$ .

**D1.**  $\mathfrak{K}$  has finite coproducts and pullbacks.

**D2.**  $\mathfrak{K}$  has comma objects of the form  $(u \downarrow b)$  and  $(b \downarrow u)$  for all morphisms  $u : A \rightarrow B$  and  $b : 1 \rightarrow B$ .

A **subdomain** of a derivator domain is a 2-full 2-subcategory that is closed under constructions specified in the above axioms.

**Definition 7.I.2.** Let  $\mathbf{U}$  be a universe. A **U-small prederivator** on  $\mathfrak{K}$  is a 2-functor  $\mathcal{D} : \mathfrak{K}^{\text{op}} \rightarrow \mathbf{Cat}$ , where  $\mathfrak{K}$  is a derivator domain and  $\mathbf{Cat}$  is the 2-category of  $\mathbf{U}$ -small categories. A **prederivator** is a 2-functor that is a  $\mathbf{U}$ -small prederivator for some universe  $\mathbf{U}$ .

We write  $\mathcal{D}^A$  for the value of  $\mathcal{D}$  at an object  $A$  in  $\mathfrak{K}$ , and we write either  $\mathcal{D}^u$  or  $u^*$  for the functor  $\mathcal{D}^B \rightarrow \mathcal{D}^A$  induced by a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$ . If  $f : x \rightarrow y$  is a morphism in  $\mathcal{D}^B$ , then we may sometimes write  $f \upharpoonright u : x \upharpoonright u \rightarrow y \upharpoonright u$  instead of  $u^*(f) : u^*(x) \rightarrow u^*(y)$ . The **underlying category** of a prederivator  $\mathcal{D}$  is the category  $\mathcal{D}^1$ , where  $1$  is any terminal object of  $\mathfrak{K}$ .

**REMARK 7.I.3.** While it is true that  $\mathfrak{K}$  is a derivator domain if and only if  $\mathfrak{K}^{\text{co}}$  is a derivator domain, the duality principle for general prederivators is somewhat subtle: because  $(-)^{\text{op}}$  is a 2-functor  $\mathbf{Cat}^{\text{co}} \rightarrow \mathbf{Cat}$ , the opposite of a prederivator on  $\mathfrak{K}$  is a prederivator on  $\mathfrak{K}^{\text{co}}$ , which is in general not isomorphic or even equivalent to  $\mathfrak{K}$ .

One should be aware that some authors (e.g. Cisinski [2003]) define prederivators to be 2-functors  $\mathfrak{K}^{\text{coop}} \rightarrow \mathbf{Cat}$ ; readers should take care to dualise results appropriately when translating between the two conventions.

**Definition 7.I.4.** A **semiderivator** on  $\mathfrak{K}$  is prederivator  $\mathcal{D} : \mathfrak{K}^{\text{op}} \rightarrow \mathbf{Cat}$  satisfying the following axioms:

**Der1.**  $\mathcal{D}$  sends coproducts of finite families of objects in  $\mathfrak{K}$  to products in  $\mathbf{Cat}$ .

**Der2.** Let  $A$  be an object in  $\mathfrak{K}$  and let  $f : x \rightarrow y$  be a morphism in  $\mathcal{D}^A$ . Then,  $f$  is an isomorphism in  $\mathcal{D}^A$  if and only if, for all morphisms  $a : 1 \rightarrow A$  in  $\mathfrak{K}$ , the morphism  $f \upharpoonright a : x \upharpoonright a \rightarrow y \upharpoonright a$  is an isomorphism in  $\mathcal{D}^1$ .

**Example 7.I.5.** If  $B$  is an object in  $\mathfrak{K}$  and  $\mathfrak{K}$  is a locally  $\mathbf{U}$ -small 2-category, then the 2-functor  $\mathfrak{K}(-, B) : \mathfrak{K}^{\text{op}} \rightarrow \mathbf{Cat}$  is a prederivator. We say  $\mathfrak{K}(-, B)$  is the **prederivator represented by  $B$** .

**Definition 7.I.6.** Let  $C$  be a  $\mathbf{U}$ -small relative category. The **prederivator** of  $C$ , denoted by  $\mathcal{D}(C)$ , is the  $\mathbf{U}$ -small prederivator on  $\mathbf{RelCat}$  (or any subdomain thereof) defined by  $\mathcal{D}(C)^A = \text{Ho} [A, C]_{\text{h}}$ .

**Proposition 7.I.7.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ . If  $A$  is an object in  $\mathfrak{K}$  and  $\mathbb{C}$  is a category for which the tensor  $\mathbb{C} \odot A$  exists, then there is a canonical comparison functor  $\mathcal{D}^{\mathbb{C} \odot A} \rightarrow [\mathbb{C}, \mathcal{D}^A]$ .

*Proof.* By definition, the object  $\mathbb{C} \odot A$  in  $\mathfrak{K}$  induces isomorphisms

$$\mathfrak{K}(\mathbb{C} \odot A, B) \cong [\mathbb{C}, \mathfrak{K}(A, B)]$$

that are 2-natural in  $B$ . Since  $\mathcal{D}$  is a prederivator on  $\mathfrak{K}$ , it induces a functor  $\mathfrak{K}(A, B) \rightarrow [\mathcal{D}^B, \mathcal{D}^A]$  that is 2-natural in  $A$  and in  $B$ , so we obtain a 2-natural functor  $\mathfrak{K}(\mathbb{C} \odot A, B) \rightarrow [\mathbb{C}, [\mathcal{D}^B, \mathcal{D}^A]]$  by composition; but we have 2-natural isomorphisms

$$[\mathbb{C}, [\mathcal{D}^B, \mathcal{D}^A]] \cong [\mathbb{C} \times \mathcal{D}^B, \mathcal{D}^A] \cong [\mathcal{D}^B, [\mathbb{C}, \mathcal{D}^A]]$$

so, taking  $B = \mathbb{C} \odot A$ , we obtain the required functor  $\mathcal{D}^{\mathbb{C} \odot A} \rightarrow [\mathbb{C}, \mathcal{D}^A]$ . ■

**Definition 7.1.8.** A **strong semiderivator** on  $\mathfrak{K}$  is a semiderivator that satisfies the additional axiom below:

**Der5.** For any object  $A$  in  $\mathfrak{K}$ , the canonical functor  $\mathcal{D}^{2 \odot A} \rightarrow [2, \mathcal{D}^A]$  is full and essentially surjective on objects (but not necessarily faithful).

**REMARK 7.1.9.** If  $\mathcal{D}$  is the prederivator represented by an object in  $\mathfrak{K}$ , then  $\mathcal{D}$  automatically satisfies axioms Der1 and Der5; and if  $\mathfrak{K}$  is a 2-full 2-subcategory of  $\mathfrak{Cat}$  with the same terminal object, then  $\mathcal{D}$  will also satisfy axiom Der2.

**Lemma 7.1.10.** *If  $\mathcal{C}$  is a uni-fractionable category, then the canonical comparison functor  $\mathrm{Ho} [\min 2, \mathcal{C}]_{\mathrm{h}} \rightarrow [2, \mathrm{Ho} \mathcal{C}]$  is full and essentially surjective on objects.*

*Proof.* Let  $\mathcal{U}$  and  $\mathcal{V}$  be subcategories of  $\mathrm{weq} \mathcal{C}$  such that  $\mathcal{C}$  admits a three-arrow calculus with respect to  $(\mathcal{U}, \mathcal{V})$ , and let  $\bar{f} : X \rightarrow Y$  be any morphism in  $\mathrm{Ho} \mathcal{C}$ . By the fundamental theorem of three-arrow calculi (3.6.9), there exist  $u : Y \rightarrow \hat{Y}$  in  $\mathcal{U}$ ,  $v : \tilde{X} \rightarrow X$  in  $\mathcal{V}$ , and  $f : \tilde{X} \rightarrow \hat{Y}$  such that  $\bar{f} = u^{-1} \circ f \circ v^{-1}$  in  $\mathrm{Ho} \mathcal{C}$ , i.e. such that the following diagram in  $\mathrm{Ho} \mathcal{C}$  commutes:

$$\begin{array}{ccc} X & \xrightarrow{v^{-1}} & \tilde{X} \\ \bar{f} \downarrow & & \downarrow f \\ Y & \xrightarrow{u} & \hat{Y} \end{array}$$

It immediately follows that  $\mathrm{Ho} [\min 2, \mathcal{C}]_{\mathrm{h}} \rightarrow [2, \mathcal{C}]$  is essentially surjective on objects.

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It remains to be shown that  $\text{Ho}[\min \mathbb{2}, C]_{\text{h}} \rightarrow [\mathbb{2}, C]$  is a full functor. Let  $x : X_1 \rightarrow X_2$  and  $y : Y_1 \rightarrow Y_2$  be morphisms in  $C$ , let  $\bar{f}_1 : X_1 \rightarrow Y_1$  and  $\bar{f}_2 : X_2 \rightarrow Y_2$  be morphisms in  $\text{Ho } C$ , and suppose we have  $\bar{f}_2 \circ x = y \circ \bar{f}_1$ ; note this constitutes a morphism in  $[\mathbb{2}, C]$  between objects in the image of the functor  $\text{Ho}[\min \mathbb{2}, C]_{\text{h}} \rightarrow [\mathbb{2}, C]$ . As before, we may choose  $u_1 : Y_1 \rightarrow \hat{Y}_1$  and  $u_2 : Y_2 \rightarrow \hat{Y}_2$  in  $\mathcal{U}$ ,  $v_1 : \tilde{X}_1 \rightarrow X_1$  and  $v_2 : \tilde{X}_2 \rightarrow X_2$  in  $\mathcal{V}$ , and  $f_1 : \tilde{X}_1 \rightarrow \hat{Y}_1$  and  $f_2 : \tilde{X}_2 \rightarrow \hat{Y}_2$  in  $C$  such that the equations below hold in  $\text{Ho } C$ :

$$\bar{f}_1 = u_1^{-1} \circ f_1 \circ v_1^{-1} \qquad \bar{f}_2 = u_2^{-1} \circ f_2 \circ v_2^{-1}$$

Using axioms A2 and A3, there exist  $u'_2 : Y_2 \rightarrow Z$  in  $\mathcal{U}$ ,  $v'_1 : W \rightarrow X_1$  in  $\mathcal{V}$ , and  $z : \hat{Y}_1 \rightarrow Z$  and  $w : W \rightarrow \tilde{X}_2$  making the following diagrams in  $C$  commute,

$$\begin{array}{ccccc} X_1 & \xleftarrow{v_1} & \tilde{X}_1 & \xrightarrow{f_1} & \hat{Y}_1 & \xleftarrow{u_1} & Y_1 \\ \parallel & & \parallel & & \downarrow z & & \downarrow y \\ X_1 & \xleftarrow{v_1} & \tilde{X}_1 & \xrightarrow{z \circ f_1} & Z & \xleftarrow{u'_2} & Y_2 \end{array}$$

$$\begin{array}{ccccc} X_1 & \xleftarrow{v'_1} & W & \xrightarrow{f_2 \circ w} & \hat{Y}_2 & \xleftarrow{u_2} & Y_2 \\ x \downarrow & & \downarrow w & & \parallel & & \parallel \\ X_2 & \xleftarrow{v_2} & \tilde{X}_2 & \xrightarrow{f_2} & \hat{Y}_2 & \xleftarrow{u_2} & Y_2 \end{array}$$

and since  $\bar{f}_2 \circ x = y \circ \bar{f}_1$ , the fundamental theorem says there exist a commutative diagram in  $C$  of the form below,

$$\begin{array}{ccccc} X_1 & \xleftarrow{v'_1} & W & \xrightarrow{f_2 \circ w} & \hat{Y}_2 & \xleftarrow{u_2} & Y_2 \\ \parallel & & \uparrow v_3 & & \uparrow v_4 & & \parallel \\ X_1 & \xleftarrow{v_5} & \bullet & \xrightarrow{f_5} & \bullet & \xleftarrow{u_6} & Y_2 \\ \parallel & & \downarrow w_3 & & \downarrow w_4 & & \parallel \\ X_1 & \xleftarrow{v_6} & \bullet & \xrightarrow{f_6} & \bullet & \xleftarrow{u_6} & Y_2 \\ \parallel & & \uparrow u_3 & & \uparrow u_4 & & \parallel \\ X_1 & \xleftarrow{v_1} & \tilde{X}_1 & \xrightarrow{z \circ f_1} & Z & \xleftarrow{u'_2} & Y_2 \end{array}$$

where  $u_3, u_4, u_5, u_6$  are in  $\mathcal{U}$ ,  $v_3, v_4, v_5, v_6$  are in  $\mathcal{V}$ , and  $w_3, w_4$  are weak equivalences in  $C$ .

It is easy to verify that the following diagram in  $\mathcal{C}$  commutes,

$$\begin{array}{ccccccccccccccc}
X_1 & \xleftarrow{v'_1} & W & \xlongequal{\quad} & W & \xleftarrow{v_3} & \bullet & \xrightarrow{w_3} & \bullet & \xleftarrow{u_3} & \tilde{X}_1 & \xrightarrow{f_1} & \hat{Y}_1 & \xleftarrow{u_1} & Y_1 \\
x \downarrow & & w \downarrow & & f_2 \circ w \downarrow & & f_5 \downarrow & & f_6 \downarrow & & z \circ f_1 \downarrow & & z \downarrow & & y \downarrow \\
X_2 & \xleftarrow{v_2} & \tilde{X}_2 & \xrightarrow{f_2} & \hat{Y}_2 & \xleftarrow{v_4} & \bullet & \xrightarrow{w_4} & \bullet & \xleftarrow{u_4} & Z & \xlongequal{\quad} & Z & \xleftarrow{u'_2} & Y_2
\end{array}$$

and this is the required lift of  $(\bar{f}_1, \bar{f}_2)$  to  $\text{Ho}[\min 2, \mathcal{C}]_{\text{h}}$ , because the diagram in  $\mathcal{C}$  shown below commutes:

$$\begin{array}{ccccccccccccccccccc}
X_1 & \xlongequal{\quad} & X_1 & \xlongequal{\quad} & X_1 & \xlongequal{\quad} & X_1 & \xlongequal{\quad} & X_1 & \xleftarrow{v_1} & \tilde{X}_1 & \xrightarrow{f_1} & \hat{Y}_1 & \xleftarrow{u_1} & Y_1 \\
\parallel & & v'_1 \uparrow & & v'_1 \uparrow & & v_5 \uparrow & & v_6 \uparrow & & \parallel & & \parallel & & \parallel \\
X_1 & \xleftarrow{v'_1} & W & \xlongequal{\quad} & W & \xleftarrow{v_3} & \bullet & \xrightarrow{w_3} & \bullet & \xleftarrow{u_3} & \tilde{X}_1 & \xrightarrow{f_1} & \hat{Y}_1 & \xleftarrow{u_1} & Y_1 \\
x \downarrow & & w \downarrow & & f_2 \circ w \downarrow & & f_5 \downarrow & & f_6 \downarrow & & z \circ f_1 \downarrow & & z \downarrow & & y \downarrow \\
X_2 & \xleftarrow{v_2} & \tilde{X}_2 & \xrightarrow{f_2} & \hat{Y}_2 & \xleftarrow{v_4} & \bullet & \xrightarrow{w_4} & \bullet & \xleftarrow{u_4} & Z & \xlongequal{\quad} & Z & \xleftarrow{u'_2} & Y_2 \\
\parallel & & \parallel & & \parallel & & u_5 \uparrow & & u_6 \uparrow & & u'_2 \uparrow & & u'_2 \uparrow & & \parallel \\
X_2 & \xleftarrow{v_2} & \tilde{X}_2 & \xrightarrow{f_2} & \hat{Y}_2 & \xleftarrow{u_2} & Y_2 & \xlongequal{\quad} & Y_2 & \xlongequal{\quad} & Y_2 & \xlongequal{\quad} & Y_2 & \xlongequal{\quad} & Y_2
\end{array}$$

We may therefore conclude that  $\text{Ho}[\min 2, \mathcal{C}]_{\text{h}} \rightarrow [2, \mathcal{C}]$  is indeed full.  $\blacksquare$

**Proposition 7.1.11.** *Let  $\mathcal{D}$  be the prederivator of a  $\mathbf{U}$ -small relative category  $\mathcal{M}$ .*

- (i)  $\mathcal{D}$  satisfies axiom *Der1*.
- (ii) Moreover, if  $\mathcal{M}$  is a (necessarily saturated) homotopical category and each homotopical functor category  $[\mathcal{A}, \mathcal{M}]_{\text{h}}$  admits a three-arrow calculus, then  $\mathcal{D}$  is a strong semiderivator.

*Proof.* (i). [Proposition A.4.19](#) implies  $\mathcal{D}$  sends finite coproducts in  $\mathfrak{RelCat}$  to products in  $\mathfrak{Cat}^+$ , so axiom *Der1* is satisfied.

(ii). Suppose  $f : X \rightarrow Y$  is a morphism in  $\text{Ho}[\mathcal{A}, \mathcal{M}]_{\text{h}}$  such that all its components are isomorphisms in  $\text{Ho} \mathcal{M}$ . The fundamental theorem of three-arrow calculi ([3.6.9](#)) says  $f : X \rightarrow Y$  may be represented by a zigzag in  $[\mathcal{A}, \mathcal{M}]_{\text{h}}$  of the form below,

$$X \xleftarrow{\psi} \tilde{X} \xrightarrow{\theta} \hat{Y} \xleftarrow{\varphi} Y$$

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where  $\psi$  and  $\varphi$  are natural weak equivalences. Thus, if  $A$  is an object in  $\mathcal{A}$ , then the following zigzag represents an isomorphism in  $\text{Ho } \mathcal{M}$ :

$$XA \xleftarrow{\psi_A} \tilde{X}A \xrightarrow{\theta_A} \hat{Y}A \xleftarrow{\varphi_A} YA$$

However, [proposition 3.6.10](#) says  $\mathcal{M}$  is a saturated homotopical category, so  $\theta_A$  must be a weak equivalence in  $\mathcal{M}$  as well; hence,  $f : X \rightarrow Y$  is an isomorphism in  $\text{Ho } [\mathcal{A}, \mathcal{M}]_h$ . This shows that  $\mathcal{D}$  satisfies axiom Der2.

Finally, observe that  $[\min 2 \times \mathcal{A}, \mathcal{M}]_h \cong [\min 2, [\mathcal{A}, \mathcal{M}]_h]_h$ , and the hypothesis says  $[\mathcal{A}, \mathcal{M}]_h$  admits a three-arrow calculus, so we apply [lemma 7.1.10](#) to deduce that axiom Der5 is satisfied. ■

**Definition 7.1.12.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ , let  $u : A \rightarrow B$  be a morphism in  $\mathfrak{K}$ , and let  $X$  be an object in  $\mathcal{D}^A$ .

- A **left  $\mathcal{D}$ -extension** of  $X$  along  $u$  is an initial object in the comma category  $(X \downarrow u^*)$ .
- A **right  $\mathcal{D}$ -extension** of  $X$  along  $u$  is a terminal object in the comma category  $(u^* \downarrow X)$ .
- We say  $\mathcal{D}$  has **left extensions** along  $u$  if the functor  $u^* : \mathcal{D}^B \rightarrow \mathcal{D}^A$  has a left adjoint, which we denote by  $u_! : \mathcal{D}^A \rightarrow \mathcal{D}^B$ .
- We say  $\mathcal{D}$  has **right extensions** along  $u$  if the functor  $u^* : \mathcal{D}^B \rightarrow \mathcal{D}^A$  has a right adjoint, which we denote by  $u_* : \mathcal{D}^A \rightarrow \mathcal{D}^B$ .

We may refer to left and right  $\mathcal{D}$ -extensions generically as **homotopy Kan extensions** in  $\mathcal{D}$ .

**REMARK 7.1.13.** It is straightforward to check that  $\mathcal{D}$  has left (resp. right) extensions along  $u$  if and only if, for every object  $X$  in  $\mathcal{D}^A$ , there exists a left (resp. right)  $\mathcal{D}$ -extension of  $X$  along  $u$ .

**Example 7.1.14.** If  $\mathfrak{K}$  is a 2-full 2-subcategory of  $\mathfrak{Cat}$  and  $\mathcal{D}$  is the prederivator represented by an object in  $\mathfrak{K}$ , then  $\mathcal{D}$ -extensions are exactly the same thing as Kan extensions in the usual sense.

As we saw in [theorem A.5.15](#), pointwise left (resp. right) Kan extensions can be computed as colimits (resp. limits) of certain diagrams whose shapes are comma categories. We shall shortly see that more is true.



**Definition 7.1.15.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$  and suppose we have a diagram in  $\mathfrak{K}$  of the following form:

$$\begin{array}{ccc} D & \xrightarrow{q} & B \\ p \downarrow & \theta \nearrow & \downarrow v \\ A & \xrightarrow{u} & C \end{array}$$

- We say the square is a **left  $\mathcal{D}$ -exact square** if  $\mathcal{D}$  has left extensions along  $u : A \rightarrow C$  and  $q : D \rightarrow B$  and the induced diagram shown below satisfies the left Beck–Chevalley condition:

$$\begin{array}{ccc} \mathcal{D}^C & \xrightarrow{v^*} & \mathcal{D}^B \\ u^* \downarrow & \theta^* \nearrow & \downarrow q^* \\ \mathcal{D}^A & \xrightarrow{p^*} & \mathcal{D}^D \end{array}$$

- We say the square is a **right  $\mathcal{D}$ -exact square** if  $\mathcal{D}$  has right extensions along  $v : B \rightarrow C$  and  $p : D \rightarrow A$  and the induced diagram shown below satisfies the right Beck–Chevalley condition:

$$\begin{array}{ccc} \mathcal{D}^C & \xrightarrow{u^*} & \mathcal{D}^A \\ v^* \downarrow & \lrcorner \theta^* & \downarrow p^* \\ \mathcal{D}^B & \xrightarrow{q^*} & \mathcal{D}^D \end{array}$$

- A  **$\mathcal{D}$ -exact square** in  $\mathfrak{K}$  is a diagram in  $\mathfrak{K}$  that is both left  $\mathcal{D}$ -exact and right  $\mathcal{D}$ -exact.

**Proposition 7.1.16.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ . Given the following diagram in  $\mathfrak{K}$ ,

$$\begin{array}{ccc} D & \xrightarrow{q} & B \\ p \downarrow & \theta \nearrow & \downarrow v \\ A & \xrightarrow{u} & C \end{array}$$

if  $\mathcal{D}$  has left extensions along  $u : A \rightarrow C$  and  $q : D \rightarrow B$ , and  $\mathcal{D}$  has right extensions along  $v : B \rightarrow C$  and  $p : D \rightarrow A$ , then the following are equivalent:

- (i) The diagram is a  $\mathcal{D}$ -exact square.

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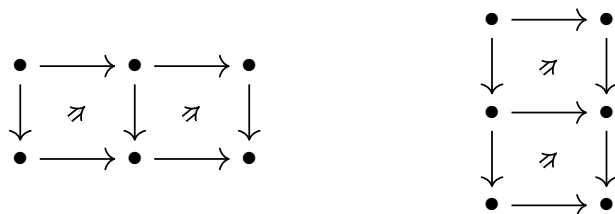
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(ii) *The diagram is a left  $\mathcal{D}$ -exact square.*

(iii) *The diagram is a right  $\mathcal{D}$ -exact square.*

*Proof.* Statement (i) is just the conjunction of statements (ii) and (iii), and when the required left and right adjoints exist, [proposition A.I.12](#) implies that statements (ii) and (iii) are equivalent. ■

**Lemma 7.1.17** (Pasting exact squares). *Let  $\mathcal{D}$  be a prederivator, and consider pasting diagrams of the following forms in  $\mathfrak{K}$ :*



*In either diagram, if both squares are left (resp. right)  $\mathcal{D}$ -exact squares, then the rectangle obtained by pasting the two squares is also a left (resp. right)  $\mathcal{D}$ -exact square.*

*Proof.* Apply [lemma A.I.11](#). ■

**Lemma 7.1.18.** *Let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets. If  $\mathcal{D}$  is the prederivator of  $\mathbf{Set}$  restricted to the subdomain  $\mathfrak{Cat}$ , then every comma square in  $\mathfrak{Cat}$  is a right  $\mathcal{D}$ -exact square.*

*Proof.* Suppose we have the following comma square in  $\mathfrak{Cat}$ :

$$\begin{array}{ccc} (u \downarrow v) & \xrightarrow{q} & \mathbb{B} \\ p \downarrow & \theta \nearrow & \downarrow v \\ \mathbb{A} & \xrightarrow{u} & \mathbb{C} \end{array}$$

Let  $Y : \mathbb{B} \rightarrow \mathbf{Set}$  be a functor and let  $(Z, \varepsilon)$  be a right Kan extension of  $Y$  along  $v$ , i.e. a terminal object in the comma category  $(v^* \downarrow Y)$ . In view of [lemma A.I.10](#), to deduce the claim, it is enough to show that  $(u^*(Z), q^*(\varepsilon) \bullet \theta_Z^*)$  is a terminal object in the comma category  $(p^* \downarrow q^*(Y))$ , i.e. a right Kan extension of  $Yq$  along  $p$ ; but this was done in [lemma A.5.8](#). ■

**Proposition 7.1.19.** *Let  $\mathcal{M}$  be a locally  $\mathbf{U}$ -small category, and let  $\mathcal{D}$  be the prederivator of  $\mathcal{M}$  restricted to  $\mathfrak{Cat}$ .*

- *If  $\mathcal{M}$  has colimits for all  $\mathbf{U}$ -small diagrams, then every comma square in  $\mathfrak{Cat}$  is a left  $\mathcal{D}$ -exact square.*
- *If  $\mathcal{M}$  has limits for all  $\mathbf{U}$ -small diagrams, then every comma square in  $\mathfrak{Cat}$  is a right  $\mathcal{D}$ -exact square.*

*Proof.* The two claims are formally dual; we will prove the first version.

Consider a comma square in  $\mathfrak{Cat}$ :

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{q} & \mathbb{B} \\ p \downarrow & \theta \nearrow & \downarrow v \\ \mathbb{A} & \xrightarrow{u} & \mathbb{C} \end{array}$$

If  $\mathcal{M}$  has colimits for all  $\mathbf{U}$ -small diagrams, then [theorem A.5.15](#) implies that, for any functor  $X : \mathbb{A} \rightarrow \mathcal{M}$ , the left Kan extension of  $X$  along  $u$  exists and is pointwise, and same is true for the left Kan extension of  $p^*(X)$  along  $q$ . Thus, for any object  $M$  in  $\mathcal{M}$ , if  $h_M : \mathcal{M}^{\text{op}} \rightarrow \mathbf{Set}^+$  is the representable functor  $\mathcal{M}(-, M)$ , we may use [lemma A.1.10](#) to deduce that the following (commutative!) diagrams satisfy the *right* Beck–Chevalley condition:

$$\begin{array}{ccc} [\mathbb{C}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{[\mathbb{C}^{\text{op}}, h_M]} & [\mathbb{C}^{\text{op}}, \mathbf{Set}^+] & & [\mathbb{B}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{[\mathbb{A}^{\text{op}}, h_M]} & [\mathbb{B}^{\text{op}}, \mathbf{Set}^+] \\ (u^{\text{op}})^* \downarrow & & \downarrow (u^{\text{op}})^* & & (q^{\text{op}})^* \downarrow & & \downarrow (q^{\text{op}})^* \\ [\mathbb{A}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{[\mathbb{A}^{\text{op}}, h_M]} & [\mathbb{A}^{\text{op}}, \mathbf{Set}^+] & & [\mathbb{D}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{[\mathbb{B}^{\text{op}}, h_M]} & [\mathbb{D}^{\text{op}}, \mathbf{Set}^+] \end{array}$$

On the other hand, [lemma 7.1.18](#) says the diagram below satisfies the right Beck–Chevalley condition,

$$\begin{array}{ccc} [\mathbb{C}^{\text{op}}, \mathbf{Set}^+] & \xrightarrow{(v^{\text{op}})^*} & [\mathbb{B}^{\text{op}}, \mathbf{Set}^+] \\ (u^{\text{op}})^* \downarrow & \swarrow_{(\theta^{\text{op}})^*} & \downarrow (q^{\text{op}})^* \\ [\mathbb{A}^{\text{op}}, \mathbf{Set}^+] & \xrightarrow{(p^{\text{op}})^*} & [\mathbb{D}^{\text{op}}, \mathbf{Set}^+] \end{array}$$

and the family  $\{h_M : \mathcal{M}^{\text{op}} \rightarrow \mathbf{Set}^+ \mid M \in \text{ob } \mathcal{M}\}$  is jointly conservative, so we deduce that the right Beck–Chevalley condition for the following diagram is sat-

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isfied,

$$\begin{array}{ccc} [\mathbb{C}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{(v^{\text{op}})^*} & [\mathbb{B}^{\text{op}}, \mathcal{M}^{\text{op}}] \\ (u^{\text{op}})^* \downarrow & \swarrow_{(\theta^{\text{op}})^*} & \downarrow (q^{\text{op}})^* \\ [\mathbb{A}^{\text{op}}, \mathcal{M}^{\text{op}}] & \xrightarrow{(p^{\text{op}})^*} & [\mathbb{D}^{\text{op}}, \mathcal{M}^{\text{op}}] \end{array}$$

and therefore this diagram satisfies the *left* Beck–Chevalley condition:

$$\begin{array}{ccc} [\mathbb{C}, \mathcal{M}] & \xrightarrow{v^*} & [\mathbb{B}, \mathcal{M}] \\ u^* \downarrow & \swarrow_{\theta^*} & \downarrow q^* \\ [\mathbb{A}, \mathcal{M}] & \xrightarrow{p^*} & [\mathbb{D}, \mathcal{M}] \end{array}$$

We then conclude that every comma square in  $\mathfrak{Cat}$  is a left  $\mathcal{D}$ -exact square.  $\blacksquare$

**Definition 7.1.20.** A  $\mathfrak{K}$ -cocomplete semiderivator is a semiderivator  $\mathcal{D}$  on  $\mathfrak{K}$  satisfying these additional axioms:

**Der3L.**  $\mathcal{D}$  has left extensions along every morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$ .

**Der4L.** Every comma square in  $\mathfrak{K}$  of the form below is a left  $\mathcal{D}$ -exact square:

$$\begin{array}{ccc} (u \downarrow c) & \longrightarrow & 1 \\ \downarrow & \swarrow & \downarrow c \\ A & \xrightarrow{u} & C \end{array}$$

Dually, a  $\mathfrak{K}$ -complete U-semiderivator is one satisfying these axioms:

**Der3R.**  $\mathcal{D}$  has right extensions along every morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$ .

**Der4R.** Every comma square in  $\mathfrak{K}$  of the form below is a right  $\mathcal{D}$ -exact square:

$$\begin{array}{ccc} (c \downarrow v) & \longrightarrow & B \\ \downarrow & \swarrow & \downarrow v \\ 1 & \xrightarrow{c} & C \end{array}$$

**Theorem 7.1.21.** *Let  $\mathbf{U}^+$  be a universe with  $\mathbf{U} \subseteq \mathbf{U}^+$ , let  $\mathcal{M}$  be a  $\mathbf{U}^+$ -small category, and let  $\mathcal{D}$  be the prederivator of  $\mathcal{M}$  restricted to  $\mathbf{Cat}$ .*

- (i)  $\mathcal{D}$  is a strong semiderivator.
- (ii)  $\mathcal{D}$  is  $\mathbf{Cat}$ -cocomplete (resp.  $\mathbf{Cat}$ -complete) if and only if  $\mathcal{M}$  is  $\mathbf{U}$ -complete (resp.  $\mathbf{U}$ -complete).

*Proof.* (i). This can be shown using the same arguments as [remark 7.1.9](#).

(ii). This is the content of [proposition 7.1.19](#). ■

Finally, we come to the definition of the subject of this chapter:

**Definition 7.1.22.** A **derivator** on  $\mathfrak{K}$  is a semiderivator that is  $\mathfrak{K}$ -cocomplete and  $\mathfrak{K}$ -complete, and a **strong derivator** is one that satisfies axiom Der5.

**REMARK 7.1.23.** The definition of ‘subdomain’ ensures that the restriction of any derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator) on  $\mathfrak{K}$  to any subdomain of  $\mathfrak{K}$  is again a derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator).

**Proposition 7.1.24.** *Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ , and let  $u \dashv v : B \rightarrow A$  be an adjunction in  $\mathfrak{K}$ , with unit  $\eta : \text{id}_A \Rightarrow v \circ u$  and counit  $\varepsilon : u \circ v \Rightarrow \text{id}_B$ .*

- (i) *We have an adjunction  $v^* \dashv u^* : \mathcal{D}^B \rightarrow \mathcal{D}^A$ , with unit  $\eta^* : \text{id}_{\mathcal{D}^A} \Rightarrow u^* \circ v^*$  and counit  $\varepsilon^* : v^* \circ u^* \Rightarrow \text{id}_{\mathcal{D}^B}$ ; in particular,  $\mathcal{D}$  has left extensions along  $u : A \rightarrow B$  and right extensions along  $v : B \rightarrow A$ .*
- (ii) *Consider the following commutative diagrams in  $\mathfrak{K}$ :*

$$\begin{array}{ccc}
 A & \xrightarrow{u} & B \\
 p \downarrow & & \downarrow q \\
 1 & \xrightarrow{\text{id}} & 1
 \end{array}
 \qquad
 \begin{array}{ccc}
 B & \xrightarrow{q} & 1 \\
 v \downarrow & & \downarrow \text{id} \\
 A & \xrightarrow{p} & 1
 \end{array}$$

*The diagram on the left is a left  $\mathcal{D}$ -exact square, and the diagram on the right is a right  $\mathcal{D}$ -exact square.*

- (iii) *Moreover, if  $\mathcal{D}$  has left extensions along  $p : A \rightarrow 1$  and  $q : A \rightarrow 1$ , then the diagram on the right is a left  $\mathcal{D}$ -exact square; and if  $\mathcal{D}$  has right extensions along  $p : A \rightarrow 1$  and  $q : A \rightarrow 1$ , then the diagram on the left is a right  $\mathcal{D}$ -exact square.*

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*Proof.* (i). Since  $\mathcal{D}$  is a 2-functor, it preserves the triangle identities; thus  $v^* \dashv u^*$  is indeed an adjunction. (The left and right adjoints are exchanged because  $\mathcal{D}$  is contravariant.)

(ii). The two halves of the claim are formally dual; we will prove the first version. By claim (i), we may take  $u_1 = v^*$ ; but the left Beck–Chevalley transformation

$$u_1 p^* \Rightarrow u_1 p^* \text{id}^* \text{id}_1 \Rightarrow u_1 u^* q^* \text{id}_1 \Rightarrow q^* \text{id}_1$$

is then equal to  $\varepsilon^* q^* : v^* u^* q^* \Rightarrow q^*$ , and  $\varepsilon^* q^* = (q\varepsilon)^* = \text{id}$ , because 1 is a terminal object in  $\mathfrak{K}$ . Thus the left Beck–Chevalley condition is satisfied.

(iii). This is a special case of [proposition 7.1.16](#). ■

**Theorem 7.1.25.** *Let  $\mathcal{D}$  be a semiderivator on  $\mathfrak{K}$  that satisfies axioms *Der3L* and *Der3R*, and let 1 be a terminal object in  $\mathfrak{K}$ . The following are equivalent:*

- (i)  $\mathcal{D}$  is a derivator.
- (ii)  $\mathcal{D}$  satisfies axiom *Der4L*.
- (iii) Every comma square in  $\mathfrak{K}$  is left  $\mathcal{D}$ -exact.
- (iv)  $\mathcal{D}$  satisfies axiom *Der4R*.
- (v) Every comma square in  $\mathfrak{K}$  is right  $\mathcal{D}$ -exact.

*Proof.* Obviously, statement (i) implies statements (ii)–(v), and the conjunction of statements (iii) and (v) implies statement (i). We are assuming that  $\mathcal{D}$  has left and right extensions along all morphisms in  $\mathfrak{K}$ , so the equivalence of statements (iii) and (v) is just [proposition 7.1.16](#). It remains to be shown that (ii)  $\Rightarrow$  (iii) and (iv)  $\Rightarrow$  (v), but the two implications are formally dual, so it is enough to prove just one; we prove the former.

Consider a general comma square in  $\mathfrak{K}$ :

$$\begin{array}{ccc} (u \downarrow v) & \xrightarrow{q} & B \\ p \downarrow & \theta \nearrow & \downarrow v \\ A & \xrightarrow{u} & C \end{array}$$

Let  $b : 1 \rightarrow B$  be a morphism in  $\mathfrak{K}$ , and let  $c = v \circ b$ , and consider the following pasting diagrams,

$$\begin{array}{ccc}
 (q \downarrow b) & \xrightarrow{t} & 1 \\
 s \downarrow & \tau \nearrow & \downarrow b \\
 (u \downarrow v) & \xrightarrow{q} & B \\
 p \downarrow & \theta \nearrow & \downarrow v \\
 A & \xrightarrow{u} & C
 \end{array}
 \qquad
 \begin{array}{ccc}
 (u \downarrow c) & \xrightarrow{r} & 1 \\
 j \downarrow & \parallel & \downarrow b \\
 (u \downarrow v) & \xrightarrow{q} & B \\
 p \downarrow & \theta \nearrow & \downarrow v \\
 A & \xrightarrow{u} & C
 \end{array}$$

where the upper square of the diagram on the left is a comma square, and the upper square of the diagram on the right is a 2-pullback square; note that the pasting lemma for comma squares implies that the outer rectangle of the diagram on the right is also a comma square.

Let  $\pi = p \circ j$  and let  $\lambda = \theta \circ \text{id}_j$ . By the universal property of comma objects, there is a unique morphism  $f : (q \downarrow b) \rightarrow (u \downarrow c)$  such that  $\pi \circ f = p \circ s$ ,  $r \circ f = t$ , and  $\lambda \circ \text{id}_f = (\text{id}_v \circ \tau) \bullet (\theta \circ \text{id}_s)$ ; and similarly there is a unique morphism  $g : (u \downarrow c) \rightarrow (q \downarrow b)$  such that  $s \circ g = j$ ,  $t \circ g = r$ , and  $\tau \circ \text{id}_g = \text{id}_{q \circ j} = \text{id}_{b \circ r}$ . Then,

$$\pi \circ (f \circ g) = p \circ s \circ g = p \circ j = \pi \qquad r \circ (f \circ g) = r$$

so  $f \circ g = \text{id}_{(u \downarrow c)}$ ; and since  $p \circ s = p \circ s \circ g \circ f$ , we may think of  $\text{id}_{p \circ s}$  as a 2-cell  $\beta : p \circ s \Rightarrow p \circ s \circ g \circ f$ , whereas  $b \circ t = q \circ s \circ g \circ f$ , so  $\tau : q \circ s \Rightarrow b \circ t$  is also a 2-cell  $\gamma : q \circ s \Rightarrow q \circ s \circ g \circ f$ , but then

$$(\theta \circ \text{id}_{s \circ g \circ f}) \bullet (\text{id}_u \circ \beta) = \theta \circ \text{id}_j \circ \text{id}_f = \lambda \circ \text{id}_f = (\text{id}_v \circ \gamma) \bullet (\theta \circ \text{id}_s)$$

so by the 2-universal property of  $(u \downarrow v)$ , there is a unique 2-cell  $\alpha : s \Rightarrow s \circ g \circ f$  such that  $\text{id}_p \circ \alpha = \beta$  and  $\text{id}_q \circ \alpha = \gamma$ ; and furthermore,

TODO: Justify this more carefully...

$$(\tau \circ \text{id}_{g \circ f}) \bullet (\text{id}_q \circ \alpha) = (\text{id}_b \circ \text{id}_{t \circ g \circ f}) \bullet \tau$$

therefore there is a unique 2-cell  $\eta : \text{id}_{(q \downarrow b)} \Rightarrow g \circ f$  such that  $\text{id}_s \circ \eta = \alpha$  and  $\text{id}_t \circ \eta = \text{id}_{t \circ g \circ f}$ .

We will now show that we have an adjunction  $f \dashv g : (u \downarrow c) \rightarrow (q \downarrow b)$  in  $\mathfrak{K}$ ; since  $f \circ g = \text{id}_{(u \downarrow c)}$ , it is enough to check that  $\text{id}_f \circ \eta = \text{id}_f$  and  $\eta \circ \text{id}_g = \text{id}_g$ . By construction,  $\text{id}_\pi \circ (\text{id}_f \circ \eta) = \text{id}_p \circ \text{id}_s \circ \eta = \text{id}_{p \circ s}$ , and  $\text{id}_r \circ (\text{id}_f \circ \eta) = \text{id}_t$ , so indeed  $\text{id}_f \circ \eta = \text{id}_f$ ; and  $\text{id}_s \circ (\eta \circ \text{id}_g) = \text{id}_s$  and  $\text{id}_t \circ (\eta \circ \text{id}_g) = \text{id}_t$ , so

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$\eta \circ \text{id}_g = \text{id}_g$  as well. Thus, by [proposition 7.1.24](#), the commutative diagram in  $\mathfrak{K}$  shown below on the left is a left  $\mathcal{D}$ -exact square,

$$\begin{array}{ccc} (u \downarrow c) & \xrightarrow{r} & 1 \\ g \downarrow & & \downarrow \text{id} \\ (q \downarrow b) & \longrightarrow & 1 \end{array} \qquad \begin{array}{ccc} (q \downarrow b) & \xrightarrow{t} & 1 \\ s \downarrow & \nearrow \tau & \downarrow b \\ (u \downarrow v) & \xrightarrow{q} & B \end{array}$$

and the diagram on the right is a left  $\mathcal{D}$ -exact square by hypothesis, so by the pasting lemma ([7.1.17](#)), the following commutative diagram is also a left  $\mathcal{D}$ -exact square:

$$\begin{array}{ccc} (u \downarrow c) & \longrightarrow & 1 \\ j \downarrow & & \downarrow b \\ (u \downarrow v) & \xrightarrow{q} & B \end{array}$$

The hypothesis also implies that this diagram satisfies the left Beck–Chevalley condition,

$$\begin{array}{ccc} \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 \\ u^* \downarrow & \nearrow \lambda^* & \downarrow r^* \\ \mathcal{D}^A & \xrightarrow{\pi^*} & \mathcal{D}^{(u \downarrow c)} \end{array}$$

but the pasting lemma ([A.1.11](#)) says that the left Beck–Chevalley transformation  $r_! \pi^* \Rightarrow c^* u_!$  is obtained by pasting together the left Beck–Chevalley transformations of the squares in the diagram below,

$$\begin{array}{ccccc} \mathcal{D}^C & \xrightarrow{v^*} & \mathcal{D}^B & \xrightarrow{b^*} & \mathcal{D}^1 \\ u^* \downarrow & \swarrow \theta^* & \downarrow q^* & \parallel & \downarrow r^* \\ \mathcal{D}^A & \xrightarrow{p^*} & \mathcal{D}^{(u \downarrow v)} & \xrightarrow{j^*} & \mathcal{D}^{(u \downarrow c)} \end{array}$$

and so, allowing  $b : 1 \rightarrow B$  to vary, we deduce that every component of the left Beck–Chevalley transformation  $v^* u_! \Rightarrow q_! p^*$  is an isomorphism in  $\mathcal{D}^1$ . We may then apply axiom Der2 to conclude that the comma square we started with is a left  $\mathcal{D}$ -exact square. ■



## 7.2 Homotopy limits and colimits

**Prerequisites.** §§3.3, 4.1, 4.6, 7.1.

¶ 7.2.1. In this section, we use the **two-universe convention**: we assume that there are two universes  $\mathbf{U}$  and  $\mathbf{U}^+$ , with  $\mathbf{U} \in \mathbf{U}^+$ . We refer to  $\mathbf{U}$ -sets,  $\mathbf{U}$ -small categories, etc. as ‘small’, and we refer to  $\mathbf{U}^+$ -sets,  $\mathbf{U}^+$ -small categories, etc. as ‘moderate’.

**Definition 7.2.2.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ , let  $A$  be an object in  $\mathfrak{K}$ , let  $1$  be a terminal object in  $\mathfrak{K}$ , let  $\Delta_A : \mathcal{D}^1 \rightarrow \mathcal{D}^A$  be the functor induced by the unique morphism  $A \rightarrow 1$  in  $\mathfrak{K}$ , and let  $X$  be an object in  $\mathcal{D}^A$ .

- A  $\mathcal{D}$ -**colimit** for  $X$  is an initial object in the comma category  $(X \downarrow \Delta_A)$ .
- A  $\mathcal{D}$ -**limit** for  $X$  is a terminal object in the comma category  $(\Delta_A \downarrow X)$ .
- We say  $\mathcal{D}$  **has colimits for diagrams of shape  $A$**  if  $\Delta_A : \mathcal{D}^1 \rightarrow \mathcal{D}^A$  has a left adjoint, which we denote by  $\text{holim}_{\rightarrow A} : \mathcal{D}^A \rightarrow \mathcal{D}^1$ .
- We say  $\mathcal{D}$  **has limits for diagrams of shape  $A$**  if  $\Delta_A : \mathcal{D}^1 \rightarrow \mathcal{D}^A$  has a right adjoint, which we denote by  $\text{holim}_{\leftarrow A} : \mathcal{D}^A \rightarrow \mathcal{D}^1$ .

We may refer to  $\mathcal{D}$ -colimits (resp.  $\mathcal{D}$ -limits) generically as **homotopy colimits** (resp. homotopy limits) in  $\mathcal{D}$ .

**REMARK 7.2.3.** Of course, homotopy colimits (resp. homotopy limits) in  $\mathcal{D}$  are a special case of homotopy left (resp. right) Kan extensions in  $\mathcal{D}$ ; in particular,  $\mathcal{D}$  has colimits (resp. limits) for diagrams of shape  $A$  if and only if, for every object  $X$  in  $\mathcal{D}^A$ , there exists a  $\mathcal{D}$ -colimit (resp.  $\mathcal{D}$ -limit) for  $X$ .

**Proposition 7.2.4.** *Let  $\mathcal{M}$  be a moderate model category and let  $\mathcal{D}$  be the prederivator of  $\mathcal{M}$  restricted along  $\text{min} : \mathfrak{Cat} \rightarrow \mathfrak{RelCat}$ .*

- (i)  $\mathcal{D}$  satisfies axiom *Der1*.
- (ii)  $\mathcal{D}$  satisfies axiom *Der5* at the terminal category  $\mathbb{1}$ , i.e. the canonical comparison functor  $\mathcal{D}^2 \rightarrow [\mathbb{2}, \mathcal{D}^1]$  is full and essentially surjective on objects.
- (iii) Moreover, if  $\mathcal{M}$  satisfies axiom *CM5\**, then  $\mathcal{D}$  is a strong semiderivator.

*Proof.* (i). [Proposition A.4.19](#) implies  $\mathcal{D}$  sends finite coproducts in  $\mathfrak{RelCat}$  to products in  $\mathfrak{Cat}^+$ , and the embedding  $\text{min} : \mathfrak{Cat} \rightarrow \mathfrak{RelCat}$  preserves finite coproducts, so axiom Der1 is satisfied.

(ii). By [theorem 4.1.30](#),  $\mathcal{M}$  admits a three-arrow calculus, so the claim follows from [lemma 7.1.10](#).

(iii). Moreover, if  $\mathcal{M}$  satisfies axiom CM5\*, then  $\mathcal{M}$  admits a *functorial* three-arrow calculus, so by [proposition 3.6.8](#), each  $[\mathbb{A}, \mathcal{M}]_h$  admits a component-wise three-arrow calculus. [Theorem 4.4.1](#) implies  $\mathcal{M}$  is a saturated homotopical category, so we deduce that  $\mathcal{D}$  is a strong semiderivator using [proposition 7.1.11](#). ■

**Theorem 7.2.5.** *If  $\mathcal{M}$  is a locally small DHK model category, then the restriction of  $\mathcal{D}(\mathcal{M})$  to  $\mathfrak{Cat}$  is a strong derivator.*

*Proof.* Let  $\mathcal{D}$  be the restriction of  $\mathcal{D}(\mathcal{M})$  to  $\mathfrak{Cat}$ . We have already shown in [proposition 7.2.4](#) that  $\mathcal{D}$  is a strong semiderivator, so it remains to be proven that  $\mathcal{D}$  is cocomplete and complete. Cocompleteness and completeness are formally dual, so it suffices to demonstrate just one half of the claim; we will show that  $\mathcal{D}$  is cocomplete.

By [theorem 4.6.16](#), for every functor  $u : \mathbb{A} \rightarrow \mathbb{B}$  between small categories, the functor  $\text{Lan}_u : [\mathbb{A}, \mathcal{M}] \rightarrow [\mathbb{B}, \mathcal{M}]$  is left deformable, so [theorem 3.3.20](#) implies the functor  $\text{Ho } u^* : \text{Ho } [\mathbb{B}, \mathcal{M}] \rightarrow \text{Ho } [\mathbb{A}, \mathcal{M}]$  has a left adjoint, namely the total left derived functor  $\mathbf{L}(\text{Lan}_u) : \text{Ho } [\mathbb{A}, \mathcal{M}] \rightarrow \text{Ho } [\mathbb{B}, \mathcal{M}]$ . Thus  $\mathcal{D}$  satisfies axiom Der3L.

Finally, to conclude, we note that [proposition 4.6.18](#) is precisely the statement that axiom Der4L is satisfied. This completes the proof that  $\mathcal{D}$  is cocomplete. ■

**Theorem 7.2.6 (Cisinski).** *Let  $\mathcal{M}$  be a locally small model category and let  $\mathcal{D}(\mathcal{M})$  be its associated prederivator. If  $\mathcal{M}$  has colimits and limits for all small diagrams, then the restriction of  $\mathcal{D}(\mathcal{M})$  to the 2-category of small categories is a derivator.*

*Proof.* See Theorem 6.11 in [Cisinski, 2003]. □

**Definition 7.2.7.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ .

- A  $\mathcal{D}$ -**cofinal morphism** is a morphism  $v : B \rightarrow A$  in  $\mathfrak{K}$  such that the diagram below is a left  $\mathcal{D}$ -exact square,

$$\begin{array}{ccc} B & \longrightarrow & 1 \\ v \downarrow & & \downarrow \text{id} \\ A & \longrightarrow & 1 \end{array}$$

i.e. such that the left Beck–Chevalley transformation

$$\text{holim}_{\longrightarrow B} \circ v^* \Rightarrow \text{holim}_{\longrightarrow A}$$

is a natural isomorphism.

- A  $\mathcal{D}$ -**coinitial morphism** is a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  such that the diagram below is a right  $\mathcal{D}$ -exact square,

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\text{id}} & 1 \end{array}$$

i.e. such that the right Beck–Chevalley transformation

$$\text{holim}_{\longleftarrow B} \Rightarrow \text{holim}_{\longleftarrow A} \circ u^*$$

is a natural isomorphism.

**Example 7.2.8.** For any derivator  $\mathcal{D}$  on  $\mathfrak{K}$ , every right adjoint (resp. left adjoint) in  $\mathfrak{K}$  is a  $\mathcal{D}$ -cofinal (resp.  $\mathcal{D}$ -coinitial) morphism: this is the content of [proposition 7.1.24](#).

**Example 7.2.9.** A category  $\mathbb{A}$  has a terminal object if and only if the unique functor  $\mathbb{A} \rightarrow \mathbb{1}$  has a right adjoint  $t : \mathbb{1} \rightarrow \mathbb{A}$ ; thus, for any derivator on  $\mathfrak{Cat}$ , if  $\mathbb{A}$  is a small category with a terminal object, then the left Beck–Chevalley transformation  $t^* \Rightarrow \text{holim}_{\longrightarrow \mathbb{A}}$  is a natural isomorphism.

**Definition 7.2.10.** Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$ . A  $\mathcal{D}$ -**equivalence** is a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  satisfying the following condition:

- For all  $X$  and  $Y$  in  $\mathcal{D}^1$ , the map  $\mathcal{D}^B(\Delta_B X, \Delta_B Y) \rightarrow \mathcal{D}^A(\Delta_A X, \Delta_A Y)$  induced by  $u^* : \mathcal{D}^B \rightarrow \mathcal{D}^A$  is a bijection.

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**Proposition 7.2.II.** *Let  $\mathcal{D}$  be a prederivator on  $\mathfrak{K}$  and let  $u : A \rightarrow B$  be a morphism in  $\mathfrak{K}$ . If  $\mathcal{D}$  is a  $\mathfrak{K}$ -cocomplete semiderivator, then the following are equivalent:*

(i) *The morphism  $u : A \rightarrow B$  is a  $\mathcal{D}$ -equivalence.*

(ii) *For  $\eta^B$  the unit of  $\text{holim}_{\rightarrow B} \dashv \Delta_B$  and  $\varepsilon^A$  the counit of  $\text{holim}_{\rightarrow A} \dashv \Delta_A$ , the natural transformation*

$$\left( \varepsilon^A \circ \text{holim}_{\rightarrow B} \circ \Delta_B \right) \bullet \left( \text{holim}_{\rightarrow A} \circ u^* \circ \eta^B \circ \Delta_B \right) : \text{holim}_{\rightarrow A} \circ \Delta_A \Rightarrow \text{holim}_{\rightarrow B} \circ \Delta_B$$

*is a natural isomorphism.*

(iii) *For  $\varepsilon^u$  the counit of  $u_! \dashv u^*$ , the natural transformation*

$$\text{holim}_{\rightarrow B} \circ \varepsilon^u \circ \Delta_B : \text{holim}_{\rightarrow A} \circ \Delta_A \Rightarrow \text{holim}_{\rightarrow B} \circ \Delta_B$$

*is a natural isomorphism.*

*Dually, if  $\mathcal{D}$  is a  $\mathfrak{K}$ -complete semiderivator, then the following are equivalent:*

(i') *The morphism  $u : A \rightarrow B$  is a  $\mathcal{D}$ -equivalence.*

(ii') *For  $\eta^A$  the unit of  $\Delta_A \dashv \text{holim}_{\leftarrow A}$  and  $\varepsilon^B$  the counit of  $\Delta_B \dashv \text{holim}_{\leftarrow B}$ , the natural transformation*

$$\left( \text{holim}_{\leftarrow A} \circ u^* \circ \varepsilon^B \circ \Delta_B \right) \bullet \left( \eta^A \circ \text{holim}_{\leftarrow B} \circ \Delta_B \right) : \text{holim}_{\leftarrow B} \circ \Delta_B \Rightarrow \text{holim}_{\leftarrow A} \circ \Delta_A$$

*is a natural isomorphism.*

(iii') *For  $\eta^u$  the unit of  $u^* \dashv u_*$ , the natural transformation*

$$\text{holim}_{\leftarrow B} \circ \eta^u \circ \Delta_B : \text{holim}_{\leftarrow B} \circ \Delta_B \Rightarrow \text{holim}_{\leftarrow A} \circ \Delta_A$$

*is a natural isomorphism.*

*Proof.* The two sets of claims are formally dual; we will prove the first version.

Observe that every morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  induces a commutative diagram of the following form:

$$\begin{array}{ccc} \mathcal{D}^1\left(\text{holim}_{\rightarrow B} \Delta_B X, Y\right) & \xrightarrow{\cong} & \mathcal{D}^B(\Delta_B X, \Delta_B Y) \\ \downarrow & & \downarrow \\ \mathcal{D}^1\left(\text{holim}_{\rightarrow A} \Delta_A X, Y\right) & \xrightarrow{\cong} & \mathcal{D}^A(\Delta_A X, \Delta_A Y) \end{array}$$

Thus, a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  satisfies condition (ii) if and only if it is a  $\mathcal{D}$ -equivalence. By factoring the counit  $\varepsilon^A : \operatorname{holim}_{\rightarrow A} \Delta_A \Rightarrow \operatorname{id}_{\mathcal{D}^1}$  in terms of the counit  $\varepsilon^u : u_! u^* \Rightarrow \operatorname{id}_{\mathcal{D}^B}$  and using the left triangle identity, we deduce that

$$\left( \varepsilon^A \circ \operatorname{holim}_{\rightarrow B} \circ \Delta_B \right) \bullet \left( \operatorname{holim}_{\rightarrow A} \circ u^* \circ \eta^B \circ \Delta_B \right) = \operatorname{holim}_{\rightarrow B} \circ \varepsilon^u \circ \Delta_B$$

and so condition (ii) is satisfied if and only if condition (iii) is satisfied.  $\blacksquare$

**Corollary 7.2.12.**

- If  $\mathcal{D}$  is a  $\mathfrak{K}$ -cocomplete semiderivator, then every  $\mathcal{D}$ -cofinal morphism in  $\mathfrak{K}$  is a  $\mathcal{D}$ -equivalence.
- If  $\mathcal{D}$  is a  $\mathfrak{K}$ -complete semiderivator, then every  $\mathcal{D}$ -coinitial morphism in  $\mathfrak{K}$  is a  $\mathcal{D}$ -equivalence.  $\blacksquare$

REMARK 7.2.13. In particular:

- If  $\mathcal{D}$  is a  $\mathfrak{K}$ -cocomplete semiderivator, then every right adjoint morphism in  $\mathfrak{K}$  is a  $\mathcal{D}$ -equivalence.
- If  $\mathcal{D}$  is a  $\mathfrak{K}$ -complete semiderivator, then every left adjoint morphism in  $\mathfrak{K}$  is a  $\mathcal{D}$ -equivalence.

**Proposition 7.2.14.** *Let  $\mathfrak{K}$  be a derivator domain and let  $\mathcal{K}$  be the underlying 1-category of  $\mathfrak{K}$ . For any prederivator  $\mathcal{D}$  on  $\mathfrak{K}$ , the category  $\mathcal{K}$  with the class of  $\mathcal{D}$ -equivalences in  $\mathfrak{K}$  constitute a saturated homotopical category.*

*Proof.* We will assume that, for every object  $A$  in  $\mathfrak{K}$ , the category  $\mathcal{D}^A$  is locally small, but there is no loss of generality in doing so because we may always enlarge the universe.

Observe that, for all objects  $X$  and  $Y$  in  $\mathcal{D}^1$ , the functor  $\mathcal{K}^{\text{op}} \rightarrow \mathbf{Set}$  defined by  $C \mapsto \mathcal{D}^C(\Delta_C X, \Delta_C Y)$  sends every  $\mathcal{D}$ -equivalence in  $\mathfrak{K}$  to a bijection. Thus, if  $u : A \rightarrow B$  is a morphism in  $\mathfrak{K}$  that becomes invertible in the localisation of  $\mathcal{K}$  at  $\mathcal{D}$ -equivalences, then for all objects  $X$  and  $Y$  in  $\mathcal{D}^1$ , the map  $\mathcal{D}^B(\Delta_B X, \Delta_B Y) \rightarrow \mathcal{D}^A(\Delta_A X, \Delta_A Y)$  induced by  $u$  must be a bijection, so  $u$  must be a  $\mathcal{D}$ -equivalence.  $\blacksquare$

**Proposition 7.2.15.** *Let  $\mathcal{D}$  be a semiderivator on  $\mathfrak{K}$ .*

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- Given a commutative triangle in  $\mathfrak{K}$  as below,

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & \searrow p & \swarrow q \\ & C & \end{array}$$

if  $\mathcal{D}$  is  $\mathfrak{K}$ -cocomplete and, for every morphism  $c : 1 \rightarrow C$  in  $\mathfrak{K}$ , the morphism  $u_c : (p \downarrow c) \rightarrow (q \downarrow c)$  induced by  $u : A \rightarrow B$  is a  $\mathcal{D}$ -equivalence, then  $u : A \rightarrow B$  is itself a  $\mathcal{D}$ -equivalence.

- Given a commutative triangle in  $\mathfrak{K}$  as below,

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & \searrow p & \swarrow q \\ & C & \end{array}$$

if  $\mathcal{D}$  is  $\mathfrak{K}$ -complete and, for every morphism  $c : 1 \rightarrow C$  in  $\mathfrak{K}$ , the morphism  ${}^c u : (c \downarrow p) \rightarrow (c \downarrow q)$  induced by  $u : A \rightarrow B$  is a  $\mathcal{D}$ -equivalence, then  $u : A \rightarrow B$  is itself a  $\mathcal{D}$ -equivalence.

*Proof.* We will use the characterisation of  $\mathcal{D}$ -equivalences afforded by [proposition 7.2.II](#). We wish to show that the natural transformation defined by the following pasting diagram is a natural isomorphism:

$$(1) \quad \begin{array}{ccccccc} \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{\text{ho lim}} & \mathcal{D}^1 & \xrightarrow{\text{id}} & \mathcal{D}^1 \\ & \searrow \text{id} & & \searrow \cong & \Delta \downarrow & \cong & \downarrow \Delta \\ & & \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{u^*} & \mathcal{D}^A \xrightarrow{\text{ho lim}} \mathcal{D}^1 \\ & & & & & & \swarrow \cong \\ & & & & & & \downarrow \Delta \\ & & & & & & \mathcal{D}^1 \end{array}$$

By factoring  $A \rightarrow 1$  and  $B \rightarrow 1$  through  $C \rightarrow 1$  and applying the left triangle identity, we see that it is enough to show that the natural transformation defined below is a natural isomorphism:

$$(2) \quad \begin{array}{ccccccc} \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{q_!} & \mathcal{D}^C & \xrightarrow{\text{id}} & \mathcal{D}^C \\ & \searrow \text{id} & & \searrow \cong & q^* \downarrow & \cong & \downarrow p^* \\ & & \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{u^*} & \mathcal{D}^A \xrightarrow{p_!} \mathcal{D}^C \\ & & & & & & \swarrow \cong \\ & & & & & & \downarrow \Delta \\ & & & & & & \mathcal{D}^1 \end{array}$$

Axiom Der4L says that the following comma square in  $\mathfrak{K}$  is left  $\mathcal{D}$ -exact,

$$\begin{array}{ccc} (p \downarrow c) & \longrightarrow & 1 \\ r \downarrow & \nearrow & \downarrow c \\ A & \xrightarrow{p} & C \end{array}$$

i.e. the left Beck–Chevalley transformation it induces is a natural isomorphism:

$$(3) \quad \begin{array}{ccccc} \mathcal{D}^A & \xrightarrow{p_!} & \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 \\ \text{id} \searrow & \nearrow p^* \downarrow & \nearrow & \downarrow \Delta & \searrow \text{id} \\ & \mathcal{D}^A & \xrightarrow{r^*} & \mathcal{D}^{(p \downarrow c)} & \xrightarrow{\text{ho lim}} & \mathcal{D}^1 \end{array}$$

Similarly, the comma square in  $\mathfrak{K}$  shown below

$$\begin{array}{ccc} (q \downarrow c) & \longrightarrow & 1 \\ s \downarrow & \nearrow & \downarrow c \\ B & \xrightarrow{q} & C \end{array}$$

induces a left Beck–Chevalley transformation that is a natural isomorphism:

$$(4) \quad \begin{array}{ccccc} \mathcal{D}^B & \xrightarrow{q_!} & \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 \\ \text{id} \searrow & \nearrow q^* \downarrow & \nearrow & \downarrow \Delta & \searrow \text{id} \\ & \mathcal{D}^B & \xrightarrow{s^*} & \mathcal{D}^{(q \downarrow c)} & \xrightarrow{\text{ho lim}} & \mathcal{D}^1 \end{array}$$

Our hypothesis is that unique morphism  $u_c : (p \downarrow c) \rightarrow (q \downarrow c)$  making the following diagram commute is a  $\mathcal{D}$ -equivalence,

$$\begin{array}{ccc} (p \downarrow c) & \xrightarrow{u_c} & (q \downarrow c) \\ r \downarrow & & \downarrow s \\ A & \xrightarrow{u} & B \end{array}$$

i.e. the natural transformation defined below is a natural isomorphism:

$$(5) \quad \begin{array}{ccccccc} \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^{(q \downarrow c)} & \xrightarrow{\text{ho lim}} & \mathcal{D}^1 & \xrightarrow{\text{id}} & \mathcal{D}^1 \\ \text{id} \searrow & & \searrow & \nearrow & \downarrow \Delta & \searrow \text{id} & \searrow \text{id} \\ & \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^{(q \downarrow c)} & \xrightarrow{(u_c)^*} & \mathcal{D}^{(p \downarrow c)} & \xrightarrow{\text{ho lim}} & \mathcal{D}^1 \end{array}$$

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However, the natural transformations defined by the following pasting diagrams are equal,

$$\begin{array}{ccc}
 \mathcal{D}^C & \xrightarrow{\text{id}} & \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 \\
 q^* \downarrow & \cong & \downarrow p^* & \cong & \downarrow \Delta \\
 \mathcal{D}^B & \xrightarrow{u^*} & \mathcal{D}^A & \xrightarrow{r^*} & \mathcal{D}^{(p \downarrow c)}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 & \xrightarrow{\text{id}} & \mathcal{D}^1 \\
 q^* \downarrow & \cong & \Delta \downarrow & \cong & \downarrow \Delta \\
 \mathcal{D}^B & \xrightarrow{s^*} & \mathcal{D}^{(q \downarrow c)} & \xrightarrow{(u_c)^*} & \mathcal{D}^{(p \downarrow c)}
 \end{array}$$

so, the natural transformation obtained by pasting together (2) and (3) is equal to the natural transformation obtained by pasting together (4) and (5); but the latter is a natural isomorphism, so we deduce that the former is a natural isomorphism as well. Thus,

$$\begin{array}{ccccccccccc}
 \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{q_!} & \mathcal{D}^C & \xrightarrow{\text{id}} & \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1 & & \\
 & \searrow \text{id} & & \searrow \cong & \searrow q^* \downarrow & \cong & \searrow \downarrow p^* & \cong & \searrow \text{id} & \cong & \searrow \text{id} \\
 & & \mathcal{D}^1 & \xrightarrow{\Delta} & \mathcal{D}^B & \xrightarrow{u^*} & \mathcal{D}^A & \xrightarrow{p_!} & \mathcal{D}^C & \xrightarrow{c^*} & \mathcal{D}^1
 \end{array}$$

defines a natural isomorphism. Since  $c : 1 \rightarrow C$  was arbitrary, we may use axiom Der2 to deduce that (1) itself defines a natural isomorphism, as claimed.  $\blacksquare$

**Lemma 7.2.16.** *Let  $\mathcal{D}$  be a semiderivator on  $\mathfrak{K}$ . Consider a diagram of the following form in  $\mathfrak{K}$ :*

$$\begin{array}{ccc}
 E & \longrightarrow & 1 \\
 \downarrow & \theta \nearrow & \downarrow b \\
 1 & \xrightarrow{a} & C
 \end{array}$$

*If  $\mathcal{D}$  is  $\mathfrak{K}$ -cocomplete (resp.  $\mathfrak{K}$ -cocomplete), then the following are equivalent:*

- (i) *The diagram above is a left  $\mathcal{D}$ -exact square (resp. right  $\mathcal{D}$ -exact square).*
- (ii) *The morphism  $w : E \rightarrow (a \downarrow b)$  induced by the universal property of  $(a \downarrow b)$  is a  $\mathcal{D}$ -equivalence.*

*Proof.* The two claims are formally dual; we will prove the first version.

By definition, the diagram above is a left  $\mathcal{D}$ -exact square if and only if the left Beck–Chevalley transformation

$$\text{holim}_{\rightarrow_E} \Delta_E \Rightarrow \text{holim}_{\rightarrow_E} \Delta_E a^* a_! \Rightarrow \text{holim}_{\rightarrow_E} \Delta_E b^* a_! \Rightarrow b^* a_!$$



is a natural isomorphism. However,  $\Delta_E = w^* \Delta_{(a \downarrow b)}$ , and axiom Der4L says the left Beck–Chevalley transformation

$$\operatorname{holim}_{\rightarrow(a \downarrow b)} \Delta_{(a \downarrow b)} \Rightarrow \operatorname{holim}_{\rightarrow(a \downarrow b)} \Delta_{(a \downarrow b)} a^* a_! \Rightarrow \operatorname{holim}_{\rightarrow(a \downarrow b)} \Delta_{(a \downarrow b)} b^* a_! \Rightarrow b^* a_!$$

is a natural isomorphism, so using the counit of the adjunction  $w_! \dashv w^*$ , [proposition 7.2.11](#), and the 2-out-of-3 property of natural isomorphisms, we may deduce that conditions (i) and (ii) are equivalent.  $\blacksquare$

**Theorem 7.2.17.** *Let  $\mathcal{D}$  be a semiderivator on  $\mathfrak{K}$ . Consider the following diagram in  $\mathfrak{K}$ :*

$$(□) \quad \begin{array}{ccc} D & \xrightarrow{q} & B \\ p \downarrow & \theta \nearrow & \downarrow v \\ A & \xrightarrow{u} & C \end{array}$$

*If  $\mathcal{D}$  is  $\mathfrak{K}$ -cocomplete (resp.  $\mathfrak{K}$ -cocomplete), then the following are equivalent:*

- (i) *Diagram (□) is a left  $\mathcal{D}$ -exact square (resp. right  $\mathcal{D}$ -exact square).*
- (ii) *For all morphisms  $a : 1 \rightarrow A$  and  $b : 1 \rightarrow B$ , for all diagrams of the form below in  $\mathfrak{K}$ ,*

$$(*) \quad \begin{array}{ccccc} E & \longrightarrow & (q \downarrow b) & \longrightarrow & 1 \\ \downarrow & \text{p.b.} & \downarrow & \beta \nearrow & \downarrow b \\ (a \downarrow p) & \longrightarrow & D & \xrightarrow{q} & B \\ \downarrow & \alpha \nearrow & p \downarrow & \theta \nearrow & \downarrow v \\ 1 & \xrightarrow{a} & A & \xrightarrow{u} & C \end{array}$$

*where the top-left square is a pullback square and the squares inhabited by  $\alpha$  and  $\beta$  are comma squares, the outer square is a left  $\mathcal{D}$ -exact square (resp. right  $\mathcal{D}$ -exact square).*

- (iii) *For all diagrams of the form (\*) in  $\mathfrak{K}$ , the morphism  $E \rightarrow (u \circ a \downarrow v \circ b)$  induced by the universal property of  $(u \circ a \downarrow v \circ b)$  is a  $\mathcal{D}$ -equivalence.*

*Proof.* (i)  $\Rightarrow$  (ii). The pasting lemma for comma diagrams implies the left rectangle of (\*) is a comma diagram, and we may apply [lemma 7.1.17](#) and [theorem 7.1.25](#) to deduce that the outer square of (\*) is a left  $\mathcal{D}$ -exact square.

(ii)  $\Leftrightarrow$  (iii). This is a special case of the previous lemma.

(ii)  $\Rightarrow$  (i). Using axioms Der2 and Der4L as well as the 2-out-of-3 property for natural isomorphisms, we may deduce that diagram ( $\square$ ) is left  $\mathcal{D}$ -exact if every diagram of the form (\*) is left  $\mathcal{D}$ -exact.  $\blacksquare$

**Corollary 7.2.18.** *Let  $\mathcal{D}$  be a semiderivator on  $\mathfrak{K}$ . If  $\mathcal{D}$  is  $\mathfrak{K}$ -cocomplete, then the following are equivalent for a morphism  $v : B \rightarrow A$  in  $\mathfrak{K}$ :*

- (i) *The morphism  $v : B \rightarrow A$  is a  $\mathcal{D}$ -cofinal morphism.*
- (ii) *For every morphism  $a : 1 \rightarrow A$  in  $\mathfrak{K}$ , the unique morphism  $(a \downarrow v) \rightarrow 1$  is a  $\mathcal{D}$ -equivalence.*

*Dually, if  $\mathcal{D}$  is  $\mathfrak{K}$ -complete, then the following are equivalent for a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$ :*

- (i) *The morphism  $u : A \rightarrow B$  is a  $\mathcal{D}$ -coinitial morphism.*
- (ii) *For every morphism  $b : 1 \rightarrow B$  in  $\mathfrak{K}$ , the unique morphism  $(u \downarrow b) \rightarrow 1$  is a  $\mathcal{D}$ -equivalence.*  $\blacksquare$

### 7.3 Basic localisers

**Prerequisites.** §§3.1, 7.1.

**Definition 7.3.1.** Let  $\mathfrak{K}$  be a derivator domain and let  $\mathcal{K}$  be its underlying 1-category. A **basic right localiser** (resp. **basic left localiser**) for  $\mathfrak{K}$  is a subcategory  $\mathcal{W}$  of  $\mathcal{K}$  satisfying these axioms:

**LF1.** Every identity morphism in  $\mathcal{K}$  is also in  $\mathcal{W}$ ,  $\mathcal{W}$  has the 2-out-of-3 property in  $\mathcal{K}$ , and  $\mathcal{W}$  is closed under retracts in  $\mathcal{K}$ .

**LF2.** For any object  $A$  in  $\mathfrak{K}$ , if the unique morphism  $A \rightarrow 1$  has a right adjoint (resp. left adjoint), then  $A \rightarrow 1$  is in  $\mathcal{W}$ .

**LF3.** Given a commutative triangle in  $\mathfrak{K}$ ,

$$\begin{array}{ccc}
 A & \xrightarrow{u} & B \\
 & \searrow p & \swarrow q \\
 & & C
 \end{array}$$

if, for every morphism  $c : 1 \rightarrow C$  in  $\mathfrak{K}$ , the morphism  $u_c : (p \downarrow c) \rightarrow (q \downarrow c)$  (resp.  ${}^c u : (c \downarrow p) \rightarrow (c \downarrow q)$ ) induced by  $u : A \rightarrow B$  is in  $\mathcal{W}$ , then  $u : A \rightarrow B$  itself is in  $\mathcal{W}$ .

A **basic localiser** for  $\mathfrak{K}$  is a subcategory of  $\mathcal{K}$  that is both a basic left localiser and a basic right localiser.

**Definition 7.3.2.** Let  $\mathfrak{K}$  be a derivator domain and let  $\mathcal{W}$  be either a basic left localiser or a basic right localiser for  $\mathfrak{K}$ . A  **$\mathcal{W}$ -equivalence** is a morphism that is in  $\mathcal{W}$ . A  **$\mathcal{W}$ -aspherical object** is an object  $A$  in  $\mathfrak{K}$  such that the unique morphism  $A \rightarrow 1$  is a  $\mathcal{W}$ -equivalence.

¶ 7.3.3. The above terminology is non-standard: it is more conventional to refer to basic right localisers as ‘basic localisers’ and ignore basic left localisers; cf. [Cisinski, 2004]. However, this is unproblematic in the case where  $\mathfrak{K} = \mathfrak{Cat}$ : one can show that all three notions coincide then. The chirality of the above terminology is chosen to agree with the chirality of the induced asphericity structures (cf. [Maltsiniotis, 2005]).

**Proposition 7.3.4.** *Let  $\mathfrak{K}$  be a derivator domain.*

- *If  $\mathcal{D}$  is a  $\mathfrak{K}$ -cocomplete semiderivator, then the class of  $\mathcal{D}$ -equivalences is a basic right localiser for  $\mathfrak{K}$ .*
- *If  $\mathcal{D}$  is a  $\mathfrak{K}$ -complete semiderivator, then the class of  $\mathcal{D}$ -equivalences is a basic left localiser for  $\mathfrak{K}$ .*

*Proof.* The two claims are formally dual; we will prove the first version.

**Proposition 7.2.14** implies that the class of  $\mathcal{D}$ -equivalences satisfies axiom LF1, and **proposition 7.2.15** says that axiom LF3 is satisfied. Axiom LF2 remains to be verified, so suppose  $A$  is an object in  $\mathfrak{K}$  such that the unique morphism  $p : A \rightarrow 1$  has a right adjoint, say  $t : 1 \rightarrow A$ . By **remark 7.2.13**,  $t$  is a  $\mathcal{D}$ -equivalence; but  $p \circ t = \text{id}_1$  since  $1$  is a terminal object in  $\mathfrak{K}$ , so we may deduce that  $p : A \rightarrow 1$  is also a  $\mathcal{D}$ -equivalence by using axiom LF1. ■

**Corollary 7.3.5.** *If  $\mathcal{D}$  is a derivator on  $\mathfrak{K}$ , then the class of  $\mathcal{D}$ -equivalences is a basic localiser for  $\mathfrak{K}$ .* ■

**Example 7.3.6.** Let  $\mathcal{D}$  be the prederivator of  $\mathbf{Set}$  (restricted to  $\mathbf{Cat}$ ). By [theorem 7.1.21](#),  $\mathcal{D}$  is a derivator, and it is straightforward to verify that the  $\mathcal{D}$ -equivalences are precisely the functors  $u : \mathbb{A} \rightarrow \mathbb{B}$  that induce bijections  $\pi_0 u : \pi_0 \mathbb{A} \rightarrow \pi_0 \mathbb{B}$ , where  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$  is the connected components functor.<sup>[1]</sup>

**REMARK 7.3.7.** It is not hard to see that the intersection of any family of basic localisers (resp. basic left localisers, basic right localisers) for a derivator domain  $\mathfrak{K}$  is automatically a basic localiser (resp. basic left localiser, basic right localiser) for  $\mathfrak{K}$ ; thus, there is a unique minimal basic localiser (resp. basic left localiser, basic right localiser) for  $\mathfrak{K}$ .

**Definition 7.3.8.** Let  $\mathfrak{K}$  be a derivator domain and let  $\mathcal{W}$  be either a basic left localiser or a basic right localiser for  $\mathfrak{K}$ .

- A **right  $\mathcal{W}$ -aspherical morphism** is a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  such that, for all morphisms  $b : 1 \rightarrow B$  in  $\mathfrak{K}$ , the unique morphism  $(u \downarrow b) \rightarrow 1$  is a  $\mathcal{W}$ -equivalence.
- A **left  $\mathcal{W}$ -aspherical morphism** is a morphism  $v : B \rightarrow A$  in  $\mathfrak{K}$  such that, for all morphisms  $a : 1 \rightarrow A$  in  $\mathfrak{K}$ , the unique morphism  $(a \downarrow v) \rightarrow 1$  is a  $\mathcal{W}$ -equivalence.

**REMARK 7.3.9.** In view of [corollary 7.2.18](#), one might also call right (resp. left)  $\mathcal{W}$ -aspherical morphisms  **$\mathcal{W}$ -coinitial** (resp.  **$\mathcal{W}$ -cofinal**).

**Lemma 7.3.10.** *Let  $\mathfrak{K}$  be a derivator domain.*

- *If a morphism  $u : A \rightarrow B$  in  $\mathfrak{K}$  has a right adjoint, then for any morphism  $b : 1 \rightarrow B$ , the unique morphism  $(u \downarrow b) \rightarrow 1$  has a right adjoint.*
- *If a morphism  $v : B \rightarrow A$  in  $\mathfrak{K}$  has a left adjoint, then for any morphism  $a : 1 \rightarrow A$ , the unique morphism  $(a \downarrow v) \rightarrow 1$  has a left adjoint.*

*Proof.* The two claims are formally dual; we will prove the first version.

Suppose the following diagram is a comma square in  $\mathfrak{K}$ :

$$\begin{array}{ccc} (u \downarrow b) & \xrightarrow{q} & 1 \\ p \downarrow & \theta \nearrow & \downarrow b \\ A & \xrightarrow{u} & B \end{array}$$

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[1] Recall [proposition A.2.15](#).

Let  $v : B \rightarrow A$  be a right adjoint of  $u : A \rightarrow B$ , say with counit  $\varepsilon : u \circ v \Rightarrow \text{id}_B$ . Consider the morphism  $t : 1 \rightarrow (u \downarrow b)$  induced by the diagram in  $\mathfrak{K}$  shown below:

$$\begin{array}{ccc} 1 & \xrightarrow{v \circ b} & 1 \\ \text{id} \downarrow & \varepsilon \circ b \nearrow & \downarrow b \\ A & \xrightarrow{u} & B \end{array}$$

Via the 2-dimensional universal property of  $(u \downarrow b)$ ,  $\theta : u \circ p \Rightarrow b \circ q$  induces a 2-cell  $\eta : \text{id}_{(u \downarrow b)} \Rightarrow t \circ q$ , and using the 2-dimensional Yoneda lemma, it is straightforward to check that  $\eta$  is the unit of an adjunction  $q \dashv t : 1 \rightarrow (u \downarrow b)$ . Thus, the unique morphism  $(u \downarrow b) \rightarrow 1$  indeed has a right adjoint. ■

**Corollary 7.3.11.** *Let  $\mathfrak{K}$  be a derivator domain.*

- *If  $\mathcal{W}$  is a basic right localiser for  $\mathfrak{K}$ , then every morphism in  $\mathfrak{K}$  that has a right adjoint is a right  $\mathcal{W}$ -spherical morphism.*
- *If  $\mathcal{W}$  is a basic left localiser for  $\mathfrak{K}$ , then every morphism in  $\mathfrak{K}$  that has a left adjoint is a left  $\mathcal{W}$ -spherical morphism.* ■

**Proposition 7.3.12.** *Let  $\mathfrak{K}$  be a derivator domain.*

- *If  $\mathcal{W}$  is a basic right localiser for  $\mathfrak{K}$ , then every right  $\mathcal{W}$ -spherical morphism is a  $\mathcal{W}$ -equivalence; in particular every morphism in  $\mathfrak{K}$  that has a right adjoint is a  $\mathcal{W}$ -equivalence.*
- *If  $\mathcal{W}$  is a basic left localiser for  $\mathfrak{K}$ , then every left  $\mathcal{W}$ -spherical morphism is a  $\mathcal{W}$ -equivalence; in particular every morphism in  $\mathfrak{K}$  that has a left adjoint is a  $\mathcal{W}$ -equivalence.*

*Proof.* The two claims are formally dual; we will prove the first version.

Suppose  $u : A \rightarrow B$  is a right  $\mathcal{W}$ -spherical morphism. Consider the following commutative triangle in  $\mathfrak{K}$ :

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & \searrow u & \swarrow \text{id} \\ & B & \end{array}$$

Let  $b : 1 \rightarrow B$  be a morphism in  $\mathfrak{K}$ . Since the unique morphism  $(u \downarrow b) \rightarrow 1$  is a  $\mathcal{W}$ -equivalence, axioms LF1 and LF2 and [lemma 7.3.10](#) imply the induced

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morphism  $u_b : (u \downarrow b) \rightarrow (\text{id}_B \downarrow b)$  is also a  $\mathcal{W}$ -equivalence. We may then apply axiom LF3 to deduce that  $u : A \rightarrow B$  itself is a  $\mathcal{W}$ -equivalence. ■

**Lemma 7.3.13.** *Let  $A$  be an object in a derivator domain  $\mathfrak{K}$ . If  $\mathcal{W}$  is a basic left or right localiser for  $\mathfrak{K}$ , then the morphism  $p : \mathbb{2} \odot A \rightarrow \mathbb{1} \odot A \cong A$  induced by the unique functor  $\mathbb{2} \rightarrow \mathbb{1}$  is a  $\mathcal{W}$ -equivalence.*

*Proof.* The unique functor  $\mathbb{2} \rightarrow \mathbb{1}$  has both a left adjoint and a right adjoint, so the induced morphism  $p : \mathbb{2} \odot A \rightarrow A$  has both a left adjoint and a right adjoint.

[Proposition 7.3.12](#) then implies that it is a  $\mathcal{W}$ -equivalence. ■

**Proposition 7.3.14.** *Let  $u_0, u_1 : A \rightarrow B$  be a parallel pair of morphisms in a derivator domain  $\mathfrak{K}$  and let  $\mathcal{W}$  be either a basic left localiser or a basic right localiser for  $\mathfrak{K}$ . If there exists a 2-cell  $\alpha : u_0 \Rightarrow u_1$ , then the following are equivalent:*

- (i) *The morphism  $u_0 : A \rightarrow B$  is a  $\mathcal{W}$ -equivalence.*
- (ii) *The morphism  $u_1 : A \rightarrow B$  is a  $\mathcal{W}$ -equivalence.*

*Proof.* Let  $i_0, i_1 : A \rightarrow \mathbb{2} \odot A$  be the morphisms induced by the left and right adjoints of the unique functor  $\mathbb{2} \rightarrow \mathbb{1}$ ; note that functoriality yields  $p \circ i_0 = \text{id}_A = p \circ i_1$ . The previous lemma says that  $p$  is a  $\mathcal{W}$ -equivalence, so we may then use axiom LF1 to deduce that  $i_0$  and  $i_1$  are both  $\mathcal{W}$ -equivalences.

By definition, there is a bijection

$$\mathcal{K}(\mathbb{2} \odot A, B) \cong \text{Fun}(\mathbb{2}, \mathfrak{K}(A, B))$$

that is natural in  $B$ ; thus, the 2-cell  $\alpha : u_0 \Rightarrow u_1$  corresponds to a morphism  $h : \mathbb{2} \odot A \rightarrow B$  such that  $h \circ i_0 = u_0$  and  $h \circ i_1 = u_1$ . Axiom LF1 then implies that  $u_0$  is a  $\mathcal{W}$ -equivalence if and only if  $u_1$  is a  $\mathcal{W}$ -equivalence. ■

**Corollary 7.3.15.** *If  $\mathcal{W}$  is a basic left or right localiser for a derivator domain  $\mathfrak{K}$ , then every left or right adjoint in  $\mathfrak{K}$  is a  $\mathcal{W}$ -equivalence.*

*Proof.* One half of the claim was proved in [proposition 7.3.12](#); it now suffices to show that, if  $\mathcal{W}$  is a basic right localiser for  $\mathfrak{K}$ , then every right adjoint in  $\mathfrak{K}$  is a  $\mathcal{W}$ -equivalence. We already know that every left adjoint in  $\mathfrak{K}$  is a  $\mathcal{W}$ -equivalence, so axiom LF1 and the above proposition together imply that right adjoints are also  $\mathcal{W}$ -equivalences. ■

## 7.4 The minimal basic localiser

**Prerequisites.** §§ 1.1, 1.2, 1.5, 1.3, 1.8, 7.1, 7.2, 7.3.

In this section, we follow [Cisinski, 2004, §2.2].

**Proposition 7.4.1.** *Let  $\mathcal{W}$  be a basic left or right localiser for  $\mathfrak{Cat}$ . For any functor  $u : \mathbb{A} \rightarrow \mathbb{B}$ , the following are equivalent:*

- (i) *The functor  $u : \mathbb{A} \rightarrow \mathbb{B}$  is a  $\mathcal{W}$ -equivalence.*
- (ii) *The functor  $u^{\text{op}} : \mathbb{A}^{\text{op}} \rightarrow \mathbb{B}^{\text{op}}$  is a  $\mathcal{W}$ -equivalence.*

*Proof.* See Proposition 1.2.6 in [Cisinski, 2004]. □

**Corollary 7.4.2.** *Let  $\mathcal{W}$  be a subcategory of  $\mathfrak{Cat}$ . The following are equivalent:*

- (i)  *$\mathcal{W}$  is a basic localiser for  $\mathfrak{Cat}$ .*
- (ii)  *$\mathcal{W}$  is a basic right localiser for  $\mathfrak{Cat}$ .*
- (iii)  *$\mathcal{W}$  is a basic left localiser for  $\mathfrak{Cat}$ .* ■

¶ 7.4.3. Throughout this section, let  $\mathcal{W}$  be any basic localiser for  $\mathfrak{Cat}$ .

**Definition 7.4.4.** A **weak homotopy equivalence of categories** is a functor  $f : \mathbb{A} \rightarrow \mathbb{B}$  such that the induced morphism  $N(f) : N(\mathbb{A}) \rightarrow N(\mathbb{B})$  is a weak homotopy equivalence of simplicial sets. We write  $\mathcal{W}_\infty$  for the class of weak homotopy equivalences of categories.

¶ 7.4.5. Weak homotopy equivalences of categories are also called  **$\infty$ -equivalences**, but we should avoid this term as it conflicts with the terminology of higher category theory.

**Definition 7.4.6.** The **category of simplices** of a simplicial set  $X$  is the category  $\Delta(X)$  defined below:

- The objects are simplices of  $X$ .
- For  $x \in X_n$  and  $x' \in X_{n'}$ , the morphisms  $x \rightarrow x'$  are simplicial operators  $f : [n] \rightarrow [n']$  such that  $f(x') = x$ .
- Composition and identities are the obvious ones.

We write  $\pi_{\Delta} : \Delta(X) \rightarrow \Delta$  for the evident projection functor that sends an  $n$ -simplex of  $X$  to the object  $[n]$  in  $\Delta$ .

¶ 7.4.7. For brevity, if  $\mathbb{A}$  is a small category, then we write  $\Delta(\mathbb{A})$  instead of  $\Delta(\mathbf{N}(\mathbb{A}))$ . This is consistent with the notation of §4.6. We will also use the left and right projection functors defined in definition 4.6.9.

**Lemma 7.4.8.** *If  $\mathbb{A}$  is a small category with a terminal object, then the category  $\Delta(\mathbb{A})$  is  $\mathcal{W}$ -aspherical.*

*Proof.* Straightforward. (This is Lemme 2.2.2 in [Cisinski, 2004].) □

**Proposition 7.4.9** (Grothendieck). *For all small categories  $\mathbb{A}$ , the right projection  $\pi_{\mathbb{R}} : \Delta(\mathbb{A}) \rightarrow \mathbb{A}$  is right  $\mathcal{W}$ -aspherical; in particular, it is a  $\mathcal{W}$ -equivalence.*

*Proof.* Let  $a$  be an object in  $\mathbb{A}$ . Lemma 4.6.12 says that the canonical comparison functor  $\Delta(\mathbb{A}_{/a}) \rightarrow (\pi_{\mathbb{R}} \downarrow a)$  is an isomorphism, and lemma 7.4.8 implies  $\Delta(\mathbb{A}_{/a})$  is  $\mathcal{W}$ -aspherical, so the induced functor  $(\pi_{\mathbb{R}} \downarrow a) \rightarrow \mathbb{A}_{/a}$  is a  $\mathcal{W}$ -equivalence. Thus,  $\pi_{\mathbb{R}} : \Delta(\mathbb{A}) \rightarrow \mathbb{A}$  is right  $\mathcal{W}$ -aspherical. ■

**Corollary 7.4.10.** *A functor  $u : \mathbb{A} \rightarrow \mathbb{B}$  is a  $\mathcal{W}$ -equivalence if and only if the functor  $\Delta(u) : \Delta(\mathbb{A}) \rightarrow \Delta(\mathbb{B})$  is a  $\mathcal{W}$ -equivalence.*

*Proof.* Use the naturality of  $\pi_{\mathbb{R}}$  and axiom LF1. ■

¶ 7.4.11. Now, let  $\mathcal{W}_{\Delta}$  be the subcategory of  $\mathbf{sSet}$  consisting of those morphisms  $f : X \rightarrow Y$  such that  $\Delta(f) : \Delta(X) \rightarrow \Delta(Y)$  are  $\mathcal{W}$ -equivalences.

**Proposition 7.4.12.** *For all simplicial sets  $X$  and all natural numbers  $n$ , the projection  $\pi : X \times \Delta^n \rightarrow X$  is a  $\mathcal{W}_{\Delta}$ -equivalence.*

*Proof.* Since  $\Delta^m \times \Delta^n \cong \mathbf{N}([m] \times [n])$ , lemma 7.4.8 implies  $\Delta(\Delta^m \times \Delta^n)$  is  $\mathcal{W}$ -aspherical. Now, let  $x$  be an  $m$ -simplex of  $X$ , and consider the comma category  $(\Delta(\pi) \downarrow x)$ . It is not hard to see that  $(\Delta(\pi) \downarrow x)$  is isomorphic to  $\Delta(\Delta^m \times \Delta^n)$ , and so the induced functor  $(\Delta(\pi) \downarrow x) \rightarrow \Delta(X)_{/x}$  is a  $\mathcal{W}$ -equivalence. Thus,  $\Delta(\pi) : \Delta(X \times \Delta^n) \rightarrow \Delta(X)$  is right  $\mathcal{W}$ -aspherical, and in particular  $\pi : X \times \Delta^n \rightarrow X$  is a  $\mathcal{W}_{\Delta}$ -equivalence. ■

**Corollary 7.4.13.** *Every trivial Kan fibration is a  $\mathcal{W}_{\Delta}$ -equivalence.*

*Proof.* Apply proposition 1.5.13. ■



**Proposition 7.4.14.** *Every trivial cofibration in  $\mathbf{sSet}$  is a  $\mathcal{W}_\Delta$ -equivalence.*

*Proof.* See Proposition 2.2.9 in [Cisinski, 2004]. □

**Theorem 7.4.15** (Cisinski). *Any  $\mathcal{W}_\infty$ -equivalence is also a  $\mathcal{W}$ -equivalence.*

*Proof.* Propositions 1.4.7 and 1.5.9 together imply that every weak homotopy equivalence in  $\mathbf{sSet}$  can be factored as a trivial cofibration followed by a trivial Kan fibration, so applying corollaries 7.4.10 and 7.4.13 and proposition 7.4.14, we deduce that every  $\mathcal{W}_\infty$ -equivalence is a  $\mathcal{W}$ -equivalence. ■

**Proposition 7.4.16.** *Let  $\mathcal{D}$  be the prederivator of  $\mathbf{sSet}$  (with the Kan–Quillen model structure) restricted to  $\mathbf{Cat}$ . Then a functor  $u : \mathbb{A} \rightarrow \mathbb{B}$  is a  $\mathcal{D}$ -equivalence if and only if  $u : \mathbb{A} \rightarrow \mathbb{B}$  is a  $\mathcal{W}_\infty$ -equivalence.*

*Proof.* Theorem 7.2.5 says  $\mathcal{D}$  is a derivator, so we may use the characterisation of  $\mathcal{D}$ -equivalences afforded by proposition 7.2.11. Let  $X$  be any simplicial set. By theorem 1.8.9,  $\text{holim}_{\rightarrow \mathbb{A}} \Delta_{\mathbb{A}} X$  can be computed using the bar construction  $B(\Delta 1, \mathbb{C}, X)$ , which is readily seen to be isomorphic to the simplicial set  $N(\mathbb{C}) \times X$ . Using the explicit description of the derived unit and counit in the proof of theorem 3.3.20, it can be shown that the diagram shown below commutes,

$$\begin{array}{ccc} \text{holim}_{\rightarrow \mathbb{A}} \Delta_{\mathbb{A}} X & \longrightarrow & \text{holim}_{\rightarrow \mathbb{B}} \Delta_{\mathbb{B}} X \\ \cong \downarrow & & \downarrow \cong \\ N(\mathbb{A}) \times X & \xrightarrow{N(u) \times \text{id}_X} & N(\mathbb{B}) \times X \end{array}$$

but by proposition 1.5.11,  $N(u) \times \text{id}_X$  is a weak homotopy equivalence for all  $X$  if and only if  $N(u) : N(\mathbb{A}) \rightarrow N(\mathbb{B})$  is a weak homotopy equivalence, i.e. if and only if  $u : \mathbb{A} \rightarrow \mathbb{B}$  is a  $\mathcal{W}_\infty$ -equivalence. ■

We thus obtain a proof of Grothendieck’s conjecture ([1983, §81]):

**Corollary 7.4.17.** *The minimal basic localiser for  $\mathbf{Cat}$  is  $\mathcal{W}_\infty$ .* ■



## HOMOTOPY TOPOSES

### 8.1 Internal Kan complexes

**Prerequisites.** §§ 1.1, 1.4, 1.5, 2.3, A.7.

To do homotopy theory inside a topos, we need a model for homotopy types. One of the simplest options is to internalise the theory of Kan complexes. This was first done in the case of sheaves on a topological space by Brown [1973], then extended to the general case of an effective regular category by van Osdol [1977].

**Definition 8.1.1.** An **internal Kan fibration** (resp. **internal trivial Kan fibration**) in a regular category  $\mathcal{S}$  is a morphism  $p : X \rightarrow Y$  in  $\mathbf{sS}$  with the following property:

- If  $i : Z \rightarrow W$  is a horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$  (resp. a boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$ ) and the square in the diagram below is a weak pullback square in  $\mathcal{S}$ :

$$\begin{array}{ccc} \{W, X\} & \xrightarrow{\{W,p\}} & \{W, Y\} \\ \{i,X\} \downarrow & & \downarrow \{i,Y\} \\ \{Z, X\} & \xrightarrow{\{Z,p\}} & \{Z, Y\} \end{array}$$

**REMARK 8.1.2.** If  $\mathcal{S} = \mathbf{Set}$ , then an internal Kan fibration (resp. internal trivial Kan fibration) is just a Kan fibration (resp. trivial Kan fibration) in the usual sense, by lemma A.3.2. If  $\mathcal{S} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$  for a small category  $\mathbb{C}$ , then an internal Kan fibration (resp. internal trivial Kan fibration) in  $\mathcal{S}$  is the same thing as a componentwise Kan fibration (resp. trivial Kan fibration), because limits and colimits in  $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$  are computed componentwise.

**Definition 8.1.3.** An **internal Kan complex** in a regular category  $\mathcal{S}$  is an object  $X$  in  $\mathfrak{s}\mathcal{S}$  such that the unique morphism  $X \rightarrow 1$  in  $\mathfrak{s}\mathcal{S}$  is an internal Kan fibration. We write  $\mathbf{Kan}(\mathcal{S})$  for the full subcategory of  $\mathfrak{s}\mathcal{S}$  spanned by the internal Kan complexes in  $\mathcal{S}$ .

**Proposition 8.1.4.** *Let  $\mathcal{S}$  be a regular category.*

- (i) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in  $\mathcal{S}$  contains all isomorphisms in  $\mathfrak{s}\mathcal{S}$ .*
- (ii) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in  $\mathcal{S}$  is closed under composition.*
- (iii) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in  $\mathcal{S}$  is closed under pullbacks.*
- (iv) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in  $\mathcal{S}$  is closed under retracts.*
- (v) *The class of internal Kan fibrations (resp. internal trivial Kan fibrations) in  $\mathcal{S}$  is closed under finite products.*

*Proof.* (i). Obvious.

(ii) and (iii). Apply the weak pullback lemma (A.7.17).

(iv) and (v). By [proposition A.7.15](#), the class of regular epimorphisms in  $\mathcal{S}$  is closed under retracts (resp. finite products), so the class of weak pullback squares in  $\mathcal{S}$  is also closed under retracts (resp. finite products). ■

**Proposition 8.1.5.** *Let  $\mathcal{S}$  be a regular category, let  $i : Z \rightarrow W$  be a morphism between finite simplicial sets, and let  $p : X \rightarrow Y$  be a morphism in  $\mathfrak{s}\mathcal{S}$ . Consider the following commutative diagram in  $\mathcal{S}$ :*

$$\begin{array}{ccc} \{W, X\} & \xrightarrow{\{W,p\}} & \{W, Y\} \\ \{i,X\} \downarrow & & \downarrow \{i,Y\} \\ \{Z, X\} & \xrightarrow{\{Z,p\}} & \{Z, Y\} \end{array}$$

- (i) *If  $i : Z \rightarrow W$  is an anodyne extension and  $p : X \rightarrow Y$  is an internal Kan fibration, then the diagram is a weak pullback square in  $\mathcal{S}$ .*

- (ii) If  $i : Z \rightarrow W$  is a monomorphism and  $p : X \rightarrow Y$  is an internal trivial Kan fibration, then the diagram is a weak pullback square in  $\mathcal{S}$ .

*Proof.* The proofs of the two claims are similar; we will prove claim (i).

By [proposition 1.4.12](#), the class of anodyne extensions between finite simplicial sets is the smallest class of morphisms containing the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  that is closed under composition, pushouts, and retracts; but the class of regular epimorphisms  $\mathcal{S}$  is closed under composition, pullbacks, and retracts (by [proposition A.7.15](#)), and  $\{-, X\}$  sends colimits in  $\mathbf{sSet}$  to limits in  $\mathcal{S}$ , so we are done. ■

**Proposition 8.1.6.** *Let  $\mathcal{S}$  be a regular category, let  $i : Z \rightarrow W$  be a monomorphism between finite simplicial sets, and let  $p : X \rightarrow Y$  be an internal Kan fibration in  $\mathcal{S}$ . Consider the following commutative diagram in  $\mathbf{sS}$ ,*

$$\begin{array}{ccc}
 W \pitchfork X & \xrightarrow{\quad \text{id}_W \pitchfork p \quad} & W \pitchfork Y \\
 \downarrow \text{dashed } i \boxplus p & \searrow & \downarrow \text{dashed } i \boxplus p \\
 (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y) & \xrightarrow{\quad} & W \pitchfork Y \\
 \downarrow & & \downarrow \text{dashed } i \boxplus p \\
 Z \pitchfork X & \xrightarrow{\quad \text{id}_Z \pitchfork p \quad} & Z \pitchfork Y \\
 \downarrow \text{dashed } i \boxplus p & & \downarrow \text{dashed } i \boxplus p \\
 W \pitchfork X & \xrightarrow{\quad \text{id}_W \pitchfork p \quad} & W \pitchfork Y
 \end{array}$$

where the square is a pullback.

- (i) The morphism  $i \boxplus p : W \pitchfork X \rightarrow (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)$  is an internal Kan fibration.
- (ii) If  $i : Z \rightarrow W$  is an anodyne extension, then  $i \boxplus p$  is an internal trivial Kan fibration.
- (iii) If  $p : X \rightarrow Y$  is an internal trivial Kan fibration, then  $i \boxplus p$  is an internal trivial Kan fibration.

*Proof.* The proofs of the three claims are similar; we will prove claim (i).

Let  $j : K \rightarrow L$  be a morphism of finite simplicial sets, and consider the following commutative diagram in  $\mathcal{S}$ ,

$$\begin{array}{ccccc}
 & & \{L, W \pitchfork X\} & \longrightarrow & \{L, W \pitchfork Y\} \\
 & \swarrow & \downarrow & & \swarrow \downarrow \\
 \{K, W \pitchfork X\} & \longrightarrow & \{K, W \pitchfork Y\} & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & \{L, Z \pitchfork X\} & \longrightarrow & \{L, Z \pitchfork Y\} \\
 \swarrow & & \downarrow & & \swarrow \\
 \{K, Z \pitchfork X\} & \longrightarrow & \{K, Z \pitchfork Y\} & & 
 \end{array}$$

where the horizontal arrows are induced by  $p : X \rightarrow Y$ , the vertical arrows are induced by  $i : Z \rightarrow W$ , and the diagonal arrows are induced by  $j : K \rightarrow L$ . We wish to show that the following diagram is a weak pullback square in  $\mathcal{S}$  when  $j : K \rightarrow L$  is any horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ :

$$\begin{array}{ccc}
 \{L, W \pitchfork X\} & \xrightarrow{\{L, i \square p\}} & \{L, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\} \\
 \downarrow \{j, W \pitchfork X\} & & \downarrow \{j, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\} \\
 \{K, W \pitchfork X\} & \xrightarrow{\{K, i \square p\}} & \{K, (Z \pitchfork X) \times_{Z \pitchfork Y} (W \pitchfork Y)\}
 \end{array}$$

It is not hard to see that this amounts to showing that the comparison morphism

$$\{L, W \pitchfork X\} \rightarrow \{K, W \pitchfork X\} \times_{\{K, Z \pitchfork Y\}} \{L, Z \pitchfork X\} \times_{\{K, Z \pitchfork Y\}} \{L, W \pitchfork Y\}$$

is a regular epimorphism in  $\mathcal{S}$ . Via the natural isomorphism  $\{K, Z \pitchfork X\} \cong \{K \times Z, X\}$ , this is in turn equivalent to showing that the following diagram is a weak pullback square in  $\mathcal{S}$ ,

$$\begin{array}{ccc}
 \{L \times W, X\} & \xrightarrow{\{j \square i, X\}} & \{(K \times W) \cup^{K \times Z} (L \times Z), X\} \\
 \downarrow \{L \times W, p\} & & \downarrow \{(K \times W) \cup^{K \times Z} (L \times Z), p\} \\
 \{L \times W, Y\} & \xrightarrow{\{j \square i, Y\}} & \{(K \times W) \cup^{K \times Z} (L \times Z), Y\}
 \end{array}$$

where  $j \square i : (K \times W) \cup^{K \times Z} (L \times Z) \rightarrow L \times W$  is the evident monomorphism of finite simplicial sets. But propositions 1.4.15 and 2.5.4 say  $j \square i$  is an anodyne extension, so we may apply proposition 8.1.5 to deduce the claim. ■

**Corollary 8.1.7** (Internal path spaces). *Let  $\mathcal{S}$  be a regular category and let  $X$  be an internal Kan complex in  $\mathcal{S}$ .*

- (i)  $\Delta^1 \pitchfork X$  is an internal Kan complex in  $\mathcal{S}$ .
- (ii) The morphism  $\Delta^1 \pitchfork X \rightarrow \partial\Delta^1 \pitchfork X$  induced by the boundary inclusion  $\partial\Delta^1 \hookrightarrow \Delta^1$  is an internal Kan fibration in  $\mathcal{S}$ .
- (iii) The morphisms  $\Delta^1 \pitchfork X \rightarrow \Delta^0 \pitchfork X$  induced by the two vertex inclusions  $\Delta^0 \rightarrow \Delta^1$  are internal trivial Kan fibrations in  $\mathcal{S}$ . ■

Let  $D^{n+1}$  be the relative cylinder  $C(\Delta^n, \partial\Delta^n)$ , as in [definition 1.3.25](#), and let  $j_0, j_1 : \Delta^n \rightarrow D^{n+1}$  be the two canonical embeddings.

**Definition 8.1.8.** A **Dugger–Isaksen weak equivalence** in a regular category  $\mathcal{S}$  is a morphism  $f : X \rightarrow Y$  in  $\mathbf{Kan}(\mathcal{S})$  such that the morphism

$$\{\Delta^n, X\} \times_{\{\Delta^n, Y\}} \{D^{n+1}, Y\} \longrightarrow \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

induced by the commutative diagram

$$\begin{array}{ccccc}
 \bullet & \xrightarrow{\quad} & \{D^{n+1}, Y\} & & \\
 \downarrow & \dashrightarrow & \downarrow & \searrow & \\
 & & \bullet & \xrightarrow{\quad} & \{\Delta^n, Y\} \\
 & & \downarrow & \downarrow & \downarrow \\
 \{\Delta^n, X\} & \xrightarrow{\quad} & \{\Delta^n, Y\} & & \{\Delta^n, Y\} \\
 \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & \{\partial\Delta^n, X\} & \xrightarrow{\quad} & \{\partial\Delta^n, Y\}
 \end{array}$$

$\{\Delta^n, X\} \xrightarrow{\{\Delta^n, f\}} \{\Delta^n, Y\}$   
 $\{\partial\Delta^n, X\} \xrightarrow{\{\partial\Delta^n, f\}} \{\partial\Delta^n, Y\}$

is a regular epimorphism in  $\mathcal{S}$ .

**REMARK 8.1.9.** If  $\mathcal{S} = \mathbf{Set}$ , then a Dugger–Isaksen weak equivalence is precisely a weak homotopy equivalence of Kan complexes in the usual sense, by [theorem 1.4.32](#). If  $\mathcal{S} = [\mathbb{C}^{\text{op}}, \mathbf{Set}]$  for a small category  $\mathbb{C}$ , then an Dugger–Isaksen weak equivalence of internal Kan complexes in  $\mathcal{S}$  is the same thing as a componentwise weak homotopy equivalence between componentwise Kan complexes, because limits and colimits in  $[\mathbb{C}^{\text{op}}, \mathbf{Set}]$  are computed componentwise.

**Proposition 8.1.10.**

- (i) *The class of Dugger–Isaksen weak equivalences in  $\mathcal{S}$  contains all isomorphisms in  $\mathbf{Kan}(\mathcal{S})$ .*

- (ii) *The class of Dugger–Isaksen weak equivalences in  $\mathcal{S}$  is closed under retracts.*
- (iii) *The class of Dugger–Isaksen weak equivalences in  $\mathcal{S}$  is closed under finite products.*

*Proof.* The class of Dugger–Isaksen weak equivalences (considered as a class of objects in the category  $[2, \mathbf{Kan}(\mathcal{S})]$ ) is defined by an internal right lifting property, so we may use the same methods used in the proof of [proposition 8.1.4](#). ■

**Proposition 8.1.11.** *Let  $F : \mathcal{S} \rightarrow \mathcal{T}$  be a regular functor.*

- (i) *The induced functor  $\mathbf{s}F : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{T}$  preserves internal Kan fibrations and internal trivial Kan fibrations.*
- (ii) *The induced functor  $\mathbf{Kan}(F) : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathbf{Kan}(\mathcal{T})$  preserves Brown factorisations and Dugger–Isaksen weak equivalences.*
- (iii) *If  $F : \mathcal{S} \rightarrow \mathcal{T}$  is conservative, then  $\mathbf{s}F : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{T}$  reflects internal Kan fibrations and internal trivial Kan fibrations, and  $\mathbf{Kan}(F) : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathbf{Kan}(\mathcal{T})$  reflects Dugger–Isaksen weak equivalences.*

*Proof.* These are immediate consequences of the fact that these definitions can be phrased in terms of properties of constructions using only finite limits and regular epimorphisms. ■

**Theorem 8.1.12.** *Let  $\mathcal{S}$  be a regular category.*

- (i)  *$\mathbf{Kan}(\mathcal{S})$ , equipped with the class of Dugger–Isaksen weak equivalences, is a saturated homotopical category.*
- (ii) *An internal Kan fibration of internal Kan complexes in  $\mathcal{S}$  is an internal trivial Kan fibration if and only if it is also a Dugger–Isaksen weak equivalence.*
- (iii)  *$\mathbf{Kan}(\mathcal{S})$  is a category of fibrant objects where the weak equivalences are the Dugger–Isaksen weak equivalences and the fibrations are the internal Kan fibrations.*

*Proof.* (i) and (ii). The claims are known in the case where  $\mathcal{S} = \mathbf{Set}$ , by [remark 8.1.2](#), [theorem 1.4.32](#), [proposition 1.5.9](#), and [lemma 3.1.8](#). Clearly, the



same is true for  $\mathcal{S} = \mathbf{Set}^B$  for any set  $B$ . On the other hand, if  $\mathcal{S}$  is any small regular category, we may apply the classical completeness theorem (A.7.22) and proposition 8.1.11 to reduce to the case of  $\mathbf{Set}^B$ , so we are done in this case. In general, we may assume  $\mathcal{S}$  is small by appealing to the universe axiom<sup>[1]</sup> and the fact that the properties of being an internal Kan fibration, internal trivial Kan fibration, or Dugger–Isaksen weak equivalence are defined without reference to a choice of universe.

(iii). We have just verified axiom A, proposition 8.1.4 implies axioms B and C are satisfied, corollary 8.1.7 is axiom D, and axiom E is satisfied by definition. ■

We have so far avoided discussing homotopy groups for internal Kan complexes. This is because they need not exist in a general regular category: clearly, if we are able to take quotients of internal equivalence relations, then we can construct  $\pi_0$ ; and conversely, because internal equivalence relations define internal Kan complexes, being able to construct  $\pi_0$  implies we can take quotients of internal equivalence relations. This suggests that the right setting for these constructions is an effective regular category.

**Proposition 8.1.13.** *If  $X$  is an internal Kan complex in a regular category  $\mathcal{S}$ , then the regular image of the morphism  $\langle d_1, d_0 \rangle : X_1 \rightarrow X_0 \times X_0$  defines an equivalence relation on  $X_0$ .*

*Proof.* For  $\mathcal{S} = \mathbf{Set}$ , this is a special case of proposition 1.4.17; and in general, we appeal to proposition 8.1.11 and the classical completeness theorem (A.7.22). ■

**Proposition 8.1.14.** *Let  $\mathcal{S}$  be a regular category and let  $(-)_0 : \mathfrak{s}\mathcal{S} \rightarrow \mathcal{S}$  be the functor that sends a simplicial object  $X$  in  $\mathcal{S}$  to the object  $X_0$  in  $\mathcal{S}$ .*

- (i) *The functor  $(-)_0 : \mathfrak{s}\mathcal{S} \rightarrow \mathcal{S}$  has both a left adjoint  $\text{disc} : \mathcal{S} \rightarrow \mathfrak{s}\mathcal{S}$  and a right adjoint  $\text{codisc} : \mathcal{S} \rightarrow \mathfrak{s}\mathcal{S}$ .*
- (ii) *The functor  $\text{disc} : \mathcal{S} \rightarrow \mathfrak{s}\mathcal{S}$  is fully faithful.*
- (iii) *For each morphism  $f : X \rightarrow Y$  in  $\mathcal{S}$ , the morphism  $\text{disc } f : \text{disc } X \rightarrow \text{disc } Y$  is an internal Kan fibration in  $\mathcal{S}$ . In particular, each  $\text{disc } X$  is an internal Kan complex in  $\mathcal{S}$ .*

[1] See §0.1.

*Proof.* (i). Let  $\text{disc} : \mathcal{S} \rightarrow \mathbf{sS}$  be the functor that sends an object  $X$  to the constant simplicial object defined by  $X$  and let  $\text{codisc} : \mathcal{S} \rightarrow \mathbf{sS}$  be the functor that sends an object  $X$  to the simplicial object defined by the formula  $[n] \mapsto X^{n+1}$ . It is straightforward to check that  $\text{disc}$  is a left adjoint for  $(-)_0$  and  $\text{codisc}$  is a right adjoint for  $(-)_0$ .

(ii). The functor  $\text{disc} : \mathcal{S} \rightarrow \mathbf{sS}$  is fully faithful because  $\mathbf{\Delta}$  is connected.

(iii). Since the face and degeneracy operators of  $\text{disc } Y$  are isomorphisms, the morphisms  $\{\Delta^n, \text{disc } Y\} \rightarrow \{\Lambda_k^n, \text{disc } Y\}$  induced by the horn inclusions  $\Lambda_k^n \hookrightarrow \Delta^n$  must also be isomorphisms. Thus, the induced morphism

$$\{\Delta^n, \text{disc } X\} \rightarrow \{\Lambda_k^n, X\} \times_{\{\Lambda_k^n, Y\}} \{\Delta^n, Y\}$$

is an isomorphism (and a regular epimorphism *a fortiori*). ■

**Proposition 8.1.15.** *Let  $\mathcal{S}$  be an effective regular category and let  $\text{disc} : \mathcal{S} \rightarrow \mathbf{Kan}(\mathcal{S})$  be a left adjoint for the functor  $(-)_0 : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathcal{S}$ .*

(i) *The functor  $\text{disc} : \mathcal{S} \rightarrow \mathbf{Kan}(\mathcal{S})$  has a left adjoint,  $\pi_0 : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathcal{S}$ .*

(ii) *If  $f : X \rightarrow Y$  is a Dugger–Isaksen weak equivalence in  $\mathcal{S}$ , then the morphism  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is an isomorphism.*

*Proof.* (i). We define the functor  $\pi_0 : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathcal{S}$  by the following coequaliser diagram:

$$X_1 \begin{array}{c} \xrightarrow{d_1} \\ \xrightarrow{d_0} \end{array} X_0 \longrightarrow \pi_0 X$$

Note that such coequalisers exist, by [proposition 8.1.13](#) and [lemma A.7.25](#). It is straightforward to verify that  $\pi_0 : \mathbf{Kan}(\mathcal{S}) \rightarrow \mathcal{S}$  is indeed a left adjoint for  $\text{disc} : \mathcal{S} \rightarrow \mathbf{Kan}(\mathcal{S})$ .

(ii). In the case  $\mathcal{S} = \mathbf{Set}$ , we may apply [corollary 1.3.14](#) and [theorem 1.4.32](#); and in general, we appeal to [proposition 8.1.11](#) and the classical completeness theorem ([A.7.22](#)). ■

**Definition 8.1.16.** Let  $n$  be a positive integer and let  $X$  be an internal Kan complex in an effective regular category  $\mathcal{S}$ .

- The **internal based  $n$ -loop fibration** on  $X$  is the internal Kan fibration  $\Omega^n(X) \rightarrow X$  defined by the following pullback diagram in  $\mathbf{Kan}(\mathcal{S})$ ,

$$\begin{array}{ccc} \Omega^n(X) & \longrightarrow & \Delta^n \pitchfork X \\ \downarrow & & \downarrow \\ X & \longrightarrow & \partial\Delta^n \pitchfork X \end{array}$$

where  $\Delta^n \pitchfork X \rightarrow \partial\Delta^n \pitchfork X$  is the internal Kan fibration induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$  and  $X \rightarrow \partial\Delta^n \pitchfork X$  is the morphism induced by  $\partial\Delta^n \rightarrow \Delta^0$ .

- Let  $x$  be a morphism  $\text{disc } T \rightarrow X$  in  $\mathbf{Kan}(\mathcal{S})$ . The **internal based  $n$ -loop space** of  $(X, x)$  is the internal Kan complex  $\Omega^n(X, x)$  in  $\mathcal{S}_{/T}$  defined by the following pullback diagram in  $\mathbf{Kan}(\mathcal{S})$ :

$$\begin{array}{ccc} \Omega^n(X, x) & \longrightarrow & \Omega^n(X) \\ \downarrow & & \downarrow \\ \text{disc } T & \xrightarrow{x} & X \end{array}$$

The **internal  $n$ -th homotopy group** of  $(X, x)$  is the object  $\pi_n(X, x) = \pi_0\Omega^n(X, x)$  in  $\mathcal{S}_{/T}$ .

**REMARK 8.1.17.** It is clear that the above constructions are functorial. Moreover,  $\pi_n(X, x)$  admits a natural internal group structure (in  $\mathcal{S}_{/T}$ ); but we do not need this fact.

**Theorem 8.1.18.** Let  $\mathcal{S}$  be an effective regular category and let  $f : X \rightarrow Y$  be a morphism in  $\mathbf{Kan}(\mathcal{S})$ . The following are equivalent:

- (i)  $f : X \rightarrow Y$  is a Dugger–Isaksen weak equivalence.
- (ii)  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is an isomorphism in  $\mathcal{S}$  and, for all positive integers  $n$ , all objects  $T$  in  $\mathcal{S}$ , and all morphisms  $x : \text{disc } T \rightarrow X$  in  $\mathbf{Kan}(\mathcal{S})$ ,  $\pi_n f : \pi_n(X, x) \rightarrow \pi_n(Y, f \circ x)$  is an isomorphism in  $\mathcal{S}_{/T}$ .
- (iii)  $\pi_0 f : \pi_0 X \rightarrow \pi_0 Y$  is an isomorphism in  $\mathcal{S}$  and, for all positive integers  $n$ ,  $\pi_n f : \pi_n(X, \bar{x}) \rightarrow \pi_n(Y, f \circ \bar{x})$  is an isomorphism in  $\mathcal{S}_{/X_0}$ , where  $\bar{x} : \text{disc } X_0 \rightarrow X$  is the component of the adjunction counit.

*Proof.* Apply the classical completeness theorem (A.7.22) to theorem 1.4.32. ■

REMARK 8.1.19. It should be emphasised that the weak equivalences in  $\mathbf{Kan}(\mathcal{S})$  really are *weak* equivalences: Bezem and Coquand [2013] have constructed an internal Kan fibration over  $\Delta^1 \odot 1$  in the presheaf topos  $\mathcal{S} = [\mathbb{3}, \mathbf{Set}]$  such that the two canonical fibres are not homotopy equivalent. (Of course, the two fibres are weakly homotopy equivalent, because internal weak homotopy equivalences are closed under pullbacks along internal Kan fibrations, by proposition 3.7.12 and theorem 8.1.12.) Furthermore, since 1 is projective in  $\mathcal{S}$ , (non-)existence in the naïve sense coincides with (non-)existence in sheaf semantics.

We now move on to the problem of internal fibrant replacement. Recall the simplicial sets  $N(P^n)$  defined in paragraph 1.5.16.

**Definition 8.1.20.** Let  $\mathcal{S}$  be a (locally small) category with finite limits. An **extension** of a simplicial object  $X$  in  $\mathcal{S}$  is a simplicial object  $\mathrm{Ex}(X)$  equipped with bijections

$$\mathcal{S}(T, \mathrm{Ex}(X)_n) \cong \mathrm{Ex}(\mathcal{S}(T, X))_n = \mathbf{sSet}(N(P^n), \mathcal{S}(T, X))$$

that are natural in both  $n$  and  $T$ . The **canonical embedding**  $i_X : X \rightarrow \mathrm{Ex}(X)$  is the unique morphism in  $\mathbf{s}\mathcal{S}$  making the diagram below commute:

$$\begin{array}{ccc} \mathcal{S}(T, X) & \xlongequal{\quad} & \mathcal{S}(T, X) \\ \mathcal{S}(T, i_X) \downarrow & & \downarrow i \\ \mathcal{S}(T, \mathrm{Ex}(X)) & \xrightarrow{\cong} & \mathrm{Ex}(\mathcal{S}(T, X)) \end{array}$$

where  $i : \mathcal{S}(T, X) \rightarrow \mathrm{Ex}(\mathcal{S}(T, X))$  is the canonical embedding in  $\mathbf{sSet}$ .

REMARK 8.1.21. The above definition makes sense because finite weighted limits exist in  $\mathcal{S}$  and each  $N(P^n)$  is a finite simplicial set. Note that  $\mathrm{Ex}(X)$  is unique up to unique isomorphism, so we obtain a functor  $\mathrm{Ex} : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{S}$ ; and the natural transformation  $i : \mathrm{id} \Rightarrow \mathrm{Ex}$  in  $\mathbf{sSet}$  induces a natural transformation of the same form in  $\mathbf{s}\mathcal{S}$ . Also note that  $\mathrm{Ex} : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{S}$  preserves all degreewise limits that exist in  $\mathcal{S}$ .

**Proposition 8.1.22.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be categories with finite limits. If  $F : \mathcal{S} \rightarrow \mathcal{T}$  is a functor that preserves finite limits, then the induced functor  $\mathbf{s}F : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{T}$  preserves extensions and canonical embeddings.*

*Proof.* Obvious. ◆

**Lemma 8.1.23.** *Let  $X$  be a simplicial object in a regular category  $\mathcal{S}$ . Consider the following pullback diagram in  $\mathcal{S}$ ,*

$$\begin{array}{ccc} \{\Lambda_k^n, \text{Ex}(X)\} \times_{\{\Lambda_k^n, \text{Ex}^2(X)\}} \{\Delta^n, \text{Ex}^2(X)\} & \longrightarrow & \{\Delta^n, \text{Ex}^2(X)\} \\ \downarrow & & \downarrow \\ \{\Lambda_k^n, \text{Ex}(X)\} & \xrightarrow{\{\Lambda_k^n, i_{\text{Ex}(X)}\}} & \{\Lambda_k^n, \text{Ex}^2(X)\} \end{array}$$

where the morphism  $\{\Delta^n, \text{Ex}^2(X)\} \rightarrow \{\Lambda_k^n, \text{Ex}^2(X)\}$  is induced by the horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ . Then  $\{\Lambda_k^n, \text{Ex}(X)\} \times_{\{\Lambda_k^n, \text{Ex}^2(X)\}} \{\Delta^n, \text{Ex}^2(X)\} \rightarrow \{\Lambda_k^n, \text{Ex}(X)\}$  is a regular epimorphism in  $\mathcal{S}$ .

*Proof.* In the case where  $\mathcal{S} = \mathbf{Set}$ , the claim is a reformulation of lemma 1.5.21; and in general, we appeal to proposition 8.1.22 and the classical completeness theorem (A.7.22). ■

**Lemma 8.1.24.** *Let  $\mathcal{S}$  be a regular category. The functor  $\text{Ex} : \mathbf{sS} \rightarrow \mathbf{sS}$  preserves internal Kan fibrations and internal trivial Kan fibrations. In particular, it preserves internal Kan complexes.*

*Proof.* The claim is known in the case where  $\mathcal{S} = \mathbf{Set}$ , by lemma 1.5.22 and corollary 1.5.24; and in general, we appeal to propositions 8.1.11 and 8.1.22 and the classical completeness theorem (A.7.22). ■

**Lemma 8.1.25.** *For any internal Kan complex  $X$  in a regular category  $\mathcal{S}$ , the canonical embedding  $i_X : X \rightarrow \text{Ex}(X)$  is a Dugger–Isaksen weak equivalence in  $\mathcal{S}$ .*

*Proof.* The claim is known in the case where  $\mathcal{S} = \mathbf{Set}$ , by lemma 1.5.23; and in general, we appeal to propositions 8.1.11 and 8.1.22 and the classical completeness theorem (A.7.22). ■

¶ 8.1.26. Let  $\mathcal{S}$  be a (locally small) category with limits for finite diagrams and colimits for  $\omega$ -sequences. For each simplicial object  $X$  in  $\mathcal{S}$ , we define  $\text{Ex}^\infty(X)$  to be the colimit of the diagram below:

$$X \xrightarrow{i_X} \text{Ex}(X) \xrightarrow{i_{\text{Ex}(X)}} \text{Ex}^2(X) \xrightarrow{i_{\text{Ex}^2(X)}} \text{Ex}^3(X) \longrightarrow \dots$$

The above defines a functor  $\text{Ex}^\infty : \mathbf{sS} \rightarrow \mathbf{sS}$  and a natural transformation  $i^\infty : \text{id}_{\mathbf{sS}} \Rightarrow \text{Ex}^\infty$ .

**Proposition 8.1.27.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be categories with limits for finite diagrams and colimits for  $\omega$ -sequences. If  $F : \mathcal{S} \rightarrow \mathcal{T}$  is a functor that preserves finite limits, then:*

- (i) *For each simplicial object  $X$  in  $\mathcal{S}$ , there is a natural comparison morphism of the form  $\mathrm{Ex}(\mathbf{s}F(X)) \rightarrow \mathbf{s}F(\mathrm{Ex}(X))$ , and it is compatible with the natural transformation  $i^\infty : \mathrm{id} \Rightarrow \mathrm{Ex}^\infty$ .*
- (ii) *Moreover, if  $F : \mathcal{S} \rightarrow \mathcal{T}$  preserves colimits for  $\omega$ -sequences, then the above morphism is an isomorphism.*

*Proof.* Obvious. ◆

**Lemma 8.1.28.** *Let  $\mathcal{S}$  be a regular category with colimits for  $\omega$ -sequences. If  $\varinjlim : [\omega, \mathcal{S}] \rightarrow \mathcal{S}$  preserves finite limits, then the following classes of morphisms are closed under colimits for  $\omega$ -sequences in  $\mathbf{s}\mathcal{S}$ :*

- *The class of internal Kan fibrations in  $\mathcal{S}$ .*
- *The class of internal trivial Kan fibrations in  $\mathcal{S}$ .*
- *The class of Dugger–Isaksen weak equivalences in  $\mathcal{S}$ .*

*Proof.* Colimits commute with colimits, so  $\varinjlim : [\omega, \mathcal{S}] \rightarrow \mathcal{S}$  always preserves regular epimorphisms. Thus, the hypothesis implies  $\varinjlim : [\omega, \mathcal{S}] \rightarrow \mathcal{S}$  is a regular functor. On the other hand, the functor  $[\omega, \mathcal{S}] \rightarrow [\mathrm{ob} \omega, \mathcal{S}]$  induced by restriction along the inclusion  $\mathrm{ob} \omega \hookrightarrow \omega$  is a conservative regular functor, so [proposition 8.1.11](#) implies the internal Kan fibrations (resp. internal trivial Kan fibrations, Dugger–Isaksen weak equivalences) in  $[\omega, \mathcal{S}]$  are just the componentwise ones. We use the same proposition again to deduce that the indicated classes of morphisms in  $\mathbf{s}\mathcal{S}$  are closed under colimits for  $\omega$ -sequences. ■

**Theorem 8.1.29.** *Let  $\mathcal{S}$  be a regular category with colimits for  $\omega$ -sequences. If  $\varinjlim : [\omega, \mathcal{S}] \rightarrow \mathcal{S}$  preserves finite limits, then:*

- (i) *For any simplicial object  $X$  in  $\mathcal{S}$ , the simplicial object  $\mathrm{Ex}^\infty(X)$  is an internal Kan complex.*
- (ii) *For any internal Kan complex  $X$  in  $\mathcal{S}$ , the morphism  $i_X^\infty : X \rightarrow \mathrm{Ex}^\infty(X)$  is a Dugger–Isaksen weak equivalence.*

(iii) *The functor  $\text{Ex}^\infty : \mathbf{sS} \rightarrow \mathbf{sS}$  preserves internal Kan fibrations and finite limits.*

*Proof.* (i). If  $\varinjlim : [\omega, \mathcal{S}] \rightarrow \mathcal{S}$  preserves finite limits, then for any finite simplicial set  $Z$ , the functor  $\{Z, -\} : \mathbf{sS} \rightarrow \mathcal{S}$  preserves colimits for  $\omega$ -sequences. In particular, we have a commutative diagram of the form below,

$$\begin{array}{ccc} \varinjlim_{m:\omega} \{\Delta^n, \text{Ex}^{m+2}(X)\} & \xrightarrow{\cong} & \{\Delta^n, \text{Ex}^\infty(X)\} \\ \downarrow & & \downarrow \\ \varinjlim_{m:\omega} \{\Lambda_k^n, \text{Ex}^{m+2}(X)\} & \xrightarrow{\cong} & \{\Lambda_k^n, \text{Ex}^\infty(X)\} \end{array}$$

where the horizontal arrows are the canonical comparisons and the vertical arrows are induced by the horn inclusion  $\Lambda_k^n \hookrightarrow \Delta^n$ . [Lemma 8.1.23](#) gives us the following pullback square in  $\mathcal{S}$ ,

$$\begin{array}{ccc} \{\Lambda_k^n, \text{Ex}^{m+1}(X)\} \times_{\{\Lambda_k^n, \text{Ex}^{m+2}(X)\}} \{\Delta^n, \text{Ex}^{m+2}(X)\} & \longrightarrow & \{\Delta^n, \text{Ex}^{m+2}(X)\} \\ \downarrow & & \downarrow \\ \{\Lambda_k^n, \text{Ex}^{m+1}(X)\} & \xrightarrow{\{\Lambda_k^n, i_{\text{Ex}^{m+1}(X)}\}} & \{\Lambda_k^n, \text{Ex}^{m+2}(X)\} \end{array}$$

where  $\{\Lambda_k^n, \text{Ex}^{m+1}(X)\} \times_{\{\Lambda_k^n, \text{Ex}^{m+2}(X)\}} \{\Delta^n, \text{Ex}^{m+2}(X)\} \rightarrow \{\Lambda_k^n, \text{Ex}^{m+1}(X)\}$  is a regular epimorphism in  $\mathcal{S}$ . It is easy to see that

$$\varinjlim_{m:\omega} \{\Lambda_k^n, i_{\text{Ex}^{m+1}(X)}\} : \varinjlim_{m:\omega} \{\Lambda_k^n, \text{Ex}^{m+1}(X)\} \rightarrow \varinjlim_{m:\omega} \{\Lambda_k^n, \text{Ex}^{m+2}(X)\}$$

is an isomorphism in  $\mathcal{S}$ , so  $\{\Delta^n, \text{Ex}^\infty(X)\} \rightarrow \{\Lambda_k^n, \text{Ex}^\infty(X)\}$  is indeed a regular epimorphism in  $\mathcal{S}$ , as required.

(ii). [Theorem 8.1.12](#) and [lemma 8.1.25](#) imply that the composite morphism

$$X \xrightarrow{i_X} \text{Ex}(X) \longrightarrow \cdots \longrightarrow \text{Ex}^m(X) \xrightarrow{i_{\text{Ex}^m(X)}} \text{Ex}^{m+1}(X)$$

is a Dugger–Isaksen weak equivalence, and since  $i_X^\infty : X \rightarrow \text{Ex}^\infty(X)$  is a colimit for the  $\omega$ -sequence of these composites, we may apply [lemma 8.1.28](#) to deduce that it is also an internal weak homotopy equivalence.

(iii). Recalling [remark 8.1.21](#) and [lemma 8.1.24](#), an argument similar to the above proves the claim. ■

**Definition 8.1.30.** Let  $\mathcal{S}$  be a  $\sigma$ -pretopos. An **internal weak homotopy equivalence** of simplicial objects in  $\mathcal{S}$  is a morphism  $f : X \rightarrow Y$  in  $\mathbf{s}\mathcal{S}$  such that  $\mathrm{Ex}^\infty(f) : \mathrm{Ex}^\infty(X) \rightarrow \mathrm{Ex}^\infty(Y)$  is a Dugger–Isaksen weak equivalence.

REMARK 8.1.31. Recalling the 2-out-of-3 property of Dugger–Isaksen weak equivalences, [theorem 8.1.29](#) implies that the internal weak homotopy equivalences of internal Kan complexes are precisely the Dugger–Isaksen weak equivalences.

REMARK 8.1.32. [Lemma 3.1.8](#) and [theorem 8.1.12](#) imply that  $\mathbf{s}\mathcal{S}$ , equipped with the class of internal weak homotopy equivalences, is a saturated homotopical category.

REMARK 8.1.33. [Theorems 8.1.12](#) and [8.1.29](#) imply that an internal Kan fibration  $p : X \rightarrow Y$  is an internal weak homotopy equivalence if and only if  $\mathrm{Ex}^\infty(p) : \mathrm{Ex}^\infty(X) \rightarrow \mathrm{Ex}^\infty(Y)$  is an internal trivial Kan fibration (of internal Kan complexes). In particular, if  $p : X \rightarrow Y$  is an internal trivial Kan fibration, then so is  $\mathrm{Ex}^\infty(p) : \mathrm{Ex}^\infty(X) \rightarrow \mathrm{Ex}^\infty(Y)$ , and therefore  $p : X \rightarrow Y$  is an internal weak homotopy equivalence.

REMARK 8.1.34. If  $\mathcal{S} = \mathbf{Set}$ , then an internal weak homotopy equivalence of simplicial objects in  $\mathcal{S}$  is precisely a weak homotopy equivalence of simplicial sets in the usual sense, by [proposition 1.5.28](#). Similarly, if  $\mathcal{S} = [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$  for a small category  $\mathbb{C}$ , then an internal weak homotopy equivalence of simplicial objects in  $\mathcal{S}$  is the same thing as a componentwise weak homotopy equivalence of simplicial presheaves on  $\mathbb{C}$ .

**Proposition 8.1.35.** *Let  $\mathcal{S}$  and  $\mathcal{T}$  be  $\sigma$ -pretoposes and let  $F : \mathcal{S} \rightarrow \mathcal{T}$  be a regular functor that preserves colimits for countable diagrams.*

- (i) *The induced functor  $\mathbf{s}F : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{T}$  preserves internal weak homotopy equivalences.*
- (ii) *If  $F : \mathcal{S} \rightarrow \mathcal{T}$  is conservative, then  $\mathbf{s}F : \mathbf{s}\mathcal{S} \rightarrow \mathbf{s}\mathcal{T}$  reflects internal weak homotopy equivalences.*

*Proof.* Combine [propositions 8.1.11](#) and [8.1.27](#). ■

**Theorem 8.1.36.** *Let  $\mathcal{S}$  be a  $\sigma$ -pretopos. For any simplicial object  $X$  in  $\mathcal{S}$ , the morphism  $i_X^\infty : X \rightarrow \mathrm{Ex}^\infty(X)$  is an internal weak homotopy equivalence.*

*Proof.* First, we prove the claim in the case of a Grothendieck topos  $\mathcal{E}$ . Let  $\mathcal{C}$  be the *opposite* of the category of finite simplicial sets. It is not hard to see



that the functor  $[C, \mathcal{E}] \rightarrow \mathbf{s}\mathcal{E}$  obtained by restricting along  $\Delta^\bullet : \Delta^{\text{op}} \rightarrow C$  itself restricts to an equivalence between the full subcategory of finite-limit-preserving functors  $C \rightarrow \mathcal{E}$  and the category  $\mathbf{s}\mathcal{E}$  itself. Thus by Diaconescu’s theorem,<sup>[2]</sup> for each simplicial object  $X$  in  $\mathcal{E}$ , there is a functor  $x^* : [C^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{E}$  such that  $x^*$  is a finite-limit-preserving left adjoint and  $X \cong x^*M$ , where  $M$  is the simplicial object in  $[C^{\text{op}}, \mathbf{Set}]$  obtained by composing  $\Delta^\bullet : \Delta^{\text{op}} \rightarrow C$  and the Yoneda embedding  $C \rightarrow [C^{\text{op}}, \mathbf{Set}]$ .

Recalling [proposition 8.1.35](#), we see that the canonical embedding  $i_X^\infty : X \rightarrow \text{Ex}^\infty(X)$  is an internal weak homotopy equivalence of simplicial objects in  $\mathcal{E}$  as soon as the universal canonical embedding  $i_M^\infty : M \rightarrow \text{Ex}^\infty(M)$  is an internal weak homotopy equivalence in  $[C^{\text{op}}, \mathbf{Set}]$ . But by [remark 8.1.34](#), this is a special case of [theorem 1.5.27](#). Thus the claim is proved in the case where  $\mathcal{E}$  is a Grothendieck topos.

Now, let  $S$  be small  $\sigma$ -pretopos. Then there exist a Grothendieck topos  $\mathcal{E}$  and a fully faithful functor  $S \rightarrow \mathcal{E}$  that preserves limits for all diagrams and colimits for countable diagrams: see [proposition A.7.31](#). Applying [proposition 8.1.35](#), we then deduce the claim for  $S$  from the result for  $\mathcal{E}$  proved above. In general, we may assume  $S$  is small by appealing to the universe axiom.<sup>[3]</sup> ■

**Corollary 8.1.37.** *Let  $S$  be a  $\sigma$ -pretopos. There exist a functor  $R : \mathbf{s}S \rightarrow \mathbf{s}S$  and a natural transformation  $i : \text{id}_{\mathbf{s}S} \Rightarrow R$  such that, for all simplicial objects  $X$  in  $S$ ,  $RX$  is an internal Kan complex and  $i_X : X \rightarrow RX$  is an internal weak homotopy equivalence. Moreover, any such functor  $R$  preserves and reflects weak homotopy equivalences.*

*Proof.* By [theorems 8.1.29](#) and [8.1.36](#), we may take  $(R, i)$  to be  $(\text{Ex}^\infty, i^\infty)$ . The 2-out-of-3 property of internal weak homotopy equivalences implies the remainder of the claim. ■

## 8.2 The Heller–Joyal model structure

**Prerequisites.** §§ [0.5](#), [1.4](#), [1.5](#), [4.1](#), [5.1](#), [8.1](#), [A.7](#).

In this section, we study a model structure on the category of simplicial objects in a Grothendieck topos. This model structure was first constructed by Joyal

[2] See Lemma 3.2.5 and Theorem 3.2.7 in [Johnstone, 2002a, Part B], or Corollary 3 in [ML–M, Ch. VII, §9].

[3] See §0.1.

[1984] and generalises the injective model structure of Heller [1988] on the category of simplicial presheaves. We will mostly follow Joyal’s original proof, but it should be noted that a different proof appears in [John F. Jardine, 1987].

**Definition 8.2.1.** Let  $\mathcal{E}$  be a Grothendieck topos and let  $\mathbf{s}\mathcal{E}$  be the category of simplicial objects in  $\mathcal{E}$ .

- A **weak equivalence** in  $\mathbf{s}\mathcal{E}$  is an internal weak homotopy equivalence of simplicial objects in  $\mathcal{E}$ .
- An **injective cofibration** in  $\mathbf{s}\mathcal{E}$  is a monomorphism in  $\mathbf{s}\mathcal{E}$ .
- An **injective trivial cofibration** in  $\mathbf{s}\mathcal{E}$  is an injective cofibration in  $\mathbf{s}\mathcal{E}$  that is also a weak equivalence.
- An **injective fibration** in  $\mathbf{s}\mathcal{E}$  is a morphism in  $\mathbf{s}\mathcal{E}$  with the right lifting property with respect to the injective trivial cofibrations.
- An **injective trivial fibration** in  $\mathbf{s}\mathcal{E}$  is a morphism in  $\mathbf{s}\mathcal{E}$  with the right lifting property with respect to the injective cofibrations.

**REMARK 8.2.2.** It is well known that  $\mathbf{s}\mathcal{E}$  is a Grothendieck topos if  $\mathcal{E}$  is, so  $\mathbf{s}\mathcal{E}$  with the Heller–Joyal model structure (once it is shown to exist) is a Cisinski model category.<sup>[4]</sup> In particular, injective trivial fibrations in  $\mathbf{s}\mathcal{E}$  are (by definition) the same thing as Cisinski trivial fibrations in  $\mathbf{s}\mathcal{E}$ .

**Proposition 8.2.3.** *Let  $\mathcal{E}$  and  $\mathcal{F}$  be Grothendieck toposes and let  $u : \mathcal{E} \rightarrow \mathcal{F}$  be a geometric morphism. Then the inverse image functor  $u^* : \mathcal{F} \rightarrow \mathcal{E}$  induces a functor  $su^* : \mathbf{s}\mathcal{F} \rightarrow \mathbf{s}\mathcal{E}$  that preserves injective cofibrations, injective trivial cofibrations, and weak equivalences.*

*Proof.* By definition,  $u^* : \mathcal{F} \rightarrow \mathcal{E}$  is a functor that preserves finite limits and has a right adjoint. Thus,  $su^* : \mathbf{s}\mathcal{F} \rightarrow \mathbf{s}\mathcal{E}$  preserves injective cofibrations (i.e. monomorphisms), and to complete the proof, it suffices to show that  $su^* : \mathbf{s}\mathcal{F} \rightarrow \mathbf{s}\mathcal{E}$  preserves weak equivalences. But by [proposition 8.1.27](#),  $su^*$  commutes with  $\text{Ex}^\infty$ , so we may use the argument in the proof of [lemma 8.1.28](#) to deduce that  $su^*$  indeed preserves weak equivalences. ■

**Lemma 8.2.4.** *Let  $\mathcal{E}$  be a Grothendieck topos. The class of weak equivalences in  $\mathbf{s}\mathcal{E}$  is closed under colimits for small filtered diagrams (in  $[2, \mathbf{s}\mathcal{E}]$ ).*

---

[4] See [definition 5.3.1](#).

*Proof.* Let  $\mathcal{J}$  be a small filtered category. Since  $\mathcal{E}$  is a Grothendieck topos,  $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathcal{E}] \rightarrow \mathcal{E}$  preserves finite limits. Thus, by [proposition 8.2.3](#),  $\varinjlim_{\mathcal{J}}$  preserves weak equivalences of simplicial objects, and it is not hard to see that a weak equivalence of simplicial objects in  $[\mathcal{J}, \mathcal{E}]$  is precisely a morphism that is componentwise a weak equivalence of simplicial objects in  $\mathcal{E}$ . ■

**Lemma 8.2.5.** *Let  $\mathcal{E}$  be a Grothendieck topos.*

- (i) *The class of injective cofibrations (resp. injective trivial cofibrations) in  $\mathbf{s}\mathcal{E}$  contains all isomorphisms in  $\mathbf{s}\mathcal{E}$ .*
- (ii) *The class of injective cofibrations (resp. injective trivial cofibrations) in  $\mathbf{s}\mathcal{E}$  is closed under (finite and) transfinite composition.*
- (iii) *The class of injective cofibrations (resp. injective trivial cofibrations) in  $\mathbf{s}\mathcal{E}$  is closed under retracts.*
- (iv) *The class of injective cofibrations (resp. injective trivial cofibrations) in  $\mathbf{s}\mathcal{E}$  is closed under pushouts.*
- (v) *The class of injective cofibrations (resp. injective trivial cofibrations) in  $\mathbf{s}\mathcal{E}$  is closed under coproducts.*

*Proof.* The claims for the class of injective cofibrations are well known; we will prove the claims for the class of injective trivial cofibrations.

(i). Obvious.

(ii) and (iii). It suffices to show that the class of weak equivalences is closed under the relevant operations; but this was done in [remark 8.1.32](#) and [lemma 8.2.4](#).

TODO: Fill in the details...

(iv). Use the classical completeness theorem for separable theories.

(v). By [lemma 0.5.6](#), a class of morphisms that is closed under transfinite composition and pushouts is also closed under coproducts. ■

**Lemma 8.2.6.** *Let  $\mathcal{E}$  be a Grothendieck topos. Every injective trivial fibration in  $\mathbf{s}\mathcal{E}$  is an internal trivial Kan fibration of simplicial objects in  $\mathcal{E}$ .*

*Proof.* Let  $p : X \rightarrow Y$  be an injective trivial fibration in  $\mathbf{sE}$  and let  $n$  be a natural number. We wish to show that the following diagram is a weak pullback square in  $\mathcal{E}$ ,

$$\begin{array}{ccc} \{\Delta^n, X\} & \xrightarrow{\{\Delta^n, p\}} & \{\Delta^n, Y\} \\ \downarrow & & \downarrow \\ \{\partial\Delta^n, X\} & \xrightarrow{\{\partial\Delta^n, p\}} & \{\partial\Delta^n, Y\} \end{array}$$

where the vertical arrows are induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$ , so it suffices to verify that the morphism

$$\{\Delta^n, X\} \rightarrow \{\partial\Delta^n, X\} \times_{\{\partial\Delta^n, Y\}} \{\Delta^n, Y\}$$

is a *split* epimorphism in  $\mathcal{E}$ . Let  $L$  denote the codomain of this morphism. By adjointness, we find that  $\{\Delta^n, X\} \rightarrow L$  splits if and only if there is a morphism  $\Delta^n \odot L \rightarrow X$  in  $\mathbf{sE}$  making the diagram below commute,

$$\begin{array}{ccc} \partial\Delta^n \odot L & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n \odot L & \longrightarrow & Y \end{array}$$

where  $\partial\Delta^n \odot L \rightarrow \Delta^n \odot L$  is induced by the boundary inclusion  $\partial\Delta^n \hookrightarrow \Delta^n$ ,  $\partial\Delta^n \odot L \rightarrow X$  is the left adjoint transpose of the projection  $L \rightarrow \partial\Delta^n \pitchfork X$ , and  $\Delta^n \odot L \rightarrow Y$  is the left adjoint transpose of the projection  $L \rightarrow \Delta^n \pitchfork Y$ . But  $\partial\Delta^n \odot L \rightarrow \Delta^n \odot L$  is an injective cofibration and  $p : X \rightarrow Y$  is an injective trivial cofibration, hence the required morphism  $\Delta^n \odot L \rightarrow X$  indeed exists. ■

**Lemma 8.2.7.** *Let  $\mathcal{S}$  be a Grothendieck topos. There exist a set  $\mathcal{I}$  of monomorphisms in  $\mathcal{S}$  such that the relative  $\mathcal{I}$ -cell complexes are precisely the monomorphisms in  $\mathcal{S}$ .*

*Proof.* We may assume without loss of generality that  $\mathcal{S} = \mathbf{Sh}(C, J)$ , where  $C$  is a small category and  $J$  is a subcanonical Grothendieck topology on  $C$ . Consider the class of all monomorphisms  $U \rightarrow V$  in  $\mathcal{S}$  where  $V$  is a quotient (in  $\mathcal{S}$ ) of some representable sheaf: since  $\mathcal{S}$  is well-powered and well-copowered, we may choose a set  $\mathcal{I}$  of representatives of isomorphism classes of such monomorphisms. The argument in the proof of [proposition 0.5.20](#) then shows that every monomorphism in  $\mathcal{S}$  is a relative  $\mathcal{I}$ -cell complex. ■

**Lemma 8.2.8.** *Let  $\mathcal{E}$  be a Grothendieck topos. The full subcategory of  $[2, \mathbf{s}\mathcal{E}]$  spanned by the weak equivalences in  $\mathbf{s}\mathcal{E}$  is an accessible subcategory of  $[2, \mathbf{s}\mathcal{E}]$ .*

*Proof.* It is not hard to see that  $\text{Ex} : \mathbf{s}\mathcal{E} \rightarrow \mathbf{s}\mathcal{E}$  is an accessible functor, so the same is true for  $\text{Ex}^\infty : \mathbf{s}\mathcal{E} \rightarrow \mathbf{s}\mathcal{E}$ . Thus, by applying [proposition 0.3.30](#), it suffices to show that the full subcategory of  $[2, \mathbf{s}\mathcal{E}]$  spanned by the Dugger–Isaksen weak equivalences is an accessible subcategory of  $[2, \mathbf{s}\mathcal{E}]$ . Clearly, this subcategory is the category of models (in  $\mathcal{E}$ ) of a small (geometric) sketch, so it is closed in  $[2, \mathbf{s}\mathcal{E}]$  under colimits for small filtered diagrams, and we may apply [Theorem 2.60](#) in [\[LPAC\]](#) to deduce that it is indeed an accessible category.  $\square$

**Theorem 8.2.9** (Heller, Joyal). *Let  $\mathcal{E}$  be a Grothendieck topos. The following data constitute a cofibrantly generated model structure on  $\mathbf{s}\mathcal{E}$ :*

- *The weak equivalences are the internal weak homotopy equivalences of simplicial objects in  $\mathcal{E}$ .*
- *The cofibrations are the injective cofibrations, i.e. the monomorphisms.*
- *The fibrations are the injective fibrations, i.e. the morphisms that have the right lifting property with respect to monomorphisms that are weak equivalences.*

*This model structure is called the **Heller–Joyal model structure**.*

*Proof.* We have shown the following:

- The class of weak equivalences in  $\mathbf{s}\mathcal{E}$  has the 2-out-of-3 property and is closed under retracts, by [remark 8.1.32](#).
- The class of injective cofibrations is the class of relative  $\mathcal{I}$ -cell complexes for a subset  $\mathcal{I} \subset \text{mor } \mathbf{s}\mathcal{E}$ , by [lemma 8.2.7](#), and every injective fibration is a weak equivalence, by [remark 8.1.33](#) and [lemma 8.2.6](#).
- The class of injective trivial cofibrations is closed under pushouts and transfinite composition, by [lemma 8.2.5](#).
- The full subcategory of  $[2, \mathbf{s}\mathcal{E}]$  is accessible, by [lemma 8.2.8](#).

Thus, we may apply Smith’s recognition principle ([5.1.8](#)) to deduce that we have the required cofibrantly generated model structure.  $\blacksquare$



## GENERALITIES

### A.I Adjoint and mates

We begin by recalling a standard definition:

**Definition A.I.1.** An **adjunction of categories** consists of the following data:

- A functor  $F : C \rightarrow D$ , called the **left adjoint**.
- A functor  $G : D \rightarrow C$ , called the **right adjoint**.
- A natural transformation  $\eta : \text{id}_C \Rightarrow GF$ , called the **unit**.
- A natural transformation  $\varepsilon : FG \Rightarrow \text{id}_D$ , called the **counit**.

These are moreover required to satisfy the **triangle identities**:

$$\varepsilon F \cdot F \eta = \text{id}_F \qquad G \varepsilon \cdot \eta G = \text{id}_G$$

If such data exist, we write

$$F \dashv G : D \rightarrow C$$

and say that  $F$  is a left adjoint of  $G$ , and  $G$  is a right adjoint of  $F$ .

**Proposition A.I.2.** *Let  $F \dashv G : D \rightarrow C$  be an adjunction, with unit  $\eta : \text{id}_C \Rightarrow GF$  and counit  $\varepsilon : FG \Rightarrow \text{id}_D$ . The following are equivalent for an object  $X$  in  $C$ :*

- (i) *The morphism  $\eta_X : X \rightarrow GFX$  is a monomorphism.*

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- (ii) For all objects  $T$  in  $\mathcal{C}$ , the hom-set map  $\mathcal{C}(T, X) \rightarrow \mathcal{D}(FT, FX)$  induced by  $F : \mathcal{C} \rightarrow \mathcal{D}$  is injective.

Dually, the following are equivalent for an object  $A$  in  $\mathcal{D}$ :

- (i') The morphism  $\varepsilon_A : FGA \rightarrow A$  is an epimorphism.
- (ii') For all objects  $B$  in  $\mathcal{D}$ , the hom-set map  $\mathcal{D}(A, B) \rightarrow \mathcal{C}(GA, GB)$  is injective.

*Proof.* Consider the hom-set map  $\mathcal{C}(T, X) \rightarrow \mathcal{C}(T, GFX)$  defined by  $x \mapsto \eta_X \circ x$ . By naturality,

$$\eta_X \circ x = GFx \circ \eta_X$$

but the left triangle identity implies

$$\varepsilon_{FX} \circ F(\eta_X \circ x) = Fx$$

and so  $\eta_X \circ x_0 = \eta_X \circ x_1$  if and only if  $Fx_0 = Fx_1$ . ■

**Proposition A.I.3.** Let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  be an adjunction with unit  $\eta$  and counit  $\varepsilon$ . The following are equivalent:

- (i) The left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$  is fully faithful.
- (ii) The adjunction unit  $\eta : \text{id}_{\mathcal{C}} \Rightarrow GF$  is a natural isomorphism.
- (iii) The natural transformation  $F\eta G : FG \Rightarrow FGFG$  is a natural isomorphism,  $F : \mathcal{C} \rightarrow \mathcal{D}$  is conservative, and  $G : \mathcal{D} \rightarrow \mathcal{C}$  is essentially surjective on objects.

Dually, the following are equivalent:

- (i') The right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$  is fully faithful.
- (ii') The adjunction counit  $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{D}}$  is a natural isomorphism.
- (iii') The natural transformation  $G\varepsilon F : GFGF \Rightarrow GF$  is a natural isomorphism,  $G : \mathcal{D} \rightarrow \mathcal{C}$  is conservative, and  $F : \mathcal{C} \rightarrow \mathcal{D}$  is essentially surjective on objects.



*Proof.* (i)  $\Leftrightarrow$  (ii). Let  $f : X \rightarrow Y$  be a morphism in  $C$ . By naturality, we have  $\eta_Y \circ f = GFf \circ \eta_X$ ; but the triangle identities imply the hom-set map  $D(FX, B) \rightarrow C(X, GB)$  given by  $g \mapsto Gg \circ \eta_X$  is also a bijection, so we deduce that the hom-set map  $C(X, Y) \rightarrow C(X, GFY)$  given by  $f \mapsto \eta_Y \circ f$  is a bijection if and only if the hom-set map  $C(X, Y) \rightarrow D(FX, FY)$  given by  $f \mapsto Ff$  is a bijection because  $F$  is fully faithful. We may then deduce that  $\eta$  is a natural isomorphism if and only if  $F$  is fully faithful.

(i)  $\Rightarrow$  (iii). We have already shown that  $\eta : \text{id}_C \Rightarrow GF$  is a natural isomorphism, so in particular  $F\eta G : FG \Rightarrow FGFG$  is a natural isomorphism. Fully faithful functors are conservative, so  $F$  is conservative. On the other hand, since  $\eta$  is a natural isomorphism,  $G$  is essentially surjective on objects.

(iii)  $\Rightarrow$  (ii). If  $F$  is conservative and  $F\eta G$  is a natural isomorphism, then  $\eta G$  is also a natural isomorphism. Since every object in  $C$  is isomorphic to one in the image of  $G$ , it follows that  $\eta$  is a natural isomorphism.  $\blacksquare$

**Proposition A.I.4.** *Let  $F \dashv G : D \rightarrow C$  be an adjunction.*

- $G : D \rightarrow C$  is fully faithful if and only if, for all categories  $\mathcal{E}$ , the induced functor  $F^* : [C, \mathcal{E}] \rightarrow [D, \mathcal{E}]$  is fully faithful.
- $F : C \rightarrow D$  is fully faithful if and only if, for all categories  $\mathcal{E}$ , the induced functor  $G^* : [D, \mathcal{E}] \rightarrow [C, \mathcal{E}]$  is fully faithful.

*Proof.* The two claims are formally dual; we will prove the first version.

Suppose  $G : D \rightarrow C$  is fully faithful. By [proposition A.I.3](#), the adjunction counit  $\varepsilon : FG \Rightarrow \text{id}_D$  must be a natural isomorphism. On the other hand, we have an induced adjunction  $G^* \dashv F^* : [C, \mathcal{E}] \rightarrow [D, \mathcal{E}]$  with counit induced by  $\varepsilon$ , so the same proposition implies  $F^*$  must be fully faithful.

Conversely, suppose  $F^* : [C, \mathcal{E}] \rightarrow [D, \mathcal{E}]$  is a fully faithful functor for all categories  $\mathcal{E}$ . Then the induced adjunction counit  $\varepsilon^* : G^*F^* \Rightarrow \text{id}_{[D, \mathcal{E}]}$  is a natural isomorphism. In particular, this is true when  $\mathcal{E} = D$ , so by considering the component of  $\varepsilon^*$  at  $\text{id}_D$ , we see that  $\varepsilon : FG \Rightarrow \text{id}_D$  itself is a natural isomorphism. Thus  $G : D \rightarrow C$  must be fully faithful.  $\blacksquare$

**Proposition A.I.5.** *Let  $F \dashv G : D \rightarrow C$  and  $F' \dashv G' : D' \rightarrow C'$  be adjunctions, let  $\eta : \text{id}_C \Rightarrow GF$  and  $\eta' : \text{id}_{C'} \Rightarrow G'F'$  be the respective units, and let  $\varepsilon : FG \Rightarrow \text{id}_D$  and  $\varepsilon' : F'G' \Rightarrow \text{id}_{D'}$  be the respective counits. Let  $H : C \rightarrow C'$*

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and  $K : D \rightarrow D'$  be functors, and let  $\varphi$  and  $\psi$  be natural transformations as in the diagrams below:

$$\begin{array}{ccc} C & \xrightarrow{H} & C' \\ F \downarrow & \wr_{\varphi} & \downarrow F' \\ D & \xrightarrow{K} & D' \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{K} & D' \\ G \downarrow & \wr_{\psi} & \downarrow G' \\ C & \xrightarrow{H} & C' \end{array}$$

Then, the following are equivalent:

- (i)  $\varepsilon' K F \cdot F' \psi F \cdot F' H \eta = \varphi$ .
- (ii)  $\psi F \cdot H \eta = G' \varphi \cdot \eta' H$ .
- (iii)  $\psi = G' K \varepsilon \cdot G' \varphi G \cdot \eta' H G$ .
- (iv)  $\varepsilon' K \cdot F' \psi = K \varepsilon \cdot \varphi G$ .

*Proof.*

$$\begin{aligned} \text{(i)} \Rightarrow \text{(ii)}. \quad G' \varphi \cdot \eta' H &= G' \varepsilon' K F \cdot G' F' \psi F \cdot G' F' H \eta \cdot \eta' H \\ &= G' \varepsilon' K F \cdot \eta' G' K F \cdot \psi F \cdot H \eta \\ &= \psi F \cdot H \eta \end{aligned}$$

$$\begin{aligned} \text{(ii)} \Rightarrow \text{(iii)}. \quad G' K \varepsilon \cdot G' \varphi G \cdot \eta' H G &= G' K \varepsilon \cdot \psi F G \cdot H \eta G \\ &= \psi \cdot H G \varepsilon \cdot H \eta G \\ &= \psi \end{aligned}$$

$$\begin{aligned} \text{(iii)} \Rightarrow \text{(iv)}. \quad \varepsilon' K \cdot F' \psi &= \varepsilon' K \cdot F' G' K \varepsilon \cdot F' G' \varphi G \cdot F' \eta' H G \\ &= K \varepsilon \cdot \varphi G \cdot \varepsilon' H G \cdot F' \eta' H G \\ &= K \varepsilon \cdot \varphi G \end{aligned}$$

$$\begin{aligned} \text{(iv)} \Rightarrow \text{(i)}. \quad \varepsilon' K F \cdot F' \psi F \cdot F' H \eta &= K \varepsilon F \cdot \varphi G F \cdot F' H \eta \\ &= K \varepsilon F \cdot K F \eta \cdot \varphi \\ &= \varphi \end{aligned} \quad \blacksquare$$

**Definition A.I.6.** A **conjugate pair of natural transformations** is a pair  $(\varphi, \psi)$  satisfying the equivalent conditions of the above proposition. Given such, we say  $\varphi$  is the **left mate** of  $\psi$ , and  $\psi$  is the **right mate** of  $\varphi$ .

**Definition A.I.7.** Let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$  and  $F' \dashv G' : \mathcal{D}' \rightarrow \mathcal{C}'$  be adjunctions, let  $H : \mathcal{C} \rightarrow \mathcal{C}'$  and  $K : \mathcal{D} \rightarrow \mathcal{D}'$  be functors, and let  $\varphi$  and  $\psi$  be a conjugate pair of natural transformations as in the diagrams below:

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\
 F \downarrow & \wr_{\varphi} & \downarrow F' \\
 \mathcal{D} & \xrightarrow{K} & \mathcal{D}'
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\
 G \downarrow & \wr_{\psi} & \downarrow G' \\
 \mathcal{C} & \xrightarrow{H} & \mathcal{C}'
 \end{array}$$

We say the diagram on the right satisfies the **left Beck–Chevalley condition** if the left mate  $\varphi$  is a natural isomorphism, and we say the diagram on the left satisfies the **right Beck–Chevalley condition** if the right mate  $\psi$  is a natural isomorphism. More generally, the **local left Beck–Chevalley condition** is satisfied at an object  $C$  in  $\mathcal{C}$  if the component  $\varphi_C : F'HC \rightarrow KFC$  is an isomorphism, and the **local right Beck–Chevalley condition** is satisfied at an object  $D$  in  $\mathcal{D}$  if the component  $\psi_D : HGD \rightarrow G'KD$  is an isomorphism.

**REMARK A.I.8.** Unfortunately, the Beck–Chevalley conditions are not vacuous. For example, consider the following (strictly!) commutative diagram of forgetful functors:

$$\begin{array}{ccc}
 \mathbf{CRing} & \longrightarrow & \mathbf{Ab} \\
 \downarrow & & \downarrow \\
 \mathbf{Set} & \xrightarrow{\text{id}} & \mathbf{Set}
 \end{array}$$

The left mate of the trivial natural transformation in the above diagram is the group homomorphism  $\mathbb{Z}X \rightarrow \mathbb{Z}[X]$  that sends a generator in  $\mathbb{Z}X$  to the corresponding generator in  $\mathbb{Z}[X]$ ; clearly, this is never an isomorphism. However, this is unsurprising: we do not expect the free abelian group generated by  $X$  to be naturally isomorphic to the additive group of free commutative ring generated by  $X$ .

**Example A.I.9.** Let  $\mathcal{C}$  be a category with pullbacks, and suppose the following diagram is a pullback square in  $\mathcal{C}$ :

$$\begin{array}{ccc}
 Z & \xrightarrow{z} & X \\
 \text{sg} \downarrow & & \downarrow f \\
 W & \xrightarrow{w} & Y
 \end{array}$$

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Let  $\Sigma_f : \mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$  etc. be the functor that sends an object  $p : E \rightarrow X$  in  $\mathcal{C}_{/X}$  to the object  $f \circ p : E \rightarrow Y$  in  $\mathcal{C}_{/Y}$ , and consider the induced (strictly!) commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{C}_{/Z} & \xrightarrow{\Sigma_z} & \mathcal{C}_{/X} \\ \Sigma_g \downarrow & & \downarrow \Sigma_f \\ \mathcal{C}_{/W} & \xrightarrow{\Sigma_w} & \mathcal{C}_{/Y} \end{array}$$

Since  $\mathcal{C}$  has pullbacks,  $\Sigma_g$  and  $\Sigma_f$  have right adjoints,<sup>[1]</sup> and the pullback pasting lemma then implies that the above square satisfies the right Beck–Chevalley condition.

**Lemma A.1.10.** *Given a diagram of functors and natural transformations of the form below,*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ G \downarrow & \psi \nearrow & \downarrow G' \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \end{array}$$

where  $\psi : HG \Rightarrow G'K$  is a natural isomorphism,  $F \dashv G$ , and  $F' \dashv G'$ , for each object  $C$  in  $\mathcal{C}$ , the following are equivalent:

- (i) *The diagram satisfies the local left Beck–Chevalley condition at  $C$ .*
- (ii) *The functor  $(C \downarrow G) \rightarrow (HC \downarrow G')$  sending an object  $(D, f)$  in the comma category  $(C \downarrow G)$  to the object  $(KD, \psi_D \circ Hf)$  in  $(HC \downarrow G')$  preserves initial objects.*

*Proof.* We know  $(FC, \eta_C)$  is an initial object of  $(C \downarrow G)$  and  $(F'HC, \eta'_{HC})$  is an initial object of  $(HC \downarrow G')$ , so there is a unique morphism  $\varphi_C : F'HC \rightarrow KFC$  such that  $G'\varphi_C \circ \eta'_{HC} = \psi_{FC} \circ H\eta_C$ . However, we observe that

$$\begin{aligned} \varphi_C &= \varphi_C \circ \varepsilon'_{F'HC} \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'G'\varphi_C \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'\psi_{FC} \circ F'H\eta_C \end{aligned}$$

so  $\varphi_C$  is precisely the component at  $C$  of the left mate of  $\psi$ . ■

[1] See lemma A.2.17.

**Lemma A.I.11** (Pasting conjugate pairs).

- (i) Let  $F \dashv G : \mathcal{D} \rightarrow \mathcal{C}$ ,  $F' \dashv G' : \mathcal{D}' \rightarrow \mathcal{C}'$ , and  $F'' \dashv G'' : \mathcal{D}'' \rightarrow \mathcal{C}''$  be adjunctions, let  $H : \mathcal{C} \rightarrow \mathcal{C}'$ ,  $H' : \mathcal{C}' \rightarrow \mathcal{C}''$ ,  $K : \mathcal{D} \rightarrow \mathcal{D}'$ , and  $K' : \mathcal{D}' \rightarrow \mathcal{D}''$  be functors, and let  $\varphi, \varphi', \psi, \psi'$  be natural transformations as in the following pasting diagrams:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \\ F \downarrow & \wr_{\varphi} & \downarrow F' & \wr_{\varphi'} & \downarrow F'' \\ \mathcal{D} & \xrightarrow{K} & \mathcal{D}' & \xrightarrow{K'} & \mathcal{D}'' \end{array} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' & \xrightarrow{K'} & \mathcal{D}'' \\ G \downarrow & \wr_{\psi} & \downarrow G' & \wr_{\psi'} & \downarrow G'' \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \end{array}$$

Let  $\bar{\varphi} = K' \varphi \cdot \varphi' H$  and  $\bar{\psi} = \psi' K \cdot H' \psi$ . If  $(\varphi, \psi)$  and  $(\varphi', \psi')$  are conjugate pairs, then  $(\bar{\varphi}, \bar{\psi})$  is also a conjugate pair.

- (ii) Let  $F_1 \dashv G_1 : \mathcal{D} \rightarrow \mathcal{C}$ ,  $F_2 \dashv G_2 : \mathcal{E} \rightarrow \mathcal{D}$ ,  $F'_1 \dashv G'_1 : \mathcal{D}' \rightarrow \mathcal{C}'$ , and  $F'_2 \dashv G'_2 : \mathcal{E}' \rightarrow \mathcal{D}'$  be adjunctions, let  $H : \mathcal{C} \rightarrow \mathcal{C}'$ ,  $K : \mathcal{D} \rightarrow \mathcal{D}'$ , and  $L : \mathcal{E} \rightarrow \mathcal{E}'$  be functors, and let  $\varphi_1, \varphi_2, \psi_1, \psi_2$  be natural transformations as in the following pasting diagrams:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ F_1 \downarrow & \wr_{\varphi_1} & \downarrow F'_1 \\ \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ F_2 \downarrow & \wr_{\varphi_2} & \downarrow F'_2 \\ \mathcal{E} & \xrightarrow{L} & \mathcal{E}' \end{array} \quad \begin{array}{ccc} \mathcal{E} & \xrightarrow{L} & \mathcal{E}' \\ G_2 \downarrow & \wr_{\psi_2} & \downarrow G'_2 \\ \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ G_1 \downarrow & \wr_{\psi_1} & \downarrow G'_1 \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \end{array}$$

Let  $\varphi = \varphi_2 F_1 \cdot F'_2 \varphi_1$  and  $\psi = G'_1 \psi_2 \cdot \psi_1 G_2$ . If  $(\varphi_1, \psi_1)$  and  $(\varphi_2, \psi_2)$  are conjugate pairs, then  $(\varphi, \psi)$  is also a conjugate pair.

*Proof.* These are straightforward exercises in using the triangle identities.  $\blacklozenge$

**Proposition A.I.12.** Let  $u_1 \dashv u^* : \mathcal{C} \rightarrow \mathcal{A}$ ,  $q_1 \dashv q^* : \mathcal{B} \rightarrow \mathcal{D}$ ,  $v^* \dashv v_* : \mathcal{B} \rightarrow \mathcal{C}$ , and  $p^* \dashv p_* : \mathcal{D} \rightarrow \mathcal{A}$  be adjunctions, and let  $\theta : u^* p^* \Rightarrow v^* q^*$  be a natural transformation.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{v^*} & \mathcal{B} \\ u^* \downarrow & \wr_{\theta} & \downarrow q^* \\ \mathcal{A} & \xrightarrow{p^*} & \mathcal{D} \end{array} \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{u^*} & \mathcal{A} \\ v^* \downarrow & \wr_{\theta} & \downarrow p^* \\ \mathcal{B} & \xrightarrow{q^*} & \mathcal{D} \end{array}$$

The following are equivalent:

(i) *The diagram on the left satisfies the left Beck–Chevalley condition.*

(ii) *The diagram on the right satisfies the right Beck–Chevalley condition.*

*Proof.* Let  $\varphi : q_! p^* \Rightarrow v^* u_!$  be the left mate of  $\theta$ , and let  $\psi : u^* v_* \Rightarrow p_* q^*$  be the right mate of  $\theta$ . Then, by [proposition A.1.5](#),

$$\begin{aligned} \theta u_! \cdot p^* \eta^u &= q^* \varphi \cdot \eta^q p^* & \varepsilon^q v^* \cdot q_! \theta &= v^* \varepsilon^u \cdot \varphi u^* \\ \psi v^* \cdot u^* \eta^v &= p_* \theta \cdot \eta^p u^* & \varepsilon^p q^* \cdot p^* \psi &= q^* \varepsilon^v \cdot \theta v_* \end{aligned}$$

where the  $\eta$  denote the various adjunction units and the  $\varepsilon$  denote the various adjunction counits, thus:

$$\begin{aligned} \psi v^* u_! \cdot (u^* \eta^v u_! \cdot \eta^u) &= p_* \theta u_! \cdot \eta^p u^* u_! \cdot \eta^u \\ p_* \theta u_! \cdot p_* p^* \eta^u \cdot \eta^p &= p_* q^* \varphi \cdot (p_* \eta^q p^* \cdot \eta^p) \\ (\varepsilon^q \cdot q_! \varepsilon^p q^*) \cdot q_! p^* \psi &= \varepsilon^q \cdot q_! q^* \varepsilon^v \cdot q_! \theta v_* \\ \varepsilon^v \cdot \varepsilon^q v^* v_* \cdot q_! \theta v_* &= (\varepsilon^v \cdot v^* \varepsilon^u v_*) \cdot \varphi u^* v_* \end{aligned}$$

Thus,  $(\varphi, \psi)$  is a conjugate pair of natural transformations between the adjunctions  $v^* u_! \dashv u^* v_*$  and  $q_! p^* \dashv p_* q^*$ . It follows (by [lemma A.1.11](#)) that  $\varphi$  is a natural isomorphism if and only if  $\psi$  is a natural isomorphism. ■

## A.2 Cartesian closed categories

**Definition A.2.1.** Let  $C$  be a category with binary products, and let  $Y$  and  $Z$  be objects in  $C$ . An **exponential object** for  $Y$  and  $Z$  is an object  $[Y, Z]_C$  in  $C$  and a morphism  $\text{ev}_{Y,Z} : [Y, Z]_C \times Y \rightarrow Z$  with the following universal property:

- For all morphisms  $f : X \times Y \rightarrow Z$  in  $C$ , there exists a unique morphism  $\bar{f} : X \rightarrow [Y, Z]_C$  such that  $\text{ev}_{Y,Z} \circ (\bar{f} \times \text{id}_Y) = f$ .

An **exponentiable object** in  $C$  is an object  $Y$  such that, for all objects  $Z$  in  $C$ , the exponential object  $[Y, Z]_C$  exists. We may write  $[Y, Z]$  or  $Z^Y$  instead of  $[Y, Z]_C$  if there is no risk of confusion.

**Lemma A.2.2.** *Let  $Y$  be an object in a category  $C$  with binary products. The following are equivalent:*

- (i)  *$Y$  is an exponentiable object in  $C$ .*

- (ii) The functor  $- \times Y : \mathcal{C} \rightarrow \mathcal{C}$  has a right adjoint  $[Y, -]_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ , and the counit of this adjunction is  $\text{ev}_{Y, -}$ .

*Proof.* Immediate from the definitions. ◆

**Definition A.2.3.** A **cartesian closed category** is a category with finite products, in which every object is exponentiable. A **locally cartesian closed category** is a category  $\mathcal{C}$  such that, for every object  $I$ , the slice category  $\mathcal{C}_{/I}$  is a cartesian closed category.

**Example A.2.4.** **Set** is cartesian closed category; in fact, it is even a locally cartesian closed category.

**Proposition A.2.5.** Let  $\mathcal{C}$  be a cartesian closed category.

- (i) The assignment  $(Y, Z) \mapsto [Y, Z]_{\mathcal{C}}$  extends to a functor  $\mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$ .
- (ii) For each object  $Z$ , the functor  $[-, Z]_{\mathcal{C}} : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$  is a contravariant right adjoint for itself.

*Proof.* (i). This is an instance of the parametrised adjunction theorem.<sup>[2]</sup>

(ii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}(X, [Y, Z]) &\cong \mathcal{C}(X \times Y, Z) \\ &\cong \mathcal{C}(Y \times X, Z) \\ &\cong \mathcal{C}(Y, [X, Z]) \end{aligned}$$

**Lemma A.2.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be cartesian closed categories. If  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor that preserves binary products, then:

- (i) For any two objects  $X$  and  $Y$  in  $\mathcal{C}$ , there is a unique morphism  $\varphi_{X, Y} : F[X, Y]_{\mathcal{C}} \rightarrow [FX, FY]_{\mathcal{D}}$  such that the following diagram commutes:

$$\begin{array}{ccc} F[X, Y]_{\mathcal{C}} \times FX & \xrightarrow{\cong} & F([X, Y]_{\mathcal{C}} \times X) \\ \varphi_{X, Y} \times \text{id} \downarrow & & \downarrow F\text{ev}_{X, Y} \\ [FX, FY]_{\mathcal{D}} \times FX & \xrightarrow{\text{ev}_{FX, FY}} & FY \end{array}$$

[2] See Theorem 3 in [CWM, Ch. IV, §7].

(ii) The morphism  $\varphi_{Y,Z}$  is natural in both  $Y$  and  $Z$ .

*Proof.* The existence and uniqueness of  $\varphi_{X,Y}$  follows from the universal property of  $[FX, FY]_D$  as an exponential object, and a standard argument proves naturality.  $\blacklozenge$

**Definition A.2.7.** A **cartesian closed functor** is a functor  $F : C \rightarrow D$  between cartesian closed categories such that the canonical comparison morphisms  $\varphi_{X,Y} : F[X, Y]_C \rightarrow [FX, FY]_D$  described above are isomorphisms.

**Proposition A.2.8.** Let  $C$  and  $D$  be cartesian closed categories, and let  $Y$  be an object in  $C$  and let  $Z$  be an object in  $D$ . Suppose we have an adjunction  $F \dashv G : D \rightarrow C$  with unit  $\eta : \text{id}_C \Rightarrow GF$  and counit  $\varepsilon : \text{id}_D \Rightarrow FG$ ; then:

(i) If  $\psi_{FY,Z} : G[FY, Z]_D \rightarrow [GFY, GZ]_C$  is the canonical comparison morphism, then  $\theta_{Y,Z} = [\eta_Y, GZ]_C \circ \psi_{FY,Z}$  is the unique morphism in  $C$  making the following diagram commute:

$$\begin{array}{ccc}
 G[FY, Z]_D \times Y & \xrightarrow{\text{id} \times \eta_Y} & G[FY, Z]_D \times GFY \\
 \theta_{Y,Z} \times \text{id} \downarrow & & \downarrow \cong \\
 [Y, GZ]_C \times Y & & G([FY, Z]_D \times FY) \\
 \text{ev}_{Y, GZ} \downarrow & & \downarrow G\text{ev}_{FY, Z} \\
 GZ & \xlongequal{\quad\quad\quad} & GZ
 \end{array}$$

(ii) If the canonical comparison morphism  $F(X \times Y) \rightarrow FX \times FY$  is an isomorphism for all objects  $X$  in  $C$ , and  $\varphi_{Y, GZ} : F[Y, GZ]_C \rightarrow [FY, FGZ]_D$  is the canonical comparison morphism, then  $\chi_{Y,Z} = [FY, \varepsilon_Z]_D \circ \varphi_{Y, GZ}$  is the unique morphism in  $D$  making the following diagram commute:

$$\begin{array}{ccc}
 F[Y, GZ]_C \times FY & \xrightarrow{\cong} & F([Y, GZ]_C \times Y) \\
 \chi_{Y,Z} \times \text{id} \downarrow & & \downarrow \text{ev}_{Y, GZ} \\
 [FY, Z]_D \times FY & & FGZ \\
 \text{ev}_{FY, Z} \downarrow & & \downarrow \varepsilon_Z \\
 Z & \xlongequal{\quad\quad\quad} & Z
 \end{array}$$

Moreover, under this hypothesis,  $G\chi_{Y,Z} \circ \eta_{[Y, GZ]_C}$  is a two-sided inverse for  $\theta_{Y,Z}$ .



(iii) If  $\theta_{Y,Z}$  is an isomorphism for all objects  $Z$  in  $\mathcal{D}$ , then for all objects  $X$  in  $\mathcal{C}$ , the canonical comparison morphism  $F(X \times Y) \rightarrow FX \times FY$  is an isomorphism.

*Proof.* (i). The claim is proven by the commutativity of the following diagram:

$$\begin{array}{ccccc}
 G[FY, Z]_{\mathcal{D}} \times Y & \xrightarrow{\text{id} \times \eta_Y} & G[FY, Z]_{\mathcal{D}} \times GFY & \xrightarrow{\cong} & G([FY, Z]_{\mathcal{D}} \times FY) \\
 \psi_{FY,Z} \times \text{id} \downarrow & & \downarrow \psi_{FY,Z} \times \text{id} & & \downarrow G\text{ev}_{FY,Z} \\
 [GFY, GZ]_{\mathcal{C}} \times Y & \xrightarrow{\text{id} \times \eta_Y} & [GFY, GZ]_{\mathcal{C}} \times GFY & \xrightarrow{\text{ev}_{GFY,GZ}} & GZ \\
 [\eta_Y, Z]_{\mathcal{C}} \downarrow & & & & \downarrow \text{ev}_{FY,Z} \\
 [Y, GZ]_{\mathcal{C}} & \xrightarrow{\text{ev}_{Y,GZ}} & & & GZ
 \end{array}$$

(ii). To show that  $\chi_{Y,Z}$  makes the diagram commute, one uses the fact that  $\text{ev}_{FY,Z} : [FY, Z]_{\mathcal{D}} \times FY \rightarrow Z$  is natural in  $Z$ . Since  $F$  preserves products with  $Y$ , we have the following natural bijections:

$$\begin{aligned}
 \mathcal{C}(X, G[FY, Z]_{\mathcal{D}}) &\cong \mathcal{D}(FX, [FY, Z]_{\mathcal{D}}) \cong \mathcal{D}(FX \times FY, Z) \\
 &\cong \mathcal{D}(F(X \times Y), Z) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{C}(X, [Y, GZ]_{\mathcal{C}})
 \end{aligned}$$

One obtains explicit isomorphisms by chasing  $\text{id}_X$  in both directions. Taking  $X = [Y, GZ]_{\mathcal{C}}$ , we find that the isomorphism  $[Y, GZ]_{\mathcal{C}} \rightarrow G[FY, Z]_{\mathcal{D}}$  is precisely  $G\chi_{Y,Z} \circ \eta_{[Y, GZ]_{\mathcal{C}}}$ , and taking  $X = G[FY, Z]_{\mathcal{D}}$ , we find that the inverse is the right exponential transpose of

$$G(\text{ev}_{FY,Z} \circ (\epsilon_{[FY, Z]_{\mathcal{D}}} \times \text{id}_Y)) \circ \eta_{G[FY, Z]_{\mathcal{D}} \times Y}$$

where we have suppressed the comparison isomorphism  $F(G[FY, Z]_{\mathcal{D}} \times Y) \cong FG[FY, Z]_{\mathcal{D}} \times FY$ ; but naturality of the comparison morphisms for binary products gives us the commutative diagram below,

$$\begin{array}{ccccc}
 G[FY, Z]_{\mathcal{D}} \times Y & \xrightarrow{\eta} & GF(G[FY, Z]_{\mathcal{D}} \times Y) & & \\
 \parallel & & \cong \downarrow & & \\
 & & G(FG[FY, Z]_{\mathcal{D}} \times FY) & \xrightarrow{G(\epsilon \times \text{id})} & G([FY, Z]_{\mathcal{D}} \times FY) \\
 & & \cong \downarrow & & \downarrow \cong \\
 G[FY, Z]_{\mathcal{D}} \times Y & \xrightarrow{\eta \times \eta} & GFG[FY, Z]_{\mathcal{D}} \times GFY & \xrightarrow{G\epsilon \times \text{id}} & G[FY, Z]_{\mathcal{D}} \times GFY \\
 & \searrow \text{id} \times \eta & & & 
 \end{array}$$

so, suppressing the comparison isomorphisms, we obtain the following equation:

$$G(\varepsilon_{[FY, Z]_D} \times \text{id}_{FY}) \circ \eta_{G[FY, Z]_D \times Y} = \text{id}_{G[FY, Z]_D} \times \eta_Y$$

Thus, the isomorphism  $G[FY, Z]_D \rightarrow [GY, Z]_C$  is indeed  $\theta_{Y, Z}$ , as claimed.

(iii). Now, suppose  $\theta_{Y, Z} : G[FY, Z]_D \rightarrow [GY, Z]_C$  is an isomorphism for all  $Z$ . Then, we have the natural bijections

$$\begin{aligned} D(FX \times FY, Z) &\cong D(FX, [FY, Z]_D) \cong C(X, G[FY, Z]_D) \\ &\cong C(X, [Y, GZ]_C) \cong C(X \times Y, GZ) \cong D(F(X \times Y), Z) \end{aligned}$$

and by chasing  $\text{id}_Z$  for  $Z = FX \times FY$ , we conclude that the *canonical* comparison morphism  $F(X \times Y) \rightarrow FX \times FY$  is an isomorphism. ■

**Definition A.2.9.** A **Frobenius adjunction of cartesian closed categories** is an adjunction  $F \dashv G : D \rightarrow C$  where  $C$  and  $D$  are cartesian closed categories, such that the natural morphisms  $\theta_{Y, Z} : G[FY, Z]_D \rightarrow [Y, GZ]_C$  described above are isomorphisms, or equivalently, such that the left adjoint  $F : C \rightarrow D$  preserves binary products.

**REMARK A.2.10.** If  $C$  and  $D$  are cartesian closed categories and  $G : D \rightarrow C$  is any functor that preserves finite products, then  $G$  induces a  $C$ -enrichment of  $D$  from the cartesian closed structure of  $D$ , and the exponential comparison morphisms  $\psi_{Y, Z} : G[Y, Z]_D \rightarrow [GY, GZ]_C$  makes  $G : D \rightarrow C$  into a  $C$ -enriched functor.

Now, suppose  $G$  has a left adjoint  $F : C \rightarrow D$ . The adjunction  $F \dashv G$  is a Frobenius adjunction precisely when it is compatible with the  $C$ -enrichments of  $C$  and  $D$ . (Of course, this means  $F$  is also a  $C$ -enriched functor.)

However, not all enriched adjunctions between cartesian closed categories are of the above form.

**Proposition A.2.11.** *Let  $X, Y$ , and  $Z$  be any three objects in a cartesian closed category  $C$ .*

- (i) *There is a unique morphism  $\lambda_{X, Y, Z} : [X \times Y, Z] \rightarrow [X, [Y, Z]]$  making*

the following diagram commute:

$$\begin{array}{ccc}
([X \times Y, Z] \times X) \times Y & \xrightarrow{\cong} & [X \times Y, Z] \times (X \times Y) \\
(\lambda_{X,Y,Z} \times \text{id}_X) \times \text{id}_Y \downarrow & & \downarrow \text{ev}_{X \times Y, Z} \\
([X, [Y, Z]] \times X) \times Y & & \\
\text{ev}_{X, [Y, Z]} \times \text{id}_X \downarrow & & \downarrow \\
[Y, Z] \times Y & \xrightarrow{\text{ev}_{Y, Z}} & Z
\end{array}$$

(ii) The morphisms  $\lambda_{X,Y,Z} : [X \times Y, Z] \rightarrow [X, [Y, Z]]$  constitute a natural isomorphism.

*Proof.* The existence and uniqueness of  $\lambda_{X,Y,Z}$  follows from the universal property of  $[X, [Y, Z]]$  and  $[Y, Z]$  as exponential objects, and a standard argument shows that  $\lambda_{X,Y,Z}$  is natural in  $X$ ,  $Y$ , and  $Z$ . By the associativity of cartesian products, we have the following natural bijections:

$$\begin{aligned}
C(T, [X \times Y, Z]) &\cong C(T \times (X \times Y), Z) \\
&\cong C((T \times X) \times Y, Z) \cong C(T \times X, [Y, Z]) \cong C(T, [X, [Y, Z]])
\end{aligned}$$

Chasing  $\text{id}_T$  for  $T = [X \times Y, Z]$ , we find that  $\lambda_{X,Y,Z}$  is an isomorphism.  $\blacksquare$

**Definition A.2.12.** Let  $\mathcal{C}$  be a cartesian closed category. An **exponential ideal** of  $\mathcal{C}$  is a full subcategory  $\mathcal{D} \subseteq \mathcal{C}$  such that, for all objects  $Y$  in  $\mathcal{C}$ , if  $Z$  is in  $\mathcal{D}$ , then the exponential object  $[Y, Z]_{\mathcal{C}}$  is (isomorphic to) an object in  $\mathcal{D}$ . A **reflective exponential ideal** of  $\mathcal{C}$  is an exponential ideal  $\mathcal{D}$  such that the inclusion  $\mathcal{D} \hookrightarrow \mathcal{C}$  has a left adjoint.

**Proposition A.2.13.** Let  $\mathcal{C}$  be a cartesian closed category, let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be the inclusion of a full subcategory, and suppose  $G$  has a left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$ . The following are equivalent:

- (i)  $F$  preserves finite products.
- (ii)  $F$  preserves binary products.
- (iii)  $\mathcal{D}$  is a reflective exponential ideal of  $\mathcal{C}$ .
- (iv)  $\mathcal{D}$  is a cartesian closed category,  $G : \mathcal{D} \rightarrow \mathcal{C}$  is a cartesian closed functor, and the canonical morphisms  $G[FY, Z]_{\mathcal{D}} \rightarrow [Y, GZ]_{\mathcal{C}}$  are isomorphisms.

*Proof.* (i)  $\Rightarrow$  (ii). Immediate.

(ii)  $\Rightarrow$  (iii). Under our hypotheses, the product of two objects  $X$  and  $Y$  in  $\mathcal{D}$  can be computed as  $F(GX \times GY)$ . Let  $\eta : \text{id}_{\mathcal{C}} \rightarrow GF$  be the unit of the adjunction. We have the following natural bijections:

$$\begin{aligned} \mathcal{C}(X, [Y, GZ]_{\mathcal{C}}) &\cong \mathcal{C}(X \times Y, GZ) \\ &\cong \mathcal{D}(FX \times FY, Z) \\ &\cong \mathcal{D}(FGFX \times FY, Z) \\ &\cong \mathcal{C}(GF X \times Y, GZ) \\ &\cong \mathcal{C}(GF X, [Y, GZ]_{\mathcal{C}}) \end{aligned}$$

By chasing these maps explicitly, we find that every morphism  $X \rightarrow [Y, GZ]_{\mathcal{C}}$  factors through  $\eta_X : X \rightarrow GF X$  in a unique way. In particular, we have

$$\text{id}_{[Y, GZ]_{\mathcal{C}}} = r_{Y, Z} \circ \eta_{[Y, GZ]_{\mathcal{C}}}$$

for a unique  $r_{Y, Z} : GF[Y, GZ]_{\mathcal{C}} \rightarrow [Y, GZ]_{\mathcal{C}}$ . The triangle identity then implies  $Fr_{Y, Z} = \varepsilon_{F[Y, GZ]_{\mathcal{C}}}$ , thus,

$$\eta_{[Y, GZ]_{\mathcal{C}}} \circ r_{Y, Z} = GF r_{Y, Z} \circ \eta_{GF[Y, GZ]_{\mathcal{C}}} = G\varepsilon_{F[Y, GZ]_{\mathcal{C}}} \circ \eta_{GF[Y, GZ]_{\mathcal{C}}} = \text{id}_{GF[Y, GZ]_{\mathcal{C}}}$$

and therefore  $r_{Y, Z}$  is an isomorphism.

(iii)  $\Rightarrow$  (iv). It is a standard fact that a reflective subcategory is closed under all limits that exist in  $\mathcal{C}$ , so  $\mathcal{D}$  must have finite products and  $G : \mathcal{D} \rightarrow \mathcal{C}$  preserves them. If  $\mathcal{D}$  is an exponential ideal, then  $\eta_{[Y, GZ]_{\mathcal{C}}} : [Y, GZ]_{\mathcal{C}} \rightarrow GF[Y, GZ]_{\mathcal{C}}$  must be an isomorphism, so we obtain natural bijections

$$\begin{aligned} \mathcal{D}(X \times Y, Z) &\cong \mathcal{C}(GX \times GY, GZ) \\ &\cong \mathcal{C}(GX, [GY, GZ]_{\mathcal{C}}) \\ &\cong \mathcal{C}(GX, GF[GY, GZ]_{\mathcal{C}}) \\ &\cong \mathcal{D}(FGX, F[GY, GZ]_{\mathcal{C}}) \\ &\cong \mathcal{D}(X, F[GY, GZ]_{\mathcal{C}}) \end{aligned}$$

and therefore we may take  $[Y, Z]_{\mathcal{D}} = F[GY, GZ]_{\mathcal{C}}$ . Obviously, this makes  $G : \mathcal{D} \rightarrow \mathcal{C}$  into a cartesian closed functor. We also have

$$\mathcal{C}(X, G[FY, Z]_{\mathcal{D}}) = \mathcal{C}(X, GF[GFY, GZ]_{\mathcal{C}})$$

$$\begin{aligned}
 &\cong \mathcal{C}(X, [GFY, GZ]_{\mathcal{C}}) \\
 &\cong \mathcal{C}(GFY, [X, GZ]_{\mathcal{C}}) \\
 &\cong \mathcal{C}(GFY, GF[X, GZ]_{\mathcal{C}}) \\
 &\cong \mathcal{C}(Y, GF[X, GZ]_{\mathcal{C}}) \\
 &\cong \mathcal{C}(Y, [X, GZ]_{\mathcal{C}}) \\
 &\cong \mathcal{C}(X, [Y, GZ]_{\mathcal{C}})
 \end{aligned}$$

and so the canonical morphism  $G[FY, Z]_{\mathcal{D}} \rightarrow [Y, GZ]_{\mathcal{C}}$  is an isomorphism.

(iv)  $\Rightarrow$  (i). Since  $\mathcal{D}$  has a terminal object and  $G : \mathcal{D} \rightarrow \mathcal{C}$  preserves it,  $F1$  must be a terminal object in  $\mathcal{D}$ . Now apply [proposition A.2.8](#).  $\blacksquare$

**Corollary A.2.14.** *If  $\mathcal{E}$  is a reflective exponential ideal of  $\mathcal{D}$ , and  $\mathcal{D}$  is a reflective exponential ideal of  $\mathcal{C}$ , then  $\mathcal{E}$  is also a reflective exponential ideal of  $\mathcal{C}$ .*  $\blacksquare$

**Proposition A.2.15.** *Let  $\mathbf{Cat}$  be the category of small categories, and let  $\mathbf{Grpd}$  be the full subcategory of groupoids.*

(i) *There exist adjunctions*

$$\pi_0 \dashv \text{disc} \dashv \text{ob} \dashv \text{codisc} : \mathbf{Set} \rightarrow \mathbf{Cat}$$

where  $\text{ob } \mathbb{C}$  is the set of objects in a category  $\mathbb{C}$ ,  $\text{disc } X$  is the category with  $\text{ob } \text{disc } X = X$  and all arrows trivial, and  $\text{codisc } X$  is the category with  $\text{ob } \text{disc } X = X$  and a unique arrow between any two objects.

(ii) *The functor  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{Cat}$  is fully faithful and exhibits  $\mathbf{Set}$  as a reflective exponential ideal of  $\mathbf{Cat}$ .*

(iii) *The functor  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$  preserves finite products.*

(iv) *There exist adjunctions*

$$\mathbf{I} \dashv \text{und} \dashv \text{iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$$

where  $\text{und} : \mathbf{Grpd} \rightarrow \mathbf{Cat}$  is the inclusion and  $\text{iso } \mathbb{C}$  is the maximal subgroupoid of a category  $\mathbb{C}$ .

(v)  *$\mathbf{Grpd}$  is a reflective exponential ideal of  $\mathbf{Cat}$ .*

(vi) *The functor  $\mathbf{I} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  preserves finite products.*

(vii) *The adjunctions in (i) factor through **Grpd**, yielding adjunctions*

$$\pi_0 \dashv \text{disc} \dashv \text{ob} \dashv \text{codisc} : \mathbf{Set} \rightarrow \mathbf{Grpd}$$

where  $\pi_0 : \mathbf{Grpd} \rightarrow \mathbf{Set}$  again preserves finite products.

(viii) *The functor  $\mathbf{Cat} \rightarrow \mathbf{Set}$  that sends a category  $\mathbb{C}$  to the set of isomorphism classes in  $\mathbb{C}$  preserves finite products.*

*Proof.* (i). The functor  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{Cat}$  obviously satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$ ; the existence of the other adjunctions is obvious.

(ii). It is clear that  $\text{disc} : \mathbf{Set} \rightarrow \mathbf{Cat}$  is fully faithful, and direct computation shows that  $[\mathbb{C}, \text{disc } X]$  is a discrete category for any  $\mathbb{C}$ , so  $\mathbf{Set}$  is indeed a reflective exponential ideal of  $\mathbf{Cat}$ .

(iii). Thus, by [proposition A.2.13](#),  $\pi_0 : \mathbf{Cat} \rightarrow \mathbf{Set}$  must preserve finite products.

(iv). It is not hard to check that the inclusion  $\mathbf{Grpd} \rightarrow \mathbf{Cat}$  satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint  $\mathbf{I} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$ ; the fact that  $\text{iso} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  is right adjoint to the inclusion is obvious.

(v). Direct computation shows that  $[\mathbb{C}, \mathbb{G}]$  is a groupoid whenever  $\mathbb{G}$  is, so  $\mathbf{Grpd}$  is indeed a reflective exponential ideal of  $\mathbf{Cat}$ .

(vi). Thus,  $\mathbf{I} : \mathbf{Cat} \rightarrow \mathbf{Grpd}$  must preserve finite products.

(vii). Clearly,  $\text{disc } X$  and  $\text{codisc } X$  are already groupoids, so the adjunctions do indeed factor through  $\mathbf{Grpd}$ .

(viii). The set of isomorphism classes of objects in  $\mathbb{C}$  is precisely  $\pi_0 \text{ iso } \mathbb{C}$ . ■

**Definition A.2.16.** The **dependent sum** of an object  $p : X \rightarrow I$  in  $\mathcal{C}_{/I}$  along a morphism  $j : I \rightarrow J$  in  $\mathcal{C}$  is the object  $j \circ p : X \rightarrow J$  in  $\mathcal{C}_{/J}$ , and we write  $\Sigma_j : \mathcal{C}_{/I} \rightarrow \mathcal{C}_{/J}$  for the functor sending an object to its dependent sum along  $j$ .

**Lemma A.2.17.** *Let  $j : I \rightarrow J$  be a morphism in a category  $\mathcal{C}$ . The following are equivalent:*

- (i)  $\mathcal{C}$  has pullbacks along  $j$ .
- (ii) There exists an adjunction

$$\Sigma_j \dashv j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$$

where  $\Sigma_j$  is the dependent sum functor, and the right adjoint  $j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$  is the pullback functor.

*Proof.* This is just a matter of unwinding the definitions. ◆

**Definition A.2.18.** Let  $\mathcal{C}$  be a category with pullbacks. A **dependent product** of an object  $p : X \rightarrow I$  in  $\mathcal{C}_{/I}$  along a morphism  $j : I \rightarrow J$  in  $\mathcal{C}$  is an object  $\Pi_j p$  in  $\mathcal{C}_{/J}$  and a morphism  $\text{ev}_{j,p} : j^* \Pi_j p \rightarrow p$  in  $\mathcal{C}_{/I}$  with the following universal property:

- For all morphisms  $f : j^* q \rightarrow p$  in  $\mathcal{C}_{/I}$ , there exists a unique morphism  $\bar{f} : q \rightarrow \Pi_j p$  in  $\mathcal{C}_{/J}$  such that  $\text{ev}_{j,p} \circ j^* \bar{f} = f$ .

A  **$\Sigma\Pi$ -category** is a category  $\mathcal{C}$  with finite limits such that, for every morphism  $j : I \rightarrow J$  in  $\mathcal{C}$ , dependent products along  $j$  exist.

**Lemma A.2.19.** Let  $j : I \rightarrow J$  be a morphism in a category  $\mathcal{C}$  with pullbacks. The following are equivalent:

- (i) For all objects  $p : X \rightarrow I$  in  $\mathcal{C}$ , a dependent product of  $p$  along  $j$  exists.
- (ii) The pullback functor  $j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$  has a right adjoint  $\Pi_j : \mathcal{C}_{/I} \rightarrow \mathcal{C}_{/J}$  that sends an object to its dependent product along  $j$ , and the counit of this adjunction is  $\text{ev}_{j,-}$ .

*Proof.* This is just a matter of unwinding the definitions. ◆

**Corollary A.2.20.** If  $j : I \rightarrow J$  is a morphism in a  $\Sigma\Pi$ -category  $\mathcal{C}$ , then the pullback functor  $j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$  preserves all limits and colimits. ■

**Proposition A.2.21.** Let  $\mathcal{C}$  be a category with a terminal object. The following are equivalent:

- (i)  $\mathcal{C}$  is a  $\Sigma\Pi$ -category.
- (ii)  $\mathcal{C}$  is a locally cartesian closed category.

*Proof.* See Proposition 9.20 in [Awodey, 2010]. □

**Theorem A.2.22.** *Let  $\mathbb{D}$  be a small category, and let  $\mathcal{C} = [\mathbb{D}^{\text{op}}, \mathbf{Set}]$ . Then:*

- (i)  *$\mathcal{C}$  has limits and colimits for all small diagrams, and these can be constructed componentwise in  $\mathbf{Set}$ : a cone (resp. cocone) in  $\mathcal{C}$  over (resp. under) a diagram in  $\mathcal{C}$  is a limiting cone (resp. colimiting cocone) if and only if it is so in every component.*
- (ii) *Every internal equivalence relation in  $\mathcal{C}$  is the kernel pair of its coequaliser.*
- (iii) *For all morphisms  $j : I \rightarrow J$  in  $\mathcal{C}$ , the pullback functor  $j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$  preserves all limits and colimits.*
- (iv) *The Yoneda embedding  $h_{\bullet} : \mathbb{D} \rightarrow \mathcal{C}$  is a dense functor, i.e. for every presheaf  $X : \mathbb{D}^{\text{op}} \rightarrow \mathbf{Set}$ , the tautological cocone<sup>[3]</sup> from the canonical diagram  $(h_{\bullet} \downarrow X) \rightarrow \mathcal{C}$  to  $X$  is a colimiting cocone.*
- (v)  *$\mathcal{C}$  is a locally finitely presentable category.*
- (vi)  *$\mathcal{C}$  is a  $\Sigma\Pi$ -category.*

*Proof.* (i). This is a standard fact about presheaf categories.

(ii) and (iii). The claims are true for  $\mathbf{Set}$ , and hence for  $\mathcal{C}$  by claim (i).

(iv). See proposition A.5.25.

(v). See theorem 0.2.39.

(vi). Apply theorem 0.2.49 to construct a right adjoint for  $j^* : \mathcal{C}_{/J} \rightarrow \mathcal{C}_{/I}$ . ■

**REMARK A.2.23.** The Yoneda lemma gives us an explicit description of the exponential objects in  $[\mathbb{D}^{\text{op}}, \mathbf{Set}]$ : given two presheaves  $Y, Z : \mathbb{D}^{\text{op}} \rightarrow \mathbf{Set}$ , if  $Z^Y$  is their exponential object, then we must have

$$Z^Y(d) \cong [\mathbb{D}^{\text{op}}, \mathbf{Set}](h_d, Z^Y) \cong [\mathbb{D}^{\text{op}}, \mathbf{Set}](h_d \times Y, Z)$$

and so we may define  $Z^Y$  by  $Z^Y(d) = [\mathbb{D}^{\text{op}}, \mathbf{Set}](h_d \times Y, Z)$ .

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[3] See definition A.5.7.



**Definition A.2.24.** Let  $Y$  and  $Z$  be topological spaces, and let  $[Y, Z]$  be the set of all *continuous* maps  $Y \rightarrow Z$ . The **compact–open topology** on  $[Y, Z]$  is the coarsest topology such that the subsets

$$V(K, U) = \{f \in [Y, Z] \mid K \subseteq f^{-1}U\}$$

are open in  $[Y, Z]$  for all compact subsets  $K \subseteq X$  and all open subsets  $U \subseteq Y$ .

**REMARK A.2.25.** If  $Y$  is a discrete space, then the compact–open topology on  $[Y, Z]$  coincides with the product topology on  $Z^Y$ .

**Definition A.2.26.** A **compactly-generated Hausdorff space** is a Hausdorff topological space  $X$  such that a subset  $U \subseteq X$  is open if and only if, for every continuous map  $f : K \rightarrow X$  where  $K$  is a compact Hausdorff space,  $f^{-1}U$  is an open subset of  $K$ . We write **CGHaus** for the category of compactly-generated Hausdorff spaces and continuous maps.

**Proposition A.2.27.**

- (i) *If  $Y$  is a locally compact Hausdorff space, then for all topological spaces  $Z$ , the set of all continuous maps  $Y \rightarrow Z$ , equipped with the compact–open topology, is an exponential object  $[Y, Z]$  in **Top**.*
- (ii) **Top** is not a cartesian closed category.
- (iii) **CGHaus** is a cartesian closed category.

*Proof.* Claim (i) follows from Theorems 46.10 and 46.11 in [Munkres, 2000], and claim (ii) is Proposition 7.1.2 in [Borceux, 1994a], and claim (iii) is proved in [GZ, Ch. III, §2]. □

## A.3 Factorisation systems

**Definition A.3.1.** Let  $\mathcal{C}$  be a category.

- Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be morphisms in  $\mathcal{C}$ . Given a commutative square in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

a **lift** is a morphism  $h : W \rightarrow X$  such that  $f \circ h = w$  and  $h \circ g = z$ .

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- We say  $g$  has the **left lifting property** with respect to  $f$  and  $f$  has the **right lifting property** with respect to  $g$ , and we write  $g \sqsupseteq f$ , if every commutative square in  $\mathcal{C}$  of the form above has a lift.
- We say  $f$  is **left orthogonal** to  $g$  and  $g$  is **right orthogonal** to  $f$ , and we write  $g \perp f$  if lifts exist *and* are unique.
- Given  $\mathcal{I} \subseteq \text{mor } \mathcal{C}$ , we define the following subensembles of  $\text{mor } \mathcal{C}$ :

$$\sqsupseteq \mathcal{I} = \{f \in \text{mor } \mathcal{C} \mid \forall g \in \mathcal{I}. f \sqsupseteq g\}$$

$$\mathcal{I} \sqsupseteq = \{g \in \text{mor } \mathcal{C} \mid \forall f \in \mathcal{I}. f \sqsupseteq g\}$$

$$\perp \mathcal{I} = \{f \in \text{mor } \mathcal{C} \mid \forall g \in \mathcal{I}. f \perp g\}$$

$$\mathcal{I}^\perp = \{g \in \text{mor } \mathcal{C} \mid \forall f \in \mathcal{I}. f \perp g\}$$

**Lemma A.3.2.** *Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be morphisms in a locally small category  $\mathcal{C}$ . Consider the commutative diagram in **Set** shown below,*

$$\begin{array}{ccc}
 \mathcal{C}(W, X) & \xrightarrow{f_*} & \mathcal{C}(W, Y) \\
 \downarrow \sigma_{g^*} & \dashrightarrow & \bullet \longrightarrow \mathcal{C}(W, Y) \\
 & & \downarrow g_* \\
 \mathcal{C}(Z, X) & \xrightarrow{f_*} & \mathcal{C}(Z, Y)
 \end{array}$$

where the inner square is a pullback diagram.

- The dashed arrow is a surjection if and only if  $g \sqsupseteq f$ .
- The dashed arrow is a bijection if and only if  $g \perp f$ .

*Proof.* This is just a restatement of the definition. ■

**Proposition A.3.3.** *Let  $\mathcal{C}$  be a category.*

- If  $\mathcal{R} \subseteq \text{mor } \mathcal{C}$ , then  $\perp \mathcal{R} \subseteq \sqsupseteq \mathcal{R}$ .
- If  $\mathcal{R}' \subseteq \mathcal{R} \subseteq \text{mor } \mathcal{C}$ , then  $\sqsupseteq \mathcal{R}' \supseteq \sqsupseteq \mathcal{R}$ .
- If  $\mathcal{R}' \subseteq \mathcal{R} \subseteq \text{mor } \mathcal{C}$ , then  $\perp \mathcal{R}' \supseteq \perp \mathcal{R}$ .

Dually:

(i') If  $\mathcal{L} \subseteq \text{mor } C$ , then  $\mathcal{L}^\perp \subseteq \mathcal{L}^\square$ .

(ii') If  $\mathcal{L}' \subseteq \mathcal{L} \subseteq \text{mor } C$ , then  $\mathcal{L}'^\square \supseteq \mathcal{L}^\square$ .

(iii') If  $\mathcal{L}' \subseteq \mathcal{L} \subseteq \text{mor } C$ , then  $\mathcal{L}'^\perp \supseteq \mathcal{L}^\perp$ .

Moreover, we have the following antitone Galois connections:

$$\mathcal{L} \subseteq \square \mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^\square$$

$$\mathcal{L} \subseteq {}^\perp \mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^\perp$$

*Proof.* Obvious. ◆

**Corollary A.3.4.** We have the following identities:

$$\square((\square \mathcal{R})\square) = \square \mathcal{R} \qquad {}^\perp(({}^\perp \mathcal{R})^\perp) = {}^\perp \mathcal{R}$$

$$(\square(\mathcal{L}^\square))\square = \mathcal{L}^\square \qquad ({}^\perp(\mathcal{L}^\perp))^\perp = \mathcal{L}^\perp$$

*Proof.* This is a standard fact about (antitone) Galois connections. ■

**Definition A.3.5.** Let  $C$  be a category.

- A **strong monomorphism** in  $C$  is a monomorphism that is right orthogonal to all epimorphisms in  $C$ .
- A **strong epimorphism** in  $C$  is a monomorphism that is left orthogonal to all monomorphisms in  $C$ .

**Lemma A.3.6.** Let  $f : X \rightarrow Y$  be a morphism in a category  $C$ . The following are equivalent:

- (i)  $f$  is an isomorphism.
- (ii)  $f$  is right orthogonal to any morphism in  $C$ .
- (iii)  $f$  has the right lifting property with respect to any morphism in  $C$ .
- (iv)  $f$  has the right lifting property with respect to itself.

Dually, the following are equivalent:

- (i')  $f$  is an isomorphism.

(ii')  $f$  is left orthogonal to any morphism in  $\mathcal{C}$ .

(iii')  $f$  has the left lifting property with respect to any morphism in  $\mathcal{C}$ .

(iv')  $f$  has the left lifting property with respect to itself.

*Proof.* (i)  $\Rightarrow$  (ii). Suppose  $r : Y \rightarrow X$  is a morphism such that  $r \circ f = \text{id}_X$ . Then, for any commutative square as below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

we have  $(r \circ w) \circ g = r \circ f \circ z = z$ ; but if  $f \circ r = \text{id}_Y$  as well, then  $f \circ (r \circ w) = w$ ; thus  $r \circ w : W \rightarrow X$  is the required lift. It is clearly unique, as  $f$  is monic.

(ii)  $\Rightarrow$  (iii), (iii)  $\Rightarrow$  (iv). Obvious.

(iv)  $\Rightarrow$  (i). Consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ f \downarrow & & \downarrow f \\ Y & \xrightarrow{\text{id}} & Y \end{array}$$

Since  $f$  has the right lifting property with respect to itself, there exists a morphism  $h : Y \rightarrow X$  such that  $h \circ f = \text{id}_X$  and  $f \circ h = \text{id}_Y$ . ■

**Corollary A.3.7.**

- A morphism is both a monomorphism and a strong epimorphism if and only if it is an isomorphism.
- A morphism is both a epimorphism and a strong monomorphism if and only if it is an isomorphism. ■

**Definition A.3.8.** A **weak factorisation system** for a category  $\mathcal{C}$  is a pair  $(\mathcal{L}, \mathcal{R})$  of subensembles of  $\text{mor } \mathcal{C}$  satisfying these conditions:

- For each morphism  $f$  in  $\mathcal{C}$  there exists a pair  $(g, h)$  with  $g \in \mathcal{L}$  and  $h \in \mathcal{R}$  such that  $f = h \circ g$ . Such a pair is a  $(\mathcal{L}, \mathcal{R})$ -**factorisation** of  $f$ .

- A morphism is in  $\mathcal{L}$  if and only if it has the left lifting property with respect to every morphism in  $\mathcal{R}$ , i.e.  $\mathcal{L} = \square\mathcal{R}$ .
- A morphism is in  $\mathcal{R}$  if and only if it has the right lifting property with respect to every morphism in  $\mathcal{L}$ , i.e.  $\mathcal{R} = \mathcal{L}\square$ .

An **orthogonal factorisation system** is defined like a weak factorisation system, except for replacing ‘... has the left/right lifting property with respect to ...’ with ‘... is left/right orthogonal to ...’.

REMARK A.3.9. Obviously,  $(\mathcal{L}, \mathcal{R})$  is a weak (resp. orthogonal) factorisation system for  $\mathcal{C}$  if and only if  $(\mathcal{R}^{\text{op}}, \mathcal{L}^{\text{op}})$  is a weak (resp. orthogonal) factorisation system for  $\mathcal{C}^{\text{op}}$ .

**Proposition A.3.10.** *Let  $(\mathcal{L}, \mathcal{R})$  be a weak factorisation system on  $\mathcal{C}$ . If either*

- *every morphism in  $\mathcal{R}$  is a monomorphism in  $\mathcal{C}$ , or*
- *every morphism in  $\mathcal{L}$  is an epimorphism in  $\mathcal{C}$ ,*

*then  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system.*

*Proof.* The two hypotheses are formally dual, so it is enough to check the first case. Observe that, given a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & \nearrow h & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

where  $f : X \rightarrow Y$  is a monomorphism, for any  $h' : W \rightarrow X$  such that  $f \circ h' = w$ , we must have  $h = h'$ . Thus, for any monomorphism  $f : X \rightarrow Y$ ,  $g \square f$  if and only if  $g \perp f$ . Hence,  $\mathcal{L} = \square\mathcal{R} = {}^{\perp}\mathcal{R}$ . On the other hand,  $\mathcal{L}^{\perp} \subseteq \mathcal{L}\square = \mathcal{R}$ , so  $\mathcal{R} = \mathcal{L}^{\perp}$  as well. ■

**Definition A.3.11.** A **proper factorisation system** on a category  $\mathcal{C}$  is an orthogonal factorisation system  $(\mathcal{E}, \mathcal{M})$  on  $\mathcal{C}$  such that every morphism in  $\mathcal{E}$  is an epimorphism *and* every morphism in  $\mathcal{M}$  is a monomorphism.

**Example A.3.12.** In **Set**, if  $\mathcal{E}$  is the class of surjective maps and  $\mathcal{M}$  is the class of injective maps, then  $(\mathcal{E}, \mathcal{M})$  is a proper factorisation system.

**Lemma A.3.13.** *Let  $A$  be an object in a category  $\mathcal{C}$  with a weak (resp. orthogonal) factorisation system  $(\mathcal{L}, \mathcal{R})$ . Then the slice category  $\mathcal{C}_{/A}$  has a weak (resp. orthogonal) factorisation system where a morphism is in the left or right class if and only if it is so in  $\mathcal{C}$ .*

*Proof.* The projection  $\mathcal{C}_{/A} \rightarrow \mathcal{C}$  induces a bijection between solutions for lifting problems in  $\mathcal{C}_{/A}$  and solutions for the corresponding lifting problems in  $\mathcal{C}$ . ■

**Proposition A.3.14** (Closure properties). *Let  $\mathcal{R} \subseteq \text{mor } \mathcal{C}$  and suppose either  $\mathcal{L} = \square\mathcal{R}$  or  $\mathcal{L} = {}^\perp\mathcal{R}$ .*

(i) *Given a pushout diagram in  $\mathcal{C}$  as below,*

$$\begin{array}{ccc} Z' & \xrightarrow{i_Z} & Z \\ g' \downarrow & & \downarrow g \\ W' & \xrightarrow{i_W} & W \end{array}$$

*if the morphism  $g'$  is in  $\mathcal{L}$ , then  $g$  is also in  $\mathcal{L}$ .*

(ii) *Let  $I$  be a set. If  $g_i : Z_i \rightarrow W_i$  is a morphism in  $\mathcal{L}$  for all  $i$  in  $I$  and the coproduct  $\coprod_i g_i : \coprod_i Z_i \rightarrow \coprod_i W_i$  exists in  $\mathcal{C}$ , then  $\coprod_i g_i$  is also in  $\mathcal{L}$ .*

(iii) *Given a commutative diagram of the form*

$$\begin{array}{ccccc} & & \text{id} & & \\ & & \curvearrowright & & \\ Z' & \xrightarrow{i_Z} & Z & \xrightarrow{r_Z} & Z' \\ g' \downarrow & & g \downarrow & & \downarrow g' \\ W' & \xrightarrow{i_W} & W & \xrightarrow{r_W} & W' \\ & & \curvearrowleft & & \\ & & \text{id} & & \end{array}$$

*if  $g$  is in  $\mathcal{L}$ , then so is  $g'$ ; in other words,  $\mathcal{L}$  is closed under retracts.*

(iv)  $\mathcal{L}$  is closed under composition.

(v) *Let  $\gamma$  be an ordinal and let  $Z : \gamma \rightarrow \mathcal{C}$  be a colimit-preserving functor. We write  $Z_\alpha$  for  $Z(\alpha)$ , where  $\alpha < \gamma$ , and  $g_{\alpha,\beta} : Z_\alpha \rightarrow Z_\beta$  for the morphism  $Z(\alpha \rightarrow \beta)$ , where  $\alpha < \beta < \gamma$ . If  $\lambda$  is a colimiting cocone from  $Z$  to  $W$  and each  $g_{\alpha,\beta}$  is in  $\mathcal{L}$ , then each component  $\lambda_\alpha : Z_\alpha \rightarrow W$  is also in  $\mathcal{L}$ .*

*Proof.* (i). Suppose  $f$  is in  $\mathcal{R}$ , and consider the following commutative diagram:

$$\begin{array}{ccccc} Z' & \xrightarrow{i_Z} & Z & \xrightarrow{z} & X \\ g' \downarrow & & \downarrow g & & \downarrow f \\ W' & \xrightarrow{i_W} & W & \xrightarrow{w} & Y \end{array}$$

There exists  $h' : W' \rightarrow X$  such that  $h' \circ g' = z \circ i_Z$  and  $f \circ h' = w \circ i_W$ . In particular, there exists a unique morphism  $h : W \rightarrow X$  such that  $h \circ g = z$  and  $h \circ i_W = h'$ , by the universal property of pullbacks. Thus  $f \circ h \circ i_W = f \circ h' = w \circ i_W$  and  $f \circ h \circ g = f \circ z = w \circ g$ , but  $i_W$  and  $g$  are jointly epic, so  $f \circ h = w$ . This shows  $h$  is the required lift, and  $h$  is unique if  $h'$  is.

(ii). We may construct the required lift componentwise.

(iii). Suppose  $f$  is in  $\mathcal{R}$ , and consider the following commutative diagram:

$$\begin{array}{ccccccc} Z' & \xrightarrow{i_Z} & Z & \xrightarrow{r_Z} & Z' & \xrightarrow{z} & X \\ g' \downarrow & & \downarrow g & & \downarrow g' & & \downarrow f \\ W' & \xrightarrow{i_W} & W & \xrightarrow{r_W} & W' & \xrightarrow{w} & Y \end{array}$$

There exists  $h : W \rightarrow X$  such that  $h \circ g = z \circ r_Z$  and  $f \circ h = w \circ r_W$ , and so for  $h' = h \circ i_W$ :

$$\begin{aligned} f \circ h' &= f \circ h \circ i_W = w \circ r_W \circ i_W = w \\ h' \circ g' &= h \circ i_W \circ g' = h \circ g \circ i_Z = z \circ r_Z \circ i_Z = z \end{aligned}$$

Thus  $h' : W' \rightarrow X$  is the required lift, and  $h'$  is unique if  $h$  is (because  $r_W$  is split epic).

(iv). Suppose  $g' : Z' \rightarrow Z$  and  $g : Z \rightarrow W$  are in  $\mathcal{L}$  and  $f : X \rightarrow Y$  is in  $\mathcal{R}$ . Consider the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{z'} & X \\ g' \downarrow & & \downarrow f \\ Z & & \\ g \downarrow & & \downarrow \\ W & \xrightarrow{w} & Y \end{array}$$

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There must exist a morphism  $z : Z \rightarrow X$  such that  $z \circ g' = z'$  and  $f \circ z' = w \circ g$ , and hence a morphism  $h : W \rightarrow X$  such that  $h \circ g = z$  and  $f \circ h = w$ . Obviously,  $h \circ (g' \circ g) = z'$ , so  $h$  is the required lift. Moreover,  $h$  unique if  $\mathcal{L} = {}^\perp\mathcal{R}$ .

(v). We may assume without loss of generality that  $\alpha = 0$ , since any non-empty terminal segment of  $\gamma$  is cofinal in  $\gamma$ . Suppose  $f : X \rightarrow Y$  is in  $\mathcal{R}$  and consider the following commutative diagram:

$$\begin{array}{ccc} Z_0 & \xrightarrow{z_0} & X \\ \lambda_0 \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

For each  $\alpha < \gamma$ , given  $z_\alpha$  making the following diagram commute,

$$\begin{array}{ccccc} Z_\alpha & \xrightarrow{z_\alpha} & X & & \\ g_{\alpha,\alpha+1} \downarrow & & \downarrow f & & \\ Z_{\alpha+1} & \xrightarrow{\lambda_{\alpha+1}} & W & \xrightarrow{w} & Y \end{array}$$

choose a lift  $z_{\alpha+1} : Z_{\alpha+1} \rightarrow X$ ; for each limit ordinal  $\beta < \gamma$ , let  $z_\beta : Z_\beta \rightarrow X$  be the unique morphism such that  $z_\beta \circ g_{\alpha,\beta} = z_\alpha$  for all  $\alpha < \beta$ . (Such  $z_\beta$  exist and are unique because  $Z_\beta = \varinjlim_{\alpha < \beta} Z_\alpha$ .) Note that the universal property of  $W$  then guarantees that  $w \circ \lambda_\beta = f \circ z_\beta$ .

Having constructed morphisms  $z_\alpha : Z_\alpha \rightarrow X$  for all  $\alpha < \gamma$  as above, we may now obtain  $h : W \rightarrow X$  as the unique morphism such that  $h \circ \lambda_\alpha = z_\alpha$  for all  $\alpha < \gamma$ , and again we automatically have  $f \circ h = w$ . It is also clear that  $h$  is unique if  $\mathcal{L} = {}^\perp\mathcal{R}$ . ■

**Proposition A.3.15** (Cancellation properties). *Let  $\mathcal{R} \subseteq \text{mor } C$ .*

- (i) *Let  $\mathcal{L}$  be either  $\square\mathcal{R}$  or  ${}^\perp\mathcal{R}$ , let  $e : A \rightarrow Z$  be an epimorphism in  $C$ , and let  $g : Z \rightarrow W$  be a morphism in  $C$ . If  $g \circ e$  is in  $\mathcal{L}$ , then so is  $g$ .*
- (ii) *Let  $\mathcal{L}$  be  ${}^\perp\mathcal{R}$ , let  $f : A \rightarrow Z$  be any morphism in  $C$ , and let  $g : Z \rightarrow W$  be a morphism in  $C$ . Assuming every morphism that is in  $\mathcal{R}$  is a monomorphism in  $C$ , if  $g \circ f$  is in  $\mathcal{L}$ , then so is  $g$ .*
- (iii) *Suppose  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on  $\mathcal{R}$ , and let  $e : A \rightarrow Z$  be in  $\mathcal{L}$ . Then, a morphism  $g : Z \rightarrow W$  is in  $\mathcal{L}$  if and only if  $g \circ e$  is in  $\mathcal{L}$ .*



Dually, let  $\mathcal{L} \subseteq \text{mor } \mathcal{C}$ .

(i') Let  $\mathcal{R}$  be either  $\mathcal{L}^\square$  or  $\mathcal{L}^\perp$ , let  $m : Y \rightarrow B$  be an monomorphism in  $\mathcal{C}$ , and let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . If  $m \circ f$  is in  $\mathcal{R}$ , then so is  $f$ .

(ii') Let  $\mathcal{R}$  be  $\mathcal{L}^\perp$ , let  $g : Y \rightarrow B$  be any morphism in  $\mathcal{C}$ , and let  $f : X \rightarrow Y$  be a morphism in  $\mathcal{C}$ . Assuming every morphism that is in  $\mathcal{L}$  is an epimorphism in  $\mathcal{C}$ , if  $g \circ f$  is in  $\mathcal{R}$ , then so is  $f$ .

(iii') Suppose  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on  $\mathcal{R}$ , and let  $m : Y \rightarrow B$  be in  $\mathcal{L}$ . Then, a morphism  $f : X \rightarrow Y$  is in  $\mathcal{L}$  if and only  $g \circ e$  is in  $\mathcal{L}$ .

*Proof.* (i). The epimorphism  $e : A \rightarrow Z$  induces a bijection between solutions of lifting problems in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

and solutions of lifting problems of the form

$$\begin{array}{ccc} A & \xrightarrow{z \circ e} & X \\ g \circ e \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

so  $g \square f$  (resp.  $g \perp f$ ) if and only if  $g \circ e \square f$  (resp.  $g \circ e \perp f$ ).

(ii). The proof is similar to that of claim (i).

(iii). By [proposition A.3.14](#), we know  $g \circ e$  is in  $\mathcal{L}$  if both  $g$  and  $e$  are in  $\mathcal{L}$ ; the converse remains to be shown. Let  $r \circ l$  be an  $(\mathcal{L}, \mathcal{R})$ -factorisation of  $g$ . If  $g \circ e$  is in  $\mathcal{L}$ , then there exists a unique  $s$  making the diagram below commute,

$$\begin{array}{ccc} A & \xrightarrow{l \circ e} & M \\ g \circ e \downarrow & \nearrow s & \downarrow r \\ W & \xrightarrow{\text{id}} & W \end{array}$$

so  $r \circ s = \text{id}_W$ , but then we also have

$$r \circ (s \circ r) = r$$

$$(s \circ r) \circ (l \circ e) = s \circ (g \circ e) = l \circ e$$

and  $l \circ e \perp r$ , so we must have  $s \circ r = \text{id}_M$ . Hence,  $g$  is also in  $\mathcal{L}$ . ■

**Proposition A.3.16** (The retract argument). *Let  $\mathcal{C}$  be a category and let  $(\mathcal{L}, \mathcal{R})$  be a pair of subclasses of  $\text{mor } \mathcal{C}$  such that  $\mathcal{L} \subseteq \square \mathcal{R}$  and  $\mathcal{R} \subseteq \mathcal{L} \square$ . If every morphism in  $\mathcal{C}$  admits an  $(\mathcal{L}, \mathcal{R})$ -factorisation, then the following are equivalent:*

- (i)  $(\mathcal{L}, \mathcal{R})$  is a weak factorisation system.
- (ii)  $\mathcal{L}$  and  $\mathcal{R}$  are both closed under retracts in  $\mathcal{C}$ .

*Proof.* (i)  $\Rightarrow$  (ii). This is a special case of [proposition A.3.14](#).

(ii)  $\Rightarrow$  (i). Suppose  $f : X \rightarrow Y$  is in  $\mathcal{L} \square$ . Let  $p \circ i$  be an  $(\mathcal{L}, \mathcal{R})$ -factorisation of  $f$ . Then, there must exist a morphism  $r$  such that  $r \circ i = \text{id}_X$  and  $f \circ r = p$ , as in the diagram below:

$$\begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ i \downarrow & \nearrow r & \downarrow f \\ Z & \xrightarrow{p} & Y \end{array}$$

Hence, we have the following commutative diagram:

$$\begin{array}{ccccc} & & \text{id} & & \\ & & \curvearrowright & & \\ X & \xrightarrow{i} & Z & \xrightarrow{r} & X \\ f \downarrow & & p \downarrow & & \downarrow f \\ Y & \xrightarrow{\text{id}} & Y & \xrightarrow{\text{id}} & Y \end{array}$$

Since  $\mathcal{R}$  is closed under retracts, we deduce that  $f$  is in  $\mathcal{R}$ . Thus,  $\mathcal{L} \square \subseteq \mathcal{R}$ . The dual argument proves that  $\square \mathcal{R} \subseteq \mathcal{L}$ , so  $(\mathcal{L}, \mathcal{R})$  is indeed a weak factorisation system. ■

**Corollary A.3.17.** *Every orthogonal factorisation system is also a weak factorisation system.*

*Proof.* Let  $(\mathcal{L}, \mathcal{R})$  be an orthogonal factorisation system on a category  $\mathcal{C}$ . [Proposition A.3.3](#) implies  $\mathcal{L} \subseteq \square \mathcal{R}$  and  $\mathcal{R} \subseteq \mathcal{L} \square$ , and [proposition A.3.14](#) says  $\mathcal{L}$  and  $\mathcal{R}$  are both closed under retracts, so we may use the earlier proposition to deduce that  $(\mathcal{L}, \mathcal{R})$  is a weak factorisation system. ■

**Definition A.3.18.** A weak factorisation system  $(\mathcal{L}, \mathcal{R})$  on a category  $C$  is **cofibrantly generated** by a subensemble  $\mathcal{I} \subseteq \text{mor } C$  if  $\mathcal{R} = \mathcal{I}^\square$ . Dually,  $(\mathcal{L}, \mathcal{R})$  is **fibrantly generated** by a subensemble  $\mathcal{F} \subseteq \text{mor } C$  if  $\mathcal{L} = \square\mathcal{F}$ .

REMARK A.3.19. Of course,  $(\mathcal{L}, \mathcal{R})$  is always cofibrantly generated by  $\mathcal{L}$ . The condition is most useful when  $(\mathcal{L}, \mathcal{R})$  is cofibrantly generated by a (small) subset of  $\mathcal{L}$ , but it is convenient to have the more general definition available.

**Definition A.3.20.** Let  $(\mathcal{L}, \mathcal{R})$  be a weak factorisation system on a category  $C$ . An **extension** of  $(\mathcal{L}, \mathcal{R})$  along a functor  $i : C \rightarrow C^+$  is a weak factorisation system  $(\mathcal{L}^+, \mathcal{R}^+)$  on  $C^+$  with the following properties:

- A morphism  $f : X \rightarrow Y$  in  $C$  is in  $\mathcal{R}$  if and only if  $if : iX \rightarrow iY$  is in  $\mathcal{R}^+$ .
- A morphism  $g : Z \rightarrow W$  in  $C$  is in  $\mathcal{L}$  if and only if  $ig : iZ \rightarrow iW$  is in  $\mathcal{L}^+$ .

**Proposition A.3.21.** Let  $C$  be a full subcategory of a category  $C^+$ , let  $(\mathcal{L}, \mathcal{R})$  be a weak factorisation system on  $C$ , and let  $(\mathcal{L}^+, \mathcal{R}^+)$  be a weak factorisation system on  $C^+$ .

- (i) If  $\mathcal{L} \subseteq \mathcal{L}^+$ , then  $\mathcal{R} \supseteq \mathcal{R}^+ \cap \text{mor } C$ .
- (ii) If  $(\mathcal{L}, \mathcal{R})$  and  $(\mathcal{L}^+, \mathcal{R}^+)$  are both cofibrantly generated by the same ensemble  $\mathcal{I}$ , then  $\mathcal{R} = \mathcal{R}^+ \cap \text{mor } C$ .

Dually:

- (i') If  $\mathcal{R} \subseteq \mathcal{R}^+$ , then  $\mathcal{L} \supseteq \mathcal{L}^+ \cap \text{mor } C$ .
- (ii') If  $(\mathcal{L}, \mathcal{R})$  and  $(\mathcal{L}^+, \mathcal{R}^+)$  are both fibrantly generated by the same ensemble  $\mathcal{F}$ , then  $\mathcal{L} = \mathcal{L}^+ \cap \text{mor } C$ .

Moreover:

- (iii) If  $\mathcal{L} \subseteq \mathcal{L}^+$  and  $\mathcal{R} \subseteq \mathcal{R}^+$ , then  $(\mathcal{L}^+, \mathcal{R}^+)$  is an extension of  $(\mathcal{L}, \mathcal{R})$ .

*Proof.* Since  $C$  is a full subcategory of  $C^+$ , if  $g : Z \rightarrow W$  and  $f : X \rightarrow Y$  are morphisms in  $C$ , then any lifting problem of the following form in  $C^+$  is already in  $C$ ,

$$\begin{array}{ccc} Z & \longrightarrow & X \\ g \downarrow & & \downarrow f \\ W & \longrightarrow & Y \end{array}$$

and moreover any solution to the above lifting problem in  $C^+$  is also a solution in  $C$ . Thus,  $g \boxtimes f$  in  $C$  if and only if  $g \boxtimes f$  in  $C^+$ .

(i). Suppose  $f$  is in  $\mathcal{R}^+ \cap \text{mor } C$ . Then  $f$  has the right lifting property in  $C^+$  with respect to every morphism in  $\mathcal{L}^+$ , and in particular,  $f$  has the right lifting property in  $C$  with respect to every morphism in  $\mathcal{L}$ ; hence  $f$  is in  $\mathcal{R}$ , and therefore  $\mathcal{R} \supseteq \mathcal{R}^+ \cap \text{mor } C$ .

(ii). A morphism is in  $\mathcal{R}$  (resp.  $\mathcal{R}^+$ ) if and only if it has the right lifting property in  $C$  (resp.  $C^+$ ) with respect to every morphism in  $\mathcal{I}$ , so by our initial observation, we must have  $\mathcal{R} = \mathcal{R}^+ \cap \text{mor } C$ .

(iii). Immediately follows from claims (i) and (i'). ■

**Proposition A.3.22.** *Let  $(\mathcal{L}, \mathcal{R})$  be a weak (resp. orthogonal) factorisation system for a category  $C$ , and let  $(\mathcal{L}', \mathcal{R}')$  be a weak (resp. orthogonal) factorisation system for a category  $C'$ . Given an adjunction*

$$F \dashv U : C' \rightarrow C$$

*the following are equivalent:*

- (i)  $F$  sends morphisms in  $\mathcal{L}$  to morphisms in  $\mathcal{L}'$ .
- (ii)  $U$  sends morphisms in  $\mathcal{R}'$  to morphisms in  $\mathcal{R}$ .

*Proof.* The adjunction induces a bijection between solutions to the two lifting problems shown below:

$$\begin{array}{ccc} FZ & \longrightarrow & X \\ Fg \downarrow & \nearrow ? & \downarrow f \\ FW & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} Z & \longrightarrow & UX \\ g \downarrow & \nearrow ? & \downarrow Uf \\ W & \longrightarrow & UY \end{array}$$

Thus,  $Fg \boxtimes f$  (resp.  $Fg \perp f$ ) if and only if  $g \boxtimes Uf$  (resp.  $g \perp Uf$ ). ■

¶ A.3.23. Let  $\mathbb{2}$  be the category  $\{0 \rightarrow 1\}$  and let  $\mathbb{3}$  be  $\{0 \rightarrow 1 \rightarrow 2\}$ . Thus, given a category  $C$ , the functor category  $[\mathbb{2}, C]$  is the category of arrows and commutative squares in  $C$ . There are three embeddings  $d^0, d^1, d^2 : \mathbb{2} \rightarrow \mathbb{3}$ :

$$\begin{array}{lll} d^0(0) = 1 & d^1(0) = 0 & d^2(0) = 0 \\ d^0(1) = 2 & d^1(1) = 2 & d^2(1) = 1 \end{array}$$

These then induce (by precomposition) three functors  $d_0, d_1, d_2 : [\mathbb{3}, C] \rightarrow [\mathbb{2}, C]$ .

**Definition A.3.24.** A **functorial factorisation system** on a category  $C$  is a pair of functors  $L, R : [2, C] \rightarrow [2, C]$  for which there exists a (necessarily unique) functor  $F : [2, C] \rightarrow [3, C]$  satisfying the following equations:

$$d_2 F = L \qquad d_1 F = \text{id}_{[2, C]} \qquad d_0 F = R$$

A **functorial weak** (resp. **orthogonal**) **factorisation system** on  $C$  is a weak (resp. orthogonal) factorisation system  $(\mathcal{L}, \mathcal{R})$  together with a functorial factorisation system  $(L, R)$  such that  $Lf \in \mathcal{L}$  and  $Rf \in \mathcal{R}$  for all morphisms  $f$  in  $C$ . We will often abuse notation and refer to the functorial factorisation system  $(L, R)$  as a functorial weak (resp. orthogonal) factorisation system, omitting mention of the weak (resp. orthogonal) factorisation system  $(\mathcal{L}, \mathcal{R})$ .

**Lemma A.3.25.** *Let  $A$  be an object in a category  $C$  and let  $\Sigma_A : C_{/A} \rightarrow C$  be the projection from the slice category.*

- (i) *For each functorial factorisation system  $(L, R)$  on  $C$ , there exists a unique functorial factorisation system  $(L_A, R_A)$  on  $C_{/A}$  such that*

$$[2, \Sigma_A] \circ L_A = L \circ [2, \Sigma_A] \qquad [2, \Sigma_A] \circ R_A = R \circ [2, \Sigma_A]$$

where  $[2, \Sigma_A] : [2, C_{/A}] \rightarrow [2, C]$  is the evident induced functor.

- (ii) *If  $(L, R)$  is part of a functorial weak or orthogonal factorisation system on  $C$ , then  $(L_A, R_A)$  is compatible with the induced weak or orthogonal factorisation system on  $C_{/A}$  as well.*

*Proof.* Obvious. ◆

**Proposition A.3.26.** *Any orthogonal factorisation system can be extended to a functorial one.*

*Proof.* For each morphism  $f$  in a category  $C$  with an orthogonal factorisation system  $(\mathcal{L}, \mathcal{R})$ , choose a factorisation  $f = Rf \circ Lf$  with  $Lf \in \mathcal{L}$  and  $Rf \in \mathcal{R}$ . Given a commutative square in  $C$ , say

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

the lifting property ensures that the dashed arrow in the diagram below exists,

$$\begin{array}{ccc}
 Z & \xrightarrow{z} & X \\
 Lg \downarrow & & \downarrow Lf \\
 \bullet & \dashrightarrow & \bullet \\
 Rg \downarrow & & \downarrow Rf \\
 W & \xrightarrow{w} & Y
 \end{array}$$

and orthogonality ensures uniqueness and hence functoriality. ■

**Corollary A.3.27.** *If  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on a category  $\mathcal{C}$ , then, for any category  $\mathcal{J}$ , there exists an orthogonal factorisation system on the functor category  $[\mathcal{J}, \mathcal{C}]$  where a natural transformation is in the left (resp. right) class if and only if all its components are in  $\mathcal{L}$  (resp.  $\mathcal{R}$ ).*

*Proof.* Obviously, every morphism in  $[\mathcal{J}, \mathcal{C}]$  admits such a factorisation, since  $(\mathcal{L}, \mathcal{R})$ -factorisations in  $\mathcal{C}$  are functorial. By considering a commutative diagram in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccccc}
 Z' & \longrightarrow & X' & & \\
 \downarrow g' & \searrow & \downarrow & \searrow & \\
 & & Z & \xrightarrow{\quad} & X \\
 & & \downarrow g & & \downarrow f' \\
 W' & \longrightarrow & Y' & & \\
 & \searrow & \downarrow & \searrow & \\
 & & W & \longrightarrow & Y
 \end{array}$$

where  $f$  and  $f'$  are in  $\mathcal{R}$  while  $g$  and  $g'$  are in  $\mathcal{L}$ , using the fact that  $(\mathcal{E}, \mathcal{M})$  is an *orthogonal* factorisation system, one may show that lifting problems in  $[\mathcal{J}, \mathcal{C}]$  admit unique solutions, and that these solutions are moreover constructed componentwise. Thus,  $(\mathcal{L}, \mathcal{R})$  induces an orthogonal factorisation system on  $[\mathcal{J}, \mathcal{C}]$ . ■

The following characterisation of functorial orthogonal factorisation systems is due to Grandis and Tholen [2006]:

**Theorem A.3.28.** *Let  $(L, R)$  be a functorial factorisation system on a category  $C$ . The following are equivalent:*

- (i)  *$L$  is the underlying endofunctor of an idempotent comonad on  $[2, C]$  with counit given by  $\varepsilon_k = (\text{id}_{\text{dom } k}, Rk)$ , and  $R$  is the underlying endofunctor of an idempotent monad on  $[2, C]$  with unit given by  $\eta_h = (h, \text{id}_{\text{codom } h})$ .*
- (ii) *For all morphisms  $h$  in  $C$ ,  $RLh$  and  $LRh$  are isomorphisms in  $C$ .*
- (iii) *For any two morphisms in  $C$ , say  $h$  and  $k$ , we have  $Lk \perp Rh$ .*
- (iv)  *$(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on  $C$  extending  $(L, R)$ , where:*

$$\begin{aligned}\mathcal{L} &= \{g \in \text{mor } C \mid Rg \text{ is an isomorphism in } C\} \\ \mathcal{R} &= \{f \in \text{mor } C \mid Lf \text{ is an isomorphism in } C\}\end{aligned}$$

- (v) *There exists an orthogonal factorisation system  $(\mathcal{L}, \mathcal{R})$  extending  $(L, R)$ .*

*Proof.* (i)  $\Leftrightarrow$  (ii). This is a standard fact about idempotent (co)monads.

(ii)  $\Rightarrow$  (iii). Now, consider the following lifting problem:

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

Since  $(L, R)$  is a functorial factorisation system, we get a commutative diagram of the form below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ Lg \downarrow & & \downarrow Lf \\ W' & \xrightarrow{t} & X' \\ Rg \downarrow & & \downarrow Rf \\ W & \xrightarrow{w} & Y \end{array}$$

but  $Rg$  and  $Lf$  are isomorphisms, so  $(Lf)^{-1} \circ t \circ (Rg)^{-1}$  is the required lift  $W \rightarrow X$ . On the other hand, if  $s : W \rightarrow X$  is any morphism such that  $f \circ s = w$

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and  $s \circ g = z$ , then by taking  $(L, R)$ -factorisations of the vertical arrows in the following diagram,

$$\begin{array}{ccccccc} Z & \xrightarrow{g} & W & \xrightarrow{s} & X & \xrightarrow{\text{id}} & X \\ g \downarrow & & \text{id} \downarrow & & \downarrow \text{id} & & \downarrow f \\ W & \xrightarrow{\text{id}} & W & \xrightarrow{s} & X & \xrightarrow{f} & Y \end{array}$$

we find it must be the case that  $Lf \circ s \circ Rg = t$ , so we indeed have  $g \perp f$ .

(iii)  $\Rightarrow$  (iv). In particular,  $g \perp Rg$  and  $Lf \perp f$ , so there must exist morphisms  $i$  and  $r$  making the diagrams below commute:

$$\begin{array}{ccc} Z & \xrightarrow{Lg} & W' \\ g \downarrow & \nearrow i & \downarrow Rg \\ W & \xrightarrow{\text{id}} & W \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\text{id}} & X \\ Lf \downarrow & \nearrow r & \downarrow f \\ X' & \xrightarrow{Rf} & Y \end{array}$$

We then obtain the following equations,

$$\begin{aligned} (i \circ Rg) \circ Lg &= Lg & (Lf \circ r) \circ Lf &= Lf \\ Rg \circ (i \circ Rg) &= Rg & Rf \circ (Lf \circ r) &= Rf \end{aligned}$$

and since  $Lg \perp Rg$  and  $Lf \perp Rf$ , we must have  $i \circ Rg = \text{id}_{W'}$  and  $Lf \circ r = \text{id}_{X'}$ . Thus,  $g \in \mathcal{L}$  and  $f \in \mathcal{R}$ , and the same argument now shows that  ${}^\perp\mathcal{R} \subseteq \mathcal{L}$  and  $\mathcal{L}^\perp \subseteq \mathcal{R}$ .

It remains to be shown that  $\mathcal{L} \subseteq {}^\perp\mathcal{R}$  and  $\mathcal{R} \subseteq \mathcal{L}^\perp$ . First, suppose  $g \in \mathcal{L}$  and  $f \in \mathcal{R}$ , and consider the following lifting problem:

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

With  $r$  and  $i$  as in the previous paragraph, we obtain a commutative diagram of the form below,

$$\begin{array}{ccccccc} Z & \xrightarrow{\text{id}} & Z & \xrightarrow{z} & X & \xrightarrow{\text{id}} & X \\ g \downarrow & & Lg \downarrow & & \downarrow Lf & \nearrow r & \downarrow f \\ & & W' & \xrightarrow{t} & X' & & \\ & \nearrow i & \downarrow Rg & & \downarrow Rf & & \\ W & \xrightarrow{\text{id}} & W & \xrightarrow{w} & Y & \xrightarrow{\text{id}} & Y \end{array}$$



where the arrow  $t$  is obtained by the functoriality of  $(L, R)$ -factorisations. Thus,  $r \circ t \circ i$  is the required lift  $W \rightarrow X$ , and it is unique, since  $Rg$  and  $Lf$  are isomorphisms. (Recall the proof of (ii)  $\Rightarrow$  (iii).) We conclude that  $\mathcal{L} = {}^\perp\mathcal{R}$  and  $\mathcal{R} = \mathcal{L}^\perp$ .

(iv)  $\Rightarrow$  (v). Immediate.

(v)  $\Rightarrow$  (iii). If  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on  $\mathcal{C}$  such that  $Lf \in \mathcal{L}$  and  $Rf \in \mathcal{R}$  for all morphisms  $f$  in  $\mathcal{C}$ , then we must have  $Lk \perp Rh$  for all  $h$  and  $k$  in  $\text{mor } \mathcal{C}$ , as required.

(iv)  $\Rightarrow$  (ii). Immediate. ■

**REMARK A.3.29.** It is clear that a functorial factorisation system is associated with *at most one* orthogonal factorisation system: indeed, if  $(\mathcal{L}', \mathcal{R}')$  is any orthogonal factorisation system extending a functorial factorisation system  $(L, R)$ , and  $(\mathcal{L}, \mathcal{R})$  is the induced orthogonal factorisation system as in the theorem, then each morphism in  $\mathcal{L}$  (resp.  $\mathcal{R}$ ) is a retract of some morphism in  $\mathcal{L}'$  (resp.  $\mathcal{R}'$ ); but by [proposition A.3.14](#), this implies  $\mathcal{L} \subseteq \mathcal{L}'$  and  $\mathcal{R} \subseteq \mathcal{R}'$ , and applying [proposition A.3.3](#), we also get  $\mathcal{L} \supseteq \mathcal{L}'$  and  $\mathcal{R} \supseteq \mathcal{R}'$ .

**Corollary A.3.30.** *If  $(\mathcal{L}, \mathcal{R})$  is an orthogonal factorisation system on a category  $\mathcal{C}$ , then:*

- (i)  $\mathcal{L}$ , considered as a full subcategory of  $[2, \mathcal{C}]$ , is replete and coreflective.
- (ii)  $\mathcal{L}$  is closed under all colimits in  $[2, \mathcal{C}]$ .
- (iii) If a diagram in  $\mathcal{L}$  has a limit in  $[2, \mathcal{C}]$ , then it also has a limit in  $\mathcal{L}$ .

*Dually:*

- (i')  $\mathcal{R}$ , considered as a full subcategory of  $[2, \mathcal{C}]$ , is replete and reflective.
- (ii')  $\mathcal{R}$  is closed under all limits in  $[2, \mathcal{C}]$ .
- (iii') If a diagram in  $\mathcal{R}$  has a colimit in  $[2, \mathcal{C}]$ , then it also has a colimit in  $\mathcal{R}$ .

*Proof.* Using [proposition A.3.26](#) and [theorem A.3.28](#), the above claims amount to standard facts about the Eilenberg–Moore category for idempotent (co)monads. ■

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There is a similar characterisation of functorial weak factorisation systems:

**Theorem A.3.31.** *Let  $(L, R)$  be a functorial factorisation system on a category  $C$ . The following are equivalent:*

- (i) *For any two morphisms in  $C$ , say  $h$  and  $k$ ,  $Lk \sqsupseteq Rh$ .*
- (ii)  *$(\mathcal{L}, \mathcal{R})$  is an weak factorisation system on  $C$  extending  $(L, R)$ , where:*

$$\mathcal{L} = \{g \in \text{mor } C \mid \exists i \in \text{mor } C. i \circ g = Lg \wedge Rg \circ i = \text{id}_{\text{codom } g}\}$$

$$\mathcal{R} = \{f \in \text{mor } C \mid \exists r \in \text{mor } C. f \circ r = Rf \wedge r \circ Lf = \text{id}_{\text{dom } f}\}$$

- (iii) *There exists a weak factorisation system  $(\mathcal{L}, \mathcal{R})$  extending  $(L, R)$ .*

*Proof.* The proof is essentially the same as that of [theorem A.3.28](#). ■

**REMARK A.3.32.** As with orthogonal factorisation systems, there is *at most one* weak factorisation system extending any functorial factorisation system.

**Proposition A.3.33.** *Let  $(L, R)$  be a functorial factorisation system on  $C$  and let  $\lambda : \text{id}_{[2, C]} \Rightarrow R$  and  $\rho : L \Rightarrow \text{id}_{[2, C]}$  be the natural transformations whose component at an object  $f$  in  $[2, C]$  correspond to the following commutative squares in  $C$ :*

$$\begin{array}{ccc} \bullet & \xrightarrow{Lf} & \bullet \\ f \downarrow & & \downarrow Rf \\ \bullet & \xlongequal{\quad} & \bullet \end{array} \qquad \begin{array}{ccc} \bullet & \xlongequal{\quad} & \bullet \\ Lf \downarrow & & \downarrow f \\ \bullet & \xrightarrow{Rf} & \bullet \end{array}$$

*Suppose  $(L, R)$  extends to a functorial weak factorisation system. Then the following are equivalent for a morphism  $g : Z \rightarrow W$  in  $C$ :*

- (i) *The morphism  $g$  is in the left class of the induced weak factorisation system.*
- (ii) *There exists a morphism  $i$  in  $C$  such that the diagram below commutes:*

$$\begin{array}{ccccc} Z & \xlongequal{\quad} & Z & \xlongequal{\quad} & Z \\ g \downarrow & & Lg \downarrow & & \downarrow g \\ W & \xrightarrow{i} & \bullet & \xrightarrow{Rg} & W \\ & \searrow & \text{id} & \nearrow & \\ & & & & \end{array}$$

(iii) *The object  $g$  in  $[2, C]$  admits a coalgebra structure for the copointed endofunctor  $(L, \rho)$ .*

*Dually, the following are equivalent for a morphism  $f : X \rightarrow Y$  in  $C$ :*

(i') *The morphism  $f$  is in the right class of the induced weak factorisation system.*

(ii') *There exists a morphism  $r$  in  $C$  such that the diagram below commutes:*

$$\begin{array}{ccccc}
& & \text{id} & & \\
& & \curvearrowright & & \\
X & \xrightarrow{Lf} & \bullet & \xrightarrow{r} & X \\
f \downarrow & & \downarrow Rf & & \downarrow f \\
Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y
\end{array}$$

(iii') *The object  $f$  in  $[2, C]$  admits an algebra structure for the pointed endofunctor  $(R, \lambda)$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Consider the following commutative diagram in  $C$ :

$$\begin{array}{ccc}
Z & \xrightarrow{Lg} & \bullet \\
g \downarrow & & \downarrow Rg \\
W & \xrightarrow{\text{id}} & W
\end{array}$$

Thus, a morphism  $i$  of the required form exists in  $C$  as soon as  $g \sqsupseteq Rg$ .

(ii)  $\Leftrightarrow$  (iii). This is simply the definition of  $(L, \rho)$ -coalgebra.

(ii)  $\Rightarrow$  (i). By definition, the morphism  $Lf$  is in the left class of the induced weak factorisation system; but the given diagram exhibits  $f$  as a retract of  $Lf$ , so we may apply [proposition A.3.14](#) to deduce that  $f$  is also in the left class.  $\blacksquare$

The results above motivate the following definition:

**Definition A.3.34.** A **natural weak factorisation system**<sup>[4]</sup> on a category  $C$  is a pair  $(\mathbf{L}, \mathbf{R})$  satisfying the following conditions:

- $\mathbf{L} = (L, \varepsilon, \delta)$  is a comonad on  $[2, C]$ , where  $\varepsilon_k = (\text{id}_{\text{dom } k}, Rk)$ .

[4] — in the sense of Grandis and Tholen [2006], *not* Garner [2009].

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- $\mathbf{R} = (R, \eta, \mu)$  is a monad on  $[2, C]$ , where  $\eta_h = (Lh, \text{id}_{\text{codom } h})$ .
- $(L, R)$  constitute a functorial factorisation system on  $C$ .

Given natural weak factorisation systems  $(\mathbf{L}', \mathbf{R})$  and  $(\mathbf{L}, \mathbf{R}')$  on  $C$ , a **morphism**  $\theta : (\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R}')$  is a pair  $(\theta^L, \theta^R)$ , where  $\theta^L : L' \Rightarrow L$  and  $\theta^R : R \Rightarrow R'$  are natural transformations such that the equations below hold,

$$\begin{array}{ll} \varepsilon \cdot \theta^L = \varepsilon' & (\theta^L \circ \theta^L) \cdot \delta' = \delta \cdot \theta^L \\ \theta^R \cdot \eta = \eta' & \mu' \cdot (\theta^R \circ \theta^R) = \theta^R \cdot \mu \end{array}$$

and furthermore we require  $d_0\theta^L = d_1\theta^R$ .

**REMARK A.3.35.** In other words, a morphism of natural weak factorisation systems is a natural transformation of functors  $[2, C] \rightarrow [3, C]$  such that the left half is a morphism of comonads and the right half is a morphism of monads. In particular, we must have  $d_1\theta^L = \text{id}$  and  $d_0\theta^R = \text{id}$ ; so for every object  $f$  in  $[2, C]$ , we obtain a commutative diagram in  $C$  of the form below:

$$\begin{array}{ccc} \bullet & \xlongequal{\quad} & \bullet \\ L'f \downarrow & \theta_f \rightarrow & \downarrow Lf \\ \bullet & & \bullet \\ Rf \downarrow & & \downarrow R'f \\ \bullet & \xlongequal{\quad} & \bullet \end{array}$$

**Proposition A.3.36.** *Any functorial orthogonal factorisation system extends to a natural weak factorisation system in a unique way; conversely, a natural weak factorisation system induces an orthogonal factorisation system if and only if the underlying comonad and monad are both idempotent.*

*Proof.* This follows from the definition above and [theorem A.3.28](#). ■

**Proposition A.3.37.** *Let  $(\mathbf{L}, \mathbf{R})$  be an natural weak factorisation system on a category  $C$ .*

- (i) *Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be objects in  $[2, C]$ . If  $\alpha : Rf \rightarrow f$  is a  $\mathbf{R}$ -algebra structure and  $\beta : g \rightarrow Lg$  is a  $\mathbf{L}$ -coalgebra structure, then  $d_0(\alpha) : Y \rightarrow Y$  and  $d_1(\beta) : Z \rightarrow Z$  are identity morphisms, and we have the following identities:*

$$\begin{array}{ll} d_1(\alpha) \circ Lf = \text{id}_X & Rg \circ d_0(\beta) = \text{id}_W \\ f \circ d_1(\alpha) = Rf & d_0(\beta) \circ g = Lg \end{array}$$

- (ii) If  $f$  admits a  $\mathbf{L}$ -coalgebra structure and  $g$  admits an  $\mathbf{R}$ -algebra structure, then  $f \square g$ .
- (iii) There exists a (unique) weak factorisation system  $(\mathcal{L}, \mathcal{R})$  on  $\mathcal{C}$  such that  $Lk \in \mathcal{L}$  and  $Rh \in \mathcal{R}$  for all  $h$  and  $k$  in  $\text{mor } \mathcal{C}$ .

*Proof.* (i). The claim follows from the  $\mathbf{L}$ -coalgebra counitality axiom and the  $\mathbf{R}$ -algebra unitality axiom:

$$\alpha \circ \eta_f = \text{id}_f \qquad \varepsilon_g \circ \beta = \text{id}_g$$

- (ii). It then follows that the diagram below commutes,

$$\begin{array}{ccccccc}
Z & \xrightarrow{\text{id}} & Z & \xrightarrow{z} & X & \xrightarrow{\text{id}} & X \\
\downarrow g & & \downarrow Lg & & \downarrow Lf & \nearrow \alpha_0 & \downarrow f \\
& & W' & \xrightarrow{t} & X' & & \\
& \nearrow \beta_1 & \downarrow Rg & & \downarrow Rf & & \\
W & \xrightarrow{\text{id}} & W & \xrightarrow{w} & Y & \xrightarrow{\text{id}} & Y
\end{array}$$

where the arrow  $t$  is obtained by the functoriality of  $(L, R)$ -factorisations; clearly,  $\alpha_0 \circ t \circ \beta_1$  is the required lift.

- (iii). Finally, for any two morphisms in  $\mathcal{C}$ , say  $h$  and  $k$ , we simply note that  $\delta_k : Lk \rightarrow LLk$  is an  $\mathbf{L}$ -coalgebra structure and  $\mu_h : RRh \rightarrow Rh$  is an  $\mathbf{R}$ -algebra structure, so we may apply [theorem A.3.31](#) to obtain the conclusion.  $\blacksquare$

**Proposition A.3.38.** *Let  $(\mathbf{L}', \mathbf{R})$  and  $(\mathbf{L}, \mathbf{R}')$  be natural weak factorisation systems on a category  $\mathcal{C}$ . If there exists a morphism  $(\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R}')$ , then:*

- Every morphism in the left class of the weak factorisation system induced by  $(\mathbf{L}', \mathbf{R})$  is also in the left class of the weak factorisation system induced by  $(\mathbf{L}, \mathbf{R}')$ .
- Every morphism in the right class of the weak factorisation system induced by  $(\mathbf{L}, \mathbf{R}')$  is also in the right class of the weak factorisation system induced by  $(\mathbf{L}', \mathbf{R})$ .

*Proof.* The two claims are formally dual; we will prove the first version.

Let  $L$  (resp.  $L'$ ) be the underlying endofunctor of  $\mathbf{L}$  (resp.  $\mathbf{L}'$ ) and let  $\varepsilon$  (resp.  $\varepsilon'$ ) be the counit of  $\mathbf{L}$  (resp.  $\mathbf{L}'$ ). Suppose we have a morphism  $\theta : (\mathbf{L}', \mathbf{R}) \rightarrow (\mathbf{L}, \mathbf{R})$ . By [proposition A.3.33](#), it suffices to show that every morphism that admits a  $(L', \varepsilon')$ -coalgebra structure also admits a  $(L, \varepsilon)$ -coalgebra structure. But if  $i$  is a  $(L', \varepsilon')$ -coalgebra structure on  $g$ , then  $\theta_g^L \circ i$  is a  $(L, \varepsilon)$ -coalgebra structure on  $g$ , because  $\varepsilon_g \circ \theta_g^L = \varepsilon'_g$ .  $\blacksquare$

**REMARK A.3.39.** Let  $(\mathbf{L}, \mathbf{R})$  be a natural weak factorisation system. Then, for each morphism  $f : X \rightarrow Y$ , we have a commutative diagram of the following form in  $\mathcal{C}$ ,

$$\begin{array}{ccc}
X & \xlongequal{\quad} & X \\
L_f \downarrow & \xrightarrow{d_0(\delta_f)} & \downarrow LL_f \\
\bullet & & \bullet \\
LR_f \downarrow & \searrow \text{id} & \downarrow RL_f \\
\bullet & \xrightarrow{d_1(\mu_f)} & \bullet \\
RR_f \downarrow & & \downarrow R_f \\
Y & \xlongequal{\quad} & Y
\end{array}$$

where the upper square corresponds to  $\delta_f : Lf \rightarrow LLf$  and the lower square corresponds to  $\mu_f : RRf \rightarrow Rf$ ; note that the middle square commutes because  $(\varepsilon \circ \text{id}_L) \cdot \delta = \text{id}_L$  and  $\mu \cdot (\eta \circ \text{id}_R) = \text{id}_R$ . Thus, we obtain a canonical natural transformation  $\xi : LR \Rightarrow RL$ .

The following definition is due to Garner [\[2009\]](#):

**Definition A.3.40.** Let  $\mathcal{C}$  be a category. An **algebraic factorisation system** on  $\mathcal{C}$  is a pair  $(\mathbf{L}, \mathbf{R})$  satisfying the following conditions:

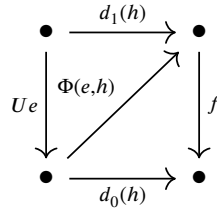
- $(\mathbf{L}, \mathbf{R})$  is a natural weak factorisation system; in particular,  $\mathbf{L} = (L, \varepsilon, \delta)$  is a comonad on  $[2, \mathcal{C}]$  and  $\mathbf{R} = (R, \eta, \mu)$  is a monad on  $[2, \mathcal{C}]$ .
- The canonical natural transformation  $\xi : LR \Rightarrow RL$  is a **distributive law**, i.e.

$$(\text{id}_{d_0} \circ \delta) \cdot (\text{id}_{d_1} \circ \mu) = (\text{id}_{d_1} \circ \mu \circ \text{id}_L) \cdot (\text{id}_M \circ \xi) \cdot (\text{id}_{d_0} \circ \delta \circ \text{id}_R)$$

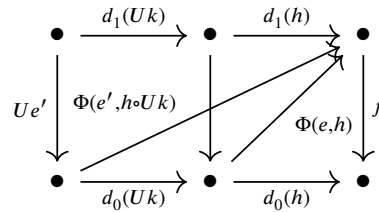
where  $M = d_0L = d_1R$ .

**¶ A.3.41.** Let  $\mathcal{C}$  be a category and let  $U : \mathcal{L} \rightarrow [2, \mathcal{C}]$  be a functor. We define a category  $\mathbf{RLP}_{\mathcal{C}}(U)$  over  $[2, \mathcal{C}]$  as follows:

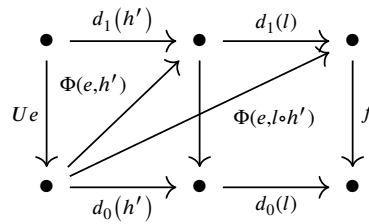
- The objects in  $\mathbf{RLP}_C(U)$  are morphisms in  $C$  equipped with a coherent choice of liftings, i.e. a pair  $(f, \Phi)$  where  $f$  is a morphism in  $C$  equipped with a chosen morphism  $\Phi(e, h) : d_0(Ue) \rightarrow d_1(f)$  in  $C$  for each morphism  $h : Ue \rightarrow f$  in  $[2, C]$  such that the following diagram in  $C$  commutes,



and furthermore, for each morphism  $k : e' \rightarrow e$  in  $\mathcal{I}$ , we require that the following diagram commute:



- The morphisms in  $\mathbf{RLP}_C(U)$  are commutative squares in  $C$  that are compatible with the chosen liftings, i.e. a morphism  $l : (f', \Phi') \rightarrow (f, \Phi)$  is a morphism  $l : f' \rightarrow f$  in  $[2, C]$  such that, for all morphisms  $h' : Ue \rightarrow f'$  in  $[2, C]$ , the following diagram commutes:



- Composition and identities are inherited from  $[2, C]$ .
- The structure functor  $\mathbf{RLP}_C(U) \rightarrow [2, C]$  is the evident forgetful functor sending  $(f, \Phi)$  to  $f$ .

Note that the construction of  $\mathbf{RLP}_C(U)$  is contravariantly functorial in  $U$ .

**Proposition A.3.42.** *Let  $\mathcal{C}$  be a category, let  $(\mathbf{L}, \mathbf{R})$  be a natural weak factorisation system on  $\mathcal{C}$ , let  $\mathcal{L}$  be the category of  $\mathbf{L}$ -coalgebras, and let  $\mathcal{R}$  be the category of  $\mathbf{R}$ -algebras in  $[\mathbf{2}, \mathcal{C}]$ . Then there is a natural functor  $T : \mathcal{R} \rightarrow \mathbf{RLP}_{\mathcal{C}}(U_{\mathbf{L}})$  making the diagram below commute,*

$$\begin{array}{ccc} \mathcal{R} & \overset{T}{\dashrightarrow} & \mathbf{RLP}_{\mathcal{C}}(U_{\mathbf{L}}) \\ & \searrow^{U^{\mathbf{R}}} & \swarrow_U \\ & & [\mathbf{2}, \mathcal{C}] \end{array}$$

where  $U_{\mathbf{L}} : \mathcal{L} \rightarrow [\mathbf{2}, \mathcal{C}]$ ,  $U^{\mathbf{R}} : \mathcal{R} \rightarrow [\mathbf{2}, \mathcal{C}]$  and  $U : \mathbf{RLP}_{\mathcal{C}}(U_{\mathbf{L}}) \rightarrow [\mathbf{2}, \mathcal{C}]$  are the respective forgetful functors.

*Proof.* Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be morphisms in  $\mathcal{C}$ , let  $(r, \text{id}) : \mathbf{R}f \rightarrow f$  be an  $\mathbf{R}$ -algebra structure on  $f$ , and let  $(\text{id}, i) : g \rightarrow Lf$  be an  $\mathbf{L}$ -coalgebra structure on  $g$ . Given a commutative square in  $\mathcal{C}$  of the form below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ g \downarrow & & \downarrow f \\ W & \xrightarrow{w} & Y \end{array}$$

we choose the lifting  $W \rightarrow X$  defined by the following commutative diagram,

$$\begin{array}{ccccccc} Z & \xrightarrow{\text{id}} & Z & \xrightarrow{z} & X & \xrightarrow{\text{id}} & X \\ \downarrow g & & \downarrow Lg & & \downarrow Lf & \nearrow r & \downarrow f \\ & & W' & \longrightarrow & X' & & \\ & \nearrow i & \downarrow Rg & & \downarrow Rf & & \\ W & \xrightarrow{\text{id}} & W & \xrightarrow{w} & Y & \xrightarrow{\text{id}} & Y \end{array}$$

where the morphism  $W' \rightarrow X'$  is the one given by the functorial factorisation. It is not hard to see that this choice of liftings is compatible with the morphisms in  $\mathcal{L}$ , so we have an object in  $\mathbf{RLP}_{\mathcal{C}}(U_{\mathbf{L}})$ . Similarly, one may verify that the liftings are compatible with the morphisms in  $\mathcal{R}$ . Thus, we have the required functor  $T : \mathcal{R} \rightarrow \mathbf{RLP}_{\mathcal{C}}(U_{\mathbf{L}})$  compatible with the forgetful functors, and it is clearly natural in  $(\mathbf{L}, \mathbf{R})$ . ■



**Definition A.3.43.** Let  $C$  be a category and let  $U : \mathcal{I} \rightarrow [2, C]$  be a functor. An **algebraically free natural weak factorisation system** on  $C$  cofibrantly generated by  $U$  is a natural weak factorisation system  $(\mathbf{L}, \mathbf{R})$  on  $C$  equipped with a functor  $E : \mathcal{I} \rightarrow \mathcal{L}$  making the following diagram commute,

$$\begin{array}{ccc} \mathcal{I} & \overset{E}{\dashrightarrow} & \mathcal{L} \\ & \searrow U & \swarrow U_{\mathbf{L}} \\ & [2, C] & \end{array}$$

where  $\mathcal{L}$  is the category of  $\mathbf{L}$ -coalgebras in  $[2, C]$  and  $U_{\mathbf{L}} : \mathcal{L} \rightarrow [2, C]$  is the forgetful functor, such that the composite functor shown below is an isomorphism,

$$\mathcal{R} \xrightarrow{T} \mathbf{RLP}_C(U_{\mathbf{L}}) \xrightarrow{E^*} \mathbf{RLP}_C(U)$$

where  $\mathcal{R}$  is the category of  $\mathbf{R}$ -algebras and  $T : \mathcal{R} \rightarrow \mathbf{LLP}(U_{\mathbf{L}})$  is the canonical functor given in [proposition A.3.42](#).

**REMARK A.3.44.** If  $C$  admits an algebraically free natural weak factorisation system  $(\mathbf{L}, \mathbf{R})$  cofibrantly generated by  $U : \mathcal{I} \rightarrow [2, C]$ , then the forgetful functor  $\mathbf{RLP}_C(U) \rightarrow [2, C]$  is monadic, and the induced monad is isomorphic to  $\mathbf{R}$ . Garner's small object argument ([theorem 0.5.24](#)) gives sufficient conditions for the existence of algebraically free natural weak factorisation systems; note that natural weak factorisation systems so constructed also satisfy the distributive law and are therefore algebraic factorisation systems.

**Proposition A.3.45.** *Let  $C$  be a category, let  $\mathcal{I}$  be a subensemble of  $\text{mor } C$ , and let  $U : \mathcal{I} \rightarrow [2, C]$  be the evident embedding. If  $(\mathbf{L}, \mathbf{R})$  is an algebraically free natural weak factorisation system cofibrantly generated by  $U$ , then the underlying weak factorisation system of  $(\mathbf{L}, \mathbf{R})$  is cofibrantly generated by  $\mathcal{I}$ .*

*Proof.* This follows from the definitions and [proposition A.3.37](#). ■

**Definition A.3.46.** Let  $C$  be a category and let  $U : \mathcal{I} \rightarrow [2, C]$  be a functor. A **free algebraic factorisation system** on  $C$  cofibrantly generated by  $U$  is an algebraic factorisation system  $(\mathbf{L}, \mathbf{R})$  equipped with a functor  $E : \mathcal{I} \rightarrow \mathcal{L}$  making the following diagram commute,

$$\begin{array}{ccc} \mathcal{I} & \overset{E}{\dashrightarrow} & \mathcal{L} \\ & \searrow U & \swarrow U_{\mathbf{L}} \\ & [2, C] & \end{array}$$

where  $\mathcal{L}$  is the category of  $\mathbf{L}$ -algebras in  $[2, C]$  and  $U_{\mathbf{L}} : \mathcal{L} \rightarrow [2, C]$  is the forgetful functor, such that  $(\mathbf{L}, \mathbf{R})$  and  $E$  have the following universal property:

- For all algebraic factorisation systems  $(\mathbf{L}', \mathbf{R}')$  and all functors  $E' : \mathcal{I} \rightarrow \mathcal{L}'$  where  $\mathcal{L}'$  is the category of  $\mathbf{L}'$ -coalgebras and  $E'$  is compatible with the forgetful functors, there exists a unique morphism  $\theta : (\mathbf{L}, \mathbf{R}) \rightarrow (\mathbf{L}', \mathbf{R}')$  such that  $E' = \theta_*^L E$ , where  $\theta_*^L : \mathcal{L} \rightarrow \mathcal{L}'$  is the functor induced by the comonad morphism  $\theta^L : \mathbf{L} \rightarrow \mathbf{L}'$ .

**Theorem A.3.47.** *Let  $C$  be a category and let  $U : \mathcal{I} \rightarrow [2, C]$  be a functor. If  $(\mathbf{L}, \mathbf{R})$  is an algebraic factorisation system on  $C$  and also an algebraically free natural weak factorisation system cofibrantly generated by  $U$ , then  $(\mathbf{L}, \mathbf{R})$  is a free algebraic factorisation system cofibrantly generated by  $U$ .*

*Proof.* See Theorem A.1 in [Garner, 2009]. □

REMARK. The cited proof of the theorem above uses the distributive law for algebraic factorisation systems.

## A.4 Relative categories

**Prerequisites.** §0.1.

In this section we use the explicit universe convention.

**Definition A.4.1.** A **relative category**  $C$  consists of a category  $\text{und } C$  and a subcategory  $\text{weq } C$  such that  $\text{ob und } C = \text{ob weq } C$ . We say  $\text{und } C$  is the **underlying category** of  $C$ , and that the morphisms in  $\text{weq } C$  are the **weak equivalences** in  $C$ . A **relative subcategory** of a relative category  $C$  is a relative category  $C'$  such that  $\text{und } C'$  is a subcategory of  $\text{und } C$ , and we further demand that  $\text{weq } C' = \text{weq } C \cap \text{und } C'$ .

REMARK A.4.2. The subcategory  $\text{weq } C$  is entirely determined by  $\text{mor weq } C$ , so a relative category may equivalently be defined as a category equipped with a distinguished subset of morphisms closed under composition and containing all the identity morphisms.

For brevity, we will write  $\text{ob } C$  for  $\text{ob und } C$ ,  $\text{mor } C$  for  $\text{ob und } C$ , and we may occasionally abuse notation and write  $\text{weq } C$  instead of  $\text{mor weq } C$ .

REMARK A.4.3. Every category  $C$  can be endowed with the structure of a relative category in two ways: we can make it into a **minimal relative category**  $\min C$  by taking  $\text{weq } \min C$  to be the set of identity morphisms in  $C$ ; or we could make it into a **maximal relative category**  $\max C$  by taking  $\text{weq } \max C = \text{mor } C$ . We may also define the **minimal saturated relative category**  $\min^+ C$  by taking  $\text{weq } \min^+ C$  to be the set of all isomorphisms in  $C$ .

**Definition A.4.4.** Given a relative category  $C$ , the **opposite relative category**  $C^{\text{op}}$  is defined by  $\text{und } C^{\text{op}} = (\text{und } C)^{\text{op}}$  and  $\text{weq } C^{\text{op}} = (\text{weq } C)^{\text{op}}$ .

**Definition A.4.5.** Let  $C$  and  $D$  be relative categories. A **relative functor**  $C \rightarrow D$  is a functor  $\text{und } C \rightarrow \text{und } D$  that sends weak equivalences in  $C$  to weak equivalences in  $D$ . The **relative functor category**  $[C, D]_{\text{h}}$  is the full subcategory of  $[\text{und } C, \text{und } D]$  spanned by the relative functors, and the weak equivalences in  $[C, D]_{\text{h}}$  are defined to be the natural transformations that are componentwise weak equivalences in  $D$ .

**Definition A.4.6.** Let  $C$  be a category and let  $\mathcal{W} \subseteq \text{mor } C$ . A **localisation of  $C$  at  $\mathcal{W}$**  is a category  $C[\mathcal{W}^{-1}]$  equipped with a functor  $\gamma : C \rightarrow C[\mathcal{W}^{-1}]$  with the following universal property:

- Given a functor  $F : C \rightarrow D$  such that  $Ff$  is an isomorphism for all  $f$  in  $\mathcal{W}$ , there exists a unique functor  $\overline{F} : C[\mathcal{W}^{-1}] \rightarrow D$  such that  $\overline{F}\gamma = F$ .

The functor  $\gamma : C \rightarrow \text{Ho } C$  is called the **localising functor**.

REMARK A.4.7. The universal property in the above definition is strict; as such,  $C[\mathcal{W}^{-1}]$  is unique up to unique isomorphism. Nonetheless,  $C[\mathcal{W}^{-1}]$  automatically has a 2-universal property: if  $F, G : C \rightarrow D$  both factor through  $C[\mathcal{W}^{-1}]$ , then so do all natural transformations  $F \Rightarrow G$ .

**Proposition A.4.8.** *If  $C$  is a  $\mathbf{U}$ -small category, then there exists a  $\mathbf{U}$ -small category with the universal property of  $C[\mathcal{W}^{-1}]$ .*

*Proof.* Use the general adjoint functor theorem. □

**Definition A.4.9.** The **homotopy category** of a relative category  $C$  is a localisation of  $\text{und } C$  at  $\text{weq } C$  and is denoted  $\text{Ho } C$ .

**Definition A.4.10.**

- A **semi-saturated relative category** is a relative category in which every isomorphism is a weak equivalence.

- A **saturated relative category** is a relative category  $C$  such that the weak equivalences in  $C$  are precisely the ones that become isomorphisms in  $\text{Ho } C$ .

REMARK A.4.11. Obviously, there is no loss of generality in considering semi-saturated relative categories and their homotopy categories instead of localisations  $C[\mathcal{W}^{-1}]$  for arbitrary subsets  $\mathcal{W} \subseteq \text{mor } C$ .

REMARK A.4.12. Clearly, every saturated relative category is semi-saturated, and a minimal saturated relative category is indeed saturated in the sense above.

**Definition A.4.13.** Let  $C$  be a category and let  $\mathcal{W}$  be a subset of  $\text{mor } C$ . The **2-out-of-3 property** for  $\mathcal{W}$  says:

- Given any two morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$  in  $C$ , if any two of  $f$ ,  $g$ , or  $g \circ f$  are in  $\mathcal{W}$ , then all of them are.

The **2-out-of-6 property** for  $\mathcal{W}$  says:

- Given any three morphisms  $f : X \rightarrow Y$ ,  $g : Y \rightarrow Z$ ,  $h : Y \rightarrow Z$  in  $C$ , if both  $h \circ g$  and  $g \circ f$  are in  $\mathcal{W}$ , then so too are  $f$ ,  $g$ ,  $h$ , and  $h \circ g \circ f$ .

**Lemma A.4.14.** Let  $C$  be a category and let  $\mathcal{W} \subseteq \text{mor } C$ .

- If  $\mathcal{W}$  has the 2-out-of-6 property, then it also has the 2-out-of-3 property.
- The set of all isomorphisms in  $C$  has the 2-out-of-6 property.
- If  $F : C' \rightarrow C$  is a functor and  $\mathcal{W}$  has either the 2-out-of-3 property or the 2-out-of-6 property, then  $F^{-1}\mathcal{W}$  has the same property.

*Proof.* (i). Consider the three cases  $f = \text{id}$ ,  $g = \text{id}$ ,  $h = \text{id}$  in turn.

(ii). If  $h \circ g$  and  $g \circ f$  are isomorphisms, then  $g$  must be split epic and split monic; thus  $g$  itself is an isomorphism, hence so too are  $f$  and  $h$ .

(iii). Obvious. ■

**Corollary A.4.15.** If  $C$  is a saturated relative category, then  $\text{weq } C$  has the 2-out-of-6 property. ■

**Definition A.4.16.** Let  $C$  be a category and let  $\mathcal{W}$  be a subset of  $\text{mor } C$ . The **2-out-of-4 property** for  $\mathcal{W}$  says:

- Given any two morphisms  $f : X \rightarrow Y, g : Y \rightarrow X$  in  $\mathcal{C}$ , if  $f \circ g$  and  $g \circ f$  are in  $\mathcal{W}$ , then both  $f$  and  $g$  are in  $\mathcal{W}$ .

The **special 2-out-of-4 property** for  $\mathcal{W}$  says:

- Given any two morphisms  $f : X \rightarrow Y, g : Y \rightarrow X$  in  $\mathcal{C}$ , if  $f \circ g$  is in  $\mathcal{W}$  and  $g \circ f = \text{id}_X$ , then both  $f$  and  $g$  are in  $\mathcal{W}$ .

**Lemma A.4.17.** *Let  $\mathcal{C}$  be a relative category.*

- (i) *If  $\text{weq } \mathcal{C}$  has the 2-out-of-4 property, then  $\text{weq } \mathcal{C}$  has the special 2-out-of-4 property.*
- (ii) *If  $\text{weq } \mathcal{C}$  has the 2-out-of-6 property, then  $\text{weq } \mathcal{C}$  has the 2-out-of-4 property.*
- (iii) *If  $\text{weq } \mathcal{C}$  has the 2-out-of-3 property and is closed under retracts, then  $\text{weq } \mathcal{C}$  has the special 2-out-of-4 property.*

*Proof.* (i) and (ii). Obvious.

(iii). Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  be morphisms in  $\mathcal{C}$  such that  $f \circ g$  is a weak equivalence and  $g \circ f = \text{id}_X$ . Consider the following diagram:

$$\begin{array}{ccccc}
 Y & \xrightarrow{\text{id}} & Y & \xrightarrow{\text{id}} & Y \\
 g \downarrow & & f \circ g \downarrow & & \downarrow g \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & X
 \end{array}$$

Since  $g \circ f = \text{id}_X$ , the diagram commutes, so we see that  $g : Y \rightarrow X$  is a retract of  $f \circ g : Y \rightarrow Y$ . We deduce that  $g$  is a weak equivalence in  $\mathcal{C}$  using the fact that  $\text{weq } \mathcal{C}$  is closed under retracts, and then we deduce that  $f$  is a weak equivalence using the the 2-out-of-3 property of  $\text{weq } \mathcal{C}$ . ■

**Proposition A.4.18.** *Let  $\mathbf{RelCat}$  be the category of  $\mathbf{U}$ -small relative categories and relative functors, let  $\mathbf{SsRelCat}$  be the full subcategory of semi-saturated relative categories, and let  $\mathbf{Cat}$  be the category of  $\mathbf{U}$ -small categories and functors.*

- (i) **RelCat** is a cartesian closed category, where the product of  $\mathcal{C}$  and  $\mathcal{D}$  is the cartesian product  $\mathcal{C} \times \mathcal{D}$  with weak equivalences taken componentwise, and the exponential of  $\mathcal{E}$  by  $\mathcal{D}$  is the relative functor category  $[\mathcal{D}, \mathcal{E}]_{\text{h}}$ .

- (ii) **RelCat** is a locally finitely presentable **U**-category,<sup>[5]</sup> and the two functors  $\text{und}, \text{weq} : \mathbf{RelCat} \rightarrow \mathbf{Cat}$  are  $\aleph_0$ -accessible<sup>[6]</sup> and jointly conservative.
- (iii) **SsRelCat** is a locally finitely presentable **U**-category, and the inclusion  $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$  is  $\aleph_0$ -accessible and has a left adjoint.
- (iv) **SsRelCat** is an exponential ideal in **RelCat**.
- (v) The full subcategory spanned by the minimal relative categories is an exponential ideal in **RelCat**.
- (vi) The full subcategory spanned by the minimal saturated relative categories is an exponential ideal in **SsRelCat**.

*Proof.* (i). This is straightforward from the definitions.

(ii). Obviously, a relative functor  $F : C \rightarrow D$  such that  $\text{und } F : \text{und } C \rightarrow \text{und } D$  and  $\text{weq } F : \text{weq } C \rightarrow \text{weq } D$  are both isomorphisms is itself an isomorphism, so  $\text{und}, \text{weq} : \mathbf{RelCat} \rightarrow \mathbf{Cat}$  are indeed jointly conservative.

It is also not hard to check that limits for all **U**-small diagrams and colimits for **U**-small filtered diagrams in **RelCat** exist and can be computed componentwise in **Cat**, so (by [theorem 0.2.39](#)) it is enough to show that **RelCat** is a  $\aleph_0$ -accessible **U**-category. Clearly, a relative category  $C$  such that  $\text{und } C$  is finitely presentable in **Cat** and  $\text{weq } C$  is a finitely-generated subcategory of  $\text{und } C$  is itself finitely presentable in **RelCat**, so **RelCat** is indeed  $\aleph_0$ -accessible.

(Alternatively, one may appeal to the sketchability theorem<sup>[7]</sup> and the fact that a relative category is manifestly a model for a certain finite-limit sketch.)

(iii). It is clear that **SsRelCat** is closed in **RelCat** under limits for all **U**-small diagrams and colimits for all **U**-small filtered diagrams, and we know that **RelCat** is a locally finitely presentable category, so (by [proposition 0.2.30](#)) it is enough to construct a left adjoint for the inclusion  $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$ . This may be done using the general adjoint functor theorem.

(iv) – (vi). All straightforward. ■

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[5] See [definition 0.2.35](#).

[6] See [definition 0.2.27](#).

[7] See Proposition 1.51 in [LPAC] or Proposition 5.6.4 in [Borceux, 1994b].

**Proposition A.4.19.** *Let  $\mathbf{RelCat}$  be the category of  $\mathbf{U}$ -small relative categories and relative functors, let  $\mathbf{SsRelCat}$  be the full subcategory of semi-saturated relative categories and relative functors, and let  $\mathbf{Cat}$  be the category of  $\mathbf{U}$ -small categories and functors. We have the following strings of adjoint functors:*

$$\begin{aligned} \min \dashv \text{und} \dashv \max \dashv \text{weq} &: \mathbf{RelCat} \rightarrow \mathbf{Cat} \\ \text{Ho} \dashv \min^+ \dashv \text{und} \dashv \max \dashv \text{weq} &: \mathbf{SsRelCat} \rightarrow \mathbf{Cat} \end{aligned}$$

*The functors  $\min$ ,  $\min^+$ , and  $\max$  are moreover fully faithful, and  $\text{Ho}$  preserves finite products.*

*Proof.* All but the last of the above claims are obvious; for the preservation of finite products under  $\text{Ho}$ , we refer to [proposition A.2.13](#). ■

**Corollary A.4.20.**  $\text{Ho} : \mathbf{SsRelCat} \rightarrow \mathbf{Cat}$  is 2-functorial.

*Proof.* Apply [remark A.2.10](#). ■

**Proposition A.4.21.** *Let  $C$  be a relative category and let  $\gamma : C \rightarrow \text{Ho } C$  be the localising functor.*

- (i) *For all categories  $D$ , the induced functor  $\gamma^* : [\text{Ho } C, D] \rightarrow [C, D]$  is fully faithful and injective on objects.*
- (ii) *Any left or right adjoint for  $\gamma : C \rightarrow \text{Ho } C$  is a fully faithful functor.*

*Proof.* (i). It is an immediate consequence of the universal property of  $\text{Ho } C$  that  $\gamma^* : [\text{Ho } C, D] \rightarrow [C, D]$  is injective on objects. It is moreover fully faithful because we have the following natural isomorphism,

$$[\text{Ho } C, D] \cong \text{und} [C, \min^+ D]_{\text{h}}$$

and  $\text{und} [C, \min^+ D]_{\text{h}}$  is manifestly a full subcategory of  $[C, D]$ .

(ii). Apply [proposition A.1.4](#). ■

**Definition A.4.22.** A **zigzag type** is a relative category  $T$  where  $\text{und } T$  is the free category on an inhabited finite planar graph of the form



where the edges are arrows that point either leftwards or rightwards, and  $\text{weq } T$  is generated by the leftward-pointing arrows. A **morphism of zigzag types** is a relative functor that maps the leftmost object to the leftmost object and the rightmost object to the rightmost object. We write  $\mathbf{T}$  for the category of zigzag types.<sup>[8]</sup>

A **zigzag** of type  $T$  in a relative category  $\mathcal{C}$  is a relative functor  $T \rightarrow \mathcal{C}$ . Given objects  $X$  and  $Y$  in  $\mathcal{C}$ , we denote by  $\mathcal{C}^T(X, Y)$  the category whose objects are the zigzags starting at  $X$  and ending at  $Y$  and whose morphisms are commutative diagrams in  $\mathcal{C}$  of the form

$$\begin{array}{ccccccc} X & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & Y \\ \parallel & & \downarrow & & & & \downarrow & & \parallel \\ X & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & \cdots & \xrightarrow{\quad} & \bullet & \xrightarrow{\quad} & Y \end{array}$$

where the rows are zigzags of type  $T$  and the unmarked vertical arrows are weak equivalences.

**Example A.4.23.** If  $f : X \rightarrow Y$  is a weak equivalence in a relative category  $\mathcal{C}$ , then we have commutative diagrams

$$\begin{array}{ccc} X \xlongequal{\quad} X \xlongequal{\quad} X & & Y \xleftarrow{f} X \xrightarrow{f} Y \\ \parallel & \begin{array}{c} f \downarrow \\ \parallel \end{array} & \parallel \\ X \xrightarrow{f} Y \xleftarrow{f} X & & Y \xlongequal{\quad} Y \xlongequal{\quad} Y \end{array}$$

and these correspond to morphisms of zigzags in  $\mathcal{C}$ .

**REMARK A.4.24.** It is clear that  $\mathcal{C}^T(X, Y)$  is a subcategory of the relative functor category  $[T, \mathcal{C}]_{\text{h}}$ . Thus, if  $\mathcal{C}$  is a  $\mathbf{U}$ -small relative category, precomposition makes the assignment  $T \mapsto \mathcal{C}^T(X, Y)$  into a functor  $\mathbf{T}^{\text{op}} \rightarrow \mathbf{Cat}$ , which we denote by  $\mathcal{C}^*(X, Y)$ . A Grothendieck construction applied to this functor yields the following  $\mathbf{U}$ -small category  $\mathcal{C}^{(\mathbf{T})}(X, Y)$ :

- Its objects are pairs  $(T, f)$ , where  $T$  is a zigzag type and  $f$  is a zigzag of type  $T$  in  $\mathcal{C}$ .
- A morphism  $(T', f') \rightarrow (T, f)$  is a pair  $(\alpha, \beta)$  where  $\alpha : T \rightarrow T'$  is a morphism in  $\mathbf{T}$  and  $\beta : \alpha^* f' \rightarrow f$  is a morphism in  $\mathcal{C}^T(X, Y)$ .

[8] **Warning.** This is the *opposite* of the category  $\mathbf{T}$  defined in [DHKS, §34].



- The composite of a pair of morphisms  $(\alpha', \beta') : (T'', f'') \rightarrow (T', f')$  and  $(\alpha, \beta) : (T', f') \rightarrow (T, f)$  is given by  $(\alpha' \circ \alpha, \beta \circ \alpha' \beta')$ .

There is an evident projection functor  $C^{(\mathbf{T})}(X, Y) \rightarrow \mathbf{T}^{\text{op}}$ , and by construction it is a Grothendieck opfibration with a canonical splitting.

**Lemma A.4.25.** *Given a commutative diagram of the form below in a relative category  $C$ ,*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ a \downarrow & & \downarrow b \\ X' & \xrightarrow{f'} & Y' \end{array}$$

*if  $a$  and  $b$  are weak equivalences in  $C$ , then we obtain the following morphisms of zigzags:*

$$\begin{array}{ccccccc} X & \xlongequal{\quad} & X & \xrightarrow{f} & Y & \xrightarrow{b} & Y' \\ \parallel & & a \downarrow & & \downarrow b & & \parallel \\ X' & \xrightarrow{a} & X' & \xrightarrow{f'} & Y' & \xlongequal{\quad} & Y' \end{array}$$
  

$$\begin{array}{ccccccc} X' & \xleftarrow{a} & X & \xrightarrow{f} & Y & \xlongequal{\quad} & Y \\ \parallel & & a \downarrow & & \downarrow b & & \parallel \\ X' & \xlongequal{\quad} & X' & \xrightarrow{f'} & Y' & \xleftarrow{b} & Y \end{array}$$

*In particular,  $X \xrightarrow{f} Y \xrightarrow{b} Y'$  and  $X \xrightarrow{a} X' \xrightarrow{f'} Y'$  are in the same connected component of  $C^{(\mathbf{T})}(X, Y')$ ; and  $X' \xleftarrow{a} X \xrightarrow{f} Y$  and  $X' \xrightarrow{f'} Y' \xleftarrow{b} Y$  are in the same connected component of  $C^{(\mathbf{T})}(X', Y)$ . ■*

**Theorem A.4.26.** *Let  $X$  and  $Y$  be objects in a relative category  $C$ .*

- (i) *For each zigzag type  $T$ , the map that sends an object in  $C^T(X, Y)$  to the corresponding composite in  $\text{Ho } C(X, Y)$  is a functor when the latter is regarded as a discrete category.*
- (ii) *The functors described above constitute a jointly surjective cocone from the diagram  $C^*(X, Y)$  to  $\text{Ho } C(X, Y)$ .*

(iii) *The induced functor  $C^{(\mathbf{T})}(X, Y) \rightarrow \text{Ho } C(X, Y)$  is surjective, and moreover two objects in  $C^{(\mathbf{T})}(X, Y)$  become equal in  $\text{Ho } C$  if and only if they are in the same connected component.*

*Proof.* All obvious except for the last part of claim (iii), for which we refer to paragraphs 33.8 and 33.10 in [DHKS]. ■

## A.5 Kan extensions

**Prerequisites.** §§0.1, A.1.

In this section we use the explicit universe convention.

**Definition A.5.1.** Let  $F : C \rightarrow D$  and  $G : C \rightarrow \mathcal{E}$  be two functors. A **left Kan extension** (resp. **right Kan extension**) of  $G$  along  $F$  is an initial (resp. terminal) object of the category  $(G \downarrow F^*)$  (resp.  $(F^* \downarrow G)$ ) described below:

- The objects are pairs  $(H, \alpha)$  where  $H$  is a functor  $D \rightarrow \mathcal{E}$  and  $\alpha$  is a natural transformation of type  $G \Rightarrow HF$  (resp.  $HF \Rightarrow G$ ).
- The morphisms  $(H', \alpha') \rightarrow (H, \alpha)$  are those natural transformations  $\beta : H' \Rightarrow H$  such that  $\beta F \bullet \alpha' = \alpha$  (resp.  $\alpha \bullet \beta F = \alpha'$ ).

**REMARK A.5.2.** Clearly, Kan extensions are unique up to unique isomorphism if they exist. We write  $(\text{Lan}_F G, \eta)$  for the left Kan extension of  $G$  along  $F$  and say  $\eta$  is the **unit** of  $\text{Lan}_F G$ ; dually, we write  $(\text{Ran}_F G, \varepsilon)$  for the right Kan extension of  $G$  along  $F$  and say  $\varepsilon$  is the **counit** of  $\text{Ran}_F G$ .

**Lemma A.5.3.** *Let  $\mathbf{U}$  be a pre-universe and let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets. Let  $B$  be a  $\mathbf{U}$ -small category and let  $C$  be a locally  $\mathbf{U}$ -small category. Given functors  $F : B \rightarrow C$  and  $G : B \rightarrow \mathbf{Set}$ , if  $H : C \rightarrow \mathbf{Set}$  is the functor defined by the formula below,*

$$H(C) = [B, \mathbf{Set}](C(C, F-), G-)$$

*and  $\varepsilon_B : H(FB) \rightarrow G(B)$  is defined by evaluation at  $\text{id}_{FB}$ , then  $(H, \varepsilon)$  is the right Kan extension of  $G$  along  $F$ .*

*Proof.* Note that  $H(C)$  so defined is indeed a  $\mathbf{U}$ -set, because  $B$  is  $\mathbf{U}$ -small and  $C$  is locally  $\mathbf{U}$ -small. The claim amounts to saying that  $(H, \varepsilon)$  is a terminal object in the comma category  $(F^* \downarrow G)$ , so that is what we must show.

Let  $\varphi : (X, \alpha) \rightarrow (H, \varepsilon)$  be a morphism in  $(F^* \downarrow G)$ , i.e. a natural transformation  $\varphi : X \Rightarrow H$  such that  $\varepsilon \bullet \varphi F = \alpha$ . Let  $C$  be an object in  $\mathcal{C}$ , let  $x$  be an element of  $X(C)$ , and consider the element  $\varphi_C(x)$  of  $H(C)$ . By definition, this is a natural transformation  $C(C, F) \Rightarrow G$ , so we may consider its component at an object  $B$  in  $\mathcal{B}$ , which will be a map  $C(C, FB) \rightarrow G(B)$ . Let  $f : C \rightarrow FB$  be an arrow in  $\mathcal{C}$ . By hypothesis,

$$\alpha_C(x) = \varepsilon_C(\varphi_C(x)_B \circ C(f, FB)) = \varphi_C(x)_B(f)$$

thus the action of  $\varphi$  is entirely determined by  $\alpha$ . Conversely, given any object  $(X, \alpha)$  in the comma category  $(F^* \downarrow G)$ , it is easily verified that the above equation defines a morphism  $\varphi : (X, \alpha) \rightarrow (H, \varepsilon)$ , so  $(H, \varepsilon)$  is indeed a terminal object in  $(F^* \downarrow G)$ . ■

**Corollary A.5.4.** *For any two functors  $F : \mathcal{B} \rightarrow \mathcal{C}$  and  $G : \mathcal{B} \rightarrow \mathbf{Set}$ , if  $\mathcal{B}$  is  $\mathbf{U}$ -small and  $\mathcal{C}$  is locally  $\mathbf{U}$ -small, then the following are equivalent:*

- (i)  $(\mathbf{Ran}_F G, \varepsilon)$  is a right Kan extension of  $G$  along  $F$ .
- (ii) The maps  $(\mathbf{Ran}_F G)(C) \rightarrow [\mathcal{B}, \mathbf{Set}](C(C, F), G)$  defined by  $x \mapsto \varepsilon \bullet \theta_x F$ , where  $\theta_x : C(C, -) \Rightarrow G$  is the unique natural transformation such that  $(\theta_x)_C(\text{id}_C) = x$ , are bijections that are natural in  $C$ . ■

**Definition A.5.5.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{C} \rightarrow \mathcal{E}$  be two functors.

- A functor  $L : \mathcal{E} \rightarrow \mathcal{F}$  **preserves** left Kan extensions of  $G$  along  $F$  if, given any left Kan extension  $(H, \alpha)$  of  $G$  along  $F$ ,  $(LH, L\alpha)$  is a left Kan extension of  $LG$  along  $F$ .
- A functor  $R : \mathcal{E} \rightarrow \mathcal{F}$  **preserves** right Kan extensions of  $G$  along  $F$  if, given any right Kan extension  $(H, \alpha)$  of  $G$  along  $F$ ,  $(RH, R\alpha)$  is a right Kan extension of  $LG$  along  $F$ .

If a Kan extension is preserved by *all* functors, then it is said to be **absolute**.

**Definition A.5.6.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -small sets, let  $\mathcal{E}$  be a locally  $\mathbf{U}$ -small category, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{C} \rightarrow \mathcal{E}$  be two functors.

- A **pointwise left Kan extension** of  $G$  along  $F$  is one that is preserved by all functors of the form  $\mathcal{E}(-, E) : \mathcal{E} \rightarrow \mathbf{Set}^{\text{op}}$ .

- A **pointwise right Kan extension** of  $G$  along  $F$  is one that is preserved by all functors of the form  $\mathcal{E}(E, -) : \mathcal{E} \rightarrow \mathbf{Set}$ .

**Definition A.5.7.** Let  $F : \mathcal{B} \rightarrow \mathcal{C}$  be a functor and let  $C$  be an object in  $\mathcal{C}$ .

- The **tautological cocone** to  $C$  induced by  $F$  is the cocone  $\theta : FP_C \Rightarrow \Delta C$ , where  $P_C : (F \downarrow C) \rightarrow \mathcal{B}$  is the projection functor sending an object  $(B, f)$  in the comma category  $(F \downarrow C)$  to the object  $B$  in  $\mathcal{B}$ , and  $\theta_{(B,f)} = f$ .
- The **tautological cone** from  $C$  induced by  $F$  is the cone  $\theta : \Delta C \Rightarrow FP^C$ , where  $P^C : (C \downarrow F) \rightarrow \mathcal{C}$  is the projection functor sending an object  $(B, f)$  in the comma category  $(C \downarrow F)$  to the object  $B$  in  $\mathcal{B}$ , and  $\theta_{(B,f)} = f$ .

**Lemma A.5.8.** Let  $\mathcal{A}$  be any category, let  $\mathcal{B}$  be a  $\mathbf{U}$ -small category, let  $\mathcal{C}$  be locally  $\mathbf{U}$ -small category, and let  $U : \mathcal{A} \rightarrow \mathcal{C}$ ,  $V : \mathcal{B} \rightarrow \mathcal{C}$ , and  $Y : \mathcal{B} \rightarrow \mathbf{Set}$  be functors. Consider the following diagram of functors and natural transformations,

$$\begin{array}{ccc} (U \downarrow V) & \xrightarrow{Q} & \mathcal{B} \\ P \downarrow & \theta \nearrow & \downarrow V \\ \mathcal{A} & \xrightarrow{U} & \mathcal{C} \end{array}$$

where  $(U \downarrow V)$  is the comma category,  $P : (U \downarrow V) \rightarrow \mathcal{A}$  and  $Q : (U \downarrow V) \rightarrow \mathcal{B}$  are the two projections, and  $\theta : UP \Rightarrow VQ$  is the tautological natural transformation defined by  $\theta_{(A,B,f)} = f$ . If  $(Z, \varepsilon)$  is a right Kan extension of  $Y$  along  $V$ , then  $(ZU, \varepsilon Q \cdot Z\theta)$  is a right Kan extension of  $YQ$  along  $P$ .

*Proof.* By lemma A.5.3, we may take  $Z : \mathcal{C} \rightarrow \mathbf{Set}$  to be the functor defined by the formula below,

$$Z(C) = [\mathbb{B}, \mathbf{Set}](C(C, F-), Y-)$$

with  $\varepsilon : V^*(Z) \Rightarrow Y$  being the natural transformation obtained by evaluating elements of  $Z(VB)$  at  $\text{id}_{VB}$ .

Let  $\varphi : (X, \alpha) \rightarrow (ZU, \varepsilon Q \cdot Z\theta)$  be a morphism in  $(P^* \downarrow YQ)$ , i.e. a natural transformation  $\varphi : X \Rightarrow ZU$  such that  $\varepsilon Q \cdot Z\theta \cdot \varphi P = \alpha$ . Let  $A$  be an object in  $\mathcal{A}$ , let  $x$  be an element of  $X(A)$ , and consider the element  $\varphi_A(x)$  of  $Z(UA)$ . By definition, this is a natural transformation  $N^V(C) \Rightarrow Y$ , so we may consider its component at an object  $B$  in  $\mathcal{B}$ , which will be a map  $C(UA, VB) \rightarrow Y(B)$ .

Let  $f : UA \rightarrow VB$  be an arrow in  $\mathcal{C}$ ; then  $(A, B, f)$  is an object in the comma category  $(U \downarrow V)$ , and  $\theta_{(A,B,f)} = f$  by definition. By hypothesis,

$$\alpha_{(A,B,f)}(x) = \varepsilon_B(\varphi_A(x)_B \circ C(f, VB)) = \varphi_A(x)_B(f)$$

thus the action of  $\varphi$  is entirely determined by  $\alpha$ . Conversely, given any object  $(X, \alpha)$  in the comma category  $(P^* \downarrow YQ)$ , it is easily verified that the above equation defines a morphism  $\varphi : (X, \alpha) \rightarrow (ZU, \varepsilon Q \bullet Z\theta)$ , so  $(ZU, \varepsilon Q \bullet Z\theta)$  is indeed a terminal object in  $(P^* \downarrow YQ)$ . ■

**Corollary A.5.9.** *Let  $\mathcal{B}$  be a  $\mathbf{U}$ -small category and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. Given functors  $F : \mathcal{B} \rightarrow \mathcal{C}$  and  $G : \mathcal{B} \rightarrow \mathbf{Set}$ , if  $(H, \varepsilon)$  is a right Kan extension of  $G$  along  $F$ , then, for each object  $C$  in  $\mathcal{C}$ , the image under  $H$  of the tautological cone from  $C$  induced by  $F$  is a limiting cone in  $\mathbf{Set}$ .*

*Proof.* In the lemma, take  $\mathcal{A}$  to be the terminal category  $\mathbb{1}$ , take  $U : \mathbb{1} \rightarrow \mathcal{C}$  to be the functor sending the unique object in  $\mathbb{1}$  to  $C$ , and take  $V = F$ ; then  $(HU, \varepsilon Q \bullet H\theta)$  is a right Kan extension of  $GQ : (\mathcal{C} \downarrow F) \rightarrow \mathbf{Set}$  along the unique functor  $P : (\mathcal{C} \downarrow F) \rightarrow \mathbb{1}$ , but it is clear that a right Kan extension of  $GQ$  along  $P$  amounts to a limit for the diagram  $GQ$  in  $\mathbf{Set}$ . ■

It is convenient at this juncture to introduce a concept borrowed from enriched category theory. The notation below follows [Kelly, 2005, §3.1].

**Definition A.5.10.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. Given functors  $W : \mathcal{J} \rightarrow \mathbf{Set}$  and  $A : \mathcal{J} \rightarrow \mathcal{C}$ , a  $W$ -**weighted limit** of  $A$  is an object  $\{W, A\}^{\mathcal{J}}$  in  $\mathcal{C}$  together with bijections

$$C(\mathcal{C}, \{W, A\}^{\mathcal{J}}) \cong [\mathcal{J}, \mathbf{Set}](W, C(\mathcal{C}, A))$$

that are natural in  $\mathcal{C}$ . We may also write  $\lim_{\leftarrow j:\mathcal{J}}^{Wj} Aj$  instead of  $\{W, A\}^{\mathcal{J}}$ , if we wish to use an explicit variable  $j$ .

Dually, given functors  $W : \mathcal{J}^{\text{op}} \rightarrow \mathbf{Set}$  and  $A : \mathcal{J} \rightarrow \mathcal{C}$ , a  $W$ -**weighted colimit** of  $A$  is an object  $W \star_{\mathcal{J}} A$  in  $\mathcal{C}$  together with bijections

$$C(W \star_{\mathcal{J}} A, \mathcal{C}) \cong [\mathcal{J}^{\text{op}}, \mathbf{Set}](W, C(A, \mathcal{C}))$$

that are natural in  $\mathcal{C}$ . We may also write  $\lim_{\rightarrow j:\mathcal{J}}^{Wj} Aj$  instead of  $W \star_{\mathcal{J}} A$ , if we wish to use an explicit variable  $j$ .

REMARK A.5.11. Clearly, weighted limits and colimits are unique up to unique isomorphism if they exist.

It is also not hard to spell out the above definition in elementary terms; for example, one notes that to give a natural transformation  $W \Rightarrow C(C, A)$ , one must give a morphism  $\lambda_{j,x} : C \rightarrow Aj$  for each object  $j$  in  $\mathcal{J}$  and each element  $x$  of  $Wj$ , and these are required to make various diagrams commute. This is a  $W$ -**weighted cone** from  $C$  to  $A$ , and  $\{W, A\}^{\mathcal{J}}$  is an object equipped with a universal  $W$ -weighted cone to  $A$ . Similarly, one may define the notion of a  $W$ -**weighted cocone** from  $A$  to  $C$ , and then  $W \star_{\mathcal{J}} A$  is an object equipped with a universal  $W$ -weighted cocone from  $A$ . In particular, if  $Wj = 1$  for all  $j$ , then  $W$ -weighted limits and colimits reduce to ordinary limits and colimits.

The above discussion also shows that the concept of a weighted limit or colimit (within a fixed category!) does not depend on  $\mathbf{U}$  in any essential way.

**Lemma A.5.12.** *Let  $\mathcal{J}$  be a  $\mathbf{U}$ -small category. Given functors  $F, G : \mathcal{J} \rightarrow \mathbf{Set}$ , the  $F$ -weighted limit of  $G$  exists in  $\mathbf{Set}$ , and we have bijections*

$$\{F, G\}^{\mathcal{J}} \cong [\mathcal{J}, \mathbf{Set}](F, G)$$

that are natural in  $F$  and  $G$ .

*Proof.* One simply has to check that this works. ◆

**Proposition A.5.13.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $F : C \rightarrow D$  be any functor where  $C$  and  $D$  are locally  $\mathbf{U}$ -small categories.*

- (i) *For each weight  $W : \mathcal{J} \rightarrow \mathbf{Set}$  and each diagram  $A : \mathcal{J} \rightarrow C$ , if the weighted limits  $\{W, A\}^{\mathcal{J}}$  and  $\{W, FA\}^{\mathcal{J}}$  both exist, then there is a canonical comparison morphism*

$$F\{W, A\}^{\mathcal{J}} \rightarrow \{W, FA\}^{\mathcal{J}}$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, C(C, A)) \rightarrow [\mathcal{J}, \mathbf{Set}](W, D(FC, FA))$$

induced by the functor  $F$ .

- (ii) *For any object  $C$  in  $C$ , the functor  $C(C, -) : C \rightarrow \mathbf{Set}$  preserves all weighted limits.*

(iii) The functors  $C(C, -) : C \rightarrow \mathbf{Set}$  jointly reflect weighted limits.

(iv) If  $F$  has a left adjoint, then  $F$  preserves weighted limits.

*Dually:*

(i') For each weight  $W : \mathcal{J}^{\text{op}} \rightarrow \mathbf{Set}$  and each diagram  $A : \mathcal{J} \rightarrow C$ , if the weighted colimits  $W \star_{\mathcal{J}} A$  and  $W \star_{\mathcal{J}} FA$  both exist, then there is a canonical comparison morphism

$$W \star_{\mathcal{J}} FA \rightarrow F(W \star_{\mathcal{J}} A)$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, C(A, C)) \rightarrow [\mathcal{J}, \mathbf{Set}](W, D(FA, FC))$$

induced by the functor  $F$ .

(ii') For any object  $C$  in  $C$ , the functor  $C(-, C) : C^{\text{op}} \rightarrow \mathbf{Set}$  sends any weighted colimit in  $C$  to the corresponding weighted limit in  $\mathbf{Set}$ .

(iii') The functors  $C(-, C) : C \rightarrow \mathbf{Set}^{\text{op}}$  jointly reflect weighted colimits.

(iv') If  $F$  has a right adjoint, then  $F$  preserves weighted colimits.

*Proof.* All straightforward. ◆

**Definition A.5.14.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{D}$  be a locally  $\mathbf{U}$ -small category. Given a functor  $F : C \rightarrow \mathcal{D}$ , the  **$F$ -nerve functor**  $N^F : \mathcal{D} \rightarrow [C^{\text{op}}, \mathbf{Set}]$  is defined by

$$N^F(\mathcal{D})(C) = D(FC, D)$$

i.e.  $N^F = F^*h_{\bullet}$ , where  $h_{\bullet} : \mathcal{D} \rightarrow [C^{\text{op}}, \mathbf{Set}]$  is the usual Yoneda embedding.

**Theorem A.5.15.** Let  $C$ ,  $\mathcal{D}$  and  $\mathcal{E}$  be locally  $\mathbf{U}$ -small categories. Given functors  $F : C \rightarrow \mathcal{D}$  and  $G : C \rightarrow \mathcal{E}$ , the following are equivalent:

(i)  $(H, \alpha)$  is a pointwise right Kan extension of  $G$  along  $F$ .

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- (ii) For each object  $d$  in  $\mathcal{D}$ , the weighted limit  $\{\mathbf{N}^{F^{\text{op}}}(d), G\}^C$  exists in  $\mathcal{E}$ , and there are isomorphisms

$$Hd \cong \{\mathbf{N}^{F^{\text{op}}}(d), G\}^C$$

natural in  $d$ , with  $\alpha_c : HFc \rightarrow Gc$  corresponding to the element  $\text{id}_{F_c}$  of  $\mathbf{N}^{F^{\text{op}}}(Fc)(c) = \mathcal{D}(Fc, Fc)$ .

- (iii) (Assuming  $\mathcal{C}$  is  $\mathbf{U}$ -small.) For each object  $d$  in  $\mathcal{D}$ , if  $P^d : (d \downarrow F) \rightarrow \mathcal{C}$  is the projection sending  $(c, f)$  in the comma category  $(d \downarrow F)$  to  $c$ , and  $\varphi : \Delta d \Rightarrow FP^d$  is the tautological cone in  $\mathcal{D}$ , then the cone  $\alpha P^d \bullet H\varphi : \Delta Hd \Rightarrow GP^d$  is limiting; and for each  $g : d \rightarrow d'$  in  $\mathcal{D}$ , the morphism  $Hg : Hd \rightarrow Hd'$  is the one induced by the functor  $(d' \downarrow F) \rightarrow (d \downarrow F)$  sending  $(c', f')$  to  $(c', f' \circ g)$ . In particular,  $\alpha_c : HFc \rightarrow Gc$  must be (equal to) the component of the limiting cone  $\Delta Fc \Rightarrow GP^d$  at the object  $(c, \text{id}_{F_c})$  of  $(Fc \downarrow F)$ .

In particular, if  $\mathcal{C}$  is a  $\mathbf{U}$ -small category and  $\mathcal{E}$  is  $\mathbf{U}$ -complete, then the right Kan extension of  $G$  along  $F$  exists and is pointwise.

Dually, the following are equivalent:

- (i')  $(H, \alpha)$  is a pointwise left Kan extension of  $G$  along  $F$ .
- (ii') For each object  $d$  in  $\mathcal{D}$ , the weighted colimit  $\mathbf{N}^F(d) \star_c G$  exists in  $\mathcal{E}$ , and there are isomorphisms

$$Hd \cong \mathbf{N}^F(d) \star_c G$$

natural in  $d$ , with  $\alpha_c : Gc \rightarrow HFc$  corresponding to the element  $\text{id}_{F_c}$  of  $\mathbf{N}^F(Fc)(c) = \mathcal{D}(Fc, Fc)$ .

- (iii') (Assuming  $\mathcal{C}$  is  $\mathbf{U}$ -small.) For each object  $d$  in  $\mathcal{D}$ , if  $P_d : (F \downarrow d) \rightarrow \mathcal{C}$  is the projection sending  $(c, f)$  in the comma category  $(F \downarrow d)$  to  $c$ , and  $\varphi : FP_d \Rightarrow \Delta d$  is the tautological cocone in  $\mathcal{D}$ , then the cocone  $H\varphi \bullet \alpha P_d : GP_d \Rightarrow \Delta Hd$  is colimiting; and for each  $g : d \rightarrow d'$  in  $\mathcal{D}$ , the morphism  $Hg : Hd \rightarrow Hd'$  is the one induced by the functor  $(F \downarrow d) \rightarrow (F \downarrow d')$  sending  $(c, f)$  to  $(c, g \circ f)$ . In particular,  $\alpha_c : Gc \rightarrow HFc$  must be (equal to) the component of the colimiting cocone  $GP_d \Rightarrow \Delta Fc$  at the object  $(c, \text{id}_{F_c})$  of  $(F \downarrow Fc)$ .



In particular, if  $\mathcal{C}$  is a  $\mathbf{U}$ -small category and  $\mathcal{E}$  is  $\mathbf{U}$ -cocomplete, then the left Kan extension of  $G$  along  $F$  exists and is pointwise.

*Proof.* (i)  $\Leftrightarrow$  (ii). This is just a matter of unwinding the definitions.

(i)  $\Leftrightarrow$  (iii). [Corollary A.5.9](#) implies that the construction in (iii) does indeed define a right Kan extension in the special case  $\mathcal{E} = \mathbf{Set}$ , so we deduce that statements (i) and (iii) are equivalent by applying the Yoneda lemma; see also [CWM, Ch. X, §§3 and 5].  $\square$

**REMARK A.5.16.** It is possible to extract an elementary characterisation of pointwise Kan extensions from the results above, thereby showing that the property of being pointwise does not depend on the choice of universe  $\mathbf{U}$ .

**Corollary A.5.17.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. If  $\mathcal{C}$  is  $\mathbf{U}$ -small and  $\mathcal{D}$  is locally  $\mathbf{U}$ -small, then the functor  $F^* : [\mathcal{D}, \mathbf{Set}] \rightarrow [\mathcal{C}, \mathbf{Set}]$  has both a left adjoint  $\text{Lan}_F$  and a right adjoint  $\text{Ran}_F$ .*  $\blacksquare$

**Corollary A.5.18.** *If  $(H, \alpha)$  is a pointwise right Kan extension of  $G : \mathcal{C} \rightarrow \mathcal{E}$  along  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and  $R : \mathcal{E} \rightarrow \mathcal{F}$  is a functor, then  $(RH, R\alpha)$  is a pointwise right Kan extension of  $RG$  along  $F$ , provided either:*

- (i)  $R$  preserves all weighted limits, or
- (ii)  $R$  preserves limits for  $\mathbf{U}$ -small diagrams and  $\mathcal{C}$  is  $\mathbf{U}$ -small.

*If  $(H, \alpha)$  is a pointwise left Kan extension of  $G : \mathcal{C} \rightarrow \mathcal{E}$  along  $F : \mathcal{C} \rightarrow \mathcal{D}$ , and  $L : \mathcal{E} \rightarrow \mathcal{F}$  is a functor, then  $(LH, L\alpha)$  is a pointwise left Kan extension of  $LG$  along  $F$ , provided either:*

- (i')  $L$  preserves all weighted colimits, or
- (ii')  $L$  preserves colimits for  $\mathbf{U}$ -small diagrams and  $\mathcal{C}$  is  $\mathbf{U}$ -small.  $\blacksquare$

**Corollary A.5.19.** *If  $(H, \alpha)$  is a pointwise right (resp. left) Kan extension of  $G : \mathcal{C} \rightarrow \mathcal{E}$  along a fully faithful functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , then  $\alpha : HF \Rightarrow G$  (resp.  $\alpha : G \Rightarrow HF$ ) is a natural isomorphism.*

*Proof.* If  $F$  is fully faithful, then the comma category  $(Fc \downarrow F)$  (resp.  $(F \downarrow Fc)$ ) has an initial (resp. terminal) object, namely  $(c, \text{id}_{Fc})$ , so the component  $\alpha_c : HFc \rightarrow Gc$  (resp.  $\alpha_c : Gc \rightarrow HFc$ ) must be an isomorphism.  $\blacksquare$

**Theorem A.5.20.** *Let  $F : \mathcal{A} \rightarrow \mathcal{C}$  and  $G : \mathcal{A} \rightarrow \mathcal{D}$  be functors, and let  $i : \mathcal{C} \rightarrow \mathcal{C}^+$  and  $j : \mathcal{D} \rightarrow \mathcal{D}^+$  be fully faithful functors. Consider the following (not necessarily commutative) diagram:*

$$\begin{array}{ccccc}
 \mathcal{A} & \xrightarrow{G} & \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \\
 F \downarrow & \nearrow H & & \nearrow H^+ & \\
 \mathcal{C} & & & & \\
 i \downarrow & & & & \\
 \mathcal{C}^+ & & & & 
 \end{array}$$

- (i) *If  $H^+$  is a pointwise right Kan extension of  $jG$  along  $iF$ , and  $H^+i \cong jH$ , then  $H$  is a pointwise right Kan extension of  $G$  along  $F$ .*
- (ii) *Suppose  $jH$  is a pointwise right Kan extension of  $jG$  along  $F$ . If  $H^+$  is a pointwise right Kan extension of  $jH$  along  $i$ , then the counit  $H^+i \Rightarrow jH$  is a natural isomorphism, and  $H^+$  is also a pointwise right Kan extension of  $jG$  along  $iF$ ; conversely, if  $H^+$  is a pointwise right Kan extension of  $jG$  along  $iF$ , then it is also a pointwise right Kan extension of  $jH$  along  $i$ .*
- (iii) *If  $\mathbf{U}$  is a pre-universe such that  $\mathcal{A}$  is  $\mathbf{U}$ -small and  $j$  preserves limits for all  $\mathbf{U}$ -small diagrams, and  $H$  is a pointwise right Kan extension of  $G$  along  $F$ , then a pointwise right Kan extension of  $jG$  along  $iF$  can be computed as a pointwise right Kan extension of  $jH$  along  $i$  (if either one exists).*

*Dually:*

- (i') *If  $H^+$  is a pointwise left Kan extension of  $jG$  along  $iF$ , and  $H^+i \cong jH$ , then  $H$  is a pointwise left Kan extension of  $G$  along  $F$ .*
- (ii') *Suppose  $jH$  is a pointwise left Kan extension of  $jG$  along  $F$ . If  $H^+$  is a pointwise left Kan extension of  $jH$  along  $i$ , then the unit  $jH \Rightarrow H^+i$  is a natural isomorphism, and  $H^+$  is also a pointwise left Kan extension of  $jG$  along  $iF$ ; conversely, if  $H^+$  is a pointwise left Kan extension of  $jG$  along  $iF$ , then it is also a pointwise left Kan extension of  $jH$  along  $i$ .*
- (iii') *If  $\mathbf{U}$  is a pre-universe such that  $\mathcal{A}$  is  $\mathbf{U}$ -small and  $j$  preserves colimits for all  $\mathbf{U}$ -small diagrams, and  $H$  is a pointwise left Kan extension of  $G$  along  $F$ , then a pointwise left Kan extension of  $jG$  along  $iF$  can be computed as a pointwise left Kan extension of  $jH$  along  $i$  (if either one exists).*

*Proof.* (i). [Theorem A.5.15](#) gives an explicit description of  $H^+ : C^+ \rightarrow D^+$  as a weighted limit:

$$H^+(C') \cong \{C^+(C', iF), jG\}^A$$

Since  $i$  is fully faithful, the weights  $C(C, F)$  and  $C^+(iC, iF)$  are naturally isomorphic, hence,

$$jH(C) \cong H^+(iC) \cong \{C^+(iC, iF), jG\}^A \cong \{C(C, F), jG\}^A$$

but, since  $j$  is fully faithful,  $j$  reflects *all* weighted limits, therefore  $H$  must be a pointwise right Kan extension of  $G$  along  $F$ .

(ii). Let  $\mathbf{U}^+$  be a pre-universe such that  $\mathcal{A}$  and  $\mathcal{C}$  are  $\mathbf{U}^+$ -small categories and  $\mathcal{D}, \mathcal{C}^+, \mathcal{D}^+$  are locally  $\mathbf{U}^+$ -small categories, and let  $\mathbf{Set}^+$  be the category of  $\mathbf{U}^+$ -sets. Using the interchange law ([theorem A.6.16](#)) and propositions [A.6.10](#) and [A.6.17](#), we obtain the following natural bijections:

$$\begin{aligned} D^+(D', H^+(C')) &\cong D^+(D', \{C^+(C', i), jH\}^C) \\ &\cong \int_{C:C} \mathbf{Set}^+(C^+(C', iC), D^+(D', jHC)) \\ &\cong \int_{C:C} \mathbf{Set}^+(C^+(C', iC), D^+(D', \{C(C, F), jG\}^A)) \\ &\cong \int_{C:C} \int_{A:A} \mathbf{Set}^+(C^+(C', iC), \mathbf{Set}^+(C(C, FA), D^+(D', jGA))) \\ &\cong \int_{C:C} \int_{A:A} \mathbf{Set}^+(C(C, FA), \mathbf{Set}^+(C^+(C', iC), D^+(D', jGA))) \\ &\cong \int_{A:A} \int_{C:C} \mathbf{Set}^+(C(C, FA), \mathbf{Set}^+(C^+(C', iC), D^+(D', jGA))) \\ &\cong \int_{A:A} \mathbf{Set}^+(C^+(C', iFA), D^+(D', jGA)) \\ &\cong D^+(D', \{C^+(C', iF), jG\}^A) \end{aligned}$$

Thus,  $H^+$  is a pointwise right Kan extension of  $jG$  along  $iF$  if and only if  $H^+$  is a pointwise right Kan extension of  $jH$  along  $i$ . The fact that the counit  $H^+i \Rightarrow jH$  is a natural isomorphism is just [corollary A.5.19](#).

(iii). Apply [corollary A.5.18](#) to claim (ii). ■

**Proposition A.5.21.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be any two categories, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  be any two functors. The following are equivalent:*

- (i)  $F \dashv G$ , with unit  $\eta : \text{id}_C \Rightarrow GF$  and counit  $\varepsilon : FG \Rightarrow \text{id}_D$ .
- (ii)  $(F, \varepsilon)$  is an absolute right Kan extension of  $\text{id}_D$  along  $G$ .
- (iii)  $(F, \varepsilon)$  is a right Kan extension of  $\text{id}_D$  along  $G$  that is preserved by  $F$ .
- (iv)  $(G, \eta)$  is an absolute left Kan extension of  $\text{id}_C$  along  $F$ .
- (v)  $(G, \eta)$  is a left Kan extension of  $\text{id}_C$  along  $F$  that is preserved by  $G$ .

*Proof.* See [CWM, Ch. X, §7]. □

**Proposition A.5.22.**

- Left adjoints preserve all left Kan extensions.
- Right adjoints preserve all right Kan extensions.

*Proof.* See Theorem I in [CWM, Ch. X, §5]. □

**Definition A.5.23.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. A **dense functor** is a functor  $F : \mathcal{B} \rightarrow \mathcal{C}$  such that the  $F$ -nerve functor  $N^F : \mathcal{C} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$  is fully faithful. A **dense subcategory** of  $\mathcal{C}$  is a subcategory  $\mathcal{B}$  such that the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is a dense functor.

Dually, a **codense functor** is a functor  $F : \mathcal{B} \rightarrow \mathcal{C}$  such that the opposite functor  $F^{\text{op}} : \mathcal{B}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$  is dense, and a **codense subcategory** of  $\mathcal{C}$  is a subcategory  $\mathcal{B}$  such that the inclusion  $\mathcal{B} \hookrightarrow \mathcal{C}$  is a codense functor.

**Example A.5.24.** The Yoneda lemma implies  $\text{id}_C : \mathcal{C} \rightarrow \mathcal{C}$  is a dense and codense functor.

One may extract an elementary definition for ‘(co)dense functor’ from the following proposition.

**Proposition A.5.25.** *With notation as in definition A.5.23, the following are equivalent:*

- (i)  $F : \mathcal{B} \rightarrow \mathcal{C}$  is a dense functor.
- (ii) For each object  $C$  in  $\mathcal{C}$ , the maps

$$C(\mathcal{C}, C') \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}](N^F(C), C(F, C'))$$

*induced by  $N^F : \mathcal{C} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$  are natural bijections, exhibiting  $C$  as a weighted colimit  $N^F(C) \star_{\mathcal{B}} F$  in  $\mathcal{C}$ .*

(iii) For each object  $C$  in  $\mathcal{C}$ , the tautological cocone to  $C$  induced by  $F$  is a colimiting cocone.

(iv)  $(\text{id}_C, \text{id}_F)$  is a pointwise left Kan extension of  $F$  along  $F$ .

Dually, the following are equivalent:

(i')  $F : \mathcal{B} \rightarrow \mathcal{C}$  is a codense functor.

(ii') For each object  $C$  in  $\mathcal{C}$ , the maps

$$\mathcal{C}(C', C) \rightarrow [\mathcal{B}, \mathbf{Set}](\mathbf{N}^{F^{\text{op}}}(C), \mathcal{C}(C', F))$$

induced by  $\mathbf{N}^{F^{\text{op}}} : \mathcal{C}^{\text{op}} \rightarrow [\mathcal{B}, \mathbf{Set}]$  are natural bijections, exhibiting  $\mathcal{C}$  as a weighted limit  $\{\mathbf{N}^{F^{\text{op}}}(C), F\}^B$  in  $\mathcal{C}$ .

(iii') For each object  $C$  in  $\mathcal{C}$ , the tautological cone from  $C$  induced by  $F$  is a limiting cone.

(iv')  $(\text{id}_C, \text{id}_F)$  is a pointwise right Kan extension of  $F$  along  $F$ .

*Proof.* (i)  $\Leftrightarrow$  (ii). The indicated maps are bijections for all  $C$  and  $C'$  if and only if  $\mathbf{N}^F$  is fully faithful, by definition.

(ii)  $\Leftrightarrow$  (iii)  $\Leftrightarrow$  (iv). This is an application of [theorem A.5.15](#). ■

**Definition A.5.26.** Let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a functor. A **densely-defined partial left adjoint** for  $G$  is a triple  $(F, i, \eta)$ , where  $F : \mathcal{B} \rightarrow \mathcal{D}$  is a functor,  $i : \mathcal{B} \rightarrow \mathcal{C}$  is a dense functor, and  $\eta : i \Rightarrow GF$  is a natural transformation such that the maps

$$\begin{aligned} \mathcal{D}(FB, D) &\rightarrow \mathcal{C}(iB, GD) \\ g &\mapsto Gg \circ \eta_B \end{aligned}$$

are bijections that are natural in  $B$  and  $D$ .

Dually, given a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ , a **codensely-defined partial right adjoint** for  $F$  is a triple  $(G, j, \varepsilon)$ , where  $G : \mathcal{B} \rightarrow \mathcal{C}$  is a functor,  $j : \mathcal{B} \rightarrow \mathcal{C}$  is a codense functor, and  $\varepsilon : FG \Rightarrow j$  is a natural transformation such that the maps

$$\begin{aligned} \mathcal{C}(C, GB) &\rightarrow \mathcal{D}(FC, jB) \\ f &\mapsto \varepsilon_B \circ Ff \end{aligned}$$

are bijections that are natural in  $B$  and  $C$ .

**Example A.5.27.** The Yoneda embedding  $h_{\bullet} : \mathcal{B} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$  has a densely-defined partial left adjoint, namely  $(\text{id}_{\mathcal{B}}, h_{\bullet}, \text{id}_{h_{\bullet}})$ .

**REMARK A.5.28.**  $(F, \text{id}_C, \eta)$  is a densely-defined partial left adjoint for  $G$  if and only if  $F$  is a left adjoint for  $G$  in the usual sense, with  $\eta$  being the adjunction unit.

**Proposition A.5.29.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  and  $\mathcal{D}$  be locally  $\mathbf{U}$ -small categories. Given functors  $G : \mathcal{D} \rightarrow \mathcal{C}$ ,  $F : \mathcal{B} \rightarrow \mathcal{D}$ , and  $i : \mathcal{B} \rightarrow \mathcal{C}$ , the following are equivalent:*

- (i)  $(F, i, \eta)$  is a densely-defined partial left adjoint for  $G$ .
- (ii) The functor  $i : \mathcal{B} \rightarrow \mathcal{C}$  is dense, and there exists a diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h_{\bullet}} & [\mathcal{D}^{\text{op}}, \mathbf{Set}] \\ G \downarrow & \swarrow_{\alpha} & \downarrow_{(F^{\text{op}})^*} \\ \mathcal{C} & \xrightarrow{N^i} & [\mathcal{B}^{\text{op}}, \mathbf{Set}] \end{array}$$

where  $\alpha$  factors through  $\eta^* : N^{GF} \Rightarrow N^i$  and is a natural isomorphism.

- (iii) The functor  $i : \mathcal{B} \rightarrow \mathcal{C}$  is dense, and the diagram

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h_{\bullet}} & [\mathcal{D}^{\text{op}}, \mathbf{Set}] \\ G \downarrow & & \downarrow_{(F^{\text{op}})^*} \\ \mathcal{C} & \xrightarrow{N^i} & [\mathcal{B}^{\text{op}}, \mathbf{Set}] \end{array}$$

commutes up to natural isomorphism.

Dually, given functors  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,  $G : \mathcal{B} \rightarrow \mathcal{C}$ , and  $j : \mathcal{B} \rightarrow \mathcal{D}$ , the following are equivalent:

- (i')  $(G, j, \varepsilon)$  is a codensely-defined partial right adjoint for  $F$ .
- (ii') The functor  $j : \mathcal{B} \rightarrow \mathcal{D}$  is codense, and there exists a diagram

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{h^{\bullet}} & [\mathcal{C}, \mathbf{Set}] \\ F^{\text{op}} \downarrow & \swarrow_{\beta} & \downarrow_{G^*} \\ \mathcal{D}^{\text{op}} & \xrightarrow{N^{j^{\text{op}}}} & [\mathcal{B}, \mathbf{Set}] \end{array}$$

where  $\beta$  factors through  $(\epsilon^{\text{op}})^* : \mathbf{N}^{F^{\text{op}}G^{\text{op}}} \Rightarrow \mathbf{N}^{j^{\text{op}}}$  and is a natural isomorphism.

(iii') The functor  $j : \mathcal{B} \rightarrow \mathcal{D}$  is codense, and the diagram

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & \xrightarrow{h^*} & [\mathcal{C}, \mathbf{Set}] \\ F^{\text{op}} \downarrow & & \downarrow G^* \\ \mathcal{D}^{\text{op}} & \xrightarrow{N^{j^{\text{op}}}} & [\mathcal{B}, \mathbf{Set}] \end{array}$$

commutes up to natural isomorphism.

*Proof.* (i)  $\Rightarrow$  (ii). This immediately follows from the definition.

(ii)  $\Rightarrow$  (iii). Obvious.

(iii)  $\Rightarrow$  (i). The displayed diagram commutes up to natural isomorphism precisely when there are bijections

$$\alpha_{B,D} : \mathcal{D}(FB, D) \rightarrow \mathcal{C}(iB, GD)$$

that are natural in both  $B$  and  $D$ . Taking  $D = FB$ , let  $\eta_B : iB \rightarrow GFB$  be the morphism corresponding to  $\text{id}_{FB} : FB \rightarrow FB$ . Applying the Yoneda lemma, we see that the natural bijection  $\alpha_{B,D}$  must be the map  $g \mapsto Gg \circ \eta_B$ . ■

**Corollary A.5.30.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be any two categories. If a functor  $G : \mathcal{D} \rightarrow \mathcal{C}$  has a densely-defined partial left adjoint, then  $G$  preserves:*

- (i) *limits for all diagrams in  $\mathcal{D}$ ,*
- (ii) *weighted limits, and*
- (iii) *pointwise right Kan extensions.*

*Dually, if a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  has a codensely-defined partial right adjoint, then  $F$  preserves:*

- (i') *colimits for all diagrams in  $\mathcal{C}$ ,*
- (ii') *weighted colimits, and*
- (iii') *pointwise left Kan extensions.*

*Proof.* Choose a universe  $\mathbf{U}$  such that the domain of  $i : \mathcal{B} \rightarrow \mathcal{C}$  is  $\mathbf{U}$ -small and both  $\mathcal{C}$  and  $\mathcal{D}$  are locally  $\mathbf{U}$ -small, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h_\bullet} & [\mathcal{D}^{\text{op}}, \mathbf{Set}] \\ G \downarrow & & \downarrow (F^{\text{op}})^* \\ \mathcal{C} & \xrightarrow{N^i} & [\mathcal{B}^{\text{op}}, \mathbf{Set}] \end{array}$$

Since  $i$  is dense, the  $i$ -nerve functor  $N^i : \mathcal{C} \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$  is fully faithful. [Corollary A.5.17](#) implies  $(F^{\text{op}})^* : [\mathcal{D}^{\text{op}}, \mathbf{Set}] \rightarrow [\mathcal{B}^{\text{op}}, \mathbf{Set}]$  is a right adjoint, and the Yoneda embedding  $h_\bullet : \mathcal{D} \rightarrow [\mathcal{D}^{\text{op}}, \mathbf{Set}]$  preserves all limits and weighted limits (see [proposition A.5.13](#)), so we use the fact that  $N^i$  reflects limits and weighted limits to conclude that  $G$  preserves them. We then apply [corollary A.5.18](#).  $\blacksquare$

**Definition A.5.31.** A **cofinal functor** (resp. **coinitial functor**) is a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  such that, for each object  $D$  in  $\mathcal{D}$ , the comma category  $(D \downarrow F)$  (resp.  $(F \downarrow D)$ ) is connected.

**Theorem A.5.32.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between  $\mathbf{U}$ -small categories. The following are equivalent:

- (i)  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a coinitial functor.
- (ii) The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\Delta} & [D, \mathbf{Set}] \\ \text{id} \downarrow & & \downarrow F^* \\ \mathbf{Set} & \xrightarrow{\Delta} & [C, \mathbf{Set}] \end{array}$$

- (iii) The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{id}} & \mathbf{Set} \\ \Delta \downarrow & & \downarrow \Delta \\ [D, \mathbf{Set}] & \xrightarrow{F^*} & [C, \mathbf{Set}] \end{array}$$



- (iv) For all locally small categories  $\mathcal{E}$  and all diagrams  $G : \mathcal{D} \rightarrow \mathcal{E}$ ,  $\varprojlim_{\mathcal{C}} GF$  exists if and only if  $\varprojlim_{\mathcal{D}} G$  exists, in which case the canonical comparison morphism  $\varprojlim_{\mathcal{D}} G \rightarrow \varprojlim_{\mathcal{C}} GF$  is an isomorphism.

Dually, the following are equivalent:

- (i')  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a cofinal functor.  
(ii') The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\Delta} & [D, \mathbf{Set}] \\ \text{id} \downarrow & & \downarrow F^* \\ \mathbf{Set} & \xrightarrow{\Delta} & [C, \mathbf{Set}] \end{array}$$

- (iii') The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\text{id}} & \mathbf{Set} \\ \Delta \downarrow & & \downarrow \Delta \\ [D, \mathbf{Set}] & \xrightarrow{F^*} & [C, \mathbf{Set}] \end{array}$$

- (iv') For all locally small categories  $\mathcal{E}$  and all diagrams  $G : \mathcal{D} \rightarrow \mathcal{E}$ ,  $\varinjlim_{\mathcal{C}} GF$  exists if and only if  $\varinjlim_{\mathcal{D}} G$  exists, in which case the canonical comparison morphism  $\varinjlim_{\mathcal{C}} GF \rightarrow \varinjlim_{\mathcal{D}} G$  is an isomorphism.

*Proof.* (i)  $\Leftrightarrow$  (ii). Using the colimit formula for  $\text{Lan}_F : [C, \mathbf{Set}] \rightarrow [D, \mathbf{Set}]$  indicated in [theorem A.5.15](#), it is clear that the comma categories  $(F \downarrow D)$  is connected if and only if the left Beck–Chevalley transformation  $\Delta \Rightarrow \text{Ran}_F(\Delta -)$  is a natural isomorphism.

(ii)  $\Leftrightarrow$  (iii). Apply [proposition A.1.12](#).

(iii)  $\Rightarrow$  (iv). We have the following natural bijections:

$$\begin{aligned} [C, \mathcal{E}](\Delta E, GF) &\cong \varinjlim_{\mathcal{C}} \mathcal{E}(E, GF) \\ &\cong \varinjlim_{\mathcal{D}} \mathcal{E}(E, G) \end{aligned}$$

$$\cong [D, \mathcal{E}](\Delta E, G)$$

Thus, there is a natural bijection between cones from  $E$  to  $GF$  and cones from  $E$  to  $G$ ; this implies that limits for  $GF$  exist in  $\mathcal{E}$  if and only if limits for  $G$  exist in  $\mathcal{E}$  and that they are canonically isomorphic.

(iv)  $\Rightarrow$  (iii). Obvious. ■

## A.6 Ends and coends

**Prerequisites.** §§0.1, A.5

In this section we use the explicit universe convention.

**Definition A.6.1.** Let  $F, G : C^{\text{op}} \times C \rightarrow D$  be functors. A **dinatural transformation**  $\alpha : F \overset{\diamond}{\rightarrow} G$  is a family  $(\alpha_C : F(C, C) \rightarrow G(C, C) \mid C \in \text{ob } C)$  such that the diagram

$$\begin{array}{ccccc}
 & & F(C, C) & \xrightarrow{\alpha_C} & G(C, C) & & \\
 & \nearrow^{F(\text{id}_C, f)} & & & & \searrow^{G(f, \text{id}_C)} & \\
 F(C, C') & & & & & & G(C', C) \\
 & \searrow_{F(f, \text{id}_{C'})} & & & & & \nearrow_{G(\text{id}_{C'}, f)} \\
 & & F(C', C') & \xrightarrow{\alpha_{C'}} & G(C', C') & & 
 \end{array}$$

commutes for all morphisms  $f : C' \rightarrow C$  in  $C$ .

**Example A.6.2.** Let  $\mathbf{U}$  be a pre-universe, let  $C$  be a locally  $\mathbf{U}$ -small category, and let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets. Consider the functor  $\text{Hom}_C : C^{\text{op}} \times C \rightarrow \mathbf{Set}$  that sends a pair of objects in  $C$  to their hom-set. For each natural number  $n$ , we have an dinatural transformation  $\text{Hom}_C \overset{\diamond}{\rightarrow} \text{Hom}_C$  defined by  $e \mapsto e^n$ , where  $e^n$  denotes the  $n$ -fold iterate of the endomorphism  $e$ .

**Definition A.6.3.** A **wedge** from an object  $D$  in  $\mathcal{D}$  to a functor  $G : C^{\text{op}} \times C \rightarrow \mathcal{D}$  is a dinatural transformation  $\Delta D \overset{\diamond}{\rightarrow} G$ , where  $\Delta D : C^{\text{op}} \times C \rightarrow \mathcal{D}$  is the constant functor with value  $D$ ; dually, a **cowedge** from a functor  $F : C^{\text{op}} \times C \rightarrow \mathcal{D}$  to an object  $D$  in  $\mathcal{D}$  is a dinatural transformation  $F \overset{\diamond}{\rightarrow} \Delta D$ .

**Definition A.6.4.** An **end** for a functor  $G : C^{\text{op}} \times C \rightarrow \mathcal{D}$  is an object  $E$  and a wedge  $\lambda : \Delta E \overset{\diamond}{\rightarrow} G$  with the following universal property:

- For each wedge  $\varphi : \Delta D \xrightarrow{\diamond} G$ , there is a unique morphism  $f : D \rightarrow E$  in  $\mathcal{D}$  such that  $\varphi_C = \lambda_C \circ f$  for all objects  $C$  in  $\mathcal{C}$ .

We define the following formula to mean that  $E$  is an end for  $G$ :

$$E = \int_{C:C} G(C, C)$$

Dually, a **coend** for a functor  $F : C^{\text{op}} \times C \rightarrow D$  is an object  $E$  and a cowedge  $\lambda : F \xrightarrow{\diamond} \Delta E$  with the following universal property:

- For each cowedge  $\varphi : F \xrightarrow{\diamond} \Delta D$ , there is a unique morphism  $f : E \rightarrow D$  in  $\mathcal{D}$  such that  $\varphi_C = f \circ \lambda_C$  for all objects  $C$  in  $\mathcal{C}$ .

We define the following formula to mean that  $E$  is a coend for  $F$ :

$$E = \int^{C:C} F(C, C)$$

REMARK A.6.5. Let  $\mathbf{U}$  be a pre-universe, let  $\mathbb{D}$  be a  $\mathbf{U}$ -small category, and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. Then, for all functors  $F, G : \mathbb{D} \rightarrow \mathcal{C}$ , we have a bijection

$$[\mathbb{D}, \mathcal{C}](F, G) \cong \int_{d:\mathbb{D}} \mathcal{C}(Fd, Gd)$$

and this is natural in both  $F$  and  $G$ . (The size restriction ensures that the LHS is a  $\mathbf{U}$ -set.) See also [lemma A.5.12](#).

**Definition A.6.6.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Cat}$  be the category of  $\mathbf{U}$ -small categories, and let  $\mathbb{S}$  be the (non-full) subcategory of  $\mathbf{Cat}$  consisting of the two embeddings  $1 \rightarrow 2$ . Given a  $\mathbf{U}$ -small category  $\mathbb{D}$ , the **Mac Lane subdivision category**  $\mathbb{D}^{\mathbb{S}}$  is the comma category  $(\mathbb{S} \downarrow \mathbb{D})$ , where we regard  $\mathbb{D}$  as an object in  $\mathbf{Cat}$ .

REMARK A.6.7. More explicitly,  $\mathbb{D}^{\mathbb{S}}$  is the following category:

- The objects are either objects in  $\mathbb{D}$  or morphisms in  $\mathbb{D}$ .
- The non-identity morphisms are of the form  $X \rightarrow f$  or  $Y \rightarrow f$ , where  $f : X \rightarrow Y$  is a morphism in  $\mathbb{D}$ .
- The only composable pairs of morphisms are trivial.

It is clear from this description that  $\mathbb{D}^\S$  is (isomorphic to) a poset regarded as a category. Note also that, for any regular cardinal  $\kappa$ , the category  $\mathbb{D}$  is  $\kappa$ -small if and only if  $\mathbb{D}^\S$  is  $\kappa$ -small.

**Proposition A.6.8.** *Let  $\mathbb{D}$  be a category and let  $\mathbb{D}^\S$  be the Mac Lane subdivision category. Then there is a natural coinital functor  $\pi_R : \mathbb{D}^\S \rightarrow \mathbb{D}$ .*

*Proof.* We define the functor  $\pi_R : \mathbb{D}^\S \rightarrow \mathbb{D}$  as follows: given an object  $X$  in  $\mathbb{D}$ , we set  $\pi_R X = X$ , and given a morphism  $f : X \rightarrow Y$  in  $\mathbb{D}$ , we set  $\pi_R f = Y$ ,  $\pi_R(X \rightarrow f) = f$ , and  $\pi_R(Y \rightarrow f) = \text{id}_Y$ . This functor is clearly natural in  $\mathbb{D}$ . It remains to be shown that  $\pi_R : \mathbb{D}^\S \rightarrow \mathbb{D}$  is a cofinal functor. Let  $Y$  be an object in  $\mathbb{D}$  and consider the comma category  $(\pi_R \downarrow Y)$ . It is not hard to see that  $(\pi_R \downarrow Y)$  is isomorphic to the Mac Lane subdivision category  $(\mathbb{D}_{/Y})^\S$ ; but  $\mathbb{D}_{/Y}$  has a terminal object (so is a connected category *a fortiori*), therefore  $(\mathbb{D}_{/Y})^\S$  must be a connected category, as required.  $\blacksquare$

**Proposition A.6.9.** *Let  $\mathbf{U}$  be a pre-universe and let  $\mathbb{D}$  be a  $\mathbf{U}$ -small category. If  $\mathcal{C}$  is a  $\mathbf{U}$ -complete category, then  $\mathcal{C}$  has ends for all functors  $A : \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathcal{C}$ . Dually, if  $\mathcal{C}$  is a  $\mathbf{U}$ -cocomplete category, then  $\mathcal{C}$  has coends for all functors  $A : \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathcal{C}$ .*

*Proof.* It is clear from the definition that an end is a special kind of limit, and a coend is a special kind of colimit. To make this precise, one can use the Mac Lane subdivision category  $\mathcal{C}^\S$ : see [CWM, Ch. IX, §5].  $\square$

**Proposition A.6.10.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be any functor where  $\mathcal{C}$  and  $\mathcal{D}$  are locally  $\mathbf{U}$ -small categories.*

- (i) *For any functor  $A : \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ , if the ends  $\int_{\mathcal{J}} A$  and  $\int_{\mathcal{J}} FA$  both exist, with  $\lambda$  being the universal wedge in  $\mathcal{C}$ , then there is a canonical comparison morphism*

$$F \int_{\mathcal{J}} A \rightarrow \int_{\mathcal{J}} FA$$

*induced by the wedge  $F\lambda$ .*

- (ii) *For any object  $C$  in  $\mathcal{C}$ , the functor  $C(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$  preserves all ends.*  
 (iii) *The functors  $C(C, -)$  jointly reflect ends.*  
 (iv) *If  $F$  has a left adjoint, then  $F$  preserves ends.*

Dually:

- (i') For any functor  $A : \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ , if the coends  $\int^{\mathcal{J}} A$  and  $\int^{\mathcal{J}} FA$  both exist, with  $\lambda$  being the universal cowedge in  $\mathcal{C}$ , then there is a canonical comparison morphism

$$\int^{\mathcal{J}} FA \rightarrow F \int^{\mathcal{J}} A$$

induced by the cowedge  $F\lambda$ .

- (ii') For any object  $C$  in  $\mathcal{C}$ , the functor  $C(-, C) : \mathcal{C} \rightarrow \mathbf{Set}$  sends any coend in  $\mathcal{C}$  to the corresponding end in  $\mathbf{Set}$ .
- (iii') The functors  $C(-, C) : \mathcal{C} \rightarrow \mathbf{Set}^{\text{op}}$  jointly reflect coends.
- (iv') If  $F$  has a right adjoint, then  $F$  preserves coends.

*Proof.* All straightforward. ◆

**Definition A.6.II.** Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathbb{1}$  be the trivial category with  $*$  as its only object. A **tensoring  $\mathbf{U}$ -category** is a locally  $\mathbf{U}$ -small category  $\mathcal{C}$  such that, for all weights  $W : \mathbb{1} \rightarrow \mathbf{Set}$  and all diagrams  $A : \mathbb{1} \rightarrow \mathbf{Set}$ , a  $W$ -weighted colimit for  $A$  exists in  $\mathcal{C}$ ; if  $\mathcal{C}$  is a tensoring  $\mathbf{U}$ -category, then we write  $X \odot C$  for the weighted colimit  $W \star_{\mathbb{1}} A$ , where  $X = W(*)$  and  $C = A(*)$ .

Dually, a **cotensoring  $\mathbf{U}$ -category** is a locally  $\mathbf{U}$ -small category  $\mathcal{C}$  such that, for all weights  $W : \mathbb{1} \rightarrow \mathbf{Set}$  and all diagrams  $A : \mathbb{1} \rightarrow \mathbf{Set}$ , a  $W$ -weighted limit for  $A$  exists in  $\mathcal{C}$ ; if  $\mathcal{C}$  is a cotensoring  $\mathbf{U}$ -category, then we write  $X \pitchfork C$  for the weighted limit  $\{W, A\}^{\mathbb{1}}$ , where  $X = W(*)$  and  $C = A(*)$ .

**Proposition A.6.I2** (Tensor–hom–cotensor adjunction). *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category.*

- (i) *If  $\mathcal{C}$  is a tensoring  $\mathbf{U}$ -category, then the assignment  $(X, C) \mapsto X \odot C$  can be extended to a functor  $\mathbf{Set} \times \mathcal{C} \rightarrow \mathcal{C}$  such that, for each object  $C$ , we have the following adjunction:*

$$- \odot C \dashv C(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$$

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- (ii) If  $\mathcal{C}$  is a cotensored  $\mathbf{U}$ -category, then the assignment  $(X, C) \mapsto X \pitchfork C$  can be extended to a functor  $\mathbf{Set}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{C}$  such that, for each object  $C$ , the functors  $- \pitchfork C : \mathbf{Set}^{\text{op}} \rightarrow \mathcal{C}$  and  $\mathcal{C}(-, C) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  are contravariantly adjoint on the right.
- (iii) If  $\mathcal{C}$  is a tensored and cotensored  $\mathbf{U}$ -category, then for each set  $X$ , we have the following adjunction:

$$X \odot - \dashv X \pitchfork - : \mathcal{C} \rightarrow \mathcal{C}$$

*Proof.* Claims (i) and (ii) are formally dual and are straightforward applications of the parametrised adjunction theorem.<sup>[9]</sup> For claim (iii), simply observe that we have bijections

$$\mathcal{C}(X \odot A, B) \cong \mathbf{Set}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, X \pitchfork B)$$

and these are natural in  $A$ ,  $B$ , and  $X$ . ■

**Theorem A.6.13.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  be a locally  $\mathbf{U}$ -small category. The following are equivalent:*

- (i)  $\mathcal{C}$  is a  $\mathbf{U}$ -complete category.
- (ii)  $\mathcal{C}$  is a cotensored  $\mathbf{U}$ -category and, for all  $\mathbf{U}$ -small categories  $\mathbb{D}$  and all functors  $B : \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathcal{C}$ , an end for  $A$  exists in  $\mathcal{C}$ .
- (iii) For all weights  $W : \mathbb{D}^{\text{op}} \rightarrow \mathbf{Set}$  and all diagrams  $A : \mathbb{D} \rightarrow \mathbf{Set}$ ,  $\mathcal{C}$  has a  $W$ -weighted limit for  $A$ , provided  $\mathbb{D}$  is a  $\mathbf{U}$ -small category.

Dually, the following are equivalent:

- (i')  $\mathcal{C}$  is a  $\mathbf{U}$ -cocomplete category.
- (ii')  $\mathcal{C}$  is a tensored  $\mathbf{U}$ -category and, for all  $\mathbf{U}$ -small categories  $\mathbb{D}$  and all functors  $B : \mathbb{D}^{\text{op}} \times \mathbb{D} \rightarrow \mathcal{C}$ , a coend for  $A$  exists in  $\mathcal{C}$ .
- (iii') For all weights  $W : \mathbb{D}^{\text{op}} \rightarrow \mathbf{Set}$  and all diagrams  $A : \mathbb{D} \rightarrow \mathbf{Set}$ ,  $\mathcal{C}$  has a  $W$ -weighted colimit for  $A$ , provided  $\mathbb{D}$  is a  $\mathbf{U}$ -small category.

[9] See Theorem 3 in [CWM, Ch. IV, §7].

*Proof.* (i)  $\Rightarrow$  (ii). It is clear that  $X \pitchfork C$  is nothing more than an  $X$ -fold product of copies of  $C$ , so  $C$  is certainly  $\mathbf{U}$ -cotensored if it is  $\mathbf{U}$ -complete, and [proposition A.6.9](#) says  $C$  also has the required ends in that case.

(ii)  $\Rightarrow$  (iii). We have the following natural bijections:

$$\begin{aligned} C(C, \{W, A\}^{\mathbb{D}}) &\cong [\mathbb{D}, \mathbf{Set}](W, C(C, A)) \\ &\cong \int_{d:\mathbb{D}} \mathbf{Set}(Wd, C(C, Ad)) \\ &\cong \int_{d:\mathbb{D}} C(C, Wd \pitchfork Ad) \\ &\cong C\left(C, \int_{d:\mathbb{D}} Wd \pitchfork Ad\right) \end{aligned}$$

Thus, using the Yoneda lemma and *assuming  $C$  is a cotensored  $\mathbf{U}$ -category*, the weighted limit  $\{W, A\}^{\mathbb{D}}$  exists if and only if the end  $\int_{d:\mathbb{D}} Wd \pitchfork Ad$  exists.

(iii)  $\Rightarrow$  (i). Ordinary limits are a special case of weighted limits, as remarked in [A.5.II](#). ■

**Proposition A.6.14.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, let  $C$  be a locally  $\mathbf{U}$ -small category, and let  $\mathcal{J}$  be any category. If  $C$  is a tensored  $\mathbf{U}$ -category and has weighted limits for all weights  $W : \mathcal{J} \rightarrow \mathbf{Set}$  and diagrams  $A : \mathcal{J} \rightarrow C$ , then:*

- (i)  $(W, A) \mapsto \{W, A\}^{\mathcal{J}}$  extends to a functor  $[\mathcal{J}, \mathbf{Set}]^{\text{op}} \times C \rightarrow C$ .
- (ii) For each diagram  $A : \mathcal{J} \rightarrow C$ , the functors  $\{-, A\}^{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}]^{\text{op}} \rightarrow C$  and  $C(-, A) : C^{\text{op}} \rightarrow [\mathcal{J}, \mathbf{Set}]$  are contravariantly adjoint on the right.
- (iii) For each weight  $W : \mathcal{J} \rightarrow \mathbf{Set}$ , we have the following adjunction:

$$W \odot - \dashv \{W, -\}^{\mathcal{J}} : [\mathcal{J}, C] \rightarrow C$$

Here,  $W \odot C : \mathcal{J} \rightarrow C$  is the diagram  $j \mapsto Wj \odot C$ .

Dually, if  $C$  is a cotensored  $\mathbf{U}$ -category and has weighted colimits for all weights  $W : \mathcal{J}^{\text{op}} \rightarrow \mathbf{Set}$  and diagrams  $A : \mathcal{J} \rightarrow C$ , then:

- (i')  $(W, A) \mapsto W \star_{\mathcal{J}} A$  extends to a functor  $[\mathcal{J}^{\text{op}}, \mathbf{Set}] \times C \rightarrow C$ .

(ii') For each diagram  $A : \mathcal{J} \rightarrow \mathcal{C}$ , we have the following adjunction:

$$- \star_{\mathcal{J}} A \dashv C(A, -) : \mathcal{C} \rightarrow [\mathcal{J}^{\text{op}}, \mathbf{Set}]$$

(iii') For each weight  $W : \mathcal{J}^{\text{op}} \rightarrow \mathbf{Set}$ , we have the following adjunction:

$$W \star_{\mathcal{J}} - \dashv W \pitchfork - : \mathcal{C} \rightarrow [\mathcal{J}, \mathcal{C}]$$

Here,  $W \pitchfork C : \mathcal{J} \rightarrow \mathcal{C}$  is the diagram  $j \mapsto W j \pitchfork C$ .

*Proof.* Claim (i) is straightforward, and for claims (ii) and (iii), observe that we have bijections

$$\begin{aligned} \mathcal{C}(C, \{W, A\}^{\mathcal{J}}) &\cong [\mathcal{J}, \mathbf{Set}](W, C(C, A)) \\ &\cong \int_{j:\mathcal{J}} \mathbf{Set}(W j, C(C, A j)) \\ &\cong \int_{j:\mathcal{J}} C(W j \odot C, A j) \\ &\cong [\mathcal{J}, \mathcal{C}](W \odot C, A) \end{aligned}$$

and these are natural in  $W$ ,  $A$ , and  $C$ . ■

**Lemma A.6.15.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathbb{I}$  and  $\mathbb{J}$  be  $\mathbf{U}$ -small categories. For all functors  $A : \mathbb{I}^{\text{op}} \times \mathbb{J}^{\text{op}} \times \mathbb{I} \times \mathbb{J} \rightarrow \mathbf{Set}$ :*

- (i) *The assignment  $(i', i) \mapsto \int_{j:\mathbb{J}} A(i', j, i, j)$  extends to a functor  $\mathbb{I}^{\text{op}} \times \mathbb{I} \rightarrow \mathbf{Set}$ .*
- (ii) *There is a unique morphism  $\theta$  making the diagram below commute for all  $i$  and  $j$ ,*

$$\begin{array}{ccc} \int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') & \longrightarrow & \int_{j':\mathbb{J}} A(i, j', i, j') \\ & \theta \downarrow & \downarrow \\ \int_{(i', j'):\mathbb{I} \times \mathbb{J}} A(i', j', i', j') & \longrightarrow & A(i, j, i, j) \end{array}$$

where the unlabelled arrows are the components of the respective universal wedges, and  $\theta$  is moreover an isomorphism.



(iii) *There is a unique morphism  $\sigma$  making the diagram below commute for all  $i$  and  $j$ ,*

$$\begin{array}{ccc}
 \int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') & \longrightarrow & \int_{j':\mathbb{J}} A(i, j', i, j') \\
 \downarrow \sigma & & \searrow \\
 \int_{j':\mathbb{J}} \int_{i':\mathbb{I}} A(i', j', i', j') & \longrightarrow & \int_{i':\mathbb{I}} A(i, j', i, j') \\
 & & \nearrow \\
 & & A(i, j, i, j)
 \end{array}$$

where the unmarked arrows are the components of the respective universal wedges, and  $\sigma$  is moreover an isomorphism.

*Proof.* See [CWM, Ch. IX, §8]. □

**Theorem A.6.16** (Interchange law for ends and coends). *Let  $\mathcal{C}$  be any category and let  $A : \mathcal{I}^{\text{op}} \times \mathcal{J}^{\text{op}} \times \mathcal{I} \times \mathcal{J} \rightarrow \mathbf{Set}$  be any functor. If the end  $\int_{i:\mathcal{I}} A(i, j', i, j)$  exists in  $\mathcal{C}$  for all  $j'$  and  $j$  in  $\mathcal{J}$ , and the end  $\int_{j:\mathcal{J}} A(i', j, i, j)$  exists in  $\mathcal{C}$  for all  $i'$  and  $i$  in  $\mathcal{I}$ , then the following are equivalent:*

- (i) *The end  $\int_{(i,j):\mathcal{I} \times \mathcal{J}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*
- (ii) *The iterated end  $\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*
- (iii) *The iterated end  $\int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*

*In this case, we have a canonical isomorphism in  $\mathcal{C}$ :*

$$\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i, j, i, j) \cong \int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i, j, i, j)$$

*Dually, if the coend  $\int^{i:\mathcal{I}} A(i, j', i, j)$  exists in  $\mathcal{C}$  for all  $j'$  and  $j$  in  $\mathcal{J}$ , and the coend  $\int^{j:\mathcal{J}} A(i', j, i, j)$  exists in  $\mathcal{C}$  for all  $i'$  and  $i$  in  $\mathcal{I}$ , then the following are equivalent:*

- (i') *The coend  $\int^{(i,j):\mathcal{I} \times \mathcal{J}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*
- (ii') *The iterated coend  $\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*
- (iii') *The iterated coend  $\int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i, j, i, j)$  exists in  $\mathcal{C}$ .*

In this case, we have a canonical isomorphism in  $\mathcal{C}$ :

$$\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i, j, i, j) \cong \int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i, j, i, j)$$

*Proof.* Choose a pre-universe  $\mathbf{U}$  such that  $\mathcal{I}$  and  $\mathcal{J}$  are  $\mathbf{U}$ -small categories and  $\mathcal{C}$  is a locally  $\mathbf{U}$ -small category, and use the Yoneda lemma to reduce the claims to the previous lemma.  $\blacksquare$

**Proposition A.6.17.** *Let  $\mathbf{U}$  be a pre-universe, let  $\mathbf{Set}$  be the category of  $\mathbf{U}$ -sets, and let  $\mathcal{C}$  and  $\mathcal{J}$  be locally  $\mathbf{U}$ -small categories.*

- (i) *For all  $j$  in  $\mathcal{J}$  and all functors  $A : \mathcal{J} \rightarrow \mathcal{C}$ , the Yoneda bijection*

$$\mathcal{C}(\mathcal{C}, A_j) \cong [\mathcal{J}, \mathbf{Set}](h^j, \mathcal{C}(\mathcal{C}, A))$$

*exhibits  $A_j$  as the weighted limit  $\{h^j, A\}^{\mathcal{J}}$  in  $\mathcal{C}$ .*

- (ii) *If  $\mathcal{C}$  is a cotensored  $\mathbf{U}$ -category, then the end  $\int_{j':\mathcal{J}} \mathcal{J}(j, j') \pitchfork A_{j'}$  exists in  $\mathcal{C}$  and can be canonically identified with  $A_j$ .*
- (iii) *For all functors  $H : \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ , the weighted limit  $\{\text{Hom}_{\mathcal{J}}, H\}^{\mathcal{J}^{\text{op}} \times \mathcal{J}}$  exists in  $\mathcal{C}$  if and only if the end  $\int_{j:\mathcal{J}} H(j, j)$  exists in  $\mathcal{C}$ , and there is a canonical identification of the two.*

*Dually:*

- (i') *For all  $j$  in  $\mathcal{J}$  and all functors  $A : \mathcal{J} \rightarrow \mathcal{C}$ , the Yoneda bijection*

$$\mathcal{C}(A_j, \mathcal{C}) \cong [\mathcal{J}^{\text{op}}, \mathbf{Set}](h_j, \mathcal{C}(A, \mathcal{C}))$$

*exhibits  $A_j$  as the weighted colimit  $h_j \star_{\mathcal{J}} A$  in  $\mathcal{C}$ .*

- (ii') *If  $\mathcal{C}$  is a tensored  $\mathbf{U}$ -category, then the coend  $\int^{j':\mathcal{J}} \mathcal{J}(j', j) \odot A_{j'}$  exists in  $\mathcal{C}$  and can be canonically identified with  $A_j$ .*
- (iii') *For all functors  $H : \mathcal{J}^{\text{op}} \times \mathcal{J} \rightarrow \mathcal{C}$ , the weighted colimit  $\text{Hom}_{\mathcal{J}^{\text{op}}} \star_{\mathcal{J}^{\text{op}} \times \mathcal{J}} H$  exists in  $\mathcal{C}$  if and only if the coend  $\int^{j:\mathcal{J}} H(j, j)$  exists in  $\mathcal{C}$ , and there is a canonical identification of the two.*

*Proof.* (i). This is an immediate consequence of the Yoneda lemma and the definition of weighted limit.

(ii). Use the identification constructed in the proof of [theorem A.6.13](#).

(iii). For all objects  $C$  in  $\mathcal{C}$ , using claim (ii) and the interchange law for ends ([theorem A.6.16](#)), there are bijections

$$\begin{aligned} [\mathcal{J}^{\text{op}} \times \mathcal{J}, \mathbf{Set}](\text{Hom}_{\mathcal{J}}, \mathcal{C}(C, H)) &\cong \int_{(j', j): \mathcal{J}^{\text{op}} \times \mathcal{J}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(C, H(j', j))) \\ &\cong \int_{j: \mathcal{J}} \int_{j': \mathcal{J}^{\text{op}}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(C, H(j', j))) \\ &\cong \int_{j: \mathcal{J}} \mathcal{C}(C, H(j, j)) \end{aligned}$$

and these are natural in  $C$ ; now apply propositions [A.5.13](#) and [A.6.10](#). ■

## A.7 Familial regularity and exactness

**Prerequisites.** §A.3.

**Definition A.7.1.** A **strict initial object** in a category  $\mathcal{C}$  is an initial object  $0$  in  $\mathcal{C}$  such that every morphism  $X \rightarrow 0$  in  $\mathcal{C}$  is an isomorphism.

**Example A.7.2.** The empty set is a strict initial object in  $\mathbf{Set}$ .

**Proposition A.7.3.** *Let  $\mathcal{C}$  be a category with a strict initial object  $0$ .*

- (i) *For any object  $Y$  in  $\mathcal{C}$ , the unique morphism  $0 \rightarrow Y$  is a monomorphism.*
- (ii) *For any object  $Y$  in  $\mathcal{C}$ ,  $0 \rightarrow Y$  is a strict initial object in the slice category  $\mathcal{C}_{/Y}$ .*

*Proof.* Obvious. ◆

**Definition A.7.4.** Let  $\kappa$  be a (not necessarily small) regular cardinal. A  **$\kappa$ -ary extensive category** is a category  $\mathcal{E}$  satisfying the following axioms:

- $\mathcal{E}$  has coproducts for  $\kappa$ -small families of objects (including the empty family).

- Given a  $\kappa$ -small family of objects in  $\mathcal{E}$ , say  $(A_i \mid i \in I)$ , a morphism  $f : B \rightarrow \coprod_{i \in I} A_i$ , and commutative diagrams as below,

$$\begin{array}{ccc} B_j & \longrightarrow & B \\ \downarrow & & \downarrow f \\ A_j & \longrightarrow & \coprod_{i \in I} A_i \end{array}$$

where the morphisms  $A_j \rightarrow \coprod_{i \in I} A_i$  are the coproduct insertions, the family of morphisms  $B_j \rightarrow B$  is a coproduct cocone if and only if each of the above commutative diagrams are pullback squares.

An **extensive category** is an  $\aleph_0$ -ary extensive category, and an **infinitary extensive category** is a  $\kappa$ -ary extensive category where  $\kappa$  is the cardinality of the universe.

**Examples A.7.5.**

- The category **Set** is an infinitary extensive category.
- More generally, if  $\mathcal{E}$  is a locally cartesian closed category with coproducts for  $\kappa$ -small families of objects, then  $\mathcal{E}$  is a  $\kappa$ -ary extensive category.
- The category **Top** is an infinitary extensive category but is neither cartesian closed nor locally cartesian closed.

**Proposition A.7.6.** *Let  $\kappa$  be a (not necessarily small) regular cardinal. If  $\mathcal{E}$  is a  $\kappa$ -ary extensive category, then for any object  $A$  in  $\mathcal{E}$ , the slice category  $\mathcal{E}_{/A}$  is also a  $\kappa$ -ary extensive category.*

*Proof.* This is an immediate consequence of the fact that the projection functor  $\mathcal{E}_{/A} \rightarrow \mathcal{E}$  creates all colimits and pullbacks. ■

**Theorem A.7.7.** *Let  $\kappa$  be a (not necessarily small) regular cardinal and let  $\mathcal{C}$  be a category with coproducts for  $\kappa$ -small families of objects. The following are equivalent:*

- $\mathcal{C}$  is a  $\kappa$ -ary extensive category.
- For any  $\kappa$ -small family of objects in  $\mathcal{C}$ , say  $(X_i \mid i \in I)$ , the functor

$$\prod_{i \in I} \mathcal{C}_{/X_i} \rightarrow \mathcal{C}_{/\coprod_{i \in I} X_i}$$

induced by  $\coprod_{i \in I}$  is fully faithful and essentially surjective on objects.

*Proof.* See Proposition 2.14 in [Carboni, Lack and Walters, 1993]. □

**Proposition A.7.8.** *Let  $\mathcal{C}$  be a category with an initial object  $0$ . The following are equivalent:*

- (i)  $0$  is a strict initial object in  $\mathcal{C}$ .
- (ii) The slice category  $\mathcal{C}_{/0}$  is equivalent to the terminal category  $\mathbb{1}$ .
- (iii) For any morphism  $f : X \rightarrow Y$ , the following diagram is a pullback square:

$$\begin{array}{ccc} 0 & \xrightarrow{\text{id}} & 0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

*Proof.* (i)  $\Rightarrow$  (ii). If  $f : X \rightarrow 0$  is an object in  $\mathcal{C}_{/0}$ , then  $f$  is an isomorphism in  $\mathcal{C}$ , and  $X$  is also an initial object in  $\mathcal{C}$ . Thus, any two objects in  $\mathcal{C}_{/0}$  are connected by a unique isomorphism, and the unique functor  $\mathcal{C}_{/0} \rightarrow \mathbb{1}$  is fully faithful and surjective on objects, with quasi-inverse the functor  $\mathbb{1} \rightarrow \mathcal{C}_{/0}$  sending the unique object in  $\mathbb{1}$  to the object  $\text{id} : 0 \rightarrow 0$ .

(ii)  $\Rightarrow$  (i), (i)  $\Rightarrow$  (iii). Obvious.

(iii)  $\Rightarrow$  (i). Take  $Y = 0$ ; then  $0 \rightarrow X$  must be an isomorphism with  $f : X \rightarrow 0$  as its inverse. ■

**Corollary A.7.9.** *Initial objects in extensive categories are strict.*

*Proof.* The second axiom in the case where  $I = \emptyset$  says that the initial object is preserved by pullbacks. ■

**Definition A.7.10.** Let  $\mathcal{C}$  be a category.

- An **extremal epimorphism** in  $\mathcal{C}$  is an epimorphism  $e$  in  $\mathcal{C}$  such that, for any morphisms  $m$  and  $z$  in  $\mathcal{C}$ , such that  $e = m \circ z$ , if  $m$  is a monomorphism in  $\mathcal{C}$ , then  $m$  is an isomorphism.
- A **regular epimorphism** in  $\mathcal{C}$  is a morphism  $e$  in  $\mathcal{C}$  for which there exist morphisms  $f_0, f_1$  in  $\mathcal{C}$  such that  $e$  is their coequaliser in  $\mathcal{C}$ .

- An **effective epimorphism** in  $\mathcal{C}$  is a morphism  $e$  in  $\mathcal{C}$  such that  $e$  has a kernel pair in  $\mathcal{C}$  and is their coequaliser in  $\mathcal{C}$ .

**Proposition A.7.II.** *Let  $\mathcal{C}$  be a category and let  $e$  be a morphism in  $\mathcal{C}$ . Consider the following statements:*

- (i)  $e$  is an effective epimorphism.
- (ii)  $e$  is a regular epimorphism.
- (iii)  $e$  is a strong epimorphism.
- (iv)  $e$  is an extremal epimorphism.
- (v)  $e$  is an epimorphism.

We always have the implications (i)  $\Rightarrow$  (ii), (ii)  $\Rightarrow$  (iii), (iii)  $\Rightarrow$  (iv), and (iv)  $\Rightarrow$  (v); if  $\mathcal{C}$  has kernel pairs, then (ii)  $\Rightarrow$  (i); and if  $\mathcal{C}$  has pullbacks of monomorphisms, then (iv)  $\Rightarrow$  (iii).

*Proof.* (i)  $\Rightarrow$  (ii), (iv)  $\Rightarrow$  (v). Immediate.

(ii)  $\Rightarrow$  (iii). Suppose  $e : Z \rightarrow W$  is a coequaliser for  $f_0, f_1 : T \rightarrow Z$  in  $\mathcal{C}$ , and consider a commutative diagram of the form below in  $\mathcal{C}$ :

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ e \downarrow & & \downarrow m \\ W & \xrightarrow{w} & Y \end{array}$$

It is clear that  $e : Z \rightarrow W$  is an epimorphism in  $\mathcal{C}$ . If  $m : X \rightarrow Y$  is a monomorphism in  $\mathcal{C}$ , then  $z \circ f_0 = z \circ f_1$ , so there must exist a unique morphism  $h : W \rightarrow X$  in  $\mathcal{C}$  such that  $z = h \circ e$ ; and  $m \circ z = w \circ e$ , so we must have  $m \circ h = w$  as well. Thus  $e : Z \rightarrow W$  has the required orthogonality property.

(iii)  $\Rightarrow$  (iv). This is the special case where  $w = \text{id}$ ; the existence of  $h$  such that  $m \circ h = \text{id}$  implies  $m$  is both a monomorphism and a split epimorphism, so  $m$  must be an isomorphism in this case.

(ii)  $\Rightarrow$  (i). Suppose  $e : Z \rightarrow W$  is a coequaliser for  $f_0, f_1 : T \rightarrow Z$  in  $\mathcal{C}$ . Let  $k_0, k_1 : K \rightarrow Z$  be a kernel pair for  $e : Z \rightarrow W$ . There is then a unique morphism  $r : T \rightarrow K$  in  $\mathcal{C}$  such that  $k_0 \circ r = f_0$  and  $k_1 \circ r = f_1$ . Consider

any morphism  $z : Z \rightarrow X$  in  $\mathcal{C}$  such that  $z \circ k_0 = z \circ k_1$ . Then  $z \circ f_0 = z \circ f_1$  as well, so there is a unique morphism  $h : W \rightarrow X$  such that  $z = h \circ e$ . By definition, we have  $e \circ k_0 = e \circ k_1$ , so it follows that  $e : Z \rightarrow W$  is a coequaliser for  $k_0, k_1 : K \rightarrow Z$  in  $\mathcal{C}$  as well.

(iv)  $\Rightarrow$  (iii). Suppose  $e : Z \rightarrow W$  is a strong epimorphism in  $\mathcal{C}$ , and consider a commutative diagram of the form below in  $\mathcal{C}$ :

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ e \downarrow & & \downarrow m \\ W & \xrightarrow{w} & Y \end{array}$$

There is then a comparison morphism  $Z \rightarrow W \times_Y X$ , and if  $m : X \rightarrow Y$  is a monomorphism in  $\mathcal{C}$ , then so is the projection  $W \times_Y X \rightarrow W$ . Since  $e : Z \rightarrow W$  is a strong epimorphism, the projection  $W \times_Y X \rightarrow W$  must be an isomorphism, so we obtain a (unique) morphism  $h : W \rightarrow X$  in  $\mathcal{C}$  such that  $h \circ e = z$  and  $m \circ h = w$ , as required. ■

**Definition A.7.12.** A **regular category** is a category  $\mathcal{C}$  that satisfies the following axioms:

- $\mathcal{C}$  has finite limits.
- $\mathcal{C}$  has coequalisers for kernel pairs.
- The class of regular epimorphisms in  $\mathcal{C}$  is closed under pullbacks.

**Example A.7.13.** **Set** is a regular category. More generally, any  $\Sigma\Pi$ -category<sup>[10]</sup> with coequalisers for kernel pairs is a regular category.

**Theorem A.7.14.** *Let  $\mathcal{C}$  be a regular category.*

- (i) *Every morphism in  $\mathcal{C}$  admits a (regular epi, mono)-factorisation.*
- (ii) *Every extremal epimorphism in  $\mathcal{C}$  is a regular epimorphism.*
- (iii) *The class of regular epimorphisms in  $\mathcal{C}$  is closed under composition.*

[10] See [definition A.2.18](#).

*Proof.* (i). See Theorem 2.1.3 in [Borceux, 1994b].

(ii). Let  $f$  be an extremal epimorphism in  $\mathcal{C}$ , and suppose  $f = m \circ e$  where  $m$  is a monomorphism and  $e$  is a regular epimorphism. Then  $m$  must be an isomorphism, so  $f$  is indeed a regular epimorphism.  $\square$

**Proposition A.7.15.** *In a regular category:*

- (i) *The class of regular epimorphisms is closed under composition, pullbacks, and retracts.*
- (ii) *If  $g \circ f$  is a regular epimorphism, then  $g$  is also a regular epimorphism.*
- (iii) *The class of regular epimorphisms is closed under finite products.*

*Proof.* (i). Recalling proposition A.7.11, theorem A.7.14 implies that strong epimorphisms in a regular category are the same as regular epimorphisms, so we may apply proposition A.3.14.

(ii). Similarly, we may apply proposition A.3.15.

(iii). Let  $f : X \rightarrow Y$  and  $g : Z \rightarrow W$  be regular epimorphisms in a regular category. Then  $f \times g = (f \times \text{id}_W) \circ (\text{id}_X \times g)$ , and both  $f \times \text{id}_W$  and  $\text{id}_X \times g$  are regular epimorphisms (because the class of regular epimorphisms is closed under pullbacks), so their composite is a regular epimorphism as well.  $\blacksquare$

**Definition A.7.16.** A **weak pullback square** in a regular category  $\mathcal{C}$  is a commutative diagram in  $\mathcal{C}$ , say

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

such that the comparison morphism  $Z \rightarrow W \times_Y X$  is a regular epimorphism.

**Lemma A.7.17.** *Let  $\mathcal{C}$  be a regular category, and consider a commutative diagram in  $\mathcal{C}$  of the form below:*

$$\begin{array}{ccccc} X'' & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$



- (i) *If the two squares are weak pullback squares in  $\mathcal{C}$ , then the outer rectangle is a weak pullback diagram in  $\mathcal{C}$ .*
- (ii) *If the right square is an ordinary pullback square and outer rectangle is weak pullback diagram in  $\mathcal{C}$ , then the left square is a weak pullback square in  $\mathcal{C}$ .*

*Proof.* (i). First, form the following pullback diagram in  $\mathcal{C}$ :

$$\begin{array}{ccc} T & \longrightarrow & Y'' \times_Y X \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Y' \times_Y X \end{array}$$

Since  $X' \rightarrow Y' \times_Y X$  is a regular epimorphism, so is  $T \rightarrow Y'' \times_Y X$ . Next, form a pullback diagram in  $\mathcal{C}$  of the form below:

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' \times_{Y'} X' \end{array}$$

Since  $X'' \rightarrow Y'' \times_{Y'} X'$  is a regular epimorphism, so is  $S \rightarrow T$ . We thus obtain a regular epimorphism  $S \rightarrow Y'' \times_Y X$  that factors through the comparison morphism  $X'' \rightarrow Y'' \times_Y X$ , so we may use [proposition A.7.15](#) to deduce that  $X'' \rightarrow Y'' \times_Y X$  is a regular epimorphism.

(ii). Start by forming the following pullback diagram in  $\mathcal{C}$ :

$$\begin{array}{ccc} T & \longrightarrow & Y'' \times_{Y'} X' \\ \downarrow & & \downarrow \\ X'' & \longrightarrow & Y'' \times_Y X \end{array}$$

Since  $X'' \rightarrow Y'' \times_Y X$  is a regular epimorphism, so is  $T \rightarrow Y'' \times_{Y'} X'$ . On the other hand, we have a commutative diagram of the form below in  $\mathcal{C}$ ,

$$\begin{array}{ccccc} Y'' \times_Y X & \longrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ Y'' & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

so  $T \rightarrow Y'' \times_{Y'} X'$  factors through the comparison morphism  $X'' \rightarrow Y'' \times_Y X$ . Thus,  $X'' \rightarrow Y'' \times_{Y'} X'$  is a regular epimorphism, as required. ■

**Definition A.7.18.** A **regular functor** (or **Barr-exact functor**) is a functor between regular categories that preserves finite limits and regular epimorphisms.

REMARK A.7.19. By [proposition A.7.11](#), a regular functor is the same thing as a functor between regular categories that preserves finite limits and coequalisers of kernel pairs. Thus, a regular functor between abelian categories automatically preserves finite colimits.

**Definition A.7.20.** An **exact fork** in a category  $C$  is a diagram in  $C$  of the form below,

$$X \begin{array}{c} \xrightarrow{f_0} \\ \xrightarrow{f_1} \end{array} \rightrightarrows Y \xrightarrow{g} Z$$

where  $g : Y \rightarrow Z$  is a coequaliser for  $f_0, f_1 : X \rightarrow Y$  in  $C$  and  $f_0, f_1 : X \rightarrow Y$  is a kernel pair for  $g : Y \rightarrow Z$  in  $C$ .

**Theorem A.7.21** (Regular embedding theorem). *Let  $C$  be a small regular category and let  $\mathcal{E}$  be the full subcategory of  $[C^{\text{op}}, \mathbf{Set}]$  spanned by those presheaves  $C^{\text{op}} \rightarrow \mathbf{Set}$  that send exact forks in  $C$  to equaliser diagrams in  $\mathbf{Set}$ .*

- (i)  $\mathcal{E}$  is a reflective subcategory of  $[C^{\text{op}}, \mathbf{Set}]$ , and the reflector  $[C^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{E}$  preserves finite limits.
- (ii) The Yoneda embedding  $h_{\bullet} : C \rightarrow [C^{\text{op}}, \mathbf{Set}]$  factors through the inclusion  $\mathcal{E} \hookrightarrow [C^{\text{op}}, \mathbf{Set}]$ , and the resulting functor  $C \rightarrow \mathcal{E}$  is fully faithful, regular, and preserves all limits.

*Proof.* (i). This follows from the fact that  $\mathcal{E}$  is the category of sheaves for a Grothendieck topology on  $C$ : see Lemma 2.7.2 in [Borceux, 1994b] and Theorem 3.3.12 in [Borceux, 1994c].

(ii). See Theorem 2.7.3 in [Borceux, 1994b]. □

**Theorem A.7.22** (Barr). *For each small regular category  $C$ , there exist a set  $B$  and a conservative regular functor  $C \rightarrow \mathbf{Set}^B$ .*

*Proof.* See Theorem 1.6 in [Barr, 1971, Ch. III] or Corollary 1.5.4 in [Johnstone, 2002b, Part D]. □

**Definition A.7.23.**

- An **effective equivalence relation** in a category  $C$  is an (internal) equivalence relation in  $C$  that appears as part of an exact fork in  $C$ , i.e. is a kernel pair for an effective epimorphism in  $C$ .

- An **effective regular category** (or **Barr-exact category**) is a regular category in which all (internal) equivalence relations are effective.

**REMARK A.7.24.** A regular category with coequalisers for all parallel pairs of morphisms is automatically an effective regular category, but effective regular categories need not have coequalisers in general.

**Lemma A.7.25.** *Let  $\mathcal{C}$  be an effective regular category. Given a parallel pair of morphisms in  $\mathcal{C}$ , say  $p_0, p_1 : X \rightarrow Y$ , if the regular image of the morphism  $\langle p_0, p_1 \rangle : X \rightarrow Y \times Y$  defines an equivalence relation  $R$  on  $Y$ , then  $p_0, p_1 : X \rightarrow Y$  have a coequaliser in  $\mathcal{C}$ , and the kernel pair of the coequaliser is the equivalence relation  $R$ .*

*Proof.* By definition of  $R$ , there exist a regular epimorphism  $e : X \rightarrow R$  and two projections  $r_0, r_1 : R \rightarrow Y$  such that  $p_0 = r_0 \circ e$  and  $p_1 = r_1 \circ e$ . Let  $q : Y \rightarrow Z$  be the coequaliser of  $r_0$  and  $r_1$  in  $\mathcal{C}$ ; such exists because  $\mathcal{C}$  is an effective regular category. Note that the kernel pair of  $q : Y \rightarrow Z$  is  $r_0, r_1 : R \rightarrow Y$ . Now,  $q \circ r_0 = q \circ r_1$ , so we must have  $q \circ p_0 = q \circ p_1$  as well; but if  $f : Y \rightarrow T$  is any morphism in  $\mathcal{C}$  such that  $f \circ p_0 = f \circ p_1$ , then we must have  $f \circ q_0 = f \circ q_1$  (because  $e : X \rightarrow R$  is an epimorphism), and so there is a unique morphism  $\bar{f} : Z \rightarrow T$  such that  $\bar{f} \circ q = f$ . Thus,  $q : Y \rightarrow Z$  is also the coequaliser of  $p_0$  and  $p_1$  in  $\mathcal{C}$ . ■

**Definition A.7.26.** Let  $\kappa$  be a (not necessarily small) regular cardinal. A  $\kappa$ -**ary pretopos** is a category that is both  $\kappa$ -ary extensive and effective regular. A  $\sigma$ -**pretopos** is an  $\aleph_1$ -ary pretopos.

**Proposition A.7.27.** *Let  $\kappa$  be a (not necessarily small) regular cardinal. A  $\kappa$ -ary pretopos is a (positive)  $\kappa$ -ary coherent category.*

*Proof.* See Theorem 5.15 in [Shulman, 2012]. □

**REMARK A.7.28.** The above proposition implies that our definition of ‘pretopos’ agrees with the one given by Johnstone [2002a, Part A, §1.4].

**Proposition A.7.29.** *Any epimorphism in a pretopos is a regular epimorphism.*

*Proof.* See Corollary 1.4.9 in [Johnstone, 2002a, Part A]. □

**Proposition A.7.30.**

- (i) *Every  $\sigma$ -pretopos has coequalisers for all parallel pairs; hence, they have colimits for all countable diagrams.*

(ii) *Every regular functor between  $\sigma$ -pretoposes that preserves coproducts for countable families also preserves coequalisers.*

*Proof.* (i). See Lemma 1.4.19 in [Johnstone, 2002a, Part A].

(ii). See Lemma 2.5.7 in [Johnstone, 2002a, Part A]. □

**Proposition A.7.31.** *Let  $\kappa$  be a small regular cardinal. If  $C$  is a small  $\kappa$ -ary pretopos, then there exist a Grothendieck topos  $\mathcal{E}$  and a fully faithful regular functor  $C \rightarrow \mathcal{E}$  that preserves coproducts for  $\kappa$ -small families of objects. Moreover, if  $\kappa$  is uncountable, then the embedding  $C \rightarrow \mathcal{E}$  also preserves coequalisers.*

*Proof.* By Theorem 5.15 in [Shulman, 2012], or Example 2.1.11(b) in [Johnstone, 2002a, Part A], we may take  $\mathcal{E}$  to be the category of sheaves for the  $\kappa$ -ary coherent topology on  $C$ ; then apply proposition A.7.30. □

**Theorem A.7.32** (Deligne). *For each small pretopos  $C$ , there exist a set  $B$  and a conservative regular functor  $C \rightarrow \mathbf{Set}^B$  that preserves coproducts for finite families of objects.*

*Proof.* See Proposition 9.0 in [SGA 4b, Exposé VI], Corollary 3 in [ML–M, Ch. IX, § 11], or Proposition 3.3.13 in [Johnstone, 2002b, Part D]. □

## HIGHER GENERALITIES

### B.I Monoidal categories

Standard references for monoidal categories include [CWM, Ch. VII and Ch. XI] and [Kelly, 2005, Ch. 1]. To fix notation, we will quickly review the main definitions in the theory of monoidal categories.

**Definition B.I.1.** A **strict monoidal category** is a category  $\mathcal{C}$  together with an object  $I$  and a functor  $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  satisfying the following axioms:

- (Left unit).  $I \otimes (-) = \text{id}_{\mathcal{C}}$ .
- (Right unit).  $(-) \otimes I = \text{id}_{\mathcal{C}}$ .
- (Associativity). For all objects  $X, Y,$  and  $Z$  in  $\mathcal{C}$ ,

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

and similarly for morphisms in  $\mathcal{C}$ .

$I$  is called the **monoidal unit**, and  $\otimes$  is called the **monoidal product**.

In short, a strict monoidal category is an internal monoid in the metacategory of all categories.

**Example B.I.2.** For any category  $\mathcal{C}$ , the endofunctor category  $[\mathcal{C}, \mathcal{C}]$  is a strict monoidal category with  $\text{id}_{\mathcal{C}}$  as the monoidal unit and endofunctor composition as the monoidal product.

## B. HIGHER GENERALITIES

Despite the above example, strict monoidal categories turn out to be less useful than one might hope: not even **Set** equipped with the usual cartesian product is a strict monoidal category.<sup>[1]</sup> The problem is in the *equations* we have imposed in the axioms above: in naturally-occurring examples, we do not get *identities* but only natural isomorphisms. This observation led Bénabou [1963] to propose the following notion instead:

**Definition B.I.3.** A **monoidal category** is a category  $\mathcal{C}$  together with an object  $I$ , a functor  $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and three natural isomorphisms  $\lambda$ ,  $\rho$ , and  $\alpha$ ,<sup>[2]</sup> of type

$$\begin{aligned}\lambda_X &: I \otimes X \xrightarrow{\cong} X \\ \rho_X &: X \otimes I \xrightarrow{\cong} X \\ \alpha_{X,Y,Z} &: (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)\end{aligned}$$

such that the following diagrams commute for all choices of objects in  $\mathcal{C}$ :

$$\begin{array}{ccccc} & & (W \otimes (X \otimes Y)) \otimes Z & & \\ & \nearrow \alpha_{W,X,Y} \otimes \text{id}_Z & & \searrow \alpha_{W,X \otimes Y,Z} & \\ ((W \otimes X) \otimes Y) \otimes Z & & & & (W \otimes ((X \otimes Y) \otimes Z)) \\ \alpha_{W \otimes X,Y,Z} \downarrow & & & & \downarrow \text{id}_W \otimes \alpha_{X,Y,Z} \\ (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{\alpha_{W,X,Y \otimes Z}} & & \xrightarrow{\alpha_{W,X,Y \otimes Z}} & W \otimes (X \otimes (Y \otimes Z)) \\ & & (X \otimes I) \otimes Y & \xrightarrow{\alpha_{X,I,Y}} & X \otimes (I \otimes Y) \\ & & \searrow \rho_X \otimes \text{id}_Y & & \swarrow \lambda_X \otimes \text{id}_Y \\ & & X \otimes Y & & \end{array}$$

The natural isomorphisms  $\lambda$ ,  $\rho$ , and  $\alpha$  are called, respectively, the **left unitor**, **right unitor**, and **associator** of the monoidal category  $\mathcal{C}$ .

**REMARK B.I.4.** Since  $\lambda$ ,  $\rho$ , and  $\alpha$  are natural *isomorphisms*, a monoidal structure on  $\mathcal{C}$  induces a monoidal structure on  $\mathcal{C}^{\text{op}}$ . Less obviously, we can define a monoidal category  $\mathcal{C}^{\text{rev}}$  whose underlying category is the same as  $\mathcal{C}$ , but  $X \otimes^{\text{rev}} Y = Y \otimes X$ ,  $\lambda^{\text{rev}} = \rho$ ,  $\rho^{\text{rev}} = \lambda$ , and  $\alpha^{\text{rev}} = \alpha^{-1}$ .

- [1] In fact, even if we identify all isomorphic objects, there is still a problem: see the closing remarks in [CWM, Ch. VII, §1].  
 [2] Beware: Mac Lane [CWM, Ch. VII] uses the opposite convention for  $\alpha$ .

¶ B.I.5. A fairly non-trivial theorem of Mac Lane [1963] and Kelly [1964] essentially states that these two axioms are enough to prove that “all diagrams involving only  $\lambda$ ,  $\rho$ , and  $\alpha$  commute”. For example, using the pentagon axiom and the triangle axiom, we may derive

$$\begin{array}{ccc} (I \otimes X) \otimes Y & \xrightarrow{\alpha_{I,X,Y}} & I \otimes (X \otimes Y) \\ & \searrow \lambda_{X \otimes \text{id}_Y} & \swarrow \lambda_{X \otimes Y} \\ & X \otimes Y & \end{array}$$

from which the equation (!) below can be obtained:

$$\lambda_I = \rho_I$$

**Definition B.I.6.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be monoidal categories. A **lax monoidal functor**  $\mathcal{C} \rightarrow \mathcal{D}$  consists of a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  of the underlying categories, together with a morphism  $\eta : I_{\mathcal{D}} \rightarrow FI_{\mathcal{C}}$  in  $\mathcal{D}$  and a natural transformation  $\mu$  of type  $F(-) \otimes_{\mathcal{D}} F(-) \Rightarrow F(- \otimes_{\mathcal{C}} -)$  making these diagrams commute:

$$\begin{array}{ccc} I_{\mathcal{D}} \otimes_{\mathcal{D}} FX & \xrightarrow{\eta \otimes_{\mathcal{D}} \text{id}_{FX}} & FI_{\mathcal{C}} \otimes_{\mathcal{D}} FX \\ \lambda_{FX} \downarrow & & \downarrow \mu_{I_{\mathcal{C}}, X} \\ FX & \xleftarrow{F\lambda_X} & F(I_{\mathcal{C}} \otimes_{\mathcal{C}} X) \end{array} \quad \begin{array}{ccc} FX \otimes_{\mathcal{D}} I_{\mathcal{D}} & \xrightarrow{\text{id}_{FX} \otimes_{\mathcal{D}} \eta} & FX \otimes_{\mathcal{D}} FI_{\mathcal{C}} \\ \rho_{FX} \downarrow & & \downarrow \mu_{X, I_{\mathcal{C}}} \\ FX & \xleftarrow{F\rho_X} & F(X \otimes_{\mathcal{C}} I_{\mathcal{C}}) \end{array}$$

$$\begin{array}{ccc} (FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} FZ & \xrightarrow{\alpha_{FX, FY, FZ}} & FX \otimes_{\mathcal{D}} (FY \otimes_{\mathcal{D}} FZ) \\ \mu_{X, Y} \otimes_{\mathcal{D}} \text{id}_{FZ} \downarrow & & \downarrow \text{id}_{FX} \otimes_{\mathcal{D}} \mu_{Y, Z} \\ F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} FZ & & FX \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \\ \mu_{X \otimes_{\mathcal{C}} Y, Z} \downarrow & & \downarrow \mu_{X, Y \otimes_{\mathcal{C}} Z} \\ F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) & \xrightarrow{F\alpha_{X, Y, Z}} & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z)) \end{array}$$

An **oplax monoidal functor**  $\mathcal{C} \rightarrow \mathcal{D}$  is a lax monoidal functor  $\mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ . A **strong monoidal functor** is a lax monoidal functor such that  $\eta$  and  $\mu$  are isomorphisms. A **strict monoidal functor** is a lax monoidal functor such that  $\eta$  and  $\mu$  are *identities*.

**Definition B.I.7.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be monoidal categories and let  $F, F' : \mathcal{C} \rightarrow \mathcal{D}$  be lax monoidal functors. A **monoidal natural transformation**  $\varphi : F \Rightarrow F'$  is a

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natural transformation making the following diagrams commute:

$$\begin{array}{ccc}
 I_D & \xrightarrow{\eta} & FI_C \\
 \text{id} \downarrow & & \downarrow \varphi_{I_C} \\
 I_D & \xrightarrow{\eta'} & F'I_C
 \end{array}
 \qquad
 \begin{array}{ccc}
 FX \otimes_D FY & \xrightarrow{\mu_{X,Y}} & F(X \otimes_C Y) \\
 \varphi_X \otimes_D \varphi_Y \downarrow & & \downarrow \varphi_{X \otimes_C Y} \\
 F'X \otimes_D F'Y & \xrightarrow{\mu'_{X,Y}} & F'(X \otimes_C Y)
 \end{array}$$

REMARK B.I.8. Note that if  $\mathcal{C}$  and  $\mathcal{D}$  are both strict monoidal categories, then the diagrams above simplify to more familiar ones:

$$\begin{array}{ccc}
 FX & \xrightarrow{\eta \otimes_D \text{id}_{FX}} & FI_C \otimes_D FX \\
 \text{id}_{FX} \searrow & & \downarrow \mu_{I_C, X} \\
 & & FX
 \end{array}
 \qquad
 \begin{array}{ccc}
 FX & \xrightarrow{\text{id}_{FX} \otimes_D \eta} & FX \otimes_D FI_C \\
 \text{id}_{FX} \searrow & & \downarrow \mu_{X, I_C} \\
 & & FX
 \end{array}$$

$$\begin{array}{ccc}
 & FX \otimes_D FY \otimes_D FZ & \\
 \mu_{X,Y} \otimes_D \text{id}_{FZ} \swarrow & & \searrow \text{id}_{FX} \otimes_D \mu_{Y,Z} \\
 F(X \otimes_C Y) \otimes_D FZ & & FX \otimes_D F(Y \otimes_C Z) \\
 \mu_{X \otimes_C Y, Z} \searrow & & \swarrow \mu_{X, Y \otimes_C Z} \\
 & F(X \otimes_C Y \otimes_C Z) & 
 \end{array}$$

Thus, we see one reason for defining lax monoidal functors as we have done: if  $\mathbb{1}$  is the terminal category, then a lax monoidal functor  $\mathbb{1} \rightarrow \mathcal{D}$  is the same thing as an internal monoid<sup>[3]</sup> in  $\mathcal{D}$ , and a monoidal natural transformation of such lax monoidal functors is the same thing as a homomorphism of internal monoids.

Many natural examples of monoidal categories have a “commutative” monoidal product. For example, the cartesian product in any category satisfies  $X \times Y \cong Y \times X$ . As usual, to do anything useful, we must demand not only the existence of such isomorphisms but also that they be natural and coherent in the following sense:

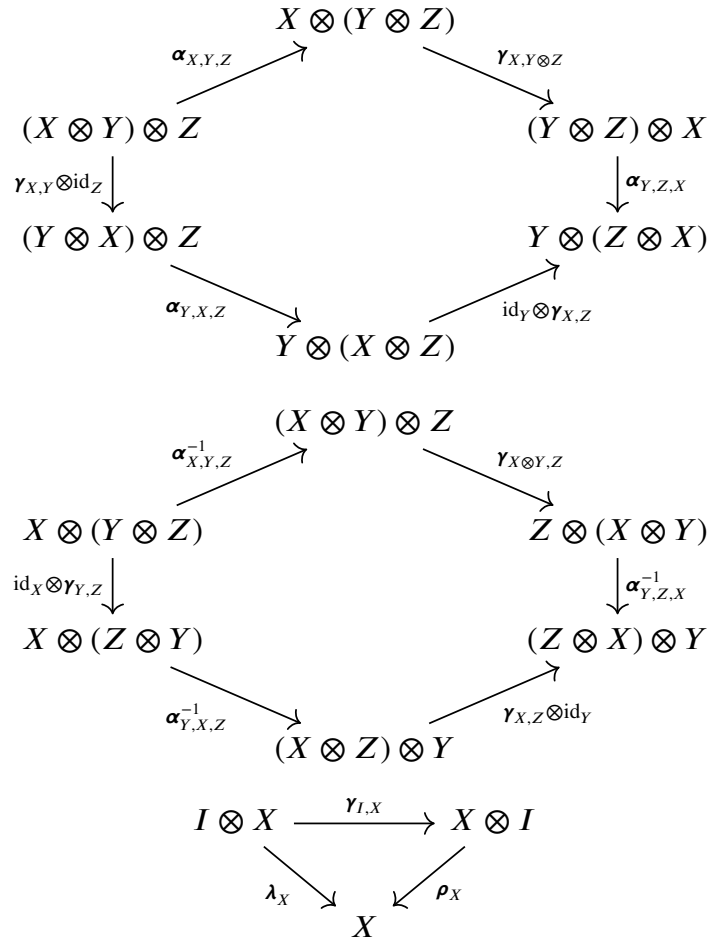
**Definition B.I.9.** A **braided monoidal category** is a monoidal category  $\mathcal{C}$  together with a natural isomorphism  $\gamma$  of type

$$\gamma_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

[3] — in the monoidal category sense, of course.



such that the following diagrams commute for all choices of objects in  $C$ :



The natural isomorphism  $\gamma$  is called the **braiding** of  $C$ . A **symmetric monoidal category** is a braided monoidal category  $C$  satisfying the following additional axiom:

$$\gamma \circ \gamma = \text{id}_C$$

A **braided / symmetric strict monoidal category** is a braided / symmetric monoidal category that is strict as a monoidal category.

There is a coherence theorem for braided and symmetric monoidal categories as well, but in the braided case it is somewhat subtle compared to the coherence theorem for monoidal categories – we cannot be so cavalier as to say that “all diagrams commute” in a braided monoidal category. Instead, just as before, every braided / symmetric monoidal category is equivalent to a strict one via functors respecting the various structural isomorphisms.

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**Definition B.I.10.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be braided monoidal categories. A **lax / oplax / strong / strict braided monoidal functor**  $\mathcal{C} \rightarrow \mathcal{D}$  is a lax / oplax / strong / strict monoidal functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  making the diagram below commute:

$$\begin{array}{ccc} FX \otimes_D FY & \xrightarrow{\mu_{X,Y}} & F(X \otimes_C Y) \\ \gamma_{FX,FY} \downarrow & & \downarrow F\gamma_{X,Y} \\ FY \otimes_D FX & \xrightarrow{\mu_{Y,X}} & F(Y \otimes_C X) \end{array}$$

**REMARK B.I.11.** The appropriate notion of natural transformation for lax braided monoidal functors is precisely that of a monoidal natural transformation: we need not impose any extra conditions.

Here is an example of an equation that does *not* necessarily hold in a braided monoidal category, even though they have the same domain and codomain:

$$\gamma_{X,Y} \stackrel{?}{=} \gamma_{Y,X}^{-1}$$

Indeed, if it were true, then every braided monoidal category would be a symmetric monoidal category! On the other hand, in a symmetric strict monoidal category, it is true that any two composites of braiding operations with the same domain and codomain are equal – provided each object is identified with a different letter, so that we do not get absurdities like this:

$$\gamma_{X,X} \stackrel{?}{=} \text{id}_{X \otimes X}$$

A similar restriction applies to our claim that “all diagrams commute” in a monoidal category, so it is not unreasonable to say the same is true in a symmetric monoidal category.

We pause briefly to indicate an important special case of a symmetric monoidal category.

**Definition B.I.12.** A **cartesian monoidal category** is a category with products for all finite families of objects, and a **cartesian monoidal functor** is a functor between cartesian monoidal categories that preserves all finite products.

**Proposition B.I.13.**

- (i) *A category with all finite products is automatically a symmetric monoidal category, with the terminal object 1 as its monoidal unit and the cartesian product  $\times$  as the monoidal product.*

- (ii) If  $\mathcal{C}$  and  $\mathcal{D}$  are two categories with finite products regarded as symmetric monoidal categories, then every functor  $\mathcal{C} \rightarrow \mathcal{D}$  can be equipped with a canonical oplax braided monoidal functor structure.
- (iii) A cartesian monoidal functor is canonically equipped with the structure of a strong braided monoidal functor.

*Proof.* (i). The verification of the axioms is straightforward and left to the reader as an exercise.

(ii). Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The universal property of the terminal object gives a unique morphism  $\varepsilon : F1 \rightarrow 1$  in  $\mathcal{D}$ , and the universal property of binary products gives a canonical morphism  $\delta_{X,Y} : F(X \times Y) \rightarrow FX \times FY$ . It can be shown that the diagrams below commute,

$$\begin{array}{ccc}
 F(1_{\mathcal{C}} \times_{\mathcal{C}} X) & \xrightarrow{\delta_{1_{\mathcal{C}},X}} & F1_{\mathcal{C}} \times_{\mathcal{D}} FX \\
 F\lambda_X \downarrow & & \downarrow \varepsilon \times_{\mathcal{D}} \text{id}_{FX} \\
 FX & \xleftarrow{\lambda_{FX}} & 1_{\mathcal{D}} \times_{\mathcal{D}} FX
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(X \times_{\mathcal{C}} 1_{\mathcal{C}}) & \xrightarrow{\delta_{X,1_{\mathcal{C}}}} & FX \times_{\mathcal{D}} F1_{\mathcal{C}} \\
 F\rho_X \downarrow & & \downarrow \text{id}_{FX} \times_{\mathcal{D}} \varepsilon \\
 FX & \xleftarrow{\rho_{FX}} & FX \times_{\mathcal{D}} 1_{\mathcal{D}}
 \end{array}$$

$$\begin{array}{ccc}
 F((X \times_{\mathcal{C}} Y) \times_{\mathcal{C}} Z) & \xrightarrow{F\alpha_{X,Y,Z}} & F(X \times_{\mathcal{C}} (Y \times_{\mathcal{C}} Z)) \\
 \delta_{X \times_{\mathcal{C}} Y, Z} \downarrow & & \downarrow \delta_{X, Y \times_{\mathcal{C}} Z} \\
 F(X \times_{\mathcal{C}} Y) \times_{\mathcal{D}} FZ & & FX \times_{\mathcal{D}} F(Y \times_{\mathcal{C}} Z) \\
 \delta_{X,Y} \times_{\mathcal{D}} \text{id}_{FZ} \downarrow & & \downarrow \text{id}_{FX} \times_{\mathcal{D}} \delta_{Y,Z} \\
 (FX \times_{\mathcal{D}} FY) \times_{\mathcal{D}} FZ & \xrightarrow{\alpha_{FX,FY,FZ}} & FX \times_{\mathcal{D}} (FY \times_{\mathcal{D}} FZ)
 \end{array}$$

$$\begin{array}{ccc}
 F(X \times_{\mathcal{C}} Y) & \xrightarrow{\delta_{X,Y}} & FX \times_{\mathcal{D}} FY \\
 F\gamma_{X,Y} \downarrow & & \downarrow \gamma_{FX,FY} \\
 F(Y \otimes_{\mathcal{C}} X) & \xrightarrow{\delta_{Y,X}} & FY \otimes_{\mathcal{D}} FX
 \end{array}$$

making  $F$  into an oplax braided monoidal functor  $\mathcal{C} \rightarrow \mathcal{D}$ .

(iii). A functor is cartesian monoidal precisely if  $\varepsilon$  and  $\delta$  as defined above are isomorphisms.  $\diamond$

**Definition B.I.14.** Let  $Y$  and  $Z$  be objects in a monoidal category  $C$ .

- A **right internal hom object** for  $Y$  and  $Z$  is an object  $\text{Hom}(Y, Z)$  in  $C$  together with a morphism  $\text{ev}_{Y,Z} : \text{Hom}(Y, Z) \otimes Y \rightarrow Z$  having the following universal property: for all morphisms  $f : X \otimes Y \rightarrow Z$  in  $C$ , there is a unique morphism  $\tilde{f} : X \rightarrow \text{Hom}(Y, Z)$  in  $C$  such that  $\text{ev}_{Y,Z} \circ (\tilde{f} \otimes \text{id}_Y) = f$ ; equivalently,  $\text{Hom}(Y, Z)$  is an object in  $C$  equipped with bijections

$$C(X \otimes Y, Z) \cong C(X, \text{Hom}(Y, Z))$$

that are natural for each object  $X$  in  $C$ . We may also write  $[Y, Z]$  or  $Y \multimap Z$  for a right internal hom object for  $Y$  and  $Z$ .

- A **left internal hom object** for  $Y$  and  $Z$  is a right internal hom object  $Y \multimap Z$  in the reverse monoidal structure on  $C$ ; equivalently,  $Y \multimap Z$  is an object equipped with bijections

$$C(Y \otimes X, Z) \cong C(X, Y \multimap Z)$$

that are natural for each object  $X$  in  $C$ . We may also write  $Z^Y$  or  $Z \multimap Y$  for a left internal hom object for  $Y$  and  $Z$ .

- A **right-closed monoidal category** is a monoidal category that has right internal hom object for all pairs of objects.
- A **left-closed monoidal category** is a monoidal category that has left internal hom objects for all pairs of objects.
- A **biclosed monoidal category** is a monoidal category that is both left-closed and right-closed.

Note that in a symmetric monoidal category,  $Y \multimap Z$  and  $\text{Hom}(Y, Z)$  are naturally isomorphic if they exist; a **closed symmetric monoidal category** is a symmetric monoidal category that is biclosed.

**Proposition B.I.15.** *Let  $C$  be a right-closed monoidal category.*

- (i) *The assignment  $(Y, Z) \mapsto \text{Hom}(Y, Z)$  extends to a functor  $C^{\text{op}} \times C \rightarrow C$  making the bijection*

$$C(X \otimes Y, Z) \cong C(X, \text{Hom}(Y, Z))$$

*natural in  $X, Y$ , and  $Z$ .*

(ii) For each object  $Y$ , we have an adjunction

$$(-) \otimes Y \dashv \text{Hom}(Y, -) : \mathcal{C} \rightarrow \mathcal{C}$$

whose counit is  $\text{ev}_{Y,-} : \text{Hom}(Y, -) \otimes Y \Rightarrow \text{id}_{\mathcal{C}}$ .

(iii) If  $I$  is the monoidal unit of  $\mathcal{C}$ , then there is a bijection

$$\mathcal{C}(Y, Z) \cong \mathcal{C}(I, \text{Hom}(Y, Z))$$

that is natural in  $Y$  and  $Z$ .

*Proof.* (i). This is a straightforward example of an adjunction with a parameter.<sup>[4]</sup>

(ii). This is clear from the definition of  $\text{Hom}(Y, Z)$  and  $\text{ev}_{Y,-}$ .

(iii). The left unitor  $\lambda_Y : Y \xrightarrow{\cong} I \otimes Y$  induces the required bijection. ■

REMARK B.1.16. A cartesian monoidal category is a closed symmetric monoidal category if and only if it is a cartesian closed category (definition A.2.3).

## B.2 Categories with actions

**Prerequisites.** §B.1.

**Definition B.2.1.** Let  $\mathcal{V}$  be a monoidal category. A **left  $\mathcal{V}$ -action** on a category  $\mathcal{C}$  is a strong monoidal functor  $\mathcal{V} \rightarrow [\mathcal{C}, \mathcal{C}]$ , where  $[\mathcal{C}, \mathcal{C}]$  is regarded as a strict monoidal category under composition. Similarly, a **right  $\mathcal{V}$ -action** on  $\mathcal{C}$  is a strong monoidal functor  $\mathcal{V} \rightarrow [\mathcal{C}, \mathcal{C}]^{\text{rev}}$ .

REMARK B.2.2. We can unfold the above definition somewhat by taking the left exponential transpose of the strong monoidal functor  $\mathcal{V} \rightarrow [\mathcal{C}, \mathcal{C}]$ : let  $\otimes$  be the corresponding functor  $\mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ . Since the original functor was strong monoidal, we get a natural isomorphism  $\eta : \text{id}_{\mathcal{C}} \Rightarrow I \otimes (-)$  and a natural isomorphism  $\mu_{X,Y} : X \otimes (Y \otimes (-)) \Rightarrow (X \otimes Y) \otimes (-)$  for each pair of objects  $X$  and  $Y$  in  $\mathcal{V}$ ; these moreover satisfy the following coherence laws:

$$\begin{array}{ccc} X \otimes (Y \otimes (-)) & \xrightarrow{\mu_{X,Y}} & (X \otimes Y) \otimes (-) \\ f \otimes (g \otimes \text{id}) \downarrow & & \downarrow (f \otimes g) \otimes \text{id} \\ X' \otimes (Y' \otimes (-)) & \xrightarrow{\mu_{X',Y'}} & (X' \otimes Y') \otimes (-) \end{array}$$

[4] See [CWM, Ch. IV, §7].

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$$\begin{array}{ccc}
X \otimes (-) \xrightarrow{\eta^{(X \otimes (-))}} I \otimes (X \otimes (-)) & & X \otimes (-) \xrightarrow{(X \otimes (-)) \cdot \eta} X \otimes (I \otimes (-)) \\
\text{id} \downarrow & & \text{id} \downarrow \\
X \otimes (-) \xleftarrow{\lambda_X \otimes \text{id}} (I \otimes X) \otimes (-) & & X \otimes (-) \xleftarrow{\rho_X \otimes \text{id}} (X \otimes I) \otimes (-) \\
& & \downarrow \mu_{X,I} \\
& & X \otimes (I \otimes (-)) \\
& & \downarrow \mu_{I,X} \\
& & I \otimes (X \otimes (-)) \\
& & \downarrow \eta^{(X \otimes (-))} \\
& & X \otimes (-)
\end{array}$$

$$\begin{array}{ccc}
& W \otimes (X \otimes (Y \otimes (-))) & \\
\mu_{W,X} \circ (Y \otimes (-)) \swarrow & & \searrow \text{id}_W \circ \mu_{X,Y} \\
(W \otimes X) \otimes (Y \otimes (-)) & & W \otimes ((X \otimes Y) \otimes (-)) \\
\mu_{W \otimes X, Y} \downarrow & & \downarrow \mu_{W, X \otimes Y} \\
((W \otimes X) \otimes Y) \otimes (-) & \xrightarrow{\alpha_{W,X,Y} \otimes \text{id}} & (W \otimes (X \otimes Y)) \otimes (-)
\end{array}$$

Conversely, any functor  $\otimes : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$  equipped with such a collection of natural isomorphisms defines a left  $\mathcal{V}$ -action on  $\mathcal{C}$ .

**Proposition B.2.3** (Bénabou). *For any monoidal category  $\mathcal{C}$ , there is a faithful strong monoidal functor  $F : \mathcal{C} \rightarrow [\mathcal{C}, \mathcal{C}]$  defined by the following data:*

$$\begin{aligned}
FX &= X \otimes (-) \\
\eta &= \lambda^{-1} \\
(\mu_{X,Y})_Z &= \alpha_{X,Y,Z}^{-1}
\end{aligned}$$

In particular, this defines a left  $\mathcal{C}$ -action on  $\mathcal{C}$ , called the **left regular representation** of  $\mathcal{C}$ .

*Proof.*  $F$  is clearly a faithful functor. In this case, the strong monoidal functor axioms become the following diagrams:

$$\begin{array}{ccc}
X \otimes Y \xrightarrow{\lambda_{X \otimes Y}^{-1}} I \otimes (X \otimes Y) & & X \otimes Y \xrightarrow{\text{id}_Y \otimes \lambda_X^{-1}} X \otimes (I \otimes Y) \\
\text{id} \downarrow & & \text{id} \downarrow \\
X \otimes Y \xleftarrow{\lambda_X \otimes \text{id}_Y} (I \otimes X) \otimes Y & & X \otimes Y \xleftarrow{\rho_X \otimes \text{id}_Y} (X \otimes I) \otimes Y \\
& & \downarrow \alpha_{X,I,Y}^{-1} \\
& & X \otimes (I \otimes Y) \\
& & \downarrow \alpha_{I,X,Y}^{-1} \\
& & I \otimes (X \otimes Y) \\
& & \downarrow \lambda_{X \otimes Y}^{-1} \\
& & X \otimes Y
\end{array}$$

$$\begin{array}{ccc}
& W \otimes (X \otimes (Y \otimes Z)) & \\
\alpha_{W,X,Y \otimes Z}^{-1} \swarrow & & \searrow \text{id}_W \otimes \alpha_{X,Y,Z}^{-1} \\
(W \otimes X) \otimes (Y \otimes Z) & & W \otimes ((X \otimes Y) \otimes Z) \\
\alpha_{W \otimes X, Y, Z}^{-1} \downarrow & & \downarrow \alpha_{W, X \otimes Y, Z}^{-1} \\
((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{\alpha_{W,X,Y} \otimes \text{id}_Z} & (W \otimes (X \otimes Y)) \otimes Z
\end{array}$$

The left square commutes by the coherence theorem, while the right square and the pentagon are seen to be immediate consequences of the triangle and pentagon axioms, respectively. ■

**Proposition B.2.4.** *Let  $\mathcal{V}$  be a monoidal category and let  $C$  be a category.*

- *If  $\otimes : \mathcal{V} \times C \rightarrow C$  defines a left  $\mathcal{V}$ -action on  $C$  such that, for each object  $X$  in  $\mathcal{V}$ , the endofunctor  $X \otimes (-)$  has a right adjoint  $(-) \circlearrowleft X$ , then the functor  $\circlearrowleft : C \times \mathcal{V}^{\text{op}} \rightarrow C$  defines a right  $\mathcal{V}^{\text{op}}$ -action on  $C$ .*
- *If  $\otimes : C \times \mathcal{V} \rightarrow C$  defines a right  $\mathcal{V}$ -action on  $C$  such that, for each object  $X$  in  $\mathcal{V}$ , the endofunctor  $(-) \otimes X$  has a right adjoint  $X \multimap (-)$ , then the functor  $\multimap : \mathcal{V}^{\text{op}} \times C \rightarrow C$  defines a left  $\mathcal{V}^{\text{op}}$ -action on  $C$ .*
- *If  $\circlearrowleft : C \times \mathcal{V}^{\text{op}} \rightarrow C$  defines a right  $\mathcal{V}^{\text{op}}$ -action on  $C$  such that, for each object  $X$  in  $\mathcal{V}$ , the endofunctor  $X \circlearrowleft (-)$  has a left adjoint  $X \otimes (-)$ , then the functor  $\otimes : \mathcal{V} \times C \rightarrow C$  defines a left  $\mathcal{V}$ -action on  $C$ .*
- *If  $\multimap : \mathcal{V}^{\text{op}} \times C \rightarrow C$  defines a left  $\mathcal{V}^{\text{op}}$ -action on  $C$  such that, for each object  $X$  in  $\mathcal{V}$ , the endofunctor  $X \multimap (-)$  has a left adjoint  $(-) \otimes X$ , then the functor  $\otimes : C \times \mathcal{V} \rightarrow C$  defines a right  $\mathcal{V}$ -action on  $C$ .*

*Proof.* The four statements are related by applying  $(-)^{\text{op}}$  and  $(-)^{\text{rev}}$  at the appropriate points, so it suffices to prove the first claim.

First, note that  $\circlearrowleft$  is indeed a functor  $C \times \mathcal{V}^{\text{op}} \rightarrow C$ , by the parameter theorem for adjunctions.<sup>[5]</sup> Let  $\text{ev}_{X,A} : X \otimes (A \circlearrowleft X) \rightarrow A$  denote the component of the counit of the adjunction  $X \otimes (-) \dashv (-) \circlearrowleft X$  at an object  $A$  in  $C$ . For each pair of objects  $X$  and  $Y$  in  $\mathcal{V}$  and each object  $A$  in  $C$ , we define the morphism  $(\delta_{X,Y})_A : A \circlearrowleft (X \otimes Y) \rightarrow (A \circlearrowleft X) \circlearrowleft Y$  to be the right adjoint transpose of  $\text{ev}_{X \otimes Y, A} \circ (\mu_{X,Y})_{(A \circlearrowleft X) \circlearrowleft Y}$ , and for each  $A$ , we define  $\varepsilon_A : A \circlearrowleft I \rightarrow A$  to be the composite  $\text{ev}_{I,A} \circ \eta_{A \circlearrowleft I}$ . These are clearly natural in  $A$ , and it is straightforward to check that  $\delta_{X,Y}$  is also natural in  $X$  and  $Y$ . One may then use the calculus of mates to show that  $\varepsilon$  and  $\delta_{X,Y}$  are natural isomorphisms and that they satisfy the axioms for making the right exponential transpose of  $\circlearrowleft : C \times \mathcal{V}^{\text{op}} \rightarrow C$  into a strong monoidal functor  $\mathcal{V}^{\text{op}} \rightarrow [C, C]^{\text{rev}}$ , i.e. a right  $\mathcal{V}^{\text{op}}$ -action on  $C$ . ■

**Example B.2.5.**  $\mathcal{V}$  is a left-closed (resp. right-closed) monoidal category if and only if the left (resp. right) self-action of  $\mathcal{V}$  has a parametrised right adjoint as

[5] See [CWM, Ch. IV, §7].

in the proposition, and the right adjoint right (resp. left)  $\mathcal{V}^{\text{op}}$ -action so obtained is precisely a left (resp. right) internal hom functor.

**Definition B.2.6.** Let  $\mathcal{V}$  be a monoidal category and let  $\mathcal{C}$  be a category.

- A **right  $\mathcal{V}$ -hom system** for  $\mathcal{C}$  consists of a left  $\mathcal{V}$ -action  $\otimes : \mathcal{V} \times \mathcal{C} \rightarrow \mathcal{C}$ , a functor  $\underline{C} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ , and a right  $\mathcal{V}^{\text{op}}$ -action  $\circ - : \mathcal{C} \times \mathcal{V}^{\text{op}} \rightarrow \mathcal{V}$  together with natural bijections of the types below,

$$\begin{aligned}\mathcal{V}(X, \underline{C}(A, B)) &\cong \mathcal{C}(A, B \circ - X) \\ \mathcal{C}(X \otimes A, B) &\cong \mathcal{C}(A, B \circ - X) \\ \mathcal{C}(X \otimes A, B) &\cong \mathcal{V}(X, \underline{C}(A, B))\end{aligned}$$

where  $X$  varies over the objects in  $\mathcal{V}$ , and  $A$  and  $B$  vary over the objects in  $\mathcal{C}$ , such that the cyclic composition of the three bijections is the identity.

- A **left  $\mathcal{V}$ -hom system** for  $\mathcal{C}$  consists of a right  $\mathcal{V}$ -action  $\otimes : \mathcal{C} \times \mathcal{V} \rightarrow \mathcal{C}$ , a functor  $\underline{C} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ , and a left  $\mathcal{V}^{\text{op}}$ -action  $- \circ : \mathcal{V}^{\text{op}} \times \mathcal{C} \rightarrow \mathcal{V}$ , together with natural bijections of the types below,

$$\begin{aligned}\mathcal{V}(X, \underline{C}(A, B)) &\cong \mathcal{C}(A, X \circ - B) \\ \mathcal{C}(A \otimes X, B) &\cong \mathcal{C}(A, X \circ - B) \\ \mathcal{C}(A \otimes X, B) &\cong \mathcal{V}(X, \underline{C}(A, B))\end{aligned}$$

where  $X$  varies over the objects in  $\mathcal{V}$ , and  $A$  and  $B$  vary over the objects in  $\mathcal{C}$ , such that the cyclic composition of the three bijections is the identity.

**Example B.2.7.** If  $\mathcal{V}$  is a biclosed monoidal category with right internal hom functor  $\text{Hom}$  and left internal hom functor  $\pitchfork$ , then  $(\otimes, \pitchfork, \text{Hom})$  is a left  $\mathcal{V}$ -hom system for  $\mathcal{V}$ :

$$\begin{aligned}\mathcal{V}(Y, X \pitchfork Z) &\cong \mathcal{V}(X, \text{Hom}(Y, Z)) \\ \mathcal{V}(X \otimes Y, Z) &\cong \mathcal{V}(X, \text{Hom}(Y, Z)) \\ \mathcal{V}(X \otimes Y, Z) &\cong \mathcal{V}(Y, X \pitchfork Z)\end{aligned}$$

**Example B.2.8.** If  $\mathcal{C}$  is a locally small category that has products and coproducts for all small families of objects, then  $\mathcal{C}$  admits a left **Set**-action and a right **Set**<sup>op</sup>-action that are related by the following adjunctions:

$$\mathbf{Set}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, B \circ - X)$$



$$\begin{aligned} C(X \otimes A, B) &\cong C(A, B \circlearrowleft X) \\ C(X \otimes A, B) &\cong \mathbf{Set}(X, C(A, B)) \end{aligned}$$

(The adjointness claim was checked in [proposition A.6.12](#), and the coherence laws are straightforwardly verified.) Thus,  $(\otimes, C, \circlearrowleft)$  is a right **Set**-hom system for  $C$ .

**Theorem B.2.9.** *Let  $\mathcal{V}$  be a monoidal category and let  $C$  be a category.*

- (i) *If  $\otimes$  is a left  $\mathcal{V}$ -action on  $C$  and  $\underline{C} : C^{\text{op}} \times C \rightarrow C$  is a functor with natural bijections of the form below,*

$$C(X \otimes A, B) \cong \mathcal{V}(X, \underline{C}(A, B))$$

*then  $\underline{C}$  is the hom functor of a  $\mathcal{V}$ -enriched category  $\underline{C}$  whose underlying ordinary category is isomorphic to  $C$ .*

- (ii) *If  $\circlearrowleft$  is a right  $\mathcal{V}^{\text{op}}$ -action on  $C$  and  $\underline{C} : C^{\text{op}} \times C \rightarrow C$  is a functor with natural bijections of the form below,*

$$C(A, B \circlearrowleft X) \cong \mathcal{V}(X, \underline{C}(A, B))$$

*then  $\underline{C}$  is the hom functor of a  $\mathcal{V}$ -enriched category  $\underline{C}$  whose underlying ordinary category is isomorphic to  $C$ .*

*Proof.* (i). The natural isomorphism  $A \cong I \otimes A$  induces a family of bijections

$$C(A, B) \cong \mathcal{V}(I, \underline{C}(A, B))$$

natural in  $A$  and  $B$ , so we have a morphism  $e_A : I \rightarrow \underline{C}(A, A)$  in  $\mathcal{V}$  for every object  $A$  in  $C$  corresponding to  $\text{id}_A : A \rightarrow A$  in  $C$ . Let  $\text{ev}_{A,B} : \underline{C}(A, B) \otimes A \rightarrow B$  be the component at  $B$  of the counit of the adjunction  $(-) \otimes A \dashv \underline{C}(A, -)$ , and define  $c_{A,B,C} : \underline{C}(B, C) \otimes \underline{C}(A, B) \rightarrow \underline{C}(A, C)$  to be the right adjoint transpose of the following morphism in  $C$ :

$$\text{ev}_{B,C} \circ (\text{id}_{\underline{C}(B,C)} \otimes \text{ev}_{A,B}) \circ (\mu_{\underline{C}(B,C), \underline{C}(A,B)})_A^{-1} : (\underline{C}(B, C) \otimes \underline{C}(A, B)) \otimes A \rightarrow C$$

By definition, the left adjoint transpose of  $e_B$  is  $\eta_B^{-1}$ , so the left and right unit axioms are satisfied:

$$c_{A,B,B} \circ (e_B \otimes \text{id}_{\underline{C}(A,B)}) = \lambda_{\underline{C}(A,B)}$$

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---

$$c_{B,B,C} \circ (\text{id}_{\underline{C}(B,C)} \otimes e_B) = \rho_{\underline{C}(B,C)}$$

One may similarly verify the associativity axiom:

$$c_{A,B,D} \circ (c_{B,C,D} \otimes \text{id}_{\underline{C}(A,B)}) = c_{A,C,D} \circ (\text{id}_{\underline{C}(C,D)} \otimes c_{A,B,C}) \circ \alpha_{\underline{C}(C,D), \underline{C}(B,C), \underline{C}(A,B)}$$

(ii). By duality and symmetry,  $\circlearrowleft$  induces a left  $\mathcal{V}^{\text{rev}}$ -action on  $C^{\text{op}}$ , so we may construct a  $\mathcal{V}^{\text{rev}}$ -enriched category  $\underline{C}^{\text{op}}$  using claim (i) and thence a  $\mathcal{V}$ -enriched category  $\underline{C} = (\underline{C}^{\text{op}})^{\text{op}}$ .  $\blacksquare$

**Definition B.2.10.** Let  $\mathcal{V}$  be a monoidal category, and let  $C$  and  $D$  be categories with left  $\mathcal{V}$ -actions. A  $\mathcal{V}$ -**strength** for a functor  $F : C \rightarrow D$  is a natural transformation  $\sigma : (-) \otimes F(-) \Rightarrow F(- \otimes -)$  making these diagrams commute:

$$\begin{array}{ccc} & FA & \\ \eta_{FA} \swarrow & & \searrow F\eta_A \\ I \otimes FA & \xrightarrow{\sigma_{I,A}} & F(I \otimes A) \end{array}$$

$$\begin{array}{ccc} & X \otimes (Y \otimes FA) & \\ \text{id}_X \otimes \sigma_{Y,A} \swarrow & & \searrow (\mu_{X,Y})_A \\ X \otimes F(Y \otimes A) & & (X \otimes Y) \otimes FA \\ \sigma_{X,Y \otimes A} \downarrow & & \downarrow \sigma_{X \otimes Y, A} \\ F(X \otimes (Y \otimes A)) & \xrightarrow{F(\mu_{X,Y})_A} & F((X \otimes Y) \otimes A) \end{array}$$

A  $\mathcal{V}$ -**strong functor** is a functor equipped with a  $\mathcal{V}$ -strength.

**Definition B.2.11.** Let  $\mathcal{V}$  be a monoidal category, let  $C$  and  $D$  be categories with left  $\mathcal{V}$ -actions, and let  $F, F' : C \rightarrow D$  be functors with  $\mathcal{V}$ -strengths  $\sigma$  and  $\sigma'$  respectively. A  $\mathcal{V}$ -**strong natural transformation**  $\varphi : F \Rightarrow F'$  is a natural transformation making the following diagram commute:

$$\begin{array}{ccc} X \otimes FA & \xrightarrow{\sigma_{X,A}} & F(X \otimes A) \\ \text{id}_X \otimes \varphi_A \downarrow & & \downarrow \varphi_{X \otimes A} \\ X \otimes F'A & \xrightarrow{\sigma'_{X,A}} & F'(X \otimes A) \end{array}$$

## B.3 Indexed categories

**Prerequisites.** §§ A.1, A.2.

**Definition B.3.1.** Let  $\mathcal{B}$  be a category. A  $\mathcal{B}$ -indexed category  $\mathbb{E}$  consists of the following data:

- For each object  $B$  in  $\mathcal{B}$ , a category  $\mathcal{E}^B$ , called the **fibre** of  $\mathbb{E}$  over  $B$ .
- For each morphism  $f : B' \rightarrow B$  in  $\mathcal{B}$ , a functor  $f^* : \mathcal{E}^B \rightarrow \mathcal{E}^{B'}$ , called the **reindexing functor** along  $f : B' \rightarrow B$ .
- For each object  $B$  in  $\mathcal{B}$ , a natural isomorphism  $\eta_B : \text{id}_{\mathcal{E}^B} \Rightarrow (\text{id}_B)^*$ .
- For each commutative triangle in  $\mathcal{B}$  of the form below,

$$\begin{array}{ccc} B'' & & \\ & \searrow^{f \circ f'} & \\ & f' \searrow & \searrow f \\ & B' & \longrightarrow B \end{array}$$

a natural isomorphism  $\mu_{f,f'} : f'^* f^* \Rightarrow (f \circ f')^*$ .

These data are moreover required to satisfy the equations shown below:

$$\begin{aligned} \mu_{f, \text{id}_{B'}} \bullet \eta_{B'} f^* &= \text{id}_{f^*} \\ \mu_{\text{id}_B, f} \bullet f^* \eta_B &= \text{id}_{f^*} \\ \mu_{f \circ f', f''} \bullet f''^* \mu_{f, f'} &= \mu_{f, f' \circ f''} \bullet \mu_{f', f''} f^* \end{aligned}$$

**Definition B.3.2.** Let  $\mathcal{B}$  be a category and let  $\mathbb{D}$  and  $\mathbb{E}$  be  $\mathcal{B}$ -indexed categories. An **oplax  $\mathcal{B}$ -indexed functor**  $F : \mathbb{D} \rightarrow \mathbb{E}$  consists of the following data:

- For each object  $B$  in  $\mathcal{B}$ , a functor  $F^B : \mathcal{D}^B \rightarrow \mathcal{E}^B$ .
- For each morphism  $f : B' \rightarrow B$  in  $\mathcal{B}$ , a natural transformation  $\theta_f$  as in the diagram below:

$$\begin{array}{ccc} \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\ f^* \downarrow & \theta_f \nearrow & \downarrow f^* \\ \mathcal{D}^{B'} & \xrightarrow{F^{B'}} & \mathcal{E}^{B'} \end{array}$$

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These data are required to satisfy the equations shown below:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \text{id} \downarrow & \eta_B \nearrow & \downarrow \theta_{\text{id}} \\
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \text{id} \downarrow & \eta_B \nearrow & \downarrow (\text{id}_B)^*
 \end{array} & = & \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \text{id} \downarrow & \eta_B \nearrow & \downarrow (\text{id}_B)^* \\
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \text{id} \downarrow & \eta_B \nearrow & \downarrow (\text{id}_B)^*
 \end{array}
 \end{array}$$
  

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \downarrow \theta_f & \nearrow \mu_{f,f'} & \downarrow f^* \\
 \mathcal{D}^{B'} & \xrightarrow{F^{B'}} & \mathcal{E}^{B'} \\
 \downarrow \theta_{f'} & \nearrow \mu_{f',f''} & \downarrow f'^* \\
 \mathcal{D}^{B''} & \xrightarrow{F^{B''}} & \mathcal{E}^{B''}
 \end{array} & = & \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \downarrow \theta_f & \nearrow \mu_{f,f'} & \downarrow f^* \\
 \mathcal{D}^{B'} & \xrightarrow{F^{B'}} & \mathcal{E}^{B'} \\
 \downarrow \theta_{f'} & \nearrow \mu_{f',f''} & \downarrow f'^* \\
 \mathcal{D}^{B''} & \xrightarrow{F^{B''}} & \mathcal{E}^{B''}
 \end{array}
 \end{array}$$

A  **$\mathcal{B}$ -indexed functor** is an oplax  $\mathcal{B}$ -indexed functor such that the natural transformations  $\theta_f$  are natural isomorphisms.

**Definition B.3.3.** Let  $\mathcal{B}$  be a category and let  $F, G : \mathbb{D} \rightarrow \mathbb{E}$  be a parallel pair of  $\mathcal{B}$ -indexed functors. A  **$\mathcal{B}$ -indexed natural transformation**  $\alpha : F \Rightarrow G$  consists of a natural transformation  $\alpha_B : F^B \Rightarrow G^B$  for each object  $B$  in  $\mathcal{B}$ , such that the following equation holds:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{G^B} & \mathcal{E}^B \\
 f^* \downarrow & \theta_f \nearrow & \downarrow f^* \\
 \mathcal{D}^{B'} & \xrightarrow{G^{B'}} & \mathcal{E}^{B'} \\
 \parallel & \alpha_{B'} \nearrow & \parallel \\
 \mathcal{D}^{B'} & \xrightarrow{F^{B'}} & \mathcal{E}^{B'}
 \end{array} & = & \begin{array}{ccc}
 \mathcal{D}^B & \xrightarrow{F^B} & \mathcal{E}^B \\
 \parallel & \alpha_B \nearrow & \parallel \\
 \mathcal{D}^B & \xrightarrow{G^B} & \mathcal{E}^B \\
 f^* \downarrow & \theta_f \nearrow & \downarrow f^* \\
 \mathcal{D}^{B'} & \xrightarrow{G^{B'}} & \mathcal{E}^{B'}
 \end{array}
 \end{array}$$

REMARK B.3.4. In other words, a  $\mathcal{B}$ -indexed category is a contravariant pseudo-functor from the category  $\mathcal{B}$  to the meta-2-category of all (not necessarily small) categories, a  $\mathcal{B}$ -indexed functor is a pseudonatural transformation between such pseudofunctors, and in turn, a  $\mathcal{B}$ -indexed natural transformation is a modification between such pseudonatural transformations.

**Example B.3.5.** Any contravariant (strict) functor from  $\mathcal{B}$  to the meta-category of all (not necessarily small) categories is a  $\mathcal{B}$ -indexed category in a trivial way. In particular, every presheaf on  $\mathcal{B}$  can be regarded as a  $\mathcal{B}$ -indexed category.

¶ B.3.6. Given two  $\mathcal{B}$ -indexed categories, say  $\mathbb{D}$  and  $\mathbb{E}$ , we may form a category  $[\mathbb{D}, \mathbb{E}]$  whose objects are  $\mathcal{B}$ -indexed functors  $\mathbb{D} \rightarrow \mathbb{E}$  and whose morphisms are  $\mathcal{B}$ -indexed natural transformations. Of course, these are the hom-categories of the evident meta-2-category of  $\mathcal{B}$ -indexed categories,  $\mathcal{B}$ -indexed functors, and  $\mathcal{B}$ -indexed natural transformations.

**Lemma B.3.7.** *Let  $\mathcal{B}$  be a category, let  $B$  be an object in  $\mathcal{B}$ , and let  $\mathbb{E}$  be a  $\mathcal{B}$ -indexed category. Then the following functor is (half of) an equivalence of categories,*

$$\begin{aligned} [\mathfrak{h}_B, \mathbb{E}] &\rightarrow \mathcal{E}^B \\ F &\mapsto F^B(\text{id}_B) \end{aligned}$$

where  $\mathfrak{h}_B$  is the representable presheaf on  $\mathcal{B}$  regarded as a  $\mathcal{B}$ -indexed category.

*Proof.* First, we show that the given functor is fully faithful. Let  $F, G : \mathfrak{h}_B \rightarrow \mathbb{E}$  be two  $\mathcal{B}$ -indexed functors, and let  $\alpha : F \Rightarrow G$  be a  $\mathcal{B}$ -indexed natural transformation. Then, for any morphism  $f : B' \rightarrow B$  in  $\mathcal{B}$ , we have the following commutative diagram:

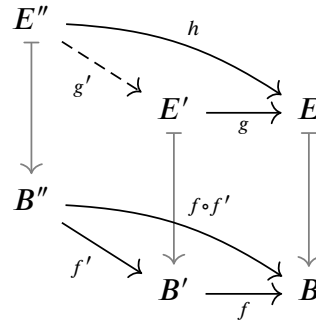
$$\begin{array}{ccc} F^{B'}(f) & \xrightarrow{(\alpha_{B'})_f} & G^{B'}(f) \\ (\theta_f)_{\text{id}} \downarrow & & \downarrow (\theta_f)_{\text{id}} \\ f^* F^B(\text{id}_B) & \xrightarrow{f^*((\alpha_B)_{\text{id}})} & f^* G^B(\text{id}_B) \end{array}$$

Since  $\theta_f : F^{B'} f^* \Rightarrow f^* F^B$  is a natural isomorphism, we may deduce that all the components of  $\alpha$  are uniquely determined by  $(\alpha_B)_{\text{id}} : F^B(\text{id}_B) \rightarrow G^B(\text{id}_B)$ . Conversely, given any morphism  $g : F^B(\text{id}_B) \rightarrow G^B(\text{id}_B)$ , we may define a  $\mathcal{B}$ -indexed natural transformation  $\alpha : F \Rightarrow G$  such that  $(\alpha_B)_{\text{id}} = g$ : the components of  $\alpha$  are determined as above, and it is straightforward to check that the various axioms are satisfied.

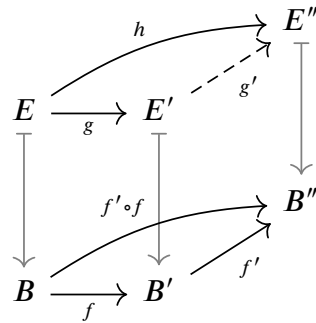
It now suffices to show that the given functor  $[\mathfrak{h}_B, \mathbb{E}] \rightarrow \mathcal{E}^B$  is essentially surjective on objects. Let  $E$  be an object in  $\mathcal{E}^B$ . We define a  $\mathcal{B}$ -indexed functor  $F : \mathfrak{h}_B \rightarrow \mathbb{E}$  as follows: given a morphism  $f : B' \rightarrow B$  in  $\mathcal{B}$ , set  $F^{B'}(f) = f^* E$ , and given another morphism  $f' : B'' \rightarrow B'$  in  $\mathcal{B}$ , set  $(\theta_{f'})_f = (\mu_{f, f'})_E^{-1}$ . It is easy to verify that these data indeed constitute a  $\mathcal{B}$ -indexed functor, and by construction, we have a canonical isomorphism  $E \rightarrow F^B(\text{id}_B)$ , namely  $(\eta_B)_E : E \rightarrow (\text{id}_B)^* E$ . ■

**Definition B.3.8.** Let  $P : \mathcal{E} \rightarrow \mathcal{B}$  be a functor.

- A  **$P$ -prone morphism** (or  **$P$ -cartesian morphism**) is a morphism  $g : E' \rightarrow E$  in  $\mathcal{E}$  with the following property: if  $f : B' \rightarrow B$  is the image of  $g$  under  $P$ , and  $h : E'' \rightarrow E$  is any morphism in  $\mathcal{E}$  such that  $Ph = f \circ f'$  for some  $f' : B'' \rightarrow B'$  in  $\mathcal{B}$ , then there is a unique morphism  $g' : E'' \rightarrow E'$  in  $\mathcal{E}$  such that  $Pg' = f'$  and  $h = g \circ g'$ :



- A  **$P$ -supine morphism** (or  **$P$ -opcartesian morphism**) is a morphism  $g : E \rightarrow E'$  in  $\mathcal{E}$  with the following property: if  $f : B \rightarrow B'$  is the image of  $g$  under  $P$ , and  $h : E \rightarrow E''$  is any morphism in  $\mathcal{E}$  such that  $Ph = f' \circ f$  for some  $f' : B' \rightarrow B''$  in  $\mathcal{B}$ , then there is a unique morphism  $g' : E \rightarrow E''$  in  $\mathcal{E}$  such that  $Pg' = f'$  and  $h = g' \circ g$ :



- A  **$P$ -vertical morphism** is a morphism in  $\mathcal{E}$  whose image in  $\mathcal{B}$  is an identity morphism.
- The **fibre** of  $P$  over an object  $B$  in  $\mathcal{B}$  is the (non-full) subcategory  $\mathcal{E}^B \subseteq \mathcal{E}$  whose objects are those  $E$  such that  $PE = B$  and whose morphisms are the  $P$ -vertical morphisms.

**Example B.3.9.** Let  $\mathcal{B}$  be a category, let  $[2, \mathcal{B}]$  be the arrow category, and let  $P : [2, \mathcal{B}] \rightarrow \mathcal{B}$  be the evident functor that sends an object in  $[2, \mathcal{B}]$  to its codomain (considered as a morphism in  $\mathcal{B}$ ). Then a morphism in  $[2, \mathcal{B}]$  is  $P$ -prone if and only if it is a pullback square in  $\mathcal{B}$ . This is the reason why  $P$ -prone morphisms are often called  $P$ -cartesian. However, a morphism in  $[2, \mathcal{B}]$  is  $P$ -supine if and only if the top arrow is an isomorphism in  $\mathcal{B}$ —not pushout, as one might expect!

**Lemma B.3.10.** *Let  $P : \mathcal{E} \rightarrow \mathcal{B}$  be a functor. If both  $g : E' \rightarrow E$  and  $h : E'' \rightarrow E$  are  $P$ -prone morphisms with  $Pg = Pg'$ , then there is a unique vertical morphism  $g' : E'' \rightarrow E'$  such that  $h = g \circ g'$ , and it is an isomorphism in  $\mathcal{E}$ .*

*Proof.* This is a straightforward exercise. ◇

**Definition B.3.11.**

- A **Grothendieck fibration** is a functor  $P : \mathcal{E} \rightarrow \mathcal{B}$  with the following lifting property: for every object  $E$  in  $\mathcal{E}$  and every morphism  $f : B' \rightarrow PE$  in  $\mathcal{B}$ , there exists a  $P$ -prone morphism  $g : f^*E \rightarrow E$  in  $\mathcal{E}$  with  $Pg = f$ .
- A **Grothendieck opfibration** is a functor  $P : \mathcal{E} \rightarrow \mathcal{B}$  with the following lifting property: for every object  $E$  in  $\mathcal{E}$  and every morphism  $f : PE \rightarrow B'$  in  $\mathcal{B}$ , there exists a  $P$ -supine morphism  $g : E \rightarrow f_*E$  in  $\mathcal{E}$  with  $Pg = f$ .





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