Notes on homotopical algebra

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PREFACE

These notes are intended as a kind of annotated index to the various standard references in homotopical algebra: the focus is on definitions and statements of results, *not* proofs.

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FOUNDATIONS

0.1 Set theory

In category theory it is often convenient to invoke a certain set-theoretic device commonly known as a 'Grothendieck universe', but we shall say simply 'universe', so as to simplify exposition and proofs by eliminating various circumlocutions involving cardinal bounds, proper classes etc.

Definition 0.1.1. A **pre-universe** is a set $\mathbf{U}$ satisfying these axioms:

1. If $x \in y$ and $y \in \mathbf{U}$, then $x \in \mathbf{U}$.
2. If $x \in \mathbf{U}$ and $y \in \mathbf{U}$ (but not necessarily distinct), then $\{x, y\} \in \mathbf{U}$.
3. If $x \in \mathbf{U}$, then $\mathscr{P}(x) \in \mathbf{U}$, where $\mathscr{P}(x)$ denotes the set of all subsets of x.
4. If $x \in \mathbf{U}$ and $f : x \to \mathbf{U}$ is a map, then $\bigcup_{i \in x} f(i) \in \mathbf{U}$.

A **universe** is a pre-universe $\mathbf{U}$ with this additional property:

5. $\omega \in \mathbf{U}$, where ω is the set of all finite (von Neumann) ordinals.

Example 0.1.2. The empty set is a pre-universe, and with very mild assumptions, so is the set **HF** of all hereditarily finite sets.

¶ 0.1.3. The notion of universe makes sense in any material set theory, but their existence must be postulated. We adopt the following:

- **Grothendieck–Verdier universe axiom.** For each set x, there exists a universe $\mathbf{U}$ with $x \in \mathbf{U}$.

For definiteness, we may take our base theory to be Mac Lane set theory, which is a weak subsystem of Zermelo–Fraenkel set theory with choice (ZFC). Readers interested in the details of Mac Lane set theory are referred to [Mathias, 2001], but in practice, as long as one is working at all times *inside some universe*, one may as well be working in ZFC. Indeed:

Proposition 0.1.4. *With the assumptions of Mac Lane set theory, any universe is a transitive model of ZFC.*

Proof. Let $\mathbf{U}$ be a universe. By definition, $\mathbf{U}$ is a transitive set containing pairs, power sets, unions, and ω, so the axioms of extensionality, empty set, pairs, power sets, unions, choice, and infinity are all automatically satisfied. We must show that the axiom schemas of separation and replacement are also satisfied, and in fact it is enough to check that replacement is valid; but this is straightforward using axioms 2 and 4. ■

Definition 0.1.5. Let $\mathbf{U}$ be a pre-universe. A **U-set** is a member of $\mathbf{U}$, a **U-class** is a subset of $\mathbf{U}$, and a **proper U-class** is a $\mathbf{U}$-class that is not a $\mathbf{U}$-set.

Lemma 0.1.6. *A* $\mathbf{U}$*-class* X *is a* $\mathbf{U}$*-set if and only if there exists a* $\mathbf{U}$*-class* Y *such that* $X \in Y$. ■

Proposition 0.1.7. *If* $\mathbf{U}$ *is a universe in Mac Lane set theory, then the collection of all* $\mathbf{U}$*-classes is a transitive model of Morse–Kelley class–set theory (MK), and so is a transitive model of von Neumann–Bernays–Gödel class–set theory (NBG) in particular.* ■

Definition 0.1.8. A **U-small category** is a category $\mathbb{C}$ such that $\operatorname{ob}\mathbb{C}$ and $\operatorname{mor}\mathbb{C}$ are $\mathbf{U}$-sets. A **locally U-small category** is a category $\mathcal{D}$ satisfying these conditions:

- $\operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{D}$ are $\mathbf{U}$-classes, and
- for all objects x and y in $\mathcal{D}$, the hom-set $\mathcal{D}(x, y)$ is a $\mathbf{U}$-set.

An **essentially U-small category** is a category $\mathcal{D}$ for which there exist a $\mathbf{U}$-small category $\mathbb{C}$ and a functor $\mathbb{C} \to \mathcal{D}$ that is fully faithful and essentially surjective on objects.

Proposition 0.1.9. *If* $\mathbb{D}$ *is a* $\mathbf{U}$*-small category and* $\mathcal{C}$ *is a locally* $\mathbf{U}$*-small category, then the functor category* $[\mathbb{D}, \mathcal{C}]$ *is locally* $\mathbf{U}$*-small.*

Proof. Strictly speaking, this depends on the set-theoretic implementation of ordered pairs, categories, functors, etc., but at the very least $[\mathbb{D}, \mathcal{C}]$ should be isomorphic to a locally **U**-small category.

In the context of $[\mathbb{D}, \mathcal{C}]$, we may regard functors $\mathbb{D} \to \mathcal{C}$ as being the pair consisting of the *graph* of the object map $\operatorname{ob} \mathbb{D} \to \operatorname{ob} \mathcal{C}$ and the *graph* of the morphism map $\operatorname{mor} \mathbb{D} \to \operatorname{mor} \mathcal{C}$, and these are **U**-sets by the **U**-replacement axiom. Similarly, if F and G are objects in $[\mathbb{D}, \mathcal{C}]$, then we may regard a natural transformation $\alpha : F \Rightarrow G$ as being the triple (F, G, A), where A is the set of all pairs (c, α_c). ■

One complication introduced by having multiple universes concerns the existence of (co)limits.

Theorem 0.1.10 (Freyd)**.** *Let $\mathcal{C}$ be a category and let κ be a cardinal such that $|\operatorname{mor} \mathcal{C}| \leq \kappa$. If $\mathcal{C}$ has products for families of size κ, then any two parallel morphisms in $\mathcal{C}$ must be equal.*

Proof. Suppose, for a contradiction, that $f, g : X \to Y$ are distinct morphisms in $\mathcal{C}$. Let Z be the product of κ-many copies of Y in $\mathcal{C}$. The universal property of products implies there are at least 2^κ-many distinct morphisms $X \to Z$; but $\mathcal{C}(X, Z) \subseteq \operatorname{mor} \mathcal{C}$, so this is an absurdity. ■

Definition 0.1.11. Let **U** be a pre-universe. A **U-complete** (resp. **U-cocomplete**) **category** is a category $\mathcal{C}$ with the following property:

- For all **U**-small categories $\mathbb{D}$ and all diagrams $A : \mathbb{D} \to \mathcal{C}$, a limit (resp. colimit) of A exists in $\mathcal{C}$.

We may instead say $\mathcal{C}$ has all **finite limits** (resp. **finite colimits**) in the special case $\mathbf{U} = \mathbf{HF}$.

Proposition 0.1.12. *Let $\mathcal{C}$ be a category and let* **U** *be a non-empty pre-universe. The following are equivalent:*

(i) *$\mathcal{C}$ is* **U**-*complete.*

(ii) *$\mathcal{C}$ has all finite limits and products for all families of objects indexed by a* **U**-*set.*

(iii) *For each* **U***-small category* $\mathbb{D}$*, there exists an adjunction*

$$\Delta \dashv \varprojlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$$

where ΔX *is the constant functor with value* X*.*

Dually, the following are equivalent:

(i′) $\mathcal{C}$ *is* **U***-cocomplete.*

(ii′) $\mathcal{C}$ *has all finite colimits and coproducts for all families of objects indexed by a* **U***-set.*

(iii′) *For each* **U***-small category* $\mathbb{D}$*, there exists an adjunction*

$$\varinjlim_{\mathbb{D}} \dashv \Delta : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$$

where ΔX *is the constant functor with value* X*.*

Proof. This is a standard result; but we remark that we do require a sufficiently powerful form of the axiom of choice to pass from (ii) to (iii). □

¶ 0.1.13. In the **explicit universe convention**, the words ‘set’, ‘class’, etc. have their usual meanings, and in the **one-universe convention**, these instead abbreviate ‘**U**-set’, ‘**U**-class’, etc. for a fixed (but arbitrary) universe **U**. However, the word ‘category’ always refers to a category that is contained in *some* universe, which may or may not be locally **U**-small, and we shall use the word ‘ensemble’ to refer to sets which may or may not be in **U**. In subsequent chapters, the implicit universe convention should be assumed *unless otherwise stated*.

We now recall some definitions and results about ordinal and cardinal numbers. Readers familiar with axiomatic set theory may wish to skip ahead.

Definition 0.1.14. A **von Neumann ordinal** is a set α with the following properties:

- If $x \in y$ and $y \in \alpha$, then $x \in \alpha$.
- The binary relation $\in$ is strict total ordering of α.
- If S is a subset of α such that
  - $\varnothing \in S$,

- If $\beta \in S$ and $\beta \cup \{\beta\} \in \alpha$, then $\beta \cup \{\beta\} \in S$.
- If $T \subseteq S$, then $\bigcup T \in S$.

then $S = \alpha$.

We identify 0 with the von Neumann ordinal $\emptyset$, and by induction, we identify the natural number $n + 1$ with the von Neumann ordinal $\{0, \dots, n\}$.

Proposition 0.1.15.

(i) *If α is a von Neumann ordinal, then every member of α is an initial segment of α and is in particular a von Neumann ordinal.*

(ii) *If α is a von Neumann ordinal, so is $\alpha \cup \{\alpha\}$. (This is usually denoted by $\alpha + 1$ and called the* **successor** *of α.)*

(iii) *The union of a set S of von Neumann ordinals is another von Neumann ordinal. (This is usually denoted by* $\sup S$ *and called the* **supremum** *of S.)*

(iv) *If $\mathbf{U}$ is a pre-universe and $\kappa(\mathbf{U})$ is the set of von Neumann ordinals in $\mathbf{U}$, then $\kappa(\mathbf{U})$ a von Neumann ordinal, but $\kappa(\mathbf{U}) \notin \mathbf{U}$.*

Proof. Claims (i) – (iii) are all easy, and claim (iv) is Burali-Forti's paradox. ◆

Theorem 0.1.16 (Classification of well-orderings)**.**

(i) *In Zermelo–Fraenkel set theory, every well-ordered set is isomorphic to a unique von Neumann ordinal.*

(ii) *In Mac Lane set theory, if $\mathbf{U}$ is a pre-universe and X is a well-ordered set in $\mathbf{U}$, then X is isomorphic to a unique von Neumann ordinal in $\mathbf{U}$.*

Proof. Claim (i) is a standard result in axiomatic set theory, and claim (ii) is an obvious corollary. □

Definition 0.1.17. A **transitive set** is a set T such that, given $x \in y$, if $y \in T$, then $x \in T$ as well. The **transitive closure** of a set X is a set $\operatorname{tcl}(X)$ such that, for all transitive sets T with $X \subseteq T$, we have $\operatorname{tcl}(X) \subseteq T$ as well.

Lemma 0.1.18. *In Mac Lane set theory, every set has a unique transitive closure.*

Proof. One of the axioms of Mac Lane set theory states that every set X is a member of some transitive set T, and so $X \subseteq T$. Clearly, the intersection of any family of transitive sets containing X is again a transitive set containing X, so $\operatorname{tcl}(X)$ exists and is unique so long as there is at least one transitive set containing X. ■

Definition 0.1.19. A **partial rank function** from a transitive set T to a well-ordered set W is a partial function $\rho : T \to W$ with these properties:

- If $\varnothing \in T$, then $\rho(\varnothing)$ is the least element of W.
- If $y \in T$ and $\rho(x)$ is defined for all $x \in y$, then

  $$\rho(y) = \min \{ w \in W \mid \forall x \in y.\, \rho(x) < w \}$$

  provided the RHS is defined.
- Otherwise $\rho(y)$ is undefined.

A **total rank function** is a partial rank function that is defined on its entire domain. The **rank** of a set X, if it exists, the least von Neumann ordinal $\operatorname{rank}(X)$ for which there exists a total rank function $\operatorname{tcl}(X) \to \operatorname{rank}(X)$.

Proposition 0.1.20. *In Mac Lane set theory:*

(i) *If T is a transitive set and W is a well-ordered set, then there is a unique partial rank function $\rho : T \to W$.*

(ii) *If $\mathbf{U}$ is a pre-universe and $x \in \mathbf{U}$, then* $\operatorname{rank}(x)$ *can be defined by a Δ_0-formula with $\mathbf{U}$ as a parameter, and for each von Neumann ordinal α in $\mathbf{U}$, the set*

$$\mathbf{V}_\alpha = \{ x \in \mathbf{U} \mid \operatorname{rank}(x) < \alpha \}$$

is a $\mathbf{U}$-set.

(iii) *Assuming the Grothendieck–Verdier universe axiom,* $\operatorname{rank}(x)$ *is defined for all x.*

Proof. (i). This is a straightforward application of well-founded induction.

(ii). $\mathbf{U}$ is a transitive set and the set $\kappa(\mathbf{U})$ of all von Neumann ordinals in $\mathbf{U}$ is well-ordered by inclusion, so by claim (i) there is a partial rank function ρ :

$\mathbf{U} \rightharpoonup \kappa(\mathbf{U})$. ZFC proves that every set has a rank, so ρ must in fact be a total rank function; hence, for any $x \in \mathbf{U}$, rank(x) is defined. It is clear that ρ can be defined by a Δ_0-formula with only $\mathbf{U}$ as a parameter, and the rest of the claim follows.

(iii). Obvious, assuming claim (ii). □

Definition 0.1.21. Two sets are **equinumerous** if there exists a bijection between them. A **cardinality class** in a pre-universe $\mathbf{U}$ is an equivalence class under the relation of equinumerosity.

Definition 0.1.22. An **$\aleph$-number** is an infinite von Neumann ordinal κ such that, for any von Neumann ordinal λ such that κ and λ are equinumerous, we have $\kappa \subseteq \lambda$.

Example 0.1.23. The first infinite von Neumann ordinal, i.e. $\omega = \{0, 1, 2, \ldots\}$, is the $\aleph$-number $\aleph_0$.

Lemma 0.1.24. *If κ is an $\aleph$-number, then there exists a unique $\aleph$-number κ^+ with the following property:*

- *For any $\aleph$-number λ such that $\kappa < \lambda$, we have $\kappa^+ \leq \lambda$.*

The **cardinal successor** *of κ is κ^+.*

Proof. The class of $\aleph$-numbers is well-ordered and unbounded, so the class of all $\aleph$-numbers $> \kappa$ has a minimal element κ^+, as required. ■

Theorem 0.1.25 (Classification of cardinalities)**.**

(i) *In Zermelo–Fraenkel set theory, for every well-ordered infinite set X, there exists a unique $\aleph$-number κ such that X and κ are equinumerous.*

(ii) *In Zermelo–Fraenkel set theory with the axiom of choice, the same is true for any infinite set whatsoever.*

(iii) *In Mac Lane set theory, if $\mathbf{U}$ is a universe and X is an infinite set in $\mathbf{U}$, then there exists a unique $\aleph$-number κ in the cardinality class of X.*

(iv) *In Mac Lane set theory with the Grothendieck–Verdier universe axiom, if $\mathbf{U}$ is a pre-universe and κ is an $\aleph$-number not in $\mathbf{U}$, then the cardinality of $\mathbf{U}$ is at most κ.*

Proof. Claim (i) is a standard fact, whence claims (ii) and (iii), by the well-ordering theorem. Claim (iv) can be proven using axiom 4 for pre-universes. □

¶ 0.1.26. Henceforth, we identify the cardinality class of a finite set with the unique von Neumann ordinal contained in that class, and similarly we identify the cardinality class of an infinite set with the unique $\aleph$-number in that class. These are the **cardinal numbers**.

Definition 0.1.27. A **cofinal subset** of a partially-ordered set X is a subset $Y \subseteq X$ such that, for all x in X, there exists some y in Y such that $x \le y$. A **regular cardinal number** is an $\aleph$-number κ such that any cofinal subset of κ has cardinality equal to κ. A **singular cardinal number** is an $\aleph$-number that is not regular.

The following helps to motivate the definition of regular cardinal numbers.

Definition 0.1.28. Let $\mathbf{U}$ be a pre-universe. An **arity class** in $\mathbf{U}$ is a $\mathbf{U}$-class K of cardinal numbers satisfying the following conditions:

- $1 \in K$.
- If $\kappa \in K$ and $\lambda : \kappa \to K$ is a function, then the cardinal sum $\sum_{\alpha \in \kappa} \lambda(\alpha)$ is also in K.
- If $\kappa \in K$ and $\lambda : \kappa \to \mathbf{U}$ is a function such that each $\lambda(\alpha)$ is a cardinal number and $\sum_{\alpha \in \kappa} \lambda(\alpha) \in K$, then $\lambda(\alpha) \in K$ as well.

Theorem 0.1.29 (Classification of arity classes)**.** *In Mac Lane set theory, if K is an arity class in a pre-universe* $\mathbf{U}$*, then K must be either*

- $\{1\}$*, or*
- $\{0, 1\}$*, or*
- *of the form $\{\lambda \in \mathbf{U} \mid \lambda \text{ is a cardinal number and } \lambda < \kappa\}$ for some regular cardinal number κ (possibly not in* $\mathbf{U}$*).*

Proof. The notion of arity class and this result are due to Shulman [2012]. □

Definition 0.1.30. Let κ be a regular cardinal number. A **κ-small category** is a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ has cardinality $< \kappa$. A **finite category** is an $\aleph_0$-small category, i.e. a category $\mathbb{C}$ such that $\operatorname{mor}\mathbb{C}$ is finite. A **finite diagram** (resp. **κ-small diagram**, **U-small diagram**) in a category $\mathcal{C}$ is a functor $\mathbb{D} \to \mathcal{C}$ where $\mathbb{D}$ is a finite (resp. κ-small, $\mathbf{U}$-small) category.

Theorem 0.1.31. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{U}^+$ *be a universe with* $\mathbf{U} \in \mathbf{U}^+$*, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathbf{Set}^+$ *be the category of* $\mathbf{U}^+$*-sets.*

(i) *If* $X : \mathbb{D} \to \mathbf{Set}$ *is a* $\mathbf{U}$*-small diagram, then there exist a limit and a colimit for* X *in* $\mathbf{Set}$*.*

(ii) *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *is fully faithful and preserves limits and colimits for all* $\mathbf{U}$*-small diagrams.*

Proof. One can construct products, equalisers, coproducts, coequalisers, and hom-sets in a completely explicit way, making the preservation properties obvious. ♦

Corollary 0.1.32. *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Corollary 0.1.33. *For any* $\mathbf{U}$*-small category* $\mathbb{C}$*:*

(i) *The functor category* $[\mathbb{C}, \mathbf{Set}]$ *is* $\mathbf{U}$*-complete and* $\mathbf{U}$*-cocomplete, with limits and colimits for* $\mathbf{U}$*-small diagrams computed componentwise in* $\mathbf{Set}$*.*

(ii) *The inclusion* $[\mathbb{C}, \mathbf{Set}] \hookrightarrow [\mathbb{C}, \mathbf{Set}^+]$ *is fully faithful and both preserves and reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Definition 0.1.34. An **strongly inaccessible cardinal number** is a regular cardinal number κ such that, for all sets X of cardinality less than κ, the power set $\mathscr{P}(X)$ is also of cardinality less than κ.

Example 0.1.35. $\aleph_0$ is a strongly inaccessible cardinal number and is the only one that can be proven to exist in ZFC. It is more conventional to exclude $\aleph_0$ from the definition of strongly inaccessible cardinal number by demanding that they be uncountable.

Proposition 0.1.36. *In Mac Lane set theory:*

(i) *If* $\mathbf{U}$ *is a non-empty pre-universe, then there exists a strongly inaccessible cardinal number* κ *such that the members of* $\mathbf{U}$ *are all the sets of rank less than* κ*. Moreover, this* κ *is the rank and the cardinality of* $\mathbf{U}$*.*

(ii) *If* $\mathbf{U}$ *is a universe and* κ *is a strongly inaccessible cardinal number such that* $\kappa \in \mathbf{U}$*, then there exists a* $\mathbf{U}$*-set* $\mathbf{V}_\kappa$ *whose members are all the sets of rank less than* κ*, and* $\mathbf{V}_\kappa$ *is a pre-universe.*

(iii) *If* $\mathbf{U}$ *and* $\mathbf{U}'$ *are pre-universes, then either* $\mathbf{U} \subseteq \mathbf{U}'$ *or* $\mathbf{U}' \subseteq \mathbf{U}$*; and if* $\mathbf{U} \subsetneqq \mathbf{U}'$*, then* $\mathbf{U} \in \mathbf{U}'$*.*

Proof. (i). Let κ be the set of all von Neumann ordinals in $\mathbf{U}$; this exists by Δ_0-separation applied to $\mathbf{U}$. Since $\mathbf{U}$ is closed under power sets and internally-indexed unions, κ must be a strongly inaccessible cardinal.

We can construct the set all of $\mathbf{U}$-sets of rank less than κ using transfinite recursion on κ as follows: starting with $\mathbf{V}_0 = \varnothing$, for each von Neumann ordinal α less than κ, we set $\mathbf{V}_{\alpha+1} = \mathscr{P}\left(\mathbf{V}_\alpha\right)$, and for each ordinal λ that is not a successor, we set $\mathbf{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha$. The well-foundedness of $\in$ (restricted to $\mathbf{U}$) implies that in fact this must be all of $\mathbf{U}$.

Clearly, every set of rank less than κ is in fact a $\mathbf{U}$-set, and $\mathbf{U}$ is itself a set of rank κ. The cardinality of $\mathbf{U}$ is also κ, since κ is a regular cardinal number and any cardinal number less than κ is a member of $\mathbf{U}$.

(ii). We may construct $\mathbf{V}_\kappa$ using the same method as in (i). By construction $\mathbf{V}_\kappa$ satisfies axiom 1; since κ is infinite, $\mathbf{V}_\kappa$ satisfies axioms 2 and 3; and since κ is strongly inaccessible, $\mathbf{V}_\kappa$ satisfies axiom 4. Thus $\mathbf{V}_\kappa$ is a pre-universe.

(iii). Again, let κ be the rank of $\mathbf{U}$. If $\kappa \in \mathbf{U}'$ then we can show by transfinite induction that $\mathbf{V}_\kappa \in \mathbf{U}'$ and so $\mathbf{U} \subsetneqq \mathbf{U}'$; else we must have $\mathbf{U}' \subseteq \mathbf{V}_\kappa = \mathbf{U}$. ■

0.2 Accessibility and ind-completions

Prerequisites. § 0.1.

A classical technology for controlling size problems in category theory, due to Gabriel and Ulmer [1971], Grothendieck and Verdier [SGA 4a, Exposé I, § 9], and Makkai and Paré [1989], is the notion of accessibility. Though we make use of universes, accessibility remains important and is a crucial tool in verifying the stability of various universal constructions when one passes from one universe to a larger one.

Definition 0.2.1. Let κ be a regular cardinal.

- A **κ-filtered category** is a category $\mathcal{J}$ with the following property:
  - For each κ-small diagram $A : \mathbb{D} \to \mathcal{J}$, there exist an object j and a cocone $A \Rightarrow \Delta j$.

  A **κ-filtered diagram** in a category $\mathcal{C}$ is a functor $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a κ-filtered category.

- A **κ-directed preorder** is a preordered set X that is κ-filtered when considered as a category, i.e. a preorder with the following property:
  - For each κ-small subset $Y \subseteq X$, there exists an element x of X such that $y \le x$ for all y in Y.

  A **κ-directed diagram** in a category $\mathcal{C}$ is a functor $\mathcal{J} \to \mathcal{C}$ where $\mathcal{J}$ is a κ-directed category.

In both cases, it is conventional to omit mention of κ when $\kappa = \aleph_0$.

Example 0.2.2. The category with one object $*$ and only one non-trivial arrow f is filtered if and only if $f = f \circ f$.

Example 0.2.3. Let X be any set. The set of all finite subsets of X, partially ordered by inclusion, is a directed preorder. More generally, if κ is any regular cardinal, then the set of all subsets of X with cardinality strictly less than κ is a κ-directed preorder.

Definition 0.2.4. Let α be an ordinal. An **α-chain** in a category $\mathcal{C}$ is a functor $\alpha \to \mathcal{C}$, where we have identified α with the well-ordered set of ordinals $< \alpha$.

Example 0.2.5. If α is an ordinal with cofinality κ, then α is a κ-directed preorder. In particular, α-chains are κ-directed diagrams.

Lemma 0.2.6. *Let $\mathcal{I}$ be any category and let $\mathcal{J}$ be a filtered category. Given a full functor $F : \mathcal{I} \to \mathcal{J}$, the following are equivalent:*

(i) *$F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor.*[1]

[1] See definition A.5.31.

(ii) *For each object j in $\mathcal{J}$, there exist an object i in $\mathcal{I}$ and a morphism $j \to Fi$ in $\mathcal{J}$.*

Proof. (i) ⇒ (ii). Since $F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor, the comma category $(j \downarrow F)$ is connected; in particular, it is inhabited.

(ii) ⇒ (i). The hypothesis says that the comma category $(j \downarrow F)$ is inhabited for all objects j in $\mathcal{J}$; it remains to be shown that each $(j \downarrow F)$ is connected. Suppose we have morphisms $f : j \to Fi$ and $f' : j \to Fi'$ in $\mathcal{J}$. Since $\mathcal{J}$ is a filtered category, there exist morphisms $g : Fi \to j'$ and $g' : Fi' \to j'$ such that $g \circ f = g' \circ f'$. By hypothesis, there is a morphism $h : j' \to Fi''$ in $\mathcal{J}$, and since $F : \mathcal{I} \to \mathcal{J}$ is full, there exist morphisms $k : i \to i''$ and $k' : i' \to i''$ in $\mathcal{I}$ such that $Fk = h \circ g$ and $Fk' = h \circ g'$. Thus, we have $Fk \circ f = Fk' \circ f'$, so $(j \downarrow F)$ is indeed connected. ■

Lemma 0.2.7. *Let $\mathcal{I}$ be a filtered category and let $\mathcal{J}$ be any preorder. Given a functor $F : \mathcal{I} \to \mathcal{J}$, the following are equivalent:*

(i) *$F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor.*

(ii) *For each object j in $\mathcal{J}$, there exist an object i in $\mathcal{I}$ such that $j \le Fi$ in $\mathcal{J}$.*

Proof. (i) ⇒ (ii). Since $F : \mathcal{I} \to \mathcal{J}$ is a cofinal functor, the comma category $(j \downarrow F)$ is connected; in particular, it is inhabited.

(ii) ⇒ (i). The hypothesis says that the comma category $(j \downarrow F)$ is inhabited for all objects j in $\mathcal{J}$; it remains to be shown that each $(j \downarrow F)$ is connected. Suppose we have morphisms $j \le Fi$ and $j \le Fi'$ in $\mathcal{J}$. Since $\mathcal{I}$ is a filtered category, there exist an object i'' in $\mathcal{I}$ and morphisms $i \to i''$ and $i' \to i''$; thus, we have $j \le Fi \le Fi''$ and $j \le Fi' \le Fi''$, so $(j \downarrow F)$ is indeed connected. ■

Lemma 0.2.8. *Let $\mathcal{J}$ be a κ-filtered diagram. If $\mathcal{J}$ is also κ-small, then there exist an object j in $\mathcal{J}$ and an idempotent morphism $e : j \to j$ such that the subcategory of $\mathcal{J}$ generated by e is cofinal in $\mathcal{J}$.*

Proof. Since $\mathrm{id} : \mathcal{J} \to \mathcal{J}$ is a κ-small diagram in $\mathcal{J}$, there must exist an object j in $\mathcal{J}$ and a cocone $\lambda : \mathrm{id} \Rightarrow \Delta j$. Let $e = \lambda_j : j \to j$. Since λ is a cocone, we must have $e = e \circ e$, i.e. $e : j \to j$ is idempotent.

Let $\mathcal{I}$ be the subcategory of $\mathcal{J}$ generated by e and let j' be any object in $\mathcal{J}$. We must show that the comma category $(j' \downarrow \mathcal{I})$ is connected. It is inhabited:

$\lambda_{j'} : j' \to j$ is an object in $(j' \downarrow \mathcal{I})$. Moreover, given any morphism $f : j' \to j$ in $\mathcal{J}$, we must have $\lambda_{j'} = \lambda_j \circ f = e \circ f$, so $(j' \downarrow \mathcal{I})$ is indeed connected. Thus, $\mathcal{I}$ is a cofinal subcategory of $\mathcal{J}$. ∎

Theorem 0.2.9. *Let κ be a regular cardinal in a universe $\mathbf{U}$. If $\mathcal{J}$ is a $\mathbf{U}$-small κ-filtered category, then there exist a $\mathbf{U}$-small κ-directed poset $\mathcal{I}$ and a cofinal functor $P : \mathcal{I} \to \mathcal{J}$.*

Proof. See Theorem 1.5 and Remark 1.21 in [LPAC]. □

Theorem 0.2.10. *Let $\mathbf{U}$ be a universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let κ be any regular cardinal in $\mathbf{U}$. Given a $\mathbf{U}$-small category $\mathbb{D}$, the following are equivalent:*

(i) *$\mathbb{D}$ is a κ-filtered category.*

(ii) *The functor $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathbf{Set}] \to \mathbf{Set}$ preserves limits for all diagrams that are κ-small.*

Proof. The claim (i) ⇒ (ii) is very well known, and the converse is an exercise in using the Yoneda lemma and manipulating limits and colimits for diagrams of representable functors; see Satz 5.2 in [Gabriel and Ulmer, 1971]. □

Definition 0.2.11. Let κ and λ be regular cardinals in a universe $\mathbf{U}$ and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets.

- A **(κ, λ)-compact object** in a locally $\mathbf{U}$-small category $\mathcal{C}$ is an object A such that the representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ preserves colimits for all λ-small κ-filtered diagrams.

- Let $\mathbf{U}'$ be a universe with $\mathbf{U}' \subseteq \mathbf{U}$. A **$(\kappa, \mathbf{U}')$-compact object** in a locally $\mathbf{U}$-small category $\mathcal{C}$ is an object that is (κ, λ)-compact for all regular cardinals λ in $\mathbf{U}'$.

Though the above definition is stated using a universe $\mathbf{U}$, the following lemma shows there is in fact no dependence on $\mathbf{U}$.

Lemma 0.2.12. *Let A be an object in a locally* $\mathbf{U}$*-small category $\mathcal{C}$. The following are equivalent:*

(i) *A is a (κ, λ)-compact object in $\mathcal{C}$.*

(ii) *For all λ-small κ-filtered diagrams $B : \mathbb{D} \to \mathcal{C}$, if $\varepsilon : B \Rightarrow \Delta C$ is a colimiting cocone, then for any morphism $f : A \to C$, there exist an object i in $\mathbb{D}$ and a morphism $f' : A \to Bi$ in $\mathcal{C}$ such that $f = \varepsilon_i \circ f'$; and moreover if $f = \varepsilon_j \circ f''$ for some morphism $f'' : A \to Bj$ in $\mathcal{C}$, then there exists an object k and a pair of arrows $g : i \to k$, $h : i \to k$ in $\mathbb{D}$ such that $Bg \circ f' = Bh \circ f''$.*

Proof. Use the explicit description of $\varinjlim_{\mathbb{D}} \mathcal{C}(A, B)$ as a filtered colimit of sets; see Definition 1.1 in [LPAC], or Proposition 5.1.3 in [Borceux, 1994b]. □

Corollary 0.2.13. *Let $B : \mathcal{J} \to \mathcal{C}$ be a λ-small κ-filtered diagram, and let $\lambda : B \Rightarrow \Delta C$ be a colimiting cocone in $\mathcal{C}$. If C is a (κ, λ)-compact object in $\mathcal{C}$, then C is a retract of some vertex of B, i.e. there exists an object i in $\mathcal{J}$ such that $\lambda_i : Bi \to C$ is a* split *epimorphism.* ■

Lemma 0.2.14. *Let A be an object in a category $\mathcal{C}$.*

(i) *If A is a (κ, λ)-compact object in $\mathcal{C}$ and λ' is any regular cardinal $\leq \lambda$, then A is (κ, λ')-compact as well.*

(ii) *If A is (κ, λ)-compact and μ is any regular cardinal $\geq \kappa$, then A is also (μ, λ)-compact.*

Proof. Obvious. ♦

Lemma 0.2.15. *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*. If $B : \mathbb{D} \to \mathcal{C}$ is a κ-small diagram of (κ, λ)-compact objects in a locally* $\mathbf{U}$*-small category, then the colimit $\varinjlim_{\mathbb{D}} B$, if it exists, is also a (κ, λ)-compact object in $\mathcal{C}$.*

Proof. Use theorem 0.2.10 and the fact that $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^+$ maps colimits in $\mathcal{C}$ to limits in $\mathbf{Set}^+$. ■

Corollary 0.2.16. *A retract of a (κ, λ)-compact object is also a (κ, λ)-compact object.*

Proof. Suppose $r : A \to B$ and $s : B \to A$ are morphisms in $\mathcal{C}$ such that $r \circ s = \mathrm{id}_B$. Then $e = s \circ r$ is an idempotent morphism and the diagram below

$$A \underset{e}{\overset{\mathrm{id}_A}{\rightrightarrows}} A \xrightarrow{r} B$$

is a (split) coequaliser diagram in $\mathcal{C}$, so B is (κ, λ)-compact if A is. ■

Proposition 0.2.17. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. For any* $\mathbf{U}$*-set* A*, the following are equivalent:*

(i) *A has cardinality less than κ.*

(ii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-directed diagrams.*

Proof. The claim (i) ⇒ (ii) follows from theorem 0.2.10, and (ii) ⇒ (iii) is obvious. To see (iii) ⇒ (i), we may use corollary 0.2.13 and the fact that every set is the κ-directed union of its subsets of cardinality $< \kappa$. ■

Corollary 0.2.18. *A* $\mathbf{U}$*-set* X *is* $(\kappa, \mathbf{U})$*-compact if and only if* $|X| < \kappa$*.* ■

Definition 0.2.19. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible $\mathbf{U}$-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ satisfying the following conditions:

- $\mathcal{C}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams.
- There exists a $\mathbf{U}$-set $\mathcal{G}$ such that every object in $\mathcal{G}$ is $(\kappa, \mathbf{U})$-compact and, for every object B in $\mathcal{C}$, there exists a $\mathbf{U}$-small κ-filtered diagram of objects in $\mathcal{G}$ with B as its colimit in $\mathcal{C}$.

We write $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the $(\kappa, \mathbf{U})$-compact objects.

Example 0.2.20. The category of $\mathbf{U}$-sets is a κ-accessible $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Theorem 0.2.21. *Let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category, and let* κ *be a regular cardinal in* $\mathbf{U}$*. There exist a locally* $\mathbf{U}$*-small category* $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *and a functor* $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *with the following properties:*

(i) *The objects of* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *are* $\mathbf{U}$*-small* κ*-filtered diagrams* $B : \mathbb{D} \to \mathcal{C}$*, and* γ *sends an object* C *in* $\mathcal{C}$ *to the corresponding trivial diagram* $\mathbb{1} \to \mathcal{C}$ *with value* C.

(ii) *The functor* $\gamma : \mathcal{C} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is fully faithful, injective on objects, preserves all limits that exist in* $\mathcal{C}$*, and preserves all* κ*-small colimits that exist in* $\mathcal{C}$.

(iii) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *has colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iv) *For every object* C *in* $\mathcal{C}$*, the object* γC *is* $(\kappa, \mathbf{U})$*-compact in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$*, and for each* $\mathbf{U}$*-small* κ*-filtered diagram* $B : \mathbb{D} \to \mathcal{C}$*, there is a canonical colimiting cocone* $\gamma B \Rightarrow \Delta B$ *in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$.

(v) *If* $\mathcal{D}$ *is a category with colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams, then for each functor* $F : \mathcal{C} \to \mathcal{D}$*, there exists a functor* $\bar{F} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *that preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *such that* $\gamma \bar{F} = F$*, and given any functor* $\bar{G} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *whatsoever, the induced map* $\operatorname{Nat}(\bar{F}, \bar{G}) \to \operatorname{Nat}(F, \gamma \bar{G})$ *is a bijection.*

The category $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is called the* **free** $(\kappa, \mathbf{U})$**-ind-completion** *of* $\mathcal{C}$*, or the* **category of** $(\kappa, \mathbf{U})$**-ind-objects** *in* $\mathcal{C}$.

Proof. If $B : \mathbb{D} \to \mathcal{C}$ and $B' : \mathbb{D}' \to \mathcal{C}$ are two $\mathbf{U}$-small κ-filtered diagrams, then properties (ii) and (iii) together imply that

$$\operatorname{Hom}(B', B) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and so, taking the RHS as the *definition* of the LHS, we need only find a suitable notion of composition to make $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ into a locally $\mathbf{U}$-small category. However, we observe that, if $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, then

$$\operatorname{Hom}\left(\varinjlim_{\mathbb{D}'} \mathrm{N} B', \varinjlim_{\mathbb{D}} \mathrm{N} B \right) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and, assuming property (v), the Yoneda embedding $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ must extend along γ to a functor $\bar{\mathrm{N}} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that preserves colimits for $\mathbf{U}$-small κ-filtered diagram, so, in consideration of properties (i) and (iv), we may as well *define* the composition in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ so that $\bar{\mathrm{N}}$ becomes fully faithful. This completes the definition of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ as a category.

It remains to be shown that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ actually has properties (ii), (iii), (iv), and (v); see Corollary 6.4.14 in [Borceux, 1994a] and Theorem 2.26 in [LPAC]. Note that the fact that γ preserves colimits for κ-small diagrams essentially follows from theorem 0.2.10. □

Proposition 0.2.22. *Let $\mathbb{B}$ be a $\mathbf{U}$-small category and let κ be a regular cardinal in $\mathbf{U}$.*

(i) *$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is a κ-accessible $\mathbf{U}$-category.*

(ii) *Every $(\kappa, \mathbf{U})$-compact object in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is a retract of an object of the form γB, where $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is the canonical embedding.*

(iii) *$\mathbf{K}^{\mathbf{U}}_{\kappa}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big)$ is an essentially $\mathbf{U}$-small category.*

Proof. (i). This claim more-or-less follows from the properties of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ explained in the previous theorem.

(ii). Use corollary 0.2.16.

(iii). Since $\mathbb{B}$ is $\mathbf{U}$-small and $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is locally $\mathbf{U}$-small, claim (ii) implies that $\mathbf{K}^{\mathbf{U}}_{\kappa}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big)$ must be essentially $\mathbf{U}$-small. ■

Proposition 0.2.23. *Let $\mathcal{C}$ be a κ-presentable $\mathbf{U}$-category and let C be an object in $\mathcal{C}$.*

(i) *The comma category $\big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \downarrow C\big)$ is an essentially $\mathbf{U}$-small κ-filtered category.*

(ii) *If $P^C : \big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \downarrow C\big) \to \mathcal{C}$ is the canonical diagram, then the tautological cocone[2] $P^C \Rightarrow \Delta C$ is a colimiting cocone in $\mathcal{C}$.*

Proof. See Proposition 2.1.5 in [Makkai and Paré, 1989] or Proposition 2.8 in [LPAC]. □

Definition 0.2.24. Let κ be a regular cardinal in a universe $\mathbf{U}$. A $(\kappa, \mathbf{U})$**-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and
- F preserves all colimits for $\mathbf{U}$-small κ-filtered diagrams.

[2] See definition A.5.7.

We write $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of the functor category $[\mathcal{C}, \mathcal{D}]$ spanned by the $(\kappa, \mathbf{U})$-accessible functors. An **accessible functor** is a functor that is $(\kappa, \mathbf{U})$-accessible functor for some regular cardinal κ in some universe $\mathbf{U}$.

Theorem 0.2.25 (Classification of accessible categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category.*

(ii) *The inclusion $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \hookrightarrow \mathcal{C}$ extends along the embedding $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\kappa(\mathcal{C})$ to a $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}_{\mathbf{U}}^\kappa\big(\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})\big) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *There exist a $\mathbf{U}$-small category $\mathbb{B}$ and a functor $\mathbf{Ind}_{\mathbf{U}}^\kappa(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

Proof. See Theorem 2.26 in [LPAC], or Theorem 5.3.5 in [Borceux, 1994b]. □

Corollary 0.2.26. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category and $\mathcal{D}$ is any category, then:*

(i) *The restriction $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \to \big[\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}), \mathcal{D}\big]$ is fully faithful and surjective on objects.*

(ii) *In particular, if $\mathcal{D}$ is also locally $\mathbf{U}$-small, then $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D})$ is equivalent to a locally $\mathbf{U}$-small category.*

(iii) *If $\mathcal{D}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams, then the inclusion $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \hookrightarrow [\mathcal{C}, \mathcal{D}]$ has a left adjoint.* ■

Corollary 0.2.27. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category. For any $\mathbf{U}$-small κ-filtered diagram $\mathbb{D}$, $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ preserves componentwise limits for κ-small diagrams.*

Proof. The claim is certainly true when $\mathcal{C} = [\mathbb{A}, \mathbf{Set}]$, by theorem 0.2.10. In general, choose a $(\kappa, \mathbf{U})$-accessible fully faithful functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ that preserves limits for all $\mathbf{U}$-small diagrams; then R *reflects* limits for κ-small diagrams and colimits for $\mathbf{U}$-small κ-filtered diagrams, so we may deduce the claim from the corresponding fact for $[\mathbb{A}, \mathbf{Set}]$. Note that such a functor exists: the classification theorem above implies we may take $\mathbb{A}$ to be $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and R to be the induced Yoneda representation. ■

Proposition 0.2.28. *Let $\mathcal{C}$ be a κ-accessible* **U***-category and let $\mathcal{D}$ be a locally* **U***-small category. Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, if G is fully faithful and preserves colimits for all* **U***-small κ-filtered diagrams, then $\mathcal{D}$ is also a κ-accessible* **U***-category.*

Proof. Under our hypotheses, given any **U**-small κ-filtered diagram $A : \mathcal{J} \to \mathcal{D}$, we may take $F \varinjlim_{\mathcal{J}} GA$ as its colimit in $\mathcal{D}$. Our hypotheses also imply that F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U})$-compact objects in $\mathcal{D}$; thus if $\mathcal{G}$ is a **U**-small set of objects that generates $\mathcal{C}$ under **U**-small κ-filtered colimits, then $\{ FX \mid X \in \mathcal{G} \}$ is a **U**-small set of objects that generates $\mathcal{D}$ in the same sense. ■

Definition 0.2.29. Let κ and λ be regular cardinals and let $\mathscr{P}_\kappa(X)$ denote the set of all κ-small subsets of a set X. We say κ is **sharply less than** λ if

- $\kappa < \lambda$, and
- for all λ-small sets X, there exists a λ-small cofinal subposet of the poset $\mathscr{P}_\kappa(X)$.

We write $\kappa \lhd \lambda$ to mean κ is sharply less than λ.

Example 0.2.30. Let κ be a regular cardinal and let κ^+ be its cardinal successor. Then $\kappa \lhd \kappa^+$: every κ^+-small set can be mapped bijectively onto an initial segment α of κ (but possibly all of κ), and it is clear that the subposet

$$\{ \beta \mid \beta \leq \alpha \} \subseteq \mathscr{P}_\kappa(\alpha)$$

is a λ-small cofinal subposet of $\mathscr{P}_\kappa(\alpha)$: given any κ-small subset $X \subseteq \alpha$, we must have $\sup X \leq \alpha$, and $X \subseteq \sup X$ by definition.

Theorem 0.2.31. *Let κ and λ be regular cardinals in a universe* **U***, and suppose $\kappa < \lambda$. The following are equivalent:*

(i) $\kappa \lhd \lambda$.

(ii) *For any* **U***-small κ-directed poset X and any λ-small subset $Y \subseteq X$, there exists a λ-small κ-directed subposet $X' \subseteq X$ with $Y \subseteq X'$.*

(iii) *Any κ-accessible* **U***-category is also a λ-accessible* **U***-category.*

Proof. See Theorem 2.11 in [LPAC]. □

Proposition 0.2.32.
(i) *The binary relation* $\lhd$ *is transitive.*

(ii) *If* $\kappa \leq \lambda$, *then* $\kappa \lhd \left(2^{<\lambda}\right)^{+}$, *where* $2^{<\lambda} = \sup\{2^{\mu} \mid \mu$ *is a cardinal* $< \lambda\}$ *and* $2^{\mu} = |\mathscr{P}(\mu)|$, *and also* $\kappa \lhd \left(2^{\lambda}\right)^{+}$.

(iii) *For any set* K *of regular cardinals, there exists a regular cardinal* λ *such that* $\kappa \lhd \lambda$ *for all* κ *in* K.

Proof. (i). See Proposition 2.3.2 in [Makkai and Paré, 1989], or theorem 0.2.31.

(ii). See Proposition 2.3.5 in [Makkai and Paré, 1989], or Example 2.13(5) in [LPAC], or Proposition 5.4.7 in [Borceux, 1994b].

(iii). This follows from claim (ii). □

Definition 0.2.33. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **locally κ-presentable U-category** is a κ-accessible $\mathbf{U}$-category that is also $\mathbf{U}$-cocomplete. A **locally presentable U-category** is one that is a locally κ-presentable $\mathbf{U}$-category for some regular cardinal κ in $\mathbf{U}$, and we often say 'locally finitely presentable' instead of 'locally $\aleph_0$-presentable'.

Example 0.2.34. The category of $\mathbf{U}$-sets is a locally κ-presentable $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Lemma 0.2.35. *Let* $\mathcal{C}$ *be a locally* κ*-presentable* $\mathbf{U}$*-category.*

(i) *For any regular cardinal* λ *in* $\mathbf{U}$, *if* $\kappa \leq \lambda$, *then* $\mathcal{C}$ *is a locally* λ*-presentable* $\mathbf{U}$*-category.*

(ii) *With* λ *as above, if* $F : \mathcal{C} \to \mathcal{D}$ *is a* $(\kappa, \mathbf{U})$*-accessible functor, then it is also a* $(\lambda, \mathbf{U})$*-accessible functor.*

(iii) *If* $\mathbf{U}^{+}$ *is any universe with* $\mathbf{U} \in \mathbf{U}^{+}$, *and* $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}^{+}$*-category, then* $\mathcal{C}$ *must be a preorder.*

Proof. (i). See the remark after Theorem 1.20 in [LPAC], or Propositions 5.3.2 and 5.2.3 in [Borceux, 1994b].

(ii). A λ-filtered diagram is certainly κ-filtered, so if F preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$, it must also preserve colimits for all $\mathbf{U}$-small λ-filtered diagrams.

(iii). This is a corollary of theorem 0.1.10. ■

Corollary 0.2.36. *A category $\mathcal{C}$ is a locally presentable* **U***-category for* at most *one universe* **U***, provided $\mathcal{C}$ is not a preorder.*

Proof. Use proposition 0.1.36 together with the above lemma. ■

Theorem 0.2.37 (Classification of locally presentable categories)**.** *Let κ be a regular cardinal in a universe* **U***, let* **Set** *be the category of* **U***-sets, and let $\mathcal{C}$ be a locally* **U***-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a locally κ-presentable* **U***-category.*

(ii) *There exist a* **U***-small category $\mathbb{B}$ that has colimits for κ-small diagrams and a functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *The restricted Yoneda embedding $\mathcal{C} \to \left[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ is fully faithful, $(\kappa, \mathbf{U})$-accessible, and has a left adjoint.*

(iv) *There exist a* **U***-small category $\mathbb{A}$ and a fully faithful $(\kappa, \mathbf{U})$-accessible functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ such that $\mathbb{A}$ has limits for all κ-small diagrams, R has a left adjoint, and R is essentially surjective onto the full subcategory of functors $\mathbb{A} \to \mathbf{Set}$ that preserve limits for all κ-small diagrams.*

(v) *There exist a* **U***-small category $\mathbb{A}$ and a fully faithful $(\kappa, \mathbf{U})$-accessible functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ such that R has a left adjoint.*

(vi) *$\mathcal{C}$ is a κ-accessible* **U***-category and is* **U***-complete.*

Proof. See Proposition 1.27, Corollary 1.28, Theorem 1.46, and Corollary 2.47 in [LPAC], or Theorems 5.2.7 and 5.5.8 in [Borceux, 1994b]. □

Remark 0.2.38. If $\mathcal{C}$ is equivalent to $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ for some **U**-small category $\mathbb{B}$ that has colimits for all κ-small diagrams, then $\mathbb{B}$ must be equivalent to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ by proposition 0.2.22. In other words, every locally κ-presentable **U**-category is, up to equivalence, the $(\kappa, \mathbf{U})$-ind-completion of an essentially unique **U**-small κ-cocomplete category.

Example 0.2.39. Obviously, for any **U**-small category $\mathbb{A}$, the functor category $[\mathbb{A}, \mathbf{Set}]$ is locally finitely presentable. More generally, one may show that for any κ-ary algebraic theory T, possibly many-sorted, the category of T-algebras in $\mathbf{U}$ is a locally κ-presentable $\mathbf{U}$-category. The above theorem can also be used to show that $\mathbf{Cat}$, the category of $\mathbf{U}$-small categories, is a locally finitely presentable $\mathbf{U}$-small category.

Proposition 0.2.40. *If $\mathcal{C}$ is an accessible $\mathbf{U}$-category and $\mathbb{D}$ is any $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is also an accessible $\mathbf{U}$-category.*

Proof. See Theorem 2.39 in [LPAC]. ■

Proposition 0.2.41. *If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category and $\mathbb{D}$ is any $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is also a locally κ-presentable category.*

Proof. This can be proven using the classification theorem by noting that the 2-functor $[\mathbb{D}, -]$ preserves reflective subcategories, but see also Corollary 1.54 in [LPAC]. □

It is commonplace to say 'λ-presentable object' instead of 'λ-compact object', especially in algebraic contexts. The following propositions justify the alternative terminology:

Proposition 0.2.42. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category. If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \lhd \lambda$, then the following are equivalent for an object C in $\mathcal{C}$:*

(i) *C is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$.*

(ii) *There exists a λ-small κ-filtered diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} A$.*

(iii) *There exists a λ-small κ-directed diagram $A : \mathcal{J} \to \mathcal{C}$ such that each Aj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and C is a retract of $\varinjlim_{\mathcal{J}} A$.*

Proof. (i) $\Leftrightarrow$ (ii). See Proposition 2.3.11 in [Makkai and Paré, 1989].

(i) $\Leftrightarrow$ (iii). See Remark 2.15 in [LPAC]. ■

Proposition 0.2.43. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let λ be a regular cardinal in $\mathbf{U}$ with $\lambda \geq \kappa$. If $\mathcal{H}$ is a $\mathbf{U}$-small full subcategory of $\mathcal{C}$ such that*

- *every $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ is isomorphic to an object in $\mathcal{H}$, and*

- *$\mathcal{H}$ is closed in $\mathcal{C}$ under colimits for λ-small diagrams,*

then every $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$ is isomorphic to an object in $\mathcal{H}$. In particular, $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is the smallest replete full subcategory of $\mathcal{C}$ containing $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and closed in $\mathcal{C}$ under colimits for λ-small diagrams.

TODO: Simplify this argument.

Proof. Let C be any $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. Clearly, the comma category $(\mathcal{H} \downarrow C)$ is a $\mathbf{U}$-small λ-filtered category. Let $\mathcal{G} = \mathcal{H} \cap \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$. One can show that $(\mathcal{G} \downarrow C)$ is a cofinal subcategory in $(\mathcal{H} \downarrow C)$, and the classification theorem (0.2.37) plus proposition A.5.25 implies that the tautological cocone on the diagram $(\mathcal{G} \downarrow C) \to \mathcal{C}$ is colimiting, so the tautological cocone on the diagram $(\mathcal{H} \downarrow C) \to \mathcal{C}$ is also colimiting. Now, by corollary 0.2.13, C is a retract of an object in $\mathcal{H}$, and hence C must be isomorphic to an object in $\mathcal{H}$, because $\mathcal{H}$ is closed under coequalisers.

For the final claim, note that $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is certainly a replete full subcategory of $\mathcal{C}$ and contained in any replete full subcategory containing $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and closed in $\mathcal{C}$ under colimits for λ-small diagrams, so we just have to show that $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is also closed in $\mathcal{C}$ under colimits for λ-small diagrams; for this, we simply appeal to lemma 0.2.15. ■

Proposition 0.2.44. *Let $\mathcal{C}$ be a locally $\mathbf{U}$-small category and let $\mathbb{D}$ be a κ-small category in $\mathbf{U}$.*

(i) *If λ is a regular cardinal $\geq \kappa$ and $A : \mathbb{D} \to \mathcal{C}$ is componentwise $(\lambda, \mathbf{U})$-compact, then A is a $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$.*

(ii) *If $\mathcal{C}$ is a λ-accessible $\mathbf{U}$-category and has products for κ-small families of objects, then every $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$ is componentwise $(\lambda, \mathbf{U})$-compact.*

Proof. (i). First, note that Mac Lane's subdivision category[3] $\mathbb{D}^{\S}$ is also κ-small, so $[\mathbb{D}, \mathcal{C}](A, B)$ is computed as the limit of a κ-small diagram of hom-sets. More precisely, using end notation,[4]

$$[\mathbb{D}, \mathcal{C}](A, B) \cong \int_{d:\mathbb{D}} \mathcal{C}(Ad, Bd)$$

and so if $\kappa \leq \lambda$ and A is componentwise $(\lambda, \mathbf{U})$-compact, then $[\mathbb{D}, \mathcal{C}](A, -)$ preserves colimits for $\mathbf{U}$-small λ-filtered diagrams, hence A is itself $(\lambda, \mathbf{U})$-compact.

[3] See [CWM, Ch. IX, § 5].
[4] See § A.6.

(ii). Now, suppose A is a $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$. Let d be an object in $\mathbb{D}$, let $d^* : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ be evaluation at d, and let $d_* : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$ be the right adjoint, which is explicitly given by

$$\big(d_* C\big)(d') = \mathbb{D}(d', d) \pitchfork C$$

where $\pitchfork$ is defined by following adjunction:

$$\mathbf{Set}(X, \mathcal{C}(C, C')) \cong \mathcal{C}(C, X \pitchfork C')$$

The unit $\eta_A : A \to d_* d^* A$ is constructed using the universal property of $\pitchfork$ in the obvious way, and the counit $\epsilon_C : d^* d_* C \to C$ is the projection $\mathbb{D}(d, d) \pitchfork C \to C$ corresponding to $\mathrm{id}_d \in \mathbb{D}(d, d)$. Since $\mathcal{C}$ is a λ-accessible $\mathbf{U}$-category, there exist a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{C}$ consisting of $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$ and a colimiting cocone $\alpha : B \Rightarrow \Delta d^* A$, and since each $\mathbb{D}(d', d)$ has cardinality $< \kappa$, the cocone $d_* \alpha : d_* B \Rightarrow \Delta d_* d^* A$ is also colimiting, by corollary 0.2.27. Lemma 0.2.12 then implies $\eta_A : A \to d_* d^* A$ factors through $d_* \alpha_j : d_*(Bj) \to d_* d^* A$ for some j in $\mathcal{J}$, say

$$\eta_A = d_* \alpha_j \circ \sigma$$

for some $\sigma : A \to d_* Bj$. But then, by the triangle identity,

$$\mathrm{id}_{Ad} = \epsilon_{Ad} \circ d^* \eta_A = \epsilon_{Ad} \circ d^* d_* \alpha_j \circ d^* \sigma = \alpha_j \circ \epsilon_{Bj} \circ d^* \sigma$$

and so $\alpha_j : Bj \to Ad$ is a split epimorphism, hence Ad is a $(\lambda, \mathbf{U})$-compact object, by corollary 0.2.16. ■

REMARK 0.2.45. The claim in the above proposition can fail if $\kappa > \lambda$. For example, we could take $\mathcal{C} = \mathbf{Set}$, with $\mathbb{D}$ being the set ω considered as a discrete category; then the terminal object in $[\mathbb{D}, \mathbf{Set}]$ is componentwise finite, but is not itself an $\aleph_0$-compact object in $\mathbf{Set}$.

Lemma 0.2.46. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, with $\kappa \le \lambda$.*

(i) *If $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and $G : \mathcal{D} \to \mathcal{C}$ is a $(\lambda, \mathbf{U})$-accessible functor that preserves limits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, then, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object.*

(ii) *If $\mathcal{C}$ is a locally κ-presentable* $\mathbf{U}$*-category, $\mathcal{D}$ is a locally* $\mathbf{U}$*-small category, and $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves colimits for all* $\mathbf{U}$*-small diagrams in $\mathcal{C}$, then, for any object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object.*

Proof. (i). Let $\mathcal{F}$ be the full subcategory of $(C \downarrow G)$ spanned by those (D, g) where D is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$. G preserves colimits for all $\mathbf{U}$-small λ-filtered diagrams, so, by lemma 0.2.12, $\mathcal{F}$ must be a weakly initial family in $(C \downarrow G)$. Proposition 0.2.22 implies $\mathcal{F}$ is an essentially $\mathbf{U}$-small category, and since $\mathcal{D}$ has limits for all $\mathbf{U}$-small diagrams and G preserves them, $(C \downarrow G)$ is also $\mathbf{U}$-complete. Thus, the inclusion $\mathcal{F} \hookrightarrow (C \downarrow G)$ has a limit, and it can be shown that this is an initial object in $(C \downarrow G)$.[5]

(ii). Let $\mathcal{G}$ be the full subcategory of $(F \downarrow D)$ spanned by those (C, f) where C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$; note that proposition 0.2.22 implies $\mathcal{G}$ is an essentially $\mathbf{U}$-small category. Since $\mathcal{C}$ has colimits for all $\mathbf{U}$-small diagrams and F preserves them, $(F \downarrow D)$ is also $\mathbf{U}$-cocomplete.[6] Let (C, f) be a colimit for the inclusion $\mathcal{G} \hookrightarrow (F \downarrow D)$. It is not hard to check that (C, f) is a weakly terminal object in $(F \downarrow D)$, so the formal dual of Freyd's initial object lemma[7] gives us a terminal object in $(F \downarrow D)$; explicitly, it may be constructed as the joint coequaliser of all the endomorphisms of (C, f). ■

Theorem 0.2.47 (Accessible adjoint functor theorem)**.** *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*, with $\kappa \le \lambda$, let $\mathcal{C}$ be a locally κ-presentable* $\mathbf{U}$*-category, and let $\mathcal{D}$ be a locally λ-presentable* $\mathbf{U}$*-category.*

Given a functor $F : \mathcal{C} \to \mathcal{D}$, the following are equivalent:

(i) *F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and G is a $(\lambda, \mathbf{U})$-accessible functor.*

(ii) *F preserves colimits for all* $\mathbf{U}$*-small diagrams and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(iii) *F has a right adjoint and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

[5] See Theorem 1 in [CWM, Ch. X, § 2].
[6] See the Lemma in [CWM, Ch. V, § 6].
[7] See Theorem 1 in [CWM, Ch. V, § 6].

On the other hand, given a functor $G : \mathcal{D} \to \mathcal{C}$, the following are equivalent:

(iv) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(v) *G is a $(\lambda, \mathbf{U})$-accessible functor and preserves limits for all $\mathbf{U}$-small diagrams.*

(vi) *G is a $(\lambda, \mathbf{U})$-accessible functor and there exist a functor $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ and hom-set bijections*

$$\mathcal{C}(C, GD) \cong \mathcal{D}\big(F_0 C, D\big)$$

natural in D for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, where D varies in $\mathcal{D}$.

Proof. We will need to refer back to the details of the proof of this theorem later, so here is a sketch of the constructions involved.

(i) ⇒ (ii). If F is a left adjoint, then F certainly preserves colimits for all $\mathbf{U}$-small diagrams. Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{D}\left(FC, \varinjlim_{\mathcal{J}} B\right) \cong \mathcal{C}\left(C, G \varinjlim_{\mathcal{J}} B\right) \cong \mathcal{C}\left(C, \varinjlim_{\mathcal{J}} GB\right) \\ \cong \varinjlim_{\mathcal{J}} \mathcal{C}(C, GB) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(FC, B)\end{aligned}$$

and thus FC is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$.

(ii) ⇒ (iii). It is enough to show that, for each object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object $\big(GD, \varepsilon_D\big)$;[8] but this was done in the previous lemma.

(iii) ⇒ (i). Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathcal{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{C}\left(C, G \varinjlim_{\mathcal{J}} B\right) \cong \mathcal{D}\left(FC, \varinjlim_{\mathcal{J}} B\right) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(FC, B) \\ \cong \varinjlim_{\mathcal{J}} \mathcal{C}(C, GB) \cong \mathcal{C}\left(C, \varinjlim_{\mathcal{J}} GB\right)\end{aligned}$$

[8] See Theorem 2 in [CWM, Ch. IV, § 1].

because $F C$ is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$; but theorem 0.2.37 says the restricted Yoneda embedding $\mathcal{C} \to \left[\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ is fully faithful, so this is enough to conclude that G preserves colimits for $\mathbf{U}$-small λ-filtered diagrams.

(iv) ⇒ (v). If G is a right adjoint, then G certainly preserves limits for all $\mathbf{U}$-small diagrams; the rest of this implication is just (iii) ⇒ (i).

(v) ⇒ (vi). It is enough to show that, for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object $\left(F_0 C, \eta_C\right)$; but this was done in the previous lemma. It is clear how to make F_0 into a functor $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$.

(vi) ⇒ (iv). We use theorems 0.2.21 and 0.2.37 to extend $F_0 : \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ along the inclusion $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \hookrightarrow \mathcal{C}$ to get $(\kappa, \mathbf{U})$-accessible functor $F : \mathcal{C} \to \mathcal{D}$. We then observe that, for any $\mathbf{U}$-small κ-filtered diagram $A : \mathbb{I} \to \mathcal{C}$ of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}\left(\varinjlim_{\mathbb{I}} A, G D\right) &\cong \varprojlim_{\mathbb{I}} \mathcal{C}(A, G D) \cong \varprojlim_{\mathbb{I}} \mathcal{C}\left(F_0 A, D\right) \\ &\cong \mathcal{C}\left(\varinjlim_{\mathbb{I}} F A, D\right) \cong \mathcal{C}\left(F \varinjlim_{\mathbb{I}} A, D\right) \end{aligned}$$

is a series of bijections natural in D, where D varies in $\mathcal{D}$; but $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, so this is enough to show that F is a left adjoint of G. The remainder of the claim is a corollary of (i) ⇒ (ii). ■

Corollary 0.2.48. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable* $\mathbf{U}$*-categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then there exists a regular cardinal μ in* $\mathbf{U}$ *such that G is a* $(\mu, \mathbf{U})$*-accessible functor.*

Proof. Suppose $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, and $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint for G. Since $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ is an essentially $\mathbf{U}$-small category, recalling lemma 0.2.14, there certainly exists a regular cardinal μ in $\mathbf{U}$ such that $\mu \geq \lambda$ and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U})$-compact objects in $\mathcal{D}$. The above theorem, plus lemma 0.2.35, implies G is an $(\mu, \mathbf{U})$-accessible functor. ■

0.3 Accessible constructions

Prerequisites. §§ 0.1, 0.2, A.5

Definition 0.3.1. Let $\mathbf{U}$ be a universe and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The **$\mathbf{U}$-rank** of F is the smallest regular cardinal κ in $\mathbf{U}$ such that F preserves colimits for $\mathbf{U}$-small κ-filtered diagrams, provided any such cardinal exists.

Remark 0.3.2. The class of regular cardinals is well-ordered, so the definition above makes sense. Of course, every $(\kappa, \mathbf{U})$-accessible functor has $\mathbf{U}$-rank $\leq \kappa$.

Definition 0.3.3. Let $\mathbf{U}$ be a universe and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. The **compactness $\mathbf{U}$-rank** of an object A in $\mathcal{C}$ is the $\mathbf{U}$-rank of the hom-functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$, where $\mathbf{Set}$ is the category of $\mathbf{U}$-sets.

Remark 0.3.4. Lemma 0.2.15 implies that, for each object A in an accessible $\mathbf{U}$-category, there exists a regular cardinal λ in $\mathbf{U}$ such that A is $(\lambda, \mathbf{U})$-compact; in particular, every object in an accessible $\mathbf{U}$-category has a compactness $\mathbf{U}$-rank.

Definition 0.3.5. Let κ and λ be regular cardinals in a universe $\mathbf{U}$. A **(κ, λ)-compactly generated $\mathbf{U}$-category** is an essentially $\mathbf{U}$-small category $\mathcal{C}$ that satisfies the following conditions:

- $\mathcal{C}$ has colimits for all λ-small κ-filtered diagrams.
- Every object in $\mathcal{C}$ is a colimit for some λ-small κ-filtered diagram of (κ, λ)-compact objects in $\mathcal{C}$.

We write $\mathbf{K}_\kappa^\lambda(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the (κ, λ)-compact objects.

Remark 0.3.6. Lemma 0.2.8 implies an essentially $\mathbf{U}$-small category is (κ, κ)-compactly generated if and only if it is Cauchy-complete, i.e. if and only if all idempotent endomorphisms in $\mathcal{C}$ are split.

Proposition 0.3.7. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category.*

(i) *$\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ is a (κ, κ)-compactly generated $\mathbf{U}$-category, and every object in $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ is (κ, κ)-compact.*

(ii) *If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \lhd \lambda$, then $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ is a (κ, λ)-compactly generated $\mathbf{U}$-category, and the (κ, λ)-compact objects in $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ are precisely the $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$.*

Proof. (i). This follows from lemma 0.2.14, corollary 0.2.16, and remark 0.3.6.

(ii). Combine corollary 0.2.13, lemma 0.2.15, and proposition 0.2.42. ■

Proposition 0.3.8. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, let $\mathbb{A}$ and $\mathbb{B}$ be $\mathbf{U}$-small categories, and let $F : \mathbb{A} \to \mathbb{B}$ be a fully faithful functor. Assume the following hypotheses:*

- *$\kappa \le \lambda$.*
- *$\mathbb{A}$ is a Cauchy-complete category and $\mathbb{B}$ has colimits for λ-small κ-filtered diagrams.*
- *Each FA is a (κ, λ)-compact object in $\mathbb{B}$, and each object in $\mathbb{B}$ is a colimit for a λ-small κ-filtered diagram of objects in the image of F.*

Then:

(i) *Every (κ, λ)-compact object in $\mathbb{B}$ is isomorphic to an object in the image of $F : \mathbb{A} \to \mathbb{B}$.*

(ii) *There exists a functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ equipped with a natural bijection of the form below,*

$$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})(A, UB) \cong \mathbb{B}(FA, B)$$

and it is unique up to unique isomorphism.

(iii) *Moreover, the functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective onto the full subcategory of $(\lambda, \mathbf{U})$-compact objects in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$.*

(iv) *$F : \mathbb{A} \to \mathbb{B}$ is a dense functor.*

(v) *If $\kappa \lhd \lambda$, then the $(\lambda, \mathbf{U})$-accessible functor $\bar{U} : \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ induced by $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective on objects.*

Proof. (i). Let B be an object in $\mathbb{B}$. By hypothesis, there is a λ-small κ-filtered diagram $Y : \mathcal{J} \to \mathbb{B}$ such that each Yj is in the image of F and $B \cong \varinjlim_{\mathcal{J}} Y$. Thus, if B is a (κ, λ)-compact object in $\mathbb{B}$, then B must be a retract of some Yj (by corollary 0.2.13). But $\mathbb{A}$ is Cauchy-complete and $F : \mathbb{A} \to \mathbb{B}$ is fully faithful, so B must be isomorphic to some object in the image of F.

(ii). The assumptions imply each functor $\mathbb{B}(F-, B) : \mathbb{A}^{\mathrm{op}} \to \mathbf{Set}$ is a colimit for a λ-small κ-filtered diagram of functors of the form $\mathbb{A}(-, A)$ for various A

in $\mathbb{A}$. Hence, for each object B in $\mathbb{B}$, there exist an object UB in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ and bijections

$$\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})(A, UB) \cong \mathbb{B}(FA, B)$$

that are natural in A. Since the canonical embedding $\mathbb{A} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is dense, we thus obtain a functor $U : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ with the required property.

(iii). It is clear that U is a fully faithful functor that preserves colimits for λ-small κ-filtered diagrams. We may then apply proposition 0.2.42 to deduce that every $(\lambda, \mathbf{U})$-compact object in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is isomorphic to one in the image of U.

(iv). This follows from claim (iii) and the fact that the canonical embedding $\mathbb{A} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is dense.

(v). If $\kappa \lhd \lambda$, then theorem 0.2.31 says $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is a λ-accessible category, so we may apply the classification theorem (0.2.25) to deduce that $\bar{U} : \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A})$ is fully faithful and essentially surjective on objects. ■

Corollary 0.3.9 (Classification of compactly generated categories)**.** *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*. If either $\kappa = \lambda$ or $\kappa \lhd \lambda$, then the following are equivalent for a category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is a (κ, λ)-compactly generated* $\mathbf{U}$*-category.*

(ii) *$\mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{C})$ is a κ-accessible* $\mathbf{U}$*-category.*

(iii) *$\mathcal{C}$ is equivalent to $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{D})$ for some κ-accessible* $\mathbf{U}$*-category $\mathcal{D}$.*

Proof. (i) ⇒ (ii). See proposition 0.3.8.

(ii) ⇒ (iii). Apply proposition 0.2.22.

(iii) ⇒ (i). See proposition 0.3.7. ■

Definition 0.3.10. Let κ and λ be regular cardinals in a universe $\mathbf{U}$. A **(κ, λ)-compactly defined functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- $\mathcal{C}$ is a (κ, λ)-compactly generated $\mathbf{U}$-category.
- $F : \mathcal{C} \to \mathcal{D}$ preserves colimits for λ-small κ-filtered diagrams of (κ, λ)-compact objects in $\mathcal{C}$.

Lemma 0.3.11. *Let $\mathcal{C}$ be a (κ, λ)-compactly generated* **U***-category, let $\mathcal{D}$ be a locally* **U***-small category, and let* **Set** *be the category of* **U***-sets. If $F : \mathcal{C} \to \mathcal{D}$ is a (κ, λ)-compactly defined functor, then the natural maps*

$$\begin{aligned} \mathcal{D}(FC, D) &\to \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, C), \mathcal{D}(F-, D)) \\ f &\mapsto (c \mapsto f \circ Fc) \end{aligned}$$

are bijections.

Proof. Choose a λ-small κ-filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that each vertex is (κ, λ)-compact in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} X$. We then have a natural bijection

$$\mathcal{C}(A, C) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(A, X)$$

as A varies in $\mathbf{K}_\kappa^\lambda(\mathcal{C})$, so

$$\left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, C), \mathcal{D}(-, D)) \cong \varprojlim_{\mathcal{J}} \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, X), \mathcal{D}(F-, D))$$

and by applying the Yoneda lemma, we have

$$\varprojlim_{\mathcal{J}} \left[\mathbf{K}_\kappa^\lambda(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right](\mathcal{C}(-, X), \mathcal{D}(F-, D)) \cong \varprojlim_{\mathcal{J}} \mathcal{D}(FX, D)$$

but $F : \mathcal{C} \to \mathcal{D}$ preserves colimits for λ-small κ-filtered diagrams of (κ, λ)-compact objects in $\mathcal{C}$, so:

$$\varprojlim_{\mathcal{J}} \mathcal{D}(FX, D) \cong \mathcal{D}\left(\varinjlim_{\mathcal{J}} FX, D\right) \cong \mathcal{D}(FC, D)$$

We may therefore deduce that the indicated maps are bijections. ■

Proposition 0.3.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be (κ, λ)-compactly generated* **U***-categories. If $F : \mathcal{C} \to \mathcal{D}$ is a (κ, λ)-compactly defined functor, then the induced functor $\mathbf{Ind}_{\mathbf{U}}^\lambda(F) : \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C}) \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ is $(\kappa, \mathbf{U})$-accessible.*

Proof. Let $\mathcal{A} = \mathbf{K}_\kappa^\lambda(\mathcal{C})$, let $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ be the canonical embeddings and let $\bar{F} = \mathbf{Ind}_{\mathbf{U}}^\lambda(F)$. Theorems 0.2.21 and A.5.15 imply $\bar{F} : \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C}) \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ along $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$. By proposition 0.3.8, $\mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{C})$ and $\mathbf{Ind}_{\mathbf{U}}^\lambda(\mathcal{D})$ are κ-accessible **U**-categories, and to verify that $\bar{F}$ is a $(\kappa, \mathbf{U})$-accessible functor, it suffices to show that $\bar{F}$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F\big|_{\mathcal{A}}$ along $\gamma_{\mathcal{C}}\big|_{\mathcal{A}}$.

Since $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{D})$ preserves colimits for λ-small diagrams, the composite $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathbf{Ind}^{\lambda}_{\mathbf{U}}(\mathcal{D})$ is also a (κ, λ)-compactly defined functor, and so $\gamma_{\mathcal{D}} F$ is (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F|_{\mathcal{A}}$ along the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ (by lemma 0.3.11). We may therefore apply theorem A.5.20 to deduce that $\bar{F}$ is indeed (the functor part of) a pointwise left Kan extension of $\gamma_{\mathcal{D}} F|_{\mathcal{A}}$ along $\gamma_{\mathcal{C}}|_{\mathcal{A}}$. ■

Definition 0.3.13. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **strongly $(\kappa, \mathbf{U})$-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- Both $\mathcal{C}$ and $\mathcal{D}$ are κ-accessible $\mathbf{U}$-categories.
- F preserves colimits for $\mathbf{U}$-small κ-filtered diagrams.
- F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U})$-compact objects in $\mathcal{D}$.

Example 0.3.14. Given any functor $F : \mathbb{A} \to \mathbb{B}$, the induced functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(F) : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{A}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is strongly $(\kappa, \mathbf{U})$-accessible, by corollaries 0.2.13 and 0.2.16.

Proposition 0.3.15 (Products of accessible categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$. If $\left(\mathcal{C}_i \,\middle|\, i \in I\right)$ is a κ-small family of κ-accessible $\mathbf{U}$-categories, then their product $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$ is also a κ-accessible $\mathbf{U}$-category, and the projection functors $\mathcal{C} \to \mathcal{C}_i$ are strongly $(\kappa, \mathbf{U})$-accessible functors.*

Proof. It is clear that $\mathcal{C}$ has colimits for $\mathbf{U}$-small κ-filtered diagrams: indeed, they can be computed componentwise. Theorem 0.2.10 implies that an object in $\mathcal{C}$ is $(\kappa, \mathbf{U})$-compact as soon as its components are $(\kappa, \mathbf{U})$-compact objects in their respective categories; thus $\mathcal{C}$ is generated under $\mathbf{U}$-small κ-filtered colimits by a $\mathbf{U}$-small family of $(\kappa, \mathbf{U})$-compact objects, as required of a κ-compact $\mathbf{U}$-category. ■

Lemma 0.3.16. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$, let $\mathcal{C}$ be an accessible $\mathbf{U}$-category, let $\mathcal{D}$ be an accesible $\mathbf{U}^+$-category, and let $F : \mathcal{C} \to \mathcal{D}$ be a $(\kappa, \mathbf{U})$-accessible functor.*

(i) *There is a regular cardinal λ in $\mathbf{U}^+$ such that F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U}^+)$-compact objects in $\mathcal{D}$.*

(ii) *Moreover, if μ is a regular cardinal in $\mathbf{U}^+$ such that $\kappa \lhd \mu$ and $\lambda \leq \mu$, then F sends $(\mu, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U}^+)$-compact objects in $\mathcal{D}$.*

Proof. (i). Such a regular cardinal exists by remark 0.3.4 and proposition 0.2.22.

(ii). If μ is not in $\mathbf{U}$, then the claim is trivial; otherwise, proposition 0.2.42 and lemma 0.2.15 imply that F sends $(\mu, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U}^+)$-compact objects in $\mathcal{D}$, as required. ∎

Corollary 0.3.17. *Let $\mathcal{C}$ and $\mathcal{D}$ be accessible $\mathbf{U}$-categories. If $F : \mathcal{C} \to \mathcal{D}$ is a $(\kappa, \mathbf{U})$-accessible functor, then:*

(i) *There exists a regular cardinal λ in $\mathbf{U}$ such that F is strongly $(\lambda, \mathbf{U})$-accessible.*

(ii) *Moreover, if μ is a regular cardinal in $\mathbf{U}$ and $\lambda \lhd \mu$, then F is also strongly $(\mu, \mathbf{U})$-accessible.*

Proof. Combine lemma 0.3.16, theorem 0.2.31, and proposition 0.2.32. ∎

Lemma 0.3.18. *Let $\mathcal{J}$ be a κ-filtered category. If $\mathbb{A}$ is a κ-small category, then the functor category $[\mathbb{A}, \mathcal{J}]$ is also a κ-filtered category.*

Proof. There is a natural bijection between diagrams $\mathbb{D} \to [\mathbb{A}, \mathcal{J}]$ and diagrams $\mathbb{D} \times \mathbb{A} \to \mathcal{J}$; but if $\mathbb{D}$ is κ-small, then so is $\mathbb{D} \times \mathbb{A}$. Thus, every κ-small diagram in $[\mathbb{A}, \mathcal{J}]$ has a cocone, as required. ∎

Lemma 0.3.19. *Let $\mathcal{J}$ be a κ-filtered category, let $A : \mathcal{I} \to \mathcal{J}$ be a κ-small diagram, let ${}^{A/}\mathcal{J}$ be the cocone category $(A \downarrow \Delta)$, and let $P : {}^{A/}\mathcal{J} \to \mathcal{J}$ be the projection functor.*

(i) *The cocone category ${}^{A/}\mathcal{J}$ is also a κ-filtered category.*

(ii) *$P : {}^{A/}\mathcal{J} \to \mathcal{J}$ is a cofinal functor.*[9]

Proof. (i). Let $\mathbb{D}$ be a κ-small category. There exists a κ-small category $\tilde{\mathbb{D}}$ equipped with a functor $L : \mathcal{I} \to \tilde{\mathbb{D}}$ and a natural bijection between diagrams $X : \mathbb{D} \to {}^{A/}\mathcal{J}$ and diagrams $\tilde{X} : \tilde{\mathbb{D}} \to \mathcal{J}$ such that $\tilde{X} L = A$, and moreover this construction is natural in $\mathbb{D}$. Thus, every κ-small diagram in ${}^{A/}\mathcal{J}$ admits a cocone, as required.

(ii). We must show that the comma category $(b \downarrow P)$ is connected for all objects b in $\mathcal{J}$. Since $\mathcal{J}$ is filtered, there must exist an object c, a cocone $A \Rightarrow \Delta c$, and

[9] See definition A.5.31.

a morphism $b \to c$ in $\mathcal{J}$; thus, $(b \downarrow P)$ is inhabited. Moreover, any diagram in $[\mathcal{I}, \mathcal{J}]$ of the form shown below on the left can be completed to one of the form shown below on the right,

$$
\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
\Delta c & & \Delta d \\
& \Delta b \arrow[ul] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dl] \arrow[d, dashed] \arrow[dr] & \\
\Delta c \arrow[r, dashed] & \Delta e & \Delta d \arrow[l, dashed] \\
& \Delta b \arrow[ul] \arrow[u, dashed] \arrow[ur] &
\end{tikzcd}
$$

so we may conclude that $(b \downarrow P)$ is indeed connected. ■

Lemma 0.3.20. *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be strongly $(\kappa, \mathbf{U})$-accessible functors. Given an object (C, D, e) in the comma category $(F \downarrow G)$, if C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and D is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}$, then (C, D, e) is a $(\kappa, \mathbf{U})$-compact object in $(F \downarrow G)$.*

Proof. Let $\mathcal{B} = (F \downarrow G)$ and let $\varphi : FP \Rightarrow GQ$ be the canonical natural transformation. Then, given any two objects B and B' in $\mathcal{B}$, we have the following pullback diagram,

$$
\begin{tikzcd}
\mathcal{B}(B, B') \arrow[r] \arrow[d] & \mathcal{D}(QB, QB') \arrow[d] \\
\mathcal{C}(PB, PB') \arrow[r] & \mathcal{E}(FPB, GQB')
\end{tikzcd}
$$

where the map $\mathcal{C}(PB, PB') \to \mathcal{E}(FPB, GQB')$ is induced by the functor $F : \mathcal{C} \to \mathcal{E}$ and the morphism $\varphi_{B'} : FPB' \to GQB'$, and the map $\mathcal{D}(QB, QB') \to \mathcal{E}(FPB, GQB')$ is induced by the functor $G : \mathcal{D} \to \mathcal{E}$ and the morphism $\varphi_B : FPB \to GQB$. Thus, if PB and QB are $(\kappa, \mathbf{U})$-compact objects, then so are FPB and GQB, and therefore we may use theorem 0.2.10 deduce that B is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{B}$. ■

Theorem 0.3.21 (Accessibility of comma categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$ and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be $(\kappa, \mathbf{U})$-accessible functors.*

(i) *The comma category $(F \downarrow G)$ has colimits for $\mathbf{U}$-small κ-filtered diagrams, created by the projection functor $(F \downarrow G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *If F and G are strongly $(\kappa, \mathbf{U})$-accessible functors, then $(F \downarrow G)$ is a κ-accessible $\mathbf{U}$-category, and the projection functors $P : (F \downarrow G) \to \mathcal{C}$ and $Q : (F \downarrow G) \to \mathcal{D}$ are strongly $(\kappa, \mathbf{U})$-accessible.*

Proof. See Theorem 2.43 in [LPAC]. □

Corollary 0.3.22. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category and A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, then:*

- *The slice category ${}^{A/}\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and the projection functor ${}^{A/}\mathcal{C} \to \mathcal{C}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor.*
- *The slice category $\mathcal{C}_{/A}$ is a κ-accessible $\mathbf{U}$-category, and the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor.* ■

Corollary 0.3.23. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, then so is the functor category $[\mathbb{2}, \mathcal{C}]$, and moreover the $(\kappa, \mathbf{U})$-compact objects in $[\mathbb{2}, \mathcal{C}]$ are precisely the componentwise $(\kappa, \mathbf{U})$-compact objects.*

Proof. The functor category $[\mathbb{2}, \mathcal{C}]$ is isomorphic to the comma category $(\mathcal{C} \downarrow \mathcal{C})$, and $\mathrm{id} : \mathcal{C} \to \mathcal{C}$ is certainly a strongly $(\kappa, \mathbf{U})$-accessible functor. ■

Corollary 0.3.24. *If $\mathcal{C}$ is a (κ, λ)-compactly generated $\mathbf{U}$-category, then so is $[\mathbb{2}, \mathcal{C}]$.*

Proof. Combine lemma 0.3.20 and corollaries 0.3.9 and 0.3.23. ■

Lemma 0.3.25. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, with $\kappa \le \lambda$, let $\mathcal{E}$ be a locally $\mathbf{U}$-small category with colimits for $\mathbf{U}$-small κ-filtered diagrams, let $X : \mathcal{I} \to \mathcal{E}$ and $Y : \mathcal{J} \to \mathcal{E}$ be $\mathbf{U}$-small λ-filtered diagrams that are componentwise $(\lambda, \mathbf{U})$-compact, let $C = \varinjlim_{\mathcal{I}} X$ and $D = \varinjlim_{\mathcal{J}} Y$, and let $c_i : X i \to C$ and $d_j : Y j \to D$ be the components of the respective colimiting cocones.*

(i) *Given any object i_0 in $\mathcal{I}$ and any morphism $e : C \to D$, there exist an object j_0 in $\mathcal{J}$ and a morphism $f_0 : X i_0 \to Y j_0$ such that the following diagram commutes:*

$$\begin{tikzcd} X i_0 \arrow[r, "c_{i_0}"] \arrow[d, dashed, "f_0"'] & C \arrow[d, "e"] \\ Y j_0 \arrow[r, "d_{j_0}"'] & D \end{tikzcd}$$

(ii) *Given any commutative diagram of the above form, if $e : C \to D$ is an isomorphism in $\mathcal{E}$, then there exist chains $I : \kappa \to \mathcal{I}$ and $J : \kappa \to \mathcal{J}$ such that $I(0) = i_0$, $J(0) = j_0$ and a factorisation of the form below,*

$$\begin{tikzcd}
X i_0 \arrow[r] \arrow[d, "f_0"'] & C' \arrow[r] \arrow[d, "e'"] & C \arrow[d, "e"] \\
Y j_0 \arrow[r] & D' \arrow[r] & D
\end{tikzcd}$$

where $C' = \varinjlim_{\alpha<\kappa} X I(\alpha)$, $D' = \varinjlim_{\alpha<\kappa} Y J(\alpha)$, $e : C' \to D'$ is an isomorphism, and the morphisms $C' \to C$ and $D' \to D$ are the ones induced by the evident cocones.

Proof. (i). Since $X i_0$ is $(\lambda, \mathbf{U})$-compact and $Y : \mathcal{J} \to \mathcal{E}$ is a $\mathbf{U}$-small λ-filtered diagram, such a factorisation of $e \circ c_{i_0}$ must exist, by lemma 0.2.12.

(ii). We will construct I, J, and e' by transfinite induction on κ.

- Given j_α and f_α, choose a morphism $i_{\alpha\to\alpha+1} : i_\alpha \to i_{\alpha+1}$ in $\mathcal{I}$ and a morphism $g_\alpha : Y j_\alpha \to X i_{\alpha+1}$ in $\mathcal{E}$ such that the diagram below commutes:

$$\begin{tikzcd}
X i_\alpha \arrow[r, dashed, "X i_{\alpha\to\alpha+1}"] \arrow[d, "f_\alpha"'] & X i_{\alpha+1} \arrow[r, "c_{i_{\alpha+1}}"] & C \\
Y j_\alpha \arrow[ur, dashed, "g_\alpha"'] \arrow[rr, "d_{j_\alpha}"'] & & D \arrow[u, "e^{-1}"']
\end{tikzcd}$$

  Such $i_{\alpha\to\alpha+1}$ and g_α exist because $f_\alpha : X i_\alpha \to Y j_\alpha$ defines a $(\lambda, \mathbf{U})$-compact object in the slice category ${}^{X i_\alpha/}\mathcal{E}$ (by lemma 0.3.20) and there is an evident $\mathbf{U}$-small λ-filtered diagram ${}^{i_\alpha/}X : {}^{i_\alpha/}\mathcal{I} \to {}^{X i_\alpha/}\mathcal{E}$ with colimit defined by $c_{i_\alpha} : X i_\alpha \to C$ (by lemma 0.3.19).

- Given $i_{\alpha+1}$ and g_α, choose a morphism $j_{\alpha\to\alpha+1} : j_\alpha \to j_{\alpha+1}$ in $\mathcal{J}$ and a morphism $f_{\alpha+1} : X i_{\alpha+1} \to Y j_{\alpha+1}$ in $\mathcal{E}$ such that the diagram below commutes:

$$\begin{tikzcd}
Y j_\alpha \arrow[r, dashed, "Y j_{\alpha\to\alpha+1}"] \arrow[d, "g_\alpha"'] & Y j_{\alpha+1} \arrow[r, "d_{i_{\alpha+1}}"] & D \\
X i_{\alpha+1} \arrow[ur, dashed, "f_{\alpha+1}"'] \arrow[rr, "c_{i_{\alpha+1}}"'] & & C \arrow[u, "e"']
\end{tikzcd}$$

- Given a limit ordinal $\beta < \kappa$ and i_α for all ordinals $\alpha < \beta$, choose an object i_β in $\mathcal{I}$ and a cocone from the chain defined by $\left(i_\alpha \,\middle|\, \alpha < \beta\right)$ to i_β.

- Given i_β for a limit ordinal $\beta < \kappa$ and j_α for all ordinals $\alpha < \beta$, choose an object j_β in $\mathcal{J}$, a cocone from the chain defined by $\left(j_\alpha \,\middle|\, \alpha < \beta\right)$, and a morphism $f_\beta : X i_\beta \to Y j_\beta$ such that the following diagram commutes for all ordinals $\alpha < \beta$:

$$\begin{tikzcd} Y j_\alpha \arrow[r, dashed, "Y j_{\alpha \to \beta}"] \arrow[d, "g_\alpha"'] & Y j_\beta \arrow[r, "d_{j_\beta}"] & D \\ X i_{\alpha+1} \arrow[r, "X i_{\alpha+1 \to \beta}"'] & X i_\beta \arrow[u, dashed, "f_\beta"'] \arrow[r, "c_{i_\beta}"'] & C \arrow[u, "e"'] \end{tikzcd}$$

  Such data exist because the chains X' and Y' defined by $\left(X i_\alpha \,\middle|\, \alpha < \beta\right)$ and $\left(Y j_\alpha \,\middle|\, \alpha < \beta\right)$ are $(\lambda, \mathbf{U})$-compact objects in the category $[\beta, \mathcal{E}]$ (by proposition 0.2.44) and there is an evident $\mathbf{U}$-small λ-filtered diagram in ${}^{Y'}/[\beta, \mathcal{E}]$ with colimit ΔD (by lemmas 0.3.18 and 0.3.19).

Now take $I : \kappa \to \mathcal{I}$ and $J : \kappa \to \mathcal{J}$ to be the chains defined by $I(\alpha) = i_\alpha$ and $J(\alpha) = j_\alpha$. Let $C' = \varinjlim_{\alpha < \kappa} X i_\alpha$ and $D' = \varinjlim_{\alpha < \kappa} Y j_\alpha$. The above construction yields commutative diagrams of the form below for all ordinals $\alpha < \beta < \kappa$,

$$\begin{tikzcd} X i_\alpha \arrow[r, "X i_{\alpha \to \beta}"] \arrow[d, "f_\alpha"'] & X i_\beta \arrow[d, "f_\beta"] \\ Y j_\alpha \arrow[r, "Y j_{\alpha \to \beta}"'] & Y j_\beta \end{tikzcd} \qquad \begin{tikzcd} Y j_\alpha \arrow[r, "Y j_{\alpha \to \beta}"] \arrow[d, "g_\alpha"'] & Y j_\beta \arrow[d, "g_\beta"] \\ X i_{\alpha+1} \arrow[r, "X i_{\alpha+1 \to \beta+1}"'] & X i_{\beta+1} \end{tikzcd}$$

so there are induced morphisms $f : C' \to D'$ and $g : D' \to C'$; moreover, since $g_\alpha \circ f_\alpha = X i_{\alpha \to \alpha+1}$ and $f_{\alpha+1} \circ g_\alpha = Y j_{\alpha \to \alpha+1}$, we have $g \circ f = \mathrm{id}_{C'}$ and $f \circ g = \mathrm{id}_{D'}$. Thus, we have the required isomorphism $e : C' \to D'$. ■

Theorem 0.3.26 (Accessibility of iso-comma categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories with colimits for $\mathbf{U}$-small λ-filtered diagrams, and let $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ be be functors of $\mathbf{U}$-rank $\leq \kappa$.*

(i) *The iso-comma category $(F \wr G)$ has colimits for $\mathbf{U}$-small κ-filtered diagrams, created by the projection functor $(F \wr G) \to \mathcal{C} \times \mathcal{D}$.*

(ii) *Assuming F and G are strongly λ-accessible functors, given an object (C, D, e) in $(F \wr G)$, if C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and D is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}$, then (C, D, e) is a $(\kappa, \mathbf{U})$-compact object in $(F \wr G)$.*

(iii) *If F and G are strongly $(\lambda, \mathbf{U})$-accessible functors and $\kappa < \lambda$, then $(F \wr G)$ is a λ-accessible $\mathbf{U}$-category, and the projection functors $P : (F \wr G) \to \mathcal{C}$ and $Q : (F \wr G) \to \mathcal{D}$ are strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). This is a straightforward consequence of the hypothesis that both $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve colimits for $\mathbf{U}$-small κ-filtered diagrams.

(ii). Since the iso-comma category $(F \wr G)$ is a full subcategory of the comma category $(F \downarrow G)$, the claim is an immediate corollary of lemma 0.3.20.

(iii). Let $\mathcal{B} = (F \wr G)$. First, we must show that there is a $\mathbf{U}$-small set of $(\lambda, \mathbf{U})$-compact objects in $\mathcal{B}$ that generate $\mathcal{B}$ under colimits for $\mathbf{U}$-small λ-filtered colimits. Let (C, D, e) be an object in $\mathcal{B}$. Since $\kappa \lhd \lambda$, we may choose $\mathbf{U}$-small skeletons $\mathcal{I}$ and $\mathcal{J}$ of the comma categories $\left(\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C}) \downarrow C\right)$ and $\left(\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{D}) \downarrow D\right)$ and obtain $\mathbf{U}$-small λ-filtered diagrams $X : \mathcal{I} \to \mathcal{C}$ and $Y : \mathcal{J} \to \mathcal{D}$ that are componentwise $(\lambda, \mathbf{U})$-compact and have $C \cong \varinjlim_{\mathcal{I}} X$ and $D \cong \varinjlim_{\mathcal{J}} Y$ (by proposition 0.2.23 and theorem 0.2.31). Let $\mathcal{K}$ be full subcategory of the iso-comma category $(FX \wr GY)$ spanned by those objects (i, j, f) such that the following diagram commutes,

$$\begin{tikzcd} FXi \arrow[r, "Fc_i"] \arrow[d, "f"'] & FC \arrow[d, "e"] \\ GYj \arrow[r, "Gd_j"'] & GD \end{tikzcd}$$

where $c_i : Xi \to C$ and $d_j : Yj \to D$ are the components of the respective colimiting cocones. Let $P' : \mathcal{K} \to \mathcal{I}$ and $Q' : \mathcal{K} \to \mathcal{J}$ be the projection functors, and let $Z : \mathcal{K} \to \mathcal{B}$ be the evident diagram with $PZ = FXP'$ and $QZ = GYQ'$. It is clear that $\mathcal{K}$ is a $\mathbf{U}$-small category, and we claim $Z : \mathcal{K} \to \mathcal{B}$ is λ-filtered diagram with (C, D, e) as its colimit.

First, we verify that (C, D, e) is a colimit for the diagram $Z : \mathcal{K} \to \mathcal{B}$. Let i be any object in $\mathcal{I}$ and consider the comma category $(i \downarrow P')$. Lemma 0.3.25 implies it is inhabited. Suppose we have two objects in $(i \downarrow P')$, i.e. two objects (i_0, j_0, f_0) and (i_1, j_1, f_1) in $\mathcal{K}$ and two morphisms $h_0 : i \to i_0$ and $h_1 : i \to i_1$ in $\mathcal{I}$. Since $\mathcal{I}$ is a filtered category, there exist an object i' in $\mathcal{I}$ and morphisms

$h'_0 : i_0 \to i'$ and $h'_1 : i_1 \to i'$ such that $h'_0 \circ h_0 = h'_1 \circ h_1$. Similarly, $\mathcal{J}$ is a filtered category, so there exist an object j_2 in $\mathcal{J}$ and morphisms $j_0 \to j_2$ and $j_1 \to j_2$. By considering a suitable diagram of shape ${}^{j_2/}\mathcal{J}$ in the category ${}^{(GYj_0, GYj_1)/}\mathcal{E} \times \mathcal{E}$ (using the fact that $f_0 : FXi_0 \to GYj_0$ and $f_1 : FXi_1 \to GYj_1$ are isomorphisms in $\mathcal{E}$) and applying lemmas 0.3.18 and 0.3.19, we see that there is a commutative diagram in $\mathcal{E}$ of the form shown below,

$$\begin{tikzcd}
 & FXi_0 \arrow[rr, bend left, "Fc_{i_0}"] \arrow[r, "FXh'_0"] \arrow[ddd, "f_0"'] & FXi' \arrow[r, "Fc_{i'}"] \arrow[ddd, dashed, "f'"] & FC \arrow[ddd, "e"] \\
 & & & \\
FXi_1 \arrow[uur, "Fh'_1"] \arrow[uurr, bend right, "Fc_{i_1}"] \arrow[ddd, "f_1"'] & & & \\
 & GYj_0 \arrow[rr, bend left, "Gd_{j_0}"] \arrow[r, dashed, "GYk'_0"'] & GYj' \arrow[r, "Gd_{j'}"'] & GD \\
 & & & \\
GYj_1 \arrow[uur, dashed, "GYk'_1"] \arrow[uurr, bend right, "Gd_{j_1}"'] & & &
\end{tikzcd}$$

and applying lemmas 0.2.15 and 0.3.25, we may assume that $f' : FXi' \to GYj'$ is an isomorphism in $\mathcal{E}$. Thus, the comma category $(i \downarrow P')$ is connected, and therefore $P' : \mathcal{K} \to \mathcal{I}$ is a cofinal functor. The symmetric argument shows that $Q' : \mathcal{K} \to \mathcal{J}$ is also a cofinal functor, and since $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ preserve colimits for $\mathbf{U}$-small λ-filtered diagrams, we may deduce that the canonical cocone from Z to (C, D, e) in $\mathcal{B}$ is a colimiting cocone.

It remains to be shown that $\mathcal{K}$ is a $\mathbf{U}$-small λ-filtered category. Indeed, suppose $K : \mathbb{D} \to \mathcal{K}$ is a λ-small diagram. Since $\mathcal{I}$ is a λ-filtered category, there is an object i_0 in $\mathcal{I}$ with a cocone $P'K \Rightarrow \Delta i_0$, and by considering a suitable λ-filtered diagram in the category ${}^{GQ'K/}[\mathbb{D}, \mathcal{E}]$, we obtain an object j_0 in $\mathcal{J}$ and a morphism $f_0 : FXi_0 \to GYj_0$ such that the diagram below commutes,

$$\begin{tikzcd}
FXi_0 \arrow[r, "Fc_{i_0}"] \arrow[d, "f_0"'] & FC \arrow[d, "e"] \\
GYj_0 \arrow[r, "Gd_{j_0}"'] & GD
\end{tikzcd}$$

as well as a cocone from K to $\left(Xi_0, Yj_0, f_0\right)$ in the comma category $(F \downarrow G)$ that is compatible with the colimiting cocone $GY \Rightarrow \Delta GD$. Combining lemmas 0.2.15 and 0.3.25, we then obtain a cocone under P in $\mathcal{K}$, as required. This shows that every object in $\mathcal{B}$ is a colimit for a $\mathbf{U}$-small λ-filtered diagram of componentwise $(\lambda, \mathbf{U})$-compact objects in $\mathcal{B}$, and since $\mathcal{C}$ and $\mathcal{D}$ are λ-accessible $\mathbf{U}$-categories, proposition 0.2.22 implies the full subcategory of $\mathcal{B}$ spanned by such componentwise $(\lambda, \mathbf{U})$-compact objects is essentially $\mathbf{U}$-small.

Finally, observe that every $(\lambda, \mathbf{U})$-compact object in $\mathcal{B}$ is a retract of a componentwise $(\lambda, \mathbf{U})$-compact object (because the set of such objects generate $\mathcal{B}$ under colimits for $\mathbf{U}$-small λ-filtered diagrams), and thus we may apply corollary 0.2.16 to deduce that every $(\lambda, \mathbf{U})$-compact object in $\mathcal{B}$ is itself componentwise $(\lambda, \mathbf{U})$-compact. Thus the projection functors $P : \mathcal{B} \to \mathcal{C}$ and $Q : \mathcal{B} \to \mathcal{D}$ are strongly $(\lambda, \mathbf{U})$-accessible. ■

Definition 0.3.27. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible $\mathbf{U}$-subcategory** of a κ-accessible $\mathbf{U}$-category $\mathcal{C}$ is a subcategory $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathcal{B}$ is a κ-accessible $\mathbf{U}$-category and the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a $(\kappa, \mathbf{U})$-accessible functor.

Proposition 0.3.28. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category and let $\mathcal{B}$ be a replete and full κ-accessible $\mathbf{U}$-subcategory of $\mathcal{C}$.*

(i) *If A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and A is in $\mathcal{B}$, then A is also a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$.*

(ii) *If the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly κ-accessible, then $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{B}) = \mathcal{B} \cap \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$.*

Proof. (i). This is clear, since hom-sets and colimits for $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{B}$ are computed as in $\mathcal{C}$.

(ii). Given claim (i), it suffices to show that every $(\kappa, \mathbf{U})$-compact object in $\mathcal{B}$ is also $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$, but this is precisely the hypothesis that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is strongly κ-accessible. ■

Proposition 0.3.29. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ and $\mathcal{E}$ be categories with colimits for $\mathbf{U}$-small κ-filtered diagrams, let $\mathcal{D}$ be a replete and full subcategory of $\mathcal{E}$ that is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams, let $F : \mathcal{C} \to \mathcal{E}$ be a functor of $\mathbf{U}$-rank $\leq \kappa$, and let $\mathcal{B}$ be the preimage* *of $\mathcal{D}$ under F, so that we have the following strict pullback diagram:*

$$\begin{tikzcd} \mathcal{B} \arrow[r] \arrow[d, hook] & \mathcal{D} \arrow[d, hook] \\ \mathcal{C} \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

(i) *$\mathcal{B}$ is a replete and full subcategory of $\mathcal{D}$ and is closed under colimits for* $\mathbf{U}$*-small κ-filtered diagrams in $\mathcal{D}$.*

(ii) *If $F : \mathcal{C} \to \mathcal{E}$ and the inclusion $\mathcal{D} \hookrightarrow \mathcal{E}$ are strongly $(\lambda, \mathbf{U})$-accessible functors and $\kappa < \lambda$, then $\mathcal{B}$ is a λ-accessible* $\mathbf{U}$*-subcategory of $\mathcal{C}$ and the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is also strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). This is a straightforward exercise.

(ii). Consider the iso-comma category $(F \wr \mathcal{D})$ and the induced comparison functor $K : \mathcal{B} \to (F \wr \mathcal{D})$. It is clear that $\mathcal{B}$ is fully faithful; but since $\mathcal{D}$ is a replete subcategory of $\mathcal{C}$, for every object (C, D, e) in $(F \wr \mathcal{D})$, there is a canonical isomorphism $KC \to (C, D, e)$, namely the one corresponding to the following commutative diagram in $\mathcal{E}$:

$$\begin{tikzcd} FC \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & FC \arrow[d, "e"] \\ FC \arrow[r, "e"'] & D \end{tikzcd}$$

Thus, $K : \mathcal{B} \to (F \wr \mathcal{D})$ is (half of) an equivalence of categories. Theorem 0.3.26 says the projection $P : (F \wr \mathcal{D}) \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor, so we may deduce that the same is true for the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$. ■

Proposition 0.3.30. *Let κ be a regular cardinal in a universe* $\mathbf{U}$*, let $F : \mathcal{C} \to \mathcal{D}$ be a strongly $(\kappa, \mathbf{U})$-accessible functor, and let $\mathcal{D}'$ be the full subcategory of $\mathcal{D}$ spanned by the image of F.*

(i) *Every object in $\mathcal{D}'$ is a colimit for some* $\mathbf{U}$*-small κ-filtered diagram consisting of objects in $\mathcal{D}'$ that are $(\kappa, \mathbf{U})$-compact as objects in $\mathcal{D}$.*

(ii) *Every $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}'$ is also $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$.*

(iii) *If $\mathcal{D}'$ is closed under colimits for* $\mathbf{U}$*-small κ-filtered diagrams in $\mathcal{D}$, then $\mathcal{D}'$ is a κ-accessible* $\mathbf{U}$*-subcategory of $\mathcal{D}$.*

Proof. (i). Let D be any object in $\mathcal{D}$. By definition, there is an object C in $\mathcal{C}$ such that $D = FC$, and since $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, there is a $\mathbf{U}$-small κ-filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that each Xj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ and $C \cong \varinjlim_{\mathcal{J}} X$. Since $F : \mathcal{C} \to \mathcal{D}$ is a strongly $(\kappa, \mathbf{U})$-accessible functor, each FXj is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}$ and we have $D \cong \varinjlim_{\mathcal{J}} FX$.

(ii). Moreover, if D is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{D}'$, then D must be a retract of FXj for some object j in $\mathcal{J}$, and so D is also $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$.

(iii). Any object in $\mathcal{D}'$ that is $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}$ must be $(\kappa, \mathbf{U})$-compact as an object in $\mathcal{D}'$, because $\mathcal{D}'$ is a full subcategory of $\mathcal{D}$ that is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams. ■

Theorem 0.3.31 (The category of algebras for an accessible monad)**.** *Let $\mathcal{C}$ be a locally κ-presentable* $\mathbf{U}$*-category, let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$, and let $\mathcal{C}^{\mathsf{T}}$ be the category of algebras for T. If $T : \mathcal{C} \to \mathcal{C}$ is a $(\kappa, \mathbf{U})$-accessible functor, then:*

(i) *The forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ creates colimits for* $\mathbf{U}$*-small κ-filtered diagrams and creates limits for all* $\mathbf{U}$*-small diagrams.*

(ii) *$\mathcal{C}^{\mathsf{T}}$ is a locally κ-presentable* $\mathbf{U}$*-category.*

Proof. (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). See Theorem 2.78 and the following remark in [LPAC], or Theorem 5.5.9 in [Borceux, 1994b]. □

Lemma 0.3.32. *Let $\mathcal{C}$ be a locally κ-presentable category and let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. If the forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\kappa, \mathbf{U})$-accessible, then so is the functor $T : \mathcal{C} \to \mathcal{C}$.*

Proof. The accessible adjoint functor theorem (0.2.47) says the free T-algebra functor $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ is strongly $(\kappa, \mathbf{U})$-accessible if the forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is $(\kappa, \mathbf{U})$-accessible; but $T = UF$, so T is strongly $(\kappa, \mathbf{U})$-accessible when U is. ■

Theorem 0.3.33 (The category of algebras for a strongly accessible monad)**.** *Let $\mathcal{C}$ be a locally λ-presentable* $\mathbf{U}$*-category, let $\mathsf{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$ where $T : \mathcal{C} \to \mathcal{C}$ has* $\mathbf{U}$*-rank κ, and let $\mathcal{C}^{\mathsf{T}}$ be the category of algebras for T. If $T : \mathcal{C} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor and $\kappa < \lambda$, then:*

(i) *Given a coequaliser diagram in $\mathcal{C}^{\mathsf{T}}$ of the form below,*

$$(A, \alpha) \rightrightarrows (B, \beta) \longrightarrow (C, \gamma)$$

if A and B are $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$, then so is C.

(ii) *Given a λ-small family $\big((A_i, \alpha_i) \,\big|\, i \in I\big)$ of T-algebras, if each A_i is a $(\lambda, \mathbf{U})$-small object in $\mathcal{C}$, then so is the underlying object of the T-algebra coproduct $\sum_{i \in I} (A_i, \alpha_i)$.*

(iii) *The forgetful functor $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). By referring to the explicit construction of coequalisers in $\mathcal{C}^{\mathsf{T}}$ given in the proof of Proposition 4.3.6 in [Borceux, 1994b] and applying lemma 0.2.15, we see that C is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$ when A and B are, provided $T : \mathcal{C} \to \mathcal{C}$ has $\mathbf{U}$-rank κ *and* is strongly $(\lambda, \mathbf{U})$-accessible.

(ii). Let $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ be a left adjoint for $U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$. In the proof of Proposition 4.3.4 in [Borceux, 1994b], we find that the T-algebra coproduct $\sum_{i \in I} (A_i, \alpha_i)$ may be computed by a coequaliser diagram of the following form:

$$F\big(\textstyle\sum_{i \in I} T A_i\big) \rightrightarrows F\big(\textstyle\sum_{i \in I} A_i\big) \xrightarrow{h} \textstyle\sum_{i \in I} (A_i, \alpha_i)$$

Since $T : \mathcal{C} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible, the underlying objects of the T-algebras $F\big(\sum_{i \in I} T A_i\big)$ and $F\big(\sum_{i \in I} A_i\big)$ are $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$. Thus, by claim (i), the underlying object of $\sum_{i \in I} (A_i, \alpha_i)$ must also be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$.

(iii). It is shown in the proof of Theorem 5.5.9 in [Borceux, 1994b] that the full subcategory $\mathcal{F}$ of $\mathcal{C}^{\mathsf{T}}$ spanned by the image of $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ under $F : \mathcal{C} \to \mathcal{C}^{\mathsf{T}}$ is a dense subcategory. Let $\mathcal{G}$ be the smallest replete full subcategory of $\mathcal{C}^{\mathsf{T}}$ that is closed under colimits for λ-small diagrams in $\mathcal{C}$ and that contains $\mathcal{F}$. Observe that claims (i) and (ii) imply that the underlying object of every T-algebra that is in $\mathcal{G}$ must be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. To show that the forgetful functor

$U : \mathcal{C}^{\mathsf{T}} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible, it is enough to verify that every $(\lambda, \mathbf{U})$-compact object is in $\mathcal{G}$.

It is not hard to see that the comma category $(\mathcal{G} \downarrow (A, \alpha))$ is then an essentially $\mathbf{U}$-small λ-filtered category for any T-algebra (A, α), and moreover, it can be shown that the tautological cocone for the canonical diagram $(\mathcal{G} \downarrow (A, \alpha)) \to \mathcal{C}^{\mathsf{T}}$ is a colimiting cocone. Thus, if (A, α) is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}^{\mathsf{T}}$, it must be a retract of an object in $\mathcal{G}$. But $\mathcal{G}$ is closed under retracts, so (A, α) is indeed in $\mathcal{G}$. ■

Definition 0.3.34. Let $\mathcal{C}$ be any category.

- A **pointed endofunctor** on $\mathcal{C}$ is a functor $J : \mathcal{C} \to \mathcal{C}$ equipped with a natural transformation $\iota : \mathrm{id}_{\mathcal{C}} \Rightarrow J$.
- An **algebra** for a pointed endofunctor (J, ι) on $\mathcal{C}$ is an object A in $\mathcal{C}$ equipped with a morphism $\alpha : JA \to A$ such that $\alpha \circ \iota_A = \mathrm{id}_A$.
- A **homomorphism of algebras** for a pointed endofunctor (J, ι) on $\mathcal{C}$, say $f : (A, \alpha) \to (B, \beta)$, is a morphism $f : A \to B$ making the following diagram commute:

$$\begin{tikzcd} JA \arrow[r, "Jf"] \arrow[d, "\alpha"'] & JB \arrow[d, "\beta"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

We write $\mathcal{C}^{(J,\iota)}$ for the category of algebras for a pointed endofunctor (J, ι) on $\mathcal{C}$.

The following result on the existence of free algebras for a pointed endofunctor is a special case of a general construction due to [Kelly, 1980].

Theorem 0.3.35 (Free algebras for a pointed endofunctor)**.** *Let κ be a regular cardinal, let $\mathcal{C}$ be a category with pushouts and colimits for chains of length $\leq \kappa$, and let (J, ι) be a pointed endofunctor on $\mathcal{C}$ such that $J : \mathcal{C} \to \mathcal{C}$ preserves colimits for κ-chains.*

(i) *The forgetful functor $U : \mathcal{C}^{(J,\iota)} \to \mathcal{C}$ has a left adjoint, say $F : \mathcal{C} \to \mathcal{C}^{(J,\iota)}$.*

(ii) *Let λ be a regular cardinal in a universe $\mathbf{U}$. If $J : \mathcal{C} \to \mathcal{C}$ sends $(\lambda, \mathbf{U})$-compact objects to $(\lambda, \mathbf{U})$-compact objects and $\kappa < \lambda$, then the functor $UF : \mathcal{C} \to \mathcal{C}$ has the same property.*

Proof. Let X be an object in $\mathcal{C}$. We now define a chain $X_\bullet : \kappa + 2 \to \mathcal{C}$ by transfinite induction:

- Let $X_0 = X$, let $X_1 = J X_0$, let $q_0 = \mathrm{id}_{J X_0}$, and let $X_{0\to 1} : X_0 \to X_1$ be ι_{X_0}.
- Given $q_\alpha : J X_\alpha \to X_{\alpha+1}$ for an ordinal $\alpha < \kappa$, define $X_{\alpha+2}$ by the following coequaliser diagram in $\mathcal{C}$:

$$\begin{tikzcd} J X_\alpha \arrow[r, shift left, "J q_\alpha \circ J \iota_{X_\alpha}"] \arrow[r, shift right, "J q_\alpha \circ \iota_{J X_\alpha}"'] & J X_{\alpha+1} \arrow[r, "q_{\alpha+1}"] & X_{\alpha+2} \end{tikzcd}$$

  Then, for all $\alpha' < \alpha + 2$, set $X_{\alpha'\to\alpha+2} = q_{\alpha+1} \circ \iota_{X_{\alpha+1}} \circ X_{\alpha'\to\alpha+1}$; note that the diagram below commutes:

$$\begin{tikzcd} J X_\alpha \arrow[r, "J X_{\alpha\to\alpha+1}"] \arrow[d, "q_\alpha"'] & J X_{\alpha+1} \arrow[d, "q_{\alpha+1}"] \\ X_{\alpha+1} \arrow[r, "X_{\alpha+1\to\alpha+2}"'] & X_{\alpha+2} \end{tikzcd}$$

- Given a limit ordinal $\beta \leq \kappa$ and q_α for all ordinals $\alpha < \beta$, define $X_\beta = \varinjlim_{\alpha<\beta} X_\alpha$ and take $X_{\beta\to\alpha} : X_\beta \to X_\alpha$ to be the component of the colimiting cocone; then define $X_{\beta+1}$ to be the colimit of the following diagram,

$$\begin{tikzcd} J X_0 \arrow[r] \arrow[d, "q_0"'] & J X_1 \arrow[r] \arrow[d, "q_1"'] & J X_2 \arrow[r] \arrow[d, "q_2"'] & \cdots \arrow[rr] & & J X_\beta \\ X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_\beta & \end{tikzcd}$$

  and let $q_\beta : J X_\beta \to X_{\beta+1}$ and $X_{\beta\to\beta+1} : X_\beta \to X_{\beta+1}$ be the respective components of the colimiting cocone; note that the following diagram commutes,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\iota_{X_0}"'] & X_1 \arrow[r] \arrow[d, "\iota_{X_1}"'] & X_2 \arrow[r] \arrow[d, "\iota_{X_2}"'] & \cdots \arrow[rr] & & X_\beta \arrow[d, "\iota_{X_\beta}"] \\ J X_0 \arrow[r] \arrow[d, "q_0"'] & J X_1 \arrow[r] \arrow[d, "q_1"'] & J X_2 \arrow[r] \arrow[d, "q_2"'] & \cdots \arrow[rr] & & J X_\beta \arrow[d, "q_\beta"] \\ X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots \arrow[r] & X_\beta \arrow[r] & X_{\beta+1} \end{tikzcd}$$

so we have $X_{\beta\to\beta+1} = q_\beta \circ \iota_{X_\beta}$.

Our hypothesis is that J preserves colimits for κ-chains, so the canonical comparison $\varinjlim_{\alpha<\kappa} JX_\alpha \to JX_\kappa$ is an isomorphism, as is $X_{\kappa\to\kappa+1}$. However, for all ordinals $\alpha < \beta < \kappa$, we have

$$X_{\alpha+1\to\beta+1} \circ q_\alpha = q_\beta \circ JX_{\alpha\to\beta}$$

so there is a unique morphism $\gamma_X : JX_\kappa \to X_\kappa$ such that

$$\gamma_X \circ JX_{\alpha\to\kappa} = X_{\alpha+1\to\kappa} \circ q_\alpha$$

for all ordinals $\alpha < \kappa$. Moreover, we have

$$\gamma_X \circ \iota_{X_\kappa} \circ X_{\alpha\to\kappa} = \gamma_X \circ JX_{\alpha\to\kappa} \circ \iota_{X_\alpha} = X_{\alpha+1\to\kappa} \circ q_\alpha \circ \iota_{X_\alpha} = X_{\alpha\to\kappa}$$

and $\left\{X_{\alpha\to\kappa} \,\middle|\, \alpha < \kappa\right\}$ is a jointly epimorphic family, so $\gamma_X \circ \iota_{X_\kappa} = \mathrm{id}_{X_\kappa}$, i.e. $\left(X_\kappa, \gamma_X\right)$ is a (J, ι)-algebra.

It remains to be shown that $\left(X_\kappa, \gamma_X\right)$ is a free (J, ι)-algebra generated by X. Let $\eta_X = X_{0\to\kappa}$, let (D, δ) be any (J, ι)-algebra, and let $f : X \to D$ be any morphism in $\mathcal{C}$. We construct a cocone $f_\bullet : X_\bullet \Rightarrow \Delta D$ by transfinite induction:

- Let $f_0 = f$, let $f_1 = \delta \circ Jf_0$, and note that $\delta \circ Jf_0 = f_1 \circ q_0$.
- Given $f_\alpha : X_\alpha \to D$ and $f_{\alpha+1} : X_{\alpha+1} \to D$ such that $f_{\alpha+1} \circ q_\alpha = \delta \circ Jf_\alpha$, let $f_{\alpha+2} : X_{\alpha+2} \to D$ be the unique morphism satisfying the following equation:

$$f_{\alpha+2} \circ q_{\alpha+1} = \delta \circ Jf_{\alpha+1}$$

Note that such a morphism exists because the diagrams below commute,

$$\begin{tikzcd}
JX_\alpha \arrow[r, "\iota_{JX_\alpha}"] \arrow[d, "q_\alpha"'] & JJX_\alpha \arrow[d, "Jq_\alpha"] \\
X_{\alpha+1} \arrow[r, "\iota_{JX_{\alpha+1}}"] \arrow[d, "f_{\alpha+1}"'] & JX_{\alpha+1} \arrow[d, "Jf_{\alpha+1}"] \\
D \arrow[r, "\iota_A"'] \arrow[dr, "\mathrm{id}"'] & JD \arrow[d, "\delta"] \\
 & D
\end{tikzcd}
\qquad
\begin{tikzcd}
JX_\alpha \arrow[r, "Jf_\alpha"] \arrow[d, "J\iota_{X_\alpha}"'] & JD \arrow[r, equal] \arrow[d, "J\iota_D"] & JD \arrow[dd, "\delta"] \\
JJX_\alpha \arrow[r, "JJf_\alpha"] \arrow[d, "Jq_\alpha"'] & JJD \arrow[d, "J\delta"] & \\
JX_{\alpha+1} \arrow[r, "Jf_{\alpha+1}"'] & JD \arrow[r, "\delta"'] & D
\end{tikzcd}$$

i.e. because the equation below holds,

$$\left(\delta \circ J f_{\alpha+1}\right) \circ \left(J q_\alpha \circ \iota_{J X_\alpha}\right) = \left(\delta \circ J f_{\alpha+1}\right) \circ \left(J q_\alpha \circ J \iota_{X_\alpha}\right)$$

and $q_{\alpha+1} : J X_{\alpha+1} \to X_{\alpha+2}$ is the coequaliser of $J q_\alpha \circ \iota_{J X_\alpha}$ and $J q_\alpha \circ J \iota_{X_\alpha}$.

- Given a limit ordinal $\beta \leq \kappa$, we define $f_\beta : X_\beta \to D$ be the unique morphism such that $f_\beta \circ X_{\alpha\to\beta} = f_\alpha$ for all ordinals $\alpha < \beta$; we may do this because the following equation holds:

$$f_{\alpha+1} \circ X_{\alpha\to\alpha+1} = f_{\alpha+1} \circ q_\alpha \circ \iota_{X_{\alpha+1}} = \delta \circ J f_\alpha \circ \iota_{X_{i+1}} = \delta \circ \iota_D \circ f_\alpha = f_\alpha$$

  Furthermore,

$$\left(\delta \circ J f_\beta\right) \circ J X_{\alpha\to\beta} = \delta \circ J f_\alpha = f_{\alpha+1} \circ q_\alpha$$

  so there exists a unique morphism $f_{\beta+1} : X_{\beta+1} \to D$ such that $f_{\beta+1} \circ q_\beta = \delta \circ J f_\beta$ and $f_{\beta+1} \circ X_{\alpha\to\beta+1} = f_\alpha$ for all ordinals $\alpha < \beta$.

Now observe that, for all ordinals $\alpha < \kappa$,

$$\begin{aligned} \delta \circ J f_\kappa \circ J X_{\alpha\to\kappa} &= \delta \circ J f_\alpha \\ &= f_{\alpha+1} \circ q_\alpha \\ &= f_\kappa \circ X_{\alpha+1\to\kappa} \circ q_\alpha \\ &= f_\kappa \circ \gamma_X \circ J X_{\alpha\to\kappa} \end{aligned}$$

and $\left\{ J X_{\alpha\to\kappa} \,\middle|\, \alpha < \kappa \right\}$ is a jointly epimorphic family, so $\delta \circ J f_\kappa = f_\kappa \circ \gamma_X$, i.e. f_κ is a (J, ι)-algebra homomorphism $\left(X_\kappa, \gamma_X\right) \to (D, \delta)$. Finally, notice that, for any homomorphism $\bar{f} : \left(X_\kappa, \gamma_X\right) \to (D, \delta)$ such that $\bar{f} \circ \eta_X = f_0$, then,

$$\delta \circ J\left(\bar{f} \circ X_{\alpha\to\kappa}\right) = \bar{f} \circ \gamma_X \circ J X_{\alpha\to\kappa} = \left(\bar{f} \circ X_{\alpha+1\to\kappa}\right) \circ q_\alpha$$

hence we must have $\bar{f} = f_\kappa$, by transfinite induction.

The above argument shows that the comma category $(X \downarrow U)$ has an initial object, and it is well known that U has a left adjoint if and only if each comma category $(X \downarrow U)$ has an initial object, so this completes the proof of claim (i). For claim (ii), we simply observe that $\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{C})$ is closed under colimits for λ-small diagrams in $\mathcal{C}$ (by lemma 0.2.15), so the above construction can be carried out entirely in $\mathbf{K}^{\mathbf{U}}_\lambda(\mathcal{C})$. ■

Theorem 0.3.36 (The category of algebras for a accessible pointed endofunctor)**.** *Let $\mathcal{C}$ be a κ-accessible* $\mathbf{U}$*-category, let $J : \mathcal{C} \to \mathcal{C}$ be a $(\kappa, \mathbf{U})$-accessible functor, let $\imath : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, and let $\mathcal{C}^{(J,\imath)}$ be the category of algebras for the pointed endofunctor $(J, \imath)$.*

(i) *The forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ creates colimits for* $\mathbf{U}$*-small κ-filtered diagrams; and if $\mathcal{C}$ is* $\mathbf{U}$*-complete, then $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ also creates limits for all* $\mathbf{U}$*-small diagrams.*

(ii) *$\mathcal{C}^{(J,\imath)}$ is an accessible* $\mathbf{U}$*-category.*

(iii) *If $\mathcal{C}$ has pushouts and colimits for chains of length $\le \kappa$, then $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ is a monadic functor.*

Proof. (i). This is well-known: cf. Propositions 4.3.1 and 4.3.2 in [Borceux, 1994b].

(ii). We may construct $\mathcal{D}$ using inserters and equifiers, as in the proof of Theorem 2.78 in [LPAC].

(iii). Since κ-chains are $\mathbf{U}$-small κ-filtered diagrams, the hypotheses of theorem 0.3.35 are satisfied, and so the forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ has a left adjoint. It is not hard to check that the other hypotheses of Beck's monadicity theorem are satisfied, so U is indeed a monadic functor. □

Theorem 0.3.37 (The category of algebras for a strongly accessible pointed endofunctor)**.** *Let $\mathcal{C}$ be a locally λ-presentable* $\mathbf{U}$*-category, let $J : \mathcal{C} \to \mathcal{C}$ be a functor of* $\mathbf{U}$*-rank κ, let $\imath : \mathrm{id}_{\mathcal{C}} \Rightarrow J$ be a natural transformation, let $\mathcal{C}^{(J,\imath)}$ be the category of algebras for the pointed endofunctor $(J, \imath)$, and let $\mathbf{T} = (T, \eta, \mu)$ be the induced monad on $\mathcal{C}$. If $J : \mathcal{C} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor and $\kappa < \lambda$, then:*

(i) *The functor $T : \mathcal{C} \to \mathcal{C}$ is $(\kappa, \mathbf{U})$-accessible and strongly $(\lambda, \mathbf{U})$-accessible.*

(ii) *$\mathcal{C}^{(J,\imath)}$ is a locally κ-presentable* $\mathbf{U}$*-category.*

(iii) *The forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ is a strongly $(\lambda, \mathbf{U})$-accessible functor.*

Proof. (i). We know that the forgetful functor $U : \mathcal{C}^{(J,\imath)} \to \mathcal{C}$ creates colimits for $\mathbf{U}$-small κ-filtered diagrams when $J : \mathcal{C} \to \mathcal{C}$ is $(\kappa, \mathbf{U})$-accessible, so $T : \mathcal{C} \to \mathcal{C}$

must also be $(\kappa, \mathbf{U})$-accessible in this case. Moreover, theorem 0.3.35 implies $T : \mathcal{C} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible if $J : \mathcal{C} \to \mathcal{C}$ is.

(ii). Apply theorem 0.3.31.

(iii). Apply theorem 0.3.33. ■

0.4 Change of universe

Prerequisites. §§ 0.1, 0.2, A.1, A.5.

Having introduced universes into our ontology, it becomes necessary to ask whether an object with some universal property retains that property when we enlarge the universe. Though it sounds inconceivable, there do exist examples of badly-behaved constructions that are not stable under change-of-universe; for example, Waterhouse [1975] defined a functor $F : \mathbf{CRing} \to \mathbf{Set}^+$, where $\mathbf{CRing}$ is the category of commutative rings in a universe $\mathbf{U}$ and $\mathbf{Set}^+$ is the category of $\mathbf{U}^+$-sets for some universe $\mathbf{U}^+$ with $\mathbf{U} \in \mathbf{U}^+$, such that the value of F at any given commutative ring in $\mathbf{U}$ does not depend on $\mathbf{U}$, and yet the value of the fpqc sheaf associated with F at the field $\mathbb{Q}$ depends on the size of $\mathbf{U}$.

Definition 0.4.1. Let κ be a regular cardinal in a universe $\mathbf{U}$, and let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$. A $(\kappa, \mathbf{U}, \mathbf{U}^+)$**-accessible extension** is a $(\kappa, \mathbf{U})$-accessible functor $i : \mathcal{C} \to \mathcal{C}^+$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category,
- $\mathcal{C}^+$ is a κ-accessible $\mathbf{U}^+$-category,
- i sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U}^+)$-compact objects in $\mathcal{C}^+$, and
- the functor $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \to \mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$ so induced by i is fully faithful and essentially surjective on objects.

Remark 0.4.2. Let $\mathbb{B}$ be a $\mathbf{U}$-small category in which idempotents split. Then the $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}) \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ obtained by extending the embedding $\gamma^+ : \mathbb{B} \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ along $\gamma : \mathbb{B} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, by proposition 0.2.22. The classification theorem (0.2.25) implies all examples of $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions are essentially of this form.

Proposition 0.4.3. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

(i) *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category if and only if $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*

(ii) *The functor $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful.*

(iii) *If $B : \mathcal{J} \to \mathcal{C}$ is any diagram (not necessarily $\mathbf{U}$-small) and $\mathcal{C}$ has a limit for B, then i preserves this limit.*

Proof. (i). If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, then $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ has colimits for all κ-small diagrams, so $\mathbf{K}^{\mathbf{U}^+}_{\kappa}(\mathcal{C}^+)$ also has colimits for all κ-small diagrams. The classification theorem (0.2.25) then implies $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category. Reversing this argument proves the converse.

(ii). Let $A : \mathbb{I} \to \mathcal{C}$ and $B : \mathbb{J} \to \mathcal{C}$ be two $\mathbf{U}$-small κ-filtered diagrams of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$. Then,

$$\begin{aligned}\mathcal{C}\left(\varinjlim_{\mathbb{I}} A, \varinjlim_{\mathbb{J}} B\right) &\cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} \mathcal{C}(A, B) \cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} \mathcal{C}^+(iA, iB) \\ &\cong \mathcal{C}^+\left(\varinjlim_{\mathbb{I}} iA, \varinjlim_{\mathbb{J}} iB\right) \cong \mathcal{C}^+\left(i \varinjlim_{\mathbb{I}} A, i \varinjlim_{\mathbb{J}} B\right)\end{aligned}$$

because i is $(\kappa, \mathbf{U})$-accessible and is fully faithful on the subcategory $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$, and therefore $i : \mathcal{C} \to \mathcal{C}^+$ itself is fully faithful. Note that this hinges crucially on theorem 0.1.31.

(iii). Let $B : \mathcal{J} \to \mathcal{C}$ be any diagram. We observe that, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$,

$$\begin{aligned}\mathcal{C}^+\left(iC, i \varprojlim_{\mathcal{J}} B\right) &\cong \mathcal{C}\left(C, \varprojlim_{\mathcal{J}} B\right) && \text{because } i \text{ is fully faithful} \\ &\cong \varprojlim_{\mathcal{J}} \mathcal{C}(C, B) && \text{by definition of limit} \\ &\cong \varprojlim_{\mathcal{J}} \mathcal{C}^+(iC, iB) && \text{because } i \text{ is fully faithful}\end{aligned}$$

but we know the restricted Yoneda embedding $\mathcal{C}^+ \to \left[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}^+\right]$ is fully faithful, so this is enough to conclude that $i \varprojlim_{\mathcal{J}} B$ is the limit of iB in $\mathcal{C}^+$. ■

REMARK 0.4.4. Similar methods show that any fully faithful functor $\mathcal{C} \to \mathcal{C}^+$ satisfying the four bulleted conditions in the definition above is necessarily $(\kappa, \mathbf{U})$-accessible.

Lemma 0.4.5. *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$*, and let* κ *be a regular cardinal in* $\mathbf{U}$*. Suppose:*

- $\mathcal{C}$ *and* $\mathcal{D}$ *are locally* κ*-presentable* $\mathbf{U}$*-categories.*
- $\mathcal{C}^+$ *and* $\mathcal{D}^+$ *are locally* κ*-presentable* $\mathbf{U}^+$*-categories.*
- $i : \mathcal{C} \to \mathcal{C}^+$ *and* $j : \mathcal{D} \to \mathcal{D}^+$ *are* $(\kappa, \mathbf{U}, \mathbf{U}^+)$*-accessible extensions.*

Given a strictly commutative diagram of the form below,

$$\begin{tikzcd} \mathcal{D} \arrow[r, "j"] \arrow[d, "G"'] & \mathcal{D}^+ \arrow[d, "G^+"] \\ \mathcal{C} \arrow[r, "i"'] & \mathcal{C}^+ \end{tikzcd}$$

where G *is* $(\kappa, \mathbf{U})$*-accessible,* G^+ *is* $(\kappa, \mathbf{U}^+)$*-accessible, if both have left adjoints, then the diagram satisfies the left Beck–Chevalley condition.*

Proof. Let C be a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$. Inspecting the proof of theorem 0.2.47, we see that the functor $(C \downarrow G) \to (iC \downarrow G^+)$ induced by j preserves initial objects. Lemma A.1.9 says the component at C of the left Beck–Chevalley natural transformation $F^+ i \Rightarrow jF$ is an isomorphism; but $\mathcal{C}$ is generated by $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and the functors F, F^+, i, j all preserve colimits for $\mathbf{U}$-small κ-filtered diagrams, so in fact $F^+ i \Rightarrow jF$ is a natural isomorphism. ■

Proposition 0.4.6. *If* $i : \mathcal{C} \to \mathcal{C}^+$ *is a* $(\kappa, \mathbf{U}, \mathbf{U}^+)$*-accessible extension and* $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}$*-category, then* i *preserves colimits for all* $\mathbf{U}$*-small diagrams in* $\mathcal{C}$*.*

Proof. It is well-known that a functor preserves colimits for all $\mathbf{U}$-small diagrams if and only if it preserves coequalisers for all parallel pairs and coproducts for all $\mathbf{U}$-small families, but coproducts for $\mathbf{U}$-small families can be constructed in a uniform way using coproducts for κ-small families and colimits for $\mathbf{U}$-small κ-filtered diagrams. It is therefore enough to show that $i : \mathcal{C} \to \mathcal{C}^+$ preserves all colimits for κ-small diagrams, since i is already $(\kappa, \mathbf{U})$-accessible.

Let $\mathbb{D}$ be a κ-small category. Recalling proposition 0.1.12, our problem amounts to showing that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "i"] \arrow[d, "\Delta"'] & \mathcal{C}^+ \arrow[d, "\Delta^+"] \\
{[\mathbb{D}, \mathcal{C}]} \arrow[r, "i_*"'] & {[\mathbb{D}, \mathcal{C}^+]}
\end{tikzcd}$$

satisfies the left Beck–Chevalley condition. It is clear that i_* is fully faithful. Colimits for $\mathbf{U}$-small diagrams in $[\mathbb{D}, \mathcal{C}]$ and in $[\mathbb{D}, \mathcal{C}^+]$ are computed componentwise, so Δ and i_* are certainly $(\kappa, \mathbf{U})$-accessible, and Δ^+ is $(\kappa, \mathbf{U}^+)$-accessible. Using proposition 0.2.44, we see that i_* is also a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, so we apply the lemma above to conclude that the left Beck–Chevalley condition is satisfied. ■

Theorem 0.4.7 (Stability of accessible adjoint functors)**.** *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$*, and let* κ *and* λ *be regular cardinals in* $\mathbf{U}$*, with* $\kappa \leq \lambda$*. Suppose:*

- $\mathcal{C}$ *is a locally* κ*-presentable* $\mathbf{U}$*-category.*
- $\mathcal{D}$ *is a locally* λ*-presentable* $\mathbf{U}$*-category.*
- $\mathcal{C}^+$ *is a locally* κ*-presentable* $\mathbf{U}^+$*-category.*
- $\mathcal{D}^+$ *is a locally* λ*-presentable* $\mathbf{U}^+$*-category.*

Let $i : \mathcal{C} \to \mathcal{C}^+$ *be a* $(\kappa, \mathbf{U}, \mathbf{U}^+)$*-accessible extension and let* $j : \mathcal{D} \to \mathcal{D}^+$ *be a fully faithful functor.*

(i) *Given a strictly commutative diagram of the form below,*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "j"] \arrow[d, "G"'] & \mathcal{D}^+ \arrow[d, "G^+"] \\
\mathcal{C} \arrow[r, "i"'] & \mathcal{C}^+
\end{tikzcd}$$

where G *is* $(\lambda, \mathbf{U})$*-accessible and* G^+ *is* $(\lambda, \mathbf{U}^+)$*-accessible, if both have left adjoints and* j *is a* $(\lambda, \mathbf{U}, \mathbf{U}^+)$*-accessible extension, then the diagram satisfies the left Beck–Chevalley condition.*

(ii) *Given a strictly commutative diagram of the form below,*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i"] \arrow[d, "F"'] & \mathcal{C}^+ \arrow[d, "F^+"] \\ \mathcal{D} \arrow[r, "j"'] & \mathcal{D}^+ \end{tikzcd}$$

if both F and F^+ have right adjoints, then the diagram satisfies the right Beck–Chevalley condition.

Proof. (i). The proof is essentially the same as lemma 0.4.5, though we have to use proposition 0.4.6 to ensure that j preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$.

(ii). Let D be any object in $\mathcal{D}$. Inspecting the proof of theorem 0.2.47, we see that our hypotheses, plus the fact that i preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, imply that the functor $(F \downarrow D) \to (F^+ \downarrow jD)$ induced by i preserves terminal objects. Thus, lemma A.1.9 implies that the diagram satisfies the right Beck–Chevalley condition. ■

Theorem 0.4.8. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category.*

(i) *If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \le \lambda$, then $i : \mathcal{C} \to \mathcal{C}^+$ is also a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

(ii) *If μ is the cardinality of $\mathbf{U}$, then $i : \mathcal{C} \to \mathcal{C}^+$ factors through the inclusion $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+) \hookrightarrow \mathcal{C}^+$ as functor $\mathcal{C} \to \mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+)$ that is (fully faithful and) essentially surjective on objects.*

(iii) *The $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}(\mathcal{C}) \to \mathcal{C}^+$ induced by $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful and essentially surjective on objects.*

Proof. (i). Since $i : \mathcal{C} \to \mathcal{C}^+$ is a $(\kappa, \mathbf{U})$-accessible functor, it is certainly also $(\lambda, \mathbf{U})$-accessible, by lemma 0.2.35. It is therefore enough to show that i restricts to a functor $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}) \to \mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$ that is (fully faithful and) essentially surjective on objects.

Proposition 0.2.43 says $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ is the smallest replete full subcategory of $\mathcal{C}$ that contains $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and is closed in $\mathcal{C}$ under colimits for λ-small diagrams, therefore the replete closure of the image of $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ must be the smallest replete full subcategory of $\mathcal{C}^+$ that contains $\mathbf{K}_\kappa^{\mathbf{U}^+}(\mathcal{C}^+)$ and is closed in $\mathcal{C}^+$ under colimits for

λ-small diagrams, since i is fully faithful and preserves colimits for all $\mathbf{U}$-small diagrams. This proves the claim.

(ii). Since every object in $\mathcal{C}$ is $(\lambda, \mathbf{U})$-compact for some regular cardinal $\lambda < \mu$, claim (i) implies that the image of $i : \mathcal{C} \to \mathcal{C}^+$ is contained in $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C})$. To show i is essentially surjective onto $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C})$, we simply have to observe that the inaccessibility of μ (proposition 0.1.36) and proposition 0.2.43 imply that, for C' any $(\mu, \mathbf{U}^+)$-compact object in $\mathcal{C}^+$, there exists a regular cardinal $\lambda < \mu$ such that C' is also a $(\lambda, \mathbf{U}^+)$-compact object, which reduces the question to claim (i).

(iii). This is an immediate corollary of claim (ii) and the classification theorem (0.2.25) applied to $\mathcal{C}^+$, considered as a $(\mu, \mathbf{U}^+)$-accessible category. ■

REMARK 0.4.9. Although the fact $i : \mathcal{C} \to \mathcal{C}^+$ that preserves limits and colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$ is a formal consequence of the theorem above (via e.g. corollary A.5.30), it is not clear whether the theorem can be proved without already knowing this.

Corollary 0.4.10. *If $\mathbb{B}$ is a $\mathbf{U}$-small category and has colimits for all κ-small diagrams, and μ is the cardinality of $\mathbf{U}$, then the canonical $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}\left(\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})\right) \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ is fully faithful and essentially surjective on objects.* ■

Proposition 0.4.11. *Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, and let κ and λ be regular cardinals in $\mathbf{U}$. Suppose:*

- *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category.*
- *$\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.*
- *$\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*
- *$\mathcal{D}^+$ is a locally λ-presentable $\mathbf{U}^+$-category.*

Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ be functors, let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, and let $j : \mathcal{D} \to \mathcal{D}^+$ be a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension.

Consider the following (not necessarily commutative) diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{D} \arrow[r, "j"] & \mathcal{D}^+ \\ \mathcal{C} \arrow[ur, "H"'] \arrow[d, "i"'] & & \\ \mathcal{C}^+ \arrow[uurr, "H^+"'] & & \end{tikzcd}$$

(i) *If H is a pointwise right Kan extension of G along F, then jH is a pointwise right Kan extension of jG along F, and if H^+ is a pointwise right Kan extension of jH along i, then H^+ is also a pointwise right Kan extension of jG along iF.*

(ii) *Assuming $\mathcal{A}$ is* $\mathbf{U}$*-small, if H is a pointwise left Kan extension of G along F, then jH is a pointwise left Kan extension of jG along F, and if H^+ is a pointwise left Kan extension of jH along i, then H^+ is also a pointwise left Kan extension of jG along iF.*

Proof. Use theorem A.5.20 and the fact that i and j preserve limits for *all* diagrams and colimits for $\mathbf{U}$-small diagrams. ■

0.5 Small object arguments

Prerequisites. §§ 0.1, 0.2, 0.3, 0.4, A.3, A.5.

The small object argument is a recurring construction in homotopical algebra, originally due to Quillen [1967, Ch. II, § 3] but refined by many authors since—notably by Garner [2009]. Roughly speaking, the small object argument shows that, under certain hypotheses, starting from a small set $\mathcal{I}$ of morphisms in a cocomplete category $\mathcal{C}$, one can define the notions of 'relative $\mathcal{I}$-cell complex' and '$\mathcal{I}$-fibration' so that every morphism in $\mathcal{C}$ factors as a relative $\mathcal{I}$-cell complex followed by an $\mathcal{I}$-fibration.

In this section, we will study the small object argument with a view toward questions of stability under change-of-universe.

Definition 0.5.1. Let $\mathcal{C}$ be a category, and let $\mathcal{I}$ be a subset of mor $\mathcal{C}$. A **presentation for a relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ consists of the following data:

- An ordinal α. (We say the presentation is **indexed over** α.)
- A colimit-preserving functor $X_\bullet : [\alpha] \to \mathcal{C}$, where $[\alpha]$ is the well-ordered set $\{0, \dots, \alpha\}$ considered as a preorder category.
- For each ordinal $\beta < \alpha$, a (possibly empty) indexing set T_β; and for each element j of T_β, a commutative diagram of the form below,

$$\begin{tikzcd} U_{\beta,j} \arrow[r, "u_{\beta,j}"] \arrow[d, "e_{\beta,j}"'] & X_\beta \arrow[d, "X_{\beta\to\beta+1}"] \\ V_{\beta,j} \arrow[r, "v_{\beta,j}"'] & X_{\beta+1} \end{tikzcd}$$

  where $e_{\beta,j} : U_{\beta,j} \to V_{\beta,j}$ is a morphism in $\mathcal{I}$.

These data are moreover required to satisfy the following condition:

- For each ordinal $\beta < \gamma$, the coproducts $\coprod_{j\in T_\beta} S_{\beta,j}$ and $\coprod_{j\in T_\beta} D_{\beta,j}$ exist in $\mathcal{C}$, and the induced diagram

$$\begin{tikzcd} \coprod_{j\in T_\beta} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j\in T_\beta} e_{\beta,j}"'] & X_\beta \arrow[d, "X_{\beta\to\beta+1}"] \\ \coprod_{j\in T_\beta} V_{\beta,j} \arrow[r, "v_\beta"'] & X_{\beta+1} \end{tikzcd}$$

  is a pushout square in $\mathcal{C}$.

The presentation is said to be **U-small** (resp. **κ-small** for a regular cardinal κ) if α is an ordinal in $\mathbf{U}$ (resp. $|\alpha| < \kappa$) and the disjoint union $\coprod_{\beta<\alpha} T_\beta$ is in $\mathbf{U}$ (resp. has cardinality less than κ). A **sequential presentation** is one where each T_β is a singleton, in which case we suppress the index j in $e_{\beta,j}$, $u_{\beta,j}$, and $v_{\beta,j}$.

A **relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ is a morphism $f : X \to Y$ in $\mathcal{C}$ for which there exists a presentation as above with f equal to $X_0 \to X_\alpha$. Given an initial object 0 in $\mathcal{C}$, an **$\mathcal{I}$-cell complex** in $\mathcal{C}$ is an object Y for which the unique morphism $0 \to Y$ is a relative $\mathcal{I}$-cell complex.

Remark 0.5.2. For any object X in $\mathcal{C}$ and any subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, the morphism $\mathrm{id} : X \to X$ is a relative $\mathcal{I}$-cell complex in $\mathcal{C}$ (with the obvious presentation indexed over 0). More generally, every isomorphism in $\mathcal{C}$ is a relative $\mathcal{I}$-cell complex, with a presentation indexed over 1 (and $T_0 = \emptyset$); but in order to get a *sequential* presentation, one must assume that there is an isomorphism in $\mathcal{I}$.

Proposition 0.5.3. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$*, let κ be a regular cardinal, and let* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes in $\mathcal{C}$ that admit a κ-small presentation.*

(i) *Every morphism in $\mathcal{I}$ is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(ii) *For each object X in $\mathcal{C}$, the morphism* $\mathrm{id} : X \to X$ *is in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(iii) *If $f : X \to Y$ and $g : Y \to Z$ are both in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$*, then so is $g \circ f$.*

(iv) *Let α be an ordinal and let $X_\bullet : \alpha \to \mathcal{C}$ be a colimit-preserving functor. If $|\alpha| < \kappa$ and λ is a colimiting cocone from $X_\bullet$ to Y and, for $\beta \le \gamma < \alpha$, the morphism $X_{\beta \to \gamma} : X_\beta \to X_\gamma$ is in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$*, then each component $\lambda_\beta : X_\beta \to Y$ is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

(v) *Given a pushout diagram of the form below in $\mathcal{C}$,*

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

if g is in $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$ *and $\mathcal{C}$ has colimits for all κ-small diagrams, then f is also in* $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}$.

Proof. (i). Given any morphism $e : U \to V$ in $\mathcal{I}$, we have the following pushout diagram:

$$\begin{tikzcd} U \arrow[r, "\mathrm{id}"] \arrow[d, "e"'] & U \arrow[d, "e"] \\ V \arrow[r, "\mathrm{id}"'] & V \end{tikzcd}$$

Thus $e : U \to V$ is in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$.

(ii). See remark 0.5.2.

(iii). It is clear that appending any κ-small presentation for g to any κ-small presentation for f yields a κ-small presentation of $g \circ f$.

(iv). The case $\alpha = 0$ falls under claim (ii). If $\alpha = \gamma + 1$, then the component $\lambda_\gamma : X_\gamma \to Y$ must be an isomorphism, and thus $\lambda_\beta = \lambda_\gamma \circ X_{\beta\to\gamma}$ is also in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$; and if α is a positive limit ordinal, since every terminal segment of α is cofinal in α, it is clear that concatenating κ-small presentations for $X_{\gamma\to\gamma+1}$ for $\beta \le \gamma < \alpha$ yields a κ-small presentation for $\lambda_\beta : X_\beta \to Y$.

(v). Fix a κ-small presentation of $g : Z \to W$. By the pushout pasting lemma, given a commutative diagram of the form below,

$$
\begin{tikzcd}
\coprod_{j\in T_\beta} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j\in T_\beta} e_{\beta,j}"'] & Z_\beta \arrow[r] \arrow[d, "Z_{\beta\to\beta+1}"] & X_\beta \arrow[d, "X_{\beta\to\beta+1}"] \\
\coprod_{j\in T_\beta} V_{\beta,j} \arrow[r, "v_\beta"'] & Z_{\beta+1} \arrow[r] & X_{\beta+1}
\end{tikzcd}
$$

if both squares are pushout diagrams, then the outer rectangle is a pushout diagram as well. Since pushout along $z : Z \to X$ is the left adjoint of the evident functor $z^* : {}^{X/}\mathcal{C} \to {}^{Z/}\mathcal{C}$, it preserves all colimits, and thus we obtain a κ-small presentation of $f : X \to Y$. ■

Definition 0.5.4. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of $\operatorname{mor} \mathcal{C}$. An **$\mathcal{I}$-injective morphism** in $\mathcal{C}$ is a morphism that has the right lifting property with respect to every morphism in $\mathcal{I}$.[10] An **$\mathcal{I}$-cofibration** in $\mathcal{C}$ is a morphism that has the left lifting property with respect to every $\mathcal{I}$-injective morphism.

Proposition 0.5.5. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$*, and let* $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$*,* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$*, and* $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes, $\mathcal{I}$-injections, and $\mathcal{I}$-cofibrations in $\mathcal{C}$, respectively.*

(i) *We have* $\mathcal{I} \subseteq \operatorname{cell}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}$.

(ii) *A morphism is in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ *if and only if it has the right lifting property with respect to every $\mathcal{I}$-cofibration.*

[10] Equivalently, it is a morphism $f : X \to Y$ in $\mathcal{C}$ that is an $\mathcal{I}$-injective object in the slice category $\mathcal{C}_{/Y}$.

(iii) *In particular, a morphism is in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ *if and only if it has the right lifting property with respect to every relative* $\mathcal{I}$*-cell complex.*

Proof. (i). Follows immediately from the definition of 'relative $\mathcal{I}$-cell complex' and proposition A.3.12.

(ii) and (iii). See proposition A.3.3. ■

Some authors define 'relative $\mathcal{I}$-cell complex' so that every such morphism admits a *sequential* presentation. The following lemma and its corollary show that there is no loss of generality in doing so.

Lemma 0.5.6. *Let* κ *be a regular cardinal, let* $\mathcal{C}$ *be a category with colimits for all* κ*-small diagrams, and let* α *be an ordinal of cardinality less than* κ*. For each ordinal* $\beta < \alpha$*, let* $e_\beta : U_\beta \to V_\beta$ *be a morphism in* $\mathcal{C}$*, and for each ordinal* $\beta \le \alpha$*, let*

$$C_\beta = \left(\coprod_{\gamma < \beta} V_\gamma \right) \amalg \left(\coprod_{\beta \le \gamma < \alpha} U_\gamma \right)$$

be a coproduct in $\mathcal{C}$ *with coproduct insertions* $u_{\gamma,\beta} : U_\gamma \to C_\beta$ *(for* $\beta \le \gamma < \alpha$*) and* $v_{\gamma,\beta} : V_\gamma \to C_\beta$ *(for* $\gamma < \beta$*).*

Given ordinals $\beta < \beta' \le \alpha$*, there is a unique morphism* $C_\beta \to C_{\beta'}$ *such that, for* $\zeta < \beta \le \zeta' < \beta' \le \zeta''$*, the following diagrams commute:*

$$\begin{tikzcd} V_\zeta \arrow[r, "v_{\zeta,\beta}"] \arrow[d, "\mathrm{id}"'] & C_\beta \arrow[d, dashed] \\ V_\zeta \arrow[r, "v_{\zeta,\beta'}"'] & C_{\beta'} \end{tikzcd} \qquad \begin{tikzcd} U_{\zeta'} \arrow[r, "u_{\zeta',\beta}"] \arrow[d, "e_{\zeta'}"'] & C_\beta \arrow[d, dashed] \\ V_{\zeta'} \arrow[r, "v_{\zeta',\beta'}"'] & C_{\beta'} \end{tikzcd} \qquad \begin{tikzcd} U_{\zeta''} \arrow[r, "u_{\zeta'',\beta}"] \arrow[d, "\mathrm{id}"'] & C_\beta \arrow[d, dashed] \\ U_{\zeta''} \arrow[r, "u_{\zeta'',\beta'}"'] & C_{\beta'} \end{tikzcd}$$

This yields a functor $C_\bullet : [\alpha] \to \mathcal{C}$*, and it preserves colimits. Moreover, the diagrams below are pushout squares for all ordinals* $\beta < \alpha$*:*

$$\begin{tikzcd} U_\beta \arrow[r, "u_{\beta,\beta}"] \arrow[d, "e_\beta"'] & C_\beta \arrow[d] \\ V_\beta \arrow[r, "v_{\beta,\beta+1}"'] & C_{\beta+1} \end{tikzcd}$$

Proof. This is a straightforward exercise. See Proposition 10.2.7 in [Hirschhorn, 2003]. □

Corollary 0.5.7. *Let κ be a regular cardinal, let $\mathcal{C}$ be a category with colimits for κ-small diagrams, and let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$. *If $f : X \to Y$ is a relative $\mathcal{I}$-cell complex in $\mathcal{C}$ that admits a κ-small presentation, and either*

- *$X = Y$ and $f = \mathrm{id}_X$, or*
- *f is an isomorphism and $\mathcal{I}$ contains an isomorphism, or*
- *f is not an isomorphism,*

then f also admits a κ-small sequential presentation.

Proof. We have already commented on the first two cases in remark 0.5.2. The third case is proven by transfinite induction, where in the induction step we may assume that f is presented by just one pushout diagram:

$$\begin{tikzcd}
\coprod_{j \in T} U_j \arrow[r, "u"] \arrow[d, "\coprod_{j \in T} e_j"'] & X \arrow[d, "f"] \\
\coprod_{j \in T} V_j \arrow[r, "v"'] & Y
\end{tikzcd}$$

By decomposing the morphism $\coprod_{j \in T} e_j : \coprod_{j \in T} U_j \to \coprod_{j \in T} V_j$ as in the earlier lemma and applying the pushout pasting lemma, we obtain a sequential presentation of f, which is κ-small precisely if $|T| < \kappa$. ∎

Definition 0.5.8. Let $\mathbf{U}$ be a universe, let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of $\operatorname{mor} \mathcal{C}$, and let $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$ be the set of relative $\mathcal{I}$-cell complexes in $\mathcal{C}$ that have a $\mathbf{U}$-small presentation. We say $(\mathcal{I}, \mathcal{C})$ is **admissible for the U-small object argument** when the following conditions are satisfied:

- $\mathcal{I}$ is a $\mathbf{U}$-set.
- $\mathcal{C}$ be a locally $\mathbf{U}$-small category with colimits for all $\mathbf{U}$-small diagrams.
- There is a regular cardinal κ in $\mathbf{U}$ such that, for every morphism $e : U \to V$ in $\mathcal{I}$, every ordinal α in $\mathbf{U}$, and every functor $X_\bullet : \alpha \to \mathcal{C}$, if $|\alpha| \geq \kappa$, and the morphism $X_{\beta \to \gamma} : X_\beta \to X_\gamma$ is in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$ for all ordinals $\beta \leq \gamma < \alpha$, then the canonical comparison map $\varinjlim_{\beta < \alpha} \mathcal{C}(U, X_\beta) \to \mathcal{C}\left(U, \varinjlim_{\beta < \alpha} X_\beta\right)$ is a bijection.

The **sequential U-rank** of $\mathcal{I}$ in $\mathcal{C}$ is the least cardinal κ with the above property.

REMARK 0.5.9. Notice that, if $|\alpha| \geq \kappa$, then α is a κ-directed preorder. Thus, for any locally presentable $\mathbf{U}$-category $\mathcal{C}$ and any $\mathbf{U}$-subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ whatsoever, $(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument.

Definition 0.5.10. Let $\mathbf{U}$ be a universe. A **U-cofibrantly-generated factorisation system** on a category $\mathcal{C}$ on is a weak factorisation system on $\mathcal{C}$ that is cofibrantly generated by some $\mathbf{U}$-subset of $\operatorname{mor} \mathcal{C}$.

Lemma 0.5.11. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category, let A be a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, and let B be a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. If the hom-set $\mathcal{C}(A, A')$ is μ-small for all $(\kappa, \mathbf{U})$-compact objects A' in $\mathcal{C}$ and $\kappa \lhd \lambda$, then the hom-set $\mathcal{C}(A, B)$ has cardinality $< \max\{\lambda, \mu\}$.*

Proof. By proposition 0.2.42, there is a λ-small κ-filtered diagram $Y : \mathcal{J} \to \mathcal{C}$ with each vertex $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$ and $B \cong \varinjlim_{\mathcal{J}} Y$. Since A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, we have

$$\mathcal{C}(A, B) \cong \varinjlim_{\mathcal{J}} \mathcal{C}(A, Y)$$

and the RHS is a set of cardinality $< \max\{\lambda, \mu\}$ by lemma 0.2.15. ■

Theorem 0.5.12 (Quillen's small object argument)**.** *Let $\mathbf{U}$ be a universe, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category with colimits for all $\mathbf{U}$-small diagrams, and let $\mathcal{I}$ be a $\mathbf{U}$-subset of* $\operatorname{mor} \mathcal{C}$.

(i) *There exist a functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ and two natural transformations $i : \operatorname{dom} \Rightarrow M$, $p : M \Rightarrow \operatorname{codom}$ such that, for all morphisms $f : X \to Y$ in $\mathcal{C}$, the morphism $i_f : X \to M(f)$ is in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$, and we have $f = p_f \circ i_f$.*

(ii) *If $(\mathcal{I}, \mathcal{C})$ is moreover admissible for the $\mathbf{U}$-small object argument, then we may choose M, i, and p so that, for all morphisms $f : X \to Y$ in $\mathcal{C}$, the morphism $p_f : M(f) \to Y$ in $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$.*

(iii) *In particular, if $(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument, then $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ is a $\mathbf{U}$-cofibrantly-generated factorisation system on $\mathcal{C}$ and extends to a functorial weak factorisation system.*

Proof. (i). Let κ be any regular cardinal, and let α be the least ordinal of cardinality κ.[11] For each morphism $f : X \to Y$ in $\mathcal{C}$, we construct by transfinite recursion a colimit-preserving functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ and a cocone $p_{f;\bullet} : M_\bullet(f) \to Y$ satisfying the following conditions:

- $M_0(f) = X$, $p_{f;0} = p$.
- For each ordinal $\beta < \alpha$, if $T_\beta(f)$ is the set of all commutative diagrams in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
U_{\beta,j} \arrow[r, "u_{\beta,j}"] \arrow[d, "e_{\beta,j}"'] & M_\beta(f) \arrow[d, "p_{f;\beta}"] \\
V_{\beta,j} \arrow[r, "v_{\beta,j}"'] & Y
\end{tikzcd}$$

  where $e_{\beta,j} : U_{\beta,j} \to V_{\beta,j}$ is in $\mathcal{I}$, then $T_\beta(f)$ is a $\mathbf{U}$-set (because $\mathcal{I}$ is a $\mathbf{U}$-set and $\mathcal{C}$ is a locally $\mathbf{U}$-small category), and we have a pushout square of the following form,

$$\begin{tikzcd}
\coprod_{j\in T_\beta(f)} U_{\beta,j} \arrow[r, "u_\beta"] \arrow[d, "\coprod_{j\in T_\beta(f)} e_{\beta,j}"'] & M_\beta(f) \arrow[d, "X_{\beta\to\beta+1}"] \\
\coprod_{j\in T_\beta(f)} V_{\beta,j} \arrow[r, "\bar{v}_\beta"'] & M_{\beta+1}(f)
\end{tikzcd}$$

  where $u_\beta : \coprod_{j\in T_\beta(f)} U_{\beta,j} \to M_\beta(f)$ is the evident morphism induced by the universal property of coproducts. Observe that there is then a unique morphism $p_{f;\beta+1} : M_{\beta+1}(f) \to Y$ such that

$$p_{f;\beta+1} \circ M_{\beta\to\beta+1}(f) = p_\beta$$

  and

$$p_{f;\beta+1} \circ \bar{v}_{\beta,j} = v_{\beta,j}$$

  for all j in $T_\beta(f)$, where $\bar{v}_{\beta,j} : V_{\beta,j} \to M_{\beta+1}(f)$ is the evident component of $\bar{v}_\beta : \coprod_{j\in T_\beta(f)} V_{\beta,j} \to M_{\beta+1}(f)$.
- For limit ordinals $\gamma \le \alpha$, $M_\gamma(f) = \varinjlim_{\beta<\gamma} M_\beta(f)$, and $p_\gamma : M_\gamma(f) \to Y$ is defined by the universal property of X_γ.

[11] In particular, we could take $\kappa = 0$, but then the factorisation so obtained is trivial.

It is not hard to see that the functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ so defined is itself functorial in f; in particular, defining $M(f) = M_\alpha(f)$, $i_f = M_{0\to\alpha}(f)$, $p_f = p_{f;\alpha}$, we obtain a functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ with two natural transformations $i : M \Rightarrow \mathrm{dom}$ and $p : M \Rightarrow \mathrm{codom}$; by construction, we have $f = p_f \circ i_f$, and $i_f : X \to M(f)$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$.

(ii). Now, take κ to be a regular cardinal as in definition 0.5.8. We wish to show that the morphism p_f constructed above has the right lifting property with respect to all morphisms in $\mathcal{I}$. Consider a lifting problem of the form below,

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "e"'] & M(f) \arrow[d, "p_f"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

where $e : U \to V$ is in $\mathcal{I}$. Since $\mathcal{I}$ is admissible, there must exist an ordinal $\beta < \alpha$ and a morphism $u' : U \to M_\beta(f)$ such that $u = M_{\beta\to\alpha}(f) \circ u'$. We then obtain the following commutative diagram:

$$\begin{tikzcd} U \arrow[r, "u'"] \arrow[d, "e"'] & M_\beta(f) \arrow[d, "p_{f;\beta}"] \\ V \arrow[r, "v"'] & Y \end{tikzcd}$$

Since this is one of the diagrams in the set $T_\beta(f)$, it must embed in a commutative diagram of the form below,

$$\begin{tikzcd} U \arrow[r, "u'"] \arrow[d, "e"'] & M_\beta(f) \arrow[r] \arrow[d] & M_\alpha(f) \arrow[dd, "p_{f;\alpha}"] \\ V \arrow[r] \arrow[d, equal] & M_{\beta+1}(f) \arrow[ur] & \\ V \arrow[rr, "v"'] & & Y \end{tikzcd}$$

and thus we have the required lift $V \to M(f)$.

(iii). Finally, apply proposition 0.5.5 and theorem A.3.29. ■

Corollary 0.5.13. *With other notation in the theorem, a morphism $g : Z \to W$ is in $\operatorname{cof}_{\mathcal{I}} C$ if and only if there exists a commutative diagram of the following form in C,*

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "g"'] & W' \arrow[d] \\ W \arrow[ur] \arrow[r, "\mathrm{id}"'] & W \end{tikzcd}$$

where $i : Z \to W'$ is in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} C$.

Proof. (i). If $g : Z \to W$ is in $\operatorname{cof}_{\mathcal{I}} C$, then g has the left lifting property with respect to $p_g : M(g) \to W$, and so there exists a commutative diagram of the required form. Conversely, suppose we have $g = p \circ i$, $i = j \circ g$, and $\mathrm{id}_W = p \circ j$ for some $i : Z \to W'$ in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} C$ and some $j : W \to W'$ in C. Then g is a retract of i,

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Z \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & Z \arrow[d, "g"] \\ W \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}"'] & W' \arrow[r, "p"] & W \end{tikzcd}$$

but proposition 0.5.5 says i is in $\operatorname{cof}_{\mathcal{I}} C$, so by proposition A.3.12, g is also in $\operatorname{cof}_{\mathcal{I}} C$. ■

Corollary 0.5.14. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let C be a locally κ-presentable $\mathbf{U}$-category, and let $\mathcal{I}$ be a $\mathbf{U}$-small subset of $\operatorname{mor} C$. If the morphisms that are in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact as objects in $[\mathbb{2}, C]$, then there exist a $(\kappa, \mathbf{U})$-accessible functor $M : [\mathbb{2}, C] \to C$ and two natural transformations $i : \mathrm{dom} \Rightarrow M$ and $p : M \Rightarrow \mathrm{codom}$ such that, for all objects f in $[\mathbb{2}, C]$:*

- $f = p_f \circ i_f$.
- *i_f is in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} C$.*
- *p_f is in $\operatorname{inj}^{\mathcal{I}} C$.*

Moreover, if λ is a regular cardinal in $\mathbf{U}$ such that every hom-set of $\mathbf{K}^{\mathbf{U}}_{\kappa}(C)$ is λ-small, $\mathcal{I}$ is λ-small, and $\kappa \lhd \lambda$, then $M : [\mathbb{2}, C] \to C$ is also strongly $(\lambda, \mathbf{U})$-accessible.

Proof. As observed in remark 0.5.9, under these hypotheses, $(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument and the sequential $\mathbf{U}$-rank of $\mathcal{I}$ is $\leq \kappa$. By tracing the construction of the functor M in theorem 0.5.12, we see that M preserves colimits for κ-filtered $\mathbf{U}$-small diagrams, so we are done. Similarly, applying proposition 0.2.44 and lemmas 0.2.15 and 0.5.11 shows that M is strongly $(\lambda, \mathbf{U})$-accessible. ■

Corollary 0.5.15. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let $\mathcal{I}$ be a $\mathbf{U}$-small subset of* $\operatorname{mor} \mathcal{C}$. *If the morphisms that are in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact as objects in $[\mathbb{2}, \mathcal{C}]$, then there exists a $(\kappa, \mathbf{U})$-accessible functor $L : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ such that $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ is the closure of the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by the image of L under the splitting of idempotent endomorphisms.*

Proof. Take L to be the functor that sends a morphism in $\mathcal{C}$ (considered as an object in $[\mathbb{2}, \mathcal{C}]$) to the left half of its $\left(\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$-factorisation, and then apply theorem A.3.29. ■

Lemma 0.5.16. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$, *and let κ be a regular cardinal. If $\mathcal{C}$ is closed in $\mathcal{C}^+$ under colimits for all κ-small diagrams, then $\operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C} = \operatorname{cell}_{\mathcal{I},\kappa} \mathcal{C}^+ \cap \operatorname{mor} \mathcal{C}$.*

Proof. Obvious. ♦

Theorem 0.5.17 (Stability of cofibrantly-generated factorisation systems)**.** *Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$. Suppose:*

- *$\mathcal{C}$ is a locally $\mathbf{U}$-small and $\mathbf{U}$-cocomplete category.*
- *$\mathcal{C}^+$ is a locally $\mathbf{U}^+$-small and $\mathbf{U}^+$-cocomplete category.*
- *The inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ preserves colimits for all $\mathbf{U}$-small diagrams.*
- *$\mathcal{I}$ is a $\mathbf{U}$-subset of* $\operatorname{mor} \mathcal{C}$.
- *$(\mathcal{I}, \mathcal{C})$ is admissible for the $\mathbf{U}$-small object argument, and (L, R) is the functorial factorisation system on $\mathcal{C}$ constructed by Quillen's small object argument argument.*
- *$(\mathcal{I}, \mathcal{C}^+)$ is admissible for the $\mathbf{U}^+$-small object argument, and (L^+, R^+) is the functorial factorisation system on $\mathcal{C}^+$ constructed by Quillen's small object argument argument.*

Under these hypotheses, if the sequential $\mathbf{U}$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}$ *is equal to the sequential* $\mathbf{U}^+$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}^+$*, then:*

(i) *For each morphism* $f : X \to Y$ *in* $\mathcal{C}$*, we have a commutative diagram of the following form in* $\mathcal{C}^+$*,*

$$\begin{tikzcd} & M^+(f) \arrow[dr, "R^+ f"] \arrow[dd, "\cong"] & \\ X \arrow[ur, "L^+ f"] \arrow[dr, "L f"'] & & Y \\ & M(f) \arrow[ur, "R f"'] & \end{tikzcd}$$

and the isomorphism $M^+(f) \to M(f)$ *is moreover canonical and natural in* f*.*

(ii) *We have* $\mathrm{cell}_{\mathcal{I},\mathbf{U}} \, \mathcal{C} \subseteq \mathrm{cell}_{\mathcal{I},\mathbf{U}} \, \mathcal{C}^+ \subseteq \mathrm{cell}_{\mathcal{I},\mathbf{U}^+} \, \mathcal{C}^+$.

(iii) $\left(\mathrm{cof}_{\mathcal{I}} \, \mathcal{C}^+, \mathrm{inj}^{\mathcal{I}} \, \mathcal{C}^+\right)$ *is an extension of* $\left(\mathrm{cof}_{\mathcal{I}} \, \mathcal{C}, \mathrm{inj}^{\mathcal{I}} \, \mathcal{C}\right)$.

Proof. (i). This can be seen by examining the explicit construction in the proof of theorem 0.5.12.

(ii). This is implied by the lemma.

(iii). Since $\left(\mathrm{cof}_{\mathcal{I}} \, \mathcal{C}, \mathrm{inj}^{\mathcal{I}} \, \mathcal{C}\right)$ and $\left(\mathrm{cof}_{\mathcal{I}} \, \mathcal{C}^+, \mathrm{inj}^{\mathcal{I}} \, \mathcal{C}^+\right)$ are both cofibrantly generated by $\mathcal{I}$, by proposition A.3.19, we have $\mathrm{inj}^{\mathcal{I}} \, \mathcal{C} \subseteq \mathrm{inj}^{\mathcal{I}} \, \mathcal{C}^+$ and so $\mathrm{cof}_{\mathcal{I}} \, \mathcal{C} \supseteq \mathrm{cof}_{\mathcal{I}} \, \mathcal{C}^+ \cap \mathrm{mor} \, \mathcal{C}$. It remains to be shown that $\mathrm{cof}_{\mathcal{I}} \, \mathcal{C} \subseteq \mathrm{cof}_{\mathcal{I}} \, \mathcal{C}^+$, but this is implied by corollary 0.5.13 applied to claim (ii). ■

REMARK 0.5.18. Let κ be a regular cardinal in $\mathbf{U}$, let $\mathcal{B}$ be a $\mathbf{U}$-small category with colimits for all κ-small diagrams, let $\mathcal{C} = \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{B})$, and let $\mathcal{C}^+ = \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathcal{B})$. Then $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is an accessible $(\kappa, \mathbf{U}, \mathbf{U}^+)$ extension, and any $\mathbf{U}$-subset $\mathcal{I} \subseteq \mathrm{mor} \, \mathcal{C}$ whatsoever will satisfy the hypotheses of the theorem.

Proposition 0.5.19. *Let* $F \dashv U : \mathcal{D} \to \mathcal{C}$ *be an adjunction of categories, let* $\mathcal{I} \subseteq \mathrm{mor} \, \mathcal{C}$*, and let* $\mathcal{J} = \{ F f \mid f \in \mathcal{I} \}$.

(i) F *sends relative* $\mathcal{I}$*-cell complexes in* $\mathcal{C}$ *to relative* $\mathcal{J}$*-cell complexes in* $\mathcal{D}$.

(ii) U *sends* $\mathcal{J}$*-injective morphisms in* $\mathcal{D}$ *to* $\mathcal{I}$*-injective morphisms in* $\mathcal{C}$.

(iii) *F sends $\mathcal{I}$-cofibrations in $\mathcal{C}$ to $\mathcal{J}$-cofibrations in $\mathcal{D}$.*

Proof. (i). This is a corollary of the fact that F preserves all colimits.

(ii). As in the proof of proposition A.3.20, a morphism $f : X \to Y$ in $\mathcal{D}$ has the right lifting property with respect to all morphisms in $\mathcal{J}$ if and only if $Uf : UX \to UY$ has the right lifting property with respect to all morphisms in $\mathcal{I}$.

(iii). Similarly, a morphism $g : Z \to W$ in $\mathcal{C}$ has the left lifting property with respect to all morphisms of the form $Uf : UX \to UY$ where $f : X \to Y$ is a $\mathcal{J}$-injective morphism $f : X \to Y$ in $\mathcal{D}$ if and only if $Fg : FZ \to FW$ is a $\mathcal{J}$-cofibration in $\mathcal{D}$; but we know that U sends $\mathcal{J}$-injective morphisms in $\mathcal{D}$ to $\mathcal{I}$-injective morphisms in $\mathcal{C}$, so F must send $\mathcal{I}$-cofibrations in $\mathcal{C}$ to $\mathcal{J}$-cofibrations in $\mathcal{D}$. ■

Proposition 0.5.20. *Let* $\mathbf{U}$ *be a universe, let* **Set** *be the category of* $\mathbf{U}$*-sets, let* $\mathbb{B}$ *be a* $\mathbf{U}$*-small category, let* $\mathcal{C} = [\mathbb{B}^{\mathrm{op}}, \mathbf{Set}]$*, and let* $\mathcal{I}$ *be the subset of* $\operatorname{mor}\mathcal{C}$ *consisting of all* monomorphisms $e : U \to V$ *in* $\mathcal{C}$ *where* V *is a quotient of a representable presheaf.*

(i) $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ *is a* $\mathbf{U}$*-cofibrantly-generated weak factorisation system.*

(ii) $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$ *is precisely the class of all monomorphisms in* $\mathcal{C}$*.*

(iii) $\operatorname{cof}_{\mathcal{I}} \mathcal{C} = \operatorname{cell}_{\mathcal{I}} \mathcal{C}$*.*

Proof. (i). Since $\mathbb{B}$ is small and $\mathcal{C}$ is well-powered and well-copowered, the full subcategory of $[\mathbb{2}, \mathcal{C}]$ spanned by $\mathcal{I}$ is essentially $\mathbf{U}$-small. We know that $\mathcal{C}$ is locally finitely presentable, thus, taking a $\mathbf{U}$-set of representatives of the isomorphism classes in $\mathcal{I}$, and recalling remark 0.5.9, Quillen's small object argument (theorem 0.5.12) implies $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ is indeed a $\mathbf{U}$-cofibrantly-generated weak factorisation system.

(ii). It is clear that the class of injective maps is closed under pushout and transfinite composition in **Set**, so the same must be true of monomorphisms in $\mathcal{C}$, since colimits in $\mathcal{C}$ are computed componentwise. Thus every morphism in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$ is a monomorphism.

Conversely, suppose $f : X \to Y$ is a monomorphism. Fix an ordinal α and a bijection $y_\bullet : \alpha \to \coprod_{B \in \operatorname{ob}\mathbb{B}} Y(B)$, and write B_β for the object in $\mathbb{B}$ such that $y_\beta \in Y\left(B_\beta\right)$. We will construct a $\mathbf{U}$-small presentation for f by transfinite recursion on α.

- To begin, put $X_0 = X$ and $f_0 = f$.
- For each ordinal $\beta < \alpha$, the Yoneda lemma implies there is a unique morphism $a_{\beta} \hbar_{B_{\beta}} \to Y$ in $\mathcal{C}$ such that $a_{\beta}\left(\mathrm{id}_{B_{\beta}}\right) = y_{\beta}$; let $\bar{v}_{\beta} : V_{\beta} \to Y$ be the image of a_{β}, and let $e_{\beta} : U_{\beta} \to V_{\beta}$ and $u_{\beta} : U_{\beta} \to V_{\beta}$ be defined by the pullback square shown below:

$$\begin{tikzcd} U_{\beta} \arrow[r, "u_{\beta}"] \arrow[d, "e_{\beta}"'] & X_{\beta} \arrow[d, "f_{\beta}"] \\ V_{\beta} \arrow[r, "\bar{v}_{\beta}"'] & Y \end{tikzcd}$$

  Since f_{β} is a monomorphism, e_{β} must also be a monomorphism and hence is in $\mathcal{I}$. There is then a commutative diagram in $\mathcal{C}$ of the following form,

$$\begin{tikzcd} U_{\beta} \arrow[r, "u_{\beta}"] \arrow[d, "e_{\beta}"'] & X_{\beta} \arrow[d] \arrow[ddr, bend left, "f_{\beta}"] & \\ V_{\beta} \arrow[r, "v_{\beta}"'] \arrow[drr, bend right, "\bar{v}_{\beta}"'] & X_{\beta+1} \arrow[dr, "f_{\beta+1}"] & \\ & & Y \end{tikzcd}$$

  where $f_{\beta+1} : X_{\beta+1} \to Y$ is the union of $f_{\beta} : X_{\beta} \to Y$ and $\bar{v}_{\beta} : V_{\beta} \to Y$ considered as subobjects of Y; note that the inner square of the diagram is then a pushout square.
- Finally, for limit ordinals $\gamma < \alpha$, we take $f_{\gamma} : X_{\gamma} \to Y$ to be the union $\bigcup_{\beta<\gamma} f_{\beta}$.

This completes the presentation of $f : X \to Y$ as a relative $\mathcal{I}$-cell complex in $\mathcal{C}$, and it is clearly $\mathbf{U}$-small.

(iii). Corollary 0.5.13 implies that each morphism in $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ is a retract of some morphism in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$, but the class of monomorphisms is closed under retracts, so in this case we must have $\operatorname{cof}_{\mathcal{I}} \mathcal{C} = \operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$. Since $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C} \subseteq \operatorname{cell}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}$, we also deduce that $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C} = \operatorname{cell}_{\mathcal{I}} \mathcal{C}$. ■

We now turn our attention to Garner's small object argument.

Lemma 0.5.21. *Let κ be a regular cardinal in a universe $\mathbf{U}$, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category and let $F : \mathcal{A} \to \mathcal{C}$ be a functor, and let $G : \mathcal{C} \to \mathcal{C}$ be (the functor part of) a pointwise left Kan extension of F along itself. If each FA is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$, then:*

(i) *$G : \mathcal{C} \to \mathcal{C}$ preserves colimits for $\mathbf{U}$-small κ-filtered diagrams.*

(ii) *In addition, if $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, λ is a regular cardinal in $\mathbf{U}$ such that every hom-set of $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ is λ-small, $\mathcal{A}$ is a λ-small category, and $\kappa \lhd \lambda$, then $G : \mathcal{C} \to \mathcal{C}$ is strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. (i). Theorem A.5.15 says there is a natural bijection of the form below:

$$\mathcal{C}(GX, C) \cong [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}](\mathcal{C}(F-, X), \mathcal{C}(F-, C))$$

Since colimits are computed componentwise in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, the hypothesis implies $\mathcal{C}(F, -) : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves colimits for $\mathbf{U}$-small κ-filtered diagrams. By the Yoneda lemma, the functors $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ jointly reflect limits, so it follows that $G : \mathcal{C} \to \mathcal{C}$ preserves colimits for $\mathbf{U}$-small κ-filtered diagrams.

(ii). Now suppose X is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. Lemma 0.5.11 then says each hom-set $\mathcal{C}(FA, X)$ is λ-small, and since $\mathcal{A}$ is a λ-small category, this shows that the comma category $(F \downarrow X)$ is also λ-small. Thus, GX is a colimit for a λ-small diagram of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$, and so we may use lemma 0.2.15 to deduce that it is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. ■

Proposition 0.5.22. *Let $\mathcal{C}$ be a category with pushouts and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. Suppose a pointwise left Kan extension of U along itself exists.*

(i) *$\mathbf{RLP}(U)$ is isomorphic as a category over $[\mathbb{2}, \mathcal{C}]$ to the category of algebras for a pointed endofunctor $(J, \imath)$ on $[\mathbb{2}, \mathcal{C}]$.*

(ii) *Moreover, if (the functor part of) the pointwise left Kan extension of U along itself is a $(\kappa, \mathbf{U})$-accessible functor (resp. strongly $(\kappa, \mathbf{U})$-accessible functor), then so is J.*

Proof. Let $G : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ be a pointwise left Kan extension of U along itself and let $\alpha : U \Rightarrow GU$ be the unit. Then there is a unique natural transformation

$\varepsilon : G \Rightarrow \mathrm{id}_{\mathcal{E}}$ such that $\varepsilon U \bullet \alpha = \mathrm{id}_{[2,\mathcal{C}]}$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By theorem A.5.15, there is a natural bijection of the form below:

$$[2, \mathcal{C}](Gf, g) \cong [\mathcal{I}^{\mathrm{op}}, \mathbf{Set}]([2, \mathcal{C}](U-, f), [2, \mathcal{C}](U-, g))$$

It is not hard to see that a coherent choice Φ of right liftings for f with respect to $U : \mathcal{I} \to [2, \mathcal{C}]$ is the same thing as a natural transformation $[2, \mathcal{C}](U-, f) \Rightarrow [2, \mathcal{C}]\big(U-, \mathrm{id}_X\big)$ making the following diagram commute for all objects e in $\mathcal{I}$,

$$\begin{tikzcd}
{[2, \mathcal{C}](Ue, f)} \arrow[r, dashed] \arrow[dr, "\mathrm{id}"'] & {[2, \mathcal{C}]\big(Ue, \mathrm{id}_X\big)} \arrow[d] \\
& {[2, \mathcal{C}](Ue, f)}
\end{tikzcd}$$

where the map $[2, \mathcal{C}]\big(Ue, \mathrm{id}_X\big) \to [2, \mathcal{C}](Ue, f)$ is the one induced by the morphism $\big(\mathrm{id}_X, f\big) : \mathrm{id}_X \to f$ in $[2, \mathcal{C}]$. We may therefore identity choices Φ with morphisms $l : d_0(Gf) \to X$ in $\mathcal{C}$ making the diagram below commute:

(∗)
$$\begin{tikzcd}
\bullet \arrow[r, "d_1(\varepsilon_f)"] \arrow[d, "Gf"'] & X \arrow[d, "f"] \\
\bullet \arrow[r, "d_0(\varepsilon_f)"'] \arrow[ur, dashed, "l"] & Y
\end{tikzcd}$$

Now, define functors $J, K : [2, \mathcal{C}] \to [2, \mathcal{C}]$ so the square in the following diagram is a natural pushout square in $\mathcal{C}$:

$$\begin{tikzcd}
\bullet \arrow[r, "d_1(\varepsilon_f)"] \arrow[d, "Gf"'] & X \arrow[d, "Kf"] \arrow[ddr, bend left, "f"] & \\
\bullet \arrow[r] \arrow[drr, bend right, "d_0(\varepsilon_f)"'] & Mf \arrow[dr, "Jf"] & \\
& & Y
\end{tikzcd}$$

We then have a natural transformation $\iota : \mathrm{id}_{[2,\mathcal{C}]} \Rightarrow J$ where $\iota_f = \big(\mathrm{id}_X, Kf\big)$, and the universal property of pushouts yields a natural bijection between morphisms $l : d_0(Gf) \to X$ making the diagram (∗) commute and morphisms $\tilde{l} : Mf \to Y$ such that $Jf = f \circ \tilde{l}$, i.e. coalgebra structures on f for the pointed endofunctor

$(J, \imath)$. The naturality of these identifications then ensures that $\mathbf{RLP}(U)$ is indeed isomorphic to $[\mathbb{2}, \mathcal{C}]^{(J,\imath)}$ as categories over $[\mathbb{2}, \mathcal{C}]$. This proves claim (i).

For claim (ii), simply observe that pushouts preserve all colimits, so $J : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible if $G : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ is, and lemmas 0.2.15 and 0.3.20 imply J is strongly $(\kappa, \mathbf{U})$-accessible if G is. ∎

Proposition 0.5.23. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, let $\mathcal{I}$ be a $\mathbf{U}$-small category, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. If each Ue is a $(\kappa, \mathbf{U})$-compact object in $[\mathbb{2}, \mathcal{C}]$, then:*

(i) *The forgetful functor $\mathbf{RLP}(U) \to [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible and monadic.*

(ii) *In addition, if λ is a regular cardinal in $\mathbf{U}$ such that each hom-set in $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ is λ-small, $\mathcal{I}$ is a λ-small category, and $\kappa \lhd \lambda$, then the forgetful functor $\mathbf{RLP}(U) \to [\mathbb{2}, \mathcal{C}]$ is strongly $(\lambda, \mathbf{U})$-accessible.*

Proof. Use theorems 0.3.36 and 0.3.37, lemma 0.5.21, and proposition 0.5.22. ∎

Theorem 0.5.24 (Garner's small object argument)**.** *Let $\mathcal{C}$ be a locally presentable $\mathbf{U}$-category, let $\mathcal{I}$ be a $\mathbf{U}$-small category, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor.*

(i) *There exists a free algebraic factorisation system (L, R) on $\mathcal{C}$ cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$.*

(ii) *(L, R) is (part of) an algebraically free natural weak factorisation system on $\mathcal{C}$ cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$.*

(iii) *In particular, if $\mathcal{I}$ is discrete, then there exists a functorial weak factorisation system on $\mathcal{C}$ cofibrantly generated by the image of* $\operatorname{ob} \mathcal{I} \to \operatorname{mor} \mathcal{C}$.

Proof. (i). See Theorem 4.4 in [Garner, 2009].

(ii). See Theorem 5.4 in [Garner, 2009].

(iii). This is proposition A.3.43. □

Lemma 0.5.25. *Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$. *If κ is a regular cardinal in a universe $\mathbf{U}$ such that the domains of morphisms in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact in $\mathcal{C}$, then the class of $\mathcal{I}$-injective objects in $\mathcal{C}$ is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$.*

Proof. Let $\mathbb{D}$ be a $\mathbf{U}$-small κ-filtered category and let $X : \mathbb{D} \to C$ be a diagram such that each Xd is an $\mathcal{I}$-injective object in C. Suppose $\bar{X}$ is a colimit for X in C with colimiting cocone $\lambda : X \Rightarrow \Delta\bar{X}$. Let $g : Z \to W$ be in $\mathcal{I}$, and consider the induced hom-set map $g^* : C(W, \bar{X}) \to C(Z, \bar{X})$; we must show that it is surjective. Since Z is a $(\kappa, \mathbf{U})$-compact object in C, the canonical comparison $\varinjlim_{\mathbb{D}} C(Z, X) \to C(Z, \bar{X})$ is a bijection, and so every morphism $Z \to \bar{X}$ factors through $\lambda_d : Xd \to X$ for some d in $\mathbb{D}$. By hypothesis Xd is $\mathcal{I}$-injective, so we obtain an extension of $Z \to Xd$ along $g : Z \to W$, and hence, an extension of $Z \to \bar{X}$ along g. Thus X is also $\mathcal{I}$-injective. ■

Lemma 0.5.26. *Let C be a category and let $g : Z \to W$ be a morphism in C. A morphism $f : X \to Y$ has the left lifting property with respect to g if and only if f is injective as an object in $[\mathbb{2}, C]$ with respect to the singleton set $\{(g, \mathrm{id}_W) : g \to \mathrm{id}_W\}$.* ■

Corollary 0.5.27. *Let C be a category and let $\mathcal{I}$ be a subset of* $\operatorname{mor} C$. *If the domains and codomains of morphisms in $\mathcal{I}$ are $(\kappa, \mathbf{U})$-compact in C, then* $\operatorname{inj}^{\mathcal{I}} C$ *is closed under colimits for $\mathbf{U}$-small κ-filtered diagrams in $[\mathbb{2}, C]$.*

Proof. Apply proposition 0.2.44 and the two lemmas above. ■

Proposition 0.5.28. *Let C be a locally presentable $\mathbf{U}$-category, let (L, R) be a functorial weak factorisation system on C, and let let $\lambda : \mathrm{id}_{[\mathbb{2}, C]} \Rightarrow R$ be the natural transformation whose component at an object f in $[\mathbb{2}, C]$ corresponds to the following commutative square in C:*

$$\begin{tikzcd} \bullet \arrow[r, "Lf"] \arrow[d, "f"'] & \bullet \arrow[d, "Rf"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd}$$

Let $\mathcal{R}$ be the full subcategory of $[\mathbb{2}, C]$ spanned by the morphisms in C that are in the right class of the induced weak factorisation system.

(i) *$\mathcal{R}$ is also the full subcategory of $[\mathbb{2}, C]$ spanned by the image of the forgetful functor $[\mathbb{2}, C]^{(R,\lambda)} \to [\mathbb{2}, C]$, where $[\mathbb{2}, C]^{(R,\lambda)}$ is the category of algebras for the pointed endofunctor (R, λ).*

(ii) *If $R : [\mathbb{2}, C] \to [\mathbb{2}, C]$ is an accessible functor, then $[\mathbb{2}, C]^{(R,\lambda)}$ is a locally presentable $\mathbf{U}$-category, and the forgetful functor $[\mathbb{2}, C]^{(R,\lambda)} \to [\mathbb{2}, C]$ is monadic.*

(iii) *If $R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ is strongly $(\pi, \mathbf{U})$-accessible and has $\mathbf{U}$-rank $\kappa < \pi$, and $\mathcal{R}$ is closed under colimits for $\mathbf{U}$-small π-filtered diagrams in $[\mathbb{2}, \mathcal{C}]$, then $\mathcal{R}$ is a π-accessible $\mathbf{U}$-subcategory of $[\mathbb{2}, \mathcal{C}]$.*

Proof. (i). This is proposition A.3.31.

(ii). Apply theorem 0.3.36.

(iii). By theorem 0.3.37, $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)}$ is a locally π-presentable $\mathbf{U}$-category, and the forgetful functor $[\mathbb{2}, \mathcal{C}]^{(R,\lambda)} \to [\mathbb{2}, \mathcal{C}]$ is moreover strongly $(\pi, \mathbf{U})$-accessible. Thus, we may apply proposition 0.3.30 to claim (i) and deduce that $\mathcal{R}$ is a π-accessible $\mathbf{U}$-subcategory. ∎

Proposition 0.5.29. *Let $\mathcal{C}$ be a locally presentable $\mathbf{U}$-category, and let $\mathcal{I}$ be a $\mathbf{U}$-subset of* $\operatorname{mor} \mathcal{C}$. *Then* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$*, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is an accessible $\mathbf{U}$-subcategory.*

Proof. Combine corollary 0.5.14 and proposition 0.5.28. ∎

Lemma 0.5.30. *Let $\mathcal{C}$ be a κ-accessible $\mathbf{U}$-category and let $\mathcal{R}$ be a κ-accessible full subcategory of $[\mathbb{2}, \mathcal{C}]$. If $g : Z \to W$ is a morphism in $\mathcal{C}$ and Z and W are $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$, then:*

(i) *Given a morphism $f : X \to Y$ in $\mathcal{C}$ that is in $\mathcal{R}$, any morphism $g \to f$ in $[\mathbb{2}, \mathcal{C}]$ admits a factorisation of the form $g \to f' \to f$ where f' is in $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{R})$.*

(ii) *The morphism $g : Z \to W$ has the left lifting property with respect to $\mathcal{R}$ if and only if it has the left lifting property with respect to $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{R})$.*

Proof. (i). Proposition 0.2.44 says that g is a $(\kappa, \mathbf{U})$-compact object in $[\mathbb{2}, \mathcal{C}]$; but every object in $\mathcal{R}$ is the colimit of a $\mathbf{U}$-small κ-filtered diagram of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{R}$, and the inclusion $\mathcal{R} \hookrightarrow [\mathbb{2}, \mathcal{C}]$ is $(\kappa, \mathbf{U})$-accessible, so any morphism $g \to f$ must factor through some $(\kappa, \mathbf{U})$-compact object in $\mathcal{R}$.

(ii). If g has the left lifting property with respect to $\mathcal{R}$, then it certainly has the left lifting property with respect to $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{R})$. Conversely, by factorising morphisms $g \to f$ as in claim (i), we see that g has the left lifting property with respect to $\mathcal{R}$ as soon as it has the left lifting property with respect to $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{R})$. ∎

— I —

Simplicial sets

Simplicial sets, like simplicial complexes, are combinatorial models for spaces built up by gluing standard n-simplices together; unlike simplicial complexes, an n-simplex in a simplicial set need not be uniquely determined by its vertices. It is for this reason that simplicial sets were once known by the unwieldy name 'complete semi-simplicial (c.s.s.) complex'.

In the 1960s, it was discovered that one can mimic the definitions and constructions of classical homotopy theory by combinatorial means using simplicial sets, and that the resulting theory is moreover equivalent to the classical theory in a natural, functorial way. More recently, it has been shown that the homotopy theory of simplicial sets is *universal* in a precise sense,[1] so it seems fitting that we begin here.

1.1 Basics

Definition 1.1.1. The **simplex category** is the category $\mathbf{\Delta}$ whose objects are the positive finite ordinals and whose morphisms are the monotone maps. We use the geometer's convention: $[n]$ denotes the ordinal $\{0, 1, \dots, n\}$.

Definition 1.1.2. A **simplicial object** in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and a **morphism of simplicial objects** in $\mathcal{C}$ is a natural transformation of such functors. The **category of simplicial objects** in $\mathcal{C}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$ and is denoted by $\mathbf{s}\mathcal{C}$.

[1] See [Dugger, 2001a].

Definition 1.1.3. The **coface maps** in $\mathbf{\Delta}$ are the morphisms $\delta_n^i : [n-1] \to [n]$, where δ_n^i is the unique injective monotone map that misses i; and the **codegeneracy maps** in $\mathbf{\Delta}$ are the morphisms $\sigma_n^i : [n+1] \to [n]$, where σ_n^i is the unique surjective monotone map with $\sigma_n^i(i) = \sigma_n^i(i+1) = i$.

Theorem 1.1.4 (Cosimplicial identities)**.** *The following equations hold in $\mathbf{\Delta}$:*

$$
\begin{aligned}
\delta_{n+1}^{j+1} \circ \delta_n^i &= \delta_{n+1}^i \circ \delta_n^j && \text{if } 0 \le i \le j \le n \\
\sigma_n^j \circ \sigma_{n+1}^i &= \sigma_n^i \circ \sigma_{n+1}^{j+1} && \text{if } 0 \le i \le j \le n \\
\sigma_{n+1}^{j+1} \circ \delta_{n+1}^i &= \delta_n^i \circ \sigma_n^j && \text{if } 0 \le i \le j \le n \\
\delta_n^{j+1} \circ \sigma_n^i &= \sigma_{n+1}^i \circ \delta_{n+1}^{j+2} && \text{if } 0 \le i < j < n \\
\sigma_n^i \circ \delta_n^i &= \mathrm{id} && \text{if } 0 \le i \le n \\
\sigma_n^{i+1} \circ \delta_n^i &= \mathrm{id} && \text{if } 0 \le i < n
\end{aligned}
$$

Equivalently, the following diagrams commute:

$$
\begin{tikzcd}
{[n-1]} \arrow[r, "\delta^i"] \arrow[d, "\delta^j"'] & {[n]} \arrow[d, "\delta^{j+1}"] \\
{[n]} \arrow[r, "\delta^i"'] & {[n+1]}
\end{tikzcd}
\quad \text{for } 0 \le i \le j \le n
$$

$$
\begin{tikzcd}
{[n+1]} \arrow[r, "\sigma^i"] \arrow[d, "\sigma^{j+1}"'] & {[n]} \arrow[d, "\sigma^j"] \\
{[n]} \arrow[r, "\sigma^i"'] & {[n-1]}
\end{tikzcd}
\quad \text{for } 0 \le i \le j \le n
$$

$$
\begin{tikzcd}
{[n]} \arrow[r, "\delta^i"] \arrow[d, "\sigma^j"'] & {[n+1]} \arrow[d, "\sigma^{j+1}"] \\
{[n-1]} \arrow[r, "\delta^i"'] & {[n]}
\end{tikzcd}
\quad \text{for } 0 \le i \le j \le n
$$

$$
\begin{tikzcd}
{[n]} \arrow[r, "\sigma^i"] \arrow[d, "\delta^{j+2}"'] & {[n-1]} \arrow[d, "\delta^{j+1}"] \\
{[n+1]} \arrow[r, "\sigma^i"'] & {[n]}
\end{tikzcd}
\quad \text{for } 0 \le i < j < n
$$

$$\begin{tikzcd}
{[n-1]} \arrow[r, "\delta^i"] \arrow[d, "\delta^i"'] \arrow[dr, "\mathrm{id}"] & {[n]} \arrow[d, "\sigma^{i+1}"] \\
{[n]} \arrow[r, "\sigma^i"'] & {[n-1]}
\end{tikzcd} \quad \textit{for } 0 \leq i \leq n$$

Moreover, every morphism $[n] \to [m]$ *in* $\mathbf{\Delta}$ *is* uniquely *a composite of the form*

$$\delta_m^{j_1} \circ \cdots \circ \delta_k^{j_{m-k}} \circ \sigma_k^{i_{n-k}} \circ \cdots \circ \sigma_n^{i_1}$$

where $k \leq \min\{n, m\}$*, and*

$$0 \leq i_{n-k} \leq \cdots \leq i_1 \leq n$$
$$0 \leq j_{m-k} \leq \cdots \leq j_1 \leq m$$

The category $\mathbf{\Delta}$ *is uniquely characterised by these properties.*

Proof. See [May, 1967, § 2], [GZ, Ch. II, § 2], or [Weibel, 1994, § 8.1]. □

Definition 1.1.5. Let A be a simplicial object in a category $\mathcal{C}$. A **face operator** for A is a morphism of the form $A(\delta_n^i) : A([n]) \to A([n-1])$, and a **degeneracy operator** for A is a morphism of the form $A(\sigma_n^i) : A([n]) \to A([n+1])$. For brevity, we will usually write A_n instead of $A([n])$, d_i^n instead of $A(\delta_n^i)$, and s_i^n instead of $A(\sigma_n^i)$.

Corollary 1.1.6 (Simplicial identities)**.** *The face and degeneracy operators of a simplicial object satisfy the formal duals of the equations in theorem 1.1.4.* ■

Corollary 1.1.7. *A simplicial object A is uniquely determined by the sequence of objects* $A_0, A_1, A_2, \ldots$ *together with the face and degeneracy operators. Conversely, any sequence of objects equipped with face and degeneracy operators satisfying the simplicial identities defined a simplicial object.* ■

Definition 1.1.8. A **simplicial set** is a simplicial object in **Set**, and the **category of simplicial sets** is denoted by **sSet**.

Lemma 1.1.9.

(i) *Limits (resp. colimits) in* **sSet** *are constructed degreewise: a cone (resp. cocone) in* **sSet** *over a diagram is limiting (resp. colimiting) if and only if it is so in every degree.*

(ii) *A morphism of* **sSet** *is monic (resp. epic) if and only if it is degreewise injective (resp. surjective).*

Proof. These are standard facts about functor categories. □

Definition 1.1.10. The **standard n-simplex** in **sSet**, denoted by Δ^n, is the representable presheaf $\mathbf{\Delta}(-, [n])$.

Theorem 1.1.11. *Let $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ be the functor $[n] \mapsto \Delta^n$.*

(i) *For any simplicial set X, the map $\mathbf{sSet}(\Delta^n, X) \to X_n$ defined by $f \mapsto f_n\left(\mathrm{id}_{[n]}\right)$ is a bijection and is moreover natural in $[n]$ and X.*

(ii) **sSet** *has limits and colimits for all small diagrams, every epimorphism is effective, and for all morphisms $f : X \to Y$ in* **sSet**, *the pullback functor $f^* : \mathbf{sSet}_{/Y} \to \mathbf{sSet}_{/X}$ preserves colimits.*

(iii) *$\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ is a dense functor, i.e. for any simplicial set X, the tautological cocone*[2] *from the canonical diagram $(\Delta^\bullet \downarrow X) \to \mathbf{sSet}$ to X is colimiting.*

(iv) *Let $\mathcal{E}$ be a locally small category with colimits for all small diagrams. If $F : \mathbf{sSet} \to \mathcal{E}$ is a functor that preserves small colimits, then it is left adjoint to the functor $\mathcal{E} \to \mathbf{sSet}$ defined by $E \mapsto \mathcal{E}(F\Delta^\bullet, E)$.*

(v) *With $\mathcal{E}$ as above, the functor $F \mapsto F\Delta^\bullet$ from the category of colimit-preserving functors $\mathbf{sSet} \to \mathcal{E}$ to the category of all functors $\mathbf{\Delta} \to \mathcal{E}$ is fully faithful and essentially surjective on objects.*

Proof. Claim (i) is just the Yoneda lemma, claim (ii) follows from the lemma above, and claims (iii)–(v) are just facts about dense functors, pointwise left Kan extensions, weighted colimits: see proposition A.5.25, theorem A.5.15, and proposition A.6.11. ■

Definition 1.1.12. Let X be a simplicial set. An **n-simplex** of X is an element of X_n; a **vertex** is a 0-simplex, and an **edge** is a 1-simplex. This is justified by statement (i) in the above theorem. Given an edge f of X, the **source** of f is the vertex $d_1(f)$, and the **target** of f is the vertex $d_0(f)$; we write $f : x \to y$ to mean $d_1(f) = x$ and $d_2(f) = y$.

Definition 1.1.13. A **degenerate n-simplex** of a simplicial set X is an n-simplex α for which there exist an $(n-1)$-simplex β and $0 \leq i < n$ such that $s_i(\beta) = \alpha$. A **non-degenerate n-simplex** of X is an n-simplex that is not degenerate.

[2] See definition A.5.7.

Remark 1.1.14. An n-simplex of X can be non-degenerate even when the corresponding morphism $\Delta^n \to X$ is not a monomorphism! Similarly, it is possible for all the proper faces of a non-degenerate simplex to be degenerate.

Definition 1.1.15. A **finite simplicial set** is a simplicial set that has only finitely many *non-degenerate* simplices.

Proposition 1.1.16. *Let X be a simplicial set. The following are equivalent:*

(i) *X is a finite simplicial set.*

(ii) *X is an $\aleph_0$-compact object in* **sSet**.[3]

(iii) *X is in the smallest full subcategory of* **sSet** *that contains the standard simplices and is closed in* **sSet** *under (isomorphisms and) colimits for finite diagrams.*

Proof. (i) $\Rightarrow$ (ii). A morphism $f : X \to Y$ is determined uniquely by the images of the non-degenerate simplices of X, and the faces of any particular simplex can only satisfy finitely many equations, so if X is a finite simplicial set and Y is a colimit for a small filtered diagram of simplicial sets, then f must factor through one of the components of the colimiting cocone. It is straightforward to check that the factorisation of f is unique up to the appropriate equivalence relation, and we may then deduce that X is an $\aleph_0$-compact object.

(ii) $\Rightarrow$ (iii). Let $\mathcal{K}$ be the indicated full subcategory of **sSet**, and consider the comma category $(\mathcal{K} \downarrow X)$. Let $P : (\mathcal{K} \downarrow X) \to \mathbf{sSet}$ be the projection, and let $\lambda : P \Rightarrow \Delta X$ be the tautological cocone.[4] It is not hard to check that λ is a colimiting cocone. Since $\mathcal{K}$ has colimits for finite diagrams, $(\mathcal{K} \downarrow X)$ is filtered; and it is clear that $\mathcal{K}$ is essentially small, so we deduce that X is a retract of an object in $\mathcal{K}$ if X is $\aleph_0$-compact. Noting that $\mathcal{K}$ is closed under retracts, we conclude that X is in $\mathcal{K}$ if it is $\aleph_0$-compact.

(iii) $\Rightarrow$ (i). Now, let $\mathcal{K}'$ be the full subcategory of **sSet** spanned by the finite simplicial sets. It is easy to see that $\mathcal{K}'$ is closed in **sSet** under (isomorphisms and) finite colimits, and the standard simplices are all in $\mathcal{K}'$, so we must have $\mathcal{K} \subseteq \mathcal{K}'$, as required. ■

[3] See definition 0.2.11.
[4] See definition A.5.7.

Definition 1.1.17. The **standard n-simplex** in **Top**, denoted by $|\Delta^n|$, is the topological space

$$|\Delta^n| = \left\{ (x_0, \ldots, x_n) \in [0,1]^{n+1} \,\middle|\, x_0 + \cdots + x_n = 1 \right\}$$

where $[0,1]$ is the closed unit interval with the standard metric. The functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ sends $[n]$ to $|\Delta^n|$ and is defined on morphisms by linearly interpolating the obvious map of vertices.

Corollary 1.1.18. *There exists an adjunction*

$$|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$$

extending the functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ defined above, and this adjunction is unique up to unique isomorphism. Explicitly, we may take

$$\mathrm{S}(Y)_n = \mathbf{Top}\big(|\Delta^n|, Y\big)$$

with the evident face and degeneracy operators induced by the coface and codegeneracy maps in $\mathbf{\Delta}$. ■

Definition 1.1.19. The **geometric realisation** of a simplicial set X is the topological space $|X|$, and the **singular set** of a topological space Y is the simplicial set $\mathrm{S}(Y)$.

Remark 1.1.20. The geometric realisation $|X|$ is stable under universe enlargement, by theorem A.5.20.

Theorem 1.1.21. *Let* **CGHaus** *be the category of compactly-generated Hausdorff spaces*[5] *and continuous maps.*

(i) *The topological standard n-simplex $|\Delta^n|$ is a compact Hausdorff space.*

(ii) *For any simplicial set X, the geometric realisation $|X|$ is a compactly-generated Hausdorff space.*

(iii) *The previously-constructed adjunction $|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$ restricts to an adjunction between* **CGHaus** *and* **sSet**, *and moreover the functor $|-| : \mathbf{sSet} \to \mathbf{CGHaus}$ preserves finite limits and reflects isomorphisms.*

Proof. Claim (i) is a standard fact, while claims (ii) and (iii) are proven in [GZ, Ch. III, § 3]. □

[5] See definition A.2.26.

1.2 Nerves, skeletons, and coskeletons

Prerequisites. §§ 1.1, A.2.

Proposition 1.2.1. *Let* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{C})_n = \mathrm{Fun}([n], \mathbb{C})$$

where $[n]$ *here denotes the preorder category* $\{0 \to \cdots \to n\}$.

(i) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *has a left adjoint* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *such that* $\tau_1 \Delta^n = [n]$.

(ii) *The functor* N *is fully faithful and exhibits* **Cat** *as a reflective subcategory of* **sSet**.

(iii) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(iv) *The functor* τ_1 *preserves finite products.*

Proof. (i). Apply theorem 1.1.11.

(ii). A functor is entirely determined by its action on objects, arrows, and composable strings of arrows, so N is fully faithful.

(iii). N preserves binary products, so we have the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}(\Delta^n, \mathrm{N}([\mathbb{C}, \mathbb{D}])) &\cong \mathrm{Fun}([n], [\mathbb{C}, \mathbb{D}]) \\
&\cong \mathrm{Fun}([n] \times \mathbb{C}, \mathbb{D}) \\
&\cong \mathbf{sSet}(\mathrm{N}([n] \times \mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]) \times \mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]), [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]) \\
&\cong \mathbf{sSet}(\Delta^n, [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})])
\end{aligned}$$

Thus, by the Yoneda lemma, the canonical morphism $\mathrm{N}([\mathbb{C}, \mathbb{D}]) \to [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]$ is an isomorphism.

(iv). It is clear that τ_1 preserves terminal objects. Let X and Y be simplicial sets. We wish to show that the canonical morphism $\tau_1(X \times Y) \to \tau_1 X \times \tau_1 Y$ is an isomorphism; but since τ_1 is a left adjoint and both **sSet** and **Cat** are cartesian closed, it is enough to check the claim for $Y = \Delta^n$, because **sSet** is generated under colimits by $\{\Delta^n \mid n \in \mathbb{N}\}$. We have the following natural bijections:

$$\begin{aligned}
\mathrm{Fun}\big(\tau_1(X \times \Delta^n), \mathbb{C}\big) &\cong \mathbf{sSet}(X \times \Delta^n, \mathrm{N}(\mathbb{C})) \\
&\cong \mathbf{sSet}\big(X, \mathrm{N}(\mathbb{C})^{\Delta^n}\big) \\
&\cong \mathbf{sSet}(X, \mathrm{N}([[n], \mathbb{C}])) \\
&\cong \mathrm{Fun}\big(\tau_1 X, [[n], \mathbb{C}]\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times [n], \mathbb{C}\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times \tau_1 \Delta^n, \mathbb{C}\big)
\end{aligned}$$

The claim follows by the Yoneda lemma. ■

Definition 1.2.2. The **fundamental category** of a simplicial set X is the small category $\tau_1 X$, and the **nerve** of a small category $\mathbb{C}$ is the simplicial set $\mathrm{N}(\mathbb{C})$.

Remark 1.2.3. Given a simplicial set X, the fundamental category $\tau_1 X$ admits the following presentation by generators and relations: the objects are the vertices of X, and the arrows are generated by the edges of X, modulo the relation $d_0(\alpha) \bullet d_2(\alpha) = d_1(\alpha)$ for all 2-simplices α in X. This shows that $\tau_1 X$ is stable under universe enlargement.

Proposition 1.2.4. *Let* disc : **Set** → **sSet** *be the functor defined by the formula*

$$(\mathrm{disc}\, Y)_n = Y$$

with id_Y *for all the face and degeneracy maps.*

(i) disc : **Set** → **sSet** *has a left adjoint* π_0 : **sSet** → **Set** *such that* $\pi_0 \Delta^n = 1$.

(ii) *The functor* disc *is fully faithful and exhibits* **Set** *as a reflective subcategory of* **sSet**.

(iii) N : **Set** → **sSet** *is a cartesian closed functor.*

(iv) *The functor* π_0 *preserves products.*

Proof. (i). We could apply theorem 1.1.11, but it is also fairly straightforward to check that this explicit construction works: for each simplicial set X, we define $\pi_0 X$ by the coequaliser diagram in **Set** shown below,

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X$$

and for each morphism $f : X \to Y$ in **sSet**, we define $\pi_0 f$ to be the unique morphism making the evident diagram commute.

(ii). It is clear that disc is fully faithful.

(iii). By proposition A.2.15, we have an analogous adjunction $\pi_0 \dashv \mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$. It is clear that we have a natural isomorphism $\mathrm{N}(\mathrm{disc}\, Y) \cong \mathrm{disc}\, Y$ for every set Y, and we know $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ and $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ are cartesian closed functors, so $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ must also be cartesian closed.

(iv). Similarly, for any simplicial set X, we have a natural isomorphism $\pi_0 X \cong \pi_0 \tau_1 X$; but we know that $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ preserves finite products, and $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ preserves finite products by proposition 1.2.1, so $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ must also preserve finite products. ■

Definition 1.2.5. The **set of connected components** of a simplicial set X is the set $\pi_0 X$, and a **discrete simplicial set** is one that is isomorphic to $\mathrm{disc}\, Y$ for some set Y.

¶ 1.2.6. We will usually not distinguish between Y and $\mathrm{disc}\, Y$ notationally.

Proposition 1.2.7. *Let* $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{G})_n = \mathrm{Fun}(\mathbf{I}[n], \mathbb{G})$$

where $\mathbf{I}[n]$ *here denotes the groupoid obtained by freely inverting the arrows in the preorder category* $[n]$.

(i) *For any groupoid* $\mathbb{G}$*, the nerve* $\mathrm{N}(\mathbb{G})$ *is the same (up to isomorphism) whether computed for* $\mathbb{G}$ *as a groupoid or* $\mathbb{G}$ *as a category.*

(ii) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *has a left adjoint* $\pi_1 : \mathbf{sSet} \to \mathbf{Grpd}$ *such that* $\pi_1 \Delta^n = \mathbf{I}[n]$.

(iii) *The functor* N *is fully faithful and exhibits* **Grpd** *as a reflective subcategory of* **sSet**.

(iv) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(v) *The functor* π_1 *preserves finite products.*

Proof. (i). By the universal property of $\mathbf{I}[n]$, there is a natural bijection

$$\mathrm{Fun}(\mathbf{I}[n], \mathbb{G}) \cong \mathrm{Fun}([n], \mathbb{G})$$

for all groupoids $\mathbb{G}$, so the two nerve constructions do indeed agree.

(ii) and (iii). These are proven in exactly the same way as in proposition 1.2.1.

(iv) and (v). These are proven in exactly the same way as in proposition 1.2.4. ■

Definition 1.2.8. The **fundamental groupoid** of a simplicial set X is the small groupoid $\pi_1 X$.

Remark 1.2.9. Given a simplicial set X, the fundamental groupoid $\pi_1 X$ admits a presentation of the same kind as the fundamental category $\tau_1 X$, and in fact $\pi_1 X$ is isomorphic to the groupoid obtained by freely inverting the arrows in $\tau_1 X$:

$$\mathrm{Fun}\big(\pi_1 X, \mathbb{G}\big) \cong \mathbf{sSet}(X, \mathrm{N}(\mathbb{G})) \cong \mathrm{Fun}\big(\tau_1 X, \mathbb{G}\big)$$

This shows that $\pi_1 X$ is stable under universe enlargement.

Definition 1.2.10. Let n be a natural number, and let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0], \ldots, [n]$. An **n-truncated simplicial set** is a functor ${\mathbf{\Delta}_{\leq n}}^{\mathrm{op}} \to \mathbf{Set}$, and we write $\mathbf{sSet}_{\leq n}$ for the category of n-truncated simplicial sets. The **brutal n-truncation** of a simplicial set X is the n-truncated simplicial set $X_{\leq n}$ defined by the evident reduct:

$$X_{\leq n}([m]) = X([m])$$

Proposition 1.2.11. *Let n be a natural number, and let $j : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ be the inclusion.*

(i) *The functor $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$ has a left adjoint $\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$.*

(ii) *The unit $\mathrm{id} \Rightarrow j^* \mathrm{Lan}_j$ is a natural isomorphism.*

(iii) *$\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$ is a fully faithful functor.*

(i′) *The functor $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$ has a right adjoint $\mathrm{Ran}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$.*

(ii′) *The counit $j^* \mathrm{Ran}_j \Rightarrow \mathrm{id}$ is a natural isomorphism.*

(iii′) $\operatorname{Ran}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$ *is a fully faithful functor.*

Proof. (i) and (i′). Use theorem A.5.15.

(ii) and (ii′). The inclusion $j : \mathbf{\Delta}_{\le n} \to \mathbf{\Delta}$ is fully faithful, so the unit $\mathrm{id} \Rightarrow j^* \operatorname{Lan}_j$ and the counit $j^* \operatorname{Ran}_j \Rightarrow \mathrm{id}$ are natural isomorphisms, by corollary A.5.19.

(iii) and (iii′). It is a well-known fact that the unit (resp. counit) of an adjunction is a natural isomorphism if and only if the left (resp. right) adjoint is fully faithful.[6] ■

Definition 1.2.12. For each natural number n, with notation as above, let $\mathrm{sk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Lan}_j j^*$, and let $\mathrm{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Ran}_j j^*$. The ***n*-skeleton** of a simplicial set X is the simplicial set $\mathrm{sk}_n(X)$, and the ***n*-coskeleton** of a simplicial set is the simplicial set $\mathrm{cosk}_n(X)$. A ***n*-skeletal simplicial set** is one that is isomorphic to the n-skeleton of some simplicial set, and an ***n*-coskeletal simplicial set** is one that is isomorphic to the n-coskeleton of some simplicial set.

Remark 1.2.13. In the special case $n = 0$, Lan_j may be identified with the functor $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ defined in proposition 1.2.4. Thus, 0-skeletal simplicial sets are precisely the discrete simplicial sets. On the other hand, given a set X, $\operatorname{Ran}_j X$ can be identified with the simplicial set whose m-simplices are $(m+1)$-tuples of elements of X, with face and degeneracy maps induced by the appropriate projections.

Proposition 1.2.14. *Let n be a natural number.*

(i) *The full subcategory of n-skeletal simplicial sets is a coreflective subcategory of* **sSet***, with coreflector* sk_n.

(ii) sk_n *is the underlying endofunctor of an idempotent comonad on* **sSet**.

(iii) *A simplicial set X is n-skeletal if and only if the counit* $\mathrm{sk}_n(X) \to X$ *is an isomorphism.*

(iv) *If* $m \ge n$*, then any n-skeletal simplicial set is also m-skeletal.*

(i′) *The full subcategory of n-coskeletal simplicial sets is a reflective subcategory of* **sSet***, with reflector* cosk_n.

[6] See e.g. [CWM, Ch. IV, § 3].

(ii′) cosk_n *is the underlying endofunctor of an idempotent monad on* **sSet**.

(iii′) *A simplicial set X is n-coskeletal if and only if the unit $X \to \operatorname{cosk}_n(X)$ is an isomorphism.*

(iv′) *If $m \geq n$, then any n-coskeletal simplicial set is also m-coskeletal.*

Proof. All straightforward from the definitions. ♦

Proposition 1.2.15. *Let n be a natural number, and let X be a simplicial set.*

(i) *We have the following adjunction:*

$$\operatorname{sk}_n \dashv \operatorname{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$$

(ii) *The counit $\operatorname{sk}_n(X) \to X$ is a monomorphism, and X is n-skeletal if and only if all m-simplices of X are degenerate for $m > n$.*

(iii) *X is n-coskeletal if and only if, for all natural numbers m, the map*

$$X_m \cong \mathbf{sSet}(\Delta^m, X) \to \mathbf{sSet}\big(\operatorname{sk}_n(\Delta^m), X\big)$$

induced by the counit $\operatorname{sk}_n(\Delta^m) \to \Delta^m$ is a bijection.

Proof. (i). Immediate from the definition of sk_n and cosk_n.

(ii). The most straightforward way of seeing this is to construct $\operatorname{sk}_n(X)$ explicitly as the smallest simplicial subset of X containing all of its n-simplices.

(iii). Apply the Yoneda lemma in conjunction with claim (i). ■

Example 1.2.16. For any small category $\mathbb{C}$, the nerve $\mathrm{N}(\mathbb{C})$ is a 2-coskeletal simplicial set: by definition, an m-simplex of $\mathrm{N}(\mathbb{C})$ is just a functor $[m] \to \mathbb{C}$, but the property of being a functor can be detected by only inspecting the vertices, edges, and 2-cells.

Proposition 1.2.17. *The following full subcategories are exponential ideals of* **sSet***:*

(i) *Discrete simplicial sets.*

(ii) *Simplicial sets isomorphic to the nerve of some category.*

(iii) *Simplicial sets isomorphic to the nerve of some groupoid.*

(iv) *n-coskeletal simplicial sets for some natural number n.*

Proof. Apply proposition A.2.13 to propositions 1.2.4, 1.2.1, 1.2.7, and 1.2.14. ■

1.3 The Kan–Quillen model structure

Prerequisites. §§ 0.5 1.1, 4.1, A.3.

In [1967], Quillen constructed an axiomatic framework for doing homotopy theory in abstract categories, which he called 'closed model categories', and showed that **sSet** can be endowed with a model structure such that the resulting homotopy theory is equivalent in a strong sense to the homotopy theory of topological spaces.

Definition 1.3.1. A **horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where Λ^n_k is the union of the images of $\delta^0_n, \ldots, \delta^{k-1}_n, \delta^{k+1}_n, \ldots, \delta^n_n : \Delta^{n-1} \to \Delta^n$ in **sSet**. In other words, Λ^n_k is the union of all the faces of Δ^n that include the k-th vertex. The **boundary** of Δ^n is the simplicial subset $\partial\Delta^n \subseteq \Delta^n$ generated by the images of $\delta^0_n, \ldots, \delta^n_n : \Delta^{n-1} \to \Delta^n$.

Remark 1.3.2. The boundary $\partial\Delta^n$ may be identified with $\mathrm{sk}_{n-1}\Delta^n$.

Definition 1.3.3. A **cofibration** in **sSet** is a monomorphism. A **Kan fibration** is a morphism $f : X \to Y$ in **sSet** that has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, where $n \geq 1$ and $0 \leq k \leq n$. A **Kan complex** is a simplicial set X such that the unique morphism $X \to 1$ is a Kan fibration.

Remark 1.3.4. In other words, a Kan complex is a simplicial set X satisfying the **Kan condition**: every horn $\alpha' : \Lambda^n_k \to X$ has a **filler**, i.e. a morphism $\alpha : \Delta^n \to X$ (equivalently, an n-simplex of X) such that α' is the restriction along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

Lemma 1.3.5. *If X is a Kan complex, then the fundamental category $\tau_1 X$ is a groupoid, and the unit $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is an epimorphism.*

Proof. Let x, y, and z be vertices in X, and let $f : x \to y$ and $g : y \to z$ be edges in X.[7] Then the pair (f, g) defines a horn $\Lambda^2_1 \to X$, and so by the Kan

[7] Recall definition 1.1.12.

condition, there exists a 2-simplex α of X such that $d_2(\alpha) = f$ and $d_0(\alpha) = g$. By remark remark 1.2.3, the composite $g \bullet f$ defined in $\tau_1 X$ must correspond to the edge $d_1(\alpha)$. Since the arrows in $\tau_1 X$ are generated by the edges of X, we conclude by induction that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on vertices and edges.

Similarly, given an edge $f : x \to y$, the Kan condition ensures that there exist two 2-simplices β and γ such that

$$\begin{aligned} d_2(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_x \\ d_0(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_y \end{aligned}$$

where $\mathrm{id}_x : x \to x$ is the edge $s_0(x)$, and $\mathrm{id}_y : y \to y$ is the edge $s_0(y)$. Together with the argument in the previous paragraph, this shows that $\tau_1 X$ is a groupoid.

Finally, to show that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on n-simplices for $n \geq 2$, we simply observe that an n-simplex of $\mathrm{N}(\tau_1 X)$ is just a string of n composable edges of X, so we may appeal to the Kan condition again to obtain the corresponding n-simplex of X. ∎

Corollary 1.3.6. *If X is a Kan complex, then the unit $\eta_X : X \to \mathrm{N}(\pi_1 X)$ is an epimorphism.*

Proof. Since $\tau_1 X$ is already a groupoid, the canonical functor $\tau_1 X \to \pi_1 X$ must be an isomorphism. (See remark 1.2.9.) ∎

Proposition 1.3.7. *Let X be a Kan complex and let $\alpha_0, \alpha_1 : x_0 \to x_1$ be edges in X. The following are equivalent:*

(i) *$\alpha_0 = \alpha_1$ in the fundamental groupoid $\pi_1 X$.*

(ii) *There exists a 2-simplex σ of X such that $d_0(\sigma) = s_0(x_1)$, $d_1(\sigma) = \alpha_1$, and $d_2(\sigma) = \alpha_0$.*

(iii) *There exists an edge $\beta : \alpha_0 \to \alpha_1$ in the exponential object $[\Delta^1, X]$ such that $[\delta^1, X](\beta) = s_0(x_0)$ and $[\delta^0, X](\beta) = s_0(x_1)$.*

Proof. (i) ⇔ (ii). See Proposition 1.2.3.9 in [HTT].

(i) ⇔ (iii). See paragraph 5.2 in [GZ]. □

Proposition 1.3.8. *Let $\mathcal{I}$ and $\mathcal{I}'$ be the following subsets of* mor **sSet***:*

$$\mathcal{I} = \left\{ \Lambda^n_k \hookrightarrow \Delta^n \,\middle|\, n \geq 1, 0 \leq k \leq n \right\}$$
$$\mathcal{I}' = \{ \partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0 \}$$

(i) *There exist a pair of functorial factorisation systems on* **sSet***, one inducing a weak factorisation system cofibrantly generated by $\mathcal{I}$, and the other inducing a weak factorisation system cofibrantly generated by $\mathcal{I}'$.*

(ii) *A morphism is $\mathcal{I}$-injective if and only if it is a Kan fibration, and every $\mathcal{I}$-cofibration is a monomorphism (but not vice versa).*

(iii) *A morphism is a $\mathcal{I}'$-cofibration if and only if it is a monomorphism, and every $\mathcal{I}'$-injective morphism is a Kan fibration (but not vice versa).*

Proof. (i). Since **sSet** is a locally finitely presentable category, we may apply Quillen's small object argument (theorem 0.5.12).

(ii). The definition of 'Kan fibration' is exactly the definition of '$\mathcal{I}$-injective morphism'; on the other hand, the class of monomorphisms is closed under pushout, transfinite composition, and retracts in **Set**, so the same is true for **sSet**, and thus, by corollary 0.5.13, every $\mathcal{I}$-cofibration must be a monomorphism.

(iii). To prove that $\operatorname{inj}^{\mathcal{I}} \mathcal{C} \supseteq \operatorname{inj}^{\mathcal{I}'} \mathcal{C}$, it is enough to check that $\mathcal{I} \subseteq \operatorname{cof}_{\mathcal{I}'} \mathcal{C}$; since every morphism in $\mathcal{I}$ is a monomorphism, it will suffice to show that $\operatorname{cof}_{\mathcal{I}'} \mathcal{C}$ is precisely the class of all monomorphisms. For this, see the remarks at the beginning of [Joyal and Tierney, 2008, § 3.1], or Proposition 1 in [Quillen, 1967, Ch. II, § 2]. □

Definition 1.3.9. An **anodyne extension**, or **trivial cofibration** in **sSet**, is a cofibration that has the left lifting property with respect to all Kan fibrations. A **trivial Kan fibration** is a Kan fibration that has the right lifting property with respect to all cofibrations.

Proposition 1.3.10. *Let $\mathcal{K}$ be the full subcategory of* **sSet** *spanned by the finite simplicial sets.*

(i) *The class of monomorphisms that are in $\mathcal{K}$ is the smallest class containing the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ that is closed under composition, pushouts, and retracts.*

(ii) *The class of anodyne extensions that are in $\mathcal{K}$ is the smallest class containing the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ that is closed under composition, pushouts, and retracts.*

Proof. (i). Corollary 0.5.13 implies that every monomorphism in **sSet** is a retract of a relative $\mathcal{I}'$-cell complex, where $\mathcal{I}'$ is the set of all boundary inclusions. More precisely, if $g : Z \to W$ is a monomorphism, then there is a commutative diagram in **sSet** of the form below,

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Z \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & Z \arrow[d, "g"] \\ W \arrow[r, "i_W"] \arrow[rr, bend right, "\mathrm{id}"'] & W' \arrow[r, "r_W"] & W \end{tikzcd}$$

where $i : Z \to W'$ is a relative $\mathcal{I}'$-cell complex. Suppose W is a finite simplicial set. Proposition 1.1.16 says that finite simplicial sets are $\aleph_0$-compact objects in **sSet**, so by considering a sequential presentation for $i : Z \to W'$, we see that $g : Z \to W$ is a retract of some relative $\mathcal{I}'$-cell complex that admits an $\aleph_0$-small presentation. In particular, if Z is a finite simplicial set, then so is W' (by lemma 0.2.15). Hence, the class of monomorphisms in $\mathcal{K}$ is the smallest class containing $\mathcal{I}'$ that is closed under composition, pushouts, and retracts.

(ii). The proof is similar to that of claim (i), except for replacing boundary inclusions by horn inclusions. ■

Proposition 1.3.11. *Let $f : X \to Y$ be a morphism in* **sSet** *and, for each n-simplex $\sigma : \Delta^n \to Y$, let $f_\sigma : X_\sigma \to \Delta^n$ be defined by the pullback diagram in* **sSet** *shown below:*

$$\begin{tikzcd} X_\sigma \arrow[r] \arrow[d, "f_\sigma"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\sigma"'] & Y \end{tikzcd}$$

(i) *$f : X \to Y$ is a Kan fibration if and only if each $f_\sigma : X_\sigma \to \Delta^n$ is a Kan fibration.*

(ii) *$f : X \to Y$ is a trivial Kan fibration if and only if each $f_\sigma : X_\sigma \to \Delta^n$ is a trivial Kan fibration.*

Proof. This is a straightforward exercise. ◊

Corollary 1.3.12.

(i) *The coproduct of a small family of Kan fibrations is a Kan fibration.*

(ii) *The coproduct of a small family of trivial Kan fibrations is a trivial Kan fibration.*

Proof. Given the previous proposition and the fact that coproducts in **sSet** are disjoint and stable under pullback, it suffices to observe that any $\Delta^n \to \coprod_{i \in I} Y_i$ must factor through one of the coproduct insertions $Y_j \to \coprod_{i \in I} Y_i$. ■

Proposition 1.3.13. *Let $i : Z \to W$ be a cofibration in* **sSet** *and let $p : X \to Y$ be a Kan fibration. Suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[drr, bend left, "{[i,X]}"] \arrow[dr, dashed, "q"] \arrow[ddr, bend right, "{[W,p]}"'] & & \\
& L(i,p) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,p]}"] \\
& {[W,Y]} \arrow[r, "{[i,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *The unique morphism $q : [W,X] \to L(i,p)$ making the diagram commute is a Kan fibration.*

(ii) *If $i : Z \to W$ is an anodyne extension, then $q : [W,X] \to L(i,p)$ is a trivial Kan fibration.*

(iii) *If $p : Z \to W$ is a trivial Kan fibration, then so is $q : [W,X] \to L(i,p)$.*

Proof. (i). See Theorem 3.3.1 in [Hovey, 1999], or Proposition 5.2 in [GJ, Ch. I].

(ii) and (iii). See Proposition 11.5 in [GJ, Ch. I]; for a purely combinatorial proof, see Theorem 3.2.1 in [Joyal and Tierney, 2008]. □

Corollary 1.3.14.

(i) *If $p : X \to Y$ is a Kan fibration (resp. trivial Kan fibration), then for all simplicial sets W, the morphism $[W,p] : [W,X] \to [W,Y]$ is also a Kan fibration (resp. trivial Kan fibration).*

(ii) *If $i : Z \to W$ is a cofibration (resp. anodyne extension) and X is a Kan complex, then the morphism $[i, X] : [W, X] \to [Z, X]$ is a Kan fibration (resp. trivial Kan fibration).*

(iii) *If W is any simplicial set and X is a Kan complex, then $[W, X]$ is also a Kan complex.*

Proof. (i). Take $Z = \varnothing$; noting that the canonical morphism $\varnothing \to W$ is a cofibration, and that $[\varnothing, p] : [\varnothing, X] \to [\varnothing, Y]$ is an isomorphism, the proposition above then implies $[W, p] : [W, X] \to [W, Y]$ is a Kan fibration (resp. trivial Kan fibration).

(ii). Take $Y = 1$; since $[W, 1] \to [Z, 1]$ is an isomorphism, the proposition above implies $[i, X] : [W, X] \to [Z, X]$ is a Kan fibration (resp. trivial Kan fibration).

(iii). Noting that $[\varnothing, X]$ is a terminal object in **sSet**, we apply claim (ii) to the case $Z = \varnothing$ to obtain the desired conclusion. ■

The following combinatorial definition of weak homotopy equivalence is due to Joyal and Tierney [2008]. Recalling the definition of $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ from proposition 1.2.4 as the functor sending a simplicial set X to the set π_0 of its connected components,

Definition 1.3.15. A **weak homotopy equivalence of simplicial sets** is a morphism $f : W \to Z$ such that, for every Kan complex K, the induced map

$$\pi_0[f, K] : \pi_0[Z, K] \to \pi_0[W, K]$$

is a bijection of sets.

Proposition 1.3.16.

(i) *A Kan fibration $p : X \to Y$ is trivial if and only if it is a weak homotopy equivalence.*

(ii) *A cofibration $i : Z \to W$ is an anodyne extension if and only if it is a weak homotopy equivalence.*

Proof. See Propositions 3.4.1 and 3.4.2 in [Joyal and Tierney, 2008]. □

In summary, we have:

Theorem 1.3.17. **sSet**, *regarded as a* **sSet**-*enriched category via its cartesian closed structure, is a simplicial model category where*

- *the cofibrations are the monomorphisms in* **sSet**,
- *the fibrations are the Kan fibrations, and*
- *the weak equivalences are the weak homotopy equivalences.*

This is the **Kan–Quillen model structure on simplicial sets**.

Proof. We know **sSet** has limits and colimits for all small diagrams and is a cartesian closed category, so it satisfies axioms CM1 and SM0. Using the definition of weak homotopy equivalence given above, the class of weak homotopy equivalences has the 2-out-of-6 property by lemma A.4.14, hence axiom CM2 is satisfied. Proposition 1.3.8 plus theorem 4.1.12 then shows that the announced cofibrations, fibrations, and weak equivalences do indeed constitute a closed model structure on **sSet**.

Finally, we note that proposition 1.3.13 is precisely the condition required by axiom SM7. ■

Proposition 1.3.18. *There exist a functor* $R : \mathbf{sSet} \to \mathbf{sSet}$ *and a natural transformation* $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ *such that, for all simplicial sets* X*,* RX *is a Kan complex and* $i_X : X \to RX$ *is an anodyne extension. Moreover, any such functor* R *preserves weak homotopy equivalences.*

Proof. By proposition 1.3.8, for each X, there is a factorisation of the unique morphism $X \to 1$ as an anodyne extension $i_X : X \to RX$ followed by a Kan fibration $RX \to 1$, and this is moreover functorial in X. Finally, if $f : X \to Y$ is a weak homotopy equivalence in **sSet**, then the commutativity of the diagram below

$$\begin{array}{ccc} X & \xrightarrow{i_X} & RX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Rf} \\ Y & \xrightarrow[i_Y]{} & RY \end{array}$$

plus proposition 1.3.16 and the 2-out-of-3 property for weak homotopy equivalences implies Rf is also a weak homotopy equivalence. ■

Proposition 1.3.19. *Let $f : W \to Z$ be a weak homotopy equivalence of simplicial sets and let X be any simplicial set.*

(i) *The morphism $f \times \mathrm{id}_X : W \times X \to Z \times X$ is a weak homotopy equivalence.*

(ii) *If X is a Kan complex, then $[f, X] : [Z, X] \to [W, X]$ is a weak homotopy equivalence.*

(iii) *If W and Z are Kan complexes, then $[X, f] : [X, W] \to [X, Z]$ is a weak homotopy equivalence.*

Proof. (i). We must show that, for all Kan complexes K, the induced map

$$\pi_0\big[f \times \mathrm{id}_X, K\big] : \pi_0[Z \times X, K] \to \pi_0[W \times X, K]$$

is a bijection. However, we have a commutative diagram

$$\begin{tikzcd}
\pi_0[Z \times X, K] \arrow[r, "\pi_0\left[\mathrm{id}_X \times f, K\right]"] \arrow[d, "\cong"'] & \pi_0[W \times X, K] \arrow[d, "\cong"] \\
\pi_0[Z, [X, K]] \arrow[r, "\pi_0[f, [X, K]]"'] & \pi_0[W, [X, K]]
\end{tikzcd}$$

and (by corollary 1.3.14) $[X, K]$ is a Kan complex, so $\pi_0[f, [X, K]]$ is a bijection; hence, $\pi_0\big[f \times \mathrm{id}_X, K\big]$ is indeed a bijection for all Kan complexes K.

(ii). If X is a Kan complex, then corollary 1.3.14 says that $[-, X]$ is a right Quillen functor; but every simplicial set is cofibrant, so Ken Brown's lemma (4.4.6) implies $[-, X]$ preserves weak homotopy equivalences.

(iii). Similarly, for any simplicial set X, $[X, -]$ is a right Quillen functor, and so Ken Brown's lemma implies $[X, -]$ preserves weak homotopy equivalences between Kan complexes. ■

1.4 Intrinsic homotopy

Prerequisites. §§ 1.2, 1.3, 3.1, A.4.

Definition 1.4.1. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in **sSet**. An **intrinsic homotopy** $\alpha : f_0 \Rightarrow f_1$ is an edge of the exponential object $[X, Y]$

such that $d_1(\alpha) = f_0$ and $d_0(\alpha) = f_1$. (Note the subscripts!) We say f_0 and f_1 are **intrisically homotopic** if there is a zigzag of intrinsic homotopies connecting f_0 and f_1, and we write $f_0 \sim f_1$ in this case.

REMARK 1.4.2. A parallel pair $f_0, f_1 : X \to Y$ in **sSet** are intrinsically homotopic if and only if they are in the same connected component of $[X, Y]$.

REMARK 1.4.3. By the Yoneda lemma,

$$[X, Y]_1 \cong \mathbf{sSet}\left(\Delta^1, [X, Y]\right) \cong \mathbf{sSet}\left(\Delta^1 \times X, Y\right)$$

so an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$ is essentially the same thing as a morphism $\tilde{\alpha} : \Delta^1 \times X \to Y$ such that $\tilde{\alpha} \circ \left(\delta^1 \times \mathrm{id}_Y\right) = f_0$ and $\tilde{\alpha} \circ \left(\delta^0 \times \mathrm{id}_Y\right) = f_1$ (where we have suppressed the canonical isomorphism $X \cong \Delta^0 \times X$), just as in classical homotopy theory. Also,

$$\mathbf{sSet}\left(\Delta^1 \times X, Y\right) \cong \mathbf{sSet}\left(X, \left[\Delta^1, Y\right]\right)$$

so intrinsic homotopies $\alpha : f_0 \Rightarrow f_1$ correspond to morphisms $\hat{\alpha} : X \to \left[\Delta^1, Y\right]$ such that $\left[\delta^1, Y\right] \circ \hat{\alpha} = f_0$ and $\left[\delta^0, Y\right] \circ \hat{\alpha} = f_1$ (where we have suppressed the canonical isomorphism $\left[\Delta^0, Y\right] \cong Y$).

Lemma 1.4.4. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in* **sSet**. *Given an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$, for each simplicial set Z, there is an induced intrinsic homotopy* $[\alpha, Z] : \left[f_0, Z\right] \Rightarrow \left[f_1, Z\right]$.

Proof. Let $\tilde{\alpha} : \Delta^1 \times X \to Y$ be the morphism corresponding to $\alpha : f_0 \Rightarrow f_1$. Then we have a morphism $[\tilde{\alpha}, Z] : [Y, Z] \to \left[\Delta^1 \times X, Z\right]$. Proposition A.2.11 says there is a natural isomorphism

$$\left[\Delta^1 \times X, Z\right] \cong \left[\Delta^1, [X, Z]\right]$$

so $[\tilde{\alpha}, Z]$ corresponds to an intrinsic homotopy $[\alpha, Z]$ between two morphisms of type $[Y, Z] \to [X, Z]$; it is not hard to check that it is an intrinsic homotopy of type $\left[f_0, Z\right] \Rightarrow \left[f_1, Z\right]$. ■

The notion of intrinsic homotopy is not well-behaved for general simplicial sets Y. For example, the existence of an intrinsic homotopy $f_0 \Rightarrow f_1$ does not guarantee the existence of an "inverse" intrinsic homotopy $f_1 \Rightarrow f_0$, and even if we have intrinsic homotopies $f_0 \Rightarrow f_1$ and $f_1 \Rightarrow f_2$, there need not be an intrinsic homotopy $f_0 \Rightarrow f_2$. However:

Proposition 1.4.5. *For any simplicial set X and any Kan complex Y, the relation $\rightsquigarrow$ on* $\mathbf{sSet}(X, Y)$ *defined by*

$$f_0 \rightsquigarrow f_1 \text{ if and only if there exists an intrinsic homotopy } f_0 \Rightarrow f_1$$

is an equivalence relation.

Proof. The relation $\rightsquigarrow$ is certainly reflexive whether or not Y is a Kan complex. Recalling lemma 1.3.5, the transitivity of $\rightsquigarrow$ may be deduced from the fact that the unit $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is an epimorphism, and the symmetry of $\rightsquigarrow$ corresponds to the fact that $\tau_1 X$ is a groupoid. ■

Proposition 1.4.6. *Let $\mathcal{W}$ be a subcategory of* $\mathbf{sSet}$ *that satisfies these conditions:*

- *Every identity morphism in* $\mathbf{sSet}$ *is in $\mathcal{W}$.*
- *$\mathcal{W}$ has the 2-out-of-3 property in* $\mathbf{sSet}$.
- *For every simplicial set X, the projection $p_X : X \times \Delta^1 \to X$ is in $\mathcal{W}$.*

Then:

(i) *Given a parallel pair $f_0, f_1 : X \to Y$ in* $\mathbf{sSet}$ *and an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$, the morphism f_0 is in $\mathcal{W}$ if and only if f_1 is in $\mathcal{W}$.*

(ii) *If $\mathcal{W}$ has the special 2-out-of-4 property, then every trivial Kan fibration is in $\mathcal{W}$.*

(iii) *If $\mathcal{W}$ is closed under retracts* or *has the 2-out-of-6 property in* $\mathbf{sSet}$, *then every trivial Kan fibration is in $\mathcal{W}$.*

Proof. (i). This follows from remark 1.4.3.

(ii). This is a special case of proposition 5.3.32.

(iii). Apply lemma A.4.17. ■

Proposition 1.4.7. *Let $\mathcal{W}$ be a subcategory of* **sSet** *that satisfies these conditions:*

(a) *The class of monomorphisms that are in $\mathcal{W}$ is closed under pushouts, transfinite composition, and retracts.*

(b) *$\mathcal{W}$ has the 2-out-of-3 property in* **sSet**, *and for all simplicial sets X, the morphism* $\mathrm{id} : X \to X$ *is in $\mathcal{W}$.*

(c) *For all natural numbers n, the unique morphism $\Delta^n \to \Delta^0$ is in $\mathcal{W}$.*

Then every weak homotopy equivalence is in $\mathcal{W}$.

Proof. First, we should show that all the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ are in $\mathcal{W}$. We proceed by induction on n. For $n = 1$, observe that conditions (a) and (b) together imply that every isomorphism is in $\mathcal{W}$, and so we may use the 2-out-of-3 property to deduce that the horn inclusions $\Lambda^1_0 \hookrightarrow \Delta^1$ and $\Lambda^1_1 \hookrightarrow \Delta^1$ are in $\mathcal{W}$.

Now, suppose that the horn inclusions $\Lambda^m_k \hookrightarrow \Delta^m$ are in $\mathcal{W}$ for all $m < n$. It is not hard to see that the horn Λ^n_l can be constructed by adjoining m copies of Δ^m along various horn inclusions (for $0 < m < n$), so conditions (a) and (b) imply that the l-th vertex $\Delta^0 \to \Lambda^n_l$ is in $\mathcal{W}$. Condition (c) says that the unique morphism $\Delta^n \to \Delta^0$ is in $\mathcal{W}$, so we can then use the 2-out-of-3 property to deduce that the horn inclusion $\Lambda^n_l \hookrightarrow \Delta^n$ is in $\mathcal{W}$.

Having shown that the horn inclusions are in $\mathcal{W}$, we can use Quillen's small object argument (theorem 0.5.12) together with corollaries 0.5.7 and 0.5.13 and condition (a) to deduce that all anodyne extensions are in $\mathcal{W}$. Notice that, if $p : X \to Y$ is a trivial Kan fibration, then there is a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$, and by 1.3.16, $s : Y \to X$ is an anodyne extension. Hence, condition (b) implies that all trivial Kan fibrations are in $\mathcal{W}$ as well. Finally, using the fact that every weak homotopy equivalence factors as an anodyne extension followed by a trivial Kan fibration, we conclude that every weak homotopy equivalence is in $\mathcal{W}$. ∎

Corollary 1.4.8. *The subcategory of weak homotopy equivalences in* **sSet** *is the smallest subcategory satisfying the conditions in the proposition.*

Proof. Proposition 1.3.16 says that the class of monomorphisms that are weak homotopy equivalences is precisely the class of anodyne extensions, which has the required closure properties by proposition A.3.12. Thus, the class of weak homotopy equivalences satisfies condition (a), and the remaining conditions are easily verified. ■

¶ 1.4.9. We require the following special case of definition 2.2.1. Let **Kan** be the full subcategory of **sSet** spanned by the Kan complexes. For each category $\mathcal{V}$ with finite products and each functor $F : \mathbf{sSet} \to \mathcal{V}$ that preserves finite products, let $F[\underline{\mathbf{Kan}}]$ denote the following $\mathcal{V}$-enriched category:

- $\operatorname{ob} F[\underline{\mathbf{Kan}}] = \operatorname{ob} \mathbf{Kan}$.
- For each pair of Kan complexes X and Y, the hom-object is $F[X, Y]$, where $[X, Y]$ is the exponential object in **sSet**.
- Composition and identity morphisms are induced by F from the cartesian closed structure of **sSet**.

We also define $F[\underline{\mathbf{sSet}}]$ similarly. The next definition is a prime example of the above construction.

Definition 1.4.10. The **homotopy category of Kan complexes** is the category $\mathbf{H} = \pi_0[\underline{\mathbf{Kan}}]$. A **weak homotopy type** is an isomorphism class of objects in **H**.

Proposition 1.4.11. *For each simplicial set Z, let $\chi_Z : Z_0 \to \pi_0 Z$ be the map of vertices induced by the adjunction unit $\mathrm{id}_{\mathbf{sSet}} \Rightarrow \operatorname{disc} \pi_0$.*

(i) *There is a (unique) functor $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ that acts as the identity on objects and as $\chi_{[X,Y]} : [X, Y]_0 \to \pi_0[X, Y]$ on morphisms.*

(ii) *The functor $\boldsymbol{\pi}$ is full, surjective on objects, and preserves finite products and finite coproducts.*

(iii) **Kan** *is closed under products for all small families in* **sSet**, *and* **H** *has products for finite families.*

(iv) **Kan** *and* **H** *are cartesian closed categories, and $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ is a cartesian closed functor.*

(v) *A morphism $f : X \to Y$ in* **Kan** *is a weak homotopy equivalence if and only if $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ is an isomorphism in* **H**.

Proof. (i). The construction of $\mathbf{H}$ as $\pi_0[\underline{\mathbf{Kan}}]$ ensures that $\boldsymbol{\pi}$ is indeed a functor.

(ii). It is clear from the construction of $\pi_0 Z$ as a coequaliser that $\chi_Z : Z_0 \to \pi_0 Z$ is a surjection; thus $\boldsymbol{\pi}$ is a full functor. It is obviously surjective on objects, and it preserves finite products and finite coproducts because π_0 preserves finite products.

(iii). By proposition A.3.12, the class of Kan fibrations is closed under products for small families, so $\mathbf{Kan}$ is as well. By claim (ii), $\mathbf{H}$ inherits finite products from $\mathbf{Kan}$.

(iv). By proposition 1.3.13, $[Y, K]$ is a Kan complex whenever K is, which combined with claim (iii) implies $\mathbf{Kan}$ is cartesian closed. Proposition A.2.11 says we have natural isomorphisms $[X \times Y, K] \cong [X, [Y, K]]$, so it follows that we have natural bijections

$$\pi_0[X \times Y, K] \cong \pi_0[X, [Y, K]]$$

for all X, Y, and K in $\mathbf{Kan}$, and this descends along $\boldsymbol{\pi}$ to make $\mathbf{H}$ cartesian closed.

(v). The Joyal–Tierney definition says $f : X \to Y$ is a weak equivalence if and only if $\pi_0[f, K] : \pi_0[Y, K] \to \pi_0[X, K]$ is a bijection for all Kan complexes K; but this is natural in K, so the Yoneda lemma implies this happens if and only if $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ is an isomorphism in $\mathbf{H}$. ■

Proposition 1.4.12.

(i) *For each simplicial set X, there exists a Kan complex RX such that the functors $\pi_0[X, -], \pi_0[RX, -] : \mathbf{Kan} \to \mathbf{Set}$ are isomorphic.*

(ii) *For each simplicial set X, the functor $\pi_0[X, -] : \mathbf{Kan} \to \mathbf{Set}$ factors through $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ as a representable functor on $\mathbf{H}$.*

(iii) *The functor $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ extends to a functor $\boldsymbol{\pi} : \mathbf{sSet} \to \mathbf{H}$ that sends weak homotopy equivalences to isomorphisms, and this extension is unique up (not necessarily unique) isomorphism.*

Proof. (i). By proposition 1.3.18, there is an anodyne extension $i : X \to RX$ where RX is a Kan complex; but proposition 1.3.16 says that anodyne extensions are weak homotopy equivalences, so $\pi_0[i, K] : \pi_0[RX, K] \to \pi_0[X, K]$ is a bijection natural in K, as required.

(ii). The claim is certainly true if X were a Kan complex, and by claim (i), $\pi_0[X, -]$ is always isomorphic to $\pi_0[RX, -]$ for some Kan complex RX.

(iii). Formally, what we seek is a functor $F : \mathbf{sSet} \to \mathbf{H}$ such that, for all Kan complexes Y and K,

$$\mathbf{H}(FY, \boldsymbol{\pi} K) = \pi_0[Y, K]$$

and, for all weak homotopy equivalences $f : X \to Y$ in **sSet**, the induced hom-set map $\mathbf{H}(Ff, \boldsymbol{\pi} K) : \mathbf{H}(FY, \boldsymbol{\pi} K) \to \mathbf{H}(FX, \boldsymbol{\pi} K)$ is a bijection for all Kan complexes K. Clearly, for any such F and any simplicial set X, there must be bijections

$$\mathbf{H}(FX, \boldsymbol{\pi} K) \cong \pi_0[X, K]$$

that are natural in K, but by claim (ii), this is representable as a functor $\mathbf{H} \to \mathbf{Set}$ for each X, so we can certainly construct such a functor F, and it is unique up to isomorphism. ■

Corollary 1.4.13. *The homotopy category of Kan complexes is a reflective subcategory of* $\pi_0[\underline{\mathbf{sSet}}]$. ■

Proposition 1.4.14. *Let* $F : \mathbf{Kan} \to \mathcal{C}$ *be any functor that sends trivial Kan fibrations in* **Kan** *to isomorphisms in* $\mathcal{C}$.

(i) *If* $f_0, f_1 : X \to Y$ *are a parallel pair of morphisms in* **Kan** *and there exists an intrinsic homotopy* $f_0 \Rightarrow f_1$, *then* $Ff_0 = Ff_1$.

(ii) *If* $f_0, f_1 : X \to Y$ *are an intrinsically homotopic pair of morphisms in* **Kan**, *then* $Ff_0 = Ff_1$.

(iii) *There exists a unique functor* $\overline{F} : \mathbf{H} \to \mathcal{C}$ *such that* $F = \overline{F}\boldsymbol{\pi}$.

Proof. (i). By remark 1.4.3, given any intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$, we may construct a morphism $\hat{\alpha} : X \to \left[\Delta^1, Y\right]$ such that $\left[\delta^1, Y\right] \circ \hat{\alpha} = f_0$ and $\left[\delta^0, Y\right] \circ \hat{\alpha} = f_1$. Clearly, $\delta^1 : \Delta^0 \to \Delta^1$ is isomorphic to the horn inclusion $\Lambda^1_0 \hookrightarrow \Delta^1$, and $\delta^0 : \Delta^0 \to \Delta^1$ is isomorphic to the horn inclusion $\Lambda^1_1 \hookrightarrow \Delta^1$, so by proposition 1.3.13, the morphisms $\left[\delta^1, Y\right], \left[\delta^0, Y\right] : \left[\Delta^1, Y\right] \to Y$ are both trivial Kan fibrations. Thus, we must have $Ff_0 = Ff_1$.

(ii). Proposition 1.4.5 implies that f_0 and f_1 are intrinsically homotopic if and only if there exists an intrinsic homotopy $f_0 \Rightarrow f_1$, so this reduces to claim (i).

(iii). The uniqueness of $\overline{F} : \mathbf{H} \to \mathcal{C}$ is an immediate corollary of the fact that $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ is full and surjective on objects; it remains to be shown that such an $\overline{F}$ exists. However, given any parallel pair $f_0, f_1 : X \to Y$ in $\mathbf{Kan}$, by the construction of $\mathbf{H}$, we have $\boldsymbol{\pi} f_0 = \boldsymbol{\pi} f_1$ if and only if f_0 and f_1 are intrinsically homotopic, so F indeed factors through $\boldsymbol{\pi}$. ∎

Corollary 1.4.15.

(i) *Any functor $F : \mathbf{Kan} \to \mathcal{C}$ that sends trivial Kan fibrations in* **Kan** *to isomorphisms in $\mathcal{C}$ must also send weak homotopy equivalences in* **Kan** *to isomorphisms in $\mathcal{C}$.*

(ii) **H** *is the localisation of* **Kan** *at weak homotopy equivalences.*

(iii) *If* Ho **sSet** *is the localisation of* **sSet** *at weak homotopy equivalences, then the functor $\boldsymbol{\pi} : \mathbf{sSet} \to \mathbf{H}$ induces a functor* Ho **sSet** $\to$ **H** *that is fully faithful and essentially surjective on objects.*

Proof. (i). The above proposition says $F = \overline{F}\boldsymbol{\pi}$ for some $\overline{F}$, and we know from proposition 1.4.11 that $\boldsymbol{\pi}$ inverts weak homotopy equivalences, so F must also invert weak homotopy equivalences.

(ii). This is a restatement of claim (iii) of the above proposition.

(iii). Apply proposition 1.4.12. ∎

Remark 1.4.16. Fixing a fibrant replacement functor $R : \mathbf{sSet} \to \mathbf{sSet}$ as in proposition 1.3.18, we have the following explicit construction of Ho **sSet**:

- The objects are simplicial sets.
- For any two simplicial sets X and Y, $\mathrm{Ho}\,\mathbf{sSet}(X, Y) = \pi_0[RX, RY]$.
- Composition and identity morphisms are constructed as in **H**.
- The localising functor $\gamma : \mathbf{sSet} \to \mathrm{Ho}\,\mathbf{sSet}$ inverting weak homotopy equivalences is the one sending $f : X \to Y$ to the homotopy class of $Rf : RX \to RY$.

The **homotopy category of simplicial sets** is the category Ho **sSet**.

Remark 1.4.17. Freyd [1970] proved that $\mathbf{H}$ is not a concrete category, i.e. that there does not exist a faithful functor $\mathbf{H} \to \mathbf{Set}$; in particular, $\mathbf{H}$ cannot be an accessible category. Nonetheless, the notion of weak homotopy type is stable under universe enlargement in the following sense:

(i) The property of being a weak homotopy equivalence is universe-independent: indeed, it is clear that the property of being a trivial Kan fibration is universe-independent, so we may apply remark 0.5.18 to the (trivial cofibration, Kan fibration) factorisation system to test whether or not a morphism is a weak homotopy equivalence in a universe-independent way.

(ii) Moreover, the property of being a Kan complex is universe-independent, and $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ is a left adjoint between locally presentable categories, so the hom-set $\mathbf{H}(K, L)$ depends only on the choice of Kan complexes K and L and does not depend on the choice of universe. Similarly, whether or not K and L have the same weak homotopy type is universe-independent.

(iii) Thus, for any two simplicial sets X and Y, the hom-set $\operatorname{Ho}\mathbf{sSet}(X, Y)$ is well-defined up to natural bijection independently of the choice of universe, and whether or not X and Y have the same weak homotopy type is also universe-indepdent.

Definition 1.4.18. Let $f : X \to Y$ be a morphism in $\mathbf{sSet}$.

- A **forward homotopy left inverse** for f is a morphism $g : Y \to X$ for which an intrinsic homotopy $g \circ f \Rightarrow \mathrm{id}_X$ exists.
- A **backward homotopy left inverse** for f is a morphism $g : Y \to X$ for which an intrinsic homotopy $\mathrm{id}_X \Rightarrow g \circ f$ exists.
- A **simple homotopy left inverse** for f is a morphism $g : Y \to X$ that is either a forward homotopy left inverse or a backward homotopy left inverse for f.
- An **intrinsic homotopy left inverse** for f is a morphism $g : Y \to X$ such that $g \circ f$ and id_X are intrinsically homotopic.
- A **forward homotopy right inverse** for f is a morphism $g : Y \to X$ for which an intrinsic homotopy $f \circ g \Rightarrow \mathrm{id}_Y$ exists.

- A **backward homotopy right inverse** for f is a morphism $g : Y \to X$ for which an intrinsic homotopy $\mathrm{id}_Y \Rightarrow f \circ g$ exists.

- A **simple homotopy right inverse** for f is a morphism $g : Y \to X$ that is either a forward homotopy right inverse or a backward homotopy right inverse for f.

- An **intrinsic homotopy right inverse** for f is a morphism $g : Y \to X$ such that $f \circ g$ and id_Y are intrinsically homotopic.

Definition 1.4.19.

- A **simple homotopy equivalence** in **sSet** is a tuple $(f, g, \eta, \varepsilon)$ where $f : X \to Y$ and $g : Y \to X$ are morphisms in **sSet** and $\eta : \mathrm{id}_X \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathrm{id}_Y$ are intrinsic homotopies.

- An **intrinsic homotopy equivalence** in **sSet** is a pair (f, g) where g (resp. f) is both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse for f (resp. g).

Proposition 1.4.20 (Formal Whitehead theorem)**.**

(i) *If (f, g) is an intrinsic homotopy equivalence in* **sSet**, *then $\boldsymbol{\pi} f$ and $\boldsymbol{\pi} g$ are mutual inverses in* **H**.

(ii) *If a morphism in* **sSet** *has both an intrinsic homotopy left inverse and an intrinsic homotopy right inverse, then it is a weak homotopy equivalence.*

(iii) *A morphism in* **Kan** *is a weak homotopy equivalence if and only if it is part of a simple homotopy equivalence.*

Proof. The claims are consequences of lemma 1.4.4, proposition 1.4.11, and corollary 1.4.15 applied to the various definitions of homotopy equivalence. ■

Lemma 1.4.21. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in* **sSet** *and suppose $(f, g, \eta, \varepsilon)$ is a simple homotopy equivalence.*

- *If Y is a Kan complex, then there is an intrinsic homotopy $\varepsilon' : f \circ g \Rightarrow \mathrm{id}_Y$ such that*

$$\left(\varepsilon' \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_f$$

in the fundamental groupoid $\pi_1[X, Y]$.

- *If X is a Kan complex, then there is an intrinsic homotopy $\eta' : \mathrm{id}_X \Rightarrow g \circ f$ such that*
$$\left(\mathrm{id}_g \circ \varepsilon\right) \bullet \left(\eta' \circ \mathrm{id}_g\right) = \mathrm{id}_g$$
in the fundamental groupoid $\pi_1[Y, X]$.

Proof. The proofs of the two claims are similar; we will prove the first version.

Suppose Y is a Kan complex. By corollary 1.3.14, the exponential objects $[X, Y]$ and $[Y, Y]$ in **sSet** are Kan complexes, so we may apply lemma 1.3.5. In particular, there exists an intrinsic homotopy $\varepsilon' : f \circ g \Rightarrow \mathrm{id}_Y$ such that
$$\varepsilon' = \varepsilon \bullet \left(\mathrm{id}_f \circ \eta \circ \mathrm{id}_g\right)^{-1} \bullet \left(\varepsilon \circ \mathrm{id}_{f \circ g}\right)^{-1}$$
in $\pi_1[Y, Y]$, but then
$$\begin{aligned}\left(\varepsilon' \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) &= \left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta \circ \mathrm{id}_{g \circ f}\right)^{-1} \bullet \left(\varepsilon \circ \mathrm{id}_{f \circ g \circ f}\right)^{-1} \bullet \left(\mathrm{id}_f \circ \eta\right) \\ &= \left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta \circ \mathrm{id}_{g \circ f}\right)^{-1} \bullet \left(\mathrm{id}_{f \circ g \circ f} \circ \eta\right) \bullet \left(\varepsilon \circ \mathrm{id}_f\right)^{-1} \\ &= \left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) \bullet \left(\mathrm{id}_f \circ \eta\right)^{-1} \bullet \left(\varepsilon \circ \mathrm{id}_f\right)^{-1} \\ &= \mathrm{id}_f\end{aligned}$$
as required. ■

Lemma 1.4.22. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in* **sSet** *and let $(f, g, \eta, \varepsilon)$ be a simple homotopy equivalence. The following are equivalent:*

- *In the fundamental groupoid $\pi_1[X, Y]$, the left triangle identity holds:*
$$\left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_f$$

- *In the fundamental groupoid $\pi_1[Y, X]$, the right triangle identity holds:*
$$\left(\mathrm{id}_g \circ \varepsilon\right) \bullet \left(\eta \circ \mathrm{id}_g\right) = \mathrm{id}_g$$

Proof. This is a standard result in 2-category theory applied to the 2-category $\pi_1[\underline{\mathbf{sSet}}]$. □

Definition 1.4.23. A **weakly contractible simplicial set** is a simplicial set X for which the unique morphism $X \to 1$ in **sSet** is a weak homotopy equivalence.

Proposition 1.4.24. *A Kan complex K is weakly contractible if and only if the unique morphism $K \to 1$ in* **Kan** *has an intrinsic homotopy inverse.*

Proof. Apply proposition 1.4.20 and the fact that 1 is a Kan complex. ■

Definition 1.4.25. Let X be a simplicial set.

- A **forward contracting homotopy** for X consists of a set X_{-1} and maps $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$ satisfying these identities:

$$
\begin{aligned}
r \circ d^1_1 &= r \circ d^1_0 \\
r \circ s &= \mathrm{id} \\
d^1_0 \circ h^0 &= s \circ r \\
d^1_1 \circ h^0 &= \mathrm{id} \\
d^{n+1}_i \circ h^n &= h^{n-1} \circ d^n_i && \text{if } 0 \le i \le n \\
d^{n+1}_{n+1} \circ h^n &= \mathrm{id} \\
h^{n+1} \circ s^n_i &= s^{n+1}_i \circ h^n && \text{if } 0 \le i \le n \\
h^{n+1} \circ h^n &= s^{n+1}_{n+1} \circ h^n
\end{aligned}
$$

- A **backward contracting homotopy** for X consists of a set X_{-1} and maps $r : X_0 \to X_{-1}$, $s : X_{-1} \to X_0$, and $h^n : X_n \to X_{n+1}$ satisfying these identities:

$$
\begin{aligned}
r \circ d^1_1 &= r \circ d^1_0 \\
r \circ s &= \mathrm{id} \\
d^1_0 \circ h^0 &= \mathrm{id} \\
d^1_1 \circ h^0 &= s \circ r \\
d^{n+1}_0 \circ h^n &= \mathrm{id} \\
d^{n+1}_{i+1} \circ h^n &= h^{n-1} \circ d^n_i && \text{if } 0 \le i \le n \\
h^{n+1} \circ h^n &= s^{n+1}_0 \circ h^n \\
h^{n+1} \circ s^n_i &= s^{n+1}_{i+1} \circ h^n && \text{if } 0 \le i \le n
\end{aligned}
$$

Proposition 1.4.26. *Let X be a simplicial set. If X admits a forward or backward contracting homotopy, then the canonical map $X \to \operatorname{disc} \pi_0 X$ has an intrinsic homotopy inverse.*

Proof. The two cases are similar; we will assume that X has a forward contracting homotopy. First, notice that the definition implies that we have the following absolute coequaliser diagram:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d^1_1"] \arrow[r, shift right, "d^1_0"'] & X_0 \arrow[r, "r"] \arrow[l, bend left, "h^0"] & X_{-1} \arrow[l, bend left, "s"] \end{tikzcd}$$

Thus, as remarked in the proof of proposition 1.2.4, $\pi_0 X \cong X_{-1}$. As always, there is a unique morphism $s' : \operatorname{disc} X_{-1} \to X$ whose degree 0 component is $s : X_{-1} \to X_0$, and the above observation ensures that there also exists a unique morphism $r' : X \to \operatorname{disc} X_{-1}$ whose degree 0 component is $r : X_0 \to X_{-1}$.

Clearly, $r' \circ s' = \mathrm{id}_{\operatorname{disc} X_{-1}}$; we must show that $s' \circ r' \sim \mathrm{id}_X$. Let $\chi^i_n : [n] \to [1]$ denote the map in $\mathbf{\Delta}$ defined below:

$$\chi^i_n(j) = \begin{cases} 0 & \text{if } j < i \\ 1 & \text{if } j \ge i \end{cases}$$

It is not hard to see that $\mathbf{\Delta}([n],[1]) = \left\{ \chi^i_n \,\middle|\, 0 \le i \le n+1 \right\}$, and moreover we have the following identities:

$$\begin{aligned} \chi^i_{n+1} \circ \delta^j_{n+1} &= \chi^i_n && \text{if } 0 \le i \le j \le n+1 \\ \chi^j_{n+1} \circ \delta^i_{n+1} &= \chi^{j-1}_n && \text{if } 0 \le i < j \le n+2 \\ \chi^i_n \circ \sigma^j_n &= \chi^i_{n+1} && \text{if } 0 \le i \le j \le n \\ \chi^j_n \circ \sigma^i_n &= \chi^{j+1}_{n+1} && \text{if } 0 \le i < j \le n+1 \end{aligned}$$

We construct by recursion a sequence of maps $H_n : X_n \times \mathbf{\Delta}([n],[1]) \to X_n$:

- For all x in X_0:

$$\begin{aligned} H_0\left(x, \chi^1_0\right) &= x \\ H_0\left(x, \chi^0_0\right) &= s(r(x)) \end{aligned}$$

- For each x in X_{n+1}:

$$\begin{aligned} H_{n+1}\left(x, \chi^{n+2}_{n+1}\right) &= x \\ H_{n+1}\left(x, \chi^{n+1}_{n+1}\right) &= h^n\left(d^{n+1}_{n+1}(x)\right) \\ H_{n+1}\left(x, \chi^j_{n+1}\right) &= s^n_n\left(H_n\left(d^{n+1}_{n+1}(x), \chi^j_n\right)\right) && \text{for } 0 \le j \le n \end{aligned}$$

It is straightforward to check that these equations hold,

$$d_0^1 \circ H_1 = H_0 \circ d_0^1 \qquad d_1^1 \circ H_1 = H_0 \circ d_1^1 \qquad s_0^0 \circ H_0 = H_1 \circ s_0^0$$

so we assume for induction that these identities hold for some $n > 0$:

$$\begin{aligned} d_i^k \circ H_k &= H_{k-1} \circ d_i^k && \text{for } 0 < k \leq n, 0 \leq i \leq k \\ s_i^k \circ H_k &= H_{k+1} \circ s_i^k && \text{for } 0 \leq k < n, 0 \leq i \leq k \end{aligned}$$

Then, for $0 \leq i \leq n+1$,

$$\begin{aligned} d_i^{n+1}\big(H_{n+1}\big(x, \chi_{n+1}^{n+2}\big)\big) &= d_i^{n+1}(x) \\ &= H_n\big(d_i^{n+1}(x), \chi_n^{n+1}\big) = H_n\big(d_i^{n+1}(x), \chi_{n+1}^{n+2} \circ \delta_{n+1}^i\big) \end{aligned}$$

and, for $0 \leq i \leq n$,

$$\begin{aligned} d_i^{n+1}\big(H_{n+1}\big(x, \chi_{n+1}^{n+1}\big)\big) &= d_i^{n+1}\big(h^n\big(d_{n+1}^{n+1}(x)\big)\big) \\ &= h^{n-1}\big(d_i^n\big(d_{n+1}^{n+1}(x)\big)\big) = h^{n-1}\big(d_n^n\big(d_i^{n+1}(x)\big)\big) \\ &= H_n\big(d_i^{n+1}(x), \chi_n^n\big) = H_n\big(d_i^{n+1}(x), \chi_n^n \circ \delta_{n+1}^i\big) \end{aligned}$$

while, for $i = n+1$:

$$\begin{aligned} d_{n+1}^{n+1}\big(H_{n+1}\big(x, \chi_{n+1}^{n+1}\big)\big) &= d_{n+1}^{n+1}\big(h^n\big(d_{n+1}^{n+1}(x)\big)\big) \\ &= d_{n+1}^{n+1}(x) = H_n\big(d_{n+1}^{n+1}(x), \chi_n^{n+1}\big) = H_n\big(d_{n+1}^{n+1}(x), \chi_{n+1}^{n+1} \circ \delta_{n+1}^{n+1}\big) \end{aligned}$$

Similarly, for $0 \leq j < n$,

$$\begin{aligned} d_{n+1}^{n+1}\Big(H_{n+1}\Big(x, \chi_{n+1}^j\Big)\Big) &= d_{n+1}^{n+1}\Big(s_n^n\Big(H_n\Big(d_{n+1}^{n+1}(x), \chi_n^j\Big)\Big)\Big) \\ &= H_n\Big(d_{n+1}^{n+1}(x), \chi_n^j\Big) H_n\Big(d_{n+1}^{n+1}(x), \chi_{n+1}^j \circ \delta_{n+1}^{n+1}\Big) \end{aligned}$$

$$\begin{aligned} d_n^{n+1}\Big(H_{n+1}\Big(x, \chi_{n+1}^j\Big)\Big) &= d_n^{n+1}\Big(s_n^n\Big(H_n\Big(d_{n+1}^{n+1}(x), \chi_n^j\Big)\Big)\Big) = H_n\Big(d_{n+1}^{n+1}(x), \chi_n^j\Big) \\ &= s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n\big(d_{n+1}^{n+1}(x)\big), \chi_{n-1}^j\Big)\Big) = s_{n-1}^{n-1}\Big(H_{n-1}\Big(d_n^n\big(d_n^{n+1}(x)\big), \chi_{n-1}^j\Big)\Big) \\ &= H_n\Big(d_n^{n+1}(x), \chi_n^j\Big) = H_n\Big(d_n^{n+1}(x), \chi_{n+1}^j \circ \delta_{n+1}^n\Big) \end{aligned}$$

and for $0 \leq i < n$, we have:

$$\begin{aligned}
d_i^{n+1}\Big(H_{n+1}\Big(x, \chi^j_{n+1}\Big)\Big) &= d_i^{n+1}\Big(s_n^n\Big(H_n\Big(d^{n+1}_{n+1}(x), \chi^j_n\Big)\Big)\Big) \\
&= s^{n-1}_{n-1}\Big(d_i^n\Big(H_n\Big(d^{n+1}_{n+1}(x), \chi^j_n\Big)\Big)\Big) = s^{n-1}_{n-1}\Big(H_{n-1}\Big(d_i^n\big(d^{n+1}_{n+1}(x)\big), \chi^j_n \circ \delta^i_n\Big)\Big) \\
&= s^{n-1}_{n-1}\Big(H_{n-1}\Big(d_n^n\big(d^{n+1}_i(x)\big), \chi^j_n \circ \delta^i_n\Big)\Big) = H_n\Big(d_i^{n+1}(x), \chi^j_{n+1} \circ \delta^i_{n+1}\Big)
\end{aligned}$$

On the other hand, for $0 \leq i \leq n$,

$$s_i^n\big(H_n\big(x, \chi_n^{n+1}\big)\big) = s_i^n(x) = H_{n+1}\big(s_i^n(x), \chi^{n+2}_{n+1}\big) = H_{n+1}\big(s_i^n(x), \chi_n^{n+1} \circ \sigma^i_n\big)$$

and for $0 \leq i < n$,

$$\begin{aligned}
s_i^n\big(H_n\big(x, \chi_n^n\big)\big) &= s_i^n\big(h^{n-1}\big(d_n^n(x)\big)\big) \\
&= h^n\big(s_i^{n-1}\big(d_n^n(x)\big)\big) = h^n\big(d^{n+1}_{n+1}\big(s_i^n(x)\big)\big) \\
&= H_{n+1}\big(s_i^n(x), \chi^{n+1}_{n+1}\big) = H_{n+1}\big(s_i^n(x), \chi^n_n \circ \sigma^i_n\big)
\end{aligned}$$

while for $i = n$, we have:

$$\begin{aligned}
s_n^n\big(H_n\big(x, \chi_n^n\big)\big) &= s_n^n\big(H_n\big(d^{n+1}_{n+1}\big(s_n^n(x)\big), \chi_n^n\big)\big) \\
&= H_{n+1}\big(s_n^n(x), \chi^n_{n+1}\big) = H_{n+1}\big(s_n^n(x), \chi^n_n \circ \sigma^n_n\big)
\end{aligned}$$

Finally, for $0 \leq i \leq n$ and $0 \leq j < n$:

$$\begin{aligned}
s_i^n\Big(H_n\Big(x, \chi^j_n\Big)\Big) &= s_i^n\Big(s^{n-1}_{n-1}\Big(H_{n-1}\Big(d_n^n(x), \chi^j_{n-1}\Big)\Big)\Big) \\
&= s_n^n\Big(s_i^{n-1}\Big(H_{n-1}\Big(d_n^n(x), \chi^j_{n-1}\Big)\Big)\Big) = s_n^n\Big(H_n\Big(s_i^{n-1}\big(d_n^n(x)\big), \chi^j_{n-1} \circ \sigma^i_{n-1}\Big)\Big) \\
&= s_n^n\Big(H_n\Big(d^{n+1}_{n+1}\big(s_i^n(x)\big), \chi^j_{n-1} \circ \sigma^i_{n-1}\Big)\Big) = H_{n+1}\Big(s_i^n(x), \chi^j_n \circ \sigma^i_n\Big)
\end{aligned}$$

We therefore have a morphism $H : X \times \Delta^1 \to X$ such that $H \circ \big(\mathrm{id}_X \times \delta^0_1\big) = s' \circ r'$ and $H \circ \big(\mathrm{id}_X \times \delta^1_1\big) = \mathrm{id}_X$, as required. ∎

Definition 1.4.27. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in **sSet**.

- f has the **forward homotopy lifting property** with respect to g if, for every commutative diagram of the following form,

$$\begin{tikzcd}
Z \arrow[r, "z_0"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
W \arrow[r, "w_0"'] \arrow[ur, "h_0"] & Y
\end{tikzcd}$$

given intrinsic homotopies $\alpha : w_0 \Rightarrow w_1$ and $\beta : z_0 \Rightarrow z_1$ such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \beta$, there exist a morphism $h_1 : W \to X$ and an intrinsic homotopy $\gamma : h_0 \Rightarrow h_1$ such that $f \circ h_1 = w_1$, $h_1 \circ g = z_1$, $\mathrm{id}_f \circ \gamma = \alpha$, and $\gamma \circ \mathrm{id}_g = \beta$.

- f has the **backward homotopy lifting property** with respect to g if, for every commutative diagram of the following form,

$$\begin{tikzcd} Z \arrow[r, "z_1"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w_1"'] \arrow[ur, "h_1"] & Y \end{tikzcd}$$

  given intrinsic homotopies $\alpha : w_0 \Rightarrow w_1$ and $\beta : z_0 \Rightarrow z_1$ such that $\alpha \circ \mathrm{id}_g = \mathrm{id}_f \circ \beta$, there exist a morphism $h_0 : W \to X$ and an intrinsic homotopy $\gamma : h_0 \Rightarrow h_1$ such that $f \circ h_0 = w_0$, $h_0 \circ g = z_0$, $\mathrm{id}_f \circ \gamma = \alpha$, and $\gamma \circ \mathrm{id}_g = \beta$.

- f has the **intrinsic homotopy lifting property** with respect to g if f has *both* the forward and backward homotopy lifting properties with respect to g.

- f has the **forward** (resp. **backward**, **intrinsic**) **homotopy lifting property** with respect to the object W if f has the forward (resp. backward, intrinsic) homotopy lifting property with respect to the unique morphism $0 \to W$.

- g has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to f if f has the forward (resp. backward, intrinsic) homotopy lifting property with respect to g.

- g has the **forward** (resp. **backward**, **intrinsic**) **homotopy extension property** with respect to the object X if g has the forward (resp. backward, intrinsic) homotopy extension property with respect to the unique morphism $X \to 1$.

Proposition 1.4.28. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in* **sSet**, *and suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[g,X]}"] \arrow[ddr, bend right, "{[W,f]}"'] & & \\
& L(g,f) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,f]}"] \\
& {[W,Y]} \arrow[r, "{[g,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square. The following are equivalent:

(i) *f has the forward homotopy lifting property with respect to g.*

(ii) *The morphism $q : [W,X] \to L(g,f)$ has the right lifting property with respect to the horn inclusion $\Lambda^1_0 \hookrightarrow \Delta^1$.*

(iii) *Suppose we have a commutative diagram*

$$\begin{tikzcd}
\Lambda^1_0 \times Z \arrow[r, hook] \arrow[d, "\mathrm{id}_{\Lambda^1_0} \times g"'] & \Delta^1 \times Z \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_{\Delta^1} \times g"] & \\
\Lambda^1_0 \times W \arrow[r] \arrow[drr, hook, bend right] & V_0(g) \arrow[dr, dashed, "j"] & \\
& & \Delta^1 \times W
\end{tikzcd}$$

where the square in the upper left is a pushout square. Then the morphism $j : V_0(g) \to \Delta^1 \times W$ has the left lifting property with respect to $f : X \to Y$.

Symmetrically, the following are equivalent:

(i′) *f has the backward homotopy lifting property with respect to g.*

(ii′) *The morphism $q : [W,X] \to L(g,f)$ has the right lifting property with respect to the horn inclusion $\Lambda^1_1 \hookrightarrow \Delta^1$.*

(iii′) *Suppose we have a commutative diagram*

$$\begin{tikzcd}
\Lambda^1_1 \times Z \arrow[r, hook] \arrow[d, "\mathrm{id}_{\Lambda^1_1} \times g"'] & \Delta^1 \times Z \arrow[d] \arrow[ddr, bend left, "\mathrm{id}_{\Delta^1} \times g"] & \\
\Lambda^1_1 \times W \arrow[r] \arrow[drr, hook, bend right] & V_1(g) \arrow[dr, dashed, "j"] & \\
& & \Delta^1 \times W
\end{tikzcd}$$

where the square in the upper left is a pushout square. Then the morphism $j : V_1(g) \to \Delta^1 \times W$ *has the left lifting property with respect to* $f : X \to Y$.

Proof. This is a special case of proposition 5.4.1: use remark 1.4.3 and the exponential adjunction. ■

Theorem 1.4.29 (Gabriel–Zisman)**.** *Let* $p : X \to Y$ *be a morphism in* **sSet**. *The following are equivalent:*

(i) *p is a Kan fibration.*

(ii) *p has the intrinsic homotopy lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n \geq 0$.*

(iii) *p has the intrinsic homotopy lifting property with respect to any monomorphism in* **sSet**.

Proof. Combine the preceding proposition and proposition A.3.12 with either Theorem 2.1 in [GZ, Ch. IV] or Proposition 4.2 in [GJ, Ch. I]. □

Remark 1.4.30. Let B^n be the closed unit ball in the euclidean space $\mathbb{R}^n$, let ∂B^n be its boundary, and let I be the closed unit interval $[0, 1]$. It is not hard to see that the inclusion $B^n \times \{0\} \hookrightarrow B^n \times I$ is isomorphic to the inclusion $B^n \times \{0\} \cup \partial B^n \times I \hookrightarrow B^n \times I$. Thus, a continuous map $p : X \to Y$ has the homotopy lifting property with respect to all B^n if and only if it has the homotopy lifting property with respect to all boundary inclusions $\partial B^n \hookrightarrow B^n$.

Unfortunately, **sSet** does not have the analogous property. Indeed, for any simplicial set X, the unique morphism $X \to 1$ has the intrinsic homotopy lifting property with respect to the n-simplices Δ^n, yet not every simplicial set is a Kan complex.

Lemma 1.4.31.

- *If $p : X \to Y$ has the forward homotopy lifting property with respect to Y, $s_0 : Y \to X$ is a morphism, and $\varepsilon : p \circ s_0 \Rightarrow \mathrm{id}_Y$ is an intrinsic homotopy, then there exists a morphism $s_1 : Y \to X$ such that $p \circ s_1 = \mathrm{id}_Y$; if moreover X is a Kan complex and there is an intrinsic homotopy $\eta : \mathrm{id}_X \Rightarrow s_0 \circ p$, then there also exists an intrinsic homotopy $\mathrm{id}_X \Rightarrow s_1 \circ p$.*

- *If $p : X \to Y$ has the backward homotopy lifting property with respect to Y, $s_1 : Y \to X$ is a morphism, and $\eta : \mathrm{id}_Y \Rightarrow p \circ s_0$ is an intrinsic homotopy, then the exists a morphism $s_1 : Y \to X$ such that $p \circ s_1 = \mathrm{id}_Y$; if moreover X is a Kan complex and there is an intrinsic homotopy $\varepsilon : s_1 \circ p \Rightarrow \mathrm{id}_X$, then there also exists an intrinsic homotopy $s_0 \circ p \Rightarrow \mathrm{id}_X$.*

- *If $i : Z \to W$ has the forward homotopy extension property with respect to Z, $r_0 : W \to Z$ is a morphism, and $\varepsilon : r_0 \circ i \Rightarrow \mathrm{id}_Z$ is an intrinsic homotopy, then there exists a morphism $r_1 : W \to Z$ such that $r_1 \circ i = \mathrm{id}_Z$; if moreover W is a Kan complex and there is an intrinsic homotopy $\eta : \mathrm{id}_W \Rightarrow i \circ r_0$, then there also exists an intrinsic homotopy $\mathrm{id}_W \Rightarrow i \circ r_1$.*

- *If $i : Z \to W$ has the backward homotopy extension property with respect to Z, $r_0 : W \to Z$ is a morphism, and $\eta : \mathrm{id}_Z \Rightarrow r_1 \circ i$ is an intrinsic homotopy, then there exists a morphism $r_0 : W \to Z$ such that $r_0 \circ i = \mathrm{id}_Z$; if moreover W is a Kan complex and there is an intrinsic homotopy $\varepsilon : i \circ r_1 \Rightarrow \mathrm{id}_W$, then there also exists an intrinsic homotopy $i \circ r_0 \Rightarrow \mathrm{id}_W$.*

Proof. The proofs of the four claims are similar; we will prove the first version.

Let $s_0 : Y \to X$ be a morphism in **sSet** and suppose there exist an intrinsic homotopy $\varepsilon : p \circ s_0 \Rightarrow \mathrm{id}_Y$. Then the forward homotopy lifting property of $p : X \to Y$ yields a morphism $s_1 : Y \to X$ and an intrinsic homotopy $\alpha : s_0 \Rightarrow s_1$ such that $p \circ s_1 = \mathrm{id}_Y$ and $\mathrm{id}_p \circ \alpha = \varepsilon$.

If X is moreover a Kan complex, then (by corollary 1.3.14) the exponential object $[X, X]$ in **sSet** is also a Kan complex, and so we may compose the intrinsic homotopies $\eta : \mathrm{id}_X \Rightarrow s_0 \circ p$ and $\alpha \circ \mathrm{id}_p : s_0 \circ p \Rightarrow s_1 \circ p$ to obtain an intrinsic homotopy $\mathrm{id}_X \Rightarrow s_1 \circ p$, as required. ■

Proposition 1.4.32. *Let $p : X \to Y$ be a Kan fibration. If X and Y are Kan complexes, the following are equivalent:*

(i) *$p : X \to Y$ is a trivial Kan fibration.*

(ii) *$p : X \to Y$ is a weak homotopy equivalence.*

(iii) *$p : X \to Y$ is fibrewise contractible in the sense that there exist a morphism $s : Y \to X$ and an intrinsic homotopy $\eta : \mathrm{id}_X \Rightarrow s \circ p$ such that $p \circ s = \mathrm{id}_Y$ and $\mathrm{id}_p \circ \eta = \mathrm{id}_p$.*

Proof. (i) ⇔ (ii). This is proposition 1.3.16.

(ii) ⇒ (iii). Proposition 1.4.20, lemmas 1.4.21 and 1.4.22, and the preceding lemma say there exist a morphism $s : Y \to X$ and an intrinsic homotopy $\eta' : \mathrm{id}_X \Rightarrow s \circ p$ such that $p \circ s = \mathrm{id}_Y$ and $\mathrm{id}_p \circ \eta' = \mathrm{id}_p$ in the fundamental groupoid $\pi_1[X, Y]$. Since Y is a Kan complex, corollary 1.3.14 and proposition 1.3.7 imply there exists an edge β in $\left[\Delta^1, [X, Y]\right]$ from the vertex corresponding to $\mathrm{id}_p \circ \eta'$ to the vertex corresponding to id_p. Thus, we have a commutative diagram of the form below,

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "z"] \arrow[d, hook] & {[X, X]} \arrow[d, "{[X,p]}"] \\ \Delta^1 \arrow[r, "w_0"'] \arrow[ur, "h_0"] & {[X, Y]} \end{tikzcd}$$

where $h_0 : \Delta^1 \to [X, X]$ is the morphism corresponding to $\eta' : \mathrm{id}_X \Rightarrow s \circ p$. It is not hard to see that we have $\beta \circ \mathrm{id}_i = \mathrm{id}_{[X,p]} \circ \mathrm{id}_z$, and (by corollary 1.3.14) the morphism $[X, p] : [X, X] \to [X, Y]$ is a Kan fibration, so theorem 1.4.29 implies there exists a morphism $h_1 : \Delta^1 \to [X, X]$ that corresponds to an intrinsic homotopy $\eta : \mathrm{id}_X \Rightarrow s \circ p$ such that $\mathrm{id}_p \circ \eta = p$, as required.

(iii) ⇒ (ii). If $p : X \to Y$ is fibrewise contractible in the sense above, then the section $s : Y \to X$ is an intrinsic homotopy inverse for p, and hence p is a weak homotopy equivalence. ∎

Remark 1.4.33. A weaker claim holds for trivial Kan fibrations between arbitrary simplicial sets: if $p : X \to Y$ is a trivial Kan fibration, then for every vertex y of Y, the fibre of p over y is a contractible Kan complex.

1.5 Bisimplicial sets and cosimplicial simplicial sets

Prerequisites. §§ 1.1, 1.3, 4.4, 4.5, A.5, A.6.

Definition 1.5.1. A **bisimplicial set** is a simplicial object in **sSet**, i.e. a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, and a **morphism of bisimplicial sets** is a natural transformation of such functors. We write **ssSet** for the **category of bisimplicial sets**.

Definition 1.5.2. Let $X_\bullet$ be a bisimplicial set and let n be a natural number. The **n-th horizontal level** of $X_\bullet$ is the simplicial set X_n, and the **n-th vertical level** of $X_\bullet$ is the simplicial set $(X_\bullet)_n$.

Definition 1.5.3. A **Reedy weak homotopy equivalence of bisimplicial sets** is a morphism in **ssSet** that is a weak homotopy equivalence of simplicial sets in each horizontal level.

Theorem 1.5.4. **ssSet** *is a DHK model category where*

- *the cofibrations are the monomorphisms in* **ssSet***,*
- *the fibrations are the Reedy fibrations, and*
- *the weak equivalences are the Reedy weak homotopy equivalences.*

This is the **Reedy model structure on bisimplicial sets**.

Proof. Given theorem 4.5.19, it suffices to show that the Reedy cofibrations are precisely the monomorphisms in **ssSet**: see Theorem 15.8.7 in [Hirschhorn, 2003]. □

Corollary 1.5.5. *The Reedy model structure on* **ssSet** *is the injective model structure on the functor category* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{sSet}]$. ■

Definition 1.5.6. The **realisation** of a bisimplicial set $X_\bullet$ is the simplicial set $|X_\bullet|$ defined by the following coend in **sSet**:

$$|X_\bullet| = \int^{[n]:\mathbf{\Delta}} \Delta^n \times X_n$$

Lemma 1.5.7. *For each bisimplicial set $X_\bullet$, there is an isomorphism*

$$|X_\bullet| \cong \operatorname{diag} X$$

where $\operatorname{diag} X$ *is the simplicial set defined by* $(\operatorname{diag} X)_n = (X_n)_n$*, and this isomorphism is natural in X.*

Proof. The Yoneda lemma for coends (proposition A.6.14) yields natural bijections of the form below:

$$\int^{[n]:\mathbf{\Delta}} \mathbf{\Delta}([m],[n]) \times (X_n)_m \cong (X_m)_m$$

Thus, $|X_\bullet| \cong \operatorname{diag} X$. ■

Corollary 1.5.8.

(i) *If $X_\bullet$ is a bisimplicial set whose horizontal levels are discrete,*[8] *then the realisation $|X_\bullet|$ is naturally isomorphic to the simplicial set $(X_\bullet)_0$.*

(ii) *If $X_\bullet$ is a bisimplicial set whose vertical levels are discrete, then the realisation $|X_\bullet|$ is naturally isomorphic to the simplicial set $(X_0)_\bullet$.* ■

Theorem 1.5.9.

(i) *The functor* $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ *has left and right adjoints.*

(ii) $|-|$ *sends Reedy weak homotopy equivalences in* **ssSet** *to weak homotopy equivalences in* **sSet**.

(iii) *Equipping* **ssSet** *with the Reedy model structure and* **sSet** *with the Kan–Quillen model structure,* $|-| : \mathbf{ssSet} \to \mathbf{sSet}$ *is a left Quillen functor.*

Proof. (i). Using the isomorphism $\mathbf{ssSet} \cong [\mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set}]$ and lemma 1.5.7, we may identify $|-|$ as the functor δ^* induced by the diagonal embedding $\delta : \mathbf{\Delta} \to \mathbf{\Delta} \times \mathbf{\Delta}$, and corollary A.5.17 says δ^* has left and right adjoints.

(ii). See Theorem 15.11.11 in [Hirschhorn, 2003], or Proposition 1.7 in [GJ].

(iii). From claims (i) and (ii) it follows that $|-|$ is a left Quillen functor; alternatively, see Proposition 3.6 in [GJ, Ch. VII]. □

Definition 1.5.10. A **cosimplicial simplicial set** is a cosimplicial object in **sSet**, i.e. a functor $\mathbf{\Delta} \to \mathbf{sSet}$, and a morphism of cosimplicial simplicial sets is a natural transformation of such functors. We write **csSet** for the **category of cosimplicial simplicial sets**.

Definition 1.5.11. Let $X^\bullet$ be a cosimplicial simplicial set and let n be a natural number. The ***n*-th horizontal level** of $X^\bullet$ is the simplicial set X^n, and the ***n*-th vertical level** of $X^\bullet$ is the cosimplicial set $(X^\bullet)_n$.

[8] Recall definition 1.2.5.

Definition 1.5.12. A **Reedy weak homotopy equivalence of cosimplicial simplicial sets** is a morphism in **csSet** that is a weak homotopy equivalence of simplicial sets in each horizontal level.

Lemma 1.5.13. *Let $X^\bullet$ be a cosimplicial simplicial set. The limit $\varprojlim_{\mathbf{\Delta}} X^\bullet$ in* **sSet** *can be computed as the equaliser of the coface operators $\delta^0, \delta^1 : X^0 \to X^1$.*

Proof. This is a straightforward exercise. ◊

Definition 1.5.14. The **maximal augmentation** of a cosimplicial simplicial set $X^\bullet$ is the limit $\varprojlim_{\mathbf{\Delta}} X^\bullet$.

Theorem 1.5.15. **csSet** *is a DHK model category where*

- *the cofibrations are the monomorphisms in* **csSet** *that induce isomorphisms of maximal augmentations,*
- *the fibrations are the Reedy fibrations, and*
- *the weak equivalences are the Reedy weak homotopy equivalences.*

This is the **Reedy model structure on cosimplicial simplicial sets**.

Proof. Given theorem 4.5.19, it suffices to show that the Reedy cofibrations are precisely the announced ones: see Theorem 15.9.9 in [Hirschhorn, 2003]. ■

Definition 1.5.16. The **totalisation** of a cosimplicial simplicial set $X^\bullet$ is the simplicial set $\operatorname{Tot} X^\bullet$ defined by the following end in **sSet**:

$$\operatorname{Tot} X^\bullet = \int_{[n]:\mathbf{\Delta}} [\Delta^n, X^n]$$

Lemma 1.5.17. *Let $Y^\bullet : \mathbf{\Delta} \to$* **sSet** *be a cosimplicial simplicial set. Then there is a bijection*

$$\mathbf{sSet}(X, \operatorname{Tot} Y^\bullet) \cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, (Y^m)_m\big)$$

for each simplicial set X, and this bijection is natural in X and Y.

Proof. Using the Yoneda lemma for ends and the interchange law (theorem A.6.13), we have the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}\left(X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, Y^n]\right) &\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X, [\Delta^n, Y^n]) \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X \times \Delta^n, Y^n) \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m \times \mathbf{\Delta}([m],[n]), (Y^n)_m\big) \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), (Y^n)_m\big)\big) \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), (Y^n)_m\big)\right) \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, (Y^m)_m\big)
\end{aligned}$$

∎

Theorem 1.5.18.

(i) *The functor* Tot : **csSet** → **sSet** *has a left adjoint.*

(ii) *For each simplicial set X and each cosimplicial simplicial set $Y^\bullet$, the canonical comparison morphism* $\operatorname{Tot}[X, Y^\bullet] \to [X, \operatorname{Tot} Y^\bullet]$ *is an isomorphism.*

(iii) *Equipping* **csSet** *with the Reedy model structure and* **sSet** *with the Kan–Quillen model structure,* Tot : **csSet** → **sSet** *is a right Quillen functor.*

Proof. (i). It is straightforward to check that the functor sending a simplicial set X to the cosimplicial simplicial set $\Delta^\bullet \times X$ is a left adjoint for Tot; see also proposition A.6.11.

(ii). Since right adjoints preserve ends, we have the natural isomorphisms shown below:

$$\begin{aligned}
[X, \operatorname{Tot} Y^\bullet] &= \left[X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, Y^n]\right] \\
&\cong \int_{[n]:\mathbf{\Delta}} [X, [\Delta^n, Y^n]] \\
&\cong \int_{[n]:\mathbf{\Delta}} [\Delta^n, [X, Y^n]] \\
&= \operatorname{Tot}[X, Y^\bullet]
\end{aligned}$$

(iii). See Theorem 18.6.7 in [Hirschhorn, 2003]. □

1.6 Bar and cobar complexes

Prerequisites. §§ 1.1, 1.4, 1.5, A.5, A.6.

Definition 1.6.1. Let $\mathbb{C}$ be a small category.

The **bar complex** for a diagram $F : \mathbb{C} \to \mathbf{Set}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is the simplicial set $\mathrm{B}_\bullet(G, \mathbb{C}, F)$, where

$$\mathrm{B}_n(G, \mathbb{C}, F) = \coprod_{(c_0,\ldots,c_n)} \left(Gc_n \times \mathbb{C}\left(c_{n-1}, c_n\right) \times \cdots \times \mathbb{C}\left(c_0, c_1\right) \times Fc_0\right)$$

with $\left(c_0, \ldots, c_n\right)$ ranging over $(n+1)$-tuples of objects in $\mathbb{C}$, face maps defined by the following formulae,

$$\begin{aligned} d_0^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_2, F\left(f_1\right)(x)\right) \\ d_i^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1, x\right) \\ d_n^n\left(y, f_n, \ldots, f_1, x\right) &= \left(G\left(f_n\right)(y), f_{n-1}, \ldots, f_1, x\right) \end{aligned}$$

and degeneracy maps defined as below:

$$\begin{aligned} s_0^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_1, \mathrm{id}_{c_0}, x\right) \\ s_i^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, f_n, \ldots, f_{i+1}, \mathrm{id}_{c_i}, f_i, \ldots, f_1, x\right) \\ s_n^n\left(y, f_n, \ldots, f_1, x\right) &= \left(y, \mathrm{id}_{c_n}, f_n, \ldots, f_1, x\right) \end{aligned}$$

The **cobar complex** for a diagram $F : \mathbb{C} \to \mathbf{Set}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ is the cosimplicial set $\mathrm{C}^\bullet(G, \mathbb{C}, F)$, where

$$\mathrm{C}^n(G, \mathbb{C}, F) = \prod_{(c_0,\ldots,c_n)} \left[Gc_n \times \mathbb{C}\left(c_n, c_{n-1}\right) \times \cdots \times \mathbb{C}\left(c_1, c_0\right), Fc_0\right]$$

with $\left(c_0, \ldots, c_n\right)$ ranging over $(n+1)$-tuples of objects in $\mathbb{C}$, coface maps defined by the following formulae,

$$\begin{aligned} \delta_n^0(x)_{(c_0,\ldots,c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto F\left(f_1\right)\left(x_{(c_1,\ldots,c_n)}\left(y, f_n, \ldots, f_2\right)\right)\right) \\ \delta_n^i(x)_{(c_0,\ldots,c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto x_{(\ldots,\widehat{c}_i,\ldots)}\left(y, f_n, \ldots, f_i \circ f_{i+1}, \ldots, f_1\right)\right) \\ \delta_n^n(x)_{(c_0,\ldots,c_n)} &= \left(\left(y, f_n, \ldots, f_1\right) \mapsto x_{(c_0,\ldots,c_{n-1})}\left(G\left(f_n\right)(y), f_{n-1}, \ldots, f_1\right)\right) \end{aligned}$$

and codegeneracy maps defined as below:

$$\begin{aligned}
\sigma_n^0(x)_{(c_0,\ldots,c_n)} &= \big((y, f_n, \ldots, f_1) \mapsto x_{c_0,c_0,\ldots,c_n}\big(y, f_n, \ldots, f_1, \mathrm{id}_{c_0}\big)\big) \\
\sigma_n^i(x)_{(c_0,\ldots,c_n)} &= \big((y, f_n, \ldots, f_1) \mapsto x_{\ldots,c_i,c_i,\ldots}\big(y, f_n, \ldots, f_{i+1}, \mathrm{id}_{c_i}, f_i, \ldots, f_1\big)\big) \\
\sigma_n^n(x)_{(c_0,\ldots,c_n)} &= \big((y, f_n, \ldots, f_1) \mapsto x_{c_0,\ldots,c_n,c_n}\big(y, \mathrm{id}_{c_n}, f_n, \ldots, f_1\big)\big)
\end{aligned}$$

REMARK 1.6.2. It is clear that $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is covariantly functorial in both F and G, while $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is contravariantly functorial in G and covariantly functorial in F. One may also verify that there are bijections

$$\mathbf{Set}\big(\mathrm{B}_n(G, \mathbb{C}, F), X\big) \cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}(F, X))$$

that are natural in n, F, G, and X: this is one sense in which the bar complex and cobar complex are formally dual.

Example 1.6.3. The nerve $\mathrm{N}(\mathbb{C})$ of a small category $\mathbb{C}$ is isomorphic to the bar complex $\mathrm{B}_\bullet(\Delta 1, \mathbb{C}, \Delta 1)$, so there is a canonical morphism $\mathrm{B}_\bullet(G, \mathbb{C}, F) \to \mathrm{N}(\mathbb{C})$ for any $F : \mathbb{C} \to \mathbf{Set}$ and any $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$.

REMARK 1.6.4. More generally, the bar complex $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is isomorphic to the nerve of the following category:

- The objects are tuples (c, x, y), where c is an object in $\mathbb{C}$, x is an element of Fc, and y is an element of Gc.
- The morphisms $f : (c, x, y) \to (c', x', y')$ are morphisms $f : c \to c'$ in $\mathbb{C}$ such that $F(f)(x) = x'$ and $G(f)(y') = y$.
- Composition and identities are inherited from $\mathbb{C}$.

In particular, $\mathrm{B}_\bullet\big(\Delta 1, \mathbb{C}, \hbar^c\big)$ may be identified with the nerve of the slice category ${}^{c/}\mathbb{C}$, and $\mathrm{B}_\bullet\big(\hbar_c, \mathbb{C}, \Delta 1\big)$ with the nerve of the slice category $\mathbb{C}_{/c}$.

Definition 1.6.5. Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category.

- A **bar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is a simplicial object $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ in $\mathcal{M}$ with bijections

$$\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) \cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M))$$

  that are natural in both n and M.

- A **cobar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ is a cosimplicial object $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ in $\mathcal{M}$ with bijections

$$\mathcal{M}(M, \mathrm{C}^n(G, \mathbb{C}, F)) \cong \mathrm{C}^n(G, \mathbb{C}, \mathcal{M}(M, F))$$

  that are natural in both n and M.

Remark 1.6.6. Of course, this definition agrees with the previous one (up to isomorphism) in the special case $\mathcal{M} = \mathbf{Set}$, and it is clear that a cobar complex in $\mathcal{M}$ for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathbf{Set}$ becomes a bar complex in $\mathcal{M}^{\mathrm{op}}$ for $F^{\mathrm{op}} : \mathbb{C}^{\mathrm{op}} \to \mathcal{M}^{\mathrm{op}}$ weighted by the same $G : \mathbb{C} \to \mathbf{Set}$, and vice versa.

Remark 1.6.7. By general considerations about the representability of limits, we see that bar complexes exist for all small diagrams and weights if $\mathcal{M}$ has coproducts for small families of objects, while cobar complexes exist for all small diagrams and weights if $\mathcal{M}$ has products for small families of objects.

Lemma 1.6.8. *Let $\mathbb{C}$ be a small category. For each diagram $F : \mathbb{C} \to \mathbf{Set}$ and each weight $G : \mathbb{C} \to \mathbf{Set}$, we have a bijection*

$$[\mathbb{C}, \mathbf{Set}](G, F) \cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F)$$

that is natural in both F and G.

Proof. It is not hard to see that the (non-full) subcategory $\{[0] \rightrightarrows [1]\}$ is coinitial in $\mathbf{\Delta}$, so it suffices to show that there is an equaliser diagram of the following form,

$$[\mathbb{C}, \mathbf{Set}](G, F) \longrightarrow \mathrm{C}^0(G, \mathbb{C}, F) \underset{\delta^1}{\overset{\delta^0}{\rightrightarrows}} \mathrm{C}^1(G, \mathbb{C}, F)$$

However, if we take the map $[\mathbb{C}, \mathbf{Set}](G, F) \to \mathrm{C}^0(G, \mathbb{C}, F)$ to be the one sending a natural transformation $\alpha : G \Rightarrow F$ to its underlying family of maps $\left(\alpha_c : Gc \to Fc \,\middle|\, c \in \operatorname{ob} \mathbb{C}\right)$, then it is clear that the diagram is indeed an equaliser. ∎

Proposition 1.6.9. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category.*

- *If $\mathrm{B}_\bullet(G, \mathbb{C}, F)$ is a bar complex in $\mathcal{M}$, then the colimit $\varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$ exists in $\mathcal{M}$ if and only if the weighted colimit $G \star_{\mathbb{C}} F$ exists in $\mathcal{M}$, and the two are isomorphic:*

$$G \star_{\mathbb{C}} F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

- *If $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is a cobar complex in $\mathcal{M}$, then the limit $\varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F)$ exists in $\mathcal{M}$ if and only if the weighted limit $\{G, F\}^{\mathbb{C}}$ exists in $\mathcal{M}$, and the two are isomorphic:*

$$\{G, F\}^{\mathbb{C}} \cong \varprojlim_{\mathbf{\Delta}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

Proof. The two claims are formally dual; we will prove the first version.

Let M be any object in $\mathcal{M}$. Using lemma A.5.12, proposition A.5.13, and the above lemma, we obtain the following natural bijections:

$$\begin{aligned} \{G, \mathcal{M}(F, M)\}^{\mathbb{C}^{\mathrm{op}}} &\cong [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}](G, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)) \\ &\cong \varprojlim_{\mathbf{\Delta}} \mathcal{M}\big(\mathrm{B}_\bullet(G, \mathbb{C}, F), M\big) \end{aligned}$$

It follows by the Yoneda lemma that $G \star_{\mathbb{C}} F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F)$. ■

Lemma 1.6.10. *Let $\mathbb{C}$ be a small category.*

(i) *For each natural number n and each weight $G : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ preserves limits, weighted limits, and ends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to \mathbf{Set}$ sends colimits to limits, weighted colimits to weighted limits, and coends to ends.*

Proof. Obvious. ◆

Proposition 1.6.11. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category. If $\mathcal{M}$ has coproducts for small families of objects, then:*

(i) *For each natural number n and each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, the functor $\mathrm{B}_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ preserves colimits, weighted colimits, and coends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{B}_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \to \mathcal{M}$ preserves colimits, weighted colimits, and coends.*

Dually, if $\mathcal{M}$ has products for small families of objects, then:

(i) *For each natural number n and each weight $G : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{Set}] \to \mathbf{Set}$ preserves limits, weighted limits, and ends.*

(ii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to \mathbf{Set}$ sends colimits to limits, weighted colimits to weighted limits, and coends to ends.*

Proof. We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

Lemma 1.6.12. *Let $\mathbb{C}$ be a small category.*

- *Let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram and let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a weight. For all sets X, we have bijections*

$$\mathrm{B}_n(G \times X, \mathbb{C}, F) \cong X \times \mathrm{B}_n(G, \mathbb{C}, F) \cong \mathrm{B}_n(G, \mathbb{C}, X \times F)$$

that are natural in X, F, and G.

- *Let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram and let $G : \mathbb{C} \to \mathbf{Set}$ be a weight. For all sets X, we have bijections*

$$\mathrm{C}^n(X \times G, \mathbb{C}, F) \cong [X, \mathrm{C}^n(G, \mathbb{C}, F)] \cong \mathrm{C}^n(G, \mathbb{C}, [X, F])$$

that are natural in X, F, and G.

Proof. Obvious. ♦

Proposition 1.6.13. *Let $\mathbb{C}$ be a small category and let $\mathcal{M}$ be a locally small category. If $\mathcal{M}$ has coproducts for small families of objects, then:*

(i) *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ be a weight and let M be any object in $\mathcal{M}$. We then have bijections*

$$\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(Gc' \odot Fc, M)\big)$$

that are natural in n, F, G, and M.

(ii) *If $\mathcal{M}$ is cotensored, then for each natural number n and each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, the functor $\mathrm{B}_n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ has a right adjoint, namely the functor that sends an object M to the diagram $c \mapsto \mathrm{B}_n\big(G, \mathbb{C}^{\mathrm{op}}, \hbar_c\big) \pitchfork M$.*

(iii) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{B}_n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}] \to \mathcal{M}$ has a right adjoint, namely the functor that sends an object M to the weight $c \mapsto \mathrm{C}^n\big(\hbar_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big)$.*

Dually, if $\mathcal{M}$ has products for small families of objects, then:

(i′) *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{C} \to \mathbf{Set}$ be a weight, and let M be an object in $\mathcal{M}$. We then have bijections*

$$\mathcal{M}(M, \mathrm{C}^n(G, \mathbb{C}, F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(M, Gc' \pitchfork Fc)\big)$$

that are natural in n, F, G, and M.

(ii′) *If $\mathcal{M}$ is tensored, then for each natural number n and each weight $G : \mathbb{C} \to \mathbf{Set}$, the functor $\mathrm{C}^n(G, \mathbb{C}, -) : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ has a left adjoint, namely the functor that sends an object M to the diagram $c \mapsto \mathrm{B}_n\big(G, \mathbb{C}^{\mathrm{op}}, \hbar_c\big) \odot M$.*

(iii′) *For each natural number n and each diagram $F : \mathbb{C} \to \mathcal{M}$, the functor $\mathrm{C}^n(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{Set}]^{\mathrm{op}} \to \mathcal{M}$ has a left adjoint, namely the functor that sends an object M to the weight $c \mapsto \mathrm{C}^n\big(\hbar^c, \mathbb{C}, \mathcal{M}(M, F)\big)$.*

Proof. The two sets of claims are formally dual; we will prove the first version.

(i). Using the interchange law for ends (theorem A.6.13), the Yoneda lemma for ends (proposition A.6.14), and proposition 1.6.11, we obtain the following natural bijections:

$$
\begin{aligned}
\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} &\mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(Gc' \odot Fc, M)\big) \\
&\cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathrm{C}^n\big(\hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\hbar^{c}, \mathcal{M}(Gc' \odot Fc, M)\big)\big) \\
&\cong \int_{c':\mathbb{C}} \int_{c:\mathbb{C}} \mathrm{C}^n\big(\hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\hbar^{c}, \mathcal{M}(Gc' \odot Fc, M)\big)\big) \\
&\cong \int_{c':\mathbb{C}} \mathrm{C}^n\left(\hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \int_{c:\mathbb{C}} \mathbf{Set}\big(\hbar^{c}, \mathcal{M}(Gc' \odot Fc, M)\big)\right) \\
&\cong \int_{c':\mathbb{C}} \mathrm{C}^n\big(\hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(Gc' \odot F, M)\big) \\
&\cong \int_{c':\mathbb{C}} \mathrm{C}^n\big(Gc' \times \hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\big) \\
&\cong \mathrm{C}^n\left(\int^{c':\mathbb{C}} Gc' \times \hbar_{c'}, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)\right) \\
&\cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)) \\
&\cong \mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big)
\end{aligned}
$$

(ii). Similarly, we have the following natural bijections:

$$
\begin{aligned}
\mathcal{M}\big(\mathrm{B}_n(G, \mathbb{C}, F), M\big) &\cong \mathrm{C}^n(G, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F, M)) \\
&\cong \mathrm{C}^n\left(G, \mathbb{C}^{\mathrm{op}}, \int_{c:\mathbb{C}} \mathbf{Set}\big(\hbar^{c}, \mathcal{M}(Fc, M)\big)\right) \\
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(G, \mathbb{C}^{\mathrm{op}}, \mathbf{Set}\big(\hbar^{c}, \mathcal{M}(Fc, M)\big)\big) \\
&\cong \int_{c:\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(G, \mathbb{C}, \hbar^{c}\big), \mathcal{M}(Fc, M)\big) \\
&\cong \int_{c:\mathbb{C}} \mathcal{M}\big(Fc, \mathrm{B}_n\big(G, \mathbb{C}, \hbar^{c}\big) \pitchfork M\big)
\end{aligned}
$$

Now apply remark A.6.5.

(iii). Along the same lines:

$$\begin{aligned}
\mathcal{M}\big(\mathrm{B}_n(G,\mathbb{C},F),M\big) &\cong \mathrm{C}^n(G,\mathbb{C}^{\mathrm{op}},\mathcal{M}(F,M)) \\
&\cong \mathrm{C}^n\left(\int^{c:\mathbb{C}} Gc \times \mathcal{k}_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F,M)\right) \\
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(Gc \times \mathcal{k}_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F,M)\big) \\
&\cong \int_{c:\mathbb{C}} \mathrm{C}^n\big(Gc \times \mathcal{k}_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F,M)\big) \\
&\cong \int_{c:\mathbb{C}} \mathbf{Set}\Big(Gc, \mathrm{C}^n\big(\mathcal{k}_c, \mathbb{C}^{\mathrm{op}}, \mathcal{M}(F,M)\big)\Big)
\end{aligned}$$

Note that in the last step we are appealing to lemma 1.6.12. ■

Remark 1.6.14. The above proposition shows that bar complexes are a certain kind of weighted colimit, while cobar complexes are a certain kind of weighted limit.

Definition 1.6.15. Let $\mathbb{C}$ be a small category, let $\mathcal{A}$ be any category and let $\mathcal{M}$ be a locally small category.

- Given $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, a **bar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$ is a simplicial object $\mathrm{B}_\bullet(G,\mathbb{C},F)$ equipped with bijections

  $$\mathcal{M}\big(\mathrm{B}_n(G,\mathbb{C},F),M\big) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\mathcal{k}_{c'},\mathbb{C},\mathcal{k}^c\big), \mathcal{M}(Gc' \odot Fc, M)\big)$$

  that are natural in both n and M.

- Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, a **cobar complex** for a diagram $F : \mathbb{C} \to \mathcal{M}$ weighted by $G : \mathbb{C} \to \mathcal{A}$ is a cosimplicial object $\mathrm{C}^\bullet(G,\mathbb{C},F)$ equipped with bijections

  $$\mathcal{M}(M,\mathrm{C}^n(G,\mathbb{C},F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{Set}\big(\mathrm{B}_n\big(\mathcal{k}_{c'},\mathbb{C},\mathcal{k}^c\big), \mathcal{M}(M, Gc' \pitchfork Fc)\big)$$

  that are natural in both n and M.

Remark 1.6.16. Although the definition given here is stated using an end, one can also state a version that only uses products. Thus these generalised bar (resp. cobar) complexes exist in a locally small category $\mathcal{M}$ as soon as $\mathcal{M}$ has coproducts (resp. products) for small families of objects.

Remark 1.6.17. In the case where $\mathcal{A} = \mathcal{M} = \mathbf{sSet}$, we will almost always take $A \odot M = A \times M$ and $A \pitchfork M = [A, M]$. With this choice, the formulae of definition 1.6.1 (understood appropriately) can be applied verbatim.

Theorem 1.6.18. *Let $\mathbb{C}$ and $\mathbb{D}$ be two small categories, let $\mathcal{A}$ and $\mathcal{M}$ be two locally small categories, and let $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $\odot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$, and $\underline{\mathcal{M}} : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{A}$ be functors. Suppose $\mathcal{A}$ has coproducts for small families of objects, that there are bijections*

$$\mathcal{M}(A \odot M, N) \cong \mathcal{A}(A, \underline{\mathcal{M}}(M, N)) \cong \mathcal{M}(M, A \pitchfork N)$$

that are natural in A, M, and N, and that there are isomorphisms

$$(A \otimes B) \odot M \cong A \odot (B \odot M)$$
$$(A \otimes B) \pitchfork M \cong A \pitchfork (B \pitchfork M)$$

that are natural in A, B, and M.

- *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{D}^{\mathrm{op}} \to \mathcal{A}$ be a weight, and let $H : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{A}$ be a functor. If $\mathcal{M}$ has coproducts for small families of objects, then there is an isomorphism*

  $$\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big) \cong \mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big)$$

  that is natural in m, n, F, G, and H.

- *Let $F : \mathbb{C} \to \mathcal{M}$ be a diagram, let $G : \mathbb{D} \to \mathcal{A}$ be a weight, and let $H : \mathbb{D}^{\mathrm{op}} \times \mathbb{C} \to \mathcal{A}$ be a functor. If $\mathcal{M}$ has products for small families of objects, then there is an isomorphism*

  $$\mathrm{C}^m\big(\mathrm{B}_n(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F\big) \cong \mathrm{C}^n(G, \mathbb{D}, \mathrm{C}^m(H, \mathbb{C}, F))$$

  that is natural in m, n, F, G, and H.

Proof. The two claims are formally dual; we will prove the first version.

Let M be any object in $\mathcal{M}$ and let $K : \mathbb{D}^{\mathrm{op}} \times \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \times \mathbb{C} \to \mathbf{Set}$ be the functor defined below:

$$K(d', c', d, c) = \mathcal{A}(Gd' \otimes H(c', d), \underline{\mathcal{M}}(Fc, M))$$

Notice that we have the following natural bijections:

$$\begin{aligned} K(d', c', d, c) &\cong \mathcal{M}((Gd' \otimes H(c', d)) \odot Fc, M) \\ &\cong \mathcal{M}(Gd' \odot (H(c', d) \odot Fc), M) \\ &\cong \mathcal{M}(H(c', d) \odot Fc, Gd' \pitchfork M) \end{aligned}$$

Now, using the definition of the generalised bar complex, we obtain the natural bijections shown below:

$$\begin{aligned} &\mathcal{M}\big(\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big), M\big) \\ &\qquad \cong \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{M}\big(\mathrm{B}_m(G, \mathbb{D}, H(c', -)) \odot Fc, M\big)\big) \\ &\qquad \cong \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \mathcal{A}\big(\mathrm{B}_m(G, \mathbb{D}, H(c', -)), \underline{\mathcal{M}}(Fc, M)\big)\big) \\ &\qquad \cong \int_{(c',c)} \mathbf{Set}\left(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_m\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), K(d', c', d, c)\big)\right) \\ &\qquad \cong \int_{(c',c)} \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big) \times \mathrm{B}_m\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), K(d', c', d, c)\big) \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathcal{M}\big(\mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big), M\big) \\ &\qquad \cong \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), \mathcal{M}\big(Gd' \odot \mathrm{B}_m(H(-, d), \mathbb{C}, F), M\big)\big) \\ &\qquad \cong \int_{(d',d)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), \mathcal{M}\big(\mathrm{B}_m(H(-, d), \mathbb{C}, F), Gd' \pitchfork M\big)\big) \\ &\qquad \cong \int_{(d',d)} \mathbf{Set}\left(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big), \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_m\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), K(d', c', d, c)\big)\right) \\ &\qquad \cong \int_{(d',d)} \int_{(c',c)} \mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{d'}, \mathbb{D}, \hbar^d\big) \times \mathrm{B}_m\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), K(d', c', d, c)\big) \end{aligned}$$

and so, applying the interchange law for ends (theorem A.6.13), we obtain a natural bijection

$$\mathcal{M}\big(\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big), M\big) \cong \mathcal{M}\big(\mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big), M\big)$$

and the claim follows by the Yoneda lemma. ■

Definition 1.6.19. Let $\mathbb{C}$ be a small category.

- Given $\odot : \mathcal{A} \times \mathbf{sSet} \to \mathbf{sSet}$, the **bar construction** for a diagram $F : \mathbb{C} \to \mathbf{sSet}$ weighted by a functor $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$ is the following coend:

$$\mathrm{B}(G, \mathbb{C}, F) = \int^{[n]:\mathbf{\Delta}} \Delta^n \times \mathrm{B}_n(G, \mathbb{C}, F)$$

  In other words, $\mathrm{B}(G, \mathbb{C}, F)$ is the realisation $\left|\mathrm{B}_\bullet(G, \mathbb{C}, F)\right|$.

- Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathbf{sSet} \to \mathbf{sSet}$, the **cobar construction** for a diagram $F : \mathbb{C} \to \mathbf{sSet}$ weighted by a functor $G : \mathbb{C} \to \mathcal{A}$ is the following end:

$$\mathrm{C}(G, \mathbb{C}, F) = \int_{[n]:\mathbf{\Delta}} \left[\Delta^n, \mathrm{B}_n(G, \mathbb{C}, F)\right]$$

  In other words, $\mathrm{C}(G, \mathbb{C}, F)$ is the totalisation $\operatorname{Tot} \mathrm{C}^\bullet(G, \mathbb{C}, F)$.

Lemma 1.6.20. *Let $\mathbb{C}$ be a small category, let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, and let $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$ be a weight. We then have bijections*

$$(\mathrm{B}(G, \mathbb{C}, F))_n \cong \mathrm{B}_n\big(G_n, \mathbb{C}, F_n\big)$$

that are natural in n.

Proof. Apply lemma 1.5.7 to remark 1.6.17. ■

Proposition 1.6.21. *Let $\mathbb{C}$ be a small category and let $\mathcal{A}$ be any category.*

- *Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, let $G : \mathbb{C}^{\mathrm{op}} \to \mathcal{A}$ be a weight, and let M be a simplicial set. Given $\odot : \mathcal{A} \times \mathbf{sSet} \to \mathbf{sSet}$, we have bijections*

$$\mathbf{sSet}(\mathrm{B}(G, \mathbb{C}, F), M) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\big(\mathrm{B}_\bullet\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \left[Gc' \odot Fc, M\right]\big)$$

  that are natural in F, G, and M.

- *Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram, let $G : \mathbb{C} \to \mathcal{A}$ be a weight, and let M be a simplicial set. Given $\pitchfork : \mathcal{A}^{\mathrm{op}} \times \mathbf{sSet} \to \mathbf{sSet}$, we have bijections*

$$\mathbf{sSet}(M, \mathrm{C}(G, \mathbb{C}, F)) \cong \int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\big(\mathrm{B}_\bullet\big(\hbar_{c'}, \mathbb{C}, \hbar^c\big), \left[M, Gc' \pitchfork Fc\right]\big)$$

  that are natural in F, G, and M.

Proof. We will prove the first claim; the second can be proved in a similar way.

By definition, we have the natural bijection

$$\mathbf{sSet}(\mathrm{B}(G,\mathbb{C},F),M)\cong\int_{[n]:\mathbf{\Delta}}\mathbf{sSet}\big(\Delta^n\times\mathrm{B}_n(G,\mathbb{C},F),M\big)$$

and furthermore, we also have the following:

$$\begin{aligned}
&\mathbf{sSet}\big(\Delta^n\times\mathrm{B}_n(G,\mathbb{C},F),M\big)\\
&\qquad\cong\mathbf{sSet}\big(\mathrm{B}_n(G,\mathbb{C},F),[\Delta^n,M]\big)\\
&\qquad\cong\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}\mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'},\mathbb{C},\hbar^c\big),\mathbf{sSet}(Gc'\odot Fc,[\Delta^n,M])\big)\\
&\qquad\cong\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}\mathbf{Set}\big(\mathrm{B}_n\big(\hbar_{c'},\mathbb{C},\hbar^c\big),\mathbf{sSet}\big(\Delta^n,\big[Gc'\odot Fc,M\big]\big)\big)\\
&\qquad\cong\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}\mathbf{sSet}\big(\operatorname{disc}\mathrm{B}_n\big(\hbar_{c'},\mathbb{C},\hbar^c\big)\times\Delta^n,\big[Gc'\odot Fc,M\big]\big)
\end{aligned}$$

Thus, applying the interchange law for ends (theorem A.6.13) and corollary 1.5.8, we obtain

$$\mathbf{sSet}(\mathrm{B}(G,\mathbb{C},F),M)\cong\int_{(c',c):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}}\mathbf{sSet}\big(\mathrm{B}_\bullet\big(\hbar_{c'},\mathbb{C},\hbar^c\big),\big[Gc'\odot Fc,M\big]\big)$$

as required. ■

Proposition 1.6.22. *Let $\mathbb{C}$ be a small category.*

(i) *For each weight $G:\mathbb{C}^{\mathrm{op}}\to\mathbf{sSet}$, the functor $\mathrm{B}(G,\mathbb{C},-):[\mathbb{C},\mathbf{sSet}]\to\mathbf{sSet}$ has a right adjoint, namely the functor that sends a simplicial set M to the diagram $c\mapsto\big[\mathrm{B}\big(G,\mathbb{C}^{\mathrm{op}},\hbar_c\big),M\big]$.*

(ii) *For each diagram $F:\mathbb{C}\to\mathbf{sSet}$, the functor $\mathrm{B}(-,\mathbb{C},F):[\mathbb{C}^{\mathrm{op}},\mathbf{sSet}]\to\mathbf{sSet}$ has a right adjoint, namely the functor that sends an object M to the weight $c\mapsto\mathrm{C}\big(\hbar_c,\mathbb{C}^{\mathrm{op}},[F,M]\big)$.*

(iii) *For each simplicial set X, there are isomorphisms*

$$\mathrm{B}(X\times G,\mathbb{C},F)\cong X\times\mathrm{B}(G,\mathbb{C},F)\cong\mathrm{B}(G,\mathbb{C},X\times F)$$

that are natural in X, F, and G.

Dually:

(i′) *For each weight* $G : \mathbb{C} \to \mathbf{sSet}$*, the functor* $\mathrm{C}(G, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ *has a left adjoint, namely the functor that sends a simplicial set* M *to the diagram* $c \mapsto \mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\big) \times M$.

(ii′) *For each diagram* $F : \mathbb{C} \to \mathbf{sSet}$*, the functor* $\mathrm{C}(-, \mathbb{C}, F) : [\mathbb{C}, \mathbf{sSet}]^{\mathrm{op}} \to \mathbf{sSet}$ *has a right adjoint, namely the functor that sends an object* M *to the weight* $c \mapsto \mathrm{C}\big(\mathcal{h}^c, \mathbb{C}, [M, F]\big)$.

(iii′) *For each simplicial set* X*, there are isomorphisms*

$$\mathrm{C}(X \times G, \mathbb{C}, F) \cong [X, \mathrm{C}(G, \mathbb{C}, F)] \cong \mathrm{C}(G, \mathbb{C}, [X, F])$$

that are natural in X*,* F*, and* G*.*

Proof. We will prove the first set of claims; the second can be proved in a similar way.

(i). Let $F : \mathbb{C} \to \mathbf{sSet}$ be a diagram. By definition, we have the following natural bijections:

$$\begin{aligned}
[\mathbb{C}, \mathbf{sSet}]\big(F, \big[\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_\bullet\big), M\big]\big) &\cong \int_{c:\mathbb{C}} \mathbf{sSet}\big(Fc, \big[\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\big), M\big]\big) \\
&\cong \int_{c:\mathbb{C}} \mathbf{sSet}\big(\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\big), [Fc, M]\big)
\end{aligned}$$

Now, using proposition I.6.21, we also obtain these natural bijections:

$$\begin{aligned}
&\mathbf{sSet}\big(\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_c\big), [Fc, M]\big) \\
&\quad\cong \int_{(c'', c'):\mathbb{C}\times\mathbb{C}^{\mathrm{op}}} \mathbf{sSet}\Big(\mathrm{B}\Big(\mathcal{h}^{c''}, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_{c'}\Big), \big[Gc'' \times \mathbb{C}(c', c), [Fc, M]\big]\Big) \\
&\quad\quad\cong \int_{(c'', c'):\mathbb{C}\times\mathbb{C}^{\mathrm{op}}} \mathbf{sSet}\Big(\mathrm{B}\Big(\mathcal{h}^{c''}, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_{c'}\Big), \big[Gc'' \times \mathbb{C}(c', c) \times Fc, M\big]\Big)
\end{aligned}$$

Applying the interchange law for ends (theorem A.6.13) and the Yoneda lemma for coends (proposition A.6.14), we deduce that

$$\begin{aligned}
&[\mathbb{C}, \mathbf{sSet}]\big(F, \big[\mathrm{B}\big(G, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_\bullet\big), M\big]\big) \\
&\quad\cong \int_{(c'', c'):\mathbb{C}^{\mathrm{op}}\times\mathbb{C}} \mathbf{sSet}\Big(\mathrm{B}\Big(\mathcal{h}^{c''}, \mathbb{C}^{\mathrm{op}}, \mathcal{h}_{c'}\Big), \big[Gc'' \times Fc', M\big]\Big) \\
&\quad\quad\cong \mathbf{sSet}(\mathrm{B}(G, \mathbb{C}, F), M)
\end{aligned}$$

naturally in F and M, as required.

(ii). The proof is similar to that of claim (i).

(iii). Apply lemmas 1.6.12 to 1.6.20. (For the dual claim, use theorem 1.5.18 instead.) ■

Theorem 1.6.23. *Let $\mathbb{C}$ and $\mathbb{D}$ be two small categories.*

- *Let $F : \mathbb{C} \to$ **sSet** be a diagram, let $G : \mathbb{D}^{\mathrm{op}} \to$ **sSet** be a weight, and let $H : \mathbb{C}^{\mathrm{op}} \times \mathbb{D} \to$ **sSet** be a functor. There is then an isomorphism*

$$\mathrm{B}_m\big(\mathrm{B}_n(G, \mathbb{D}, H), \mathbb{C}, F\big) \cong \mathrm{B}_n\big(G, \mathbb{D}, \mathrm{B}_m(H, \mathbb{C}, F)\big)$$

  that is natural in m, n, F, G, and H.

- *Let $F : \mathbb{C} \to$ **sSet** be a diagram, let $G : \mathbb{D} \to$ **sSet** be a weight, and let $H : \mathbb{D}^{\mathrm{op}} \times \mathbb{C} \to$ **sSet** be a functor. There is then an isomorphism*

$$\mathrm{C}^m\big(\mathrm{B}_n(G, \mathbb{D}^{\mathrm{op}}, H), \mathbb{C}, F\big) \cong \mathrm{C}^n(G, \mathbb{D}, \mathrm{C}^m(H, \mathbb{C}, F))$$

  that is natural in m, n, F, G, and H.

Proof. The proof is essentially the same as that of theorem 1.6.18. ■

Proposition 1.6.24. *Let $\mathbb{C}$ be a small category.*

- *For each diagram $F : \mathbb{C} \to$ **sSet** and each functor $G : \mathbb{C}^{\mathrm{op}} \to$ **Set**, there is a morphism $\mathrm{B}(G, \mathbb{C}, F) \to G \star_{\mathbb{C}} F$, and it is natural in both F and G.*

- *For each diagram $F : \mathbb{C} \to$ **sSet** and each functor $G : \mathbb{C} \to$ **Set**, there is a morphism $\{G, F\}^{\mathbb{C}} \to \mathrm{C}(G, \mathbb{C}, F)$, and it is natural in both F and G.*

Proof. By theorem A.6.10 and proposition 1.6.9, we have the following natural isomorphisms:

$$\int^{[n]:\mathbf{\Delta}} \mathrm{B}_n(G, \mathbb{C}, F) \cong \Delta 1 \star_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F) \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \mathrm{B}_\bullet(G, \mathbb{C}, F) \cong G \star_{\mathbb{C}} F$$

$$\int_{[n]:\mathbf{\Delta}} \mathrm{C}^n(G, \mathbb{C}, F) \cong \{\Delta 1, \mathrm{C}^\bullet(G, \mathbb{C}, F)\}^{\mathbf{\Delta}} \cong \varprojlim_{\mathbf{\Delta}} \mathrm{C}^\bullet(G, \mathbb{C}, F) \cong \{G, F\}^{\mathbb{C}}$$

The claim then follows from the existence of a (unique) natural transformation $\Delta^\bullet \Rightarrow \Delta 1$. ■

Definition 1.6.25. Let $\mathbb{C}$ be a small category, let $\mathcal{M}$ be a locally small category, and let $F : \mathbb{C} \to \mathcal{M}$ be a diagram.

- The **bar resolution** of F is the diagram $\mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]$ defined by the following formula,
$$c \mapsto \mathrm{B}_\bullet\left(\hbar_c, \mathbb{C}, F\right)$$
where $\hbar_c : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$ is the representable functor $\mathbb{C}(-, c)$.
- The **cobar resolution** of F is the diagram $\mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F) : \mathbb{C} \to [\mathbf{\Delta}, \mathcal{M}]$ defined by the following formula,
$$c \mapsto \mathrm{C}^\bullet\left(\hbar^c, \mathbb{C}, F\right)$$
where $\hbar^c : \mathbb{C} \to \mathbf{Set}$ is the representable functor $\mathbb{C}(c, -)$.

Lemma 1.6.26. *Let $\mathbb{C}$ be a small category and let $F : \mathbb{C} \to \mathbf{Set}$ be a diagram.*

(i) *There is an isomorphism*
$$F \cong \varprojlim_{\mathbf{\Delta}} \circ \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$$
and it is natural in F.

(ii) *For each weight $G : \mathbb{C} \to \mathbf{Set}$, there is an isomorphism*
$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathrm{C}^\bullet(G, \mathbb{C}, F)$$
and it is natural in both F and G.

(iii) *For each object c in $\mathbb{C}$, there exist maps $\eta_c : Fc \to \mathrm{C}^0\left(\hbar^c, \mathbb{C}, F\right)$, $\varepsilon_c : \mathrm{C}^0\left(\hbar^c, \mathbb{C}, F\right) \to Fc$, and $h_{n,c} : \mathrm{C}^{n+1}\left(\hbar^c, \mathbb{C}, F\right) \to \mathrm{C}^n\left(\hbar^c, \mathbb{C}, F\right)$ satisfying these identities:*
$$\begin{aligned}
\delta^1_1 \circ \eta_c &= \delta^0_1 \circ \eta_c \\
\varepsilon_c \circ \eta_c &= \mathrm{id} \\
h_{0,c} \circ \delta^0_1 &= \eta_c \circ \varepsilon_c \\
h_{n,c} \circ \delta^i_{n+1} &= \delta^i_n \circ h_{n-1,c} && \text{if } 0 \le i \le n \\
h_{n,c} \circ \delta^{n+1}_{n+1} &= \mathrm{id} \\
\sigma^i_n \circ h_{n+1,c} &= h_{n,c} \circ \sigma^i_{n+1} && \text{if } 0 \le i \le n \\
h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma^{n+1}_{n+1}
\end{aligned}$$
These maps are moreover natural in F, and η_c is also natural in c.

Proof. (i). By lemma 1.6.8, there are bijections

$$[\mathbb{C}, \mathbf{Set}]\big(\hbar^c, F\big) \cong \varprojlim_{\Delta} \mathrm{C}^\bullet\big(\hbar^c, \mathbb{C}, F\big)$$

that are natural in c and F, so the Yoneda lemma implies $F \cong \varprojlim_{\Delta} \circ \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$, naturally in F.

(ii). Applying the Yoneda lemma for ends (proposition A.6.14), we obtain the following natural bijections:

$$\begin{aligned}
&\int_{c:\mathbb{C}} \big[G(c), \big[\mathbb{C}\big(c, c_n\big) \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), F\big(c_0\big)\big]\big] \\
&\qquad \cong \int_{c:\mathbb{C}} \big[\mathbb{C}\big(c, c_n\big), \big[G(c) \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), F\big(c_0\big)\big]\big] \\
&\qquad\qquad \cong \big[G\big(c_n\big) \times \mathbb{C}\big(c_n, c_{n-1}\big) \times \cdots \times \mathbb{C}\big(c_1, c_0\big), F\big(c_0\big)\big]
\end{aligned}$$

Theorem A.6.10 implies that there is a natural isomorphism

$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \int_{c:\mathbb{C}} [G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)]$$

and it is now clear that the claim holds.

(iii). Let η_c, ε_c, and $h_{n,c}$ be the maps defined below:

$$\begin{aligned}
\eta_c(x)_{(c_0)} &= (y \mapsto F(y)(x)) \\
\varepsilon_c(x) &= x_{(c)}\big(\mathrm{id}_c\big) \\
h_{n,c}(x)_{(c_0,\ldots,c_n)} &= \Big(\big(y, f_n, \ldots, f_1\big) \mapsto x_{(c_0,\ldots,c_n,c)}\big(\mathrm{id}_c, y, f_n, \ldots, f_1\big)\Big)
\end{aligned}$$

By construction, we have $\varepsilon_c \circ \eta_c = \mathrm{id}_{F_c}$, and it is not hard to check that the other identities are satisfied. For naturality of η_c in c, observe that, given $f : c \to c'$ in $\mathbb{C}$, we have

$$\begin{aligned}
\eta_{c'}(F(f)(x))_{(c_0)} &= (y \mapsto F(y)(F(f)(x))) \\
&= (y \mapsto F(y \circ f)(x)) \\
&= \big(y \mapsto F\big(\hbar^f(y)\big)(x)\big) \\
&= \mathrm{C}^0\big(\hbar^f, \mathbb{C}, F\big)\big(\eta_c(x)\big)_{(c_0)}
\end{aligned}$$

and so $\eta_{c'} \circ F(f) = \mathrm{C}^0\big(\hbar^f, \mathbb{C}, F\big) \circ \eta_c$, as required. ∎

Proposition 1.6.27. *Let $\mathbb{C}$ be a small category, let $\mathcal{M}$ be a locally small category, and let $F : \mathbb{C} \to \mathcal{M}$ be a diagram. If the bar resolution $\mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F)$ exists, then:*

(i) *There is an isomorphism*

$$F \cong \varinjlim_{\mathbf{\Delta}^{\mathrm{op}}} \circ \mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in F.

(ii) *For each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, there is an isomorphism*

$$G \star_{\mathbb{C}} \mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F) \cong \mathrm{B}_\bullet(G, \mathbb{C}, F)$$

and it is natural in both F and G.

(iii) *For each object c in $\mathbb{C}$, there exist morphisms $\eta_c : Fc \to \mathrm{B}_0(\hbar_c, \mathbb{C}, F)$, $\varepsilon_c : \mathrm{B}_0(\hbar_c, \mathbb{C}, F) \to Fc$, and $h_c^n : \mathrm{B}_n(\hbar_c, \mathbb{C}, F) \to \mathrm{B}_{n+1}(\hbar_c, \mathbb{C}, F)$ satisfying these identities:*

$$\begin{aligned}
\varepsilon_c \circ d_1^1 &= \varepsilon_c \circ d_0^1 && \\
\varepsilon_c \circ \eta_c &= \mathrm{id} && \\
d_0^1 \circ h_c^0 &= s \circ r && \\
d_i^{n+1} \circ h_c^n &= h_c^{n-1} \circ d_i^n && \text{if } 0 \le i \le n \\
d_{n+1}^{n+1} \circ h_c^n &= \mathrm{id} && \\
h_c^{n+1} \circ s_i^n &= s_i^{n+1} \circ h_c^n && \text{if } 0 \le i \le n \\
h_c^{n+1} \circ h_c^n &= s_{n+1}^{n+1} \circ h_c^n &&
\end{aligned}$$

These morphisms are moreover natural in F, and ε_c is also natural in c.

Dually, if the cobar resolution $\mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$ exists, then:

(i) *There is an isomorphism*

$$F \cong \varprojlim_{\mathbf{\Delta}} \circ \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)$$

and it is natural in F.

(ii) *For each weight $G : \mathbb{C} \to \mathbf{Set}$, there is an isomorphism*

$$\{G, \mathrm{C}^\bullet(\mathbb{C}, \mathbb{C}, F)\}^{\mathbb{C}} \cong \mathrm{C}^\bullet(G, \mathbb{C}, F)$$

and it is natural in both F and G.

(iii) *For each object c in $\mathbb{C}$, there exist morphisms $\eta_c : Fc \to \mathrm{C}^0\left(h^c, \mathbb{C}, F\right)$, ε_c : $\mathrm{C}^0\left(h^c, \mathbb{C}, F\right) \to Fc$, and $h_{n,c} : \mathrm{C}^{n+1}\left(h^c, \mathbb{C}, F\right) \to \mathrm{C}^n\left(h^c, \mathbb{C}, F\right)$ satisfying these identities:*

$$
\begin{aligned}
\delta_1^1 \circ \eta_c &= \delta_1^0 \circ \eta_c \\
\varepsilon_c \circ \eta_c &= \mathrm{id} \\
h_{0,c} \circ \delta_1^0 &= \eta_c \circ \varepsilon_c \\
h_{n,c} \circ \delta_{n+1}^i &= \delta_n^i \circ h_{n-1,c} && \text{if } 0 \le i \le n \\
h_{n,c} \circ \delta_{n+1}^{n+1} &= \mathrm{id} \\
\sigma_n^i \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^i && \text{if } 0 \le i \le n \\
h_{n,c} \circ h_{n+1,c} &= h_{n,c} \circ \sigma_{n+1}^{n+1}
\end{aligned}
$$

These morphisms are moreover natural in F, and η_c is also natural in c.

Proof. We may use the Yoneda lemma to reduce the claims to the case in the previous lemma. ■

1.7 Homotopy limits and colimits

Prerequisites. §§ 1.2, 1.3, 1.5, 1.6, 3.3, 3.4.

Remark 1.7.1. It is important to stress that there is an asymmetry between the theory of homotopy colimits and the theory of homotopy limits in **sSet** because not all simplicial sets are fibrant. As such, it will often be necessary to restrict our attention to Kan complexes when working with homotopy limits.

Proposition 1.7.2. *Let $\mathbb{C}$ be a small category. For any weight $G : \mathbb{C} \to$ **sSet**, if $F : \mathbb{C} \to$ **sSet** is a diagram of Kan complexes, then:*

(i) *The cobar complex $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is a Reedy-fibrant cosimplicial simplicial set, and each horizontal level is a Kan complex.*

(ii) *The cobar construction $\mathrm{C}(G, \mathbb{C}, F)$ is a Kan complex.*

Proof. (i). Recalling remark 1.6.17, we have the following formula:

$$
\mathrm{C}^n(G, \mathbb{C}, F) \cong \prod_{(c_0, \ldots, c_n)} \left[Gc_n \times \mathbb{C}\left(c_n, c_{n-1}\right) \times \cdots \times \mathbb{C}\left(c_1, c_0\right), Fc_0\right]
$$

Since Fc_0 is a Kan complex, corollary 1.3.14 and proposition 4.4.18 imply that $\mathrm{C}^n(G, \mathbb{C}, F)$ is also a Kan complex. The same results, plus the fact that the class of Kan fibrations is closed under pullbacks (proposition A.3.12), can then be used to show that the matching morphisms for $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ are Kan fibrations. Thus, the cobar complex is Reedy-fibrant.

(ii). Theorem 1.5.18 says Tot sends Reedy fibrations in $[\mathbf{\Delta}, \mathbf{sSet}]$ to Kan fibrations in **sSet**, so $\mathrm{C}(G, \mathbb{C}, F)$ is a Kan complex when $\mathrm{C}^\bullet(G, \mathbb{C}, F)$ is Reedy-fibrant. ■

Corollary 1.7.3. **Kan** *is closed in* **sSet** *under cobar complexes and cobar constructions.* ■

Definition 1.7.4. Let C be a category and let $F, F' : C \to \mathbf{sSet}$ be functors. A **natural weak homotopy equivalence** $F \Rightarrow F'$ is a natural transformation whose components are weak homotopy equivalences of simplicial sets.

Proposition 1.7.5. *Let $\mathbb{C}$ be a small category.*

(i) *For each weight $G : \mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, if $\varphi : F \Rightarrow F'$ is a natural weak homotopy equivalence of diagrams $\mathbb{C} \to \mathbf{sSet}$, then the induced morphism $\mathrm{B}(G, \mathbb{C}, \varphi) : \mathrm{B}(G, \mathbb{C}, F) \to \mathrm{B}(G, \mathbb{C}, F')$ is a weak homotopy equivalence of simplicial sets.*

(ii) *For each diagram $F : \mathbb{C} \to \mathbf{sSet}$, if $\psi : G \Rightarrow G'$ is a natural weak homotopy equivalence of weights $\mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, then the induced morphism $\mathrm{B}(\psi, \mathbb{C}, F) : \mathrm{B}(G, \mathbb{C}, F) \to \mathrm{B}(G', \mathbb{C}, F)$ is a weak homotopy equivalence of simplicial sets.*

Dually:

(i′) *For each weight $G : \mathbb{C} \to \mathbf{sSet}$, if $\varphi : F \Rightarrow F'$ is a natural weak homotopy equivalence of diagrams $\mathbb{C} \to \mathbf{Kan}$, then the induced morphism $\mathrm{C}(G, \mathbb{C}, \varphi) : \mathrm{C}(G, \mathbb{C}, F) \to \mathrm{C}(G, \mathbb{C}, F')$ is a weak homotopy equivalence of Kan complexes.*

(ii′) *For each diagram $F : \mathbb{C} \to \mathbf{Kan}$, if $\psi : G \Rightarrow G'$ is a natural weak homotopy equivalence of weights $\mathbb{C}^{\mathrm{op}} \to \mathbf{sSet}$, then the induced morphism $\mathrm{C}(\psi, \mathbb{C}, F) : \mathrm{C}(G, \mathbb{C}, F) \to \mathrm{C}(G', \mathbb{C}, F)$ is a weak homotopy equivalence of Kan complexes.*

Proof. Use proposition 1.3.19 and corollary 4.4.19 to show that the corresponding morphisms of bar/cobar complexes is a Reedy weak homotopy equivalence, and then apply theorems 1.5.9 and 1.5.18 to deduce that the induced morphism of bar/cobar constructions is a weak homotopy equivalence. ∎

Lemma 1.7.6. *Let $\mathbb{C}$ be a small category. For each object c in $\mathbb{C}$, the bar construction $\mathrm{B}\left(\Delta 1, \mathbb{C}, \mathit{ћ}^c\right)$ is a weakly contractible simplicial set.*

Proof. It is not hard to see that $\mathrm{B}\left(\Delta 1, \mathbb{C}, \mathit{ћ}^c\right)$ is naturally isomorphic to the nerve $\mathrm{N}\left({}^{c/}\mathbb{C}\right)$, and since ${}^{c/}\mathbb{C}$ has an initial object, the unique functor ${}^{c/}\mathbb{C} \to \mathbb{1}$ has a left adjoint. Using proposition 1.2.1, we may obtain a backward contracting homotopy for $\mathrm{N}\left({}^{c/}\mathbb{C}\right)$ onto Δ^0, and thus, by proposition 1.4.26, $\mathrm{B}\left(\Delta 1, \mathbb{C}, \mathit{ћ}^c\right)$ is a weakly contractible simplicial set. ∎

Proposition 1.7.7. *Let $F : \mathbb{C} \to \mathbf{sSet}$ be a small diagram.*

(i) *There is a natural weak homotopy equivalence $\bar{\varepsilon}_F : \mathrm{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$ that is natural in F.*

(ii) *The induced morphism*

$$\varinjlim_{\mathbb{C}} \bar{\varepsilon}_{\mathrm{B}(\mathbb{C},\mathbb{C},F)} : \varinjlim_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, \mathrm{B}(\mathbb{C}, \mathbb{C}, F)) \to \varinjlim_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, F)$$

is a weak homotopy equivalence of simplicial sets.

(iii) *The bar resolution functor $\mathrm{B}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ preserves natural weak homotopy equivalences.*

Dually:

(i′) *There is a natural weak homotopy equivalence $\bar{\eta}_F : F \Rightarrow \mathrm{C}(\mathbb{C}, \mathbb{C}, F)$ that is natural in F.*

(ii′) *If Fc is a Kan complex for each object c in $\mathbb{C}$, then the induced morphism*

$$\varprojlim_{\mathbb{C}} \bar{\eta}_{\mathrm{C}(\mathbb{C},\mathbb{C},F)} : \varprojlim_{\mathbb{C}} \mathrm{C}(\mathbb{C}, \mathbb{C}, F) \to \varprojlim_{\mathbb{C}} \mathrm{C}(\mathbb{C}, \mathbb{C}, \mathrm{C}(\mathbb{C}, \mathbb{C}, F))$$

is a weak homotopy equivalence of Kan complexes.

(iii′) *The cobar resolution functor $\mathrm{C}(\mathbb{C}, \mathbb{C}, -) : [\mathbb{C}, \mathbf{sSet}] \to [\mathbb{C}, \mathbf{sSet}]$ preserves natural weak homotopy equivalences.*

Proof. We will prove the first set of claims; the second set can be proved in a similar way.

(i). Proposition 1.4.26 applied to proposition 1.6.27 implies that the components of the natural transformation $\varepsilon_F : \mathrm{B}_\bullet(\mathbb{C}, \mathbb{C}, F) \Rightarrow \Delta \circ F$ are (halves of) simplicial homotopy equivalences. Lemma 1.5.7 plus corollary 2.3.4 then says that the induced morphisms $\mathrm{B}\left(h^c, \mathbb{C}, F\right) \to Fc$ are (halves of) simple homotopy equivalences, and so $\left|\varepsilon_F\right| : \mathrm{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$ is a componentwise weak equivalence, by proposition 1.4.20. It is clear that $\left|\varepsilon_F\right|$ is natural in F and is the required natural weak homotopy equivalence $\mathrm{B}(\mathbb{C}, \mathbb{C}, F) \Rightarrow F$.

(ii). Proposition 1.6.24 and remark A.5.11 imply there is a natural isomorphism $\varinjlim_{\mathbb{C}} \mathrm{B}(\mathbb{C}, \mathbb{C}, F) \cong \mathrm{B}(\Delta 1, \mathbb{C}, F)$. Lemma 1.7.6 implies the unique natural transformation $\mathrm{B}(\Delta 1, \mathbb{C}, \mathbb{C}) \Rightarrow \Delta 1$ is a natural weak homotopy equivalence, and therefore (by proposition 1.7.5) the induced morphism

$$\mathrm{B}(\mathrm{B}(\Delta 1, \mathbb{C}, \mathbb{C}), \mathbb{C}, F) \to \mathrm{B}(\Delta 1, \mathbb{C}, F)$$

is a weak homotopy equivalence. Using the fact that the functor $\mathrm{B}(-, \mathbb{C}, F)$ preserves weighted **sSet**-colimits, it can be shown that the following diagram commmutes,

$$\begin{tikzcd}
\mathrm{B}(\Delta 1, \mathbb{C}, \mathrm{B}(\mathbb{C}, \mathbb{C}, F)) \arrow[r, "\alpha_F"] \arrow[d, "\varinjlim_{\mathbb{C}} \bar{\varepsilon}_{\mathrm{B}(\Delta 1, \mathbb{C}, F)}"'] & \mathrm{B}(\mathrm{B}(\Delta 1, \mathbb{C}, \mathbb{C}), \mathbb{C}, F) \arrow[d] \\
\mathrm{B}(\Delta 1, \mathbb{C}, F) \arrow[r, equal] & \mathrm{B}(\Delta 1, \mathbb{C}, F)
\end{tikzcd}$$

where α_F is the natural isomorphism of theorem 1.6.23, and thus we may deduce that $\bar{\varepsilon}_{\mathrm{B}(\Delta 1, \mathbb{C}, F)}$ is indeed a weak homotopy equivalence.

(iii). Consider a natural transformation $\varphi : F \Rightarrow F'$ such that the components of φ are weak homotopy equivalences. Then, we have the following commutative diagram of natural transformations:

$$\begin{tikzcd}
\mathrm{B}(\mathbb{C}, \mathbb{C}, F) \arrow[r, "\left|\varepsilon_F\right|"] \arrow[d, "\mathrm{B}(\mathbb{C}, \mathbb{C}, \varphi)"'] & F \arrow[d, "\varphi"] \\
\mathrm{B}(\mathbb{C}, \mathbb{C}, F') \arrow[r, "\left|\varepsilon_{F'}\right|"'] & F'
\end{tikzcd}$$

The 2-out-of-3 property for weak homotopy equivalences then implies that the components of $\mathrm{B}(\mathbb{C}, \mathbb{C}, \varphi)$ must also be weak homotopy equivalences. ■

Definition 1.7.8. Let $\mathbb{C}$ be a small category.

- A **homotopy colimit functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ is a homotopical left approximation for the functor $\varinjlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.
- A **homotopy limit functor** for diagrams $\mathbb{C} \to \mathbf{sSet}$ is a homotopical right approximation for the functor $\varprojlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$.

Theorem 1.7.9. *Let $\mathbb{C}$ be a small category and let $R : \mathbf{sSet} \to \mathbf{sSet}$ be a fibrant replacement functor for* **sSet***.*

(i) *The functor $\varinjlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ sends natural weak homotopy equivalences between diagrams of the form $\mathrm{B}(\mathbb{C}, \mathbb{C}, F)$ to weak homotopy equivalences in* **sSet***.*

(ii) $\mathrm{B}(\mathbb{C}, \mathbb{C}, -)$ *is (the functor part of) a functorial left deformation retract for $\varinjlim_{\mathbb{C}}$.*

(iii) $\mathrm{B}(\Delta 1, \mathbb{C}, -)$ *is a homotopy colimit functor for diagrams $\mathbb{C} \to \mathbf{sSet}$.*

Dually:

(i′) *The functor $\varprojlim_{\mathbb{C}} : [\mathbb{C}, \mathbf{sSet}] \to \mathbf{sSet}$ sends natural weak homotopy equivalences between diagrams of the form $\mathrm{C}(\mathbb{C}, \mathbb{C}, RF)$ to weak homotopy equivalences in* **sSet***.*

(ii′) $\mathrm{C}(\mathbb{C}, \mathbb{C}, -)$ *is (the functor part of) a functorial right deformation retract for $\varprojlim_{\mathbb{C}}$.*

(iii′) $\mathrm{C}(\Delta 1, \mathbb{C}, R-)$ *is a homotopy limit functor for diagrams $\mathbb{C} \to \mathbf{sSet}$.*

Proof. We will prove the first set of claims; the second set can be proved in a similar way.

(i) and (ii). Propositions 1.6.24, 1.7.5, and 1.7.7 together imply that $\mathrm{B}(\mathbb{C}, \mathbb{C}, -)$ and $\bar{\varepsilon} : \mathrm{B}(\mathbb{C}, \mathbb{C}, -) \Rightarrow \mathrm{id}_{\mathbf{sSet}}$ satisfy the hypotheses of proposition 3.4.5 and so constitute a left deformation for the functor $\varinjlim_{\mathbb{C}}$.

(iii). Apply theorem 3.4.10. ■

— II —

Simplicial categories

2.1 Basics

Prerequisites. §§ 0.2, 1.1, 1.2, A.2.

In this section, we use the explicit universe convention.

Definition 2.1.1. A **simplicial category** $\mathcal{C}_\bullet$ consists of the following data:

- For each natural number n, a category $\mathcal{C}_n$.
- For each natural number n and $0 \le i \le n$, a functor $d_i^n : \mathcal{C}_n \to \mathcal{C}_{n-1}$ and a functor $s_i^n : \mathcal{C}_n \to \mathcal{C}_{n+1}$.

These functors are moreover required to satisfy the simplicial identities. The **underlying category** of $\mathcal{C}_\bullet$ is the category $\mathcal{C}_0$.

Remark 2.1.2. In short, a simplicial category is a simplicial object in the meta-category of all categories. Thus, we may refer to the functors d_i^n and s_i^n as **face operators** and **degeneracy operators**, just as in the general case.

Definition 2.1.3. Given two simplicial categories $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$, a **simplicial functor** $F_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$ consists of a functor $F_n : \mathcal{C}_n \to \mathcal{D}_n$ for each natural number n, such that the functors F_n are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n F_n = F_{n-1} d_i^n \qquad\qquad s_i^n F_n = F_{n+1} s_i^n$$

Definition 2.1.4. Given two simplicial functors $F_\bullet, F'_\bullet : \mathcal{C}_\bullet \to \mathcal{D}_\bullet$, a **simplicial natural transformation** $\varphi_\bullet : F_\bullet \Rightarrow F'_\bullet$ consists of a natural transformation

$\varphi_n : F_n \Rightarrow F'_n$ for each natural number n, such that the natural transformations φ_n are compatible with the face and degeneracy operators in the obvious sense:

$$d_i^n \varphi_n = \varphi_{n-1} d_i^n \qquad\qquad s_i^n \varphi_n = \varphi_{n+1} s_i^n$$

Definition 2.1.5. Let $\mathbf{U}$ be a universe. A **U-small** (resp. **locally U-small**) **simplicial category** is a simplicial category $\mathcal{C}_\bullet$ such that each $\mathcal{C}_n$ is $\mathbf{U}$-small (resp. locally $\mathbf{U}$-small).

Example 2.1.6. If $\mathcal{C}$ is a $\mathbf{U}$-small category, then we have a $\mathbf{U}$-small constant simplicial category $\mathcal{C}_\bullet$, where $\mathcal{C}_n = \mathcal{C}$ for all n, with the trivial face and degeneracy operators.

Definition 2.1.7. The **bisimplicial nerve** of a simplicial category $\mathcal{C}_\bullet$ is the bisimplicial set $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ defined by the following formula:

$$(\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)_n)_m = \mathrm{N}(\mathcal{C}_m)_n$$

In other words, the m-simplices of the n-th level of $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ are the composable strings of morphisms in $\mathcal{C}_m$ of length n.

Example 2.1.8. Let $\mathcal{C}$ be an ordinary category, and consider the simplicial category $\mathcal{C}_\bullet$ defined by $\mathcal{C}_n = [\mathbf{I}[n], \mathcal{C}]$, where $\mathbf{I}[n]$ denotes the groupoid obtained by freely inverting all the arrows in $[n]$. The bisimplicial nerve $\mathrm{N}^{\mathrm{ss}}(\mathcal{C}_\bullet)$ is then (isomorphic to) the **classifying diagram** of $\mathcal{C}$, in the sense of Rezk [2001].

Proposition 2.1.9. *Let* $\mathbf{U}$ *be a universe, let* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *be the category of* $\mathbf{U}$*-small simplicial categories, and let* $\mathbf{ssSet}$ *be the category of bisimplicial sets.*

(i) $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *is a locally finitely presentable* $\mathbf{U}$*-category.*

(ii) $\mathrm{N}^{\mathrm{ss}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{ssSet}$ *is a fully faithful* $\aleph_0$*-accessible functor.*

(iii) N^{ss} *has a left adjoint.*

Proof. (i). This is an instance of proposition 0.2.41.

(ii). That $\mathrm{N}^{\mathrm{ss}} : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{ssSet}$ is a fully faithful $\aleph_0$-accessible functor essentially follows from the fact that $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is so: see proposition 1.2.1 and the accessible adjoint functor theorem (0.2.47).

(iii). It is also clear that N^{ss} preserves limits for $\mathbf{U}$-small diagrams, so we may apply the accessible adjoint functor theorem to construct a left adjoint for N^{ss}. ■

Definition 2.1.10. A **simplicially enriched category** $\underline{C}$ consists of the following data:

- A set of objects, $\operatorname{ob} C$.
- A simplicial set of morphisms, $\operatorname{mor} \underline{C}$.
- A pair of simplicial maps $\operatorname{dom}, \operatorname{codom} : \operatorname{mor} \underline{C} \to \operatorname{disc} \operatorname{ob} C$.
- For each element C of $\operatorname{ob} C$, a vertex id_C in $\operatorname{mor} \underline{C}$ such that $\operatorname{dom} \mathrm{id}_C = C$ and $\operatorname{codom} \mathrm{id}_C = C$.
- A simplicial map $\underline{C}^{[2]} \to \operatorname{mor} \underline{C}$, written as $(\beta, \alpha) \mapsto \beta \circ \alpha$, where $\underline{C}^{[2]}$ is the simplicial set defined by the following pullback diagram:

$$\begin{tikzcd}
\underline{C}^{[2]} \arrow[r] \arrow[d] & \operatorname{mor} \underline{C} \arrow[d, "\operatorname{codom}"] \\
\operatorname{mor} \underline{C} \arrow[r, "\operatorname{dom}"'] & \operatorname{disc} \operatorname{ob} C
\end{tikzcd}$$

These are moreover required to satisfy the following condition:

- For each natural number n, the given identities and binary operation induce a category with $\operatorname{ob} C$ for its object-set and $(\operatorname{mor} \underline{C})_n$ for its morphism-set.

As usual, we write $\underline{C}(C, C')$ for the simplicial subset of $\operatorname{mor} \underline{C}$ consisting of those simplices α such that $\operatorname{dom} \alpha = C$ and $\operatorname{codom} \alpha = C'$.

The **underlying category** of a simplicial category $\underline{C}$ is the category C obtained by taking $C(C', C) = \underline{C}(C', C)_0$, with the evident identity morphisms and induced composition. By **object** or **morphism** in $\underline{C}$, we shall always mean an object or morphism in the underlying category C.

Remark 2.1.11. It is clear from the definition that a simplicially enriched category $\underline{C}$ induces a simplicial category $C_\bullet$, but not every simplicial category arises in this fashion: simplicially enriched categories correspond to the simplicial categories $C_\bullet$ where $\operatorname{ob} C_\bullet$ is a constant simplicial set.

Definition 2.1.12. Given two simplicially enriched categories $\underline{C} \to \underline{D}$, a **simplicially enriched functor** $\underline{F} : \underline{C} \to \underline{D}$ consists of a map $\operatorname{ob} F : \operatorname{ob} C \to \operatorname{ob} D$ and a simplicial map $\operatorname{mor} \underline{F} : \operatorname{mor} \underline{C} \to \operatorname{mor} \underline{D}$ that respect the structure of simplicially enriched categories in the obvious sense.

Remark 2.1.13. There is a natural bijection between simplicially enriched functors $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ and simplicial functors $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$, where $\mathcal{C}_\bullet$ and $\mathcal{D}_\bullet$ are the simplicial categories associated with $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$.

Of course, just as in the simplicial case, a simplicially enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ has a underlying functor $F : \mathcal{C} \to \mathcal{D}$ between the underlying categories.

Definition 2.1.14. Given two simplicially enriched functors $\underline{F}, \underline{F'} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, a **simplicially enriched natural transformation** $\varphi : \underline{F} \Rightarrow \underline{F'}$ consists of a morphism $\varphi_C : FC \to F'C$ in $\mathcal{D}$ for each object C in $\mathcal{C}$, such that the following diagram commutes for all pairs (C, C'):

$$
\begin{tikzcd}
 & \underline{\mathcal{D}}(FC, FC') \arrow[dr, "\underline{\mathcal{D}}(FC,\varphi_{C'})"] & \\
\underline{\mathcal{C}}(C, C') \arrow[ur, "\mathrm{mor}\,\underline{F}"] \arrow[dr, "\mathrm{mor}\,\underline{F'}"'] & & \underline{\mathcal{D}}(FC, F'C') \\
 & \underline{\mathcal{D}}(F'C, F'C') \arrow[ur, "\underline{\mathcal{D}}(\varphi_C,FC')"'] &
\end{tikzcd}
$$

Remark 2.1.15. It is not hard to see that any simplicially enriched natural transformation has an underlying natural transformation; but unlike simplicially enriched functors, being a simplicially enriched natural transformation merely a property, rather than an extra structure. Less obviously, the bijection between simplicially enriched functors and simplicial functors also extends to a bijection between simplicially enriched natural transformations and simplicial natural transformations.

Definition 2.1.16. Let $\mathbf{U}$ be a universe. A **$\mathbf{U}$-small simplicially enriched category** is a simplicially enriched category $\underline{\mathcal{C}}$ such that $\mathrm{ob}\,\mathcal{C}$ is a $\mathbf{U}$-set and $\mathrm{mor}\,\underline{\mathcal{C}}$ is a simplicial $\mathbf{U}$-set. A **locally $\mathbf{U}$-small simplicially enriched category** is a simplicially enriched category $\underline{\mathcal{C}}$ such that $\mathrm{ob}\,\mathcal{C}$ is a $\mathbf{U}$-class and, for each pair (C', C) of elements of $\mathrm{ob}\,\mathcal{C}$, the simplicial set $\underline{\mathcal{C}}(C', C)$ is a simplicial $\mathbf{U}$-set.

Remark 2.1.17. If $\mathbf{sSet}$ is the category of simplicial $\mathbf{U}$-sets, then a locally $\mathbf{U}$-small simplicially enriched category is essentially the same thing as a $\mathbf{sSet}$-enriched category whose object-set is a $\mathbf{U}$-class.

Example 2.1.18. Since $\mathbf{sSet}$ is a cartesian closed category (theorem A.2.22), we may make it a simplicially enriched category by taking $\underline{\mathbf{sSet}}(X, Y) = [X, Y]$.

This is a locally $\mathbf{U}$-small simplicially enriched category, since each $[X, Y]$ is a simplicial $\mathbf{U}$-set.

Definition 2.1.19. A **discrete simplicially enriched category** is a simplicially enriched category $\underline{\mathcal{C}}$ such that $\operatorname{mor} \underline{\mathcal{C}}$ is a constant simplicial set.

Proposition 2.1.20. *Let $\mathbf{U}$ be a universe. If $\mathcal{C}$ is a locally $\mathbf{U}$-small category, then there exists a locally $\mathbf{U}$-small discrete simplicially enriched category $\underline{\mathcal{C}}$ whose underlying category is $\mathcal{C}$ such that, for all simplicially enriched categories $\underline{\mathcal{D}}$, the map sending a simplicially enriched functor $\underline{\mathcal{C}} \to \underline{\mathcal{D}}$ to its underlying ordinary functor $\mathcal{C} \to \mathcal{D}$ is a bijection.*

Proof. Obvious. ♦

Definition 2.1.21. Given two simplicially enriched functors $\underline{F}, \underline{F}' : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$, the **simplicial set of natural transformations** $\underline{F} \Rightarrow \underline{F}'$ is the simplicial set $\underline{\operatorname{Nat}}(\underline{F}, \underline{F}')$ defined by the equaliser diagram shown below,

$$\underline{\operatorname{Nat}}(\underline{F}, \underline{F}') \longrightarrow \prod_{C \in \operatorname{ob} \mathcal{C}} \underline{\mathcal{D}}(FC, F'C) \rightrightarrows \prod_{\substack{C \in \operatorname{ob} \mathcal{C} \\ C' \in \operatorname{ob} \mathcal{C}}} \left[\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C')\right]$$

where the two maps are defined in components by the following composites, respectively,

$$\underline{\mathcal{D}}(FC, F'C) \longrightarrow \left[\underline{\mathcal{D}}(F'C, F'C'), \underline{\mathcal{D}}(FC, F'C')\right] \xrightarrow{(\operatorname{mor} \underline{F}')^*} \left[\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C')\right]$$

$$\underline{\mathcal{D}}(FC', F'C') \longrightarrow \left[\underline{\mathcal{D}}(FC, FC'), \underline{\mathcal{D}}(FC, F'C')\right] \xrightarrow{(\operatorname{mor} \underline{F})^*} \left[\underline{\mathcal{C}}(C, C'), \underline{\mathcal{D}}(FC, F'C')\right]$$

with the first arrow in each diagram being the exponential transpose of the composition map.

Proposition 2.1.22. *Let $\mathbf{U}$ be a universe.*

(i) *If $\underline{\mathcal{D}}$ and $\underline{\mathcal{E}}$ are $\mathbf{U}$-small simplicially enriched categories, then there exist a $\mathbf{U}$-small simplicially enriched category $\underline{\mathcal{D}} \times \underline{\mathcal{E}}$ and simplicially enriched functors $p_1 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \to \underline{\mathcal{D}}$ and $p_2 : \underline{\mathcal{D}} \times \underline{\mathcal{E}} \to \underline{\mathcal{E}}$ such that (p_1, p_2) induce a bijection between simplicially enriched functors $\langle \underline{F}, \underline{G} \rangle : \underline{\mathcal{C}} \to \underline{\mathcal{D}} \times \underline{\mathcal{E}}$*

and pairs $(\underline{F}, \underline{G})$ *of simplicially enriched functors, where* $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ *and* $\underline{G} : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$*, where* $\underline{\mathcal{C}}$ *varies over all simplicially enriched categories.*

(ii) *If* $\underline{\mathcal{D}}$ *is a* $\mathbf{U}$*-small simplicially enriched category and* $\underline{\mathcal{E}}$ *is a locally* $\mathbf{U}$*-small simplicially enriched category, then there exist a locally* $\mathbf{U}$*-small simplicially enriched category* $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$ *and a simplicially enriched functor* $\mathrm{ev} : [\underline{\mathcal{D}}, \underline{\mathcal{E}}] \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ *such that* ev *induces a bijection between simplicially enriched functors* $\underline{\mathcal{C}} \times \underline{\mathcal{D}} \to \underline{\mathcal{E}}$ *and simplicially enriched functors* $\underline{\mathcal{C}} \to [\underline{\mathcal{D}}, \underline{\mathcal{E}}]$*, where* $\underline{\mathcal{C}}$ *varies over all simplicially enriched categories.*

(iii) *If* $\underline{\mathcal{D}}$ *and* $\underline{\mathcal{E}}$ *are both* $\mathbf{U}$*-small simplicially enriched categories, then* $[\underline{\mathcal{D}}, \underline{\mathcal{E}}]$ *is also* $\mathbf{U}$*-small.*

Proof. Omitted, but straightforward. ◆

Proposition 2.1.23. *Let* $\mathbf{U}$ *be a universe, let* $\mathbf{SCat}$ *be the category of* $\mathbf{U}$*-small simplicially enriched categories, and let* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *be the category of* $\mathbf{U}$*-small simplicial categories.*

(i) $\mathbf{SCat}$*, regarded as a full subcategory of* $[\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$*, is closed under limits and colimits for all* $\mathbf{U}$*-small diagrams.*

(ii) $\mathbf{SCat}$ *is a cartesian closed category.*

(iii) *The inclusion* $\mathbf{SCat} \hookrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$ *has a left adjoint, and* $\mathbf{SCat}$ *is a locally finitely presentable* $\mathbf{U}$*-category.*

Proof. (i). The functor $[\mathbf{\Delta}^{\mathrm{op}}, \mathrm{ob}] : [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}] \to \mathbf{sSet}$ has a left adjoint and a right adjoint, so it follows that a limit or colimit for diagrams of simplicially enriched categories, computed as a simplicial category, will have object-space a discrete simplicial set and thus be isomorphic to a simplicially enriched category.

(ii). This is implied by proposition 2.1.22.

(iii). It is not hard to directly construct a left adjoint for the inclusion $\mathbf{SCat} \hookrightarrow [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Cat}]$, and once this is done, we may apply the classification theorem for locally presentable categories (0.2.37) to deduce (from proposition 2.1.9) that $\mathbf{SCat}$ is also locally finitely presentable. Alternatively, one may instead first show that $\mathbf{SCat}$ is locally finitely presentable and then use the accessible adjoint functor theorem (0.2.47) to construct a left adjoint for the inclusion. ◇

Definition 2.1.24. Let $\underline{C}$ be a simplicially enriched category.

- A **tensor product** of a simplicial set X and an object A in $\underline{C}$ is an object $X \odot A$ in $\underline{C}$ equipped with a family of isomorphisms of simplicial sets of the form
$$\underline{C}(X \odot A, B) \cong [X, \underline{C}(A, B)]$$
that are simplicially natural as B varies in $\underline{C}$. We may also say that $X \odot A$ is a **simplicial copower** of A by X.

- A **cotensor product** of a simplicial set X and an object B in $\underline{C}$ is an object $X \pitchfork B$ in $\underline{C}$ equipped with a family of isomorphisms of simplicial sets of the form
$$\underline{C}(A, X \pitchfork B) \cong [X, \underline{C}(A, B)]$$
that are simplicially natural as A varies in $\underline{C}$. We may also say that $X \pitchfork B$ is a **simplicial power** of B by X.

Definition 2.1.25. Let $\mathbf{U}$ be a universe and let $\underline{\mathbf{sSet}}$ be the simplically enriched category of simplicial $\mathbf{U}$-sets.

- A **simplicially tensored U-category** is a locally $\mathbf{U}$-small simplicially enriched category $\underline{C}$ equipped with a simplically enriched functor
$$\odot : \underline{\mathbf{sSet}} \times \underline{C} \to \underline{C}$$
and a family of isomorphisms in $\mathbf{sSet}$ of the form
$$\underline{C}(X \odot A, B) \cong \underline{\mathbf{sSet}}(X, \underline{C}(A, B))$$
constituting a simplicially enriched natural transformation in A, B, and X.

- A **simplicially cotensored U-category** is a locally $\mathbf{U}$-small simplicially enriched category $\underline{C}$ equipped with a simplically enriched functor
$$\pitchfork : \underline{\mathbf{sSet}}^{\mathrm{op}} \times \underline{C} \to \underline{C}$$
and a family of isomorphisms in $\mathbf{sSet}$ of the form
$$\underline{\mathbf{sSet}}(X, \underline{C}(A, B)) \cong \underline{C}(A, X \pitchfork B)$$
constituting a simplicially enriched natural transformation in A, B, and X.

Remark 2.1.26. The simplicially enriched functor $\odot$ (resp. $\pitchfork$) is unique up to unique natural isomorphism, so a locally **U**-small simplicially enriched category is a simplicially tensored (resp. cotensored) **U**-category in at most one way up to isomorphism.

Theorem 2.1.27. *Let $\underline{C}$ be a locally* **U***-small simplicially enriched category. The following are equivalent:*

(i) *$\underline{C}$ is a simplicially tensored* **U***-category.*

(ii) *$\underline{C}$ has tensor products $X \odot A$ for all simplicial* **U***-sets X and all objects A in C.*

(iii) *There exist a functor $\odot : \mathbf{sSet} \times C \to C$ and natural* bijections *of hom-sets*

$$C(X \odot A, B) \cong \mathbf{sSet}(X, \underline{C}(A, B))$$

together with a natural isomorphism $\boldsymbol{\eta} : \mathrm{id}_C \Rightarrow 1 \odot (-)$ and natural isomorphisms $\boldsymbol{\mu}_{X,Y} : X \odot (Y \odot (-)) \Rightarrow (X \times Y) \odot (-)$ satisfying the conditions in remark B.2.2.

Dually, the following are equivalent:

(i′) *$\underline{C}$ is a simplicially cotensored* **U***-category.*

(ii′) *$\underline{C}$ has cotensor products $X \pitchfork B$ for all simplicial* **U***-sets X and all objects B in C.*

(iii′) *There exist a functor $\pitchfork : \mathbf{sSet}^{\mathrm{op}} \times C \to C$ and natural* bijections *of hom-sets*

$$\mathbf{sSet}(X, \underline{C}(A, B)) \cong C(A, X \pitchfork B)$$

together with a natural isomorphism $\boldsymbol{\eta} : \mathrm{id}_C \Rightarrow 1 \pitchfork (-)$ and natural isomorphisms $\boldsymbol{\mu}_{X,Y} : X \pitchfork (Y \pitchfork (-)) \Rightarrow (X \times Y) \pitchfork (-)$ satisfying the conditions in remark B.2.2.

Proof. See [???]. ∎

Definition 2.1.28. Let $\underline{C}$ be a locally $\mathbf{U}$-small simplicially enriched category and let $F : \mathbb{D} \to C$ be a diagram in C.

- A **conical colimit** for F in $\underline{C}$ is an object A and a cocone $\lambda : F \Rightarrow \Delta A$ such that, for all objects B in C, the hom-functor $\underline{C}(-, B) : C^{\mathrm{op}} \to \mathbf{sSet}$ sends λ to a limiting cone in $\mathbf{sSet}$.
- A **conical limit** for F in $\underline{C}$ is an object B and a cone $\lambda : \Delta B \Rightarrow F$ such that, for all objects A in C, the hom-functor $\underline{C}(A, -) : C \to \mathbf{sSet}$ sends λ to a limiting cone in $\mathbf{sSet}$.

Remark 2.1.29. Every conical colimit (resp. limit) for F in $\underline{C}$ is a colimit (resp. limit) for F in the underlying category C, but the converse is not true in general.

Proposition 2.1.30. *Let $\underline{C}$ be a locally $\mathbf{U}$-small simplicially enriched category and let $F : \mathbb{D} \to C$ be a diagram in C. If $\underline{C}$ is simplicially cotensored, then the following are equivalent for any cocone $\lambda : F \Rightarrow \Delta A$:*

(i) *λ is a conical colimit for F in the simplicially enriched category $\underline{C}$.*

(ii) *λ is a colimit for F in the underlying category C.*

Dually, if $\underline{C}$ is simplicially tensored, then the following are equivalent for any cone $\lambda : \Delta B \Rightarrow F$:

(i′) *λ is a conical limit for F in the simplicially enriched category $\underline{C}$.*

(ii′) *λ is a limit for F in the underlying category C.*

Proof. This is a straightforward exercise in manipulating adjunctions and hom-functors. ◊

Definition 2.1.31. Let $\underline{C}$ be a locally $\mathbf{U}$-small simplicially enriched category, let $\underline{\mathbb{D}}$ be a $\mathbf{U}$-small simplicially enriched category, and let $\underline{F} : \underline{\mathbb{D}} \to \underline{C}$ be a simplicially enriched functor.

- Given a simplicially enriched functor $\underline{W} : \underline{\mathbb{D}}^{\mathrm{op}} \to \mathbf{sSet}$, a **$\underline{W}$-weighted colimit** for $\underline{F}$ is an object $\underline{W} \star_{\underline{\mathbb{D}}} \underline{F}$ equipped with a simplicially enriched natural isomorphism of the following form:

$$\underline{C}\big(\underline{W} \star_{\underline{\mathbb{D}}} \underline{F}, -\big) \cong [\underline{\mathbb{D}}^{\mathrm{op}}, \underline{\mathbf{sSet}}](\underline{W}, \underline{C}(\underline{F}, -))$$

- Given a simplicially enriched functor $\underline{W} : \underline{\mathbb{D}} \to \mathbf{sSet}$, a $\underline{W}$**-weighted limit** for $\underline{F}$ is an object $\{\underline{W}, \underline{F}\}^{\underline{\mathbb{D}}}$ equipped with a simplicially enriched natural isomorphism of the following form:

$$\underline{C}\big(-, \{\underline{W}, \underline{F}\}^{\underline{\mathbb{D}}}\big) \cong [\underline{\mathbb{D}}, \underline{\mathbf{sSet}}](\underline{W}, \underline{C}(-, \underline{F}))$$

Remark 2.1.32. When $\underline{\mathbb{D}}$ is the free simplicial enrichment of an ordinary category $\mathbb{D}$, ordinary cocones (resp. cones) on diagrams $F : \mathbb{D} \to C$ are automatically simplicially enriched, and thus conical colimits (resp. limits) for F are the same thing as $\Delta 1$-weighted colimits (resp. limits) for $\underline{F}$, where $\Delta 1$ denotes the constant functor with value 1 in **sSet**.

Definition 2.1.33. Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \subseteq \mathbf{U}^+$.

- A **U-cocomplete simplicially enriched category** is a locally $\mathbf{U}^+$-small simplicially enriched category $\underline{C}$ such that, for all $\mathbf{U}$-small simplicially enriched diagrams $F : \underline{\mathbb{D}} \to \underline{C}$ and all $\mathbf{U}$-small weights $W : \underline{\mathbb{D}}^{\mathrm{op}} \to \underline{\mathbf{sSet}}$, $\underline{C}$ has a W-weighted colimit for F.

- A **U-complete simplicially enriched category** is a locally $\mathbf{U}^+$-small simplicially enriched category $\underline{C}$ such that, for all $\mathbf{U}$-small simplicially enriched diagrams $F : \underline{\mathbb{D}} \to \underline{C}$ and all $\mathbf{U}$-small weights $W : \underline{\mathbb{D}} \to \underline{\mathbf{sSet}}$, $\underline{C}$ has a W-weighted limit for F.

Proposition 2.1.34. *Let $\underline{C}$ be a locally* $\mathbf{U}$*-small simplicially enriched category.*

- *$\underline{C}$ is* $\mathbf{U}$*-cocomplete if and only if $\underline{C}$ is simplicially tensored and has conical colimits for all* $\mathbf{U}$*-small diagrams.*

- *$\underline{C}$ is* $\mathbf{U}$*-complete if and only if $\underline{C}$ is simplicially cotensored and conical limits for all* $\mathbf{U}$*-small diagrams.*

- *$\underline{C}$ is both* $\mathbf{U}$*-cocomplete and* $\mathbf{U}$*-complete if and only if $\underline{C}$ is both simplicially tensored and cotensored and the* underlying *category C is* $\mathbf{U}$*-cocomplete and* $\mathbf{U}$*-complete.*

Proof. See [???]. □

2.2 Homotopical aspects

Prerequisites. §§ 1.2, 1.4, 2.1, A.4.

Definition 2.2.1. Let $\mathcal{V}$ be a category with finite products and let $F : \mathbf{sSet} \to \mathcal{V}$ be a functor that preserves finite products. The ***F*-localisation** of a locally small simplicially enriched category $\underline{C}$ is the following $\mathcal{V}$-enriched category $F[\underline{C}]$:

- The objects in $F[\underline{C}]$ are the objects in $\underline{C}$.
- For each pair (X, Y) of objects in $\underline{C}$, the hom-object $F[\underline{C}](X, Y)$ is the object $F(\underline{C}(X, Y))$.
- Identities and composition in $F[\underline{C}]$ are inherited from $\underline{C}$ via F.

Remark 2.2.2. It is clear that F-localisation is 2-functorial and moreover preserves finite products of simplicially enriched categories; unlike localisation of relative categories, F-localisation may or may not have a universal property. Nonetheless, there is always a localising functor $C \to F[\underline{C}]$ between the underlying categories.

Definition 2.2.3. Let $\underline{C}$ be a locally small simplicially enriched category. A parallel pair of morphisms $g_0, g_1 : A \to B$ in $\underline{C}$ are ***F*-homotopic** if their images under the localising functor $C \to F[\underline{C}]$ are equal, in which case we write $g_0 \overset{F}{\sim} g_1$.

Example 2.2.4. The notion of intrinsic homotopy in **sSet** is obtained as the special case where F is the connected components functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$.[1]

Definition 2.2.5. Let $\underline{C}$ be a locally small simplicially enriched category. A **weak *F*-homotopy equivalence** in $\underline{C}$ is a morphism in $\underline{C}$ whose image in $F[\underline{C}]$ is an isomorphism. An ***F*-homotopy equivalence** in $\underline{C}$ is a pair (f, g), where $f : A \to B$ and $g : B \to A$ are morphisms in $\underline{C}$ such that $g \circ f \overset{F}{\sim} \mathrm{id}_A$ and $f \circ g \overset{F}{\sim} \mathrm{id}_B$. Two morphisms $f : A \to B$ and $g : B \to A$ are **mutual *F*-homotopy inverses** when (f, g) constitute an F-homotopy equivalence.

Remark 2.2.6. By lemma A.4.14, the class of weak F-homotopy equivalences in $\underline{C}$ automatically has the 2-out-of-6 property in C.

[1] Recall proposition 1.2.4.

Lemma 2.2.7. *Let $\underline{C}$ be a locally small simplicially enriched category, let $\mathcal{V}$ be a cartesian closed category, and let $F : \mathbf{sSet} \to \mathcal{V}$ be a functor that preserves finite products.*

- *If $\underline{C}$ is tensored over* **sSet**, *$f : X \to Y$ is a weak F-homotopy equivalence in $\underline{\mathbf{sSet}}$, and $g : A \to B$ is a weak F-homotopy equivalence in $\underline{C}$, then the morphism $f \odot g : X \odot A \to Y \odot B$ is a weak F-homotopy equivalence in $\underline{C}$.*

- *If $\underline{C}$ is cotensored over* **sSet**, *$f : X \to Y$ is a weak F-homotopy equivalence in $\underline{\mathbf{sSet}}$, and $g : A \to B$ is a weak F-homotopy equivalence in $\underline{C}$, then the morphism $f \pitchfork g : Y \pitchfork A \to X \pitchfork B$ is a weak F-homotopy equivalence in $\underline{C}$.*

Proof. Since $\odot$ (resp. $\pitchfork$) is a simplicially enriched functor $\underline{\mathbf{sSet}} \times \underline{C} \to \underline{C}$ (resp. $\underline{\mathbf{sSet}}^{\mathrm{op}} \times \underline{C} \to \underline{C}$), it induces a $\mathcal{V}$-enriched functor $F[\underline{\mathbf{sSet}}] \times F[\underline{C}] \to F[\underline{C}]$ (resp. $F[\underline{\mathbf{sSet}}]^{\mathrm{op}} \times F[\underline{C}] \to F[\underline{C}]$ and so *a fortiori* must preserve weak F-homotopy equivalences. ■

Definition 2.2.8. A **simplicial homotopy** $\alpha : f_0 \Rightarrow f_1$ in a simplicially enriched category $\underline{C}$ is an edge α in $\operatorname{mor} \underline{C}$ such that $d^0(\alpha) = f_1$ and $d^1(\alpha) = f_0$. For each morphism $f : X \to Y$ in C, we define $\mathrm{id}_f : f \Rightarrow f$ to be the simplicial homotopy $s_0(f)$.

Remark 2.2.9. Because $\operatorname{ob} \underline{C}$ is a discrete set, we must have $\operatorname{dom} f_0 = \operatorname{dom} f_1$ and $\operatorname{codom} f_0 = \operatorname{codom} f_1$.

Lemma 2.2.10. *Let $\underline{C}$ be a locally small simplicially enriched category, and let $\alpha : f_0 \Rightarrow f_1$ be an intrinsic homotopy of morphisms in* **sSet**.

- *If $\underline{C}$ is tensored over* **sSet**, *then for any morphism $g : A \to B$ in $\underline{C}$, $\alpha \odot \mathrm{id}_g : f_0 \odot g \Rightarrow f_1 \odot g$ is a simplicial homotopy of morphisms in $\underline{C}$.*

- *If $\underline{C}$ is cotensored over* **sSet**, *then for any morphism $g : A \to B$ in $\underline{C}$, $\alpha \pitchfork \mathrm{id}_g : f_0 \pitchfork g \Rightarrow f_1 \pitchfork g$ is a simplicial homotopy of morphisms in $\underline{C}$.*

Proof. This is an immediate consequence of the fact that $\odot$ (resp. $\pitchfork$) is a simplicially enriched functor $\underline{\mathbf{sSet}} \times \underline{C} \to \underline{C}$ (resp. $\underline{\mathbf{sSet}}^{\mathrm{op}} \times \underline{C} \to \underline{C}$). ■

Proposition 2.2.11. *Let $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ be the connected components functor, let $\boldsymbol{\pi} : \mathbf{sSet} \to \mathbf{H}$ be the weak homotopy type functor,*[2] *and let $\underline{\mathcal{C}}$ be a locally small simplicially enriched category.*

(i) *A morphism in $\underline{\mathcal{C}}$ is a weak π_0-homotopy equivalence if and only if it is a weak $\boldsymbol{\pi}$-homotopy equivalence.*

(ii) *The localising functor $\gamma : \mathcal{C} \to \pi_0[\underline{\mathcal{C}}]$ induces a bijection between simplicially enriched functors $\underline{\mathcal{C}} \to \mathcal{D}$ and ordinary functors $\mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is an ordinary category (regarded as a simplicially enriched category via proposition 2.1.20).*

(iii) *If $\underline{\mathcal{C}}$ is moreover tensored or cotensored over* **sSet**, *then $\pi_0[\underline{\mathcal{C}}]$ is the localisation of $\mathcal{C}$ at the weak $\boldsymbol{\pi}$-equivalences.*

Proof. (i). The underlying category of the $\mathbf{H}$-enriched category $\boldsymbol{\pi}[\underline{\mathcal{C}}]$ is naturally isomorphic to the category $\pi_0[\underline{\mathcal{C}}]$, since $\mathbf{H}(1, \boldsymbol{\pi}X) \cong \pi_0 X$, and the property of being an isomorphism in a $\mathbf{H}$-enriched category depends only on the underlying category.

(ii). By proposition 1.2.4, a morphism from a simplicial set X to a discrete set Y must factor through $\pi_0 X$ in a unique way, so a simplicially enriched functor $\underline{\mathcal{C}} \to \mathcal{D}$ must factor through $\pi_0[\underline{\mathcal{C}}]$.

(iii). Simplicially tensored categories and simplicially cotensored categories are formally dual; we will prove the claim for case where $\underline{\mathcal{C}}$ is tensored over **sSet**.

First, consider a simplicial homotopy $\alpha : f_0 \Rightarrow f_1$ of morphisms $A \to B$ in $\underline{\mathcal{C}}$. Transposing across the tensor–hom adjunction yields $H : \Delta^1 \odot A \to B$ making the diagram below commute:

$$\begin{tikzcd}
\Delta^0 \odot A \arrow[d, "\delta^1 \odot \mathrm{id}_A"'] & A \arrow[l, "\eta_A"'] \arrow[d, "f_0"] \\
\Delta^1 \odot A \arrow[r, "H"] & B \\
\Delta^0 \odot A \arrow[u, "\delta^0 \odot \mathrm{id}_A"] & A \arrow[l, "\eta_A"] \arrow[u, "f_1"']
\end{tikzcd}$$

[2] Recall proposition 1.4.12.

Using lemma 2.2.7, it is not hard to see that $\delta^0 \odot \mathrm{id}_A$ and $\delta^1 \odot \mathrm{id}_A$ are π_0-homotopy equivalences in $\underline{C}$ with common π_0-homotopy inverse $\sigma^0 \odot \mathrm{id}_A$, so any functor that sends weak π_0-homotopy equivalences to isomorphisms must also identify f_0 and f_1, and hence, must factor through $\pi_0[\underline{C}]$. ■

Proposition 2.2.12. *Let $\underline{C}$ be a simplicially enriched category.*

(i) *The localising functor $C \to \pi_0[C]$ is full and surjective on objects.*

(ii) *A morphism in $\underline{C}$ is a weak $\boldsymbol{\pi}$-homotopy equivalence if and only if it has a $\boldsymbol{\pi}$-homotopy inverse.*

(iii) *Two objects in $\underline{C}$ are isomorphic in $\boldsymbol{\pi}[C]$ if and only if there is a $\boldsymbol{\pi}$-homotopy equivalence between them in $\underline{C}$.*

Proof. Claim (i) is just the observation that the canonical map $X_0 \to \pi_0 X$ is surjective, and the rest follows straightforwardly. ◆

Definition 2.2.13. The **simplicial homotopy category** of a locally small simplicially enriched category $\underline{C}$ is the $\mathbf{H}$-enriched category $\boldsymbol{\pi}[\underline{C}]$, and a **simplicial homotopy equivalence** in $\underline{C}$ is a $\boldsymbol{\pi}$-homotopy equivalence.

Remark 2.2.14. By remark 1.4.17, the notion of $\boldsymbol{\pi}$-equivalence is stable under universe enlargement, as is the notion of simplicial homotopy category.

Remark 2.2.15. It is sometimes convenient to consider other localisations; for example, if $\pi_1 : \mathbf{sSet} \to \mathbf{Grpd}$ is the fundamental groupoid functor,[3] then the 2-category $\pi_1[\underline{C}]$ has the following properties:

(i) The underlying category of $\pi_1[\underline{C}]$ is naturally isomorphic to the underlying category of $\underline{C}$ itself.

(ii) Given a parallel pair $f_0, f_1 : A \to B$ in C, there exists a 2-cell $f_0 \Rightarrow f_1$ if and only if f_0 and f_1 are $\boldsymbol{\pi}$-homotopic in $\underline{C}$.

(iii) A morphism is a simplicial homotopy equivalence in $\underline{C}$ if and only if it is an equivalence in the 2-category $\pi_1[\underline{C}]$.

However, if $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ is the fundamental category functor,[4] then the 2-category $\tau_1[\underline{C}]$ in general only enjoys the first of the above properties.

[3] Recall proposition 1.2.7.
[4] Recall proposition 1.2.1.

Definition 2.2.16. A **Dwyer–Kan equivalence of simplicially enriched categories** is a simplicially enriched functor $\underline{F} : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ such that the induced $\mathbf{H}$-enriched functor $\boldsymbol{\pi}[\underline{F}] : \boldsymbol{\pi}[\underline{\mathcal{C}}] \to \boldsymbol{\pi}[\underline{\mathcal{D}}]$ is fully faithful and essentially surjective on objects.

2.3 Simplicial and cosimplicial objects

Prerequisites. §§ 1.1, 2.1, A.6.

Definition 2.3.1. Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.

- A **realisation** of a simplicial object $A_\bullet$ in $\underline{\mathcal{M}}$ is an object $|A_\bullet|$ in $\underline{\mathcal{M}}$ with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}\big(|A_\bullet|, -\big) \cong [\boldsymbol{\Delta}, \underline{\mathbf{sSet}}]\big(\Delta^\bullet, \underline{\mathcal{M}}(A_\bullet, -)\big)$$

- A **totalisation** of a cosimplicial object $B^\bullet$ in $\underline{\mathcal{M}}$ is an object $|B^\bullet|$ in $\underline{\mathcal{M}}$ with a simplicially enriched natural isomorphism of the form below:

$$\underline{\mathcal{M}}(-, \operatorname{Tot} B^\bullet) \cong [\boldsymbol{\Delta}, \underline{\mathbf{sSet}}](\Delta^\bullet, \underline{\mathcal{M}}(-, B^\bullet))$$

Proposition 2.3.2. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *If $\underline{\mathcal{M}}$ is cocomplete as a simplicially enriched category, then realisations exist for all simplicial objects in $\underline{\mathcal{M}}$.*

- *If $\underline{\mathcal{M}}$ is complete as a simplicially enriched category, then totalisations exist for all cosimplicial objects in $\underline{\mathcal{M}}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $A_\bullet$ be a simplicial object in $\underline{\mathcal{M}}$. If $\underline{\mathcal{M}}$ is cocomplete, then there must exist an object $\Delta^\bullet \star_{\boldsymbol{\Delta}^{\mathrm{op}}} A_\bullet$ and a simplicially enriched natural isomorphism of the form below,

$$\underline{\mathcal{M}}\big(\Delta^\bullet \star_{\boldsymbol{\Delta}^{\mathrm{op}}} A_\bullet, B\big) \cong [\boldsymbol{\Delta}, \underline{\mathbf{sSet}}]\big(\Delta^\bullet, \underline{\mathcal{M}}(A_\bullet, B)\big)$$

so we may take $|A_\bullet| = \Delta^\bullet \star_{\boldsymbol{\Delta}^{\mathrm{op}}} A_\bullet$. ■

Proposition 2.3.3. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *Let X be a simplicial set and let $A_\bullet$ be a simplicial object in $\underline{\mathcal{M}}$. If $\mathcal{M}$ is cocomplete and $X \boxdot A_\bullet$ is the simplicial object defined below,*

$$(X \boxdot A_\bullet)_n = X_n \odot A_n$$

  then there is an isomorphism

$$\operatorname{Tot}(X \boxdot A_\bullet) \cong X \odot \operatorname{Tot} A_\bullet$$

  and it is natural in both X and $A_\bullet$.

- *Let X be a simplicial set and let $B^\bullet$ be a cosimplicial object in $\underline{\mathcal{M}}$. If $\mathcal{M}$ is complete and $X \boxed{\pitchfork} G^\bullet$ is the cosimplicial object defined below,*

$$(X \boxed{\pitchfork} B^\bullet)^n = X_n \pitchfork B^n$$

  then there is an isomorphism

$$|X \boxed{\pitchfork} B^\bullet| \cong X \pitchfork |B^\bullet|$$

  and it is natural in both X and $B^\bullet$.

Proof. The two claims are formally dual; we will prove the first version.

Using the calculus of ends (§ A.6), we have the following natural bijections:

$$\begin{aligned}
\mathcal{M}(X \odot |A_\bullet|, B) &\cong \mathbf{sSet}(X, \underline{\mathcal{M}}(|A_\bullet|, B)) \\
&\qquad \text{by definition} \\
&\cong \mathbf{sSet}\left(X, \int_{[n]:\mathbf{\Delta}} [\Delta^n, \underline{\mathcal{M}}(A_n, B)]\right) \\
&\qquad \text{by theorem A.6.10} \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X, [\Delta^n, \underline{\mathcal{M}}(A_n, B)]) \\
&\qquad \text{by proposition A.6.7} \\
&\cong \int_{[n]:\mathbf{\Delta}} \mathbf{sSet}(X \times \Delta^n, \underline{\mathcal{M}}(A_n, B)) \\
&\qquad \text{by exponential adjunction} \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}(X_m \times \mathbf{\Delta}([m],[n]), \underline{\mathcal{M}}(A_n, B)_m)
\end{aligned}$$

$$
\begin{aligned}
&\qquad \text{by remark A.6.5} \\
&\cong \int_{[n]:\mathbf{\Delta}} \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), \underline{\mathcal{M}}(A_n, B)_m\big)\big) \\
&\qquad \text{by exponential adjunction} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\left(X_m, \int_{[n]:\mathbf{\Delta}} \mathbf{Set}\big(\mathbf{\Delta}([m],[n]), \underline{\mathcal{M}}(A_n, B)_m\big)\right) \\
&\qquad \text{by the interchange law (theorem A.6.13)} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{Set}\big(X_m, \underline{\mathcal{M}}(A_m, B)_m\big) \\
&\qquad \text{by the Yoneda lemma for ends (proposition A.6.14)} \\
&\cong \int_{[m]:\mathbf{\Delta}} \underline{\mathcal{M}}\big(X_m \odot A_m, B\big)_m \\
&\qquad \text{by definition} \\
&\cong \int_{[m]:\mathbf{\Delta}} \mathbf{sSet}\big(\Delta^m, \underline{\mathcal{M}}(X_m \odot A_m, B)\big) \\
&\qquad \text{by the ordinary Yoneda lemma} \\
&\cong \mathcal{M}\big(\left|X \boxdot A_\bullet\right|, B\big)
\end{aligned}
$$

Applying the Yoneda lemma once more, we deduce that $\left|X \boxdot A_\bullet\right|$ is naturally isomorphic to $X \odot \left|A_\bullet\right|$. ■

Corollary 2.3.4. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *Let $f_\bullet, f'_\bullet : A_\bullet \to B_\bullet$ be a parallel pair of morphisms in $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]$. If $\underline{\mathcal{M}}$ is cocomplete as a simplicially enriched category and there exists a morphism $H : \Delta^1 \boxdot A_\bullet \to B_\bullet$ making the following diagram commute,*

$$
\begin{tikzcd}
\Delta^0 \boxdot A_\bullet \arrow[d, "\delta^1 \boxdot \mathrm{id}_{A_\bullet}"'] & A_\bullet \arrow[l, "\cong"'] \arrow[d, "f_\bullet"] \\
\Delta^1 \boxdot A_\bullet \arrow[r, "H"] & B_\bullet \\
\Delta^0 \boxdot A_\bullet \arrow[u, "\delta^0 \boxdot \mathrm{id}_{A_\bullet}"] & A_\bullet \arrow[l, "\cong"] \arrow[u, "f'_\bullet"']
\end{tikzcd}
$$

  then there is a simplicial homotopy $\alpha : |f| \Rightarrow |f'|$.

- *Let $f^\bullet, f'^\bullet : A^\bullet \to B^\bullet$ be a parallel pair of morphisms in $[\mathbf{\Delta}, \mathcal{M}]$. If $\underline{\mathcal{M}}$ is complete as a simplicially enriched category and there exists a morphism $H : A^\bullet \to \Delta^1 \boxed{\pitchfork} B^\bullet$ making the following diagram commute,*

$$\begin{tikzcd} B^\bullet \arrow[r, "\cong"] & \Delta^0 \boxed{\pitchfork} B^\bullet \\ A^\bullet \arrow[u, "f^\bullet"] \arrow[r, "H"] \arrow[d, "f'^\bullet"'] & \Delta^1 \boxed{\pitchfork} B^\bullet \arrow[u, "\delta^1 \boxed{\pitchfork} \mathrm{id}_{B^\bullet}"'] \arrow[d, "\delta^0 \boxed{\pitchfork} \mathrm{id}_{B^\bullet}"] \\ B^\bullet \arrow[r, "\cong"'] & \Delta^0 \boxed{\pitchfork} B^\bullet \end{tikzcd}$$

then there is a simplicial homotopy $\alpha : |f| \Rightarrow |f'|$.

Proof. The Yoneda lemma implies there are natural bijections

$$\mathcal{M}\big(\Delta^1 \odot A, B\big) \cong \underline{\mathcal{M}}(A, B)_1 \cong \mathcal{M}\big(A, \Delta^1 \pitchfork B\big)$$

so the required simplicial homotopy is obtained by applying realisation to the displayed diagrams. ■

Proposition 2.3.5. *Let $\underline{\mathcal{M}}$ be a locally small simplicially enriched category.*

- *If $\underline{\mathcal{M}}$ is cocomplete and cotensored, then we have the following adjunction of ordinary categories:*

$$|-| \dashv \Delta^\bullet \pitchfork (-) : \mathcal{M} \to [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]$$

- *If $\underline{\mathcal{M}}$ is complete and tensored, then we have the following adjunction of ordinary categories:*

$$\Delta^\bullet \odot (-) \dashv \mathrm{Tot} : [\mathbf{\Delta}, \mathcal{M}] \to \mathcal{M}$$

Proof. By definition, we have the following natural bijections:

$$\mathcal{M}\big(\big|A_\bullet\big|, B\big) \cong [\mathbf{\Delta}, \mathbf{sSet}]\big(\Delta^\bullet, \underline{\mathcal{M}}\big(A_\bullet, B\big)\big) \cong [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}]\big(A_\bullet, \Delta^\bullet \pitchfork B\big)$$

$$\mathcal{M}(A, \mathrm{Tot}\, B^\bullet) \cong [\mathbf{\Delta}, \mathbf{sSet}](\Delta^\bullet, \underline{\mathcal{M}}(A, B^\bullet)) \cong [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{M}](\Delta^\bullet \odot A, B^\bullet)$$ ■

2.4 Homotopy-coherent diagrams

Prerequisites. §§ 1.1, 1.2, 2.1, 2.2, 6.1.

Lemma 2.4.1. *Let* **Cat** *be the category of small categories and let* **Grph** $=$ $\mathbf{sSet}_{\leq 1}$ *be the category of reflexive graphs.*

(i) *The forgetful functor* $U : \mathbf{Cat} \to \mathbf{Grph}$ *has a left adjoint, say* F.

(ii) *The forgetful functor* $U : \mathbf{Cat} \to \mathbf{Grph}$ *preserves any colimiting cocone that* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *preserves; in particular it is an* $\aleph_0$*-accessible functor.*

(iii) *For any reflexive graph* X*, the unit morphism* $\eta_X : X \to UFX$ *is a bijection on vertices.*

Proof. (i). We may factor $U : \mathbf{Cat} \to \mathbf{Grph}$ as the nerve functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ followed by the brutal 1-truncation functor $(-)_{\leq 1} : \mathbf{sSet} \to \mathbf{Grph}$; but each of these has a left adjoint, by propositions 1.2.1 and 1.2.11.

(ii). We deduce this claim from the above discussion by noting that $(-)_{\leq 1}$ is *itself* a left adjoint; for accessibility, we appeal to the accessible adjoint functor theorem (0.2.47).

(iii). This follows straightforwardly from the explicit description of τ_1. ■

Definition 2.4.2. With notation as in the lemma, the **standard resolution** of a small category $\mathbb{C}$ is the small simplicial category $\mathbf{S}(\mathbb{C})_\bullet$ defined by the following formulae:

$$\begin{aligned} \mathbf{S}(\mathbb{C})_n &= (FU)^{n+1}(\mathbb{C}) \\ d_i^n &= (FU)^{n-i+1}\varepsilon_{(FU)^{i+1}(\mathbb{C})} \\ s_i^n &= (FU)^{n-i}F\eta_{U(FU)^i(\mathbb{C})} \end{aligned}$$

Here, η and ε are the unit and counit of the adjunction $F \dashv U : \mathbf{Cat} \to \mathbf{Grph}$. The **standard augmentation** for a category $\mathbb{C}$ is the unique simplicial functor $\left(\varepsilon_{\mathbb{C}}\right)_\bullet : \mathbf{S}(\mathbb{C})_\bullet \to \mathbb{C}$ given in degree 0 by the counit $\varepsilon_{\mathbb{C}} : FU(\mathbb{C}) \to \mathbb{C}$.

Remark 2.4.3. The fact that the above formulae do satisfy the simplicial identities is an instance of the general construction of simplicial objects using a comonad. More subtly, the fact that the standard resolution of $\mathbb{C}$ is stable under universe enlargement is an instance of the stability of accessible adjunctions.

Remark 2.4.4. Although the standard resolution $\mathbf{S}(\mathbb{C})_\bullet$ of a category $\mathbb{C}$ is most naturally defined as a simplicial category, the fact that $\operatorname{ob} \mathbf{S}(\mathbb{C})_\bullet$ is a constant simplicial set enables us to view it as a simplicially enriched category $\underline{\mathbf{S}}(\mathbb{C})$, per remark 2.1.11.

Proposition 2.4.5. *For any small category $\mathbb{C}$, the standard augmentation $\varepsilon_{\mathbb{C}} : \mathbf{S}(\mathbb{C}) \to \mathbb{C}$ is a Dwyer–Kan equivalence of simplicially enriched categories.*

Proof. Given any pair (A, B) of objects in $\mathbb{C}$, $\varepsilon_{\mathbb{C}} : \underline{\mathbf{S}}(\mathbb{C}) \to \mathbb{C}$ induces a hom-space morphism $\mathbf{S}(\mathbb{C})_\bullet(A, B) \to \mathbb{C}(A, B)$ which admits a backward contracting homotopy (defined in degree 0 by the adjunction unit), so it is a weak homotopy equivalence by propositions 1.4.20 and 1.4.26. ■

Corollary 2.4.6. *The functor $\pi_0[\underline{\mathbf{S}}(\mathbb{C})] \to \mathbb{C}$ induced by the standard augmentation is an isomorphism of categories.* ■

Definition 2.4.7. Let $\mathcal{J}$ be an ordinary category. A **homotopy-coherent diagram** of shape $\mathcal{J}$ in a simplicially enriched category $\underline{\mathcal{C}}$ is a simplicially enriched functor $\underline{\mathbf{S}}(\mathcal{J}) \to \underline{\mathcal{C}}$.

Remark 2.4.8. It is worth thinking about the data that comprise a homotopy-coherent diagram of shape $\mathcal{J}$: in degree 0, one must specify a morphism $F(f)$ in $\mathcal{C}$ for every non-trivial morphism f in $\mathcal{J}$ (but this assignment need not be functorial!); in degree 1, for every composable string of non-trivial morphisms of positive length, such as $f_3 \circ f_2 \circ f_1$, one has a simplicial homotopy from the "free" composition to the "true" composition, e.g.

$$\mu_{f_3, f_2, f_1} : F(f_3) \circ F(f_2) \circ F(f_1) \Rightarrow F(f_3 \circ f_2 \circ f_1)$$

and so on in higher degrees. The phrase 'homotopy-coherent' alludes to the relations imposed by the higher simplices: for instance, for each composable triple (f_3, f_2, f_1) as above, one has a pair of 2-cells in $\operatorname{mor} \underline{\mathcal{C}}$ as in the diagram below:

$$\begin{tikzcd}
& F(f_3 \circ f_2) \circ F(f_1) \arrow[dr, "\mu_{f_3 \circ f_2, f_1}"] \arrow[d, phantom, "\Downarrow"] & \\
F(f_3) \circ F(f_2) \circ F(f_1) \arrow[ur, "\mu_{f_3, f_2} \circ \mathrm{id}_{f_1}"] \arrow[rr, "\mu_{f_3, f_2, f_1}"'] \arrow[dr, "\mathrm{id}_{f_3} \circ \mu_{f_2, f_1}"'] & & F(f_3 \circ f_2 \circ f_1) \\
& F(f_3) \circ F(f_2 \circ f_1) \arrow[ur, "\mu_{f_3, f_2 \circ f_1}"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}$$

In particular, if $\underline{\mathcal{C}}$ is obtained from a 2-category $\mathfrak{C}$ by applying the nerve functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ to its hom-categories, a homotopy-coherent diagram of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$ is the same thing as a normalised lax 2-functor $\mathcal{J} \to \mathfrak{C}$.

Definition 2.4.9. The **homotopy-coherent nerve** of a simplicially enriched category $\underline{\mathcal{C}}$ is the simplicial set defined by the formula below,

$$\mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})_n = \{\text{simplicially enriched functors } \underline{\mathbf{S}}([n]) \to \underline{\mathcal{C}}\}$$

with face and degeneracy maps induced by the coface and codegeneracy maps in $\mathbf{\Delta}$.

Proposition 2.4.10. *Let* **SCat** *be the category of small simplicially enriched categories.*

(i) $\mathrm{N}^{\mathrm{hc}} : \mathbf{SCat} \to \mathbf{sSet}$ *has a left adjoint, which is the unique (up to unique isomorphism) colimit-preserving functor* $\underline{\mathbf{C}} : \mathbf{sSet} \to \mathbf{SCat}$ *such that* $\underline{\mathbf{C}}(\Delta^n) = \underline{\mathbf{S}}([n])$.

(ii) $\mathrm{N}^{\mathrm{hc}} : \mathbf{SCat} \to \mathbf{sSet}$ *and* $\underline{\mathbf{C}} : \mathbf{sSet} \to \mathbf{SCat}$ *are both accessible functors.*

(iii) *If* $\mathbb{C}$ *is a small category regarded as a simplicially enriched category, then* $\mathrm{N}^{\mathrm{hc}}(\mathbb{C})$ *is naturally isomorphic to* $\mathrm{N}(\mathbb{C})$.

Proof. (i). Apply theorem 1.1.11.

(ii). This is an instance of the accessible adjoint functor theorem (0.2.47).

(iii). This follows from proposition 2.2.11 and corollary 2.4.6. ∎

Definition 2.4.11. Given a simplicial set X, the **associated simplicially enriched category** is the simplicially enriched category $\underline{\mathbf{C}}(X)$ constructed above.

Remark 2.4.12. The stability of accessible adjunctions under universe enlargement implies that the simplicially enriched category $\underline{\mathbf{C}}(X)$ associated with a simplicial set X does not depend on the choice of universe.

Remark 2.4.13. One way of getting a good grip on the hom-spaces of $\underline{\mathbf{C}}(X)$ for a general simplicial set X is to use the formalism of necklaces introduced by Dugger and Spivak [2011].

Theorem 2.4.14 (Riehl)**.**

(i) *For any simplicial set X and any pair (a, b) of vertices of X, the hom-space $\underline{\mathbf{C}}(X)(a, b)$ is 3-coskeletal.*

(ii) *For any category $\mathbb{C}$ and any pair (A, B) of objects in $\mathbb{C}$, the hom-space $\underline{\mathbf{C}}(\mathrm{N}(\mathbb{C}))(A, B)$ is 2-coskeletal.*

(iii) *For any category $\mathbb{C}$, its associated simplicially enriched category $\underline{\mathbf{C}}(\mathrm{N}(\mathbb{C}))$ is naturally isomorphic to the standard resolution $\underline{\mathbf{S}}(\mathbb{C})$.*

Proof. See Theorems 4.1, 6.4, and 6.7 in [Riehl, 2011c]. □

Corollary 2.4.15. *For any simplicially enriched category $\underline{\mathcal{C}}$ and any ordinary category $\mathcal{J}$, there is a bijection*

$$\left\{\text{simplicial maps } \mathrm{N}(\mathcal{J}) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})\right\} \cong \{\textit{homotopy-coherent diagrams of shape } \mathcal{J} \textit{ in } \underline{\mathcal{C}}\}$$

and it is natural in $\mathcal{J}$ and in $\underline{\mathcal{C}}$. ■

Remark 2.4.16. The above result can also be proven directly, and the uniqueness of representations for functors up to unique isomorphism then implies that $\underline{\mathbf{C}}(\mathrm{N}(\mathcal{J}))$ must be isomorphic to $\underline{\mathbf{S}}(\mathcal{J})$.

Definition 2.4.17. A **fibrant simplicially enriched category** is a simplicially enriched category $\underline{\mathcal{C}}$ such that the hom-spaces $\underline{\mathcal{C}}(A, B)$ are Kan complexes for all pairs (A, B) of objects in $\underline{\mathcal{C}}$.

Definition 2.4.18. Let F and G be homotopy-coherent diagrams of shape $\mathcal{J}$ in a simplicially enriched category $\underline{\mathcal{C}}$. A **homotopy-coherent natural transformation** $F \Rightarrow G$ is a homotopy-coherent diagram of shape $\mathcal{J} \times [1]$ such that the restriction along $\underline{\mathbf{S}}\left(\mathrm{id}_{\mathcal{J}} \times \delta^1\right)$ is F and the restriction along $\underline{\mathbf{S}}\left(\mathrm{id}_{\mathcal{J}} \times \delta^0\right)$ is G.

Unfortunately, it is in general not possible to compose homotopy-coherent natural transformations, and even when it is possible, the composite is usually only well-defined up to higher homotopy. Instead, in good situations, what we get is a quasicategory:

Theorem 2.4.19. *Let $\mathcal{J}$ be a small category and let $\underline{\mathcal{C}}$ be a small simplicially enriched category. Consider the following simplicial set:*

$$[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}} = \left[\mathrm{N}(\mathcal{J}), \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})\right]$$

(i) *There is a natural identification of the vertices of $[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ as homotopy-coherent diagrams of shape $\mathcal{J}$ in $\underline{\mathcal{C}}$, and similarly, there is a natural identification of the edges as homotopy-coherent natural transformations.*

(ii) *If $\underline{\mathcal{C}}$ is fibrant, then the homotopy-coherent nerve $\mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})$ is a small quasicategory.*

(iii) *Under the same hypothesis, $[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is a small quasicategory.*

Proof. (i). Apply corollary 2.4.15 to the explicit description of exponential objects in the category of simplicial **U**-sets.

(ii). See Theorem 2.1 in [Cordier and Porter, 1986].

(iii). Use corollary 6.2.13. □

Let us say that a locally small simplicially enriched category $\underline{\mathcal{C}}$ admits **rectification for homotopy-coherent diagrams** if, for all small categories $\mathcal{J}$, we have a commutative diagram of functors of the form below,

$$\begin{tikzcd} {[\mathcal{J}, \mathcal{C}]} \arrow[r] \arrow[d] & \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}} \\ \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] \arrow[ur, dashed] & \end{tikzcd}$$

where $[\mathcal{J}, \mathcal{C}] \to \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is the functor

$$[\mathcal{J}, \mathcal{C}] \cong \tau_1\mathrm{N}([\mathcal{J}, \mathcal{C}]) \cong \tau_1[\mathrm{N}(\mathcal{J}), \mathrm{N}(\mathcal{C})] \to \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$$

induced by the canonical morphism $\mathrm{N}(\mathcal{C}) \to \mathrm{N}^{\mathrm{hc}}(\underline{\mathcal{C}})$, $[\mathcal{J}, \mathcal{C}] \to \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$ is the localising functor, and $\pi_0[[\mathcal{J}, \underline{\mathcal{C}}]] \to \tau_1[\mathcal{J}, \underline{\mathcal{C}}]_{\mathrm{hc}}$ is fully faithful and essentially surjective on objects. (Note that this functor is unique *if* it exists, because the localising functor $[\mathcal{J}, \mathcal{C}] \to \pi_0[[\mathcal{J}, \underline{\mathcal{C}}]]$ is full and bijective on objects.)

Theorem 2.4.20 (Cordier–Porter)**.** *Let $\underline{C}$ be a locally small simplicially enriched category. Consider the following conditions:*

(i) *$\underline{C}$ is fibrant and complete as a simplicially enriched category.*

(ii) *$\underline{C}$ is fibrant and cocomplete as a simplicially enriched category.*

(iii) *$\underline{C}$ is the simplicially enriched category of Kan complexes.*

If $\underline{C}$ satisfies any one of the above conditions, then $\underline{C}$ admits rectification for homotopy-coherent diagrams.

Proof. (i). See Theorem 4.7 in [Cordier and Porter, 1986].

(ii). This follows from claim (i) by duality.

(iii). See the remark following Corollary 2.3 in [Cordier and Porter, 1997]. □

2.5 Simplicial localisation

Prerequisites. §§ 1.2, 1.3, 2.1, 2.2, 2.4, 3.1, A.4.

When one passes from a relative category to its homotopy category by freely inverting the weak equivalences, one loses much of the homotopical information. Dwyer and Kan [1980a,b,c] instead proposed a more sophisticated notion of localisation that produces a simplicial category retaining all the homotopical information, at least in the case of a simplicial model category.

Definition 2.5.1. The **standard resolution** of a small relative category C is the simplicial relative category $\mathbf{S}(C)_\bullet$ where und $\mathbf{S}(C)_\bullet = \mathbf{S}(\operatorname{und} C)_\bullet$ and weq $\mathbf{S}(C)_\bullet = \mathbf{S}(\operatorname{weq} C)_\bullet$. The **standard simplicial localisation** of C is the simplicial category $\mathbf{Lo}(C)_\bullet$ obtained by applying Ho to $\mathbf{S}(C)_\bullet$ degreewise, and the **simplicial localising functor** is the induced simplicial functor $\mathbf{S}(C)_\bullet \to \mathbf{Lo}(C)_\bullet$.

Remark 2.5.2. As in remark 2.4.4, the face and degeneracy operators of the simplicial category $\mathbf{Lo}(C)_\bullet$ are trivial, so we may regard it as a simplicially enriched category $\underline{\mathbf{Lo}}(C)$.

Proposition 2.5.3. *Let C be a small relative category. The standard augmentation for C induces an isomorphism $\pi_0[\underline{\mathbf{Lo}}(C)] \to \operatorname{Ho} C$.*

Proof. Let $\mathcal{D}$ be an ordinary category and let $F : \mathcal{C} \to \mathcal{D}$ be a functor that sends weak equivalences in $\mathcal{C}$ to isomorphisms in $\mathcal{D}$. Then, composing with the standard augmentation $(\varepsilon_{\mathcal{C}})_\bullet : \mathbf{S}(\mathcal{C})_\bullet \to \mathcal{C}$ yields a simplicial functor $\mathbf{S}(\mathcal{C})_\bullet \to \mathcal{D}$ that sends weak equivalences in each $\mathbf{S}(\mathcal{C})_n$ to isomorphisms in $\mathcal{D}$, so the degreewise universal property of $\mathbf{Lo}(\mathcal{C})_\bullet$ yields a unique simplicial functor $\mathbf{Lo}(\mathcal{C})_\bullet \to \mathcal{D}$ making the diagram below commute (strictly),

$$\begin{tikzcd}
\mathbf{S}(\mathcal{C})_\bullet \arrow[r, "(\varepsilon_{\mathcal{C}})_\bullet"] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\mathbf{Lo}(\mathcal{C})_\bullet \arrow[r, dashed] & \mathcal{D}
\end{tikzcd}$$

where $\mathbf{S}(\mathcal{C})_\bullet \to \mathbf{Lo}(\mathcal{C})_\bullet$ is the simplicial localising functor. $\mathcal{D}$ is an ordinary category, so proposition 2.2.11 says the corresponding simplicially enriched functor $\underline{\mathbf{Lo}}(\mathcal{C}) \to \mathcal{D}$ factors through the π_0-localising functor $\underline{\mathbf{Lo}}(\mathcal{C}) \to \pi_0[\underline{\mathbf{Lo}}(\mathcal{C})]$ in a unique way. Thus, $\pi_0[\underline{\mathbf{Lo}}(\mathcal{C})]$ has the universal property of $\operatorname{Ho} \mathcal{C}$, and the required isomorphism $\pi_0[\underline{\mathbf{Lo}}(\mathcal{C})] \to \operatorname{Ho} \mathcal{C}$ is induced by the ordinary localising functor $\mathcal{C} \to \operatorname{Ho} \mathcal{C}$. ∎

Proposition 2.5.4. *Let $\mathcal{C}$ be a small relative category. The following are equivalent for a morphism $f : X \to Y$ in $\mathcal{C}$:*

(i) *The morphism $f : X \to Y$ is a weak equivalence in $\mathcal{C}$.*

(ii) *The morphism in $\underline{\mathbf{Lo}}(\mathcal{C})$ corresponding to $f : X \to Y$ is an isomorphism.*

(iii) *The morphism in $\mathbf{Lo}(\mathcal{C})_0$ corresponding to $f : X \to Y$ is an isomorphism.*

Proof. (i) ⇒ (ii). For each natural number n, the morphism in $\mathbf{S}(\mathcal{C})_n$ corresponding to $f : X \to Y$ is a weak equivalence (by definition), so its image in $\mathbf{Lo}(\mathcal{C})_n$ is an isomorphism. Thus, the morphism corresponding to f in the simplicially enriched category $\underline{\mathbf{Lo}}(\mathcal{C})$ is an isomorphism.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Since $\operatorname{und} \mathbf{S}(\mathcal{C})_0$ and $\operatorname{weq} \mathbf{S}(\mathcal{C})_0$ are free categories, the morphisms in $\mathbf{Lo}(\mathcal{C})_0$ can be represented by reduced composable strings generated by morphisms in $\operatorname{und} \mathcal{C}$ and the formal inverses of morphisms in $\operatorname{weq} \mathcal{C}$. Thus, a morphism in $\mathcal{C}$ corresponds to an isomorphism in $\mathbf{Lo}(\mathcal{C})_0$ if and only if it is a weak equivalence in $\mathcal{C}$. ∎

Definition 2.5.5. Let X and Y be objects in a relative category C. A **hammock** in C from X to Y of width k and length n is a commutative diagram in C of the form below,

$$
\begin{array}{ccccccccccccc}
X & \text{---} & Z_{0,1} & \text{---} & Z_{0,2} & \text{---} & \cdots & \text{---} & Z_{0,n-2} & \text{---} & Z_{0,n-1} & \text{---} & Y \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
X & \text{---} & Z_{1,1} & \text{---} & Z_{1,2} & \text{---} & \cdots & \text{---} & Z_{1,n-2} & \text{---} & Z_{1,n-1} & \text{---} & Y \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots & & \ddots & & \vdots & & \vdots & & \vdots \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
X & \text{---} & Z_{k-1,1} & \text{---} & Z_{k-1,2} & \text{---} & \cdots & \text{---} & Z_{k-1,n-2} & \text{---} & Z_{k-1,n-1} & \text{---} & Y \\
\| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
X & \text{---} & Z_{k,1} & \text{---} & Z_{k,2} & \text{---} & \cdots & \text{---} & Z_{k,n-2} & \text{---} & Z_{k,n-1} & \text{---} & Y
\end{array}
$$

such that the following conditions are satisfied:

- In each column, all horizontal arrows point in the same direction.
- All leftward-pointing arrows are weak equivalences.
- All vertical arrows are weak equivalences.

We allow both k and n to be zero; if $n = 0$ then we must have $X = Y$.

A **reduced hammock** in C is a hammock with these additional properties:

- In each column, not every horizontal arrow is an identity morphism.
- Arrows in adjacent columns point in opposite directions.

¶ 2.5.6. It is clear that we can transform any hammock into a reduced hammock by iteratively omitting any column of identity morphisms and composing any adjacent columns where possible.

Definition 2.5.7. Let C be a small relative category. The **hammock localisation** of C is the simplicial category $\underline{\mathbf{Lo}}^{\mathrm{H}}(C)$ defined below:

- The objects are those in C.

- For each pair (X, Y) of objects, the hom-space $\underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X, Y)$ is the simplicial set whose k-simplices are the reduced hammocks of width k (and any length), with face (resp. degeneracy) operators defined by omitting (resp. repeating) a row of objects and reducing the resulting hammock if necessary.

- Composition is defined by concatenation of hammocks (reducing as necessary), and identities are hammocks of zero length.

REMARK 2.5.8. The hom-space $\underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X, Y)$ can be constructed as a colimit as shown below,

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(C)(X, Y) \cong \varinjlim_{\mathbf{T}^{\mathrm{op}}} \mathrm{N}(C^*(X, Y))$$

where $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ is the nerve functor and $C^*(X, Y) : \mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$ is the functor described in remark A.4.24.

REMARK 2.5.9. Unlike the standard simplical localisation, the hammock localisation of a relative category C is equipped with a natural functor $C \to \mathbf{Lo}^{\mathrm{H}}(C)$ that is bijective on objects and faithful (but not necessarily full).

Theorem 2.5.10. *Let C be a small relative category and let $\underline{D}$ be the following simplicially enriched category:*

- *The objects are those in C.*

- *For each pair (X, Y) of objects, the hom-space $\underline{D}(X, Y)$ is given by*

$$\underline{D}(X, Y)_n = \underline{\mathbf{Lo}}^{\mathrm{H}}\big(\mathbf{S}(C)_n\big)(X, Y)_n$$

  where $\mathbf{S}(C)_\bullet$ is the standard resolution of C.

- *Composition and identities are inherited from $\underline{\mathbf{Lo}}^{\mathrm{H}}\big(\mathbf{S}(C)_\bullet\big)$.*

Let $\underline{\mathbf{Lo}}(C)$ be the standard simplicial localisation of C and let $\underline{\mathbf{Lo}}^{\mathrm{H}}(C)$ be the hammock localisation of C. Then:

(i) *The simplicially enriched functor $\underline{D} \to \underline{\mathbf{Lo}}^{\mathrm{H}}(C)$ induced by the standard augmentation $\underline{\mathbf{S}}(C) \to C$ is a Dwyer–Kan equivalence.*

(ii) *The simplicially enriched functor $\underline{D} \to \underline{\mathbf{Lo}}(C)$ induced by the localising functors $\mathbf{S}(C)_n \to \mathbf{Lo}(C)_n$ is a Dwyer–Kan equivalence.*

Proof. See Proposition 2.2 in [Dwyer and Kan, 1980b]. □

Corollary 2.5.11. *If $f : X \to Y$ is a weak equivalence in a small relative category $\mathcal{C}$, then its image in $\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})$ is a simplicial homotopy equivalence.* ■

Definition 2.5.12. A **Dwyer–Kan equivalence of relative categories** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ such that the induced simplicially enriched functor $\underline{\mathbf{Lo}}^{\mathrm{H}}(F) : \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C}) \to \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{D})$ is a Dwyer–Kan equivalence of simplically enriched categories.

Definition 2.5.13. Let $\mathcal{C}_\bullet$ be a simplicial category. The **flattening** of $\mathcal{C}_\bullet$ is the relative category $\mathcal{C}^\flat$ defined below:

- The objects are pairs (n, X), where n is a natural number and X is an object in $\mathcal{C}_n$.
- A morphism $(n, X) \to (m, Y)$ is a pair (φ, f), where $\varphi : [m] \to [n]$ is a morphism in $\mathbf{\Delta}$ and $f : \varphi^* X \to Y$ is a morphism in $\mathcal{C}_m$; a weak equivalence is any morphism of the form $(\varphi, \mathrm{id}) : (n, X) \to (m, \varphi^* X)$.
- Given morphisms $(\varphi, f) : (n, X) \to (m, Y)$ and $(\psi, g) : (m, Y) \to (l, Z)$, their composite is $(\varphi \circ \psi, g \circ \psi^* f) : (n, X) \to (l, Z)$.

In other words, $\mathcal{C}^\flat$ is the Grothendieck construction applied to $\mathcal{C}_\bullet$ considered as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Cat}$.

Theorem 2.5.14. *Let* **RelCat** *be the category of small relative categories and let* **SCat** *be the category of small simplicially enriched categories.*

(i) *There is a zigzag of natural Dwyer–Kan equivalences between* $\mathrm{id}_{\mathbf{SCat}}$ *and* $\underline{\mathbf{Lo}}^{\mathrm{H}}\big((-)^\flat\big)$.

(ii) *There is a zigzag of natural Dwyer–Kan equivalences between* $\mathrm{id}_{\mathbf{RelCat}}$ *and* $\mathbf{Lo}^{\mathrm{H}}(-)^\flat$.

(iii) *If we regard* **RelCat** *as a homotopical category where the weak equivalences are the Dwyer–Kan equivalences of relative categories, then the functors*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(-) : \mathbf{RelCat} \to \mathbf{SCat} \qquad (-)^\flat : \mathbf{SCat} \to \mathbf{RelCat}$$

are a mutually quasi-inverse pair of homotopical equivalences.

Proof. See paragraph 2.5 and Proposition 3.1 in [Barwick and Kan, 2012]. $\square$

Theorem 2.5.15 (Relative Yoneda embedding)**.** *Let $\mathcal{C}$ be a small relative category and let $\hbar_\bullet : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}$ be the relative functor defined by the formula below:*

$$\hbar_Y(X) = \underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y)$$

(i) *For each pair (X, Y) of objects in $\mathcal{C}$, the induced hom-space morphism*

$$\underline{\mathbf{Lo}}^{\mathrm{H}}(\mathcal{C})(X, Y) \to \underline{\mathbf{Lo}}^{\mathrm{H}}\big([\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}\big)(X, Y)$$

is a weak homotopy equivalence of simplicial sets.

(ii) *Let $\mathcal{D}$ be the full relative subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathbf{sSet}]_{\mathrm{h}}$ spanned by the relative functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{sSet}$ that are naturally weakly equivalent to one in the image of $\hbar_\bullet$. Then the functor $\hbar_\bullet : \mathcal{C} \to \mathcal{D}$ is a Dwyer–Kan equivalence of relative categories.*

Proof. See paragraph 4.3 in [Barwick and Kan, 2011]. $\square$

— III —

Homotopical categories

3.1 Basics

Prerequisites. § A.4.

Definition 3.1.1. A relative category $\mathcal{C}$ is a **category with weak equivalences** if it is semi-saturated and $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property, and it is a **homotopical category** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property. A **homotopical functor** is a relative functor between homotopical categories.

Remark 3.1.2. If $\mathcal{C}$ is a relative category such that $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property, then every isomorphism in $\mathcal{C}$ is automatically a weak equivalence. Indeed, suppose $f : X \to Y$ and $g : Y \to X$ are mutual inverses in $\mathcal{C}$; then the fact that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$ are in $\operatorname{weq} \mathcal{C}$ implies that f and g must also be in $\operatorname{weq} \mathcal{C}$. Recalling lemma A.4.14, it follows that every homotopical category is a category with weak equivalences.

¶ 3.1.3. To simplify notation, we will usually not distinguish between $\operatorname{und} \mathcal{C}$ and $\mathcal{C}$. For example, when $\mathcal{C}$ and $\mathcal{D}$ are relative categories, then by 'ordinary functor $\mathcal{C} \to \mathcal{D}$' we mean a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$.

Example 3.1.4. Any saturated relative category is automatically a homotopical category, by corollary A.4.15. In particular, any minimal saturated relative category is a homotopical category. On the other hand, any maximal relative category is obviously a homotopical category.

Remark 3.1.5. A relative category $\mathcal{C}$ is a category with weak equivalences or a homotopical category if and only if the opposite relative category $\mathcal{C}^{\mathrm{op}}$ is.

Lemma 3.1.6. *Let A be an object in a homotopical category (resp. category with weak equivalences) $\mathcal{C}$. Then the slice category $\mathcal{C}_{/A}$ is also a homotopical category (resp. category with weak equivalences) if we declare a morphism in $\mathcal{C}_{/A}$ to be a weak equivalence if and only if it is a weak equivalence in $\mathcal{C}$.*

Proof. Use lemma A.4.14 on the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$. ■

Lemma 3.1.7. *Any relative subcategory $\mathcal{D}$ of a homotopical category (resp. category with weak equivalences) $\mathcal{C}$ is also a homotopical category (resp. category with weak equivalences).*

Proof. Use lemma A.4.14 on the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$. ■

Lemma 3.1.8. *Let $\mathcal{C}$ be a relative category, let $\mathcal{D}$ be a saturated homotopical category, and let $F : \mathcal{C} \to \mathcal{D}$ be a relative functor. If a morphism in $\mathcal{C}$ is a weak equivalence if and only if its image under F is a weak equivalence in $\mathcal{D}$, then $\mathcal{C}$ is also a saturated homotopical category.*

Proof. Consider the induced functor $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that f is an isomorphism in $\operatorname{Ho} \mathcal{C}$. Since $\operatorname{Ho} F$ is a functor, $F f$ must be an isomorphism in $\operatorname{Ho} \mathcal{D}$; but $\mathcal{D}$ is saturated, so $F f$ is a weak equivalence in $\mathcal{D}$. We may therefore deduce that f is a weak equivalence in $\mathcal{C}$. ■

Corollary 3.1.9. *Any relative subcategory of a saturated homotopical category is a saturated homotopical category.* ■

Lemma 3.1.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be two relative categories. If $\mathcal{D}$ is a homotopical category (resp. category with weak equivalences), then the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is also a homotopical category (resp. category with weak equivalences).*

Proof. This is a straightforward check. ◆

Lemma 3.1.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be two relative categories. If $\mathcal{D}$ is a saturated homotopical category, then the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is also a saturated homotopical category.*

Proof. For each object C in $\mathcal{C}$, we have a homotopical functor $C^* : [\mathcal{C}, \mathcal{D}]_{\mathrm{h}} \to \mathcal{D}$ that evaluates an object F in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ at C. Thus, we obtain a functor $\operatorname{Ho} C^* : \operatorname{Ho} [\mathcal{C}, \mathcal{D}]_{\mathrm{h}} \to \operatorname{Ho} \mathcal{D}$.

Consider a morphism $\varphi : F \Rightarrow F'$ in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ such that φ is an isomorphism in $\operatorname{Ho}[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Since $\operatorname{Ho} C^*$ is a functor, $(\operatorname{Ho} C^*)(\varphi)$ must be an isomorphism in $\operatorname{Ho} \mathcal{C}$; but $\mathcal{C}$ is a saturated homotopical category, so that implies the component φ_C is a weak equivalence in $\mathcal{C}$. We therefore conclude that φ is a weak equivalence in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. ■

Definition 3.1.12. Two objects in a relative category are **weakly equivalent** if they can be connected by a *zigzag* of weak equivalences; we write $X \overset{\mathrm{w}}{\simeq} Y$ to mean that X and Y are weakly equivalent.

Remark 3.1.13. If X and Y are weakly equivalent in a relative category $\mathcal{C}$, then they are isomorphic in $\operatorname{Ho} \mathcal{C}$. The converse is certainly true if $\mathcal{C}$ is saturated, but is false if $\mathcal{C}$ is not semi-saturated.

Definition 3.1.14. A **homotopically replete subcategory** of a relative category $\mathcal{C}$ is a relative subcategory $\mathcal{D}$ with the following property:

- If D is an object in $\mathcal{D}$ and $f : C \to D$ is a weak equivalence in $\mathcal{C}$, then both C and f are in $\mathcal{D}$.

- If D is an object in $\mathcal{D}$ and $g : D \to C$ is a weak equivalence in $\mathcal{C}$, then both C and g are in $\mathcal{D}$.

Remark 3.1.15. Any full relative subcategory $\mathcal{D}$ of a relative category $\mathcal{C}$ is homotopically replete if and only if it has the following property:

- If D is an object in $\mathcal{D}$ and C an object in $\mathcal{C}$ that is weakly equivalent to D, then C is in $\mathcal{D}$.

Definition 3.1.16. A parallel pair of morphisms in a relative category $\mathcal{C}$ are **weakly homotopic** if they are equal in $\operatorname{Ho} \mathcal{C}$; we write $f \overset{\mathrm{w}}{\sim} g$ to mean that f and g are weakly homotopic.

Definition 3.1.17. An **equivalence** in a relative category $\mathcal{C}$ is a pair (f, g), where $f : X \to Y$ and $g : Y \to X$ are morphisms in $\mathcal{C}$ such that $g \circ f \overset{\mathrm{w}}{\sim} \mathrm{id}_X$ and $f \circ g \overset{\mathrm{w}}{\sim} \mathrm{id}_Y$. Two morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{C}$ are **mutual quasi-inverses** when (f, g) constitute an equivalence in $\mathcal{C}$.

Remark 3.1.18. It follows from the definitions that quasi-inverses are unique up to weak homotopy.

Lemma 3.1.19. *If the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ for a relative category $\mathcal{C}$ is full, then the following are equivalent for all morphisms $f : X \to Y$ in $\mathcal{C}$:*

- *f is a morphism in $\mathcal{C}$ and has a quasi-inverse.*
- *γf is an isomorphism in $\mathcal{C}$.*

Proof. Obvious. ◆

Remark 3.1.20. Clearly, any isomorphism in any relative category has a quasi-inverse; but this implies that in a relative category that is *not* semi-saturated, a morphism that has a quasi-inverse need *not* be a weak equivalence. On other hand, if f is a morphism in a *saturated* homotopical category and f has a quasi-inverse, then f must be a weak equivalence.

Definition 3.1.21. A relative category $\mathcal{C}$ has the **Whitehead property** when the following are equivalent:

- f is a weak equivalence in $\mathcal{C}$.
- f is a morphism in $\mathcal{C}$ and has a quasi-inverse.

Theorem 3.1.22. *Let $\mathcal{C}$ be a relative category. The following are equivalent:*

(i) *$\mathcal{C}$ has the Whitehead property.*

(ii) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is full, and $\mathcal{C}$ is a saturated homotopical category.*

Proof. (i) ⇒ (ii). By theorem A.4.26, every morphism $\gamma X_0 \to \gamma X_n$ in $\operatorname{Ho}\mathcal{C}$ is of the form

$$(\gamma f_n)^{-1} \circ \cdots \circ \gamma h_2 \circ (\gamma f_1)^{-1} \circ \gamma h_1$$

for some morphisms $h_1 : X_0 \to Y_1$, $f_1 : X_1 \to Y_1$, $h_2 : X_1 \to Y_2$, etc. in $\mathcal{C}$, where $f_1, \ldots, f_n$ are weak equivalences. By the Whitehead property, each $f_i : X_i \to Y_i$ has a quasi-inverse in $\mathcal{C}$, say $g_i : Y_i \to X_i$. Since $\gamma g_i = (\gamma f_i)^{-1}$, it follows that

$$(\gamma f_n)^{-1} \circ \cdots \circ h_2 \circ (\gamma f_1)^{-1} \circ \gamma h_1 = \gamma(g_n \circ \cdots \circ h_2 \circ g_1 \circ h_1)$$

and therefore $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is indeed full.

In particular, every morphism $f : X \to Y$ in $\mathcal{C}$ such that $\gamma f : \gamma X \to \gamma Y$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$ must have a quasi-inverse, and hence must be a weak equivalence, in view of the Whitehead property. We therefore conclude that $\mathcal{C}$ is a saturated homotopical category.

(ii) ⇒ (i). The converse follows from the definitions and lemma 3.1.19. ■

Remark 3.1.23. The Whitehead property is in general not inherited by slice categories or by functor categories. For example, if $q \circ f = p$ and g is a quasi-inverse for f, it is only guaranteed that $q \overset{\mathrm{w}}{\sim} p \circ g$.

Definition 3.1.24. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two ordinary functors between relative categories. A **natural weak equivalence** $\alpha : F \Rightarrow G$ is a natural transformation such that $\alpha_C : FC \to GC$ is a weak equivalence in $\mathcal{D}$ for all objects C in $\mathcal{C}$, and we say F and G are **naturally weakly equivalent** if they can be connected by a *zigzag* of natural weak equivalences.

Remark 3.1.25. This is precisely the notion of weak equivalence in the relative functor category $[\operatorname{min}\operatorname{und}\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Although the definition above applies to all functors, if $H : \mathcal{D} \to \mathcal{E}$ is an ordinary functor, then the natural transformation $H\alpha : HF \Rightarrow HG$ is only guaranteed to be a natural weak equivalence if we assume H is a relative functor.

Definition 3.1.26. A **relative equivalence** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ for which there exists a relative functor $G : \mathcal{D} \to \mathcal{C}$ such that GF is naturally weakly equivalent to $\mathrm{id}_{\mathcal{C}}$ and FG is naturally weakly equivalent to $\mathrm{id}_{\mathcal{D}}$. Such a G is said to be a **relative inverse** of F. When $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories, we may say **homotopical equivalence** and **homotopical inverse** instead of 'relative equivalence' and 'relative inverse'.

Proposition 3.1.27. *If $F : \mathcal{C} \to \mathcal{D}$ is a relative equivalence of relative categories with relative inverse $G : \mathcal{D} \to \mathcal{C}$, then $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ is an equivalence of categories, with quasi-inverse $\operatorname{Ho} G : \operatorname{Ho} \mathcal{D} \to \operatorname{Ho} \mathcal{C}$.* ■

Definition 3.1.28. An **adjoint relative equivalence** is an adjunction of the form below,

$$F \dashv G : \mathcal{D} \to \mathcal{C}$$

where $\mathcal{C}$ and $\mathcal{D}$ are relative categories, F and G are relative functors, and both the adjunction unit and counit are natural weak equivalences. When $\mathcal{C}$ and $\mathcal{D}$ are homotopical categories, we may say **adjoint homotopical equivalence** instead of 'adjoint relative equivalence'.

Proposition 3.1.29. *An adjoint relative equivalence of relative categories descends to an adjoint equivalence of homotopy categories.*

Proof. Use the 2-functoriality of $\mathrm{Ho} : \mathfrak{RelCat} \to \mathfrak{Cat}$ (corollary A.4.20). ■

Definition 3.1.30. A **homotopically contractible category** is a homotopical category C such that the unique (homotopical) functor $C \to \mathbb{1}$ is a homotopical equivalence, where $\mathbb{1}$ is the trivial category with only one object.

Proposition 3.1.31. *Let C be a homotopical category. The following are equivalent:*

(i) *C is homotopically contractible.*

(ii) *C is inhabited, and for every object A in C, the constant functor ΔA is naturally weakly equivalent to id_C.*

(iii) *There exists an object A in C such that ΔA and id_C are naturally weakly equivalent.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ♦

3.2 Homotopical Kan extensions

Prerequisites. §§ 3.1, A.4.

Definition 3.2.1. Let C be a homotopical category. A **homotopically initial object** in C is an object A for which there exists a zigzag of natural transformations of the form

$$\Delta A \rightsquigarrow F \xrightarrow{\alpha} G \rightsquigarrow \mathrm{id}_C$$

where $\Delta A : C \to C$ is the constant functor with value A, $\alpha_A : FA \to GA$ is a weak equivalence in C, and the squiggles denote (possibly trivial) zigzags of natural weak equivalences. Dually, a **homotopically terminal object** in C is a homotopically initial object in C^{op}.

Proposition 3.2.2. *Let C be a homotopical category. If A is a homotopically initial (resp. homotopically terminal) object in C, then:*

(i) *Any object in C weakly equivalent to A is also a homotopically initial (resp. homotopically terminal) object in C.*

(ii) *A is an initial (resp. terminal) object in* $\operatorname{Ho} \mathcal{C}$.

(iii) *If $\mathcal{C}$ is a minimal homotopical category, then A is an initial (resp. terminal) object in $\mathcal{C}$ as well.*

Conversely, any initial (resp. terminal) object in $\mathcal{C}$ is also homotopically initial (resp. homotopically terminal).

Proof. Obvious. (This is Proposition 38.3 in [DHKS].) ◆

Proposition 3.2.3. *If A is a homotopically initial object in a homotopical category $\mathcal{C}$, then for any object Z in $\mathcal{C}$, the zigzag category $\mathcal{C}^{(\mathbf{T})}(A, Z)$ is connected.*

Proof. By theorem A.4.26, there is a bijection between the connected components of $\mathcal{C}^{(\mathbf{T})}(A, Z)$ and the morphisms $A \to Z$ in $\operatorname{Ho} \mathcal{C}$; but we know A is an initial object in $\operatorname{Ho} \mathcal{C}$, so $\mathcal{C}^{(\mathbf{T})}(A, Z)$ has exactly one connected component. ■

Lemma 3.2.4. *Let $H : \mathcal{C} \to \mathcal{D}$ be a relative functor and let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor. If If* $\operatorname{weq} \mathcal{D}$ *has the 2-out-of-3 property and F is naturally weakly equivalent to H, then F is also a relative functor.*

Proof. Apply the 2-out-of-3 property inductively. ◆

Lemma 3.2.5. *If A and A' be homotopically initial objects in a homotopical category $\mathcal{C}$, then $A \overset{\mathrm{w}}{\simeq} A'$, and moreover every morphism $A \to A'$ in $\mathcal{C}$ is a weak equivalence.*

Proof. This is paragraph 38.5 in [DHKS].

Suppose, as in the definition, that we have endofunctors F, F', G, G' on $\mathcal{C}$ and natural transformations $\alpha : F \Rightarrow G$, $\alpha' : F' \Rightarrow G'$, such that $F \overset{\mathrm{w}}{\simeq} \Delta A$, $F' \overset{\mathrm{w}}{\simeq} \Delta A'$, $G \overset{\mathrm{w}}{\simeq} \mathrm{id}_{\mathcal{C}}$, and $G' \overset{\mathrm{w}}{\simeq} \mathrm{id}_{\mathcal{C}}$, and the morphisms $\alpha_A : FA \to GA$ and $\alpha'_{A'} : FA' \to GA'$ are both weak equivalences. Note that the previous lemma implies G and G' are both homotopical functors, while a similar argument shows that F and F' sends *all* morphisms to weak equivalences.

Let $f : A \to A'$ be a morphism in $\mathcal{C}$. By applying the 2-out-of-3 property repeatedly in the following diagram,

$$\begin{tikzcd} A \arrow[r, squiggly, no head] \arrow[d, equal] & FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[r, squiggly, no head] \arrow[d, "Gf"] & A \arrow[d, "f"] \\ A \arrow[r, squiggly, no head] & FA' \arrow[r, "\alpha_{A'}"'] & GA' \arrow[r, squiggly, no head] & A' \end{tikzcd}$$

we see that f is a weak equivalence if and only if $\alpha_{A'} : FA' \to GA'$ is a weak equivalence. Since $\alpha'_{A'} : F'A' \to G'A'$ is a weak equivalence, and $GA' \overset{\mathrm{w}}{\simeq} A'$, it follows that $\alpha'_{GA'} : FGA' \to G'GA'$ is a weak equivalence, and since G is homotopical, so $G\alpha'_{GA'} : GFGA' \to GG'GA'$ is also a weak equivalence. Similarly, $\alpha_A : FA \to GA$ is a weak equivalence, and $A \overset{\mathrm{w}}{\simeq} FA' \overset{\mathrm{w}}{\simeq} G'FA'$, so $\alpha_{G'FA'} : FG'FA' \to GG'FA'$ is a weak equivalence as well.

Now, by applying the 2-out-of-6 property to the diagram below,

$$\begin{tikzcd} FF'FA' \arrow[r, "\alpha_{F'FA'}"] \arrow[d, "F\alpha'_{FA'}"'] & GF'FA' \arrow[r, "GF'\alpha_{A'}"] \arrow[d, "G\alpha'_{FA'}"'] & GF'GA' \arrow[d, "G\alpha'_{GA'}"] \\ FG'FA' \arrow[r, "\alpha_{G'FA'}"'] & GG'FA' \arrow[r, "GG'\alpha_{A'}"'] & GG'GA' \end{tikzcd}$$

we may deduce that $GG'\alpha_{A'} : GG'FA' \to GG'GA'$ is a weak equivalence, and hence that $\alpha_{A'} : FA' \to GA'$ is a weak equivalence, as required. ■

¶ 3.2.6. We will say that an object in a homotopical category C characterised by a homotopical universal property is **homotopically unique** if the full subcategory spanned by such objects inside the homotopical category of objects in C *equipped with the relevant additional structure*.

Proposition 3.2.7. *Let C be a homotopically contractible category.*

(i) *Every morphism in C is a weak equivalence.*

(ii) *The unique functor* $\mathrm{Ho}\, C \to \mathbb{1}$ *is an equivalence of categories.*

(iii) *If C is a minimal homotopical category, then $C \to \mathbb{1}$ is also an equivalence of categories.*

(iv) *The opposite homotopical category C^{op} and the homotopical functor category $[\mathcal{D}, C]_{\mathrm{h}}$ (for any homotopical category $\mathcal{D}$) are also homotopically contractible.*

(v) *Every object in C is both homotopically initial and homotopically terminal.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ♦

Proposition 3.2.8. *Let C be a homotopical category. If $\mathcal{D}$ is the full homotopical subcategory of C spanned by the homotopically initial (or homotopically terminal) objects, then $\mathcal{D}$ is homotopically contractible.*

Proof. This follows from lemma 3.2.5. ■

Remark 3.2.9. Even if $\mathcal{C}$ is a saturated homotopical category, an object that is initial in $\operatorname{Ho}\mathcal{C}$ need not be homotopically initial in $\mathcal{C}$. Indeed, let $\mathcal{C}$ be the maximal homotopical category generated by a graph of the following form:

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \cdots$$

No object in $\mathcal{C}$ is homotopically initial, because the length of the shortest zigzag connecting two objects cannot be bounded above; yet every object in $\operatorname{Ho}\mathcal{C}$ is initial. The same argument shows that $\mathcal{C}$ is not homotopically contractible, but $\operatorname{Ho}\mathcal{C}$ is certainly contractible.

Definition 3.2.10. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two ordinary functors between homotopical categories. A **homotopical left Kan extension** (resp. **homotopical right Kan extension**) of G along F is a homotopically initial (resp. homotopically terminal) object of the homotopical category $(G \downarrow F^*)_{\mathrm{h}}$ (resp. $(F^* \downarrow G)_{\mathrm{h}}$) described below:

- The objects are pairs (H, α) where H is a homotopical functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).
- The weak equivalences are the natural weak equivalences.

Remark 3.2.11. Note that any homotopical Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $G : \mathcal{C} \to \mathcal{E}$ has, by definition, an underlying *homotopical* functor $H : \mathcal{D} \to \mathcal{E}$.

Corollary 3.2.12. *Homotopical Kan extensions are homotopically unique, any two homotopical left (resp. right) Kan extensions of G along F are naturally weakly equivalent.* ■

Definition 3.2.13. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two ordinary functors between homotopical categories, and let $L : \mathcal{E} \to \mathcal{F}$ be a homotopical functor. We say L **preserves** a homotopical left (resp. right) Kan extension (H, α) of G along F if $(LH, L\alpha)$ is a homotopical left (resp. right) Kan extension of LF along G. If a homotopical Kan extension is preserved by *all* homotopical functors, then it is said to be **absolute**.

3.3 Quillen–Verdier derived functors

Prerequisites. §§ 3.1, A.4, A.1, A.5

The fact that $\mathrm{Ho} : \mathfrak{RelCat} \to \mathfrak{Cat}$ is a 2-functor means that relative functors $F : \mathcal{C} \to \mathcal{D}$ descend to functors $\mathrm{Ho}\, F : \mathrm{Ho}\, \mathcal{C} \to \mathrm{Ho}\, \mathcal{D}$ in a very well-behaved way. However, what can we say about ordinary (i.e. not necessarily relative) functors $\mathcal{C} \to \mathcal{D}$?

In this section, we follow [DHKS, §§ 40–43]; however, we will use a weaker definition of 'deformation retract' and a stronger definition of 'total derived functor'.

Definition 3.3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories, and let $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}\, \mathcal{C}$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}\, \mathcal{D}$ be the localising functors.

- A **total left derived functor** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is an absolute right (!) Kan extension of $\gamma_{\mathcal{D}} F : \mathcal{C} \to \mathrm{Ho}\, \mathcal{D}$ along $\gamma_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ho}\, \mathcal{C}$.
- A **total right derived functor** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is an absolute left (!) Kan extension of $\gamma_{\mathcal{C}} G : \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$ along $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathrm{Ho}\, \mathcal{D}$.

Remark 3.3.2. The above definition is essentially due to Verdier [1963], but the formulation using Kan extensions is due to Quillen [1967, Ch. I, § 4]. We deviate from convention by demanding that the Kan extensions be *absolute*; this is in order to make theorem 3.3.5 true.

Remark 3.3.3. As with everything defined by a universal property, total derived functors are unique up to unique isomorphism *if* they exist.

Definition 3.3.4. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of ordinary categories. A **derived adjunction** for $F \dashv G$ consists of

- a left derived functor $(\mathbf{L}F, \alpha)$ for F,
- a right derived functor $(\mathbf{R}G, \beta)$ for G, and
- an adjunction $\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}\, \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$ with unit $\bar{\eta} : \mathrm{id}_{\mathrm{Ho}\, \mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ and counit $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\mathrm{Ho}\, \mathcal{D}}$,

such that (α, β) constitute a conjugate pair of natural transformations. We refer to $\bar{\eta}$ as the **derived unit** and $\bar{\varepsilon}$ as the **derived counit**.

Theorem 3.3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an ordinary adjunction. If $(\mathbf{L}F, \alpha)$ is a total left derived functor for F and $(\mathbf{R}G, \beta)$ is a total right derived functor for G, then there exist unique natural transformations $\bar{\eta} : \mathrm{id}_{\operatorname{Ho}\mathcal{C}} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ and $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\operatorname{Ho}\mathcal{D}}$ making $\mathbf{L}F \dashv \mathbf{R}G : \operatorname{Ho}\mathcal{D} \to \operatorname{Ho}\mathcal{C}$ a derived adjunction for $F \dashv G$ with derived unit $\bar{\eta}$ and derived counit $\bar{\varepsilon}$.*

Proof. Let η and ε be the unit and counit of the adjunction $F \dashv G$. First, we prove that $\bar{\eta}$ and $\bar{\varepsilon}$ are unique if they exist. Indeed, if they exist, then (α, β) is a conjugate pair of natural transformations, so we must have the equations shown below:

$$\beta F \bullet \gamma_{\mathcal{C}}\eta = (\mathbf{R}G)\alpha \bullet \bar{\eta}\gamma_{\mathcal{C}} \qquad\qquad \bar{\varepsilon}\gamma_{\mathcal{D}} \bullet (\mathbf{L}F)\beta = \gamma_{\mathcal{D}}\varepsilon \bullet \alpha G$$

However, $((\mathbf{R}G)(\mathbf{L}F), (\mathbf{R}G)\alpha)$ is a left Kan extension of $(\mathbf{R}G)\gamma_{\mathcal{D}}F$ along $\gamma_{\mathcal{C}}$ and $((\mathbf{L}F)(\mathbf{R}G), (\mathbf{L}F)\beta)$ is a right Kan extension of $(\mathbf{L}F)\gamma_{\mathcal{C}}G$ along $\gamma_{\mathcal{D}}$, so $\bar{\eta}$ and $\bar{\varepsilon}$ are uniquely determined as natural transformations by these equations.

Next, we prove that the natural transformations $\bar{\eta}$ and $\bar{\varepsilon}$ defined above satisfy the left and right triangle identities. Using naturality and the defining equations for $\bar{\eta}$ and $\bar{\varepsilon}$, we obtain the following:

$$\begin{aligned}
\alpha \bullet (\bar{\varepsilon}(\mathbf{L}F) \bullet (\mathbf{L}F)\bar{\eta})\gamma_{\mathcal{C}} &= \alpha \bullet \bar{\varepsilon}(\mathbf{L}F)\gamma_{\mathcal{C}} \bullet (\mathbf{L}F)\bar{\eta}\gamma_{\mathcal{C}} \\
&= \bar{\varepsilon}\gamma_{\mathcal{D}}F \bullet (\mathbf{L}F)(\mathbf{R}G)\alpha \bullet (\mathbf{L}F)\bar{\eta}\gamma_{\mathcal{C}} \\
&= \bar{\varepsilon}\gamma_{\mathcal{D}}F \bullet (\mathbf{L}F)\beta F \bullet (\mathbf{L}F)\gamma_{\mathcal{C}}\eta \\
&= \gamma_{\mathcal{D}}\varepsilon F \bullet \alpha G F \bullet (\mathbf{L}F)\gamma_{\mathcal{C}}\eta \\
&= \gamma_{\mathcal{D}}\varepsilon F \bullet \gamma_{\mathcal{D}}F\eta \bullet \alpha \\
&= \gamma_{\mathcal{D}}(\varepsilon F \bullet F\eta) \bullet \alpha
\end{aligned}$$

Since $(\mathbf{L}F, \alpha)$ is a right Kan extension of F along $\gamma_{\mathcal{C}}$, this implies that $\bar{\eta}$ and $\bar{\varepsilon}$ satisfy the left triangle identity if η and ε do. A formally dual calculation shows that the same is true for the right triangle identity. Thus, we have the required derived adjunction. ■

Definition 3.3.6. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **left deformation retract** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a triple $(\mathcal{C}^{\circ}, Q, p)$ where

- $\mathcal{C}^{\circ}$ is a full subcategory of $\mathcal{C}$ with the induced relative subcategory structure,

- Q is a pair of maps $\mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{C}$ and $\mathrm{mor}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{C}$ (but not necessarily functorial), and

- p assigns to each object X in $\mathcal{C}$ a weak equivalence $p_X : QX \to X$,

and these data are required to satisfy the following axioms:

DR1. For all objects X in $\mathcal{C}$, the object QX is in $\mathcal{C}^\circ$.

DR2. For all morphisms $f : X \to Y$ in $\mathcal{C}$, we have $p_Y \circ Qf = f \circ p_X$, i.e. the diagram in $\mathcal{C}$ shown below commutes,

$$\begin{tikzcd} QX \arrow[r, "p_X"] \arrow[d, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

and if f is a weak equivalence in $\mathcal{C}$, then so is Qf.

DR3. The inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ induces a fully faithful functor $\mathrm{Ho}\,\mathcal{C}^\circ \to \mathrm{Ho}\,\mathcal{C}$.

DR4. The restriction $F|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \to \mathcal{D}$ is a relative functor.

An ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is **left deformable** if there exists a left deformation retract for F. A **left deformation retract** of a relative category $\mathcal{C}$ is a left deformation retract for $\mathrm{id}_{\mathcal{C}}$.

Dually, a **right deformation retract** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a triple $(\mathcal{D}^\circ, R, i)$ where

- $\mathcal{D}^\circ$ is a full subcategory of $\mathcal{D}$ with the induced relative subcategory structure,

- R is a pair of maps $\mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathcal{D}$ and $\mathrm{mor}\,\mathcal{D} \to \mathrm{mor}\,\mathcal{D}$ (but not necessarily functorial), and

- i assigns to each object A in $\mathcal{D}$ a weak equivalence $i_A : A \to RA$,

and these data are required to satisfy the following axioms:

DR1. For all objects A in $\mathcal{D}$, the object RA is in $\mathcal{D}^\circ$.

DR2. For all morphisms $g : A \to B$ in $\mathcal{D}$, we have $Rg \circ i_A = i_B \circ g$, i.e. the diagram in $\mathcal{D}$ shown below commutes,

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "g"'] & RA \arrow[d, "Rg"] \\ B \arrow[r, "i_B"'] & RB \end{tikzcd}$$

and if g is a weak equivalence in $\mathcal{D}$, then so is Rg.

DR3. The inclusion $\mathcal{D}^\circ \hookrightarrow \mathcal{D}$ induces a fully faithful functor $\operatorname{Ho}\mathcal{D}^\circ \to \operatorname{Ho}\mathcal{D}$.

DR4. The restriction $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor.

An ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is **right deformable** if there exists a weak right deformation retract for G. A **right deformation retract** of a relative category $\mathcal{D}$ is a right deformation retract for $\mathrm{id}_{\mathcal{D}}$.

REMARK 3.3.7. Every relative functor is both left deformable and right deformable, with trivial left and right deformation retracts.

REMARK 3.3.8. Given any weak left (resp. right) deformation retract $(\mathcal{C}^\circ, Q, p)$ (resp. $(\mathcal{D}^\circ, R, i)$), the canonical functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ (resp. $\operatorname{Ho}\mathcal{D}^\circ \to \operatorname{Ho}\mathcal{D}$) is not only fully faithful but also essentially surjective on objects, so the categories $\operatorname{Ho}\mathcal{C}^\circ$ and $\operatorname{Ho}\mathcal{C}$ (resp. $\operatorname{Ho}\mathcal{D}^\circ$ and $\operatorname{Ho}\mathcal{D}$) are equivalent.

Proposition 3.3.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories, and let $(\mathcal{C}^\circ, Q, p)$ be a left deformation retract for $F : \mathcal{C} \to \mathcal{D}$.*

(i) *If Q is functorial, then the composite $FQ : \mathcal{C} \to \mathcal{D}$ is a relative functor.*

(ii) *If $\mathcal{C}^\circ_F$ is the full subcategory of $\mathcal{C}$ spanned by the objects X such that the morphism $Fp_X : FQX \to FX$ is weak equivalence in $\mathcal{D}$, then $\mathcal{C}^\circ \subseteq \mathcal{C}^\circ_F$.*

(iii) *If moreover $\operatorname{weq}\mathcal{D}$ has the 2-out-of-3 property in $\mathcal{D}$, then $\left(\mathcal{C}^\circ_F, Q, p\right)$ is also a left deformation retract for F.*

Dually, let $(\mathcal{D}^\circ, R, i)$ be a right deformation retract for $G : \mathcal{D} \to \mathcal{C}$.

(i′) *If Q is functorial, then the composite $GR : \mathcal{D} \to \mathcal{C}$ is a relative functor.*

(ii′) *If $\mathcal{D}^\circ_G$ is the full subcategory of $\mathcal{D}$ spanned by the objects A such that the morphism $Gi_A : GA \to GRA$ is weak equivalence in $\mathcal{C}$, then $\mathcal{D}^\circ \subseteq \mathcal{D}^\circ_G$.*

(iii′) *If moreover* weq $\mathcal{C}$ *has the 2-out-of-3 property in* $\mathcal{C}$*, then* $\left(\mathcal{D}^{\circ}_{G}, R, i\right)$ *is also a right deformation retract for* F.

Proof. (i). Immediate from the definitions.

(ii). Let $\tilde{X}$ be an object in $\mathcal{C}^{\circ}$. By definition, $Q\tilde{X}$ is also an object in $\mathcal{C}^{\circ}$, and $F|_{\mathcal{C}^{\circ}}$ is a relative functor, so $Fp_{\tilde{X}} : FQ\tilde{X} \to F\tilde{X}$ is a weak equivalence in $\mathcal{C}$.

(iii). Let X and Y be objects in $\mathcal{C}^{\circ}_{F}$ and let $f : X \to Y$ be a weak equivalence in $\mathcal{C}$. Consider the following commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd} FQX \arrow[r, "Fp_X"] \arrow[d, "FQf"'] & FX \arrow[d, "Ff"] \\ FQY \arrow[r, "Fp_Y"'] & FY \end{tikzcd}$$

FQf is a weak equivalence in $\mathcal{D}$ by claim (i), and both Fp_X and Fp_Y are weak equivalences by the definition of $\mathcal{C}^{\circ}_{F}$, so using the 2-out-of-3 property of weq $\mathcal{D}$, we may deduce that Ff is a weak equivalence in $\mathcal{D}$ too. Thus, $F|_{\mathcal{C}^{\circ}_{F}}$ is a relative functor, as required for $\left(\mathcal{C}^{\circ}_{F}, Q, p\right)$ to be a left deformation retract for F. ■

Proposition 3.3.10. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be relative categories, and let* $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *and* $\gamma_{\mathcal{D}} : \mathcal{D} \to \operatorname{Ho}\mathcal{D}$ *be the respective localising functors.*

- *If* $(\mathcal{C}^{\circ}, Q, p)$ *is a weak left deformation retract for an ordinary functor* $F : \mathcal{C} \to \mathcal{D}$*, then there exist a right Kan extension* $(\mathbf{L}F, \alpha)$ *of* $\gamma_{\mathcal{D}}F$ *along* $\gamma_{\mathcal{C}}$ *such that* $(\mathbf{L}F)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}}FQ$ *and* $\alpha = \gamma_{\mathcal{D}}Fp$*. (In particular,* $\gamma_{\mathcal{D}}FQ$ *is functorial even if* Q *is not.)*

- *If* $(\mathcal{D}^{\circ}, R, i)$ *is a weak right deformation retract for an ordinary functor* $G : \mathcal{D} \to \mathcal{C}$*, then there exist a left Kan extension* $(\mathbf{R}G, \beta)$ *of* $\gamma_{\mathcal{C}}G$ *along* $\gamma_{\mathcal{D}}$ *such that* $(\mathbf{R}G)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}}GR$ *and* $\beta = \gamma_{\mathcal{C}}Gi$*. (In particular,* $\gamma_{\mathcal{C}}GR$ *is functorial even if* R *is not.)*

Proof. The two claims are formally dual; we will prove the first version.

To simplify notation, we may assume without loss of generality that $\mathcal{D}$ is a minimal saturated relative category and that $\gamma_{\mathcal{D}} = \mathrm{id}_{\mathcal{D}}$. Henceforth, we write γ instead of $\gamma_{\mathcal{C}}$. First, observe that γQ is functorial (even if Q is not) because each $\gamma p_X : \gamma QX \to \gamma X$ is an isomorphism, so (using axioms DR1 and DR3) there is a unique functor $\tilde{Q} : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{C}^{\circ}$ such that $\tilde{Q}\gamma = \gamma Q$. Let $\gamma^{\circ} : \mathcal{C}^{\circ} \to \operatorname{Ho}\mathcal{C}^{\circ}$

be the localising functor for $\mathcal{C}^\circ$. Since $F|_{\mathcal{C}^\circ}$ is a relative functor (by axiom DR4), we must have $F|_{\mathcal{C}^\circ} = \tilde{F}\gamma^\circ$ for a unique functor $\tilde{F} : \operatorname{Ho}\mathcal{C}^\circ \to \mathcal{D}$. We may then define $\mathbf{L}F$ to be the functor $\tilde{F}\tilde{Q}$. We define $\alpha : (\mathbf{L}F)\gamma \Rightarrow F$ by taking $\alpha_X = Fp_X$; by axiom DR2, this is indeed a natural transformation.

It remains to be shown that $(\mathbf{L}F, \alpha)$ is a right Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$. Let $H : \operatorname{Ho}\mathcal{C} \to \mathcal{D}$ be a functor and let $\varphi : H\gamma \Rightarrow F$ be any natural transformation. By restricting along the inclusion $\mathcal{C}^\circ \to \mathcal{C}$, we obtain a natural transformation $\varphi|_{\mathcal{C}^\circ} : H|_{\operatorname{Ho}\mathcal{C}^\circ}\gamma^\circ \Rightarrow F|_{\mathcal{C}^\circ}$, so there is a unique natural transformation $\tilde{\varphi} : H|_{\operatorname{Ho}\mathcal{C}^\circ} \Rightarrow \tilde{F}$ such that $\tilde{\varphi}\gamma^\circ = \varphi|_{\mathcal{C}^\circ}$ (by the 2-dimensional universal property of $\operatorname{Ho}\mathcal{C}$). Since γp is a natural isomorphism, there is then a unique natural transformation $\bar{\varphi} : H \Rightarrow \mathbf{L}F$ such that $\bar{\varphi}_{\gamma X} \circ H\gamma p_X = \tilde{\varphi}_{\gamma^\circ QX}$ for all objects X in $\mathcal{C}$. We then have $\alpha \bullet \bar{\varphi}\gamma = \varphi$, and $\bar{\varphi}$ is the unique such natural transformation because the canonical functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ is essentially surjective on objects. ■

Definition 3.3.11. Let $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ be relative categories. Given a composable pair of ordinary functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, a **lax left deformation retract** for (G, F) consists of

- a left deformation retract $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ for F, and
- a left deformation retract $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ for G,

such that $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is also a left deformation retract for GF as well. A **strong left deformation retract** for (G, F) is a lax left deformation retract as above such that F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$. We say a composable pair of functors is **laxly left deformable** (resp. **strongly left deformable**) if it admits a lax left deformation (resp. strong left deformation).

Dually, given a composable pair of ordinary functors $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$, an **oplax right deformation retract** for (F, G) consists of

- a right deformation retract $\left(\mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ}\right)$ for F, and
- a right deformation retract $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ for G,

such that $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ is a right deformation retract for GF as well. A **strong right deformation retract** for (F, G) is an oplax right deformation retract as above such that G sends objects in $\mathcal{D}^\circ$ to objects in $\mathcal{C}^\circ$. We say a composable pair of functors is **oplaxly right deformable** (resp. **strongly left deformable**) if it admits an oplax right deformation (resp. strong right deformation).

Lemma 3.3.12.

- *Let $\left(C^\circ, Q^{C^\circ}, p^{C^\circ}\right)$ be a left deformation retract for $F : C \to \mathcal{D}$ and let $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ be a left deformation retract for $G : \mathcal{D} \to \mathcal{E}$. If F maps objects in C° to objects in $\mathcal{D}^\circ$, then $\left(C^\circ, Q^{C^\circ}, p^{C^\circ}\right)$ is a left deformation retract for $GF : C \to \mathcal{E}$.*

Dually:

- *Let $\left(C^\circ, R^{C^\circ}, i^{C^\circ}\right)$ be a right deformation retract for $F : C \to \mathcal{B}$ and let $\left(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ be a right deformation retract for $G : \mathcal{D} \to C$. If G maps objects in $\mathcal{D}^\circ$ to objects in C°, then $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}\right)$ is a right deformation retract for $FG : \mathcal{D} \to \mathcal{B}$.*

Proof. Our hypotheses imply that the restriction $GF|_{C^\circ} : C^\circ \to \mathcal{E}$ is a relative functor, so $\left(C^\circ, Q^{C^\circ}, p^{C^\circ}\right)$ satisfies the conditions required to be a left deformation retract for $GF : C \to \mathcal{E}$. ■

Theorem 3.3.13. *Let C, $\mathcal{D}$, and $\mathcal{E}$ be relative categories, and let $\gamma_C : C \to \operatorname{Ho} C$, $\gamma_{\mathcal{D}} : \mathcal{D} \to \operatorname{Ho} \mathcal{D}$, and $\gamma_{\mathcal{E}} : \mathcal{E} \to \operatorname{Ho} \mathcal{E}$ be the respective localising functors.*

(i) *Let $F : C \to \mathcal{D}$ be an ordinary functor. If (C°, Q, p) is any left deformation retract for F, then F has a total left derived functor $(\mathbf{L}F, \alpha)$ such that $(\mathbf{L}F)\gamma_C = \gamma_{\mathcal{D}} F Q$ and $\alpha = \gamma_{\mathcal{D}} F p$.*

(ii) *Let $F, F' : C \to \mathcal{D}$ be a parallel pair of ordinary functors. If $(\mathbf{L}F, \alpha)$ and $(\mathbf{L}F', \alpha')$ are total left derived functors for F and F' (respectively), then for any natural transformation $\varphi : F \Rightarrow F'$, there exists a unique natural transformation $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$ such that $\alpha' \bullet (\mathbf{L}\varphi)\gamma_C = \gamma_{\mathcal{D}}\varphi \bullet \alpha$.*

(iii) *Moreover, if (C°, Q, p) is a left deformation retract for both F and F', then $(\mathbf{L}\varphi)\gamma_C = \gamma_{\mathcal{D}}\varphi Q$.*

(iv) *Let $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be ordinary functors between relative categories. If $\left(\mathbf{L}F, \alpha^F\right)$, $\left(\mathbf{L}G, \alpha^G\right)$, and $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ are total left derived functors for F, G, and GF (respectively), then there is a unique natural transformation $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ such that $\alpha^{GF} \bullet \boldsymbol{\mu}_{G,F}\gamma_C = \alpha^G F \bullet (\mathbf{L}G)\alpha^F$.*

(v) *If (G, F) is moreover a strongly left deformable composable pair, then the canonical comparison $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ is an isomorphism.*

Dually:

(i′) *Let $G : \mathcal{D} \to \mathcal{C}$ be an ordinary functor. If $(\mathcal{D}^\circ, R, i)$ is any right deformation retract for G, then G has a total right derived functor $(\mathbf{R}G, \beta)$ such that $(\mathbf{R}G)\gamma_\mathcal{D} = \gamma_\mathcal{C} G R$ and $\beta = \gamma_\mathcal{C} G i$.*

(ii′) *Let $G, G' : \mathcal{D} \to \mathcal{C}$ be a parallel pair of ordinary functors. If $(\mathbf{R}G, \beta)$ and $(\mathbf{R}G', \beta')$ are total right derived functors for G and G' (respectively), then for any natural transformation $\psi : G' \Rightarrow G$, there exists a unique natural transformation $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$ such that $(\mathbf{R}\psi)\gamma_\mathcal{D} \bullet \beta' = \beta \bullet \gamma_\mathcal{C}\psi$.*

(iii′) *Moreover, if $(\mathcal{D}^\circ, R, i)$ is a right deformation retract for both G and G', then $(\mathbf{R}\psi)\gamma_\mathcal{D} = \gamma_\mathcal{C}\psi R$.*

(iv′) *Let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$ be ordinary functors between relative categories. If $\left(\mathbf{R}F, \beta^F\right)$, $\left(\mathbf{R}G, \beta^G\right)$, and $\left(\mathbf{R}(FG), \beta^{FG}\right)$ are total right derived functors for F, G, and FG (respectively), then there is a unique natural transformation $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ such that $\boldsymbol{\delta}_{F,G}\gamma_\mathcal{D} \bullet \beta^{FG} = (\mathbf{R}F)\beta^G \bullet \beta^F G$.*

(v′) *If (F, G) is moreover a strongly right deformable composable pair, then the canonical comparison $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ is an isomorphism.*

Proof. (i). By proposition 3.3.10, the functor $\gamma_\mathcal{D} F : \mathcal{C} \to \operatorname{Ho}\mathcal{D}$ has a right Kan extension along $\gamma_\mathcal{C} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$, say $(\mathbf{L}F, \alpha)$, characterised by the announced equations. We must verify that $(\mathbf{L}F, \alpha)$ is an absolute right Kan extension, i.e. that $(H(\mathbf{L}F), H\alpha)$ is a right Kan extension for any functor $H : \operatorname{Ho}\mathcal{D} \to \mathcal{E}$ whatsoever.

It is clear that $(\mathcal{C}^\circ, Q, p)$ is also a left deformation retract for $H\gamma_\mathcal{D} F : \mathcal{C} \to \mathcal{E}$, so the cited proposition yields a right Kan extension (L', α') of $H\gamma_\mathcal{D} F$ along $\gamma_\mathcal{C}$. There is then a unique natural transformation $\varphi : H(\mathbf{L}F) \Rightarrow L'$ such that $\alpha' \bullet \varphi\gamma_\mathcal{C} = H\alpha$, i.e. the following diagram commutes for all objects X in $\mathcal{C}$:

$$\begin{tikzcd} H(\mathbf{L}F)\gamma_\mathcal{C} X \arrow[r, "\varphi_{\gamma_\mathcal{C} X}"] \arrow[d, "H\alpha_X"'] & L'\gamma_\mathcal{C} X \arrow[d, "\alpha'_X"] \\ H\gamma_\mathcal{D} F X \arrow[r, equal] & H\gamma_\mathcal{D} F X \end{tikzcd}$$

However, if $\tilde{X}$ is in $\mathcal{C}^\circ$, then $\alpha_{\tilde{X}}$ and $\alpha'_{\tilde{X}}$ are isomorphisms, and so $\varphi_{\gamma_\mathcal{C} X}$ must be an isomorphism as well. Since the canonical functor $\operatorname{Ho}\mathcal{C}^\circ \to \operatorname{Ho}\mathcal{C}$ is essentially surjective on objects, $\varphi : H(\mathbf{L}F) \Rightarrow L'$ must be a natural isomorphism. In particular, $(H(\mathbf{L}F), H\alpha)$ is indeed a right Kan extension.

(ii). Noting that $\gamma_{\mathcal{D}}\varphi \bullet \alpha$ is a natural transformation $(\mathbf{L}F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}F'$, the universal property of $(\mathbf{L}F', \alpha')$ yields a unique natural transformation $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$ such that $\gamma_{\mathcal{D}}\varphi \bullet \alpha = \alpha' \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}}$, as required.

(iii). We must have

$$\gamma_{\mathcal{D}}Fp \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}}\varphi \bullet \gamma_{\mathcal{D}}F'p = \gamma_{\mathcal{D}}Fp \bullet \gamma_{\mathcal{D}}\varphi Q$$

as required.

(iv). Since $\alpha^G F \bullet (\mathbf{L}G)\alpha^F$ is a natural transformation $(\mathbf{L}G)(\mathbf{L}F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}GF$, the universal property of $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ yields the required natural transformation $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$.

(v). Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ and $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ constitute a strong left deformation retract for (G, F), and let $\left(\mathbf{L}F, \alpha^F\right)$, $\left(\mathbf{L}G, \alpha^G\right)$, $\left(\mathbf{L}(GF), \alpha^{GF}\right)$ be the total left derived functors for F and G, respectively, as constructed in claim (i). Then,

$$\begin{aligned}
\alpha^{GF} \bullet \boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} &= \alpha^G F \bullet (\mathbf{L}G)\alpha^F \\
&= \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}F \bullet \gamma_{\mathcal{E}}GQ^{\mathcal{D}^\circ}Fp^{\mathcal{C}^\circ} \\
&= \gamma_{\mathcal{E}}GFp^{\mathcal{C}^\circ} \bullet \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}
\end{aligned}$$

so we must have $\boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}$; but $\gamma_{\mathcal{E}}Gp^{\mathcal{D}^\circ}FQ^{\mathcal{C}^\circ}$ is a natural isomorphism because F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$ and G preserves weak equivalences in $\mathcal{D}^\circ$, so we deduce that $\boldsymbol{\mu}_{G,F}$ is also a natural isomorphism (using the fact that $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ is bijective on objects). ■

Corollary 3.3.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *If $F : \mathcal{C} \to \mathcal{D}$ is a relative functor, then* $(\operatorname{Ho}F, \mathrm{id})$ *is a total left derived functor for F.*
- *If $G : \mathcal{D} \to \mathcal{C}$ is a relative functor, then* $(\operatorname{Ho}G, \mathrm{id})$ *is a total right derived functor for G.*

Proof. The two claims are formally dual; we will prove the first version.

By remark 3.3.7, the trivial right deformation retract is a right deformation retract for $F : \mathcal{C} \to \mathcal{D}$. Thus, $\operatorname{Ho}F : \operatorname{Ho}\mathcal{C} \to \operatorname{Ho}\mathcal{D}$ together with $\mathrm{id} : (\operatorname{Ho}F)\gamma_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{D}}F$ constitute a total left derived functor for F. ■

Proposition 3.3.15. *Let $\mathcal{C}$ be a relative category.*

- *If $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract of $\mathcal{C}$ and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ such that* $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \operatorname{weq} \mathcal{C}$*, then the functor* $\mathcal{C}[\mathcal{W}^{-1}] \to \operatorname{Ho} \mathcal{C}$ *induced by the inclusion* $\mathcal{W} \hookrightarrow \operatorname{weq} \mathcal{C}$ *has a fully faithful left adjoint.*

- *If $(\mathcal{C}^\circ, R, i)$ is a right deformation retract of $\mathcal{C}$ and $\mathcal{W}$ is a subcategory of $\mathcal{C}$ such that* $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W} \subseteq \operatorname{weq} \mathcal{C}$*, then the functor* $\mathcal{C}[\mathcal{W}^{-1}] \to \operatorname{Ho} \mathcal{C}$ *induced by the inclusion* $\mathcal{W} \hookrightarrow \operatorname{weq} \mathcal{C}$ *has a fully faithful right adjoint.*

Proof. The two claims are formally dual; we will prove the first version.

Consider the localising functor $\gamma_{\mathcal{W}} : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$. Since $\operatorname{weq} \mathcal{C}^\circ \subseteq \mathcal{W}$, $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for $\gamma_{\mathcal{W}}$, so (by theorem 3.3.13) there exists an absolute right Kan extension (F, α) of $\gamma_{\mathcal{W}} : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ along the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$. Since γ factors through $\gamma_{\mathcal{W}}$, say $\gamma = G\gamma_{\mathcal{W}}$, the 2-dimensional universal property of $\mathcal{C}[\mathcal{W}^{-1}]$ yields a natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}[\mathcal{W}^{-1}]}$ such that $\varepsilon\gamma_{\mathcal{W}} = \alpha$; similar arguments show that (F, ε) is an absolute right Kan extension of $\mathrm{id} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]$ along $G : \mathcal{C}[\mathcal{W}^{-1}] \to \operatorname{Ho} \mathcal{C}$, so F is a left adjoint for G with counit ε, by proposition A.5.21.

It remains to be shown that $F : \operatorname{Ho} \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is fully faithful. Consider the natural transformation $G\varepsilon : GFG \Rightarrow G$. The total derived functor theorem says $\varepsilon\gamma_{\mathcal{W}} : FG\gamma_{\mathcal{W}} \Rightarrow \gamma_{\mathcal{W}}$ is given by $\gamma_{\mathcal{W}} p$, so $G\varepsilon\gamma_{\mathcal{W}}$ is given by $G\gamma_{\mathcal{W}} p$, which is a natural isomorphism. Since $\gamma_{\mathcal{W}}$ is bijective on objects, we deduce that $G\varepsilon$ itself is a natural isomorphism. Thus, $\eta G : G \Rightarrow GFG$ is a natural isomorphism (by the right triangle identity), and since G is bijective on objects, we may use proposition A.1.2 to see that F is fully faithful. ■

Definition 3.3.16. The **2-category of small left deformation retracts** is defined as follows:

- The objects are pairs $\left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ where $\mathcal{C}$ is a small relative category and $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract of $\mathcal{C}$.

- A 1-morphism $F : \left(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right) \to \left(\mathcal{D}, \mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ is an ordinary functor $F : \mathcal{C} \to \mathcal{D}$, such that $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract for F, and F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$.

- The 2-morphisms are ordinary natural transformations.

- All compositions and identities are inherited from 2-category of small categories.

We write $\mathfrak{LDef}$ for this 2-category, and we write LDefFun for its hom-sets.

The **2-category of small right deformation retracts** is defined dually:

- The objects are pairs $(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ})$ where $\mathcal{D}$ is a small relative category and $(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ})$ is a right deformation retract of $\mathcal{D}$.
- A 1-morphism $G : (\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}) \to (\mathcal{C}, \mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ})$ is an ordinary functor $G : \mathcal{D} \to \mathcal{C}$, such that $(\mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ})$ is a right deformation retract for G, and G sends objects in $\mathcal{D}^\circ$ to objects in $\mathcal{C}^\circ$.
- The 2-morphisms are ordinary natural transformations.
- All compositions and identities are inherited from 2-category of small categories.

We write $\mathfrak{RDef}$ for this 2-category, and we write RDefFun for its hom-sets.

Remark 3.3.17. The duality principle for deformation retracts can be formalised as follows: there is a 2-functor $\mathfrak{LDef}^{\mathrm{co}} \to \mathfrak{RDef}$ that sends $(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ})$ to its opposite $(\mathcal{C}^{\mathrm{op}}, (\mathcal{C}^\circ)^{\mathrm{op}}, (Q^{\mathcal{C}^\circ})^{\mathrm{op}}, (p^{\mathcal{C}^\circ})^{\mathrm{op}})$, and it has an evident strict inverse $\mathfrak{RDef}^{\mathrm{co}} \to \mathfrak{LDef}$. Note that these two 2-functors reverse the direction of 2-morphisms but preserve the direction of 1-morphisms!

Corollary 3.3.18. *There are two pseudofunctors,* $\mathbf{L}$ *and* $\mathbf{R}$*, where:*

- $\mathbf{L}$ *is a pseudofunctor* $\mathfrak{LDef} \to \mathfrak{Cat}$ *that sends an object* $(\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ})$ *to the homotopy category* $\operatorname{Ho} \mathcal{C}$*, a 1-morphism* $F : (\mathcal{C}, \mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}) \to (\mathcal{D}, \mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ})$ *to its total left derived functor* $\mathbf{L}F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$*, and a 2-morphism* $\varphi : F \Rightarrow F'$ *to the derived natural transformation* $\mathbf{L}\varphi : \mathbf{L}F \Rightarrow \mathbf{L}F'$*, and* $\mathbf{L}$ *preserves identity 1-morphisms strictly.*
- $\mathbf{R}$ *is a pseudofunctor* $\mathfrak{RDef} \to \mathfrak{Cat}$ *that sends an object* $(\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ})$ *to the homotopy category* $\operatorname{Ho} \mathcal{C}$*, a 1-morphism* $G : (\mathcal{D}, \mathcal{D}^\circ, R^{\mathcal{D}^\circ}, i^{\mathcal{D}^\circ}) \to (\mathcal{C}, \mathcal{C}^\circ, R^{\mathcal{C}^\circ}, i^{\mathcal{C}^\circ})$ *to its total right derived functor* $\mathbf{R}G : \operatorname{Ho} \mathcal{D} \to \operatorname{Ho} \mathcal{C}$*, and a 2-morphism* $\psi : G' \Rightarrow G$ *to the derived natural transformation* $\mathbf{R}\psi : \mathbf{R}G' \Rightarrow \mathbf{R}G$*, and* $\mathbf{R}$ *preserves identity 1-morphisms strictly.*

- $\mathbf{L}$ *and* $\mathbf{R}$ *are compatible with the duality principle, in the sense that the following diagrams commute (strictly):*

$$\begin{tikzcd} \mathfrak{LDef}^{\mathrm{co}} \arrow[r, "\mathbf{L}^{\mathrm{co}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathfrak{Cat}^{\mathrm{co}} \arrow[d, "(-)^{\mathrm{op}}"] \\ \mathfrak{RDef} \arrow[r, "\mathbf{R}"'] & \mathfrak{Cat} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{RDef}^{\mathrm{co}} \arrow[r, "\mathbf{R}^{\mathrm{co}}"] \arrow[d, "(-)^{\mathrm{op}}"'] & \mathfrak{Cat}^{\mathrm{co}} \arrow[d, "(-)^{\mathrm{op}}"] \\ \mathfrak{LDef} \arrow[r, "\mathbf{L}"'] & \mathfrak{Cat} \end{tikzcd}$$

Proof. The main claims follow from theorem 3.3.13; the only thing left to check is that the collection of 2-isomorphisms $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ satisfy the coherence laws for pseudofunctors; that is, we should show that the following diagrams commute:

$$\begin{tikzcd} (\mathbf{L}H)(\mathbf{L}G)(\mathbf{L}F) \arrow[r, "(\mathbf{L}H)\boldsymbol{\mu}_{G,F}"] \arrow[d, "\boldsymbol{\mu}_{H,G}(\mathbf{L}F)"'] & (\mathbf{L}H)\mathbf{L}(GF) \arrow[d, "\boldsymbol{\mu}_{H,GF}"] \\ \mathbf{L}(HG)(\mathbf{L}F) \arrow[r, "\boldsymbol{\mu}_{HG,F}"'] & \mathbf{L}(HGF) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{R}(FGH) \arrow[r, "\boldsymbol{\delta}_{F,GH}"] \arrow[d, "\boldsymbol{\delta}_{FG,H}"'] & (\mathbf{R}F)\mathbf{R}(GH) \arrow[d, "(\mathbf{R}F)\boldsymbol{\delta}_{G,H}"] \\ \mathbf{R}(FG)(\mathbf{R}H) \arrow[r, "\boldsymbol{\delta}_{F,G}(\mathbf{R}H)"'] & (\mathbf{R}F)(\mathbf{R}G)(\mathbf{R}H) \end{tikzcd}$$

However, using the explicit formulae for $\boldsymbol{\mu}$ and $\boldsymbol{\delta}$ in the proof of the theorem, it is easy to see that these diagrams do indeed commute. ∎

Definition 3.3.19. A **deformable adjunction** between two relative categories is an ordinary adjunction where the left adjoint is left deformable and the right adjoint is right deformable.

Theorem 3.3.20. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of ordinary categories, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

(i) *If $F \dashv G : \mathcal{D} \to \mathcal{C}$ is a deformable adjunction, then it admits a derived adjunction.*

(ii) *Let $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be another adjunction, with unit η' and counit ε', and let $H : \mathcal{C}' \to \mathcal{C}$ and $K : \mathcal{D}' \to \mathcal{D}$ be relative functors. If*

- *$(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for F,*
- *$(\mathcal{C}'^\circ, Q', p')$ is a left deformation retract for F',*
- *H sends objects in $\mathcal{C}'^\circ$ to objects in $\mathcal{C}^\circ$,*
- *$(\mathcal{D}^\circ, R, i)$ is a right deformation retract for G,*
- *$(\mathcal{D}'^\circ, R', i')$ is a right deformation retract for G', and*
- *K sends objects in $\mathcal{D}'^\circ$ to objects in $\mathcal{D}^\circ$,*

then for any conjugate pair of natural transformations,

$$\varphi : FH \Rightarrow KF' \qquad\qquad \psi : HG' \Rightarrow GK$$

the derived natural transformations

$$\mathbf{L}\varphi : (\mathbf{L}F)(\mathrm{Ho}\,H) \Rightarrow (\mathrm{Ho}\,K)(\mathbf{L}F') \quad \mathbf{R}\psi : (\mathrm{Ho}\,K)(\mathbf{R}G') \Rightarrow (\mathbf{R}G)(\mathrm{Ho}\,K)$$

also constitute a conjugate pair.

(iii) *Let $F' \dashv G' : \mathcal{D}' \to \mathcal{D}$ be another adjunction, with unit η' and counit ε'. If (F', F) is strongly left deformable and (G, G') is strongly right deformable, then the three derived adjunctions*

$$\mathbf{L}F \dashv \mathbf{R}G : \mathrm{Ho}\,\mathcal{D} \to \mathrm{Ho}\,\mathcal{C}$$
$$\mathbf{L}F' \dashv \mathbf{R}G' : \mathrm{Ho}\,\mathcal{D}' \to \mathrm{Ho}\,\mathcal{D}$$
$$\mathbf{L}(F'F) \dashv \mathbf{R}(GG') : \mathrm{Ho}\,\mathcal{D}' \to \mathrm{Ho}\,\mathcal{C}$$

make $\left(\boldsymbol{\mu}_{F',F}, \boldsymbol{\delta}_{G,G'}\right)$ a conjugate pair of natural transformations, i.e.

$$\left(\boldsymbol{\delta}_{G,G'}\mathbf{L}(F'F)\right) \bullet \bar{\eta}'' = \left((\mathbf{R}G)(\mathbf{R}G')\boldsymbol{\mu}_{F',F}\right) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}$$
$$\bar{\varepsilon}'' \bullet \left(\boldsymbol{\mu}_{F',F}\mathbf{R}(GG')\right) = \bar{\varepsilon}' \bullet (\mathbf{L}F')\bar{\varepsilon}(\mathbf{R}G') \bullet \left((\mathbf{L}F)(\mathbf{L}F')\boldsymbol{\delta}_{G,G'}\right)$$

where $\bar{\eta}''$ and $\bar{\varepsilon}''$ are the unit and counit for $\mathbf{L}(F'F) \dashv \mathbf{R}(GG')$.

Proof. (i). We appeal to theorems 3.3.5 and 3.3.13.

(ii). Recall the following characterisations of $\mathbf{L}\varphi$ and $\mathbf{R}\psi$:

$$\gamma_{\mathcal{D}} K F' p' \bullet (\mathbf{L}\varphi)\gamma_{\mathcal{C}'} = \gamma_{\mathcal{D}}\varphi \bullet \gamma_{\mathcal{D}} F p H$$
$$(\mathbf{R}\psi)\gamma_{\mathcal{D}'} \bullet \gamma_{\mathcal{C}} H G' i' = \gamma_{\mathcal{C}} G i K \bullet \gamma_{\mathcal{C}}\psi$$

We wish to show that these equations hold:

$$\bar{\varepsilon}(\mathrm{Ho}\,K) \bullet (\mathbf{L}F)(\mathbf{R}\psi) = (\mathrm{Ho}\,K)\bar{\varepsilon}' \bullet (\mathbf{L}\varphi)(\mathbf{R}G') \tag{1}$$

$$(\mathbf{R}G)(\mathbf{L}\varphi) \bullet \bar{\eta}(\mathrm{Ho}\,H) = (\mathbf{R}\psi)(\mathbf{L}F') \bullet (\mathrm{Ho}\,H)\bar{\eta}' \tag{2}$$

By proposition A.1.4, it suffices to show that equation (1) is satisfied, and since the canonical functor $\mathrm{Ho}\,\mathcal{D}'^{\circ} \to \mathrm{Ho}\,\mathcal{D}'$ is essentially surjective on objects, equation (1) holds if and only if the following equation holds for all $\hat{A}$ in $\mathcal{D}'^{\circ}$:

$$\bar{\varepsilon}_{\gamma_{\mathcal{D}} K\hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{\mathcal{D}'}\hat{A}} = (\mathrm{Ho}\,K)\bar{\varepsilon}'_{\gamma_{\mathcal{D}'}\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{\mathcal{C}'} G' R' \hat{A}} \tag{3}$$

We observe that $G' i'_{\hat{A}} : G'\hat{A} \to G' R' \hat{A}$ is a weak equivalence in $\mathcal{C}'$ (because $(\mathcal{D}'^{\circ}, R', i')$ is a right deformation retract for G'), so $\gamma_{\mathcal{C}} H G' i'_{\hat{A}}$ is invertible, and we must have

$$(\mathbf{R}\psi)_{\gamma_{\mathcal{D}'}\hat{A}} = \gamma_{\mathcal{C}} G i_{K\hat{A}} \circ \gamma_{\mathcal{C}} \psi_{\hat{A}} \circ \left(\gamma_{\mathcal{C}} H G' i'_{\hat{A}}\right)^{-1}$$

and hence,

$$(\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{\mathcal{D}'}\hat{A}} = \gamma_{\mathcal{D}} F Q G i_{K\hat{A}} \circ \gamma_{\mathcal{D}} F Q \psi_{\hat{A}} \circ \left(\gamma_{\mathcal{D}} F Q H G' i'_{\hat{A}}\right)^{-1}$$

therefore:

$$\begin{aligned}
\bar{\varepsilon}_{\gamma_{\mathcal{D}} K\hat{A}} \circ (\mathbf{L}F)(\mathbf{R}\psi)_{\gamma_{\mathcal{D}'}\hat{A}} &= \gamma_{\mathcal{D}} \varepsilon_{K\hat{A}} \circ \gamma_{\mathcal{D}} F p_{G\hat{A}} \circ \gamma_{\mathcal{D}} F Q \psi_{\hat{A}} \circ \left(\gamma_{\mathcal{D}} F Q H G' i'_{\hat{A}}\right)^{-1} \\
&= \gamma_{\mathcal{D}} \varepsilon_{K\hat{A}} \circ \gamma_{\mathcal{D}} F \psi_{\hat{A}} \circ \gamma_{\mathcal{D}} F p_{HG'\hat{A}} \circ \left(\gamma_{\mathcal{D}} F Q H G' i'_{\hat{A}}\right)^{-1}
\end{aligned}$$

On the other hand,

$$\bar{\varepsilon}'_{\gamma_{\mathcal{D}'}\hat{A}} = \gamma_{\mathcal{D}'} \varepsilon'_{\hat{A}} \circ \gamma_{\mathcal{D}'} F' p'_{G'\hat{A}} \circ \left(\gamma_{\mathcal{D}'} F' Q' G' i'_{\hat{A}}\right)^{-1}$$

and so,

$$\begin{aligned}
&(\mathrm{Ho}\,K)\bar{\varepsilon}'_{\gamma_{\mathcal{D}'}\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{\mathcal{C}'} G' R' \hat{A}} \\
&\qquad = \gamma_{\mathcal{D}} K \varepsilon'_{\hat{A}} \circ \gamma_{\mathcal{D}} K F' p'_{G'\hat{A}} \circ \left(\gamma_{\mathcal{D}} K F' Q' G' i'_{\hat{A}}\right)^{-1} \circ (\mathbf{L}\varphi)_{\gamma_{\mathcal{C}'} G' R' \hat{A}} \\
&\qquad = \gamma_{\mathcal{D}} K \varepsilon'_{\hat{A}} \circ \gamma_{\mathcal{D}} K F' p'_{G'\hat{A}} \circ (\mathbf{L}\varphi)_{\gamma_{\mathcal{C}'} G' \hat{A}} \circ \left(\gamma_{\mathcal{D}} F Q H G' i'_{\hat{A}}\right)^{-1} \\
&\qquad = \gamma_{\mathcal{D}} K \varepsilon'_{\hat{A}} \circ \gamma_{\mathcal{D}} \varphi_{G'\hat{A}} \circ \gamma_{\mathcal{D}} F p_{HG'\hat{A}} \circ \left(\gamma_{\mathcal{D}} F Q H G' i'_{\hat{A}}\right)^{-1}
\end{aligned}$$

but $\varepsilon_{K\hat{A}} \circ F\psi_{\hat{A}} = K\varepsilon'_{\hat{A}} \circ \varphi_{G'\hat{A}}$ by hypothesis, so equation (3) indeed holds.

(iii). Suppose

- $(\mathcal{C}^\circ, Q, p)$ is a left deformation retract for F,
- $(\mathcal{C}'^\circ, Q', p')$ is a left deformation retract for F',
- F sends objects in $\mathcal{C}^\circ$ to objects in $\mathcal{C}'^\circ$,
- $(\mathcal{D}^\circ, R, i)$ is a right deformation retract for G,
- $(\mathcal{D}'^\circ, R', i')$ is a right deformation retract for G', and
- G' sends objects in $\mathcal{D}'^\circ$ to objects in $\mathcal{D}^\circ$,

and recall that the comparison isomorphisms are characterised by the following equations:

$$\boldsymbol{\mu}_{F',F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}'} F' p' F Q \qquad\qquad \boldsymbol{\delta}_{G,G'}\gamma_{\mathcal{D}'} = \gamma_{\mathcal{C}} G i G' R'$$

Thus, $\big(\big((\mathbf{R}G)(\mathbf{R}G') \circ \boldsymbol{\mu}_{F',F}\big) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}\big)\gamma_{\mathcal{C}}$ expands to

$$\begin{aligned}
&\gamma_{\mathcal{C}} G R G' R' F' p' F Q \\
&\qquad \bullet \gamma_{\mathcal{C}}(G R G' i' F' Q' F Q \bullet G R \eta' Q' F Q) \bullet \big(\gamma_{\mathcal{C}} G R p' F Q\big)^{-1} \\
&\qquad\qquad \bullet \gamma_{\mathcal{C}}(G i F Q \bullet \eta Q) \bullet \big(\gamma_{\mathcal{C}} p\big)^{-1}
\end{aligned}$$

and a straightforward calculation then shows

$$\begin{aligned}
&\big(\big(\boldsymbol{\delta}_{G,G'}^{-1} \circ \boldsymbol{\mu}_{F',F}\big) \bullet (\mathbf{R}G)\bar{\eta}'(\mathbf{L}F) \bullet \bar{\eta}\big)\gamma_{\mathcal{C}} \\
&\qquad = \gamma_{\mathcal{C}} G i G' R' F' F Q \bullet \gamma_{\mathcal{C}}(G G' i' F' F Q \bullet G \eta F Q \bullet \eta Q) \bullet \big(\gamma_{\mathcal{C}} p\big)^{-1}
\end{aligned}$$

but the RHS is precisely the definition of $\big(\big(\boldsymbol{\delta}_{G,G'} \mathbf{L}(F'F)\big) \bullet \bar{\eta}''\big)\gamma_{\mathcal{C}}$. The dual calculation proves the other equation. ■

Corollary 3.3.21. *Let $\mathcal{C}, \mathcal{C}', \mathcal{D}, \mathcal{D}'$ be relative categories, let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be two adjunctions of ordinary categories, and let $H : \mathcal{C}' \to \mathcal{C}$ and $K : \mathcal{D}' \to \mathcal{D}$ be homotopical functors. Suppose we have a conjugate pair of natural transformations as in the diagrams below:*

$$\text{(L)} \qquad \begin{tikzcd} \mathcal{C}' \arrow[r, "H"] \arrow[d, "F'"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C} \arrow[d, "F"] \\ \mathcal{D}' \arrow[r, "K"'] & \mathcal{D} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{D}' \arrow[r, "K"] \arrow[d, "G'"'] \arrow[dr, phantom, "\psi\nearrow"] & \mathcal{D} \arrow[d, "G"] \\ \mathcal{C}' \arrow[r, "H"'] & \mathcal{C} \end{tikzcd} \qquad \text{(R)}$$

Assume the following hypotheses:

- $(\mathcal{C}^\circ, Q, p)$ *is a left deformation retract for* F.
- $(\mathcal{C}'^\circ, Q', p')$ *is a left deformation retract for* F'.
- H *sends objects in* $\mathcal{C}'^\circ$ *to objects in* $\mathcal{C}^\circ$.
- $(\mathcal{D}^\circ, R, i)$ *is a right deformation retract for* G.
- $(\mathcal{D}'^\circ, R', i')$ *is a right deformation retract for* G'.
- K *sends objects in* $\mathcal{D}'^\circ$ *to objects in* $\mathcal{D}^\circ$.

Then, considering the derived natural transformations $\mathbf{L}\varphi$ *and* $\mathbf{R}\varphi$*:*

$$\text{(L}'\text{)}\qquad \begin{tikzcd} \operatorname{Ho}\mathcal{C}' \arrow[r, "\operatorname{Ho}H"] \arrow[d, "\mathbf{L}F'"'] \arrow[dr, phantom, "\swarrow \mathbf{L}\varphi"] & \operatorname{Ho}\mathcal{C} \arrow[d, "\mathbf{L}F"] \\ \operatorname{Ho}\mathcal{D}' \arrow[r, "\operatorname{Ho}K"'] & \operatorname{Ho}\mathcal{D} \end{tikzcd} \qquad\qquad \begin{tikzcd} \operatorname{Ho}\mathcal{D}' \arrow[r, "\operatorname{Ho}K"] \arrow[d, "\mathbf{R}G'"'] \arrow[dr, phantom, "\mathbf{R}\psi \nearrow"] & \operatorname{Ho}\mathcal{D} \arrow[d, "\mathbf{R}G"] \\ \operatorname{Ho}\mathcal{C}' \arrow[r, "\operatorname{Ho}H"'] & \operatorname{Ho}\mathcal{C} \end{tikzcd} \qquad \text{(R}'\text{)}$$

- *If diagram* (R) *satisfies the left Beck–Chevalley condition, then so does* (R′).
- *If diagram* (L) *satisfies the right Beck–Chevalley condition, then so does* (L′).

Proof. The theorem says that $\mathbf{L}\varphi$ and $\mathbf{R}\psi$ constitute a conjugate pair of natural transformations, and by theorem 3.3.13 it is clear that $\mathbf{L}\varphi$ (resp. $\mathbf{R}\psi$) is a natural isomorphism if φ (resp. ψ) is a natural isomorphism. ∎

Proposition 3.3.22. *Let* $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ *be relative categories.*

- *Let* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{E}$ *be functors and suppose* (G, F) *is laxly left deformable. If the canonical comparison* $\boldsymbol{\mu}_{G,F} : (\mathbf{L}G)(\mathbf{L}F) \Rightarrow \mathbf{L}(GF)$ *is a natural isomorphism and* $\mathcal{E}$ *is a saturated homotopical category, then* (G, F) *is a left deformable composable pair.*

Dually:

- *Let* $F : \mathcal{C} \to \mathcal{B}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *be functors and suppose* (F, G) *is oplaxly right deformable. If the canonical comparison* $\boldsymbol{\delta}_{F,G} : \mathbf{R}(FG) \Rightarrow (\mathbf{R}F)(\mathbf{R}G)$ *is a natural isomorphism and* $\mathcal{C}$ *is a saturated homotopical category, then* (F, G) *is a left deformable composable pair.*

Proof. Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ and $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ constitute a lax left deformation retract for (G, F). By theorem 3.3.13, we may assume without loss of generality that $(\mathbf{L}F)\gamma_{\mathcal{C}} = \gamma_{\mathcal{D}} FQ$, $(\mathbf{L}G)\gamma_{\mathcal{D}} = \gamma_{\mathcal{C}} GQ^{\mathcal{D}^\circ}$, and $\boldsymbol{\mu}_{G,F}\gamma_{\mathcal{C}} = \gamma_{\mathcal{C}} Gp^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ}$. Our hypothesis says $\boldsymbol{\mu}_{G,F}$ is a natural isomorphism and $\mathcal{E}$ is a saturated homotopical category, so the morphisms $Gp^{\mathcal{D}^\circ}_{FQ^{\mathcal{C}^\circ} X} : GQ^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ} X \Rightarrow GFQ^{\mathcal{C}^\circ} X$ are weak equivalences, for all objects X in $\mathcal{C}$.

Now, let $\tilde{X}$ be an object in $\mathcal{C}^\circ$. The following diagram commutes,

$$\begin{tikzcd}
GQ^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ} \tilde{X} \arrow[r, "Gp^{\mathcal{D}^\circ}_{FQ^{\mathcal{C}^\circ} \tilde{X}}"] \arrow[d, "GQ^{\mathcal{D}^\circ} Fp^{\mathcal{C}^\circ}_{\tilde{X}}"'] & GFQ^{\mathcal{C}^\circ} \tilde{X} \arrow[d, "GFp^{\mathcal{C}^\circ}_{\tilde{X}}"] \\
GQ^{\mathcal{D}^\circ} F\tilde{X} \arrow[r, "Gp^{\mathcal{D}^\circ}_{F\tilde{X}}"'] & GF\tilde{X}
\end{tikzcd}$$

and since $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a left deformation retract for both F and GF, it follows that the downward-pointing arrows in the above diagrams are weak equivalences in $\mathcal{E}$; so using the 2-out-of-3 property of weq $\mathcal{E}$ and the fact that $Gp^{\mathcal{D}^\circ}_{FQ^{\mathcal{C}^\circ} \tilde{X}}$ is a weak equivalence, we deduce that $Gp^{\mathcal{D}^\circ}_{F\tilde{X}}$ is a weak equivalence in $\mathcal{E}$. Thus, recalling proposition 3.3.9, we obtain a left deformation retract $\left(\mathcal{D}^\circ_G, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ for G such that F sends every object in $\mathcal{C}^\circ$ to an object in $\mathcal{D}^\circ_G$, and so (G, F) is indeed strongly left deformable. ∎

Corollary 3.3.23. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be relative categories, and let*

$$F_! \dashv F^* : \mathcal{D} \to \mathcal{C} \qquad\qquad G_! \dashv G^* : \mathcal{E} \to \mathcal{D}$$

be adjunctions of ordinary categories. If $\mathcal{C}$ and $\mathcal{E}$ are saturated homotopical categories, then the following are equivalent:

(i) $\left(G_!, F_!\right)$ *is strongly left deformable and* (F^*, G^*) *is strongly right deformable.*

(ii) $\left(G_!, F_!\right)$ *is laxly left deformable and* (F^*, G^*) *is strongly right deformable.*

(iii) $\left(G_!, F_!\right)$ *is strongly left deformable and* (F^*, G^*) *is oplaxly right deformable.*

Proof. Theorem 3.3.20 says $\left(\boldsymbol{\mu}_{G_!,F_!}, \boldsymbol{\delta}_{F^*,G^*}\right)$ is a conjugate pair of natural transformations, and the pasting lemma (A.1.10) implies $\boldsymbol{\mu}_{G_!,F_!}$ is a natural isomorphism if and only if $\boldsymbol{\delta}_{F^*,G^*}$ is a natural isomorphism, so the equivalence of the three statements follows from the proposition above. ∎

Proposition 3.3.24. *Let C and $\mathcal{D}$ be two relative categories, let $F \dashv G : \mathcal{D} \to C$ be an adjunction of ordinary categories with unit η and counit ε, let (C°, Q, p) be a left deformation retract for F, and let $(\mathcal{D}^{\circ}, R, i)$ be a right deformation retract for G. Consider the following statements:*

(i) *For all objects $\tilde{X}$ in C° and all objects $\hat{B}$ in $\mathcal{D}^{\circ}$, if $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$, then its right adjoint transpose $\tilde{X} \to G\hat{B}$ is a weak equivalence in C.*

(ii) *For all objects X in C, The morphism $Gi_{FQX} \circ \eta_{QX} : QX \to GRFQX$ is a weak equivalence in C.*

(iii) *The derived unit $\bar{\eta} : \mathrm{id}_{\mathrm{Ho}\, C} \Rightarrow (\mathbf{R}G)(\mathbf{L}F)$ is a natural isomorphism.*

(i′) *For all objects $\tilde{X}$ in C° and all objects $\hat{B}$ in $\mathcal{D}^{\circ}$, if $\tilde{X} \to G\hat{B}$ is a weak equivalence in C, then its left adjoint transpose $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$.*

(ii′) *For all objects B in $\mathcal{D}$, the morphism $\varepsilon_{RB} \circ Fp_{GRB} : FQGRB \Rightarrow RB$ is a weak equivalence in $\mathcal{D}$.*

(iii′) *The derived counit $\bar{\varepsilon} : (\mathbf{L}F)(\mathbf{R}G) \Rightarrow \mathrm{id}_{\mathrm{Ho}\, \mathcal{D}}$ is a natural isomorphism.*

We have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*; if* $\mathrm{weq}\, C$ *has the 2-out-of-3 property, then* (ii) $\Rightarrow$ (i)*; and if C is a saturated homotopical category, then* (iii) $\Rightarrow$ (ii). *Dually,* (i′) $\Rightarrow$ (ii′) $\Rightarrow$ (iii′)*; if* $\mathrm{weq}\, \mathcal{D}$ *has the 2-out-of-3 property, then* (ii′) $\Rightarrow$ (i′)*; and if $\mathcal{D}$ is a saturated homotopical category, then* (iii′) $\Rightarrow$ (ii′).

Proof. (i) $\Rightarrow$ (ii). We have a weak equivalence $i_{FQX} : FQX \to RFQX$, and QX is an object in C°, so by the hypothesis, its right adjoint transpose $Gi_{FQX} \circ \eta_{QX}$ is also a weak equivalence.

(ii) $\Rightarrow$ (iii). The derived unit is given by $\bar{\eta}\gamma_C = \gamma_C(GiFQ \bullet \eta Q) \circ \left(\gamma_C p\right)^{-1}$, which is certainly a natural isomorphism if $Gi_{FQX} \circ \eta_{QX}$ is a weak equivalence for all X.

(ii) ⇒ (i). Assume $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property. Given $\tilde{X}$ in $\mathcal{C}^\circ$, the diagram below commutes,

$$\begin{tikzcd} Q\tilde{X} \arrow[r, "\eta_{Q\tilde{X}}"] \arrow[d, "p_{\tilde{X}}"'] & GFQ\tilde{X} \arrow[r, "Gi_{FQ\tilde{X}}"] & GRFQ\tilde{X} \arrow[d, "GRFp_{\tilde{X}}"] \\ \tilde{X} \arrow[r, "\eta_{\tilde{X}}"'] & GF\tilde{X} \arrow[r, "Gi_{F\tilde{X}}"'] & GRF\tilde{X} \end{tikzcd}$$

but the top row and the two vertical arrows are weak equivalences in $\mathcal{C}$, so the bottom row must be a weak equivalence as well, by the 2-out-of-3 property.

Let $g : F\tilde{X} \to \hat{B}$ be a weak equivalence in $\mathcal{D}$, and let $f = Gg \circ \eta_{\tilde{X}}$ be its right adjoint transpose in $\mathcal{C}$. We know $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor, so $GRg : GRF\tilde{X} \to GR\hat{B}$ is a weak equivalence in $\mathcal{C}$; but

$$Gi_{\hat{B}} \circ f = Gi_{\hat{B}} \circ Gg \circ \eta_{\tilde{X}} = GRg \circ \left(Gi_{F\tilde{X}} \circ \eta_{\tilde{X}}\right)$$

and we know $Gi_{\hat{B}} : G\hat{B} \to GR\hat{B}$ is a weak equivalence in $\mathcal{C}$, so by the 2-out-of-3 property again, f must be a weak equivalence in $\mathcal{C}$.

(iii) ⇒ (ii). Now assume $\mathcal{C}$ is a saturated homotopical category. If $\bar{\eta}$ is a natural isomorphism, then each $\gamma_{\mathcal{C}}(GiFQ \bullet \eta Q)$ must also be a natural isomorphism, and so each $Gi_{FQX} \circ \eta_{QX}$ is a weak equivalence, by the saturation hypothesis. ■

Corollary 3.3.25. *With notation as above, suppose the* **Quillen equivalence condition** *is satisfied:*

- *For all objects $\tilde{X}$ in $\mathcal{C}^\circ$ and all objects $\hat{B}$ in $\mathcal{D}^\circ$, a morphism $F\tilde{X} \to \hat{B}$ is a weak equivalence in $\mathcal{D}$ if and only if its right adjoint transpose $\tilde{X} \to G\hat{B}$ is a weak equivalence in $\mathcal{C}$.*

Then the derived adjunction is an adjoint equivalence of categories. ■

3.4 DHKS derived functors

Prerequisites. §§ 3.1, 3.2, 3.3.

Notice that in theorem 3.3.13, we constructed derived functors by restricting to a relatively equivalent full subcategory on which the functor respects weak equivalences. This suggests that, by strengthening the definition of ‘deformation retract', we may be able to construct derived functors without first passing to the homotopy category.

In this section we follow [DHKS, Ch. VII].

Definition 3.4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **functorial left deformation retract** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a triple $(\mathcal{C}^\circ, Q, p)$ where

- $\mathcal{C}^\circ$ is a full subcategory of $\mathcal{C}$ with the induced relative subcategory structure,
- $Q : \mathcal{C} \to \mathcal{C}$ is a relative functor, and
- $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction $F|_{\mathcal{C}^\circ} : \mathcal{C}^\circ \to \mathcal{D}$ is a relative functor.
- For all objects X in $\mathcal{C}$, the object QX is in $\mathcal{C}^\circ$.

An ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is **functorially left deformable** if there exists a functorial left deformation retract for F.

Dually, a **functorial right deformation retract** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a triple $(\mathcal{D}^\circ, R, i)$ where

- $\mathcal{D}^\circ$ is a full subcategory of $\mathcal{D}$ with the induced relative subcategory structure,
- $R : \mathcal{D} \to \mathcal{D}$ is a relative functor, and
- $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ is a natural weak equivalence,

and these data are required to have the following properties:

- The restriction $G|_{\mathcal{D}^\circ} : \mathcal{D}^\circ \to \mathcal{C}$ is a relative functor.
- For all objects A in $\mathcal{D}$, the object RA is in $\mathcal{D}^\circ$.

An ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is **functorially right deformable** if there exists a functorial right deformation retract for G.

Remark 3.4.2. Every relative functor is both functorially left deformable and functorially right deformable, with trivial functorial left and right deformation retracts.

Remark 3.4.3. The definition above is the one found in [DHKS, § 40] under the name 'deformation retract'; they do not consider the non-functorial version.

Lemma 3.4.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$, then $(\mathcal{C}^\circ, Q, p)$ is also a left deformation retract for F.*
- *If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$, then $(\mathcal{D}^\circ, R, i)$ is also a right deformation retract for G.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that axioms DR1, DR2, and DR4 are satisfied, so we need only check axiom DR3. For this, we simply observe that the inclusion $\mathcal{C}^\circ \hookrightarrow \mathcal{C}$ and the relative functor $Q : \mathcal{C} \to \mathcal{C}^\circ$ (together with the natural weak equivalence $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$) constitute a relative equivalence of relative categories; thus, proposition 3.1.27 implies the canonical functor $\operatorname{Ho} \mathcal{C}^\circ \to \operatorname{Ho} \mathcal{C}$ is fully faithful, as required. ■

Proposition 3.4.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories.*

- *Let $Q : \mathcal{C} \to \mathcal{C}$ be a relative functor, let $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ be a natural weak equivalence, and let $\mathcal{C}^\circ$ be the full subcategory of $\mathcal{C}$ spanned by the image of Q. If $\operatorname{weq} \mathcal{D}$ has the 2-out-of-3 property in $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ is a functor such that FQ is a relative functor and $FqQ : FQQ \Rightarrow FQ$ is a natural weak equivalence, then $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for F.*

Dually:

- *Let $R : \mathcal{D} \to \mathcal{D}$ be a relative functor, let $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ be a natural weak equivalence, and let $\mathcal{D}^\circ$ be the full subcategory of $\mathcal{D}$ spanned by the image of R. If $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property in $\mathcal{C}$ and $G : \mathcal{D} \to \mathcal{C}$ is a functor such that GR is a relative functor and $GiR : GR \Rightarrow GRR$ is a natural weak equivalence, then $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for G.*

Proof. Let $f : QX \to QY$ be a weak equivalence in $\mathcal{C}^\circ$. By naturality, the following diagram commutes:

$$\begin{tikzcd}
FQQX \arrow[r, "FQf"] \arrow[d, "Fp_{QX}"'] & FQQY \arrow[d, "Fp_{QY}"] \\
FQX \arrow[r, "Ff"'] & FQY
\end{tikzcd}$$

We know FQf, Fp_{QX}, and Fp_{QY} are weak equivalences in $\mathcal{D}$, so using the 2-out-of-3 property of $\operatorname{weq} \mathcal{D}$, we deduce that Ff is also a weak equivalence in $\mathcal{D}$. Thus $F|_{\mathcal{C}^\circ}$ is a relative functor, as required. ■

Definition 3.4.6. Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories. A **homotopical left approximation** for an ordinary functor $F : \mathcal{C} \to \mathcal{D}$ is a homotopical right (!) Kan extension of F along $\mathrm{id}_\mathcal{C}$. Dually, a **homotopical right approximation** for an ordinary functor $G : \mathcal{D} \to \mathcal{C}$ is a homotopical left (!) Kan extension of G along $\mathrm{id}_\mathcal{D}$.

Remark 3.4.7. More explicitly, a homotopical left approximation for $F : \mathcal{C} \to \mathcal{D}$ is a homotopically terminal object in the homotopical category $\left([\mathcal{C}, \mathcal{D}]_\mathrm{h} \downarrow F\right)_\mathrm{h}$ described below:

- The objects are pairs (K, α) where K is a homotopical functor $\mathcal{C} \to \mathcal{D}$ and α is a natural transformation of type $K \Rightarrow F$.
- The morphisms $(K', \alpha') \to (K, \alpha)$ are those natural transformations $\psi : K' \Rightarrow K$ such that $\alpha \bullet \psi = \alpha'$.
- The weak equivalences are the natural weak equivalences.

Dually, a homotopical right approximation for $G : \mathcal{D} \to \mathcal{C}$ is a homotopically initial object in the homotopical category $\left(F \downarrow [\mathcal{D}, \mathcal{C}]_\mathrm{h}\right)_\mathrm{h}$. By corollary 3.2.12, homotopical approximations are homotopically unique.

We have the following special case:

Proposition 3.4.8. *Let Q be a homotopical endofunctor on a homotopical category $\mathcal{C}$ and let $p : Q \Rightarrow \mathrm{id}_\mathcal{C}$ be a natural transformation. The following are equivalent:*

(i) *(Q, p) is a homotopical left approximation for* $\mathrm{id}_\mathcal{C}$.

(ii) $(\mathcal{C}, Q, p)$ *is a functorial left deformation retract for* $\mathrm{id}_{\mathcal{C}}$.

Dually, let R *be a homotopical endofunctor on a homotopical category* $\mathcal{D}$, *and let* $i : \mathrm{id}_{\mathcal{D}} \Rightarrow R$ *be a natural transformation. The following are equivalent:*

(i′) (R, i) *is a homotopical right approximation for* $\mathrm{id}_{\mathcal{C}}$.

(ii′) $(\mathcal{D}, R, i)$ *is a functorial right deformation retract for* $\mathrm{id}_{\mathcal{D}}$.

Proof. (i) ⇒ (ii). If (Q, p) is a homotopical left approximation for $\mathrm{id}_{\mathcal{C}}$, then there must exist a commutative diagram of the form below,

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{C}} \arrow[r] \arrow[d, "\mathrm{id}"'] & Q_1 \arrow[d, "p_1"'] & Q_2 \arrow[l] \arrow[r] \arrow[d, "p_2"'] & \cdots & Q \arrow[l] \arrow[d, "p"] \\
\mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \mathrm{id}_{\mathcal{C}} \arrow[r, equal] & \cdots \arrow[r, equal] & \mathrm{id}_{\mathcal{C}}
\end{tikzcd}$$

where all the arrows in the top row are natural weak equivalences. Using 2-out-of-3 property, we deduce (by induction) that $p_1, p_2, \ldots, p$ are also natural weak equivalences; thus $(\mathcal{C}, Q, p)$ is indeed a functorial left deformation retract for $\mathrm{id}_{\mathcal{C}}$.

(ii) ⇒ (i). If $(\mathcal{C}, Q, p)$ is a functorial left deformation retract for $\mathrm{id}_{\mathcal{C}}$, then $p : Q \Rightarrow \mathrm{id}_{\mathcal{C}}$ is a natural weak equivalence; but $\left(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathrm{id}_{\mathcal{C}}}\right)$ is a terminal object in $\left([\mathcal{C}, \mathcal{C}]_{\mathrm{h}} \downarrow \mathrm{id}_{\mathcal{C}}\right)_{\mathrm{h}}$, so by proposition 3.2.2, (Q, p) must be a homotopically terminal object. ■

Definition 3.4.9. Let $F, F' : \mathcal{C} \to \mathcal{D}$ be ordinary functors between homotopical categories, and let $\varphi : F \Rightarrow F'$ be a natural transformation. We define the homotopical category $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$ as follows:

- The objects are tuples $(H, H', \alpha, \alpha', \theta)$ where H and H' are homotopical functors $\mathcal{C} \to \mathcal{D}$, α and α' are natural transformations of type $H \Rightarrow F$ and $H' \Rightarrow F'$ (respectively), and $\theta : H \Rightarrow H'$ is a natural transformation such that $\varphi \bullet \alpha = \alpha' \bullet \theta$.

- The morphisms $(H, H', \alpha, \alpha', \theta) \to (K, K', \beta, \beta', \chi)$ are pairs (ζ, ζ') of natural transformations, where $\zeta : H \Rightarrow K$ and $\zeta' : H' \Rightarrow K'$, such that $\chi \bullet \zeta = \zeta' \bullet \theta$, $\beta \bullet \zeta = \alpha$, and $\beta' \bullet \zeta' = \alpha'$.

- The weak equivalences are those (ζ, ζ') where both ζ and ζ' are natural weak equivalences.

A **homotopical left approximation** for φ is a homotopically terminal object $(\mathbb{L}F, \mathbb{L}F', \delta, \delta', \mathbb{L}\varphi)$ in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$ *such that* $(\mathbb{L}F, \delta)$ is a homotopical left approximation for F and $(\mathbb{L}F', \delta')$ is a homotopical left approximation for F'.

Dually, let $G, G' : \mathcal{D} \to \mathcal{C}$ be ordinary functors between homotopical categories, and let $\psi : G' \Rightarrow G$ be a natural transformation. We define the homotopical category $\left(\psi \downarrow \left[\min \mathbb{2}, [\mathcal{D}, \mathcal{C}]_{\mathrm{h}}\right]_{\mathrm{h}}\right)_{\mathrm{h}}$ as follows:

- The objects are tuples $(H, H', \alpha, \alpha', \theta)$ where H and H' are homotopical functors $\mathcal{D} \to \mathcal{C}$, α and α' are natural transformations of type $G \Rightarrow H$ and $G' \Rightarrow H'$ (respectively), and $\theta : H' \Rightarrow H$ is a natural transformation such that $\alpha \bullet \psi = \theta \bullet \alpha'$.
- The morphisms $(K, K', \beta, \beta', \chi) \to (H, H', \alpha, \alpha', \theta)$ are pairs (ζ, ζ') of natural transformations, where $\zeta : K \Rightarrow H$ and $\zeta' : K' \Rightarrow H'$, such that $\zeta \bullet \chi = \theta \bullet \zeta'$, $\zeta \bullet \beta = \alpha$, and $\zeta' \bullet \beta' = \alpha'$.
- The weak equivalences are those (ζ, ζ') where both ζ and ζ' are natural weak equivalences.

A **homotopical right approximation** for ψ is a homotopically initial object $(\mathbb{R}G, \mathbb{R}G', \delta, \delta', \mathbb{R}\psi)$ in $\left(\psi \downarrow \left[\min \mathbb{2}, [\mathcal{D}, \mathcal{C}]_{\mathrm{h}}\right]_{\mathrm{h}}\right)_{\mathrm{h}}$ *such that* $(\mathbb{R}G, \delta)$ is a homotopical right approximation for G and $(\mathbb{R}G', \delta')$ is a homotopical right approximation for G'.

Theorem 3.4.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories.*

(i) *Let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor. If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for F, then (FQ, Fp) is a homotopical absolute right Kan extension of F along* $\mathrm{id}_{\mathcal{C}}$.

(ii) *Let $F, F' : \mathcal{C} \to \mathcal{D}$ be a parallel pair of ordinary functors. If $(\mathcal{C}^\circ, Q, p)$ is a functorial left deformation retract for both F and F', then for any natural transformation $\varphi : F \Rightarrow F'$, $(FQ, F'Q, Fp, F'p, \varphi Q)$ is a homotopical left approximation for φ.*

(iii) *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be ordinary functors between homotopical categories. If (G, F) is strongly left deformable, then, for any homotopical left approximation $\left((\mathbb{L}F), \delta^F\right)$ for F and any homotopical left approximation $\left((\mathbb{L}G), \delta^G\right)$ for G, $\left((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\right)$ is a homotopical left approximation for GF.*

Dually:

(i′) *Let $G : \mathcal{D} \to \mathcal{C}$ be an ordinary functor. If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for F, then (GR, Gi) is a homotopical absolute left Kan extension of G along* $\mathrm{id}_{\mathcal{D}}$.

(ii′) *Let $G, G' : \mathcal{D} \to \mathcal{C}$ be a parallel pair of ordinary functors. If $(\mathcal{D}^\circ, R, i)$ is a functorial right deformation retract for both G and G', then for any natural transformation $\psi : G' \Rightarrow G$, $(GR, G'R, Gi, G'i, \psi R)$ is a homotopical right approximation for ψ.*

(iii′) *Let $F : \mathcal{C} \to \mathcal{B}$ and $G : \mathcal{D} \to \mathcal{C}$ be ordinary functors between homotopical categories. If (F, G) is strongly right deformable, then, for any homotopical right approximation $\big((\mathbb{R}F), \delta^F\big)$ for F and any homotopical right approximation $\big((\mathbb{R}G), \delta^G\big)$ for G, $\big((\mathbb{R}F)(\mathbb{R}G), \delta^F \circ \delta^G\big)$ is a homotopical right approximation for FG.*

Proof. (i). Let $H : \mathcal{D} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{E}$ be any two homotopical functors, and let $\alpha : K \Rightarrow HF$ be any natural transformation. Then, we have the following commutative diagram of natural transformations,

$$\begin{tikzcd} K & KQ \arrow[l, "Kp"'] \arrow[r, "\alpha Q"] \arrow[d] & HFQ \arrow[dl, "HFp"] \\ & HF \arrow[from=1-1, "\alpha"'] & \end{tikzcd}$$

and, for any other homotopical functor $K' : \mathcal{C} \to \mathcal{E}$ and natural transformation $\psi : K' \Rightarrow K$, for $\alpha' = \alpha \bullet \psi$, the diagram

$$\begin{tikzcd} K' \arrow[d, "\psi"'] & K'Q \arrow[l, "K'p"'] \arrow[r, "\alpha' Q"] \arrow[d, "\psi Q"] & HFQ \arrow[d, equal] \\ K \arrow[dr, "\alpha"'] & KQ \arrow[l, "Kp"'] \arrow[r, "\alpha Q"] \arrow[d] & HFQ \arrow[dl, "HFp"] \\ & HF & \end{tikzcd}$$

also commutes; thus, (HFQ, HFp) is indeed a homotopically terminal object in $\big([\mathcal{C}, \mathcal{E}]_{\mathrm{h}} \downarrow HF\big)_{\mathrm{h}}$.

(ii). Suppose $(H, H', \alpha, \alpha', \theta)$ is an object in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$. The diagram below commutes,

$$\begin{tikzcd}
& HQ \arrow[rr, "\theta Q"] \arrow[dl, "Hp"'] \arrow[dd, "\alpha Q" near end] & & H'Q \arrow[dl, "H'p"'] \arrow[dd, "\alpha' Q"] \\
H \arrow[rr, crossing over, "\theta" near end] \arrow[dd, "\alpha"'] & & H' & \\
& FQ \arrow[rr, "\varphi Q"' near start] \arrow[dl, "Fp"] & & F'Q \arrow[dl, "F'p"] \\
F \arrow[rr, "\varphi"'] & & F' \arrow[from=uu, crossing over, "\alpha'" near start] &
\end{tikzcd}$$

and $(Hp, H'p)$ is a weak equivalence, so $(FQ, F'Q, Fp, F'p, \varphi Q)$ is indeed a homotopically terminal object in $\left(\left[\min \mathbb{2}, [\mathcal{C}, \mathcal{D}]_{\mathrm{h}}\right]_{\mathrm{h}} \downarrow \varphi\right)_{\mathrm{h}}$.

(iii). Let $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ and $\left(\mathcal{D}^\circ, Q^{\mathcal{D}^\circ}, p^{\mathcal{D}^\circ}\right)$ be functorial left deformation retracts for F and G respectively, and suppose F maps objects in $\mathcal{C}^\circ$ to objects in $\mathcal{D}^\circ$. To begin, observe that $Gp^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ} : GQ^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ} \Rightarrow GFQ^{\mathcal{C}^\circ}$ is a natural weak equivalence; and, as established above, both $\delta^F Q^{\mathcal{C}^\circ} : (\mathbb{L}F) Q^{\mathcal{C}^\circ} \Rightarrow FQ^{\mathcal{C}^\circ}$ and $\delta^G Q^{\mathcal{D}^\circ} : (\mathbb{L}G) Q^{\mathcal{D}^\circ} \Rightarrow GQ^{\mathcal{D}^\circ}$ are natural weak equivalences, so their horizontal composite $\left(\delta^G Q^{\mathcal{C}^\circ}\right) \circ \left(\delta^F Q^{\mathcal{D}^\circ}\right)$ is also a natural weak equivalence. We also know that $\left(\mathcal{C}^\circ, Q^{\mathcal{C}^\circ}, p^{\mathcal{C}^\circ}\right)$ is a functorial left deformation retract for GF, so $\left(GFQ^{\mathcal{C}^\circ}, GFp^{\mathcal{C}^\circ}\right)$ is a homotopical left approximation for GF. Now, noting that the following diagram commutes,

$$\begin{tikzcd}
(\mathbb{L}G) Q^{\mathcal{D}^\circ} (\mathbb{L}F) Q^{\mathcal{C}^\circ} \arrow[r, "(\delta^G Q^{\mathcal{D}^\circ}) \circ (\delta^F Q^{\mathcal{C}^\circ})"] \arrow[d, "((\mathbb{L}G) p^{\mathcal{D}^\circ}) \circ ((\mathbb{L}F) p^{\mathcal{C}^\circ})"'] & GQ^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ} \arrow[r, "Gp^{\mathcal{D}^\circ} FQ^{\mathcal{C}^\circ}"] \arrow[d, "(Gp^{\mathcal{D}^\circ}) \circ (Fp^{\mathcal{C}^\circ})"'] & GFQ^{\mathcal{C}^\circ} \arrow[d, "GFp^{\mathcal{C}^\circ}"] \\
(\mathbb{L}G)(\mathbb{L}F) \arrow[r, "\delta^G \circ \delta^F"'] & GF \arrow[r, equal] & GF
\end{tikzcd}$$

we conclude that $\left((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\right)$ and $\left(GFQ^{\mathcal{C}^\circ}, GFp^{\mathcal{C}^\circ}\right)$ are weakly equivalent in $\left([\mathcal{C}, \mathcal{E}]_{\mathrm{h}} \downarrow GF\right)_{\mathrm{h}}$, and so $\left((\mathbb{L}G)(\mathbb{L}F), \delta^G \circ \delta^F\right)$ is also a homotopical left approximation for GF, by proposition 3.2.2. ■

Remark 3.4.11. Unlike the situation we had with total derived functors, the assignment $F \mapsto FQ$ (resp. $G \mapsto GR$) is not a lax (resp. oplax) 2-functor, because we do not have a natural transformation $\mathrm{id}_{\mathcal{C}} \Rightarrow Q$ (resp. $R \Rightarrow \mathrm{id}_{\mathcal{D}}$).

Corollary 3.4.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be homotopical categories, and let $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ and $\gamma_{\mathcal{D}} : \mathcal{D} \to \operatorname{Ho}\mathcal{D}$ be the respective localising functors.*

- *If $F : \mathcal{C} \to \mathcal{D}$ is a left deformable functor and $(\mathbb{L}F, \delta)$ is any homotopical left approximation for F, then $\big(\operatorname{Ho}(\mathbb{L}F), \gamma_{\mathcal{D}}\delta\big)$ is a total left derived functor for F.*

- *If $G : \mathcal{D} \to \mathcal{C}$ is a right deformable functor and $(\mathbb{R}G, \delta)$ is any homotopical right approximation for G, then $\big(\operatorname{Ho}(\mathbb{R}G), \gamma_{\mathcal{C}}\delta\big)$ is a total right derived functor for G.*

Proof. Combine theorems 3.3.13 and 3.4.10. ∎

3.5 Two-arrow calculi

Prerequisites. §§ 3.1, A.4.

Definition 3.5.1. Let $\mathcal{C}$ be a relative category.

- We say $\mathcal{C}$ admits a **calculus of spans** if, for any morphism $f : X \to Y$ and any weak equivalence $v : \tilde{Y} \to Y$ in $\mathcal{C}$, there exists a pullback square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "f'"] \arrow[d, dashed, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  where $v' : \tilde{X} \to X$ is also a weak equivalence in $\mathcal{C}$.

- We say $\mathcal{C}$ admits a **calculus of cospans** if, for any weak equivalence $u : Y \to \hat{Y}$ and any morphism $g : Y \to Z$ in $\mathcal{C}$, there exists a pushout square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[d, dashed, "u'"] \\ \hat{Y} \arrow[r, dashed, "g'"'] & \hat{Z} \end{tikzcd}$$

  where $u' : Z \to \hat{Z}$ is also a weak equivalence in $\mathcal{C}$.

We follow J. F. Jardine [2009] in using the following terminology:

Definition 3.5.2. Let C be a relative category.

- A **cocycle** $(f, v) : X \nrightarrow Y$ in C is a span of the form below,

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} Y$$

  where $v : \tilde{X} \to X$ is a weak equivalence in C and $f : \tilde{X} \to Y$ is any morphism. The **cocycle category** $C^{\sim\to}(X, Y)$ is the category whose objects are cocycles $X \nrightarrow Y$ in C and whose morphisms are commutative diagrams of the following form,

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] \arrow[d] & Y \arrow[d, equal] \\ X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] & Y \end{tikzcd}$$

  with composition and identities inherited from C.

- A **cycle** $(u, f) : X \nrightarrow Y$ in C is a cospan of the form below,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

  where $u : Y \to \hat{Y}$ is a weak equivalence in C and $f : X \to \hat{Y}$ is any morphism. The **cycle category** $C^{\to\sim}(X, Y)$ is the category whose objects are cycles $X \nrightarrow Y$ in C and whose morphisms are commutative diagrams of the following form,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & \hat{Y} \arrow[d] & Y \arrow[l, "u"'] \arrow[d, equal] \\ X \arrow[r, "f'"'] & \hat{Y}' & Y \arrow[l, "u'"] \end{tikzcd}$$

  with composition and identities inherited from C.

Remark 3.5.3. In many cases of interest, C will be a relative category where $\operatorname{weq} C$ does *not* have the 2-out-of-3 property; as such, we cannot assume that the underlying morphism of a morphism of cocycles or cycles is a weak equivalence.

¶ 3.5.4. Let $\mathcal{C}$ be a relative category that admits a calculus of spans. Given a pair of cocycles in $\mathcal{C}$, say (f, v) and (g, v') as below,

$$\begin{tikzcd} X & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] & Y & \tilde{Y} \arrow[l, "v'"'] \arrow[r, "g"] & Z \end{tikzcd}$$

a **composition** for the pair is a commutative diagram of the following form,

$$\begin{tikzcd} & & W \arrow[dl, "v''"'] \arrow[dr, "f''"] & & \\ & \tilde{X} \arrow[dl, "v"'] \arrow[dr, "f"] & & \tilde{Y} \arrow[dl, "v'"'] \arrow[dr, "f'"] & \\ X & & Y & & Z \end{tikzcd}$$

where the diamond is a pullback square with $v'' : W \to \tilde{X}$ a weak equivalence in $\mathcal{C}$, and the **composite** is the cocycle $(f' \circ f'', v \circ v'')$. It is clear that compositions exist and are unique up to unique isomorphism (in the appropriate sense). Moreover, composition is associative and unital up to coherent natural isomorphism, so we get a **bicategory of cocycles** in $\mathcal{C}$, which we denote by $\mathcal{C}^{\sim\to}$, and we have an obvious pseudofunctor $\mathcal{C} \to \mathcal{C}^{\sim\to}$ that sends a morphism $f : X \to Y$ in $\mathcal{C}$ to the cocycle (f, id_X).

Dually, if $\mathcal{C}$ is a relative category that admits a calculus of cospans, then we get a **bicategory of cycles** in $\mathcal{C}$, which we denote by $\mathcal{C}^{\to\sim}$, and we have an obvious pseudofunctor $\mathcal{C} \to \mathcal{C}^{\to\sim}$ that sends a morphism $f : X \to Y$ in $\mathcal{C}$ to the cycle (id_Y, f).

Remark 3.5.5. If $\mathcal{C}$ is a small relative category, then the category of cocycles or cycles between any two objects is a small category; but if $\mathcal{C}$ is merely locally small, then the category of cocycles or cycles may not even be essentially small.

Theorem 3.5.6 (Fundamental theorem of calculi of spans and cospans)**.** *Let $\mathcal{C}$ be a small relative category and let π_0 : **Cat** → **Set** be the connected components functor.*[1]

- *If $\mathcal{C}$ admits a calculus of spans and $\pi_0[\mathcal{C}^{\sim\to}]$ is the category obtained by applying π_0 to the hom-categories of the bicategory of cocycles, then the pseudofunctor $\mathcal{C} \to \mathcal{C}^{\sim\to}$ induces an isomorphism $\operatorname{Ho}\mathcal{C} \to \pi_0[\mathcal{C}^{\sim\to}]$.*

[1] Recall proposition A.2.15.

- *If $\mathcal{C}$ admits a calculus of cospans and $\pi_0[\mathcal{C}^{\to\sim}]$ is the category obtained by applying π_0 to the hom-categories of the bicategory of cycles, then the pseudofunctor $\mathcal{C} \to \mathcal{C}^{\to\sim}$ induces an isomorphism* $\operatorname{Ho}\mathcal{C} \to \pi_0[\mathcal{C}^{\to\sim}]$.

Proof. The two claims are formally dual; we will prove the first version.

Let $v : \tilde{X} \to X$ be a weak equivalence in $\mathcal{C}$. We must first show that the cocycle $\left(v, \mathrm{id}_{\tilde{X}}\right) : \tilde{X} \nrightarrow X$ becomes an isomorphism in $\pi_0[\mathcal{C}^{\sim\to}]$. Consider the cocycle $\left(\mathrm{id}_{\tilde{X}}, v\right) : X \nrightarrow \tilde{X}$. The following diagram commutes,

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "v"] \arrow[d, "v"] & X \arrow[d, equal] \\
X & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & X
\end{tikzcd}$$

so $\left(v, \mathrm{id}_{\tilde{X}}\right) \circ \left(\mathrm{id}_{\tilde{X}}, v\right) = \left(\mathrm{id}_X, \mathrm{id}_X\right)$ in $\pi_0[\mathcal{C}^{\sim\to}]$. On the other hand, given a pullback square in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
K \arrow[r, "p_1"] \arrow[d, "p_0"'] & \tilde{X} \arrow[d, "v"] \\
\tilde{X} \arrow[r, "v"'] & X
\end{tikzcd}$$

where $p_0 : K \to \tilde{X}$ is a weak equivalence, the universal property of K yields a unique morphism $\Delta : X \to K$ making the diagram below commute:

$$\begin{tikzcd}
X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "\mathrm{id}"] \arrow[d, dashed, "\Delta"'] & X \arrow[d, equal] \\
X & K \arrow[l, "p_0"] \arrow[r, "p_1"'] & X
\end{tikzcd}$$

Thus, $\left(\mathrm{id}_{\tilde{X}}, v\right) \circ \left(v, \mathrm{id}_{\tilde{X}}\right) = \left(\mathrm{id}_{\tilde{X}}, \mathrm{id}_{\tilde{X}}\right)$ in $\pi_0[\mathcal{C}^{\sim\to}]$. It now follows that every morphism $X \to Y$ in $\pi_0[\mathcal{C}^{\sim\to}]$ is of the form $\left(f, \mathrm{id}_{\tilde{X}}\right) \circ \left(v, \mathrm{id}_{\tilde{X}}\right)^{-1}$ for some weak equivalence $v : \tilde{X} \to X$ in $\mathcal{C}$ and some morphism $f : \tilde{X} \to Y$; hence, the induced functor $\operatorname{Ho}\mathcal{C} \to \pi_0[\mathcal{C}^{\sim\to}]$ is a bijection on objects and full.

It remains to be shown that the functor $\operatorname{Ho} C \to \pi_0[C^{\sim\to}]$ is faithful. Suppose we have the following commutative diagram in C,

$$\begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] \arrow[d, "h"] & X \arrow[d, equal] \\ X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] & X \end{tikzcd}$$

where $v : \tilde{X} \to X$ and $v' : \tilde{X}' \to X$ are weak equivalences in C. The 2-out-of-3 property of isomorphisms in $\operatorname{Ho} C$ ensures $h : \tilde{X} \to \tilde{X}'$ is an isomorphism in $\operatorname{Ho} C$, so:

$$f \circ v^{-1} = (f' \circ h) \circ (h \circ v')^{-1} = f' \circ v'^{-1}$$

We may therefore define a functor $\pi_0[C^{\sim\to}] \to \operatorname{Ho} C$ that sends the connected component of a cocycle $(f, v) : X \nrightarrow Y$ in C to the morphism $f \circ v^{-1}$ in $\operatorname{Ho} C$; and using the fact that localising functor $C \to \operatorname{Ho} C$ is an epimorphism in **Cat**, we see that this functor is a left inverse for the functor $\operatorname{Ho} C \to \pi_0[C^{\sim\to}]$ constructed in the previous paragraph. Thus $\operatorname{Ho} C \to \pi_0[C^{\sim\to}]$ is indeed an isomorphism. ■

Proposition 3.5.7. *Let C be a relative category in which* weq C *has the 2-out-of-3 property, and let X and Y be objects in C.*

- *If C admits a calculus of spans, then the cocycle category $C^{\sim\to}(X, Y)$ (considered as a maximal relative category) also admits a calculus of spans.*
- *If C admits a calculus of cospans, then the cycle category $C^{\to\sim}(X, Y)$ (considered as a maximal relative category) also admits a calculus of cospans.*

Proof. The two claims are formally dual; we will prove the first version.

Since weq C has the 2-out-of-3 property in C, the underlying morphisms of morphisms of cocycles must be weak equivalences in C. It follows that pullbacks in $C^{\sim\to}(X, Y)$ exist and can be constructed componentwise in C. ■

Corollary 3.5.8. *Let C be a relative category in which* weq C *has the 2-out-of-3 property.*

- *Let (f, v) and (f', v') be two cocycles $X \nrightarrow Y$ in C. If C admits a calculus of spans, then (f, v) and (f', v') are in the same connected component of*

$C^{\sim\to}(X, Y)$ if and only if there exists a commutative diagram in C of the following form,

$$\begin{tikzcd}
X & \bullet \arrow[l, "v"'] \arrow[r, "f"] & Y \\
X \arrow[u, equal] \arrow[d, equal] & \bullet \arrow[l, "w_3"'] \arrow[r, "f_3"] \arrow[u, "w_1"'] \arrow[d, "w_2"] & Y \arrow[u, equal] \arrow[d, equal] \\
X & \bullet \arrow[l, "v'"] \arrow[r, "f'"'] & Y
\end{tikzcd}$$

where w_1, w_2, w_3 are weak equivalences in C.

- *Let (u, g) and (u', g') be two cycles $X ⇸ Y$ in C. If C admits a calculus of cospans, then (u, g) and (u', g') are in the same connected component of $C^{\to\sim}(X, Y)$ if and only if there exists a commutative diagram in C of the following form,*

$$\begin{tikzcd}
X \arrow[r, "g"] & \bullet \arrow[d, "w_1"] & Y \arrow[l, "u"'] \\
X \arrow[u, equal] \arrow[d, equal] \arrow[r, "g_3"] & \bullet & Y \arrow[l, "w_3"'] \arrow[u, equal] \arrow[d, equal] \\
X \arrow[r, "g'"'] & \bullet \arrow[u, "w_2"'] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where w_1, w_2, w_3 are weak equivalences in C.

Proof. Combine the fundamental theorem of calculi of spans and cospans (3.5.6) with the previous proposition. ∎

The following definition is due to Gabriel and Zisman [GZ].

Definition 3.5.9. Let C be a relative category. We say C admits a **calculus of right fractions** if the following axioms are satisfied:

- (Right Ore condition). Given any morphism $f : X \to Y$ in C and any weak equivalence $v : X \to \tilde{X}$, there exists a commutative diagram of the form below,

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "f'"] \arrow[d, dashed, "v'"'] & \tilde{Y} \arrow[d, "v"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

where $v' : \tilde{X} \to X$ is also a weak equivalence in C.

- (Right cancellability). Given any parallel pair $f_0, f_1 : X \to Y$ in C, if $t : Y \to T$ is a weak equivalence in C such that $t \circ f_0 = t \circ f_1$, then there exists a weak equivalence $s : S \to X$ such that $f_0 \circ s = f_1 \circ s$.

Dually, we say C admits a **calculus of left fractions** if the following axioms are satisfied:

- (Left Ore condition). Given any weak equivalence $u : Y \to \hat{Y}$ and any morphism $g : Y \to Z$ in C, there exists a commutative diagram of the form below,

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & Z \arrow[d, dashed, "u'"] \\ \hat{Y} \arrow[r, dashed, "g'"'] & \hat{Z} \end{tikzcd}$$

where $u' : Z \to \hat{Z}$ is also a weak equivalence in C.

- (Left cancellability). Given any parallel pair $g_0, g_1 : Y \to Z$ in C, if $s : S \to Y$ is a weak equivalence in C such that $g_0 \circ s = g_1 \circ s$, then there exists a weak equivalence $t : Z \to T$ such that $t \circ g_0 = t \circ g_1$.

Remark 3.5.10. Although we cannot compose cocycles (resp. cycles) using pullbacks (resp. pushouts) and form a bicategory of cocycles (resp. cycles) in a relative category C with a calculus of right fractions (resp. calculus of left fractions), the axioms are still enough to give a well-defined category $\pi_0[C^{\sim\to}]$ (resp. $\pi_0[C^{\to\sim}]$).

Lemma 3.5.11. *Let Y be an object in a relative category C.*

- *Let $(C_{/Y})_{\mathrm{w}}$ be the full subcategory of the slice category $C_{/Y}$ spanned by the objects $v : \tilde{Y} \to Y$ where v is a weak equivalence in C. If C admits a calculus of right fractions, then $(C_{/Y})_{\mathrm{w}}^{\mathrm{op}}$ is a filtered category.*[2]
- *Let $({}^{Y/}C)_{\mathrm{w}}$ be the full subcategory of the slice category ${}^{Y/}C$ spanned by the objects $u : Y \to \hat{Y}$ where u is a weak equivalence in C. If C admits a calculus of left fractions, then $({}^{Y/}C)_{\mathrm{w}}$ is a filtered category.*

[2] Recall definition 0.2.1.

Proof. The two claims are formally dual; we will prove the first version.

To begin, we observe that id : $Y \to Y$ is an object in $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}$, so $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}$ is indeed an inhabited category. Now suppose we have two objects in $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}$, say $v' : \tilde{Y} \to Y$ and $v : \tilde{Y}' \to Y$. Then the right Ore condition ensures there is a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} & \tilde{Y}'' \arrow[dl] \arrow[dr] \arrow[dd, "v''"] & \\ \tilde{Y} \arrow[dr, "v"'] & & \tilde{Y}' \arrow[dl, "v'"] \\ & Y & \end{tikzcd}$$

where $v'' : \tilde{Y}'' \to Y$ is a weak equivalence in $\mathcal{C}$. Finally, suppose we have a parallel pair of morphisms in $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}$, say $f_0, f_1 : \tilde{Y} \to \tilde{Y}'$ such that $v' \circ f_0 = v' \circ f_1 = v$. The right cancellability condition then yields a weak equivalence $s : S \to \tilde{Y}$ such that $f_0 \circ s = f_1 \circ s$. This completes the proof that $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}$ is a cofiltered category. ∎

Theorem 3.5.12 (Fundamental theorem of calculi of fractions)**.** *Let $\mathcal{C}$ be a relative category.*

- *Let Y and Z be objects in $\mathcal{C}$. If $\mathcal{C}$ admits a calculus of right fractions, then the hom-ensemble maps*

  $$\mathcal{C}\left(\tilde{Y}, Z\right) \to \operatorname{Ho}\mathcal{C}(Y, Z)$$
  $$f \mapsto f \circ v^{-1}$$

  defined by each weak equivalence $v : \tilde{Y} \to Y$ in $\mathcal{C}$ constitute a colimiting cocone over the evident filtered diagram of shape $\left(\mathcal{C}_{/Y}\right)_{\mathrm{w}}^{\mathrm{op}}$.

- *Let X and Y be objects in $\mathcal{C}$. If $\mathcal{C}$ admits a calculus of left fractions, then the hom-ensemble maps*

  $$\mathcal{C}\left(X, \hat{Y}\right) \to \operatorname{Ho}\mathcal{C}(X, Y)$$
  $$g \mapsto u^{-1} \circ g$$

  defined by each weak equivalence $u : Y \to \hat{Y}$ in $\mathcal{C}$ constitute a colimiting cocone over the evident filtered diagram of shape $\left({}^{Y/}\mathcal{C}\right)_{\mathrm{w}}$.

Proof. See Proposition 2.4 in [GZ, Ch. I]. □

Proposition 3.5.13. *Let C be a relative category. Let (f, v) and (f', v') be two cocycles $Y \nrightarrow Z$ in C. If C admits a calculus of right fractions, then the following are equivalent:*

(i) *The cocycles (f, v) and (f', v') are in the same connected component of $C^{\sim\to}(Y, Z)$.*

(ii) *We have $f \circ v^{-1} = f' \circ v'^{-1}$ in $\operatorname{Ho} C$.*

(iii) *There exists a commutative diagram in C of the form below,*

$$\begin{tikzcd}
Y \arrow[d, equal] & \bullet \arrow[l, "v"'] \arrow[r, "f"] & Z \arrow[d, equal] \\
Y \arrow[d, equal] & \bullet \arrow[l, "w_3"'] \arrow[r] \arrow[u] \arrow[d] & Z \arrow[d, equal] \\
Y & \bullet \arrow[l, "v'"] \arrow[r, "f'"'] & Z
\end{tikzcd}$$

where w_3 is a weak equivalence in C.

Dually, let (u, g) and (u', g') be two cocycles $X \nrightarrow Y$ in C. If C admits a calculus of left fractions, then the following are equivalent:

(i′) *The cycles (u, g) and (u', g') are in the same connected component of $C^{\to\sim}(X, Y)$.*

(ii′) *We have $u^{-1} \circ g = u'^{-1} \circ g'$ in $\operatorname{Ho} C$.*

(iii′) *There exists a commutative diagram in C of the form below,*

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & \bullet \arrow[d] & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[r] \arrow[d, equal] & \bullet & Y \arrow[l, "w_3"'] \arrow[d, equal] \\
X \arrow[r, "g'"'] & \bullet \arrow[u] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where w_3 is a weak equivalence in C.

Proof. (i) ⇒ (ii). It is clear that any two cocycles in the same connected component of $\mathcal{C}^{\sim\to}(Y, Z)$ must represent the same morphism $Y \to Z$ in $\operatorname{Ho}\mathcal{C}$.

(ii) ⇒ (iii). Suppose (f, v) and (f', v') represent the same morphism in $\operatorname{Ho}\mathcal{C}$. Using the explicit description of filtered colimits of ensembles, we deduce that there is a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
 & \tilde{Y}'' \arrow[dl, "h_1"'] \arrow[dr, "h_2"] \arrow[dd, "v''"] & \\
\tilde{Y} \arrow[dr, "v"'] & & \tilde{Y}' \arrow[dl, "v'"] \\
 & Y &
\end{tikzcd}$$

where v'' is a weak equivalence in $\mathcal{C}$ and $f \circ h_1 = f' \circ h_2$. Thus, the following diagram commutes, as required:

$$\begin{tikzcd}
Y \arrow[d, equal] & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f"] & Z \arrow[d, equal] \\
Y \arrow[d, equal] & \tilde{Y}'' \arrow[l, "v''"'] \arrow[r] \arrow[u, "h_1"'] \arrow[d, "h_2"] & Z \arrow[d, equal] \\
Y & \tilde{Y}' \arrow[l, "v'"] \arrow[r, "f'"'] & Z
\end{tikzcd}$$

(iii) ⇒ (i). Immediate. ■

Proposition 3.5.14. *Let $\mathcal{C}$ be a homotopical category. If $\mathcal{C}$ admits*

- *a calculus of spans, or*
- *a calculus of cospans, or*
- *a calculus of right fractions, or*
- *a calculus of left fractions*

then $\mathcal{C}$ is a saturated homotopical category.

Proof. The four cases are similar; we will assume that $\mathcal{C}$ admits a calculus of spans.

Suppose $f : X \to Y$ is a morphism that is invertible in $\operatorname{Ho}\mathcal{C}$. Then there exists a cocycle $(g, v) : Y \nrightarrow X$ in $\mathcal{C}$ such that $g \circ v^{-1}$ is a two-sided inverse for f in $\operatorname{Ho}\mathcal{C}$. Construct a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} \tilde{X} \arrow[r, "f'"] \arrow[d, "v'"'] & \tilde{Y} \arrow[d, "v"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

where $v' : \tilde{X} \to X$ is a weak equivalence in $\mathcal{C}$. The fundamental theorem of calculi of spans (3.5.6) implies that $(f \circ g, v) = \left(\mathrm{id}_Y, \mathrm{id}_Y\right)$ and $(g \circ f', v') = \left(\mathrm{id}_X, \mathrm{id}_X\right)$ in $\pi_0[\mathcal{C}^{\sim\to}]$, so by corollary 3.5.8, we must have commutative diagrams of the form below:

$$\begin{tikzcd} Y \arrow[d, equal] & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f \circ g"] & Y \arrow[d, equal] \\ Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[d, equal] \\ Y & Y \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, equal] & \tilde{X} \arrow[l, "v'"'] \arrow[r, "g \circ f'"] & X \arrow[d, equal] \\ X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \arrow[d, equal] \\ X & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

Thus, by repeatedly using the 2-out-of-3 property of $\operatorname{weq}\mathcal{C}$ in $\mathcal{C}$, we see that $f \circ g$ and $g \circ f'$ are weak equivalences in $\mathcal{C}$, and by using the 2-out-of-6 property, we deduce that f (as well as g and f') is indeed a weak equivalence in $\mathcal{C}$. ■

One advantage of calculi of fractions over calculi of spans and cospans is the following:

Proposition 3.5.15. *Let $\mathcal{C}$ be a relative category and let $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ be the localising functor.*

- *If $\mathcal{C}$ admits a calculus of right fractions, then $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ preserves limits for any finite diagram in $\mathcal{C}$.*
- *If $\mathcal{C}$ admits a calculus of left fractions, then $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ preserves colimits for any finite diagram in $\mathcal{C}$.*

Proof. Apply theorems 0.2.10 and 3.5.12. ■

Definition 3.5.16. Let C be a relative category.

- A **colocal object** (or **right-closed object**) in C is an object X in C such that the hom-ensemble map

$$C(X, v) : C(X, \tilde{Y}) \to C(X, Y)$$

  is a bijection for all weak equivalences $v : \tilde{Y} \to Y$ in C.

- A **local object** (or **left-closed object**) in C is an object Y in C such that the hom-ensemble map

$$C(u, Y) : C(\hat{X}, Y) \to C(X, Y)$$

  is a bijection for all weak equivalences $u : X \to \hat{X}$ in C.

Proposition 3.5.17. *Let C be a relative category. If C admits a calculus of right fractions, then the following are equivalent for an object X in C:*

(i) *X is a colocal object in C.*

(ii) *For all weak equivalences $v : \tilde{Y} \to Y$ in C, the hom-ensemble map*

$$C(X, v) : C(X, \tilde{Y}) \to C(X, Y)$$

is a surjection.

(iii) *The map $C(X, Y) \to \operatorname{Ho} C(\gamma X, \gamma Y)$ induced by the localising functor $\gamma : C \to \operatorname{Ho} C$ is a bijection.*

Dually, if C admits a calculus of left fractions, then the following are equivalent for an object Y in C:

(i′) *Y is a local object in C.*

(ii′) *For all weak equivalences $u : X \to \hat{X}$ in C, the hom-ensemble map*

$$C(u, Y) : C(\hat{X}, Y) \to C(X, Y)$$

is a surjection.

(iii′) *The map* $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ *induced by the localising functor* $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *is a bijection.*

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). The fundamental theorem of calculi of fractions (3.5.12) says that there is a natural bijection

$$\varinjlim_{v:(\mathcal{C}_{/X})_{\mathrm{w}}^{\mathrm{op}}} \mathcal{C}(\operatorname{dom} v, Y) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

where v varies over the weak equivalences in $\mathcal{C}$ with codomain X (considered as a full subcategory of the slice category $\mathcal{C}_{/X}$). Note that each weak equivalence $v : \tilde{X} \to X$ is a split epimorphism, so $\operatorname{Ho}\mathcal{C}(X, Y)$ is a filtered colimit for a diagram of injective maps. In particular, the map $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ is injective. On the other hand, if $i : X \to \tilde{X}$ is a section of a weak equivalence $v : \tilde{X} \to X$, then $\gamma(v)^{-1} = \gamma(i)$. Thus, the map $\mathcal{C}(X, Y) \to \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$ is also surjective.

(iii) ⇒ (i). Let $v : \tilde{Y} \to Y$ be any weak equivalence in $\mathcal{C}$. The hom-ensemble bijection in the hypothesis is natural, so we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{C}\big(X, \tilde{Y}\big) \arrow[r, "\gamma"] \arrow[d, "\mathcal{C}(X, v)"'] & \operatorname{Ho}\mathcal{C}\big(\gamma X, \gamma \tilde{Y}\big) \arrow[d, "\operatorname{Ho}\mathcal{C}(\gamma(v))"] \\ \mathcal{C}(X, Y) \arrow[r, "\gamma"'] & \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y) \end{tikzcd}$$

Since $\gamma(v) : \gamma\tilde{Y} \to \gamma Y$ is an isomorphism in $\operatorname{Ho}\mathcal{C}$, the map $\mathcal{C}(X, v)$ must be a bijection. Thus, X is a colocal object in $\mathcal{C}$. ■

¶ 3.5.18. Given a functor $F : \mathcal{C} \to \mathcal{D}$, an **$F$-isomorphism** is a morphism in $\mathcal{C}$ that F sends to an isomorphism in $\mathcal{D}$. Note that $\mathcal{C}$, together with the class of F-isomorphisms, is then a saturated homotopical category by lemma 3.1.8.

Proposition 3.5.19. *Let $\mathcal{C}$ be a relative category. Consider the following statements:*

(i) *The localising functor* $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *has a left adjoint.*

(ii) *The localising functor* $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ *has a fully faithful left adjoint.*

(iii) *For each object X in $\mathcal{C}$, there exists a colocal object $\tilde{X}$ and a γ-isomorphism $p : \tilde{X} \to X$.*

We always have the implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii)*, and if $\mathcal{C}$ admits a calculus of right factions, then* (iii) $\Rightarrow$ (i) *as well.*

Dually:

(i′) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a right adjoint.*

(ii′) *The localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ has a fully faithful right adjoint.*

(iii′) *For each object Y in $\mathcal{C}$, there exists a local object $\hat{Y}$ and a γ-isomorphism $i : Y \to \hat{Y}$.*

We always have the implications (i′) $\Rightarrow$ (ii′) $\Rightarrow$ (iii′)*, and if $\mathcal{C}$ admits a calculus of left fractions, then* (iii′) $\Rightarrow$ (i′) *as well.*

Proof. (i) $\Rightarrow$ (ii). This is proposition A.4.21.

(ii) $\Rightarrow$ (iii). Let $L : \operatorname{Ho}\mathcal{C} \to \mathcal{C}$ be a left adjoint for $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$. We then have the following natural bijection:

$$\mathcal{C}(L\gamma X, Y) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

Since $\gamma v : \gamma\tilde{Y} \to \gamma Y$ is an isomorphism for any weak equivalence $v : \tilde{Y} \to Y$ in $\mathcal{C}$, it follows that $L\gamma X$ is a colocal object in $\mathcal{C}$.

Now, consider the adjunction counit component $\varepsilon_X : L\gamma X \to X$. Proposition A.1.2 says the adjunction unit $\eta : \mathrm{id}_{\operatorname{Ho}\mathcal{C}} \Rightarrow \gamma L$ is a natural isomorphism, so the right triangle identity implies $\gamma\varepsilon_X : \gamma L\gamma X \to \gamma X$ is an isomorphism, i.e. ε_X is a γ-isomorphism, as required.

(iii) $\Rightarrow$ (i). Suppose $\mathcal{C}$ admits a calculus of right fractions. Proposition 3.5.17 says the localising functor $\gamma : \mathcal{C} \to \operatorname{Ho}\mathcal{C}$ induces a natural map

$$\mathcal{C}\big(\tilde{X}, Y\big) \to \operatorname{Ho}\mathcal{C}\big(\gamma\tilde{X}, \gamma Y\big)$$

that is a bijection whenever $\tilde{X}$ is a colocal object, so if $p : \tilde{X} \to X$ is a γ-isomorphism, we obtain a bijection

$$\mathcal{C}\big(\tilde{X}, Y\big) \cong \operatorname{Ho}\mathcal{C}(\gamma X, \gamma Y)$$

that is natural in Y. Since γ is bijective on objects, this implies γ has a left adjoint. ■

Theorem 3.5.20 (Reflective localisations)**.** *Let $U : \mathcal{D} \to \mathcal{C}$ be a fully faithful functor. If U has a left adjoint, say $F : \mathcal{C} \to \mathcal{D}$, then:*

(i) *Let $\mathcal{U}$ be the smallest subcategory of $\mathcal{C}$ that contains all identity morphisms and the components of the adjunction unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$. Then $(\mathcal{C}, \mathcal{U})$ admits a calculus of left fractions.*

(ii) *Any localisation of $\mathcal{C}$ at $\mathcal{U}$ is also a localisation of $\mathcal{C}$ at F-isomorphisms.*

(iii) *The canonical functor $\bar{F} : \mathcal{C}\left[\mathcal{U}^{-1}\right] \to \mathcal{D}$ induced by $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective on objects.*

Dually, if U has a right adjoint, say $H : \mathcal{C} \to \mathcal{D}$, then:

(i′) *Let $\mathcal{V}$ be the smallest subcategory of $\mathcal{C}$ that contains all identity morphisms and the components of the adjunction counit $\varepsilon : UH \Rightarrow \mathrm{id}_{\mathcal{C}}$. Then $(\mathcal{C}, \mathcal{V})$ admits a calculus of right fractions.*

(ii′) *Any localisation of $\mathcal{C}$ at $\mathcal{V}$ is also a localisation of $\mathcal{C}$ at H-isomorphisms.*

(iii′) *The canonical functor $\bar{H} : \mathcal{C}\left[\mathcal{V}^{-1}\right] \to \mathcal{D}$ induced by $H : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective on objects.*

Proof. (i). The naturality of η ensures that $(\mathcal{C}, \mathcal{U})$ satisfies the left Ore condition. Suppose $f_0, f_1 : UFX \to Y$ are morphisms in $\mathcal{C}$ such that $f_0 \circ \eta_X = f_1 \circ \eta_X$. By proposition A.1.2, the adjunction counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, so the triangle identities imply that $\eta UF = FU\eta$. But $\eta_Y \circ f_0 = UFf_0 \circ \eta_{UFX}$ and $\eta_Y \circ f_1 = UFf_1 \circ \eta_{UFX}$, so we may deduce that $\eta_Y \circ f_0 = \eta_Y \circ f_1$. Thus $(\mathcal{C}, \mathcal{U})$ is left cancellable.

(ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By naturality of η, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & UFX \arrow[d, "UFf"] \\ Y \arrow[r, "\eta_Y"'] & UFY \end{tikzcd}$$

Thus, any functor that sends the components of η to isomorphisms must also make F-isomorphisms invertible. On the other hand, $F\eta$ is a natural isomorphism because ε is, so any functor that makes F-isomorphisms invertible must also send the components of η to isomorphisms.

(iii). Since $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, the functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects, and so $\bar{F} : \mathcal{C}\left[\mathcal{V}^{-1}\right] \to \mathcal{D}$ must also be essentially surjective on objects.

It remains to be shown that $\bar{F}$ is a fully faithful functor. Let Y be an object in $\mathcal{C}$, and let $f : X \to X'$ be an F-isomorphism. Since $F \dashv U$, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{D}(FX', FY) \arrow[r, "\cong"] \arrow[d, "\mathcal{D}(Ff,FY)"'] & \mathcal{C}(X', UFY) \arrow[d, "\mathcal{C}(f,UFY)"] \\
\mathcal{D}(FX, FY) \arrow[r, "\cong"'] & \mathcal{C}(X, UFY)
\end{tikzcd}$$

We then see that UFY is a local object in $\mathcal{C}$ (with respect to F-isomorphisms). Since $\eta_Y : Y \to UFY$ is an F-isomorphism, we may then apply proposition 3.5.19 to deduce that the localising functor $\gamma : \mathcal{C} \to \mathcal{C}\left[\mathcal{V}^{-1}\right]$ has a fully faithful right adjoint that sends each object γY to UFY. Thus $\bar{F}$ is indeed fully faithful. ■

3.6 Three-arrow calculi

Prerequisites. §§ 3.1, A.4.

In this section, we follow [DHKS, § 36] and [Thomas, 2011].

Definition 3.6.1. Let $\mathcal{C}$ be a relative category, let $\mathcal{W} = \operatorname{weq} \mathcal{C}$ be the subcategory of weak equivalences in $\mathcal{C}$, and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W}$. We say $\mathcal{C}$ **admits a three-arrow calculus** for $\mathcal{C}$ with respect to $(\mathcal{U}, \mathcal{V})$ if the following conditions are satisfied:

A1. For each weak equivalence w in $\mathcal{C}$, there exist u in $\mathcal{U}$ and v in $\mathcal{V}$ such that $w = v \circ u$.

A2. Given a diagram of the form $\hat{Y} \xleftarrow{u} Y \xrightarrow{g} Z$ in $\mathcal{C}$ with u in $\mathcal{U}$, there exists a diagram of the form $\hat{Y} \xrightarrow{g'} \hat{Z} \xleftarrow{u'} Z$ such that

- $g' \circ u = u' \circ g$,
- u' is in $\mathcal{U}$, and

– given any diagram of the form $\hat{Y} \xrightarrow{y} T \xleftarrow{z} Z$ such that $y \circ u = z \circ g$, there exists a (not necessarily unique) morphism $T \to \hat{Z}$ making the diagram below commute:

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "g'"] \arrow[ddr, bend left, "y"] & \\ Z \arrow[r, "u'"'] \arrow[drr, bend right, "z"'] & \hat{Z} \arrow[dr, dashed] & \\ & & T \end{tikzcd}$$

A3. Given a diagram of the form $X \xrightarrow{f} Y \xleftarrow{v} \tilde{Y}$ in C with v in $\mathcal{V}$, there exists a diagram of the form $X \xleftarrow{v'} \tilde{X} \xrightarrow{f'} \tilde{Y}$ such that

– $f \circ v' = v \circ g'$,

– v' is in $\mathcal{V}$, and

– given any diagram of the form $X \xleftarrow{x} S \xrightarrow{y} Y$ such that $f \circ x = v \circ y$, there exists a (not necessarily unique) morphism $S \to \tilde{X}$ making the diagram below commute:

$$\begin{tikzcd} S \arrow[dr, dashed] \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] & & \\ & \tilde{X} \arrow[r, "v'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & \tilde{Y} \arrow[r, "v"'] & Y \end{tikzcd}$$

A **uni-fractionable category** is a relative category C together with a pair of subcategories $(\mathcal{U}, \mathcal{V})$ such that $\operatorname{weq} C$ has the 2-out-of-3 property in C and C admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$.

REMARK 3.6.2. Note that axiom A1 implies that $\operatorname{ob} \mathcal{U} = \operatorname{ob} \mathcal{V} = \operatorname{ob} C$; in particular, every identity morphism in C is also in $\mathcal{U}$ and $\mathcal{V}$.

Remark 3.6.3. Consider diagrams of the following forms,

$$
\begin{tikzcd}
 & Y \arrow[dl, "u"'] \arrow[dr, "g"] & \\
\hat{Y} \arrow[dr, "g'"'] & & Z \arrow[dl, "u'"] \\
 & \hat{Z} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & \tilde{X} \arrow[dl, "v'"'] \arrow[dr, "f'"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "v"] \\
 & \tilde{Y} &
\end{tikzcd}
$$

where u, u' are in $\mathcal{U}$ and v, v' are in $\mathcal{V}$. Under the assumption that $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{C}$, the morphism g is in $\mathcal{W}$ if and only if g' is in $\mathcal{W}$, and the morphism f is in $\mathcal{W}$ if and only if f' is in $\mathcal{W}$.

Definition 3.6.4. Let $\mathcal{C}$ be a relative category, let $\mathcal{W} = \operatorname{weq} \mathcal{C}$ be the subcategory of weak equivalences in $\mathcal{C}$, and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W}$. A **functorial three-arrow calculus** for $\mathcal{C}$ with respect to $(\mathcal{U}, \mathcal{V})$ consists of the following data:

FA1. A functorial factorisation system on $\mathcal{W}$ with left class contained in $\operatorname{mor} \mathcal{U}$ and right class contained in $\operatorname{mor} \mathcal{V}$.

FA2. A functor from the full subcategory of $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$ spanned by those diagrams of the form $\hat{Y} \overset{u}{\leftarrow} Y \overset{g}{\rightarrow} Z$, where u is in $\mathcal{U}$, to the category $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$, such that each diagram $\hat{Y} \overset{u}{\leftarrow} Y \overset{g}{\rightarrow} Z$ is sent to a diagram of the form $\hat{Y} \overset{g'}{\rightarrow} \hat{Z} \overset{u'}{\leftarrow} Z$, where $g' \circ u = u' \circ g$, u' is in $\mathcal{U}$, and u' is an isomorphism if u is.

FA3. A functor from the full subcategory of $[\{\bullet \rightarrow \bullet \leftarrow \bullet\}, \mathcal{C}]$ spanned by those diagrams of the form $X \overset{f}{\rightarrow} Y \overset{v}{\leftarrow} \tilde{Y}$, where v is in $\mathcal{V}$, to the category $[\{\bullet \leftarrow \bullet \rightarrow \bullet\}, \mathcal{C}]$, such that each diagram $X \overset{f}{\rightarrow} Y \overset{v}{\leftarrow} \tilde{Y}$ is sent to a diagram of the form $X \overset{v'}{\leftarrow} \tilde{X} \overset{f'}{\rightarrow} \tilde{Y}$, where $f \circ v' = v \circ g'$, v' is in $\mathcal{V}$, and v' is an isomorphism if v is.

If such data exist, then we say $\mathcal{C}$ **admits a functorial three-arrow calculus** with respect to $(\mathcal{U}, \mathcal{V})$.

Remark 3.6.5. If $\operatorname{mor} \mathcal{U}$ is closed under pushout in $\mathcal{C}$, then we may take pushouts to construct datum FA2; similarly, if $\operatorname{mor} \mathcal{V}$ is closed under pullback in $\mathcal{C}$, then we may take pullbacks to construct datum FA3.

Remark 3.6.6. A relative category $\mathcal{C}$ admits a (functorial) three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$ if and only if the opposite relative category $\mathcal{C}^{\mathrm{op}}$ admits a (functorial) three-arrow calculus with respect to $(\mathcal{V}, \mathcal{U})$.

Proposition 3.6.7. *Let $\mathcal{C}$ be a relative category and let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\mathcal{W} = \operatorname{weq} \mathcal{C}$ (itself considered as a subcategory of $\mathcal{C}$). If $\mathcal{C}$ admits a functorial three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, then $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$.*

Proof. Obviously, having datum FA1 implies axiom A1 is satisfied. Now suppose we have a commutative square of the form below in $\mathcal{C}$,

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "y"] \\
Z \arrow[r, "z"'] & T
\end{tikzcd}$$

where u is in $\mathcal{U}$. The datum FA2 then gives us the following commutative diagram,

$$\begin{tikzcd}
Y \arrow[rr, "u"] \arrow[dr] \arrow[dd, "g"'] & & \hat{Y} \arrow[dr, "y"] \arrow[dd, "g'"' near end] & \\
& T \arrow[rr, equal] \arrow[dd, equal] & & T \arrow[dd, "w"] \\
Z \arrow[rr, "u'"' near start] \arrow[dr, "z"'] & & \hat{Z} \arrow[dr, dashed] & \\
& T \arrow[rr, "w"'] & & \hat{T}
\end{tikzcd}$$

and $w : T \to \hat{T}$ is an isomorphism, thus, there exists a morphism $\hat{Z} \to T$ making the diagram below commute:

$$\begin{tikzcd}
Y \arrow[r, "u"] \arrow[d, "g"'] & \hat{Y} \arrow[d, "g'"] \arrow[ddr, bend left, "y"] & \\
Z \arrow[r, "u'"'] \arrow[drr, bend right, "z"'] & \hat{Z} \arrow[dr, dashed] & \\
& & T
\end{tikzcd}$$

This shows that axiom A2 is satisfied, and the dual argument proves axiom A3. ∎

Proposition 3.6.8. *Let $\mathcal{A}$ and $\mathcal{C}$ be relative categories. If $\mathcal{C}$ admits a functorial three-arrow calculus, and either*

- $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-3 property in $\mathcal{C}$, or*
- *$\mathcal{A}$ is a minimal relative category,*

then the relative functor category $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$ admits a functorial three-arrow calculus constructed componentwise from $\mathcal{C}$.

Proof. Let $(\mathcal{U}, \mathcal{V})$ be a functorial three-arrow calculus for $\mathcal{C}$. It is clear that, when $\mathcal{A}$ is a minimal relative category, all the data constituting a three-arrow calculus for $\mathcal{C}$ may be lifted componentwise to define a three-arrow calculus for $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$.

In general, we must check that $[\mathcal{A}, \mathcal{C}]_{\mathrm{h}}$ is closed under the various componentwise constructions. However, if $f : A \to B$ is a weak equivalence in $\mathcal{A}$ and $\theta : X \Rightarrow Y$ is a natural weak equivalence of relative functors $X, Y : \mathcal{A} \to \mathcal{M}$, and $\psi \bullet \varphi$ is the componentwise $(\mathcal{U}, \mathcal{V})$-factorisation of θ, then the diagram below commutes,

$$\begin{tikzcd}
XA \arrow[r, "\varphi_A"] \arrow[d, "Xf"'] & ZA \arrow[r, "\psi_A"] \arrow[d, "Zf"'] & YA \arrow[d, "Yf"] \\
XB \arrow[r, "\varphi_B"'] & ZB \arrow[r, "\psi_B"'] & YB
\end{tikzcd}$$

and so by the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$, Zf is also a weak equivalence in $\mathcal{C}$, thus $Z : \mathcal{A} \to \mathcal{M}$ is a relative functor. Similarly, one uses the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$ to ensure that the componentwise constructions satisfy the conditions to be data FA2 and FA3 for a functorial three-arrow calculus. ■

Theorem 3.6.9 (Fundamental theorem of three-arrow calculi)**.** *Let $\mathcal{C}$ be a relative category such that* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-3 property in $\mathcal{C}$. If $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, then:*

(i) *Every morphism in* $\operatorname{Ho} \mathcal{C}$ *can be represented by a zigzag in $\mathcal{C}$ of the form below,*

$$X \xleftarrow{v} \tilde{X} \xrightarrow{f} \hat{Y} \xleftarrow{u} Y$$

where u is in $\mathcal{U}$ and v is in $\mathcal{V}$.

(ii) *Two such zigzags represent the same morphism in* $\operatorname{Ho} \mathcal{C}$ *if and only if there exists a commutative diagram in* $\mathcal{C}$ *of the form*

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[r, "f_3"] \arrow[u, "v_1"'] \arrow[d, "w_1"'] & \bullet \arrow[u, "v_2"] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] \arrow[u, "u_1"'] & \hat{Y}' \arrow[u, "u_2"] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where u_1, u_2, u_3, u_4 *are in* $\mathcal{U}$*,* v_1, v_2, v_3, v_4 *are in* $\mathcal{V}$*, and* w_1, w_2 *are weak equivalences in* $\mathcal{C}$*.*

Proof. For the functorial case, see paragraph 36.3 in [DHKS]; for the general case, see Lemma 4.9 and Theorem 5.13 in [Thomas, 2011]. □

Proposition 3.6.10. *If* $\mathcal{C}$ *is a homotopical category that admits a three-arrow calculus, then* $\mathcal{C}$ *is a saturated homotopical category.*

Proof. Suppose $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ whose image in $\operatorname{Ho} \mathcal{C}$ is an isomorphism, with inverse represented by the following zigzag,

$$Y \xleftarrow{v} \tilde{Y} \xrightarrow{g} \hat{X} \xleftarrow{u} X$$

where u is in $\mathcal{U}$ and v is in $\mathcal{V}$. Then, by axioms A2 and A3, there exist v' in $\mathcal{V}$, f' in $\mathcal{C}$, u'' in $\mathcal{U}$, and f'' in $\mathcal{C}$ such that the diagrams below commute,

$$\begin{tikzcd}
\tilde{X} \arrow[r, "f'"] \arrow[d, "v'"'] & \tilde{Y} \arrow[d, "v"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "u''"] \\
\hat{X} \arrow[r, "f''"'] & \hat{Y}
\end{tikzcd}$$

and by theorem 3.6.9, we have commutative diagrams in $\mathcal{C}$ of the following form,

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] & X \arrow[r, equal] & X \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u] \arrow[d] & X \arrow[l] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l] \arrow[r] & \bullet & X \arrow[l] \arrow[d, equal] \\
X & \tilde{X} \arrow[l, "v'"'] \arrow[r, "g \circ f'"'] \arrow[u] & \hat{X} \arrow[u] & X \arrow[l, "u"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, equal] & Y \arrow[r, equal] & Y \arrow[r, equal] & Y \arrow[d, equal] \\
Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \bullet \arrow[u] \arrow[d] & Y \arrow[l] \arrow[d, equal] \\
Y \arrow[d, equal] & \bullet \arrow[l] \arrow[r] & \bullet & Y \arrow[l] \arrow[d, equal] \\
Y & \tilde{Y} \arrow[l, "v"'] \arrow[r, "f'' \circ g"'] \arrow[u] & \hat{Y} \arrow[u] & Y \arrow[l, "u''"']
\end{tikzcd}
$$

where all leftward- and upward-pointing arrows are weak equivalences in $\mathcal{C}$. We may then deduce that *every* arrow appearing in the above diagrams are in $\operatorname{weq} \mathcal{C}$ by iteratively applying the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$. In particular, $g \circ f'$ and $f'' \circ g$ are weak equivalences in $\mathcal{C}$, so the 2-out-of-6 property of $\operatorname{weq} \mathcal{C}$ implies that f', f'', g are all in $\operatorname{weq} \mathcal{C}$. We then conclude that f is in $\operatorname{weq} \mathcal{C}$, by using the 2-out-of-3 property again. ■

— IV —

Model categories

In [1967], Quillen introduced the notion of a 'closed model category' (but we shall say simply 'model category') for homotopy theory, so as to formalise the similarities between the homotopy theory of spaces and homological algebra. The idea was that, to do homotopy theory, one only really needs to know which morphisms are cofibrations, which are weak equivalences, and which are fibrations.

4.1 Basics

Prerequisites. §§ 3.1, 3.5, 3.6, A.3, A.4.

Definition 4.1.1. A **model structure** on a category $\mathcal{M}$ is a triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of subensembles of mor $\mathcal{M}$ satisfying the following axioms:[1]

CM2. $\mathcal{W}$ has the 2-out-of-3 property.

CM3. $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{F}$ are closed under retracts.

CM4. Given a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ W \arrow[r] & Y \end{tikzcd}$$

[1] This presentation is due to Quillen [1969].

where i is in $\mathcal{C}$ and p is in $\mathcal{F}$, if at least one of i or p is also in $\mathcal{W}$, then there exists a morphism $W \to X$ making both of the evident triangles commute.

CM5. Any morphism f in $\mathcal{M}$ may be factored in two ways:

- $f = p \circ i$ where i is in $\mathcal{C} \cap \mathcal{W}$ and p is in $\mathcal{F}$, and
- $f = q \circ j$, where j is in $\mathcal{C}$ and q is in $\mathcal{W} \cap \mathcal{F}$.

Given a model structure $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ on a category $\mathcal{M}$,

- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **cofibration** is a morphism in $\mathcal{C}$,
- a **fibration** is a morphism in $\mathcal{F}$,
- a **trivial cofibration** (or **acyclic cofibration**) is a morphism in $\mathcal{C} \cap \mathcal{W}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$;
- a **cofibrant object** is an object W that is projective with respect to the class of trivial fibrations, i.e. for every trivial fibration $p : X \to Y$ and every morphism $w : W \to Y$, there exists a morphism $W \to X$ making the diagram below commute:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ W \arrow[ur, dashed] \arrow[r, "w"'] & Y \end{tikzcd}$$

- a **fibrant object** is an object X that is injective with respect to the class of trivial cofibrations, i.e. for every trivial cofibration $i : Z \to W$ and every morphism $z : Z \to X$, there exists a morphism $W \to X$ making the diagram below commute:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "i"'] & X \\ W \arrow[ur, dashed] & \end{tikzcd}$$

- a **cofibrant–fibrant object** is an object that is both cofibrant and fibrant.

A **model category** is a locally small category $\mathcal{M}$ that is equipped with a model structure and satisfies the additional axiom below:

CM1. $\mathcal{M}$ has finite limits and finite colimits.

A **derivable category** is a locally small category $\mathcal{M}$ that is equipped with a model structure and satisfies the additional axioms below:

DC0. For each object X in $\mathcal{M}$, there exist

- a trivial cofibration $X \to \hat{X}$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$, and
- a trivial fibration $\tilde{X} \to X$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$.

DC1. $\mathcal{M}$ has pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$, and pullbacks along morphisms in $\mathcal{W} \cap \mathcal{F}$; i.e. given diagrams in $\mathcal{M}$ of the form below,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "i"'] & Z' \\ W & \end{tikzcd} \qquad\qquad \begin{tikzcd} & X \arrow[d, "p"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

if $i : Z \to W$ is in $\mathcal{C} \cap \mathcal{W}$, then the diagram on the left can be completed to a pushout square; and if $p : X \to Y$ is in $\mathcal{W} \cap \mathcal{F}$, then the diagram on the right can be completed to a pullback square.

Remark 4.1.2. Our definition of 'cofibrant object' (resp. 'fibrant object') is necessarily non-standard, because we do not always have initial objects (resp. terminal objects). Nonetheless, in a model category, our definitions agree with the standard ones: see lemma 4.1.15.

Definition 4.1.3. A **DHK model category** is a model category satisfying the following variants of CM1 and CM5:

CM1*. $\mathcal{M}$ is complete and cocomplete.

CM5*. The $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations can be chosen *functorially* in the sense of definition A.3.22.

Remark 4.1.4. Hovey [1999] and Hirschhorn [2003] attribute the stronger definition of 'model category' to Dwyer, Hirschhorn and Kan [DHK], hence the name 'DHK model category'; of course, this is the definition used in the cited works, as well as in [DHKS]. Note also that the definition in [Hovey, 1999] includes the functorial factorisations as a *structure* instead of a property. On the other hand, [DS] and [GJ] use Quillen's 1969 definition essentially verbatim.

Example 4.1.5. Let $\mathcal{M}$ be any category. The **trivial model structure** on $\mathcal{M}$ is defined by the following data:

- The weak equivalences are the isomorphisms.
- Every morphism is both a cofibration and a fibration.

It is straightforward to directly verify that the axioms are satisfied in this case. Notice that if $\mathcal{M}$ is complete and cocomplete, then the trivial model structure even makes $\mathcal{M}$ into a DHK model category.

Example 4.1.6. The **mono–epi model structure** on **Set** is defined by the following data:

- Every morphism is a weak equivalence.
- The cofibrations are the injective maps.
- The fibrations are the surjective maps.

The key observation is that **Set** admits a mono–epi weak factorisation system;[2] in fact, we can even choose the mono–epi factorisations functorially: for example, given a map $f : X \to Y$, we may take the cograph factorisation $X \to X \amalg Y \to Y$, where $X \to X \amalg Y$ is the coproduct insertion and $X \amalg Y \to Y$ is the map $(\!(f, \mathrm{id}_Y)\!)$.

Remark 4.1.7. Let $\mathcal{M}$ be a category. Then, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{M}$ if and only if $(\mathcal{F}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ is a model structure on $\mathcal{M}^{\mathrm{op}}$.

Lemma 4.1.8. *Let $\mathcal{M}$ be a category equipped with a model structure.*

- *If $i : Z \to W$ is a cofibration in $\mathcal{M}$ and Z is a cofibrant object, then W is also a cofibrant object.*

[2] — not to be confused with the epi–mono *orthogonal* factorisation system!

- *If $p : X \to Y$ is a fibration in $\mathcal{M}$ and Y is a fibrant object, then X is also a fibrant object.*

Proof. The two claims are formally dual; we will prove the first version.

Let $p : X \to Y$ be a trivial fibration in $\mathcal{M}$ and let $w : W \to Y$ be any morphism in $\mathcal{M}$. Since Z is cofibrant, there exists a morphism $z : Z \to X$ such that the diagram below commutes,

$$\begin{tikzcd} Z \arrow[r, dashed, "z"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and since $i : Z \to W$ is a cofibration, axiom CM4 gives a morphism $s : W \to X$ such that $p \circ s = w$. Thus W is also cofibrant. ■

Lemma 4.1.9. *In a category equipped with a model structure:*

- *Every trivial fibration with cofibrant codomain is a split epimorphism.*
- *Every trivial cofibration with fibrant domain is a split monomorphism.*

Proof. The two claims are formally dual; we will prove the first version.

Let $p : X \to Y$ be a trivial fibration, and suppose Y is cofibrant. Consider the following diagram in $\mathcal{M}$:

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

By definition, there exists a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$. This shows that $p : X \to Y$ is a split epimorphism. ■

Lemma 4.1.10. *Let $\mathcal{M}$ be a category equipped with a model structure. The following are equivalent for a morphism f in $\mathcal{M}$:*

(i) *f is a weak equivalence in $\mathcal{M}$.*

(ii) *For any factorisation $f = p \circ j$ in $\mathcal{M}$ where p is a fibration and j is a trivial cofibration, p must be a trivial fibration.*

(iii) *There exist a trivial cofibration j and a trivial fibration q such that $f = q \circ j$.*

Proof. (i) ⇒ (ii). Use axiom CM2.

(ii) ⇒ (iii). Use axiom CM5.

(iii) ⇒ (i). Use axiom CM2 again. ■

Lemma 4.1.11. *Let $\mathcal{M}$ be a category with a pair of weak factorisation systems $(\mathcal{C}', \mathcal{F})$ and $(\mathcal{C}, \mathcal{F}')$. Suppose $\mathcal{W}$ is a subensemble of* mor $\mathcal{C}$ *satisfying the following condition:*

$$\mathcal{W} \subseteq \{ q \circ j \mid j \in \mathcal{C}', q \in \mathcal{F}' \}$$

(i) $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$.

(ii) *If $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$, then $\mathcal{F}' \subseteq \mathcal{F}$ and $\mathcal{C} \cap \mathcal{W} = \mathcal{C}'$.*

Dually:

(i′) $\mathcal{W} \cap \mathcal{F} \subseteq \mathcal{F}'$.

(ii′) *If $\mathcal{F}' \subseteq \mathcal{W} \cap \mathcal{F}$, then $\mathcal{C}' \subseteq \mathcal{C}$ and $\mathcal{W} \cap \mathcal{F} = \mathcal{F}'$.*

In particular, $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$ if and only if $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$.

Proof. (i). Suppose $i : X \to Z$ is in $\mathcal{C} \cap \mathcal{W}$; then there must be $j : X \to Y$ in $\mathcal{C}'$ and $q : Y \to Z$ in $\mathcal{F}'$ such that $i = q \circ j$, and so we have the commutative diagram shown below:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "i"'] & Y \arrow[d, "q"] \\ Z \arrow[r, "\mathrm{id}"'] & Z \end{tikzcd}$$

Since $i \boxslash q$, i must be a retract of j; hence, by proposition A.3.12, i is in $\mathcal{C}'$, and therefore $\mathcal{C} \cap \mathcal{W} \subseteq \mathcal{C}'$.

(ii). If we know $\mathcal{C}' \subseteq \mathcal{C}$, then $\mathcal{F}' \subseteq \mathcal{F}$ by proposition A.3.3, and $\mathcal{C}' \subseteq \mathcal{C} \cap \mathcal{W}$, so from claim (i) it follows that $\mathcal{C}' = \mathcal{C} \cap \mathcal{W}$. ■

Theorem 4.1.12. *Let $\mathcal{M}$ be a category and let $\mathcal{C}, \mathcal{W}, \mathcal{F}$ be subclasses of* mor $\mathcal{M}$. *Assuming $\mathcal{M}$ has* either *pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$* or *pullbacks along morphisms in $\mathcal{W} \cap \mathcal{F}$, the following are equivalent:*

(i) $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ *is a model structure for* $\mathcal{M}$.

(ii) $\mathcal{W}$ *has the 2-out-of-3 property in* $\mathcal{M}$, *and both* $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ *and* $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ *are weak factorisation systems for* $\mathcal{M}$.

Proof. (i) ⇒ (ii). Axiom CM5 says that $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$- and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations exist, and axiom CM4 says we have the following inclusions:

$$\mathcal{C} \subseteq {}^{\boxslash}(\mathcal{W} \cap \mathcal{F}) \qquad \mathcal{W} \cap \mathcal{F} \subseteq \mathcal{C}^{\boxslash}$$
$$\mathcal{F} \subseteq (\mathcal{C} \cap \mathcal{W})^{\boxslash} \qquad \mathcal{C} \cap \mathcal{W} \subseteq {}^{\boxslash}\mathcal{F}$$

Axiom CM3 implies each one of $\mathcal{C}, \mathcal{F}, \mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F}$ is closed under retracts, so we may apply proposition A.3.14 to deduce that both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are indeed weak factorisation systems.

(ii) ⇒ (i). We may deduce from proposition A.3.12 that $\mathcal{C}$ and $\mathcal{F}$ are closed under retracts, and it remains to be shown that $\mathcal{W}$ is closed under retracts. The two cases are formally dual; we will assume that $\mathcal{M}$ has pushouts along morphisms in $\mathcal{C} \cap \mathcal{W}$.

Let $w : X \to Z$ be a morphism in $\mathcal{W}$, and consider a commutative diagram of the form below:

$$\begin{tikzcd}
X' \arrow[r, "s_X"'] \arrow[d, "w'"'] \arrow[rr, bend left, "\mathrm{id}"] & X \arrow[r, "r_X"'] \arrow[d, "w"'] & X' \arrow[d, "w'"] \\
Z' \arrow[r, "s_Z"] \arrow[rr, bend right, "\mathrm{id}"'] & Z \arrow[r, "r_Z"] & Z'
\end{tikzcd}$$

Choose a $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ factorisation for w', say $w' = p' \circ j'$, with $j' : X' \to Y'$ in $\mathcal{C} \cap \mathcal{W}$ and $p' : Y' \to Z'$ in $\mathcal{F}$. Construct the following commutative diagram,

$$\begin{tikzcd}
X' \arrow[r, "s_X"] \arrow[d, "j'"'] & X \arrow[r, "r_X"] \arrow[d, "u"'] & X' \arrow[d, "j'"] \\
Y' \arrow[r, "s_Y"] \arrow[d, "p'"'] & Y \arrow[r, "r_Y"] \arrow[d, "v"'] & Y' \arrow[d, "p'"] \\
Z' \arrow[r, "s_Z"'] & Z \arrow[r, "r_Z"'] & Z'
\end{tikzcd}$$

where the top left square is a pushout square, $v \circ u = w$, and $r_Y \circ s_Y = \mathrm{id}_Y$. Since $\mathcal{C} \cap \mathcal{W}$ is closed under pushouts, u is also in $\mathcal{C} \cap \mathcal{W}$, and by the 2-out-of-3 property, v is in $\mathcal{W}$. Thus, p' is in $\mathcal{F}$ and is a retract of v:

$$\begin{tikzcd}
Y' \arrow[r, "s_Y"'] \arrow[d, "p'"'] \arrow[rr, bend left, "\mathrm{id}"] & Y \arrow[r, "r_Y"'] \arrow[d, "v"'] & Y' \arrow[d, "p'"] \\
Z' \arrow[r, "s_Z"] \arrow[rr, bend right, "\mathrm{id}"'] & Z \arrow[r, "r_Z"] & Z'
\end{tikzcd}$$

Using the 2-out-of-3 property again, choose a $(\mathcal{C} \cap \mathcal{W}, \mathcal{W} \cap \mathcal{F})$-factorisation of v, say $v = q \circ j$. Since $j \boxslash p'$, there exists a morphism r such that $r \circ j = r_Y$ and $p' \circ r = r_Z \circ q$. Putting $s = j \circ s_Y$, we obtain $r \circ s = r_Y \circ s_Y = \mathrm{id}_Y$; thus p' is a retract of q and must therefore be in $\mathcal{F} \cap \mathcal{W}$. Hence, $w' = p' \circ j'$ is in $\mathcal{W}$. ■

Corollary 4.1.13. *Let $\mathcal{M}$ be a derivable category.*

- *Pushouts of trivial cofibrations along any morphism in $\mathcal{M}$ exist, and any such is a trivial cofibration.*
- *Pullbacks of trivial fibrations along any morphism in $\mathcal{M}$ exist, and any such is a trivial fibration.*

Proof. Apply proposition A.3.12. ■

Remark 4.1.14. May and Ponto [2012, Ch. 14] define 'model category' to mean a complete and cocomplete locally small category $\mathcal{M}$ equipped with a triple of classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ satisfying condition (ii) of the above proposition; if the two weak factorisation systems can be extended to a pair of functorial factorisation systems, then this is a DHK model category.

Lemma 4.1.15. *Let $\mathcal{M}$ be a category equipped with a model structure. If $\mathcal{M}$ has an initial object 0, then the following are equivalent for any object W in $\mathcal{M}$:*

(i) *W is a cofibrant object in $\mathcal{M}$.*

(ii) *The unique morphism $0 \to W$ has the left lifting property with respect to all trivial fibrations in $\mathcal{M}$.*

(iii) *The unique morphism $0 \to W$ is a cofibration.*

Dually, if $\mathcal{M}$ has a terminal object 1*, then the following are equivalent for any object X in $\mathcal{M}$:*

(i′) *X is a fibrant object in $\mathcal{M}$.*

(ii′) *The unique morphism $X \to 1$ has the right lifting property with respect to all trivial fibrations in $\mathcal{M}$.*

(iii′) *The unique morphism $X \to 1$ is a fibration.*

Proof. (i) ⇔ (ii). Obvious.

(ii) ⇔ (iii). By theorem 4.1.12, any morphism that has the left lifting property with respect to all trivial fibrations must be a cofibration. ■

Proposition 4.1.16. *Let $\mathcal{M}$ be a category equipped with a model structure. If $\mathcal{M}$ satisfies axiom DC1 and has both an initial object and a terminal object, then $\mathcal{M}$ is a derivable category. In particular, any model category is a derivable category.*

Proof. Use axiom CM5 to factorise the unique morphisms $0 \to X$ and $X \to 1$, and then apply lemma 4.1.15 to deduce that axiom DC0 is satisfied. ■

Lemma 4.1.17. *Let $\mathcal{M}$ be a category equipped with a model structure and let A be an object in $\mathcal{M}$.*

(i) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$) admits a* **slice model structure***, where a morphism in $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$) is a weak equivalence, cofibration, or fibration if it is so in $\mathcal{M}$.*

(ii) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$), equipped with the slice model structure, is a derivable category if $\mathcal{M}$ is a derivable category.*

(iii) *The slice category $\mathcal{M}_{/A}$ (resp. ${}^{A/}\mathcal{M}$), equipped with the slice model structure, is a model category if $\mathcal{M}$ is a model category.*

Proof. The two halves of each claim are formally dual; we will prove the versions for $\mathcal{M}_{/A}$.

(i). Use lemmas 3.1.6 and A.3.11.

(ii). $\mathcal{M}_{/A}$ always has a terminal object, so axiom CM5 and lemma 4.1.15 imply one half of axiom DC0 in $\mathcal{M}_{/A}$; for the other half, we may use axiom DC0 in $\mathcal{M}$ directly.

It is well known that the projection functor $\mathcal{M}_{/A} \to \mathcal{M}$ preserves and reflects pullbacks and pushouts, so pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in $\mathcal{M}_{/A}$ if pushouts along trivial cofibrations (resp. pullbacks along trivial fibrations) exist in $\mathcal{M}$. Thus $\mathcal{M}_{/A}$ satisfies axiom DC1 if $\mathcal{M}$ does.

(iii). The argument above also shows that $\mathcal{M}_{/A}$ has finite limits and colimits if $\mathcal{M}$ does. ■

Lemma 4.1.18. *Let $\left(\mathcal{M}_i \,\middle|\, i \in I\right)$ be a sequence of categories equipped with model structures.*

(i) *The product category $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$ admits a* **product model structure***, where a morphism in $\mathcal{M}$ is a weak equivalence, cofibration, or fibration if each component is so.*

(ii) *$\mathcal{M}$, equipped with the product model structure, is a derivable category if each $\mathcal{M}_i$ is a derivable category.*

(iii) *$\mathcal{M}$, equipped with the product model structure, is a model category if each $\mathcal{M}_i$ is a model category.*

Proof. Everything can be checked componentwise. ♦

Definition 4.1.19. Let X be an object in a category $\mathcal{M}$ equipped with a model structure.

- A **cofibrant replacement** for X is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$ and p is a weak equivalence $\tilde{X} \to X$.
- A **fibrant replacement** for X is a pair $(\hat{X}, i)$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$ and i is a weak equivalence $X \to \hat{X}$.
- A **fibrant cofibrant replacement** for X is a cofibrant replacement $(\tilde{X}, p)$ where $p : \tilde{X} \to X$ is a trivial fibration.
- A **cofibrant fibrant replacement** for X is a fibrant replacement $(\hat{X}, i)$ where $i : X \to \hat{X}$ is a trivial cofibration.

Definition 4.1.20. Let $\mathcal{M}$ be a category equipped with a model structure.

- A **cofibrant replacement functor** for $\mathcal{M}$ is a pair (Q, p), where Q is an endofunctor on $\mathcal{M}$ and p is a natural transformation $Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that, for every object X in $\mathcal{M}$, (QX, p_X) is a cofibrant replacement for X.

- A **fibrant replacement functor** for $\mathcal{M}$ is a pair (R, i), where R is an endofunctor on $\mathcal{M}$ and i is a natural transformation $\mathrm{id}_{\mathcal{M}} \Rightarrow R$ such that, for every object X in $\mathcal{M}$, (RX, i_X) is a fibrant replacement for X.

- A **fibrant cofibrant replacement functor** for $\mathcal{M}$ is a pair (Q, p), where Q is an endofunctor on $\mathcal{M}$ and p is a natural transformation $Q \Rightarrow \mathrm{id}_{\mathcal{M}}$ such that, for every object X in $\mathcal{M}$, (QX, p_X) is a fibrant cofibrant replacement for X.

- A **cofibrant fibrant replacement functor** for $\mathcal{M}$ is a pair (R, i), where R is an endofunctor on $\mathcal{M}$ and i is a natural transformation $\mathrm{id}_{\mathcal{M}} \Rightarrow R$ such that, for every object X in $\mathcal{M}$, (RX, i_X) is a cofibrant fibrant replacement for X.

Remark 4.1.21. Note that a fibrant cofibrant replacement for X is precisely a cofibrant replacement for X that is fibrant as an object in $\mathcal{M}_{/X}$, and a cofibrant fibrant replacement for X is precisely a fibrant replacement for X that is cofibrant as an object in ${}^{X/}\mathcal{M}$.

Moreover, if X is fibrant and $(\tilde{X}, p)$ is a fibrant cofibrant replacement for X, then $\tilde{X}$ is both fibrant and cofibrant in $\mathcal{M}$, and if X is cofibrant and $(\hat{X}, i)$ is a cofibrant fibrant replacement for X, then $\hat{X}$ is both cofibrant and fibrant in $\mathcal{M}$.

Proposition 4.1.22.

(i) *Any object in a derivable category has both a fibrant cofibrant replacement and a cofibrant fibrant replacement.*

(ii) *Any DHK model category has both a fibrant cofibrant replacement functor and a cofibrant fibrant replacement functor.*

Proof. (i). This is axiom DC0.

(ii). Use axiom CM5* to factorise the unique natural transformations $\Delta 0 \Rightarrow \mathrm{id}_{\mathcal{M}}$ and $\mathrm{id}_{\mathcal{M}} \Rightarrow \Delta 1$, and then apply lemma 4.1.15. ■

It should go without saying that any two cofibrant or fibrant replacements for a fixed object are weakly equivalent; however, more is true:

Lemma 4.1.23. *Let X be an object in a derivable category $\mathcal{M}$.*

- *Any two cofibrant replacements for X are weakly equivalent as objects in the slice model category $\mathcal{M}_{/X}$.*
- *Any two fibrant replacements for X are weakly equivalent as objects in the slice model category ${}^{X/}\mathcal{M}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let $(\tilde{X}, p)$ be a fibrant cofibrant replacement for X; such exist, by proposition 4.1.22. Let $(\tilde{X}', p')$ be any cofibrant replacement for X. Then, $p : \tilde{X} \to X$ is a trivial fibration, so there exists a morphism $f : \tilde{X}' \to \tilde{X}$ such that $p \circ f = p'$. The 2-out-of-3 property of weak equivalences implies any such $f : \tilde{X}' \to \tilde{X}$ is a weak equivalence, so we may deduce that every cofibrant replacement for X is weakly equivalent to $(\tilde{X}, p)$ as objects in $\mathcal{M}_{/X}$. ∎

In the presence of functorial cofibrant and fibrant replacements, we can say something stronger still:

Proposition 4.1.24. *Let X be an object in a derivable category $\mathcal{M}$.*

- *If $\mathcal{M}$ has a cofibrant replacement functor, then the full subcategory of the slice category $\mathcal{M}_{/X}$ spanned by the cofibrant replacements for X is homotopically contractible.*
- *If $\mathcal{M}$ has a fibrant replacement functor, then the full subcategory of the slice category ${}^{X/}\mathcal{M}$ spanned by the fibrant replacements for X is homotopically contractible.*

Proof. The two claims are formally dual; we will prove the first version.

Let (Q, p) be a cofibrant replacement functor for $\mathcal{M}$. Then, for each cofibrant replacement $(\tilde{X}, q)$ for X, we have the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd}
\tilde{X} \arrow[dr, "q"'] & Q\tilde{X} \arrow[l, "p_{\tilde{X}}"'] \arrow[r, "Qq"] \arrow[d] & QX \arrow[dl, "p_X"] \\
 & X &
\end{tikzcd}$$

Thus, the constant functor at (QX, p_X) is naturally weakly equivalent to the identity functor of the category of cofibrant replacements for X, and we may then apply proposition 3.1.31 to deduce that it is homotopically contractible. ■

REMARK 4.1.25. In other words, cofibrant replacements (resp. fibrant replacements) are homotopically unique in a model category with functorial cofibrant replacements (resp. functorial fibrant replacements).

Proposition 4.1.26. *Let $\mathcal{M}$ be a category with a model structure, let $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ be the model structure on $\mathcal{M}$, and let $\mathcal{N}$ be a full subcategory of $\mathcal{M}$.*

(i) *If $\mathcal{N}$ is homotopically replete in $\mathcal{M}$, then the data*

$$(\mathcal{C} \cap \operatorname{mor} \mathcal{N}, \mathcal{W} \cap \operatorname{mor} \mathcal{N}, \mathcal{F} \cap \operatorname{mor} \mathcal{N})$$

constitute a model structure on $\mathcal{N}$.

(ii) *If W is a cofibrant object in $\mathcal{M}$ and is in $\mathcal{N}$, then W is a projective object in $\mathcal{N}$ with respect to $\mathcal{W} \cap \mathcal{F} \cap \operatorname{mor} \mathcal{N}$; dually, if X is a fibrant object in $\mathcal{M}$ and is in $\mathcal{N}$, then X is an injective object in $\mathcal{N}$ with respect to $\mathcal{C} \cap \mathcal{W} \cap \operatorname{mor} \mathcal{N}$.*

(iii) *If $\mathcal{M}$ is a derivable category, then so is $\mathcal{N}$ when equipped with the above model structure.*

Proof. (i). Lemma 3.1.7 implies that axiom CM2 is satisfied. Since $\mathcal{N}$ is a full subcategory of $\mathcal{M}$, the data $(\mathcal{C} \cap \operatorname{mor} \mathcal{N}, \mathcal{W} \cap \operatorname{mor} \mathcal{N}, \mathcal{F} \cap \operatorname{mor} \mathcal{N})$ satisfy axioms CM3 and CM4 because $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ do. Finally, for axiom CM5, we appeal to the hypothesis that $\mathcal{N}$ is homotopically replete.

(ii). This follows from the assumption that $\mathcal{N}$ is a full subcategory of $\mathcal{M}$.

(iii). It remains to be shown that pushouts along trivial cofibrations and pullbacks along trivial fibrations exist in $\mathcal{N}$. For this, simply apply corollary 4.1.13 to the hypothesis that $\mathcal{N}$ is homotopically replete and full. ■

Definition 4.1.27. The **Quillen homotopy category** (or, more simply, **homotopy category**) of a derivable category $\mathcal{M}$ is the category $\operatorname{Ho} \mathcal{M}$ obtained by freely inverting the weak equivalences in $\mathcal{M}$, as in definition A.4.9.

Definition 4.1.28. A **saturated derivable category** is a derivable category that is saturated as a category with weak equivalences.

Theorem 4.1.29. *Let $\mathcal{M}$ be a derivable category and let $\gamma : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ be the localising functor.*

(i) *Let $\mathcal{U}$ and $\mathcal{V}$ be the classes of trivial cofibrations and trivial fibrations in $\mathcal{M}$, respectively. Then $\mathcal{M}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, which is functorial if $\mathcal{M}$ satisfies axiom CM5*.*

(ii) *Let X and Y be objects in $\mathcal{M}$, let $v : X \to \tilde{X}$ and $v' : X \to \tilde{X}'$ be trivial fibrations, let $u : Y \to \hat{Y}$ and $u' : Y \to \hat{Y}'$ be trivial cofibrations, and let $f : \tilde{X} \to \hat{Y}$ and $f' : \tilde{X}' \to \hat{Y}'$ be morphisms in $\mathcal{M}$. Then,*

$$\gamma(u)^{-1} \circ \gamma(f) \circ \gamma(v)^{-1} = \gamma(u')^{-1} \circ \gamma(f') \circ \gamma(v')^{-1}$$

if and only if there exists a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X \arrow[d, equal] & \tilde{X} \arrow[l, "v"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[r, "f_3"] \arrow[u, "v_1"] \arrow[d, "w_1"] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"'] & Y \arrow[l, "u_3"'] \arrow[d, equal] \\
X \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \arrow[d, equal] \\
X & \tilde{X}' \arrow[l, "v'"] \arrow[r, "f'"'] \arrow[u, "u_1"] & \hat{Y}' \arrow[u, "u_2"'] & Y \arrow[l, "u'"]
\end{tikzcd}$$

where u_1, u_2, u_3, u_4 are trivial cofibrations, v_1, v_2, v_3, v_4 are trivial fibrations, and w_1, w_2 are weak equivalences. In any such diagram, u_2 is a split monomorphism if $\tilde{X}$ is cofibrant, and v_1 is a split epimorphism if $\hat{Y}'$ is fibrant.

(iii) *$\mathcal{M}$ is a saturated derivable category if and only if the weak equivalences in $\mathcal{M}$ have the 2-out-of-6 property.*

(iv) *If X is a cofibrant object in $\mathcal{M}$ and Y is a fibrant object in $\mathcal{M}$, then the hom-set map $\mathcal{M}(X, Y) \to \operatorname{Ho} \mathcal{M}(\gamma X, \gamma Y)$ is surjective.*

(v) *$\operatorname{Ho} \mathcal{M}$ is a locally small category.*

Proof. (i). Axioms CM2 and CM5 imply axiom A1 is satisfied, and axioms A2 and A3 follow from the above claims; that we get a functorial three-arrow calculus under axiom CM5* is an obvious consequence of the universal property of pushouts and pullbacks.

(ii). This is a special case of the fundamental theorem of three-arrow calculi (3.6.9), plus lemma 4.1.9.

(iii). Apply proposition 3.6.10 and lemma A.4.14.

(iv). Consider a zigzag of the following form in $\mathcal{M}$,

$$X \xleftarrow{v} X' \xrightarrow{f'} Y' \xleftarrow{u} Y$$

where $u : Y \to Y'$ is a trivial cofibration and $v : X' \to X$ is a trivial fibration. Let $\bar{f} = \gamma(u)^{-1} \circ \gamma(f') \circ \gamma(v)^{-1}$ be the corresponding morphism in $\operatorname{Ho}\mathcal{M}$; note that the fundamental theorem of three-arrow calculi says that every morphism $\gamma X \to \gamma Y$ in $\operatorname{Ho}\mathcal{M}$ is of this form. Suppose X is cofibrant and Y is fibrant. Then lemma 4.1.9 says u is a split monomorphism and v is a split epimorphism, so choose $r : Y' \to Y$ and $s : X \to X'$ such that $r \circ u = \mathrm{id}_Y$ and $v \circ s = \mathrm{id}_X$. Since $\gamma(u)$ and $\gamma(v)$ are isomorphisms in $\operatorname{Ho}\mathcal{M}$, we must have $\gamma(u)^{-1} = \gamma(r)$ and $\gamma(v)^{-1} = \gamma(s)$. Hence, taking $f = r \circ f' \circ s$, we have $\bar{f} = \gamma(f)$, as required.

(v). By proposition 4.1.22, every object in $\mathcal{M}$ is weakly equivalent to both a cofibrant object and a fibrant object, so we may deduce that $\operatorname{Ho}\mathcal{M}$ is locally small from claim (iii). ■

Corollary 4.1.30. *Let $\mathcal{M}$ be a derivable category. For any two objects X and Y in $\mathcal{M}$, every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{M}$ can be represented by a zigzag of the following form,*

$$X \xleftarrow{p} \tilde{X} \longrightarrow \hat{Y} \xleftarrow{i} Y$$

where $(\tilde{X}, p)$ is any cofibrant replacement for X and $(\hat{Y}, i)$ is any fibrant replacement for Y. ■

Lemma 4.1.31. *Let $\mathcal{M}$ be a derivable category and let $\mathcal{C}$ be a relative category where $\operatorname{weq}\mathcal{C}$ has the special 2-out-of-4 property.*

- *Let $\mathcal{M}_c$ be the full subcategory of cofibrant objects in $\mathcal{M}$. If a functor $F : \mathcal{M}_c \to \mathcal{C}$ sends trivial cofibrations in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$, then F preserves all weak equivalences.*

- *Let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects in $\mathcal{M}$. If a functor $G : \mathcal{M}_{\mathrm{f}} \to \mathcal{C}$ sends trivial fibrations in $\mathcal{M}_{\mathrm{f}}$ to weak equivalences in $\mathcal{C}$, then G preserves all weak equivalences.*

Proof. The two claims are formally dual; we will prove the first version.

Axioms CM2 and CM5 imply that every weak equivalence in $\mathcal{M}$ can be factored as a trivial cofibration followed by a trivial fibration, so it is enough to show that F sends trivial fibrations in $\mathcal{M}_{\mathrm{c}}$ to weak equivalences in $\mathcal{C}$. Let $p : X \to Y$ be a trivial fibration in $\mathcal{M}_{\mathrm{c}}$. Y is cofibrant, so lemma 4.1.9 says $p : X \to Y$ has a section $s : Y \to X$.

Let $e = s \circ p$. Since $p : X \to Y$ is a trivial fibration, we may form a pullback square in $\mathcal{M}$ of the following form:

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_0"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"'] & Y \end{tikzcd}$$

There is then a unique morphism $\Delta : X \to K$ such that $k_0 \circ \Delta = k_1 \circ \Delta = \mathrm{id}_X$. Since $k_0 : K \to X$ is a trivial fibration (by corollary 4.1.13), $\Delta : X \to K$ is a weak equivalence in $\mathcal{M}$ and therefore factorises as $q \circ j$ for some trivial cofibration $j : X \to \tilde{K}$ and some trivial fibration $q : \tilde{K} \to K$; note that $\tilde{K}$ is a cofibrant object. There is also a unique morphism $t : X \to K$ such that $k_0 \circ t = \mathrm{id}_X$ and $k_1 \circ t = e$; and X is a cofibrant object, so there exists a morphism $h : X \to \tilde{K}$ such that $q \circ h = t$. Taking $q_0 = k_0 \circ q$ and $q_1 = k_1 \circ q$, we obtain the following commutative diagram in $\mathcal{M}_{\mathrm{c}}$:

$$\begin{tikzcd} X \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "e"] \arrow[d, "h"] & X \arrow[d, equal] \\ X \arrow[d, equal] & \tilde{K} \arrow[l, "q_0"'] \arrow[r, "q_1"] & X \arrow[d, equal] \\ X & X \arrow[l, "\mathrm{id}"] \arrow[r, "\mathrm{id}"'] \arrow[u, "j"'] & X \end{tikzcd}$$

Consider the image of the above diagram in $\mathcal{C}$. By hypothesis, $Fj : FX \to F\tilde{K}$ is a weak equivalence in $\mathcal{C}$, and by repeatedly applying the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$, we may deduce that $Fe : FX \to FX$ is a weak equivalence in $\mathcal{C}$ as well. But $\mathrm{weq}\, \mathcal{C}$ has the special 2-out-of-4 property, and $Fe = Fs \circ Fp$, so we may conclude that $Fp : FX \to FY$ is a weak equivalence in $\mathcal{C}$, as required. ■

Proposition 4.1.32. *Let $\mathcal{M}$ be a derivable category. Let $\mathcal{M}_{\mathrm{c}}$ be the full subcategory of cofibrant objects in $\mathcal{M}$.*

(i) *$\mathcal{M}_{\mathrm{c}}$, considered as a relative category with* trivial cofibrations *as weak equivalences, admits a calculus of cospans.*

(ii) *The localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to trivial cofibrations is isomorphic to the localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to all weak equivalences.*

(iii) *Every morphism $X \to Y$ in $\mathrm{Ho}\, \mathcal{M}_{\mathrm{c}}$ can be represented by a cycle in $\mathcal{M}_{\mathrm{c}}$ of the form below,*

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y$$

where $(\hat{Y}, i)$ is any *cofibrant fibrant replacement for Y.*

Dually, let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects in $\mathcal{M}$.

(i′) *$\mathcal{M}_{\mathrm{f}}$, considered as a relative category with* trivial fibrations *as weak equivalences, admits a calculus of spans.*

(ii′) *The localisation of $\mathcal{M}_{\mathrm{f}}$ with respect to trivial fibrations is isomorphic to the localisation of $\mathcal{M}_{\mathrm{f}}$ with respect to all weak equivalences.*

(iii′) *Every morphism $X \to Y$ in $\mathrm{Ho}\, \mathcal{M}_{\mathrm{f}}$ can be represented by a cocycle in $\mathcal{M}_{\mathrm{c}}$ of the form below,*

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} Y$$

where $(\tilde{X}, p)$ is any *fibrant cofibrant replacement for X.*

Proof. (i). This is an immediate consequence of 4.1.13.

(ii). Suppose $F : \mathcal{M}_{\mathrm{c}} \to \mathcal{C}$ is a functor that sends trivial cofibrations in $\mathcal{M}_{\mathrm{c}}$ to isomorphisms in $\mathcal{C}$. It is clear that isomorphisms have the special 2-out-of-4 property, so we may apply lemma 4.1.31 to deduce that F sends weak equivalences in $\mathcal{M}_{\mathrm{c}}$ to isomorphisms in $\mathcal{C}$ as well. Hence, any localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to trivial cofibrations must also be a localisation of $\mathcal{M}_{\mathrm{c}}$ with respect to weak equivalences.

(iii). The fundamental theorem of calculi of cospans (3.5.6) says every morphism $X \to Y$ in $\operatorname{Ho} \mathcal{M}_c$ can be represented by a cycle in $\mathcal{M}_c$ of the form below,

$$X \xrightarrow{g} Y' \xleftarrow{u} Y$$

where $u : Y \to Y'$ is a trivial cofibration, and that two such cycles represent the same morphism if and only if they are in the same connected component of the cycle category $\mathcal{M}_c^{\to\sim}(X, Y)$. Let $(\hat{Y}, i)$ be any cofibrant fibrant replacement for Y. Since $u : Y \to Y'$ is a trivial cofibration and $\hat{Y}$ is fibrant, axiom CM4 yields a morphism $h : Y' \to \hat{Y}$ such that $h \circ u = i$. Taking $f = h \circ g$, we have the following commutative diagram in $\mathcal{M}_c$:

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, equal] & Y' \arrow[d, "h"] & Y \arrow[l, "u"'] \arrow[d, equal] \\
X \arrow[r, "f"'] & \hat{Y} & Y \arrow[l, "i"]
\end{tikzcd}$$

Thus, the cycles (u, g) and (i, f) represent the same morphism in $\operatorname{Ho} \mathcal{M}_c$. ■

Proposition 4.1.33. *Let $\mathcal{M}$ be a derivable category.*

- *Let $\mathcal{M}_c$ be the full subcategory of cofibrant objects in $\mathcal{M}$. The canonical functor $\operatorname{Ho} \mathcal{M}_c \to \operatorname{Ho} \mathcal{M}$ induced by the inclusion $\mathcal{M}_c \hookrightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects.*
- *Let $\mathcal{M}_f$ be the full subcategory of fibrant objects in $\mathcal{M}$. The canonical functor $\operatorname{Ho} \mathcal{M}_f \to \operatorname{Ho} \mathcal{M}$ induced by the inclusion $\mathcal{M}_f \hookrightarrow \mathcal{M}$ is fully faithful and essentially surjective on objects.*

Proof. The two claims are formally dual; we will prove the first version.

It is clear that proposition 4.1.22 implies the functor $\operatorname{Ho} \mathcal{M}_c \to \operatorname{Ho} \mathcal{M}$ is essentially surjective on objects; it remains to be shown that the functor is fully faithful. Consider the full subcategory $\mathcal{M}_{cf}$ spanned by the cofibrant–fibrant objects in $\mathcal{M}$. By restricting the localising functors, we obtain the following commutative diagram,

$$\begin{tikzcd}
\mathcal{M}_{cf} \arrow[r, equal] \arrow[d] & \mathcal{M}_{cf} \arrow[d] \\
\operatorname{Ho} \mathcal{M}_c \arrow[r] & \operatorname{Ho} \mathcal{M}
\end{tikzcd}$$

where $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}_{\mathrm{c}}$ and $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}$ are essentially surjective on objects. Theorem 4.1.29 implies $\mathcal{M}_{\mathrm{cf}} \to \operatorname{Ho}\mathcal{M}$ is a full functor, so $\operatorname{Ho}\mathcal{M}_{\mathrm{c}} \to \operatorname{Ho}\mathcal{M}$ must also be full.

Now, consider a parallel pair of morphisms in $\operatorname{Ho}\mathcal{M}_{\mathrm{c}}$. Proposition 4.1.32 says they can be represented by cycles of the following form,

$$X \xrightarrow{f} \hat{Y} \xleftarrow{i} Y \qquad\qquad X \xrightarrow{f'} \hat{Y} \xleftarrow{i} Y$$

where $(\hat{Y}, i)$ is any cofibrant fibrant replacement for Y. Suppose the two morphisms are equal in $\operatorname{Ho}\mathcal{M}$. Then, there must be a commutative diagram in $\mathcal{M}$ of the form below,

$$\begin{tikzcd}
X & X \arrow[l, "\mathrm{id}"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "i"'] \\
X \arrow[u, equal] \arrow[d, equal] & \bullet \arrow[l, "v_3"'] \arrow[r, "f_3"] \arrow[u, "v_1"] \arrow[d, "w_1"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[u, equal] \arrow[d, equal] \\
X & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \\
X \arrow[u, equal] & X \arrow[l, "\mathrm{id}"] \arrow[r, "f'"'] \arrow[u, "u_1"] & \hat{Y} \arrow[u, "u_2"'] & Y \arrow[l, "i"] \arrow[u, equal]
\end{tikzcd}$$

where u_1, u_2, u_3, u_4 are trivial cofibrations, v_1, v_2, v_3, v_4 are trivial fibrations, and w_1, w_2 are weak equivalences. Since X is cofibrant, there exists a morphism s in $\mathcal{M}$ such that $v_1 \circ s = \mathrm{id}_X$, so (using lemma 4.1.8) we obtain the following commutative diagram in $\mathcal{M}_{\mathrm{c}}$:

$$\begin{tikzcd}
X & X \arrow[l, "\mathrm{id}"'] \arrow[r, "f"] & \hat{Y} & Y \arrow[l, "i"'] \\
X \arrow[u, equal] \arrow[d, equal] & X \arrow[l, "\mathrm{id}"'] \arrow[r, "f_3 \circ s"] \arrow[u, "\mathrm{id}"] \arrow[d, "w_1 \circ s"'] & \bullet \arrow[u, "v_2"'] \arrow[d, "w_2"] & Y \arrow[l, "u_3"'] \arrow[u, equal] \arrow[d, equal] \\
X & \bullet \arrow[l, "v_4"] \arrow[r, "f_4"'] & \bullet & Y \arrow[l, "u_4"] \\
X \arrow[u, equal] & X \arrow[l, "\mathrm{id}"] \arrow[r, "f'"'] \arrow[u, "u_1"] & \hat{Y} \arrow[u, "u_2"'] & Y \arrow[l, "i"] \arrow[u, equal]
\end{tikzcd}$$

Noting that axiom CM2 implies $w_1 \circ s$ is a weak equivalence in $\mathcal{M}_c$, we may then deduce that the two zigzags also represent the same morphism in $\operatorname{Ho} \mathcal{M}_c$. Thus, the functor $\operatorname{Ho} \mathcal{M}_c \to \operatorname{Ho} \mathcal{M}$ is indeed faithful. ■

4.2 Left and right homotopy

Prerequisites. § 4.1.

Definition 4.2.1. Let X be an object in a model category $\mathcal{M}$.

- A **cylinder object** for X is a quadruple $\left(\operatorname{Cyl}(X), i_0, i_1, p\right)$, where $\operatorname{Cyl}(X)$ is an object in $\mathcal{M}$, $p : \operatorname{Cyl}(X) \to X$ is a weak equivalence, and i_0 and i_1 are sections of p such that the morphism $(\!(i_0, i_1)\!) : X + X \to \operatorname{Cyl}(X)$ is a cofibration.

- A **path object** for X is a quadruple $\left(\operatorname{Path}(X), i, p_0, p_1\right)$, where $\operatorname{Path}(X)$ is an object in $\mathcal{M}$, $i : X \to \operatorname{Path}(X)$ is a weak equivalence, and p_0 and p_1 are retractions of i such that the morphism $\langle p_0, p_1 \rangle : \operatorname{Path}(X) \to X \times X$ is a fibration.

Remark 4.2.2. Let $\left(\operatorname{Cyl}(X), i_0, i_1, p\right)$ be a cylinder object for X. By definition, $p \circ i_0 = p \circ i_1 = \mathrm{id}_X$, and p is a weak equivalence, so by the 2-out-of-3 property, i_0 and i_1 must also be weak equivalences $X \to \operatorname{Cyl}(X)$.

Dually, if $\left(\operatorname{Path}(X), i, p_0, p_1\right)$ is a path object for X, then p_0 and p_1 must be weak equivalences $\operatorname{Path}(X) \to X$.

Proposition 4.2.3. *Let X be an object in a model category $\mathcal{M}$.*

- *There exists a cylinder object $\left(\operatorname{Cyl}(X), i_0, i_1, p\right)$ for X, where the morphism $p : \operatorname{Cyl}(X) \to X$ is a trivial fibration.*

- *There exists a path object $\left(\operatorname{Path}(X), i, p_0, p_1\right)$ for X, where the morphism $i : X \to \operatorname{Path}(X)$ is a trivial cofibration.*

Proof. Use axioms CM1 and CM5. ■

Definition 4.2.4. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$, let $\left(\operatorname{Cyl}(X), i_0, i_1, p\right)$ be a cylinder object for X, and let $\left(\operatorname{Path}(Y), i, p_0, p_1\right)$ be a path object for Y.

- A **left homotopy** from f_0 to f_1 with respect to $(\mathrm{Cyl}(X), i_0, i_1, p)$ is a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$.
- A **right homotopy** from f_0 to f_1 with respect to $(\mathrm{Path}(Y), i, p_0, p_1)$ is a morphism $H : X \to \mathrm{Path}(Y)$ such that $p_0 \circ H = f_0$ and $p_1 \circ H = f_1$.
- We say f_0 and f_1 are **left homotopic** if there exists a left homotopy from f_0 to f_1 with respect to some cylinder object for X.
- We say f_0 and f_1 are **right homotopic** if there exists a right homotopy from f_0 to f_1 with respect to some path object for Y.

Remark 4.2.5. If f_0 and f_1 are either left homotopic or right homotopic, then they must represent the same morphism in $\mathrm{Ho}\,\mathcal{M}$. For definiteness, let us write $\gamma : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ for the localising functor, and suppose $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1. Since i_0 and i_1 are both sections of the weak equivalence $p : \mathrm{Cyl}(X) \to X$, we must have $\gamma i_0 = (\gamma p)^{-1} = \gamma i_1$; but $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$, so indeed $\gamma f_0 = \gamma f_1$. This is one of the reasons for calling $\mathrm{Ho}\,\mathcal{M}$ the *homotopy* category of $\mathcal{M}$.

However, it is not quite true that $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are either left homotopic or right homotopic; this only happens in special cases. In general, being left/right homotopic fails to even be an equivalence relation.

Definition 4.2.6. Let $f : X \to Y$ be a morphism in a model category $\mathcal{M}$.

- A **left homotopy left inverse** for f is a morphism $g : Y \to X$ in $\mathcal{M}$ such that $g \circ f$ and id_X are left homotopic.
- A **right homotopy right inverse** for f is a morphism $h : Y \to X$ in $\mathcal{M}$ such that $f \circ h$ and id_Y are right homotopic.
- A **right homotopy left inverse** for f is a morphism $g : Y \to X$ in $\mathcal{M}$ such that $g \circ f$ and id_X are right homotopic.
- A **left homotopy right inverse** for f is a morphism $h : Y \to X$ in $\mathcal{M}$ such that $f \circ h$ and id_Y are left homotopic.

A **homotopy equivalence** in $\mathcal{M}$ is a pair (f, g) such that g (resp. f) is both a left homotopy left inverse and a right homotopy right inverse for f (resp. g). Two morphisms $f : X \to Y$ and $g : Y \to X$ in $\mathcal{M}$ are **mutual homotopy inverses** when (f, g) constitute a homotopy equivalence in $\mathcal{M}$.

Remark 4.2.7. Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a model category.

- g is a left homotopy left inverse for f if and only if f is a left homotopy right inverse for g.
- g is a right homotopy left inverse for f if and only if f is a right homotopy left inverse for g.

However, note that the dual of 'left homotopy left inverse' is 'right homotopy right inverse', and the dual of 'right homotopy left inverse' is 'left homotopy right inverse'!

Lemma 4.2.8. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category, and suppose f_0 and f_1 are either left or right homotopic. Then, f_0 is a weak equivalence if and only if f_1 is a weak equivalence.*

Proof. Assume f_0 and f_1 are left homotopic; the other case is formally dual. So, there exist a cylinder object $(\mathrm{Cyl}(X), i_0, i_1, p)$ for X and a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Suppose f_0 is a weak equivalence. By remark 4.2.2, i_0 is a weak equivalence, so the 2-out-of-3 property implies H is also a weak equivalence; but i_1 is a weak equivalence as well, so f_1 must be a weak equivalence too. A symmetrical argument proves that f_0 is a weak equivalence if f_1 is. ■

Lemma 4.2.9. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms in a model category $\mathcal{M}$.*

(i) *If $g \circ f$ is either left or right homotopic to id_X, and $f \circ g$ is either left or right homotopic to id_Y, then (f, g) is an equivalence in $\mathcal{M}$ (in the sense of definition 3.1.17).*

(ii) *If there exist morphisms $g, h : Y \to X$ such that $g \circ f$ is either left or right homotopic to id_X and $f \circ h$ is either left or right homotopic to id_Y, then (the image of) f is an isomorphism in $\mathrm{Ho}\,\mathcal{M}$.*

Proof. Obvious, given remark 4.2.5. ◆

Lemma 4.2.10. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *Given any cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, $f_0 \circ p : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to itself.*

(ii) *If $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1 with respect to a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, then the same H is a left homotopy from f_1 to f_0 for the cylinder object $\big(\mathrm{Cyl}(X), i_1, i_0, p\big)$.*

Dually:

(i′) *Given any path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, $i \circ f_0 : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to itself.*

(ii′) *If $H : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to f_1 with respect to a path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, then the same H is a right homotopy from f_1 to f_0 for the path object $\big(\mathrm{Path}(Y), i, p_1, p_0\big)$.*

Proof. Obvious. ⧫

Lemma 4.2.11. *Let $\mathcal{M}$ be a model category.*

- *If $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ is a cylinder object for a cofibrant object in $\mathcal{M}$, then the insertions $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are trivial cofibrations, and* $\mathrm{Cyl}(X)$ *is a cofibrant object in $\mathcal{M}$.*

- *If $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ is a path object for a fibrant object in $\mathcal{M}$, then the projections $p_0, p_1 : Y \to \mathrm{Path}(Y)$ are trivial fibrations, and* $\mathrm{Path}(X)$ *is a fibrant object in $\mathcal{M}$.*

Proof. See Lemmas 1.5 and 1.7 in [GJ], or Lemma 7.3.6 in [Hirschhorn, 2003]. □

Lemma 4.2.12. *Let X be a cofibrant object in a model category $\mathcal{M}$. Given two cylinder objects for X, say $\big(\mathrm{Cyl}(X)', i_0', i_1', p'\big)$ and $\big(\mathrm{Cyl}(X)'', i_0'', i_1'', p''\big)$, there exists a third cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ such that the diagram below* *commutes,*

$$\begin{tikzcd}
X \arrow[dr, "i'_0"] \arrow[ddrr, bend right, "i_0"'] & & X \arrow[dl, "i'_1"'] \arrow[dr, "i''_0"] & & X \arrow[dl, "i''_1"'] \arrow[ddll, bend left, "i_1"] \\
& \mathrm{Cyl}(X)' \arrow[dr] \arrow[ddr, "p'"'] & & \mathrm{Cyl}(X)'' \arrow[dl] \arrow[ddl, "p''"] & \\
& & \mathrm{Cyl}(X) \arrow[d, "p"'] & & \\
& & X & &
\end{tikzcd}$$

and the diamond is a pushout diagram.

Dually, if Y is a fibrant object in $\mathcal{M}$, and we have two path objects for Y, say $\left(\mathrm{Path}(Y)', i', p'_0, p'_1\right)$ and $\left(\mathrm{Path}(Y)'', i'', p''_0, p''_1\right)$, then there exists a third path object $\left(\mathrm{Path}(Y), i, p_0, p_1\right)$ such that the diagram below commutes,

$$\begin{tikzcd}
& & Y \arrow[d, "i"'] \arrow[ddl, bend right, "i'"'] \arrow[ddr, bend left, "i''"] & & \\
& & \mathrm{Path}(Y) \arrow[dl] \arrow[dr] \arrow[ddll, bend right, "p_0"'] \arrow[ddrr, bend left, "p_1"] & & \\
& \mathrm{Path}(Y)' \arrow[dl, "p'_0"] \arrow[dr, "p'_1"'] & & \mathrm{Path}(Y)'' \arrow[dl, "p''_0"] \arrow[dr, "p''_1"'] & \\
Y & & Y & & Y
\end{tikzcd}$$

and the diamond is a pullback diagram.

Proof. See Lemmas 1.5 and 1.7 in [GJ, Ch. II], or Lemma 7.4.2 in [Hirschhorn, 2003]. □

Corollary 4.2.13. *Let $f_0, f_1, f_2 : X \to Y$ be three parallel morphisms in a model category $\mathcal{M}$.*

(i) *If f_0 and f_1 are left homotopic, and f_1 and f_2 are left homotopic, then f_0 and f_2 are also left homotopic.*

(ii) *If f_0 and f_1 are right homotopic, and f_1 and f_2 are right homotopic, then f_0 and f_2 are also right homotopic.* ■

Lemma 4.2.14. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *If X is cofibrant, and f_0 and f_1 are left homotopic, given any path object $(\mathrm{Path}(Y), i, p_0, p_1)$ for Y, there is a right homotopy $H : X \to \mathrm{Path}(Y)$ from f_0 to f_1.*

(ii) *If Y is fibrant, and f_0 and f_1 are right homotopic, given any cylinder object $(\mathrm{Cyl}(X), i_0, i_1, p)$ for X, there is a left homotopy $H : \mathrm{Cyl}(X) \to Y$ from f_0 to f_1.*

Proof. See Proposition 1.8 in [GJ, Ch. II], or Proposition 7.4.7 in [Hirschhorn, 2003]. □

Proposition 4.2.15. *Let X and Y be objects in a model category $\mathcal{M}$.*

(i) *If X is cofibrant, then being left homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(ii) *If Y is fibrant, then being right homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(iii) *If X is cofibrant and Y is fibrant, then these two equivalence relations on $\mathcal{M}(X, Y)$ coincide.*

Proof. Use the preceding lemmas. ■

Lemma 4.2.16. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *If f_0 and f_1 are right homotopic and $g : W \to X$ is any morphism in $\mathcal{M}$, then $f_0 \circ g$ and $f_1 \circ g$ are also right homotopic.*

(ii) *If f_0 and f_1 are left homotopic and $g : Y \to Z$ is any morphism in $\mathcal{M}$, then $g \circ f_0$ and $g \circ f_1$ are also left homotopic.*

Proof. Obvious. ♦

Corollary 4.2.17. *Let $\mathcal{M}$ be a model category, and let $\mathcal{M}_{\mathrm{cf}}$ be the full subcategory spanned by the cofibrant–fibrant objects. Then the equivalence relation induced by homotopy is a congruence on $\mathcal{M}_{\mathrm{cf}}$; in particular, there exist a locally small category $\mathcal{M}_{\mathrm{h}}$ and a full functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ with these properties:*

- *The objects of $\mathcal{M}_{\mathrm{h}}$ are those of $\mathcal{M}_{\mathrm{cf}}$.*
- *The hom-set $\mathcal{M}_{\mathrm{h}}(X, Y)$ is $\mathcal{M}(X, Y)$ modulo homotopy.*
- *The functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}_{\mathrm{h}}$ sends each morphism in $\mathcal{M}_{\mathrm{cf}}$ to its homotopy class.* ■

The next result is a version of Whitehead's theorem; however, this is a purely formal consequence of the model category axioms and has no real content, unlike the original theorem.

Proposition 4.2.18. *Let X and Y be cofibrant–fibrant objects in a model category $\mathcal{M}$. If $f : X \to Y$ is a weak equivalence, then f has a homotopy inverse in $\mathcal{M}$.*

Proof. See Theorem 1.10 in [GJ, Ch. II], or Theorem 7.5.10 in [Hirschhorn, 2003]. □

Lemma 4.2.19. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

- *If $g : W \to X$ is a morphism with a right homotopy right inverse in $\mathcal{M}$, then $f_0 \circ g$ and $f_1 \circ g$ are right homotopic if and only if f_0 and f_1 are right homotopic.*
- *If $g : Y \to Z$ is a morphism with a left homotopy left inverse in $\mathcal{M}$, then $g \circ f_0$ and $g \circ f_1$ are left homotopic if and only if f_0 and f_1 are left homotopic.*

Proof. This follows immediately from the definitions and lemma 4.2.16. ■

Corollary 4.2.20. *Let W, X, Y, Z be cofibrant–fibrant objects in a model category $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms.*

- *If $g : W \to X$ is a weak equivalence such that $f_0 \circ g$ and $f_1 \circ g$ are homotopic, then f_0 and f_1 are homotopic.*
- *If $g : Y \to Z$ is a weak equivalence such that $g \circ f_0$ and $g \circ f_1$ are homotopic, then f_0 and f_1 are homotopic.*

Proof. Apply proposition 4.2.18 in conjunction with the above lemma. ■

4.3 The homotopy category

Prerequisites. §§ 4.1, 4.2, A.4.

Theorem 4.3.1. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the localising functor.*

(i) $\operatorname{Ho}\mathcal{M}$ *is equivalent to the locally small category $\mathcal{M}_{\mathrm{h}}$ defined in corollary 4.2.17, and $\mathcal{M}$ is a saturated homotopical category.*

(ii) *If X and Y are cofibrant–fibrant objects in $\mathcal{M}$, then the hom-set map $\mathcal{M}(X, Y) \to \operatorname{Ho}\mathcal{M}(X, Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

Proof. (i). This is Theorem 1.11 in [GJ, Ch. II], or Proposition 5.8 in [DS].

(ii). Implied by claim (i). □

Corollary 4.3.2. *Let $f : X \to Y$ be a morphism in a model category $\mathcal{M}$. If f has a quasi-inverse in $\mathcal{M}$ (in the sense of definition 3.1.17), then f is a weak equivalence in $\mathcal{M}$.*

Proof. If f has a quasi-inverse in $\mathcal{M}$, then (the image of) f is an isomorphism in $\operatorname{Ho}\mathcal{M}$; but $\mathcal{M}$ is a saturated homotopical category, so f must be a weak equivalence in $\mathcal{M}$. ■

Corollary 4.3.3. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the localising functor.*

(i) *For any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, if X is cofibrant and Y is fibrant, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

(ii) *The full subcategory $\mathcal{M}_{\mathrm{cf}}$ of cofibrant–fibrant objects in $\mathcal{M}$ has the Whitehead property (in the sense of definition 3.1.21).*

Proof. (i). As noted in remark 4.2.5, if $f_0, f_1 : X \to Y$ are homotopic, then we must have $\gamma f_0 = \gamma f_1$. Conversely, suppose $\gamma f_0 = \gamma f_1$ with X cofibrant and Y fibrant. Let (RX, i') be a cofibrant fibrant replacement for X and (QY, p') be a fibrant cofibrant replacement for Y. Then, there exists morphisms $f'_0, f'_1 : RX \to QY$ such that $f_0 = p' \circ f'_0 \circ i'$ and $f_1 = p' \circ f'_1 \circ i'$. Since $i' : X \to RX$

and $p' : QY \to Y$ are weak equivalences, we must have $\gamma f_0' = \gamma f_1'$ in $\operatorname{Ho}\mathcal{M}$. The theorem then implies f_0' and f_1' are homotopic; thus f_0 and f_1 are also homotopic, by lemmas 4.2.14 and 4.2.16.

(ii). Apply theorem 3.1.22 in conjunction with lemma 4.2.9 and the above corollary. ∎

Corollary 4.3.4. *Let $f : X \to Y$ be a morphism between two cofibrant objects in a derivable category $\mathcal{M}$. If $\mathcal{M}$ is a saturated homotopical category, then the following are equivalent:*

(i) *The morphism $f : X \to Y$ is a weak equivalence in $\mathcal{M}$.*

(ii) *The hom-set map map $\operatorname{Ho}\mathcal{M}(f, Z) : \operatorname{Ho}\mathcal{M}(Y, Z) \to \operatorname{Ho}\mathcal{M}(X, Z)$ is a bijection for all cofibrant–fibrant objects Z in $\mathcal{M}$.*

(iii) *The hom-set map $\mathcal{M}_h(f, Z) : \mathcal{M}_h(Y, Z) \to \mathcal{M}_h(X, Z)$ is a bijection for all cofibrant–fibrant objects Z in $\mathcal{M}$, where $\mathcal{M}_h(Y, Z)$ (resp. $\mathcal{M}_h(X, Z)$) denotes the set of all morphisms $Y \to Z$ (resp. $X \to Z$) in $\mathcal{M}$* modulo homotopy.

Proof. (i) ⇒ (ii). Every weak equivalence in $\mathcal{M}$ becomes an isomorphism in $\operatorname{Ho}\mathcal{M}$, so in particular $\operatorname{Ho}\mathcal{M}(f, Z) : \operatorname{Ho}\mathcal{M}(Y, Z) \to \operatorname{Ho}\mathcal{M}(X, Z)$ must be a bijection.

(ii) ⇔ (iii). The previous corollary implies that the vertical arrows in the following commutative diagram are bijections,

$$\begin{tikzcd} \mathcal{M}_h(Y, Z) \arrow[r, "\mathcal{M}'(f,Z)"] \arrow[d] & \mathcal{M}_h(X, Z) \arrow[d] \\ \operatorname{Ho}\mathcal{M}(Y, Z) \arrow[r, "\operatorname{Ho}\mathcal{M}(f,Z)"'] & \operatorname{Ho}\mathcal{M}(X, Z) \end{tikzcd}$$

and so $\mathcal{M}_h(f, Z)$ is a bijection if and only if $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection.

(ii) ⇒ (i). Suppose $(\hat{X}, i_X)$ is a cofibrant fibrant replacement for X and $(\hat{Y}, i_Y)$ is a cofibrant fibrant replacement for Y. Then, (by axiom CM4) there exists a morphism $\hat{f} : \hat{X} \to \hat{Y}$ making the diagram below commute,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ \hat{X} \arrow[r, "\hat{f}"'] & \hat{Y} \end{tikzcd}$$

and by the 2-out-of-3 property, f is a weak equivalence if and only if $\hat{f}$ is a weak equivalence. On the other hand, the following diagram also commutes,

$$\begin{tikzcd} \operatorname{Ho}\mathcal{M}(\hat{Y}, Z) \arrow[r, "\operatorname{Ho}\mathcal{M}(\hat{f},Z)"] \arrow[d, "\operatorname{Ho}\mathcal{M}(i_Y,Z)"'] & \operatorname{Ho}\mathcal{M}(\hat{X}, Z) \arrow[d, "\operatorname{Ho}\mathcal{M}(i_X,Z)"] \\ \operatorname{Ho}\mathcal{M}(Y, Z) \arrow[r, "\operatorname{Ho}\mathcal{M}(f,Z)"'] & \operatorname{Ho}\mathcal{M}(X, Z) \end{tikzcd}$$

and so $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection if and only if $\operatorname{Ho}\mathcal{M}(\hat{f}, Z)$ is a bijection; but $\hat{X}$ and $\hat{Y}$ are both cofibrant–fibrant objects, so if $\operatorname{Ho}\mathcal{M}(f, Z)$ is a bijection for all cofibrant–fibrant objects Z, then $\hat{f}$ must be a weak equivalence (because $\mathcal{M}$ is a saturated homotopical category). ■

Proposition 4.3.5 (Joyal)**.** *Let $\mathcal{M}$ and $\mathcal{M}'$ be two model categories with the same underlying category. If cofibrations in $\mathcal{M}$ are cofibrations in $\mathcal{M}'$* and vice versa*, then the following are equivalent:*

(i) *Every weak equivalence in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}'$.*

(ii) *Every fibrant object in $\mathcal{M}'$ is a fibrant object in $\mathcal{M}$.*

(iii) *Every cofibrant–fibrant object in $\mathcal{M}'$ is a cofibrant–fibrant object in $\mathcal{M}$.*

(iv) *Every weak equivalence between cofibrant objects in $\mathcal{M}$ is a weak equivalence between cofibrant objects in $\mathcal{M}'$.*

Proof. This result is due to Joyal [2010].

(i) ⇒ (ii). Since every trivial cofibration in $\mathcal{M}$ is a trivial cofibration in $\mathcal{M}'$, theorem 4.1.12 (plus the definition of weak factorisation system) implies every fibration in $\mathcal{M}'$ is a fibration in $\mathcal{M}$; in particular, every fibrant object in $\mathcal{M}'$ is a fibrant object in $\mathcal{M}$.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (iv). Let $f : X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. X and Y are also cofibrant objects in $\mathcal{M}'$, and by proposition 4.2.3, we may choose cylinder objects for X and Y in $\mathcal{M}$ that are also cylinder objects in $\mathcal{M}'$, since the trivial fibrations in $\mathcal{M}$ and $\mathcal{M}'$ are the same. Now, if Z is a cofibrant–fibrant object in $\mathcal{M}'$, then it is also a a cofibrant–fibrant object in $\mathcal{M}$, and so by lemma 4.2.10, we deduce that the homotopy relation on morphisms $X \to Z$ (resp. $Y \to Z$) in $\mathcal{M}$ agrees with the homotopy relation on morphisms $X \to Z$ (resp. $Y \to Z$) in $\mathcal{M}'$. Thus, applying corollary 4.3.4, we conclude that $f : X \to Y$ is also a weak equivalence in $\mathcal{M}'$.

(iv) ⇒ (i). Let $f : X \to Y$ be a weak equivalence in $\mathcal{M}$, let $\left(\tilde{X}, p_X\right)$ be a fibrant cofibrant replacement for X in $\mathcal{M}$, and let $\left(\tilde{Y}, p_Y\right)$ be a fibrant cofibrant replacement for Y in $\mathcal{M}$. There exists a morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ making the following diagram commute,

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "p_X"'] & \tilde{Y} \arrow[d, "p_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

and by the 2-out-of-3 property, $\tilde{f} : \tilde{X} \to \tilde{Y}$ is a weak equivalence between cofibrant objects in $\mathcal{M}$. The hypothesis says $\tilde{f}$ is also a weak equivalence between cofibrant objects in $\mathcal{M}'$, and p_X and p_Y are trivial cofibrations in $\mathcal{M}'$, so we conclude that $f : X \to Y$ is a weak equivalence in $\mathcal{M}'$ as well. ■

Theorem 4.3.6 (Determination principle)**.** *A model structure on a category with finite limits and colimits is uniquely determined by any one of the following sets of data:*

(i) *The cofibrations and the weak equivalences.*

(ii) *The cofibrations and the trivial cofibrations.*

(iii) *The cofibrations and the fibrant objects.*

(iv) *The cofibrations and the cofibrant–fibrant objects.*

(v) *The cofibrations and the weak equivalences between cofibrant objects.*

(vi) *The cofibrations and the fibrations.*

(vii) *The trivial cofibrations and the trivial fibrations.*

(i′) *The fibrations and the weak equivalences.*

(ii′) *The fibrations and the trivial fibrations.*

(iii′) *The fibrations and the cofibrant objects.*

(iv′) *The fibrations and the cofibrant–fibrant objects.*

(v′) *The fibrations and the weak equivalences between fibrant objects.*

Proof. (i) and (ii). By theorem 4.1.12, the fibrations are precisely the morphisms with the right lifting property with respect to every trivial cofibration.

(iii), (iv), and (v). Apply Joyal's result (proposition 4.3.5) and reduce to case (i).

(vi). The trivial cofibrations are precisely the morphisms with the left lifting property with respect to all fibrations, and the trivial fibrations are precisely the morphisms with the right lifting property with respect to all cofibrations, so this reduces to case (vii).

(vii). Axioms CM2 and CM5 imply that every weak equivalence is of the form $p \circ i$ where i is a trivial cofibration and p is a trivial fibration. Thus, the trivial cofibrations and the trivial fibrations together determine the weak equivalences. On the other hand, the trivial cofibrations determine the fibrations, and the trivial fibrations determine the cofibrations, thus the entire model structure is determined. ■

4.4 Quillen functors

Prerequisites. §§ 3.1, 3.3, 3.4, 4.1, A.5.

Definition 4.4.1.

- A **left Quillen functor** is a functor between derivable categories that has a right adjoint and preserves cofibrations and trivial cofibrations.

- A **right Quillen functor** is a functor between derivable categories that has a left adjoint and preserves fibrations and trivial fibrations.

- A **Quillen adjunction** is an adjunction

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

where F is a left Quillen functor and G is a right Quillen functor.

- A **Quillen equivalence** is a Quillen adjunction as above satisfying this additional condition:

  – Given a cofibrant object A in $\mathcal{N}$ and fibrant object Y in $\mathcal{M}$, a morphism $FA \to Y$ is a weak equivalence in $\mathcal{M}$ if and only if its right adjoint transpose $A \to GY$ is a weak equivalence in $\mathcal{N}$.

Proposition 4.4.2. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be an adjunction between categories with model structures. The following are equivalent:*

(i) *F preserves cofibrations and trivial cofibrations.*

(ii) *G preserves fibrations and trivial fibrations.*

(iii) *F preserves cofibrations and G preserves fibrations.*

(iv) *F preserves trivial cofibrations and G preserves trivial fibrations.*

(v) *(Assuming $\mathcal{M}$ and $\mathcal{N}$ are derivable categories.) $F \dashv G$ is a Quillen adjunction.*

Proof. Use proposition A.3.20. ∎

REMARK 4.4.3. A functor between categories with model structures that preserves both trivial cofibrations and trivial fibrations must also preserve weak equivalences, since axioms CM2 and CM5 together imply that a morphism is a weak equivalence if and only if it is of the form $p \circ i$ where i is a trivial cofibration and p is a trivial fibration. In particular, a functor that is both left and right Quillen must be homotopical.

Proposition 4.4.4. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

- *F sends cofibrant objects in $\mathcal{N}$ to cofibrant objects in $\mathcal{M}$.*

- *G sends fibrant objects in $\mathcal{M}$ to fibrant objects in $\mathcal{N}$.*

Proof. The two claims are formally dual; we will prove the first version.

Let B be a cofibrant object in $\mathcal{N}$ and let $p : X \to Y$ be a trivial fibration in $\mathcal{M}$. Since $F \dashv G$, we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{M}(FB, X) \arrow[r, "\cong"] \arrow[d, "\mathcal{M}(FB,p)"'] & \mathcal{N}(B, GX) \arrow[d, "\mathcal{M}(B,Gp)"] \\ \mathcal{M}(FB, Y) \arrow[r, "\cong"'] & \mathcal{N}(B, GY) \end{tikzcd}$$

By hypothesis, $Gp : GX \to GY$ is a trivial fibration in $\mathcal{N}$, so the hom-set map $\mathcal{N}(B, Gp)$ is a surjection. It follows that $\mathcal{M}(FB, p)$ is also a surjection, and thus FB is a cofibrant object in $\mathcal{M}$. ■

Proposition 4.4.5.

(i) *The composite of two Quillen adjunctions is also a Quillen adjunction.*

(ii) *The composite of two Quillen equivalences is also a Quillen equivalence.*

Proof. Obvious. ◆

Lemma 4.4.6 (Kenneth S. Brown)**.** *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a category with weak equivalences.*

- *Let $\mathcal{M}_c$ be the full subcategory of cofibrant objects in $\mathcal{M}$. If $F : \mathcal{M}_c \to \mathcal{C}$ sends trivial cofibrations in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$, then F also sends weak equivalences in $\mathcal{M}_c$ to weak equivalences in $\mathcal{C}$.*
- *Let $\mathcal{M}_f$ be the full subcategory of fibrant objects in $\mathcal{M}$. If $F : \mathcal{M}_f \to \mathcal{C}$ sends trivial fibrations in $\mathcal{M}_f$ to weak equivalences in $\mathcal{C}$, then F also sends weak equivalences in $\mathcal{M}_f$ to weak equivalences in $\mathcal{C}$.*

Proof. See Lemma 9.9 in [DS], Lemma 7.7.1 in [Hirschhorn, 2003], or Lemma 14.5 in [DHKS]. □

The usual proof of the Ken Brown's lemma uses binary coproducts (or binary products, as the case may be), so it cannot be used in the case where the domain is merely a derivable category. Nonetheless, we have already proved something very similar, namely lemma 4.1.31.

Proposition 4.4.7 (Dugger)**.** *Let $F \dashv G$ be an adjunction between DHK model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(iii) *G preserves fibrations between fibrant objects and all trivial fibrations.*

Proof. See Proposition 8.5.4 in [Hirschhorn, 2003], or Corollary A.2 in [Dugger, 2001b]. □

Definition 4.4.8. Let $\mathcal{M}$ be a derivable category.

- A **left Quillen deformation retract** (resp. **functorial left Quillen deformation retract**) of $\mathcal{M}$ is a left deformation retract of $\mathcal{M}$ of the form $(\mathcal{M}_c, Q, p)$ where $\mathcal{M}_c$ is the full subcategory of cofibrant objects in $\mathcal{M}$.
- A **right Quillen deformation retract** (resp. **functorial right Quillen deformation retract**) of $\mathcal{M}$ is a right deformation retract of $\mathcal{M}$ of the form $(\mathcal{M}_f, R, i)$ where $\mathcal{M}_f$ is the full subcategory of fibrant objects in $\mathcal{M}$.

Lemma 4.4.9. *Let $\mathcal{M}$ be a derivable category.*

- *Left Quillen deformation retracts of $\mathcal{M}$ exist.*
- *Right Quillen deformation retracts of $\mathcal{M}$ exist.*

Proof. The two claims are formally dual; we will prove the first version.

For each object X in $\mathcal{M}$, choose a fibrant cofibrant replacement (QX, p_X); such exist by proposition 4.1.22. Then, for each morphism $f : X \to Y$ in $\mathcal{M}$, there exists a morphism $Qf : QX \to QY$ making the diagram commute,

$$\begin{tikzcd} QX \arrow[r, "p_X"] \arrow[d, dashed, "Qf"'] & X \arrow[d, "f"] \\ QY \arrow[r, "p_Y"'] & Y \end{tikzcd}$$

because $p_Y : QY \to Y$ is a trivial fibration and QX is cofibrant; note that axiom CM2 implies Qf is a weak equivalence if (and only if!) f is. Thus, axioms DR1–2 are satisfied. For axiom DR3, we refer to proposition 4.1.33. Finally, we simply need to observe that axiom DR4 is trivial. ■

Lemma 4.4.10. *Let $\mathcal{M}$ be a derivable category.*

- $\left(\mathcal{M}_{\mathrm{c}}, Q, p\right)$ *is a functorial left Quillen deformation for $\mathcal{M}$ if and only if (Q, p) is a cofibrant replacement functor for $\mathcal{M}$.*
- $\left(\mathcal{M}_{\mathrm{f}}, R, i\right)$ *is a functorial left Quillen deformation for $\mathcal{M}$ if and only if (R, i) is a cofibrant replacement functor for $\mathcal{M}$.*

Proof. Obvious. ◆

Theorem 4.4.11. *Let $\mathcal{M}$ be a derivable category, let $\mathcal{C}$ be a relative category, and let $\gamma_{\mathcal{M}} : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ and $\gamma_{\mathcal{C}} : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ be the respective localising functors. Suppose $\operatorname{weq} \mathcal{C}$ has the special 2-out-of-4 property. If $F : \mathcal{M} \to \mathcal{C}$ is a functor that sends trivial cofibrations in $\mathcal{M}$ to weak equivalences in $\mathcal{C}$, then:*

(i) *Any left Quillen deformation retract of $\mathcal{M}$ is a left deformation retract for F; in particular, a total left derived functor for F exists.*

(ii) *If $\mathcal{M}$ has a cofibrant replacement functor, then F is functorially left deformable and has a homotopical left approximation.*

(iii) *If $(\mathbf{L}F, \alpha)$ is any total left derived functor for F, then the extension counit component $\alpha_X : (\mathbf{L}F)\gamma_{\mathcal{M}} X \to \gamma_{\mathcal{C}} F X$ is an isomorphism for all cofibrant objects X in $\mathcal{M}$.*

Dually, if $F : \mathcal{M} \to \mathcal{C}$ is a functor that sends trivial fibrations in $\mathcal{M}$ to weak equivalences in $\mathcal{C}$, then:

(i′) *Any right Quillen deformation retract of $\mathcal{M}$ is a right deformation retract for F; in particular, a total right derived functor for F exists.*

(ii′) *If $\mathcal{M}$ has a fibrant replacement functor, then F is functorially right deformable and has a homotopical right approximation.*

(iii′) *If $(\mathbf{R}F, \beta)$ is any total right derived functor for F, then the extension counit component $\beta_X : (\mathbf{R}G)\gamma_{\mathcal{M}} X \to \gamma_{\mathcal{C}} F X$ is an isomorphism for all fibrant objects X in $\mathcal{M}$.*

Proof. (i). Let $\left(\mathcal{M}_{\mathrm{c}}, Q, p\right)$ be a left Quillen deformation retract of $\mathcal{M}$. Then F sends weak equivalences in $\mathcal{M}_{\mathrm{c}}$ to weak equivalences in $\mathcal{C}$ by lemma 4.1.31, so $\left(\mathcal{M}_{\mathrm{c}}, Q, p\right)$ is indeed a left deformation retract for $\mathcal{C}$. We may then apply theorem 3.3.13 to obtain a total left derived functor.

(ii). By the same argument, if (Q, p) is a cofibrant replacement functor for $\mathcal{M}$, then $\left(\mathcal{M}_{\mathrm{c}}, Q, p\right)$ is a functorial left deformation retract for F. We then appeal to theorem 3.4.10.

(iii). The extension counit has the required property because, for all cofibrant objects X in $\mathcal{M}$, the morphism $F p_X : FQX \to FX$ is a weak equivalence in $\mathcal{C}$; but this is precisely the component of the extension counit at X. ■

Theorem 4.4.12. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

(i) *Any left Quillen deformation retract of $\mathcal{N}$ is a left deformation retract for F; dually, any right Quillen deformation retract of $\mathcal{M}$ is a right deformation retract for G.*

(ii) *$F \dashv G$ is a deformable adjunction; in particular, a derived adjunction exists.*

(iii) *If $F \dashv G$ is a Quillen equivalence, then the derived adjunction*

$$\mathbf{L}F \dashv \mathbf{R}G : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathcal{N}$$

is an adjoint equivalence of categories; and if $\mathcal{M}$ and $\mathcal{N}$ are saturated derivable categories, then the converse is true.

Proof. (i). Since weak equivalences in derivable categories are closed under retracts (by axiom CM3), we may use theorem 4.4.11.

(ii). That $F \dashv G$ is a derivable adjunction follows immediately; then apply theorem 3.3.20 for the existence of the derived adjunction.

(iii). This is a special case of proposition 3.3.24. ■

Proposition 4.4.13. *Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be derivable categories.*

- *If $F : \mathcal{N} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{L}$ are left Quillen functors, then the composite $(\mathbf{L}G)(\mathbf{L}F)$ is (the functor part of) a total left derived functor for GF.*
- *If $F : \mathcal{N} \to \mathcal{P}$ and $G : \mathcal{M} \to \mathcal{N}$ are right Quillen functors, then the composite $(\mathbf{R}F)(\mathbf{R}G)$ is (the functor part of) a total right derived functor for FG.*

Assuming $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{L}$ have fibrant and cofibrant replacement functors:

- *If $F : \mathcal{N} \to \mathcal{M}$ and $G : \mathcal{M} \to \mathcal{L}$ are left Quillen functors, then the composite $(\mathbb{L}G)(\mathbb{L}F)$ is (the functor part of) a homotopical left approximation for GF.*
- *If $F : \mathcal{N} \to \mathcal{P}$ and $G : \mathcal{M} \to \mathcal{N}$ are right Quillen functors, then the composite $(\mathbb{R}F)(\mathbb{R}G)$ is (the functor part of) a homotopical right approximation for FG.*

Proof. Use theorems 3.3.13, 3.4.10, and 4.4.11 with proposition 4.4.5. ■

Proposition 4.4.14. *Let $\mathcal{M}$ be a derivable category, let $\mathcal{U}$ be the class of trivial cofibrations in $\mathcal{M}$, and let $\mathcal{V}$ be the class of trivial fibrations in $\mathcal{M}$.*

- *Let $(\mathcal{M}_c, Q, p)$ be a left Quillen deformation retract of $\mathcal{M}$ and let $\mathcal{M}[\mathcal{U}^{-1}]$ be the localisation of $\mathcal{M}$ with respect to the trivial cofibrations. Then the inclusion $\mathcal{M}_c \hookrightarrow \mathcal{M}$ induces a fully faithful functor $\operatorname{Ho}\mathcal{M}_c \to \mathcal{M}[\mathcal{U}^{-1}]$, and (Q, p) induces a right adjoint for that functor.*
- *Let $(\mathcal{M}_f, R, i)$ be a right Quillen deformation retract of $\mathcal{M}$ and let $\mathcal{M}[\mathcal{V}^{-1}]$ be the localisation of $\mathcal{M}$ with respect to the trivial fibrations. Then the inclusion $\mathcal{M}_f \hookrightarrow \mathcal{M}$ induces a fully faithful functor $\operatorname{Ho}\mathcal{M}_f \to \mathcal{M}[\mathcal{V}^{-1}]$, and (R, i) induces a left adjoint for that functor.*

Proof. The two claims are formally dual; we will prove the first version.

By corollary 4.1.13, $(\mathcal{M}, \mathcal{U})$ admits a calculus of cospans, so we may use the fundamental theorem of calculi of cospans (3.5.6) and lemma 4.1.8 to deduce that the canonical functor $\operatorname{Ho}\mathcal{M}_c \to \mathcal{M}[\mathcal{U}^{-1}]$ is indeed fully faithful.

On the other hand, lemma 4.1.31 says the localising functor $\mathcal{M} \to \mathcal{M}[\mathcal{U}^{-1}]$ sends weak equivalences in $\mathcal{M}_c$ to isomorphisms in $\mathcal{M}[\mathcal{U}^{-1}]$, so we may apply proposition 3.3.15 to deduce that the canonical functor $\mathcal{M}[\mathcal{U}^{-1}] \to \operatorname{Ho}\mathcal{M}$ has a fully faithful left adjoint defined by Q. Proposition 4.1.33 says the canonical functor $\operatorname{Ho}\mathcal{M}_c \to \operatorname{Ho}\mathcal{M}$ is fully faithful, so we have the following hom-set bijections:

$$\begin{aligned}\operatorname{Ho}\mathcal{M}_c(X, QY) &\cong \operatorname{Ho}\mathcal{M}(X, QY)\\ &\cong \operatorname{Ho}\mathcal{M}(QX, QY)\\ &\cong \mathcal{M}[\mathcal{U}^{-1}](X, Y)\end{aligned}$$

These bijections are moreover natural in X because Q induces a well-defined functor $\bar{Q} : \operatorname{Ho} \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ and p induces a natural isomorphism $\bar{Q} \Rightarrow \mathrm{id}_{\operatorname{Ho} \mathcal{M}}$. Thus, we obtain from (Q, p) a right adjoint for the functor $\operatorname{Ho} \mathcal{M}_{\mathrm{c}} \to \mathcal{M}[\mathcal{V}^{-1}]$, as required. ■

Definition 4.4.15. Let $\mathbb{A}$ be a small category and let $\mathcal{M}$ be a category equipped with a model structure.

- The **injective model structure** on the functor category $[\mathbb{A}, \mathcal{M}]$ is a model structure such that a morphism in $[\mathbb{A}, \mathcal{M}]$ is a cofibration (resp. weak equivalence) if and only if all its components are cofibrations (resp. weak equivalences) in $\mathcal{M}$.

- The **projective model structure** on the functor category $[\mathbb{A}, \mathcal{M}]$ is a model structure such that a morphism in $[\mathbb{A}, \mathcal{M}]$ is a fibration (resp. weak equivalence) if and only if all its components are fibrations (resp. weak equivalences) in $\mathcal{M}$.

Remark 4.4.16. The injective (resp. projective) model structure on $[\mathbb{A}, \mathcal{M}]$ is unique *if it exists*, by theorem 4.1.12.

Proposition 4.4.17. *Let $\mathcal{M}$ be a derivable category, let $\mathbb{A}$ be a small category, and let $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ be the functor that sends an object X in $\mathcal{M}$ to the constant functor $\Delta X : \mathbb{A} \to \mathcal{M}$ with value X.*

- *If $\mathcal{M}$ has colimits for diagrams of shape $\mathbb{A}$, then $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a right Quillen functor with respect to the projective model structure on $[\mathbb{A}, \mathcal{M}]$ when it exists.*

- *If $\mathcal{M}$ has limits for diagrams of shape $\mathbb{A}$, then $\Delta : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a left Quillen functor with respect to the injective model structure on $[\mathbb{A}, \mathcal{M}]$ when it exists.*

Proof. Δ certainly preserves fibrations (resp. cofibrations) and weak equivalences with respect to the projective (resp. injective) model structure, so by proposition 4.4.2, $\varinjlim_{\mathbb{A}} \dashv \Delta$ (resp. $\Delta \dashv \varprojlim_{\mathbb{A}}$) is a Quillen adjunction.[3] ■

[3] Recall proposition 0.1.12.

Proposition 4.4.18. *Let $\mathcal{M}$ be a category and let I be a set.*

(i) *The functor category $[I, \mathcal{M}]$ admits a model structure that is simultaneously an injective model structure and a projective model structure.*

(ii) *If $\mathcal{M}$ is a derivable category (resp. saturated derivable category, model category), then $[I, \mathcal{M}]$ equipped with the above model structure is a derivable category (resp. saturated derivable category, model category).*

(iii) *If $\mathcal{M}$ is a derivable category and has products and coproducts for families of objects indexed by I, then $\Delta : \mathcal{M} \to [I, \mathcal{M}]$ is both a left Quillen functor and a right Quilen functor.*

(iv) *If $\mathcal{M}$ is a model category, then the canonical exponential comparison functor* $\operatorname{Ho}[I, \mathcal{M}] \to [I, \operatorname{Ho} \mathcal{M}]$ *is an isomorphism of categories.*

Proof. (i). If we declare the cofibrations (resp. weak equivalences, fibrations) in $[I, \mathcal{M}]$ to be precisely the morphisms that are cofibrations (resp. weak equivalences, fibrations) componentwise, then the axioms CM2–5 may be verified componentwise as well.

(ii). Axioms DC0, DC1, and CM1 can be verified componentwise. If $\mathcal{M}$ is saturated, then we can use lemma 3.1.11 to deduce that $[I, \mathcal{M}]$ is also saturated.

(iii). Apply proposition 4.4.17.

(iv). Use theorem 4.3.1 and the fact that the congruence of homotopy is componentwise in $[I, \mathcal{M}]$. ■

Corollary 4.4.19. *Let $\mathcal{M}$ be a saturated derivable category and let I be a set.*

- *If $\mathcal{M}$ has products for families of objects indexed by I, then the product of an I-indexed family of weak equivalences between fibrant objects is also a weak equivalence between fibrant objects.*
- *If $\mathcal{M}$ has coproducts for families of objects indexed by I, then the coproduct of an I-indexed family of weak equivalences between cofibrant objects is also a weak equivalence between cofibrant objects.*

Proof. Apply lemma 4.1.31 to the previous proposition. ■

Proposition 4.4.20. *Let $\mathcal{M}$ be a derivable category and let $\mathbb{A}$ be a small category.*

- *If $\mathcal{M}$ has coproducts for families of size $\leq |\mathrm{mor}\ \mathbb{A}|$, then the evaluation functors $[\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ are right Quillen functors with respect to the injective model structure on $[\mathbb{A}, \mathcal{M}]$ (if it exists).*

- *If $\mathcal{M}$ has products for families of size $\leq |\mathrm{mor}\ \mathbb{A}|$, then the evaluation functors $[\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ are left Quillen functors with respect to the projective model structure on $[\mathbb{A}, \mathcal{M}]$ (if it exists).*

Proof. The two claims are formally dual; we will prove the first version.

Let A be an object in $\mathbb{A}$ and let $A^* : [\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ be the functor $F \mapsto FA$. It is not hard to check that A^* has a left adjoint $A_! : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$, namely the functor $X \mapsto \mathbb{A}(A, -) \odot X$. Since the class of cofibrations and the class of trivial cofibrations are both closed under coproducts, we see that $A_! : \mathcal{M} \to [\mathbb{A}, \mathcal{M}]$ is a left Quillen functor with respect to the injective model structure. Thus, by proposition 4.4.2, $A^* : [\mathbb{A}, \mathcal{M}] \to \mathcal{M}$ is a right Quillen functor. ■

Corollary 4.4.21. *Let $\mathcal{M}$ be a derivable category and let $\mathbb{A}$ be a small category. Suppose the injective and projective model structures on $[\mathbb{A}, \mathcal{M}]$ both exist. If $\mathcal{M}$ has both coproducts and products for families of size $\leq |\mathrm{mor}\ \mathbb{A}|$, then:*

- *Every fibration (resp. trivial fibration) in the injective model structure on $[\mathbb{A}, \mathcal{M}]$ is a fibration (resp. trivial fibration) in the projective model structure.*

- *Every cofibration (resp. trivial cofibration) in the projective model structure on $[\mathbb{A}, \mathcal{M}]$ is a cofibration (resp. trivial cofibration) in the injective model structure.*

- *The trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : [\mathbb{A}, \mathcal{M}] \to [\mathbb{A}, \mathcal{M}]$$

is a Quillen equivalence between the injective and projective model structures. ■

4.5 Reedy diagrams

Prerequisites. §§ 4.1, 4.4

Definition 4.5.1. A **direct category** is a category $\mathcal{C}$ equipped with a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ such that, if $f : A \to B$ is a morphism in $\mathcal{C}$, then $\deg A \le \deg B$, with equality if and only if $f = \mathrm{id}_A = \mathrm{id}_B$. Dually, an **inverse category** is a category $\mathcal{C}$ equipped with a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ such that, if $f : A \to B$ is a morphism in $\mathcal{C}$, then $\deg A \ge \deg B$, with equality if and only if $f = \mathrm{id}_A = \mathrm{id}_B$.

Remark 4.5.2. The degree function for a direct or inverse category is not determined by the underlying category: for example, if deg is a degree function for $\mathcal{C}$, then so is $A \mapsto 1 + \deg A$. However, the partial order induced by deg *is* determined by the underlying category of a direct (resp. inverse) category: $\deg A \le \deg B$ if and only if there exists a morphism $A \to B$ (resp. $B \to A$) in $\mathcal{C}$; note that this relation is indeed antisymmetric because the only morphisms that do not change the degree are identity morphisms.

Definition 4.5.3. A **Reedy category** is a category $\mathcal{C}$ equipped with two subcategories, the **direct subcategory** $\mathcal{C}_{\rightarrow}$ and the **inverse subcategory** $\mathcal{C}_{\leftarrow}$, such that the following conditions are satisfied:

- $\operatorname{ob} \mathcal{C} = \operatorname{ob} \mathcal{C}_{\rightarrow} = \operatorname{ob} \mathcal{C}_{\leftarrow}$.
- There exists a function $\deg : \operatorname{ob} \mathcal{C} \to \mathbb{N}$ such that $(\mathcal{C}_{\rightarrow}, \deg)$ is a direct category and $(\mathcal{C}_{\leftarrow}, \deg)$ is an inverse category.
- Every morphism in $\mathcal{C}$ admits a unique factorisation of the form $s \circ d$, where d is in $\mathcal{C}_{\leftarrow}$ and s is in $\mathcal{C}_{\rightarrow}$.

A **Reedy diagram** in a category $\mathcal{M}$ is a functor $\mathcal{C} \to \mathcal{M}$, where $\mathcal{C}$ is a Reedy category.

Remark 4.5.4. Any direct (resp. inverse) category is a Reedy category in a trivial way: take the whole category as the direct (resp. inverse) subcategory, and take $\operatorname{disc} \operatorname{ob} \mathcal{C}$ as the inverse (resp. direct) subcategory.

Example 4.5.5. The simplex category $\mathbf{\Delta}$ is a Reedy category, where the direct subcategory consists of all injective maps, and the inverse subcategory consists of all surjective maps; note that the unique factorisation condition is implied by theorem 1.1.4.

REMARK 4.5.6. The opposite of any Reedy category is automatically a Reedy category, after exchanging the direct and inverse subcategories.

Definition 4.5.7. Let A be an object in a Reedy category $\mathcal{C}$.

- The **latching category** of $\mathcal{C}$ at A, denoted by $\partial\big(\mathcal{C}_{\rightarrow} \downarrow A\big)$, is the largest full subcategory of the slice category $\big(\mathcal{C}_{\rightarrow} \downarrow A\big)$ that does *not* contain the object $\mathrm{id}_A : A \to A$.
- The **matching category** of $\mathcal{C}$ at A, denoted by $\partial\big(A \downarrow \mathcal{C}_{\leftarrow}\big)$, is the largest full subcategory of the slice category $\big(A \downarrow \mathcal{C}_{\leftarrow}\big)$ that does *not* contain the object $\mathrm{id}_A : A \to A$.

REMARK 4.5.8. If $\mathcal{C}$ is a Reedy category whose direct (resp. inverse) subcategory is discrete, then all its latching (resp. matching) categories are empty.

Definition 4.5.9. Let $\mathcal{M}$ be a category with limits and colimits for all finite (resp. small) diagrams, and let $X : \mathbb{C} \to \mathcal{M}$ be a finite (resp. small) Reedy diagram.

- The **latching object** of X at A, denoted by $\mathrm{L}_A(X)$, is the colimit of the diagram $\partial\big(\mathbb{C}_{\rightarrow} \downarrow A\big) \to \mathcal{M}$ obtained by composing $X : \mathbb{C} \to \mathcal{M}$ and the projection $\partial\big(\mathbb{C}_{\rightarrow} \downarrow A\big) \to \mathbb{C}$.
- The **matching object** of X at A, denoted by $\mathrm{M}_A(X)$, is the limit of the diagram $\partial\big(A \downarrow \mathbb{C}_{\leftarrow}\big) \to \mathcal{M}$ obtained by composing $X : \mathbb{C} \to \mathcal{M}$ and the projection $\partial\big(A \downarrow \mathbb{C}_{\leftarrow}\big) \to \mathbb{C}$.
- The **latching morphism** of X at A is the morphism $\mathrm{L}_A(X) \to XA$ induced by the inclusion $\partial\big(\mathcal{C}_{\rightarrow} \downarrow A\big) \hookrightarrow \big(\mathcal{C}_{\rightarrow} \downarrow A\big)$.
- The **matching morphism** of X at A is the morphism $XA \to \mathrm{M}_A(X)$ induced by the inclusion $\partial\big(A \downarrow \mathcal{C}_{\leftarrow}\big) \hookrightarrow \big(A \downarrow \mathcal{C}_{\leftarrow}\big)$.

REMARK 4.5.10. The latching object $\mathrm{L}_A(X)$ is functorial in A (as A varies in the direct subcategory), and the matching object $\mathrm{M}_A(X)$ is functorial in A (as A varies in the inverse subcategory). Of course, it goes without saying that $\mathrm{L}_A(X)$ and $\mathrm{M}_A(X)$ are both functorial in X (as X varies in $[\mathbb{C}, \mathcal{M}]$).

Definition 4.5.11. Let $\mathcal{M}$ be a category with limits and colimits for all finite (resp. small) diagrams, and let $\varphi : X \Rightarrow Y$ be a natural transformation between two finite (resp. small) Reedy diagrams $X, Y : \mathbb{C} \to \mathcal{M}$.

- The **relative latching morphism** $XA \cup^{L_A(X)} L_A(Y) \to YA$ is the unique morphism in $\mathcal{M}$ making the diagram below commute,

$$\begin{tikzcd}
L_A(X) \arrow[r, "L_A(\varphi)"] \arrow[d] & L_A(Y) \arrow[d] \arrow[ddr, bend left] & \\
XA \arrow[r] \arrow[drr, bend right, "\varphi_A"'] & XA \cup^{L_A(X)} L_A(Y) \arrow[dr, dashed] & \\
& & YA
\end{tikzcd}$$

where the arrows $L_A(X) \to XA$ and $L_A(Y) \to YA$ are the latching morphisms and the square is a pushout square.

- The **relative matching morphism** $XA \to M_A(X) \times_{M_A(Y)} YA$ is the unique morphism in $\mathcal{M}$ making the diagram below commute,

$$\begin{tikzcd}
XA \arrow[dr, dashed] \arrow[drr, bend left, "\varphi_A"] \arrow[ddr, bend right] & & \\
& M_A(X) \times_{M_A(Y)} YA \arrow[r] \arrow[d] & YA \arrow[d] \\
& M_A(X) \arrow[r, "M_A(\varphi)"'] & M_A(Y)
\end{tikzcd}$$

where the arrows $XA \to M_A(X)$ and $YA \to M_A(Y)$ are the latching morphisms and the square is a pullback square.

REMARK 4.5.12. If the direct subcategory of $\mathbb{C}$ is discrete, then $L_A(X)$ is an initial object in $\mathcal{M}$ for all A and X, so the relative latching morphism of a natural transformation $\varphi : X \Rightarrow Y$ at any object A in $\mathbb{C}$ is (isomorphic to) $\varphi_A : XA \to YA$ itself.

Dually, if the inverse subcategory of $\mathbb{C}$ is discrete, then $M_A(X)$ is a terminal object in $\mathcal{M}$ for all A and X, so the relative matching morphism of a natural transformation $\varphi : X \Rightarrow Y$ at any object A in $\mathbb{C}$ is (isomorphic to) $\varphi_A : XA \to YA$ itself.

Definition 4.5.13. Let $\mathcal{M}$ be a model category, let $\mathbb{C}$ be a finite (resp. small) Reedy category, and assume $\mathcal{M}$ has limits and colimits for all finite (resp. small) diagrams.

- A **Reedy weak equivalence** in $[\mathbb{C}, \mathcal{M}]$ is a natural transformation such that all its components are weak equivalences in $\mathcal{M}$.

- A **Reedy cofibration** in $[\mathbb{C}, \mathcal{M}]$ is a natural transformation such that all its relative latching morphisms are cofibrations in $\mathcal{M}$.

- A **Reedy fibration** in $[\mathbb{C}, \mathcal{M}]$ is a natural transformation such that all its relative matching morphisms are fibrations in $\mathcal{M}$.

Proposition 4.5.14. *With notation as in the definition:*

- *If $\varphi : X \Rightarrow Y$ is a Reedy cofibration (resp. trivial Reedy cofibration) in $[\mathbb{C}, \mathcal{M}]$, then, for each object A in $\mathbb{C}$, the morphisms $\mathrm{L}_A(\varphi) : \mathrm{L}_A(X) \to \mathrm{L}_A(Y)$ and $\varphi_A : XA \to YA$ are cofibrations (resp trivial cofibrations).*

- *If $\varphi : X \Rightarrow Y$ is a Reedy fibration (resp. trivial Reedy fibration) in $[\mathbb{C}, \mathcal{M}]$, then, for each object A in $\mathbb{C}$, the morphisms $\mathrm{M}_A(\varphi) : \mathrm{M}_A(X) \to \mathrm{M}_A(Y)$ and $\varphi_A : XA \to YA$ are fibrations (resp trivial fibrations).*

Proof. See Propositions 15.3.11 and 15.3.14 in [Hirschhorn, 2003]. □

TODO: Check if this requires functorial factorisation. Surely not!

Proposition 4.5.15. *With notation as in the definition:*

- *A Reedy cofibration in $[\mathbb{C}, \mathcal{M}]$ is a Reedy weak equivalence if and only if all its relative latching morphisms are trivial cofibrations in $\mathcal{M}$.*

- *A Reedy fibration in $[\mathbb{C}, \mathcal{M}]$ is a Reedy weak equivalence if and only if all its relative matching morphisms are trivial fibrations in $\mathcal{M}$.*

Proof. See Theorem 15.3.15 in [Hirschhorn, 2003]. □

TODO: Check if this requires functorial factorisation. Surely not!

Proposition 4.5.16. *Let $\mathcal{M}$ be a model category, let $\mathbb{C}$ be a finite (resp. small) Reedy category, and assume $\mathcal{M}$ has limits and colimits for all finite (resp. small) diagrams.*

- *If the injective model structure on $[\mathbb{C}, \mathcal{M}]$ exists, then the trivial adjunction*

$$\mathrm{id} \dashv \mathrm{id} : [\mathbb{C}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$$

is a Quillen equivalence between the injective model structure and the Reedy model structure.

- *If the projective model structure on* $[\mathbb{C}, \mathcal{M}]$ *exists, then the trivial adjunction*
$$\mathrm{id} \dashv \mathrm{id} : [\mathbb{C}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$$
is a Quillen equivalence between the the Reedy model structure and the projective model structure.

Proof. This is an immediate consequence of proposition 4.5.14. ■

Lemma 4.5.17. *Let* $\mathcal{M}$ *be a model category, let* $\mathbb{C}$ *be a finite (resp. small) Reedy category, and assume* $\mathcal{M}$ *has limits and colimits for all finite (resp. small) diagrams.*

- *A diagram* $X : \mathbb{C} \to \mathcal{M}$ *is Reedy cofibrant if and only if every latching morphism of* X *is a cofibration in* $\mathcal{M}$*.*

- *A diagram* $X : \mathbb{C} \to \mathcal{M}$ *is Reedy fibrant if and only if every matching morphism of* X *is a fibration in* $\mathcal{M}$*.*

Proof. Let 0 be an initial object in $\mathcal{M}$ and let 1 be a terminal object in $\mathcal{M}$. It is a standard fact that $\Delta 0$ is an initial object in $[\mathbb{C}, \mathcal{M}]$ and $\Delta 1$ is a terminal object in $[\mathbb{C}, \mathcal{M}]$, so the claims follow from the observation that the latching morphism $\mathrm{L}_A(\Delta 0) \to 0$ and the matching morphism $1 \to \mathrm{M}_A(\Delta 1)$ are isomorphisms for all objects A in $\mathbb{C}$. ■

Lemma 4.5.18. *With notation as in the previous lemma:*

- *If* $X : \mathbb{C} \to \mathcal{M}$ *is a Reedy cofibrant diagram, then, for every object* A *in* $\mathbb{C}$*, the object* XA *and the latching object* $\mathrm{L}_A(X)$ *are cofibrant objects in* $\mathcal{M}$*.*

- *If* $X : \mathbb{C} \to \mathcal{M}$ *is a Reedy fibrant diagram, then, for every object* A *in* $\mathbb{C}$*, the object* XA *and the matching object* $\mathrm{M}_A(X)$ *are fibrant objects in* $\mathcal{M}$*.*

Proof. Apply proposition 4.5.14. ■

Theorem 4.5.19. *With notation as in the definition, the announced weak equivalences, cofibrations, and fibrations constitute a model structure on* $[\mathbb{C}, \mathcal{M}]$*, called the* **Reedy model structure***; moreover, if* $\mathcal{M}$ *is a DHK model category, then so is* $[\mathbb{C}, \mathcal{M}]$ *when equipped with the Reedy model structure.*

Proof. See Theorem 5.2.5 in [Hovey, 1999], or Theorem 15.3.4 in [Hirschhorn, 2003]. □

Corollary 4.5.20. *Let $\mathcal{M}$ be a model category, let $\mathbb{C}$ be a finite (resp. small) Reedy category, and assume $\mathcal{M}$ has limits and colimits for all finite (resp. small) diagrams.*

- *If the direct subcategory of $\mathbb{C}$ is discrete, then the Reedy model structure on $[\mathbb{C}, \mathcal{M}]$ is the injective model structure.*
- *If the inverse subcategory of $\mathbb{C}$ is discrete, then the Reedy model structure on $[\mathbb{C}, \mathcal{M}]$ is the projective model structure.*

Proof. This follows from the theorem and remark 4.5.12. ■

Definition 4.5.21. Let $\mathcal{C}$ be a Reedy category.

- $\mathcal{C}$ has **cofibrant constants** if, for every object A in $\mathbb{C}$, the latching category $\partial(\mathcal{C}_{\rightarrow} \downarrow A)$ has at most one connected component.
- $\mathcal{C}$ has **fibrant constants** if, for every object A in $\mathbb{C}$, the matching category $\partial(A \downarrow \mathcal{C}_{\leftarrow})$ has at most one connected component.

Example 4.5.22. Let $\mathcal{C}$ be a Reedy category.

- If the direct subcategory of $\mathcal{C}$ is discrete, then $\mathcal{C}$ has cofibrant constants. (In fact, every latching category is empty.)
- If the inverse subcategory of $\mathcal{C}$ is discrete, then $\mathcal{C}$ has fibrant constants. (In fact, every matching category is empty.)

Proposition 4.5.23. *Let $\mathcal{M}$ be a model category, let $\mathbb{C}$ be a finite (resp. small) Reedy category, and assume $\mathcal{M}$ has limits and colimits for all finite (resp. small) diagrams.*

- *If $\mathbb{C}$ has cofibrant constants, then the functor $\Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ is a left Quillen functor.*
- *If $\mathbb{C}$ has fibrant constants, then the functor $\Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ is a right Quillen functor.*

Proof. The two claims are formally dual; we will prove the second version.

If the matching category $\partial(A \downarrow \mathbb{C}_{\leftarrow})$ is empty, then the matching object of ΔX at A is a terminal object in $\mathcal{M}$, so the relative matching morphism of Δf at A is isomorphic to $f : X \to Y$ in this case.

On the other hand, if the matching category $\partial(A \downarrow \mathbb{C}_{\leftarrow})$ of $\mathbb{C}$ has only one connected component, then the matching morphism $X \to \mathrm{M}_A(\Delta X)$ must be an isomorphism, so the relative matching morphism of Δf at A is an isomorphism, hence a (trivial) fibration in particular.

We now conclude that, for any fibration $f : X \to Y$ in $\mathcal{M}$, every relative matching morphism of $\Delta f : \Delta X \to \Delta Y$ is a fibration. Clearly, the functor $\Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ preserves weak equivalences, so this completes the proof that Δ is a right Quillen functor. ■

Theorem 4.5.24 (Hirschhorn)**.** *Let $\mathbb{C}$ be a small Reedy category.*

(i) *$\mathbb{C}$ has cofibrant constants.*

(ii) *$\Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ is a left Quillen functor for all DHK model categories $\mathcal{M}$.*

(iii) *For every cofibrant object X in any DHK model category $\mathcal{M}$, the constant diagram $\Delta X : \mathbb{C} \to \mathcal{M}$ is Reedy cofibrant.*

Dually, the following are equivalent:

(i′) *$\mathbb{C}$ has fibrant constants.*

(ii′) *$\Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ is a right Quillen functor for all DHK model categories $\mathcal{M}$.*

(iii′) *For every fibrant object X in any DHK model category $\mathcal{M}$, the constant diagram $\Delta X : \mathbb{C} \to \mathcal{M}$ is Reedy fibrant.*

Proof. (i) ⇒ (ii). This is the content of the earlier proposition.

(ii) ⇒ (iii). Left Quillen functors preserve cofibrant objects.

(iii) ⇒ (i). Take $\mathcal{M}$ to be **Set** equipped with the mono–epi model structure,[4] and consider the constant diagram $\Delta 1$. Since 1 is a cofibrant object in $\mathcal{M}$, $\Delta 1$ must be

[4] See example 4.1.6.

a Reedy cofibrant object in $[\mathbb{C}, \mathcal{M}]$. It is not hard to see that the latching object $\mathrm{L}_A(\Delta 1)$ is the set of connected components of the latching category $\partial\left(\mathbb{C}_\rightarrow \downarrow A\right)$, so by lemma 4.5.17, $\partial\left(\mathbb{C}_\rightarrow \downarrow A\right)$ has at most one connected component. ■

Corollary 4.5.25. *Let $\mathcal{M}$ be a DHK model category and let $\mathbb{C}$ be a small Reedy category.*

- *If $\mathbb{C}$ has fibrant constants, then the adjunction $\varinjlim_{\mathbb{C}} \dashv \Delta : \mathcal{M} \to [\mathbb{C}, \mathcal{M}]$ is deformable.*
- *If $\mathbb{C}$ has cofibrant constants, then the adjunction $\Delta \dashv \varprojlim_{\mathbb{C}} : [\mathbb{C}, \mathcal{M}] \to \mathcal{M}$ is deformable.*

Proof. Apply theorem 4.4.12 to the above result. ■

For the remainder of this section, we follow [Barwick, 2007] and discuss the functoriality of the Reedy model structure.

Definition 4.5.26. Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories. A **morphism of Reedy categories** $\mathcal{C} \to \mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ that sends every morphism in $\mathcal{C}_\rightarrow$ to $\mathcal{D}_\rightarrow$ and every morphism in $\mathcal{C}_\leftarrow$ to $\mathcal{D}_\leftarrow$, or equivalently, a commutative diagram of functors of the form below:

$$\begin{tikzcd} \mathcal{C}_\rightarrow \arrow[r, "F_\rightarrow"] \arrow[d, hook] & \mathcal{D}_\rightarrow \arrow[d, hook] \\ \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\ \mathcal{C}_\leftarrow \arrow[r, "F_\leftarrow"'] \arrow[u, hook] & \mathcal{D}_\leftarrow \arrow[u, hook] \end{tikzcd}$$

Lemma 4.5.27. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of Reedy categories. If D is any object in $\mathcal{D}$, then:*

- *There is a unique Reedy category structure on the comma category $(F \downarrow D)$ making the projection $(F \downarrow D) \to \mathcal{C}$ a morphism of Reedy categories.*
- *There is a unique Reedy category structure on the comma category $(D \downarrow F)$ making the projection $(D \downarrow F) \to \mathcal{C}$ a morphism of Reedy categories.*

Proof. Obvious. ♦

While it is true that any functor $F : \mathbb{C} \to \mathbb{D}$ induces a homotopical functor $F^* : [\mathbb{D}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$, even if F is a morphism of Reedy categories, F^* need not be either a left Quillen functor or a right Quillen functor. Instead, we must consider the following:

Definition 4.5.28. Let $\mathcal{C}$ and $\mathcal{D}$ be Reedy categories.

- A **left fibration of Reedy categories** is a morphism $F : \mathcal{C} \to \mathcal{D}$ such that, for any object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has fibrant constants.
- A **right fibration of Reedy categories** is a morphism $F : \mathcal{C} \to \mathcal{D}$ such that, for any object D in $\mathcal{D}$, the comma category $(D \downarrow F)$ has cofibrant constants.

Remark 4.5.29. A Reedy category $\mathcal{C}$ has fibrant (resp. cofibrant) constants if and only if the unique morphism $\mathcal{C} \to \mathbb{1}$ is a left (resp. right) fibration.

Remark 4.5.30. A morphism $F : \mathcal{C} \to \mathcal{D}$ of Reedy categories is a left (resp. right) fibration if and only if $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is a right (resp. left) fibration.

Theorem 4.5.31 (Barwick)**.** *Let $F : \mathbb{C} \to \mathbb{D}$ be a morphism between small Reedy categories. The following are equivalent:*

(i) *The morphism $F : \mathbb{C} \to \mathbb{D}$ is a left fibration of Reedy categories.*

(ii) *For every object D in $\mathbb{D}$ and every object (C, h) in $(F \downarrow D)$, the matching category $\partial\big((C, h) \downarrow (F \downarrow D)_{\leftarrow}\big)$ has at most one connected component.*

(iii) *The functor $F^* : [\mathbb{D}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$ is a right Quillen functor for all DHK model categories $\mathcal{M}$.*

Dually, the following are equivalent:

(i′) *The morphism $F : \mathbb{C} \to \mathbb{D}$ is a right fibration of Reedy categories.*

(ii′) *For every object D in $\mathbb{D}$ and every object (C, h) in $(D \downarrow F)$, the latching category $\partial\big((D \downarrow F) \downarrow (C, h)_{\rightarrow}\big)$ has at most one connected component.*

(iii′) *The functor $F^* : [\mathbb{D}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$ is a left Quillen functor for all DHK model categories $\mathcal{M}$.*

Proof. See Lemma 2.6 and Theorem 2.7 in [Barwick, 2007]. ∎

4.6 Virtual cofibrancy and fibrancy

Prerequisites. §§ 1.1, 3.1, 3.3, 4.1, 4.5, A.1, A.5.

In this section, we follow [DHKS, § 23]. As usual, for each natural number n, let $[n]$ denote the category $\{0 \to \cdots \to n\}$ corresponding to the finite ordinal $\{0, \ldots, n\}$, and let $\mathbf{\Delta}$ be the category whose objects are the $[n]$ and whose morphisms are functors.

Definition 4.6.1. The **category of simplices** of a (small) category $\mathbb{C}$ is the category $\mathbf{\Delta}(\mathbb{C})$ defined below:

- The objects are functors $[n] \to \mathbb{C}$.
- The morphisms $(f : [m] \to \mathbb{C}) \to (g : [n] \to \mathbb{C})$ are functors $[m] \to [n]$ making the evident triangle commute (strictly).
- Composition and identities are the obvious ones.

We write $\pi_\Delta : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}$ for the evident projection functor that sends an object $[n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ to the object $[n]$ in $\mathbf{\Delta}$.

¶ 4.6.2. To elucidate the above definition, it is helpful to introduce some notation for the objects in $\mathbf{\Delta}(\mathbb{C})$. It is not hard to see that a functor $f : [n] \to \mathbb{C}$ is the same thing as a string of n composable morphisms in $\mathbb{C}$, e.g.

$$A_0 \xrightarrow{f_1} A_1 \longrightarrow \cdots \longrightarrow A_{n-1} \xrightarrow{f_n} A_n$$

so let us write $\left[A_0 \overset{f_1}{\to} A_1 \cdots A_{n-1} \overset{f_n}{\to} A_n\right]$ for the corresponding object in $\mathbf{\Delta}(\mathbb{C})$. Since the projection $\pi_\Delta : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}$ is faithful, we may borrow the notation of § 1.1 and write e.g. $\delta^1 : [A_0] \to \left[A_0 \overset{f_1}{\to} A_1\right]$ for the unique morphism whose image under π_Δ is $\delta^1 : [0] \to [1]$.

Observe that, given a commutative triangle in $\mathbb{C}$ of the form below,

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"'] & & C \end{tikzcd}$$

we obtain the following commutative diagram in $\mathbf{\Delta}(\mathbb{C})$:

$$\begin{tikzcd}
 & & {[B]} \arrow[dl, "\delta^1"'] \arrow[dr, "\delta^0"] \arrow[dd] & & \\
 & \left[A \xrightarrow{f} B\right] \arrow[dr, "\delta^2"] & & \left[B \xrightarrow{g} C\right] \arrow[dl, "\delta^0"] & \\
 & & \left[A \xrightarrow{f} B \xrightarrow{g} C\right] & & \\
{[A]} \arrow[uur, "\delta^1"] \arrow[urr] \arrow[rr, "\delta^1"'] & & \left[A \xrightarrow{h} C\right] \arrow[u, "\delta^1"] & & {[C]} \arrow[uul, "\delta^0"'] \arrow[ull] \arrow[ll, "\delta^0"]
\end{tikzcd}$$

Similar phenomena occur for longer strings of composable morphisms. Thus, one may think of $\mathbf{\Delta}(\mathbb{C})$ as being a kind of barycentric subdivision of $\mathbb{C}$; notice also that Mac Lane's subdivision category $\mathbb{C}^{\S}$ occurs as a subcategory of $\mathbf{\Delta}(\mathbb{C})$.

REMARK 4.6.3. There is an obvious natural isomorphism $\mathbf{\Delta}(\mathbb{C}) \cong \mathbf{\Delta}(\mathbb{C}^{\mathrm{op}})$ such that the following diagram of functors commutes,

$$\begin{tikzcd}
\mathbf{\Delta}(\mathbb{C}) \arrow[r, "\cong"] \arrow[d, "\pi_\Delta"'] & \mathbf{\Delta}(\mathbb{C}^{\mathrm{op}}) \arrow[d, "\pi_\Delta"] \\
\mathbf{\Delta} \arrow[r, "(-)^{\mathrm{op}}"'] & \mathbf{\Delta}
\end{tikzcd}$$

but in general there is no isomorphism between $\mathbf{\Delta}(\mathbb{C})$ and $\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}$.

Proposition 4.6.4. *Let X be a simplicial set, and let $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ be the inclusion of the standard simplices.*

(i) *The comma category $(\Delta^\bullet \downarrow X)$ is a Reedy category, where the direct subcategory consists of all face operators and the inverse subcategory consists of all degeneracy operators.*

(ii) *Moreover, $(\Delta^\bullet \downarrow X)$ has fibrant constants.*

Proof. (i). The evident projection $(\Delta^\bullet \downarrow X) \to \mathbf{\Delta}$ is a discrete right fibration, so the Reedy category structure on $\mathbf{\Delta}$ induces one on $(\Delta^\bullet \downarrow X)$.

(ii). See Proposition 15.10.4 in [Hirschhorn, 2003]. □

Corollary 4.6.5. *The category $\mathbf{\Delta}(\mathbb{C})$ of simplices of a (small) category $\mathbb{C}$ admits a Reedy category structure with fibrant constants.*

Proof. It is not hard to see that the category $\mathbf{\Delta}(\mathbb{C})$ as defined above is isomorphic to the comma category $(\Delta^\bullet \downarrow \mathrm{N}(\mathbb{C}))$, where $\mathrm{N}(\mathbb{C})$ is the nerve of $\mathbb{C}$. ■

Corollary 4.6.6. *If $\mathcal{M}$ is a DHK model category and $\mathbb{C}$ is a small category, then:*

- *The functor $\varinjlim_{\mathbf{\Delta}(\mathbb{C})} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to \mathcal{M}$ sends Reedy weak equivalences between Reedy-cofibrant diagrams to weak equivalences between cofibrant objects.*
- *The functor $\varprojlim_{\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to \mathcal{M}$ sends Reedy weak equivalences between Reedy-fibrant diagrams to weak equivalences between fibrant objects.*

Proof. Apply Ken Brown's lemma (4.4.6) and corollary 4.5.25. ■

Lemma 4.6.7. *Let $F : \mathbb{C} \to \mathbb{D}$ be a functor between (small) categories.*

(i) $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ *is a left fibration of Reedy categories.*

(ii) $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ *is a right fibration of Reedy categories.*

Proof. (i). Let $\left[D_0 \cdots D_n\right]$ be an object in $\mathbf{\Delta}(\mathbb{D})$, let $\left(\left[C_0 \cdots C_m\right], h\right)$ be an object in the comma category $\left(\mathbf{\Delta}(F) \downarrow \left[D_0 \cdots D_n\right]\right)$. We will show that the matching category

$$\partial\left(\left(\left[C_0 \cdots C_m\right], h\right) \downarrow \left(\mathbf{\Delta}(F) \downarrow \left[D_0 \cdots D_n\right]\right)_{\leftarrow}\right)$$

has at most one connected component.

First, note that the objects of this matching category are pairs (k, l), where k is in $\mathbf{\Delta}(\mathbb{C})_{\leftarrow}$, $k \neq \mathrm{id}_{[C_0 \cdots C_m]}$, l is in $\mathbf{\Delta}(\mathbb{D})$, and $h = l \circ \mathbf{\Delta}(F)k$. Let (σ, δ) be the codegeneracy–coface factorisation of $\pi_\Delta h$ in $\mathbf{\Delta}$.

- If $\sigma = \mathrm{id}_{[m]}$, then the matching category must be empty.

- If $\sigma \neq \mathrm{id}_{[m]}$, then we may lift (σ, δ) along the respective π_Δ projections to obtain a terminal object in the matching category, so the matching category is connected *a fortiori*.

Thus, by theorem 4.5.31, $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ is a left fibration of Reedy categories.

(ii). A similar argument shows that $\mathbf{\Delta}(F) : \mathbf{\Delta}(\mathbb{C}) \to \mathbf{\Delta}(\mathbb{D})$ is a right fibration of Reedy categories. ■

Corollary 4.6.8. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *The functor $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ is a right Quillen functor.*

(ii) *The functor $\mathbf{\Delta}(F)^* : [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ is a left Quillen functor.*

Proof. Apply theorem 4.5.31. ■

Definition 4.6.9. Let $\mathbb{C}$ be a (small) category and let $\mathbf{\Delta}(\mathbb{C})$ be its category of simplices.

- The **left projection functor** $\pi_{\mathrm{L}} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathbb{C}$ is the functor defined by evaluating objects $f : [n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ at the initial object in $[n]$.
- The **right projection functor** $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{C}) \to \mathbb{C}$ is the functor defined by evaluating objects $f : [n] \to \mathbb{C}$ in $\mathbf{\Delta}(\mathbb{C})$ at the terminal object in $[n]$.
- A **strong left equivalence** in $\mathbf{\Delta}(\mathbb{C})$ is a morphism such that the underlying map in $\mathbf{\Delta}$ preserves the initial object.
- A **strong right equivalence** in $\mathbf{\Delta}(\mathbb{C})$ is a morphism such that the underlying map in $\mathbf{\Delta}$ preserves the terminal object.
- The class of **weak left equivalences** in $\mathbf{\Delta}(\mathbb{C})$ is the smallest subcategory that has the 2-out-of-6 property and contains all the strong left equivalences.
- The class of **weak right equivalences** in $\mathbf{\Delta}(\mathbb{C})$ is the smallest subcategory that has the 2-out-of-6 property and contains all the strong right equivalences.

We write $\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}$ for the category of simplices of $\mathbb{C}$ regarded as a relative category with weak equivalences the strong left equivalences, and we write $\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}$ for the category of simplices of $\mathbb{C}$ regarded as a relative category with weak equivalences the strong right equivalences.

Remark 4.6.10. The strong left (resp. right) equivalences in $\mathbf{\Delta}(\mathbb{C})$ are closed under composition, and the left (resp. right) projection to $\mathbb{C}$ sends strong left (resp. right) equivalences to identity morphisms, so if we regard $\mathbf{\Delta}(\mathbb{C})$ as a relative category with weak equivalences the strong left (resp. right) equivalences, then the left (resp. right) projection functor becomes a relative functor.

Unfortunately, the subcategory of strong left (resp. right) equivalences in $\mathbf{\Delta}(\mathbb{C})$ does not generally have the 2-out-of-6 property, or even the 2-out-of-3 property; one may rectify this by instead considering the class of weak left (resp. right) equivalences. An example of a weak left equivalence that is not a strong left equivalence is the morphism $\delta^0 : \left[A \xrightarrow{\mathrm{id}} A\right] \to [A]$: this is a weak left equivalence because $\sigma^0 : [A] \to \left[A \xrightarrow{\mathrm{id}} A\right]$ is a strong left equivalence and $\delta^0 \circ \sigma^0 = \mathrm{id}_{[A]}$, but δ^0 is not a strong left equivalence because the underlying cosimplicial operator in $\mathbf{\Delta}$ sends 0 in [0] to 1 in [1].

Remark 4.6.11. It is not hard to see that $\mathbf{\Delta}(-)$ is a functor $\mathbf{Cat} \to \mathbf{Cat}$ and that π_{L} (resp. π_{R}) defines a natural transformation $\mathbf{\Delta}(-)^{\mathrm{op}} \Rightarrow \mathrm{id}_{\mathbf{Cat}}$ (resp. $\mathbf{\Delta}(-) \Rightarrow \mathrm{id}_{\mathbf{Cat}}$).

Lemma 4.6.12. *Let $F : \mathbb{C} \to \mathbb{D}$ be a functor, let $\pi_{\mathrm{L}} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathbb{C}$ be the left projection functor, and let $\pi_{\mathrm{R}} : \mathbf{\Delta}(\mathbb{C}) \to \mathbb{C}$ be the right projection functor. Then, for any object D in $\mathbb{D}$:*

- *The canonical comparison functor $\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \to \left(D \downarrow F\pi_{\mathrm{L}}\right)$ is an isomorphism.*
- *The canonical comparison functor $\mathbf{\Delta}((F \downarrow D)) \to \left(F\pi_{\mathrm{R}} \downarrow D\right)$ is an isomorphism.*

Proof. The two claims are formally dual; we will prove the first version.

As always, the comma category $(D \downarrow F)$ fits into a comma square,

$$\begin{tikzcd}
(D \downarrow F) \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbb{C} \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "D"'] & \mathbb{D}
\end{tikzcd}$$

and the following diagram of functors commutes,

$$\begin{tikzcd}
\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \arrow[r, "\mathbf{\Delta}(P)^{\mathrm{op}}"] \arrow[d, "\pi_{\mathrm{L}}"'] & \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \arrow[d, "\pi_{\mathrm{L}}"] \\
(D \downarrow F) \arrow[r, "P"'] & \mathbb{C}
\end{tikzcd}$$

so the universal property of $(D \downarrow F)$ gives us a canonical comparison functor $\mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \to \left(D \downarrow F\pi_{\mathrm{L}}\right)$, as claimed. It is not hard to check that the second diagram is a pullback square, so the pasting lemma for comma squares implies that the comparison functor is an isomorphism. ■

Proposition 4.6.13. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

- *The functor* $\mathrm{Ran}_{F\pi_{\mathrm{L}}} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends Reedy weak equivalences between Reedy-fibrant diagrams to componentwise weak equivalences between componentwise fibrant diagrams.*
- *The functor* $\mathrm{Lan}_{F\pi_{\mathrm{R}}} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends Reedy weak equivalences between Reedy-cofibrant diagrams to componentwise weak equivalences between componentwise cofibrant diagrams.*

Proof. The two claims are formally dual; we will prove the second version.

Using the formula for $\mathrm{Lan}_{F\pi_{\mathrm{R}}}$ given by theorem A.5.15, we see that, for each object D in $\mathbb{D}$, the functor $\left(\mathrm{Lan}_{F\pi_{\mathrm{R}}} -\right)(D) : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to \mathcal{M}$ is naturally isomorphic to the functor $\varinjlim : \left[\left(F\pi_{\mathrm{R}} \downarrow D\right), \mathcal{M}\right] \to \mathcal{M}$; but by lemma 4.6.12, there is a canonical isomorphism $\left(F\pi_{\mathrm{R}} \downarrow D\right) \cong \mathbf{\Delta}((F \downarrow D))$, so $\left(\mathrm{Lan}_{F\pi_{\mathrm{R}}} -\right)(D)$ is in turn naturally isomorphic to $\varinjlim : [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \to \mathcal{M}$. The claim now follows from corollary 4.6.6. ■

Theorem 4.6.14. *Let $\mathcal{M}$ be a DHK model category and let $\mathbb{C}$ be a small category.*

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\pi_{\mathrm{L}}^* \dashv \mathrm{Ran}_{\pi_{\mathrm{L}}} : \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}^{\mathrm{op}}, \mathcal{M}\right]_{\mathrm{h}} \to [\mathbb{C}, \mathcal{M}]$$

- *The adjunction shown below is deformable and satisfies the Quillen equivalence condition for homotopical categories:*

$$\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [\mathbb{C}, \mathcal{M}] \to \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$

Proof. See Proposition 23.2 in [DHKS]. □

Definition 4.6.15. Let $\mathcal{M}$ be a DHK model category and let $\mathbb{C}$ be a small category.

- A **virtually cofibrant diagram** $X : \mathbb{C} \to \mathcal{M}$ is one for which there exists a Reedy-cofibrant diagram $\tilde{X} : \mathbf{\Delta}(\mathbb{C}) \to \mathcal{M}$ such that $\tilde{X}$ is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$ and $X \cong \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{X}$.
- A **virtually fibrant diagram** $Y : \mathbb{C} \to \mathcal{M}$ is one for which there exists a Reedy-fibrant diagram $\hat{Y} : \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \to \mathcal{M}$ such that $\hat{Y}$ is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{L}}^{\mathrm{op}}, \mathcal{M}\right]_{\mathrm{h}}$ and $Y \cong \mathrm{Ran}_{\pi_{\mathrm{L}}} \hat{Y}$.

We write $[\mathbb{C}, \mathcal{M}]_{\mathrm{vc}}$ for the full subcategory of $[\mathbb{C}, \mathcal{M}]$ spanned by the virtually cofibrant diagrams, and we write $[\mathbb{C}, \mathcal{M}]_{\mathrm{vf}}$ for the full subcategory of $[\mathbb{C}, \mathcal{M}]$ spanned by the virtually fibrant diagrams.

Theorem 4.6.16. *Let $\mathcal{M}$ be a DHK model category and let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories.*

(i) *The functor* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends virtually cofibrant diagrams to componentwise cofibrant diagrams and preserves componentwise weak equivalences between such diagrams.*

(ii) *If* $\mathrm{Lan}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}]$ *moreover restricts to a functor* $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \to \left[\mathbf{\Delta}(\mathbb{D})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$, *then* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *preserves virtually cofibrant diagrams.*

(iii) *If (Q, p) is a cofibrant replacement functor for* $[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$, *then*

$$\left([\mathbb{C}, \mathcal{M}]_{\mathrm{vc}}, \mathrm{Lan}_{\pi_{\mathrm{R}}} \circ Q \circ \pi_{\mathrm{R}}^{*}, \varepsilon \bullet \left(\mathrm{Lan}_{\pi_{\mathrm{R}}} \circ p \circ \pi_{\mathrm{R}}^{*}\right)\right)$$

is a functorial left deformation retract for Lan_F, *where ε is the counit of the adjunction* $\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^{*}$.

(iv) *The adjunction shown below is deformable:*

$$\mathrm{Lan}_F \dashv F^{*} : [\mathbb{D}, \mathcal{M}] \to [\mathbb{C}, \mathcal{M}]$$

(v) *Given another functor $G : \mathbb{D} \to \mathbb{E}$ between small categories,* $\left(\mathrm{Lan}_G, \mathrm{Lan}_F\right)$ *is strongly left deformable.*

Dually:

(i′) *The functor* $\mathrm{Ran}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *sends virtually fibrant diagrams to componentwise fibrant diagrams and preserves componentwise weak equivalences between such diagrams.*

(ii′) *If* $\mathrm{Ran}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D})^{\mathrm{op}}, \mathcal{M}]$ *moreover restricts to a functor* $\left[\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}_{\mathrm{L}}, \mathcal{M}\right]_{\mathrm{h}} \to \left[\mathbf{\Delta}(\mathbb{D})^{\mathrm{op}}_{\mathrm{L}}, \mathcal{M}\right]_{\mathrm{h}}$, *then* $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ *preserves virtually cofibrant diagrams.*

(iii′) *If* (R, i) *is a fibrant replacement functor for* $[\mathbf{\Delta}(\mathbb{C})^{\mathrm{op}}, \mathcal{M}]$, *then*

$$\left([\mathbb{C}, \mathcal{M}]_{\mathrm{vf}}, \mathrm{Ran}_{\pi_{\mathrm{L}}} \circ R \circ \pi_{\mathrm{L}}^*, \left(\mathrm{Ran}_{\pi_{\mathrm{L}}} \circ i \circ \pi_{\mathrm{L}}^*\right) \bullet \eta\right)$$

is a functorial right deformation retract for Ran_F, *where* η *is the unit of the adjunction* $\pi_{\mathrm{L}}^* \dashv \mathrm{Ran}_{\pi_{\mathrm{L}}}$.

(iv′) *The adjunction shown below is deformable:*

$$F^* \dashv \mathrm{Ran}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$$

(v′) *Given another functor* $G : \mathbb{D} \to \mathbb{E}$ *between small categories,* $\left(\mathrm{Ran}_G, \mathrm{Ran}_F\right)$ *is strongly right deformable.*

Proof. (i). Let $\tilde{X}$ be a Reedy-cofibrant diagram $\mathbb{C} \to \mathcal{M}$ that is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$ and let $X = \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{X}$. There is a canonical isomorphism $\mathrm{Lan}_{F\pi_{\mathrm{R}}} \cong \mathrm{Lan}_F \circ \mathrm{Lan}_{\pi_{\mathrm{R}}}$, so proposition 4.6.13 implies $\mathrm{Lan}_F X$ is a componentwise cofibrant diagram $\mathbb{D} \to \mathcal{M}$.

Let $\tilde{Y}$ be another Reedy-cofibrant diagram $\mathbb{C} \to \mathcal{M}$ that is in $\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$, let $Y = \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{Y}$, and let $\varphi : X \Rightarrow Y$ be a componentwise weak equivalence. Proposition 3.3.24 applied to theorem 4.6.14 implies the adjunction unit components $\tilde{X} \to \pi_{\mathrm{R}}^* X$ and $\tilde{Y} \to \pi_{\mathrm{R}}^* Y$ are Reedy weak equivalences. Using axiom CM2 and CM5, factor $\tilde{Y} \to \pi_{\mathrm{R}}^* Y$ as a trivial cofibration $\theta : \tilde{Y} \to \tilde{Z}$ followed by a trivial fibration $\tilde{Z} \to \pi_{\mathrm{R}}^* Y$; then by axiom CM4 there exists a natural transformation $\psi : \tilde{X} \Rightarrow \tilde{Z}$ making the diagram in $[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ shown below commute:

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "\psi"] \arrow[d] & \tilde{Z} \arrow[d] & \tilde{Y} \arrow[l, "\theta"'] \arrow[d] \\
\pi_{\mathrm{R}}^*(X) \arrow[r, "\pi_{\mathrm{R}}^*(\varphi)"'] & \pi_{\mathrm{R}}^*(Y) \arrow[r, equal] & \pi_{\mathrm{R}}^*(Y)
\end{tikzcd}$$

Since $\pi_{\mathrm{R}}^*(\varphi)$ is a Reedy weak equivalence, it follows from axiom CM2 that ψ is also a Reedy weak equivalence. Transposing across the adjunction $\mathrm{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^*$, we obtain a commutative diagram in $[\mathbb{C}, \mathcal{M}]$,

$$\begin{tikzcd}
\mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{X} \arrow[r, "\mathrm{Lan}_{\pi_{\mathrm{R}}} \psi"] \arrow[d, equal] & \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{Z} \arrow[d] & \mathrm{Lan}_{\pi_{\mathrm{R}}} \tilde{Y} \arrow[l, "\mathrm{Lan}_{\pi_{\mathrm{R}}} \theta"'] \arrow[d, equal] \\
X \arrow[r, "\varphi"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

to which we may then apply Lan_F, yielding the following commutative diagram in $[\mathbb{D}, \mathcal{M}]$:

$$\begin{tikzcd}
\mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{X} \arrow[r, "\mathrm{Lan}_{F\pi_{\mathrm{R}}} \psi"] \arrow[d, "\cong"'] & \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z} \arrow[d] & \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{Y} \arrow[l, "\mathrm{Lan}_{F\pi_{\mathrm{R}}} \theta"'] \arrow[d, "\cong"] \\
\mathrm{Lan}_F X \arrow[r, "\mathrm{Lan}_F \varphi"'] & \mathrm{Lan}_F Y \arrow[r, equal] & \mathrm{Lan}_F Y
\end{tikzcd}$$

Now, $\mathrm{Lan}_{F\pi_{\mathrm{R}}} \psi : \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{X} \to \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z}$ and $\mathrm{Lan}_{F\pi_{\mathrm{R}}} \theta : \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{Y} \to \mathrm{Lan}_{F\pi_{\mathrm{R}}} \tilde{Z}$ are componentwise weak equivalences between componentwise cofibrant diagrams, by proposition 4.6.13, so we deduce that $\mathrm{Lan}_F \varphi$ is also a componentwise weak equivalence between componentwise cofibrant diagrams as claimed, using the 2-out-of-3 property of weak equivalences in $\mathcal{M}$.

(ii). The following diagram of functors is commutative,

$$\begin{tikzcd}
\mathbf{\Delta}(\mathbb{C}) \arrow[r, "\mathbf{\Delta}(F)"] \arrow[d, "\pi_{\mathrm{R}}"'] & \mathbf{\Delta}(\mathbb{D}) \arrow[d, "\pi_{\mathrm{R}}"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

so there is a canonical natural isomorphism $\mathrm{Lan}_F \circ \mathrm{Lan}_{\pi_{\mathrm{R}}} \cong \mathrm{Lan}_{\pi_{\mathrm{R}}} \circ \mathrm{Lan}_{\mathbf{\Delta}(F)}$. Corollary 4.6.8 implies $\mathrm{Lan}_{\mathbf{\Delta}(F)} : [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{D}), \mathcal{M}]$ preserves Reedy-cofibrant diagrams, so it follows from the hypothesis that the functor $\mathrm{Lan}_F : [\mathbb{C}, \mathcal{M}] \to [\mathbb{D}, \mathcal{M}]$ preserves virtually cofibrant diagrams.

(iii). Having proved claim (i), it is now enough to show that the natural transformation $\varepsilon \bullet \left(\mathrm{Lan}_{\pi_{\mathrm{R}}} \circ p \circ \pi_{\mathrm{R}}^*\right) : \mathrm{Lan}_{\pi_{\mathrm{R}}} \circ Q \circ \pi_{\mathrm{R}}^* \Rightarrow \mathrm{id}_{[\mathbb{C}, \mathcal{M}]}$ is a natural weak equivalence; but this is also a consequence of proposition 3.3.24 applied to theorem 4.6.14.

(iv). The functor F^* is a homotopical functor, hence trivially right deformable, and claim (iii) implies Lan_F is left deformable.

(v). Since F^* and G^* are both homotopical functors, (F^*, G^*) is strongly right deformable, and we may deduce from claim (i) that $(\mathrm{Lan}_G, \mathrm{Lan}_F)$ is laxly left deformable. Thus, by lemma 3.1.11, theorem 4.3.1, and corollary 3.3.23, the composable pair $(\mathrm{Lan}_G, \mathrm{Lan}_F)$ is strongly left deformable. ■

Lemma 4.6.17. *Let $\mathcal{M}$ be a DHK model category, let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, and let D be an object in $\mathbb{D}$.*

- *Given the following comma square,*

$$\begin{tikzcd} \mathbf{\Delta}((F \downarrow D)) \arrow[r] \arrow[d, "\mathbf{\Delta}(P)"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{1} \arrow[d, "D"] \\ \mathbf{\Delta}(\mathbb{C}) \arrow[r, "F\pi_{\mathrm{R}}"'] & \mathbb{D} \end{tikzcd}$$

  the derived left Beck–Chevalley transformation

$$\left(\mathbf{L} \varinjlim_{\mathbf{\Delta}((F\downarrow D))}\right) \circ (\mathrm{Ho}\, \mathbf{\Delta}(P)^*) \Rightarrow (\mathrm{Ho}\, D^*) \circ \left(\mathbf{L}\, \mathrm{Lan}_{F\pi_{\mathrm{R}}}\right)$$

  is a natural isomorphism.

Dually:

- *Given the following comma square,*

$$\begin{tikzcd} \mathbf{\Delta}((D \downarrow F))^{\mathrm{op}} \arrow[r, "\mathbf{\Delta}(P)^{\mathrm{op}}"] \arrow[d] \arrow[dr, phantom, "\swarrow"] & \mathbf{\Delta}(\mathbb{C})^{\mathrm{op}} \arrow[d, "F\pi_{\mathrm{L}}"] \\ \mathbb{1} \arrow[r, "D"'] & \mathbb{D} \end{tikzcd}$$

  the derived right Beck–Chevalley transformation

$$\left(\mathbf{R} \varprojlim_{\mathbf{\Delta}((D\downarrow F))^{\mathrm{op}}}\right) \circ (\mathrm{Ho}\, \mathbf{\Delta}(P)^*) \Rightarrow (\mathrm{Ho}\, D^*) \circ \left(\mathbf{R}\, \mathrm{Ran}_{F\pi_{\mathrm{L}}}\right)$$

  is a natural isomorphism.

Proof. Lemma 4.6.7 says $\mathbf{\Delta}(P) : \mathbf{\Delta}((F \downarrow D)) \to \mathbf{\Delta}(\mathbb{C})$ is a right fibration of Reedy categories, so by theorem 4.5.31, $\mathbf{\Delta}(P)^* : [\mathbf{\Delta}((F \downarrow D)), \mathcal{M}] \to [\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]$ preserves Reedy-cofibrant diagrams. Proposition 7.1.19 implies that the left Beck–Chevalley transformation $\varinjlim_{(F\pi_{\mathrm{R}} \downarrow D)} (-\mathbf{\Delta}(P)) \Rightarrow \left(\operatorname{Lan}_{F\pi_{\mathrm{R}}} -\right)(D)$ is a natural isomorphism, hence by corollary 3.3.21, so too is its derived natural transformation. ∎

Proposition 4.6.18. *Let $\mathcal{M}$ be a DHK model category, let $F : \mathbb{C} \to \mathbb{D}$ be a functor between small categories, and let D be an object in $\mathbb{D}$.*

- *Given the following comma square,*

$$\begin{tikzcd}
(F \downarrow D) \arrow[r] \arrow[d, "P"'] \arrow[dr, phantom, "\nearrow"] & \mathbb{1} \arrow[d, "D"] \\
\mathbb{C} \arrow[r, "F"'] & \mathbb{D}
\end{tikzcd}$$

the derived left Beck–Chevalley transformation

$$\left(\mathbf{L} \varinjlim_{(F\downarrow D)}\right) \circ (\operatorname{Ho} P^*) \Rightarrow (\operatorname{Ho} D^*) \circ \left(\mathbf{L} \operatorname{Lan}_F\right)$$

is a natural isomorphism.

Dually:

- *Given the following comma square,*

$$\begin{tikzcd}
(D \downarrow F) \arrow[r, "P"] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \mathbb{C} \arrow[d, "F"] \\
\mathbb{1} \arrow[r, "D"'] & \mathbb{D}
\end{tikzcd}$$

the derived right Beck–Chevalley transformation

$$\left(\mathbf{R} \varprojlim_{(D\downarrow F)}\right) \circ (\operatorname{Ho} P^*) \Rightarrow (\operatorname{Ho} D^*) \circ \left(\mathbf{R} \operatorname{Ran}_F\right)$$

is a natural isomorphism.

Proof. Consider the following diagram, where the 2-cells are the respective left Beck–Chevalley transformations:

$$
\begin{tikzcd}
\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[r, "\mathbf{\Delta}(P)^*"] \arrow[d, hook] & \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[d, hook] \\
{[\mathbf{\Delta}(\mathbb{C}), \mathcal{M}]} \arrow[r, "\mathbf{\Delta}(P)^*"] \arrow[d, "\mathrm{Lan}_{\pi_{\mathrm{R}}}"'] & {[\mathbf{\Delta}((F \downarrow D)), \mathcal{M}]} \arrow[d, "\mathrm{Lan}_{\pi_{\mathrm{R}}}"] \arrow[dl, phantom, "\swarrow"] \\
{[\mathbb{C}, \mathcal{M}]} \arrow[r, "P^*"] \arrow[d, "\mathrm{Lan}_F"'] & {[(F \downarrow D), \mathcal{M}]} \arrow[d, "\varinjlim"] \arrow[dl, phantom, "\swarrow"] \\
{[\mathbb{D}, \mathcal{M}]} \arrow[r, "D^*"'] & \mathcal{M}
\end{tikzcd}
$$

The pasting lemma (A.1.10) implies that left Beck–Chevalley transformations can be pasted together, and the preceding lemma says the derived left Beck–Chevalley transformation

$$\left(\mathbf{L} \varinjlim_{\mathbf{\Delta}((F \downarrow D))}\right) \circ (\operatorname{Ho} \mathbf{\Delta}(P)^*) \Rightarrow (\operatorname{Ho} D^*) \circ \left(\mathbf{L} \operatorname{Lan}_{F \pi_{\mathrm{R}}}\right)$$

is a natural isomorphism; but theorem 4.6.14 says that the adjunctions

$$\operatorname{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [\mathbb{C}, \mathcal{M}] \to \left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$
$$\operatorname{Lan}_{\pi_{\mathrm{R}}} \dashv \pi_{\mathrm{R}}^* : [(F \downarrow D), \mathcal{M}] \to \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}$$

satisfy the Quillen equivalence condition, so the commutative diagram shown below automatically satisfies the derived left Beck–Chevalley condition,

$$
\begin{tikzcd}
{[\mathbb{C}, \mathcal{M}]} \arrow[r, "P^*"] \arrow[d, "\pi_{\mathrm{R}}^*"'] & {[(F \downarrow D), \mathcal{M}]} \arrow[d, "\pi_{\mathrm{R}}^*"] \\
\left[\mathbf{\Delta}(\mathbb{C})_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}} \arrow[r, "\mathbf{\Delta}(P)^*"'] & \left[\mathbf{\Delta}((F \downarrow D))_{\mathrm{R}}, \mathcal{M}\right]_{\mathrm{h}}
\end{tikzcd}
$$

and therefore, by cancelling natural isomorphisms, we conclude that the derived left Beck–Chevalley transformation

$$\left(\mathbf{L} \varinjlim_{(F \downarrow D)}\right) \circ (\operatorname{Ho} P^*) \Rightarrow (\operatorname{Ho} D^*) \circ \left(\mathbf{L} \operatorname{Lan}_F\right)$$

is a natural isomorphism, as claimed. ■

4.7 Framings and resolutions

Prerequisites. §§ 1.1, 1.2, 1.3, 4.1, 4.2, 4.4, 4.5.

In homological algebra, one studies objects in categories without homotopical structure by embedding them in one that does, in such a way that objects in the original category become weakly equivalent to their presentations. The notion of 'resolution' in the sense of Dwyer and Kan [1980c] was invented for similar reasons: though not every model category has a simplicial enrichment, we can still replace objects with homotopically better-behaved simplicial (or cosimplicial) ones. It is also useful to simultaneously discuss the closely related notion of 'framing' introduced by Dwyer, Hirschhorn and Kan [DHK].

In this section, we follow [Hirschhorn, 2003, Ch. 16].

¶ 4.7.1. Recall that a **simplicial object** in a category is a diagram of shape $\mathbf{\Delta}^{\mathrm{op}}$, and dually, a **cosimplicial object** is a diagram of shape $\mathbf{\Delta}$. Let us write $\mathbf{s}\mathcal{M}$ for the category of simplicial objects in $\mathcal{M}$, and $\mathbf{c}\mathcal{M}$ for the category of cosimplicial objects in $\mathcal{M}$.

Proposition 4.7.2. *Let $\mathcal{M}$ be a model category and let X be a finite simplicial set.*

- *For all cosimplicial objects $A^\bullet$ in $\mathcal{M}$, there exists an object $X \star A$ in $\mathcal{M}$ equipped with bijections*

$$\mathcal{M}\big(X \star A, B\big) \cong \mathbf{sSet}(X, \mathcal{M}(A^\bullet, B))$$

that are natural in B.

- *For all simplicial objects $B_\bullet$ in $\mathcal{M}$, there exists an object $\{X, B\}$ in $\mathcal{M}$ equipped with bijections*

$$\mathcal{M}(A, \{X, B\}) \cong \mathbf{sSet}\big(X, \mathcal{M}\big(A, B_\bullet\big)\big)$$

that are natural in A.

Proof. The two claims are formally dual; we will prove the first version.

Applying the Yoneda lemma, we see that $\Delta^n \star A$ must be (isomorphic to) A^n. It is not hard to see that, if $X : \mathcal{J} \to \mathbf{sSet}$ is a diagram such that $Xj \star A$ exists for all j in $\mathcal{J}$, then $\Big(\varinjlim_{j:\mathcal{J}} Xj\Big) \star A$ must be (isomorphic to) $\varinjlim_{j:\mathcal{J}} \big(Xj \star A\big)$ when the latter exists; thus, the class of simplicial sets X for which $X \star A$ exists must be closed under finite colimits (because $\mathcal{M}$ has colimits for finite diagrams). We may then use proposition 1.1.16 to deduce that $X \star A$ exists if X is a finite simplicial set. ■

Remark 4.7.3. The same is true for a general simplicial set X when $\mathcal{M}$ has limits and colimits for all small diagrams: see theorem A.6.10.

Corollary 4.7.4. *Let $\mathcal{M}$ be a model category and let X be a finite simplicial set.*

- *For all cosimplicial objects $A^\bullet$ in $\mathcal{M}$, there exists a cosimplicial object $(X \odot A)^\bullet$ in $\mathcal{M}$ equipped with isomorphisms*

$$\mathcal{M}((X \odot A)^\bullet, B) \cong [X, \mathcal{M}(A^\bullet, B)]$$

  that are natural in B.

- *For all simplicial objects $B_\bullet$ in $\mathcal{M}$, there exists a simplicial object $(X \pitchfork B)_\bullet$ in $\mathcal{M}$ equipped with isomorphisms*

$$\mathcal{M}\big(A, (X \pitchfork A)_\bullet\big)B \cong \big[X, \mathcal{M}\big(A, B_\bullet\big)\big]$$

  that are natural in A.

Proof. The two claims are formally dual; we will prove the first version.

It is clear that $\Delta^n \times X$ is a finite simplicial set for all $n \geq 0$ when X is a finite simplicial set, so the objects $(\Delta^n \times X) \star A$ exist in $\mathcal{M}$. On the other hand, the set of n-simplices of the RHS is precisely the hom-set $\mathbf{sSet}(\Delta^n \times X, \mathcal{M}(A^\bullet, B))$, so we may define $(X \odot A)^\bullet$ by taking $(X \odot A)^n = (\Delta^n \times X) \star A$. ■

¶ 4.7.5. Since $\mathbf{\Delta}$ is a Reedy category, theorem 4.5.19 says these categories admit model structures in which the weak equivalences are degreewise, at least when $\mathcal{M}$ has enough limits and colimits. In fact, finite limits and colimits are enough, because the latching and matching categories of $\mathbf{\Delta}$ at any object are always finite. In this section, $\mathbf{s}\mathcal{M}$ and $\mathbf{c}\mathcal{M}$ will always be equipped the Reedy model structure.

Lemma 4.7.6. *Let $\mathcal{M}$ be a model category.*

- *A morphism $f : A^\bullet \to B^\bullet$ in $\mathbf{c}\mathcal{M}$ is a Reedy cofibration (resp. trivial Reedy cofibration) if and only if, for all $n \geq 0$, the morphism*

$$\big(\Delta^n \star A\big) \cup^{\partial\Delta^n \star A} \big(\partial\Delta^n \star B\big) \to \Delta^n \star B$$

  induced by the boundary inclusion $\partial\Delta^n \to \Delta^n$ is a cofibration (resp. trivial cofibration) in $\mathcal{M}$.

- *A morphism $f : A_\bullet \to B_\bullet$ in $\mathbf{s}\mathcal{M}$ is a Reedy fibration (resp. trivial Reedy fibration) if and only if, for all $n \geq 0$, the morphism*

$$\{\Delta^n, A\} \to \{\partial\Delta^n, A\} \times_{\{\partial\Delta^n, B\}} \{\Delta^n, B\}$$

induced by the boundary inclusion $\partial\Delta^n \to \Delta^n$ is a fibration (resp. trivial fibration) in $\mathcal{M}$.

Proof. It is not hard to check that the indicated morphisms are (isomorphic to) the relevant relative matching or latching morphisms; for the trivial fibration/cofibration case, apply proposition 4.5.15. ♦

Corollary 4.7.7. *Let $\mathcal{M}$ be a model category.*

- *A cosimplicial object $B^\bullet$ in $\mathcal{M}$ is Reedy-cofibrant if and only if, for all $n \geq 0$, the morphism*
$$\partial\Delta^n \star B \to \Delta^n \star B$$
induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ is a cofibration in $\mathcal{M}$.

- *A simplicial object $A_\bullet$ in $\mathcal{M}$ is Reedy-fibrant if and only if, for all $n \geq 0$, the morphism*
$$\{\Delta^n, A\} \to \{\partial\Delta^n, A\}$$
induced by the boundary inclusion $\partial\Delta^n \hookrightarrow \Delta^n$ is a fibration in $\mathcal{M}$. ■

Proposition 4.7.8. *Let $\mathcal{M}$ be a model category.*

- *A cosimplicial object $B^\bullet$ in $\mathcal{M}$ is Reedy-cofibrant if and only if, for all monomorphisms $i : Z \to W$ between finite simplicial sets, the morphism*

$$i \star \mathrm{id}_B : Z \star B \to W \star B$$

induced by i is a cofibration in $\mathcal{M}$.

- *A simplicial object $A_\bullet$ in $\mathcal{M}$ is Reedy-fibrant if and only if, for all monomorphisms $i : Z \to W$ between finite simplicial sets, the morphism*

$$\{i, A\} : \{W, A\} \to \{Z, A\}$$

induced by i is a fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to see that the class of monomorphisms between finite simplicial sets is the smallest class of morphisms between finite simplicial sets that contains the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ and is closed under composition, pushouts, and retracts: simply take an appropriate cellular decomposition. The class of cofibrations in $\mathcal{M}$ is closed under composition, pushouts, and retracts by proposition A.3.12, so we may then apply corollary 4.7.7 to deduce that $i \star A : Z \star A \to W \star A$ is a cofibration in $\mathcal{M}$ for any monomorphism $i : Z \to W$ between finite simplicial sets and any Reedy-cofibrant cosimplicial object $A^\bullet$. Conversely, any such cosimplicial object must be Reedy-cofibrant. ■

Lemma 4.7.9. *Let $\mathcal{M}$ be a model category. There exist adjunctions of the form below,*

$$\mathrm{sk}^0 \dashv (-)^0 \dashv \mathrm{cosk}^0 : \mathcal{M} \to \mathbf{c}\mathcal{M}$$
$$\mathrm{sk}_0 \dashv (-)_0 \dashv \mathrm{cosk}_0 : \mathcal{M} \to \mathbf{s}\mathcal{M}$$

where $(-)^0 : \mathbf{c}\mathcal{M} \to \mathcal{M}$ is the functor that sends a cosimplicial object $A^\bullet$ to the object A^0, and dually, $(-)_0 : \mathbf{s}\mathcal{M} \to \mathcal{M}$ is the functor that sends a simplicial object $A_\bullet$ to the object A_0.

Proof. It is straightforward to verify that the following formulae work,

$$\mathrm{sk}_0(A)_n = A \qquad \mathrm{cosk}_0(A)_n = [n] \pitchfork A$$
$$\mathrm{sk}^0(A)^n = [n] \odot A \qquad \mathrm{cosk}^0(A)^n = A$$

where $[n] \pitchfork A$ is the $(n+1)$-fold power of A, and $[n] \odot A$ is the $(n+1)$-fold copower of A. ■

Definition 4.7.10. Let A be an object in a model category $\mathcal{M}$.

- A **cosimplicial resolution** of A is a Reedy-cofibrant replacement in $\mathbf{c}\mathcal{M}$ for the cosimplicial object $\mathrm{cosk}^0(A)$.
- A **simplicial resolution** of A is a Reedy-fibrant replacement in $\mathbf{s}\mathcal{M}$ for the simplicial object $\mathrm{sk}_0(A)$.

Remark 4.7.11. Proposition 4.7.8 implies that the the above definition is equivalent to the original definition of 'resolution' given by Dwyer and Kan [1980c].

Proposition 4.7.12. *Let $\mathcal{M}$ be a model category.*

(i) *Every object in $\mathcal{M}$ has both a cosimplicial resolution and a simplicial resolution.*

(ii) *If $\mathcal{M}$ is a DHK model category, then cosimplicial resolutions and simplicial resolutions can both be chosen functorially.*

Proof. This follows from proposition 4.1.22 and theorem 4.5.19. ■

Proposition 4.7.13. *Let A be an object in a DHK model category $\mathcal{M}$.*

- *The full subcategory of the slice category $\mathbf{c}\mathcal{M}_{/\operatorname{cosk}^0(A)}$ spanned by the cosimplicial resolutions of A is homotopically contractible.*[5]

- *The full subcategory of the slice category ${}^{\operatorname{sk}_0(A)/}\mathbf{s}\mathcal{M}$ spanned by the simplicial resolutions of A is homotopically contractible.*

Proof. This follows from proposition 4.1.24 and theorem 4.5.19. ■

Lemma 4.7.14. *Let $\mathcal{M}$ be a model category.*

- $\operatorname{cosk}_0 : \mathcal{M} \to \mathbf{s}\mathcal{M}$ *is a right Quillen functor.*

- $\operatorname{sk}^0 : \mathcal{M} \to \mathbf{c}\mathcal{M}$ *is a left Quillen functor.*

Proof. The claims are formally dual; we will prove the first version.

By proposition 4.4.2, it is enough to show that $(-)_0 : \mathbf{s}\mathcal{M} \to \mathcal{M}$ preserves cofibrations and trivial cofibrations. However, the latching category at $[0]$ is empty, so if $f : A_\bullet \to B_\bullet$ is a Reedy cofibration, then $f_0 : A_0 \to B_0$ must be a cofibration in $\mathcal{M}$. Since $(-)_0$ preserves weak equivalences, it follows that $(-)_0$ preserves trivial cofibrations. ■

Lemma 4.7.15. *Let $\mathcal{M}$ be a model category.*

- *There is a unique natural transformation $\Delta : \operatorname{sk}_0 \Rightarrow \operatorname{cosk}_0$ such that $\varepsilon_A \circ \left(\Delta_A\right)_0 \circ \eta_A = \mathrm{id}_A$ for all objects A in $\mathcal{M}$, where $\eta_A : A \to \operatorname{sk}_0(A)_0$ and $\varepsilon_A : \operatorname{cosk}_0(A)_0 \to A$ are the components of the unit and counit of the respective adjunctions.*

[5] See definition 3.1.30.

- *There is a unique natural transformation* $\nabla : \mathrm{sk}^0 \Rightarrow \mathrm{cosk}^0$ *such that* $\varepsilon_A \circ (\nabla_A)_0 \circ \eta_A = \mathrm{id}_A$ *for all objects* A *in* $\mathcal{M}$*, where* $\eta_A : A \to \mathrm{sk}^0(A)^0$ *and* $\varepsilon_A : \mathrm{cosk}^0(A)^0 \to A$ *are the components of the unit and counit of the respective adjunctions.*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to check that η_A is an isomorphism, so $\varepsilon_A \circ (\Delta_A)_0$ is uniquely determined. The universal property of $\mathrm{cosk}_0(A)$ implies $\Delta_A : \mathrm{sk}_0(A) \to \mathrm{cosk}_0(A)$ is determined by its adjoint transpose $\varepsilon_A \circ (\Delta_A)_0 : \mathrm{sk}_0(A)_0 \to A$, so Δ_A is also uniquely determined. ■

Definition 4.7.16. Let A be an object in a model category $\mathcal{M}$.

- A **cosimplicial frame** on A is a pair $(\tilde{A}^\bullet, p^\bullet)$, where $\tilde{A}^\bullet$ is a cosimplicial object in $\mathcal{M}$, $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a Reedy weak equivalence with $p^0 : \tilde{A}^0 \to \mathrm{cosk}^0(A)^0$ an isomorphism, and $\tilde{A}^\bullet$ is Reedy-cofibrant if A is cofibrant.

- A **simplicial frame** on A is a pair $(\hat{A}_\bullet, i_\bullet)$, where $\hat{A}_\bullet$ is a simplicial object in $\mathcal{M}$, $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a Reedy weak equivalence with $i_0 : \mathrm{sk}_0(A)_0 \to \hat{A}_0$ an isomorphism, and $\hat{A}_\bullet$ is Reedy-fibrant if A is fibrant.

- A **left frame** on A is a tuple $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$, where $\tilde{A}^\bullet$ is a cosimplicial object in $\mathcal{M}$, $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a Reedy weak equivalence with $p^0 : \tilde{A}^0 \to \mathrm{cosk}^0(A)^0$ an isomorphism, $i^\bullet$ is a Reedy cofibration, and $p^\bullet \circ i^\bullet = \nabla_A$.

- A **right frame** on A is a tuple $(\hat{A}_\bullet, i_\bullet, p_\bullet)$, where $\hat{A}_\bullet$ is a simplicial object in $\mathcal{M}$, $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a Reedy weak equivalence with $i_0 : \mathrm{sk}_0(A)_0 \to \hat{A}_0$ an isomorphism, $p_\bullet$ is a Reedy fibration, and $p_\bullet \circ i_\bullet = \Delta_A$.

Proposition 4.7.17. *Let* A *be an object in a model category* $\mathcal{M}$*.*

(i) *If* $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ *is a left frame on* A*, then* $(\tilde{A}^\bullet, p^\bullet)$ *is a cosimplicial frame on* A*.*

(ii) *If* A *is cofibrant, then every cosimplicial frame on* A *is a cosimplicial resolution of* A*.*

(iii) *If* $(\tilde{A}^\bullet, p^\bullet)$ *is a cosimplicial resolution of* A*, then* $\tilde{A}^\bullet$ *is (the underlying cosimplicial object of) a cosimplicial frame on* $\tilde{A}^0$*, and* $(\tilde{A}^0, p^0)$ *is (isomorphic to) a cofibrant replacement for* A*.*

Dually:

(i′) *If* $(\hat{A}_\bullet, i_\bullet, p_\bullet)$ *is a right frame on* A*, then* $(\hat{A}_\bullet, i_\bullet)$ *is a simplicial frame on* A*.*

(ii′) *If* A *is fibrant, then every simplicial frame on* A *is a simplicial resolution of* A*.*

(iii′) *If* $(\hat{A}_\bullet, i_\bullet)$ *is a simplicial resolution of* A*, then* $\hat{A}_\bullet$ *is (the underlying simplicial object of) a simplicial frame on* $\hat{A}_0$*, and* $(\hat{A}_0, i_0)$ *is (isomorphic to) a fibrant replacement for* A*.*

Proof. (i). Suppose $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ is a left frame on A. Lemma 4.7.14 implies that $\operatorname{cosk}^0(A)$ is Reedy-cofibrant when A is cofibrant, so $\tilde{A}^\bullet$ is Reedy-cofibrant when A is cofibrant. Thus $(\tilde{A}^\bullet, p^\bullet)$ is indeed a cosimplicial frame on A.

(ii). If A is cofibrant and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then $\tilde{A}^\bullet$ is Reedy-cofibrant, and hence $(\tilde{A}^\bullet, p^\bullet)$ is a Reedy-cofibrant replacement for $\operatorname{cosk}^0(A)$.

(iii). Let $q^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(\tilde{A}^0)$ be the component of the adjunction unit at $\tilde{A}^\bullet$. Since $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ implies $q^\bullet$ is also a Reedy weak equivalence. Now, $\tilde{A}^\bullet$ is Reedy-cofibrant by definition, it follows that $(\tilde{A}^\bullet, q^\bullet)$ is a cosimplicial frame on $\tilde{A}^0$.

Finally, we note that proposition 4.7.8 implies that $\tilde{A}^0$ is a cofibrant object in $\mathcal{M}$, and $p^0 : \tilde{A}^0 \to \operatorname{cosk}^0(A)^0$ is a weak equivalence by definition, so $(\tilde{A}^0, p^0)$ is (isomorphic to) a cofibrant replacement for A. ■

REMARK 4.7.18. The notions of 'left frame' and 'right frame' are originally due to Hovey [1999, § 5.2], but he calls them 'cosimplicial frame' and 'simplicial frame' and does not give a name to the weaker notion. It is explained in loc. cit. that a left (resp. right) frame on A is a cosimplicial (resp. simplicial) frame that is almost Reedy-cofibrant (resp. Reedy-fibrant), in the sense that all but one its latching (resp. matching) morphisms are cofibrations (resp. fibrations). One consequence of this is given in proposition 4.7.25.

Definition 4.7.19. Let $\mathcal{M}$ be a model category.

- A **left framing** for $\mathcal{M}$ is a tuple $(Q^\bullet, i^\bullet, p^\bullet)$, where $Q^\bullet : \mathcal{M} \to \mathbf{c}\mathcal{M}$ is a functor, $i^\bullet : \mathrm{sk}^0 \Rightarrow Q^\bullet$ and $p^\bullet : Q^\bullet \Rightarrow \mathrm{cosk}^0$ are natural transformations, and $\left(Q^\bullet A, \left(i_A\right)^\bullet, \left(p_A\right)^\bullet\right)$ is a left frame for all objects A in $\mathcal{M}$.
- A **right framing** for $\mathcal{M}$ is a tuple $\left(R_\bullet, i_\bullet, p_\bullet\right)$, where $R_\bullet : \mathcal{M} \to \mathbf{s}\mathcal{M}$ is a functor, $i_\bullet : \mathrm{sk}_0 \Rightarrow R_\bullet$ and $p_\bullet : R_\bullet \Rightarrow \mathrm{cosk}_0$ are natural transformations, and $\left(R_\bullet A, \left(i_A\right)_\bullet, \left(p_A\right)_\bullet\right)$ is a right frame for all objects A in $\mathcal{M}$.

A **framed model category** is a model category equipped with a left framing and a right framing.

Theorem 4.7.20. *Let $\mathcal{M}$ be a model category.*

(i) *On each object A in $\mathcal{M}$, there exist a left frame $\left(\tilde{A}^\bullet, i^\bullet, p^\bullet\right)$ and a right frame $\left(\hat{A}_\bullet, i_\bullet, p_\bullet\right)$ such that $p^\bullet : \tilde{A}^\bullet \to \mathrm{cosk}^0(A)$ is a trivial Reedy fibration and $i_\bullet : \mathrm{sk}_0(A) \to \hat{A}_\bullet$ is a trivial Reedy cofibration.*

(ii) *If $\mathcal{M}$ satisfies axiom CM5*, then the left and right frames in (i) can be chosen functorially; in particular, left and right framings for $\mathcal{M}$ exist.*

Proof. See Theorem 5.2.8 in [Hovey, 1999]. □

Theorem 4.7.21. *Let A be an object in a DHK model category $\mathcal{M}$.*

- *The nerve of the full subcategory of the slice category $\mathbf{c}\mathcal{M}_{/\mathrm{cosk}^0(A)}$ spanned by the cosimplicial frames on A is weakly contractible.*
- *The nerve of the full subcategory of the slice category ${}^{\mathrm{sk}^0(A)/}\mathbf{s}\mathcal{A}$ spanned by the simplicial frames on A is weakly contractible.*

Proof. See Theorem 16.6.18 in [Hirschhorn, 2003]. □

Proposition 4.7.22. *Let $\mathcal{M}$ be a model category.*

- *If A is a cofibrant object in $\mathcal{M}$ and $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial frame on A, then the morphism*
$$\Lambda^n_k \star \tilde{A} \to \Delta^n \star \tilde{A}$$
induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial cofibration in $\mathcal{M}$.

- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on A, then the morphism*
$$\{\Delta^n, \hat{B}\} \to \{\Lambda^n_k, \hat{B}\}$$
induced by any horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is a trivial fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

First, note that proposition 4.7.8 implies that $\Lambda^n_k \star \tilde{A} \to \Delta^n \star \tilde{A}$ is a cofibration in $\mathcal{M}$. Since $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ is a Reedy weak equivalence, the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ implies that the morphism $\Delta^n \star \tilde{A} \to \Delta^0 \star \tilde{A}$ is a weak equivalence in $\mathcal{M}$ for all $n \geq 0$. It is clear that $- \star \tilde{A}$ preserves finite colimits of finite simplicial sets, so we may then apply Proposition 3.6.8 in [Hovey, 1999]. □

Corollary 4.7.23. *Let $\mathcal{M}$ be a model category and let $i : Z \to W$ be an anodyne extension between finite simplicial sets.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then the morphism*
$$i \star \mathrm{id}_{\tilde{A}} : Z \star \tilde{A} \to W \star \tilde{A}$$
induced by $i : Z \to W$ is a trivial cofibration in $\mathcal{M}$.

- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then the morphism*
$$\{i, \hat{B}\} : \{W, \hat{B}\} \to \{Z, \hat{B}\}$$
induced by $i : Z \to W$ is a trivial fibration in $\mathcal{M}$.

Proof. The two claims are formally dual; we will prove the first version.

By proposition 1.3.10, the class of anodyne extensions between finite simplicial sets is generated by the boundary inclusions under composition, pushouts, and retracts. We already know that $- \star \tilde{A}$ sends horn inclusions to trivial cofibrations in $\mathcal{M}$, and it is clear that $- \star \tilde{A}$ preserves composition, pushouts, and retracts, so theorem 4.1.12 and proposition A.3.12 imply that $i \star \mathrm{id}_{\tilde{A}}$ is a trivial cofibration in $\mathcal{M}$. ■

Cosimplicial frames and left frames (resp. simplicial frames and right frames) should be regarded as higher cylinder objects (resp. higher path objects). We can make this precise in two different ways:

Proposition 4.7.24. *Let $\mathcal{M}$ be a model category.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$ is a cylinder object for $\tilde{A}^0$ (and hence, isomorphic to a cylinder object for A).*
- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then $(\hat{B}_1, d_1, d_0, s_0)$ is a path object for $\hat{B}_0$ (and hence, isomorphic to a path object for B).*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism $(\!(\delta^1, \delta^0)\!) : \tilde{A}^0 + \tilde{A}^0 \to \tilde{A}^1$ is isomorphic to the morphism $\partial\Delta^1 \star \tilde{A} \to \Delta^1 \star \tilde{A}$ induced by $\partial\Delta^1 \hookrightarrow \Delta^1$, and the latter is a cofibration by proposition 4.7.8. On the other hand, the morphism $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ is a retraction for $\delta^1 : \tilde{A}^0 \to \tilde{A}^1$, and proposition 4.7.22 implies the latter is (isomorphic to) a trivial cofibration; thus, by the 2-out-of-3 property of weak equivalences, $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ must be a weak equivalence. ■

Proposition 4.7.25. *Let $\mathcal{M}$ be a model category.*

- *If $(\tilde{A}^\bullet, i^\bullet, p^\bullet)$ is a left frame on an object in $\mathcal{M}$, then $(\tilde{A}^1, \delta^1, \delta^0, \sigma^0)$ is a cylinder object for $\tilde{A}^0$.*
- *If $(\tilde{B}_\bullet, i_\bullet, p_\bullet)$ is a right frame on an object in $\mathcal{M}$, then $(\tilde{B}_1, d_1, d_0, s_0)$ is a path object for $\hat{B}_0$.*

Proof. The two claims are formally dual; we will prove the first version.

It is not hard to see that the morphism $(\!(\delta^1, \delta^0)\!) : \tilde{A}^0 + \tilde{A}^0 \to \tilde{A}^1$ is isomorphic to the relative latching morphism at $[1]$ for $i^\bullet : \operatorname{sk}^0(A) \to \tilde{A}^\bullet$, and the latter is a Reedy cofibration, so $(\!(\delta^1, \delta^0)\!)$ is a cofibration in $\mathcal{M}$. On the other hand, we have the following commutative diagram,

$$\begin{tikzcd}
\tilde{A}^1 \arrow[r, "p^1"] \arrow[d, "\sigma^0"'] & \operatorname{cosk}^0(A)^1 \arrow[d, "\sigma^0"] \\
\tilde{A}^0 \arrow[r, "p^0"'] & \operatorname{cosk}^0(A)^0
\end{tikzcd}$$

where p^0 and p^1 are weak equivalences in $\mathcal{M}$. Since $\sigma^0 : \operatorname{cosk}^0(A)^1 \to \operatorname{cosk}(A)^0$ is an isomorphism (and so a weak equivalence *a fortiori*), the 2-out-of-3 property of weak equivalences implies $\sigma^0 : \tilde{A}^1 \to \tilde{A}^0$ is also a weak equivalence. ■

Proposition 4.7.26. *Let $\mathcal{M}$ be a model category and let X be a finite simplicial set.*

- *If $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial frame on a cofibrant object A in $\mathcal{M}$, then the cosimplicial object $\left(X \odot \tilde{A}\right)^\bullet$ is (the object part of) a cosimplicial frame on $X \star \tilde{A}$.*
- *If $\left(\hat{B}_\bullet, i_\bullet\right)$ is a simplicial frame on a fibrant object B in $\mathcal{M}$, then the simplicial object $\left(X \pitchfork \hat{B}\right)_\bullet$ is (the object part of) a simplicial frame on $\left\{X, \hat{B}\right\}$.*

Proof. The two claims are formally dual; we will prove the first version.

To show that the cosimplicial object $\left(X \odot \tilde{A}\right)^\bullet$ is indeed (the object part of) a cosimplicial frame on $X \star \tilde{A}$, it suffices to verify that $\left(X \odot \tilde{A}\right)^\bullet$ is Reedy-cofibrant and all its codegeneracy operators are weak equivalences: the latter condition ensures that the counit component $\left(X \odot \tilde{A}\right)^\bullet \to \operatorname{cosk}^0\left(\left(X \odot \tilde{A}\right)^0\right)$ is a Reedy weak equivalence, and we know that $\left(X \odot \tilde{A}\right)^0 \cong X \star \tilde{A}$. By definition, we have the following natural bijections:

$$\begin{aligned}
\mathcal{M}\left(Z \star \left(X \odot \tilde{A}\right), B\right) &\cong \mathbf{sSet}\left(Z, \mathcal{M}\left(\left(X \odot \tilde{A}\right)^\bullet, B\right)\right) \\
&\cong \mathbf{sSet}\left(Z, \left[X, \mathcal{M}\left(\tilde{A}^\bullet, B\right)\right]\right) \\
&\cong \mathbf{sSet}\left(Z \times X, \mathcal{M}\left(\tilde{A}^\bullet, B\right)\right) \\
&\cong \mathcal{M}\left((Z \times X) \star \tilde{A}, B\right)
\end{aligned}$$

Since $i \times \mathrm{id}_X : Z \times X \to W \times X$ is a monomorphism between finite simplicial sets when $i : Z \to W$ is, we may then use proposition 4.7.8 to deduce that $\left(X \odot \tilde{A}\right)^\bullet$ is indeed Reedy-cofibrant.

It remains to be shown that the codegeneracy operators of $\left(X \odot \tilde{A}\right)^\bullet$ are weak equivalences. The cosimplicial identities and axiom CM2 implies it is enough to show that each coface operator $\delta_n^i : \left(X \odot \tilde{A}\right)^{n-1} \to \left(X \odot \tilde{A}\right)^n$ is a weak equivalence. Since the unique morphism $\Delta^n \to \Delta^0$ is a (weak) homotopy equivalence, we can use proposition 1.3.19 and the 2-out-of-3 property of weak homotopy equivalences to deduce that, for each $\delta_n^i : \Delta^{n-1} \to \Delta^n$, the induced morphism $\delta_n^i \times \mathrm{id}_X : \Delta^0 \times X \to \Delta^n \times X$ is a weak homotopy equivalence. Proposition 1.3.16 then says that $\delta_n^i \times \mathrm{id}_X$ is an anodyne extension, so by corollary 4.7.23, the induced morphism $\left(\Delta^{n-1} \times X\right) \star \tilde{A} \to (\Delta^n \times X) \star \tilde{A}$ is a trivial cofibration in $\mathcal{M}$. Thus, every coface operator $\left(X \odot \tilde{A}\right)^0 \to \left(X \odot \tilde{A}\right)^n$ is a weak equivalence in $\mathcal{M}$. ∎

4.8 Derived hom-spaces

Prerequisites. §§ 1.1, 1.3, 3.1, 3.3, 4.1, 4.4, 4.5, 4.7, A.4.

Given a cofibrant object A and a fibrant object B in a model category $\mathcal{M}$, there ought to be a space of morphisms $A \to B$, at least well-defined up to weak equivalence, such that the set of connected components is in natural bijection with the hom-set $\operatorname{Ho}\mathcal{M}(A, B)$, while homotopy classes of paths correspond to homotopy classes of homotopies of morphisms $A \to B$, and so on. For this, we will use the notion of 'frame' introduced in the previous section.

Definition 4.8.1. Let $\mathcal{M}$ be a model category.

- A **cosimplicial frame** in $\mathcal{M}$ is a cosimplicial object $\tilde{A}^\bullet$ in $\mathcal{M}$ for which there exist an object A and a morphism $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ such that $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A.
- A **simplicial frame** in $\mathcal{M}$ is a simplicial object $\hat{B}_\bullet$ in $\mathcal{M}$ for which there exist an object B and a morphism $i_\bullet : \operatorname{sk}_0(B) \to \hat{B}_\bullet$ such that $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B.
- A **cosimplicial resolution** in $\mathcal{M}$ is a cosimplicial object $\tilde{A}^\bullet$ in $\mathcal{M}$ for which there exist an object A and a morphism $p^\bullet : \tilde{A}^\bullet \to \operatorname{cosk}^0(A)$ such that $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial resolution on A.
- A **simplicial resolution** in $\mathcal{M}$ is a simplicial object $\hat{B}_\bullet$ in $\mathcal{M}$ for which there exist an object B and a morphism $i_\bullet : \operatorname{sk}_0(B) \to \hat{B}_\bullet$ such that $(\hat{B}_\bullet, i_\bullet)$ is a simplicial resolution on B.
- A **weakly constant cosimplicial object** in $\mathcal{M}$ is a cosimplicial object in $\mathcal{M}$ such that every coface and codegeneracy operator is a weak equivalence in $\mathcal{M}$.
- A **weakly constant simplicial object** in $\mathcal{M}$ is a simplicial object in $\mathcal{M}$ such that every face and degeneracy operator is a weak equivalence in $\mathcal{M}$.

Proposition 4.8.2. *Let $\mathcal{M}$ be a model category. Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let $p^\bullet : A^\bullet \to \operatorname{cosk}^0(A^0)$ be the component of the adjunction unit at $A^\bullet$.*

(i) *$A^\bullet$ is a cosimplicial frame in $\mathcal{M}$ if and only if $(A^\bullet, p^\bullet)$ is a cosimplicial frame on A^0.*

(ii) *$A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ if and only if $A^\bullet$ is a cosimplicial frame in $\mathcal{M}$* and *A^0 is cofibrant.*

(iii) *$A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ if and only if $(A^\bullet, p^\bullet)$ is a cosimplicial resolution of A^0.*

(iv) *$A^\bullet$ is a weakly constant cosimplicial object in $\mathcal{M}$ if and only if the morphism $p^\bullet : A^\bullet \to \operatorname{cosk}^0(A^0)$ is a Reedy weak equivalence.*

(v) *$A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ if and only if $A^\bullet$ is Reedy-cofibrant and weakly constant.*

Dually, let $B_\bullet$ be a simplicial object in $\mathcal{M}$ and let $i_\bullet : \operatorname{sk}_0(B_0) \to B$ be the component of the adjunction counit at $B_\bullet$.

(i′) *$B_\bullet$ is a simplicial frame in $\mathcal{M}$ if and only if $(B_\bullet, i_\bullet)$ is a simplicial frame on B_0.*

(ii′) *$B_\bullet$ is a simplicial resolution in $\mathcal{M}$ if and only if $B_\bullet$ is a simplicial frame in $\mathcal{M}$* and *B_0 is fibrant.*

(iii′) *$B_\bullet$ is a simplicial resolution in $\mathcal{M}$ if and only if $(B_\bullet, i_\bullet)$ is a simplicial resolution of B_0.*

(iv′) *$B_\bullet$ is a weakly constant simplicial object in $\mathcal{M}$ if and only if the morphism $i_\bullet : \operatorname{sk}_0(B_0) \to B_\bullet$ is a Reedy weak equivalence.*

(v′) *$B_\bullet$ is a simplicial resolution in $\mathcal{M}$ if and only if $B_\bullet$ is Reedy-fibrant and weakly constant.*

Proof. These are straightforward consequences of the definitions and proposition 4.7.17. ◆

Lemma 4.8.3. *Let $\mathcal{M}$ be a model category.*

- *Any cosimplicial resolution in $\mathcal{M}$ is a degreewise cofibrant cosimplicial object in $\mathcal{M}$.*
- *Any simplicial resolution in $\mathcal{M}$ is a degreewise fibrant simplicial object in $\mathcal{M}$.*

Proof. This is a corollary of proposition 4.7.8. ■

Definition 4.8.4. Let $\mathcal{M}$ be a model category.

- The **category of weakly constant cosimplicial objects** in $\mathcal{M}$ is the full subcategory $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ of $\mathbf{c}\mathcal{M}$ spanned by the weakly constant cosimplicial objects. A **weak equivalence** (resp. **cofibration**, **fibration**) in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a Reedy weak equivalence (resp. cofibration, fibration).

- The **category of weakly constant simplicial objects** in $\mathcal{M}$ is the full subcategory $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ of $\mathbf{s}\mathcal{M}$ spanned by the weakly constant simplicial objects. A **weak equivalence** (resp. **cofibration**, **fibration**) in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a Reedy weak equivalence (resp. cofibration, fibration).

Remark 4.8.5. Any cosimplicial (resp. simplicial) object that is weakly equivalent to a weakly constant cosimplicial (resp. simplicial) object must itself be a weakly constant cosimplicial (resp. simplicial) object. Thus, $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ (resp. $\mathbf{s}_{\mathrm{w}}\mathcal{M}$) together with the inherited notions of 'weak equivalence', 'cofibration', and 'fibration' is a derivable category (by proposition 4.1.26).

Proposition 4.8.6. *Let $\mathcal{M}$ be a model category.*

- *The following adjunction is an adjoint homotopical equivalence of homotopical categories:*

$$(-)^0 \dashv \operatorname{cosk}^0 : \mathcal{M} \to \mathbf{c}_{\mathrm{w}}\mathcal{M}$$

In particular, we have an adjoint equivalence of homotopy categories:

$$\operatorname{Ho}(-)^0 \dashv \operatorname{Ho}\operatorname{cosk}^0 : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{c}_{\mathrm{w}}\mathcal{M}$$

- *The following adjunction is an adjoint homotopical equivalence of homotopical categories:*

$$\operatorname{sk}_0 \dashv (-)_0 : \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathcal{M}$$

In particular, we have an adjoint equivalence of homotopy categories:

$$\operatorname{Ho}\operatorname{sk}_0 \dashv \operatorname{Ho}(-)_0 : \operatorname{Ho}\mathbf{s}_{\mathrm{w}}\mathcal{M} \to \operatorname{Ho}\mathcal{M}$$

Proof. The two claims are formally dual; we will prove the first version.

First of all, we note that $\operatorname{cosk}^0(A)$ is a weakly constant cosimplicial object in $\mathcal{M}$ for every object A in $\mathcal{M}$, so the adjunction in lemma 4.7.9 restricts to an adjunction between $\mathcal{M}$ and $\mathbf{c}_{\mathrm{w}}\mathcal{M}$. It is clear that the adjunction counit is a natural isomorphism, and proposition 4.8.2 says that the adjunction unit is a natural weak equivalence, so we indeed have an adjoint homotopical equivalence of homotopical categories. Finally, we apply proposition 3.1.29 to prove the claim about homotopy categories. ■

Corollary 4.8.7. *Let $\mathcal{M}$ be a model category.*

- *A morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a Reedy weak equivalence if and only if the component $f^0 : A^0 \to B^0$ is a weak equivalence in $\mathcal{M}$.*
- *A morphism $f_\bullet : A_\bullet \to B_\bullet$ in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ is a Reedy weak equivalence if and only if the component $f_0 : A_0 \to B_0$ is a weak equivalences in $\mathcal{M}$.* ■

Definition 4.8.8. Let $\mathcal{M}$ be a model category.

- Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let B be an object in $\mathcal{M}$. The **left hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\left(\mathcal{H}om_{\mathcal{M}}(A, B)\right)_n = \mathcal{M}(A^n, B)$$
- Let A be an object in $\mathcal{M}$ and let $B_\bullet$ be a simplicial object in $\mathcal{M}$. The **right hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\left(\mathcal{H}om_{\mathcal{M}}(A, B)\right)_m = \mathcal{M}\left(A, B_m\right)$$
- Let $A^\bullet$ be a cosimplicial object in $\mathcal{M}$ and let $B_\bullet$ be a simplicial object in $\mathcal{M}$. The **total hom-complex** $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the simplicial set defined by the formula below:
$$\left(\mathcal{H}om_{\mathcal{M}}(A, B)\right)_k = \mathcal{M}\left(A_k, B_k\right)$$

Remark 4.8.9. Let $\mathcal{M}$ be a model category.

- For each pair (A, B) of objects in $\mathcal{M}$, we have the following natural isomorphisms:
$$\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0(A), B\right) \cong \operatorname{disc} \mathcal{M}(A, B)$$
$$\mathcal{H}om_{\mathcal{M}}\left(A, \operatorname{sk}_0(B)\right) \cong \operatorname{disc} \mathcal{M}(A, B)$$

- For each cosimplicial object $A^\bullet$ in $\mathcal{M}$ and each object B in $\mathcal{M}$, we have the following natural isomorphism:

$$\mathcal{H}om_{\mathcal{M}}\big(A, \mathrm{sk}_0(B)\big) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$

- For each object A in $\mathcal{M}$ and each simplicial object $B_\bullet$ in $\mathcal{M}$, we have the following natural isomorphism:

$$\mathcal{H}om_{\mathcal{M}}\big(\mathrm{cosk}^0(A), B\big) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$

This justifies our use of the same notation for left, right, and total hom-complexes.

Proposition 4.8.10. *Let $\mathcal{M}$ be a model category.*

- *If A is a cofibrant object in $\mathcal{M}$ and $(\tilde{A}^\bullet, p^\bullet)$ is a cosimplicial frame on A, then the functor $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M} \to$* **sSet** *preserves limits, fibrations, and trivial fibrations, and for all fibrant objects B, there is a natural bijection between $\pi_0\mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$ and* $\operatorname{Ho}\mathcal{M}(A, B)$.

- *If B is a fibrant object in $\mathcal{M}$ and $(\hat{B}_\bullet, i_\bullet)$ is a simplicial frame on B, then the functor $\mathcal{H}om_{\mathcal{M}}(-, \hat{B}) : \mathcal{M}^{\mathrm{op}} \to$* **sSet** *preserves limits, fibrations, and trivial fibrations, and for all cofibrant objects A, there is a natural bijection between $\pi_0\mathcal{H}om_{\mathcal{M}}(A, \hat{B})$ and* $\operatorname{Ho}\mathcal{M}(A, B)$.

Proof. The two claims are formally dual; we will prove the first version.

It is well-known that each $\mathcal{M}(\tilde{A}^n, -) : \mathcal{M} \to$ **Set** preserves limits, so by lemma 1.1.9, $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, -) : \mathcal{M} \to$ **sSet** preserves limits as well.

Let $f : B \to C$ be a fibration in $\mathcal{M}$. To verify that $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, f)$ is a Kan fibration, it is enough to show that it has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$. Proposition 4.7.22 implies that $f : B \to C$ has the right lifting property with respect to the morphisms $\Lambda^n_k \star \tilde{A} \to \Delta^n \star \tilde{A}$ induced by the horn inclusions, so by applying proposition A.3.20, we deduce that $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, f)$ is indeed a Kan fibration.

Now, suppose $f : B \to C$ is a trivial fibration in $\mathcal{M}$. Then corollary 4.7.7 implies that $f : B \to C$ has the right lifting property with respect to the morphisms $\partial\Delta^n \star \tilde{A} \to \Delta^n \star \tilde{A}$ induced by the boundary inclusions, so we may deduce that $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, f)$ is a trivial Kan fibration in this case.

Finally, recalling that $p^0 : \tilde{A}^0 \to \mathrm{cosk}^0(A)^0$ is an isomorphism, we get a natural morphism disc $\mathcal{M}(A, B) \to \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$ for all objects B in $\mathcal{M}$, and it is a bijection on vertices. Proposition 4.7.24 says that $\left(\tilde{A}^1, \delta^1, \delta^0, \sigma^0\right)$ is a cylinder object for $\tilde{A}^0$, so if B is fibrant, we may apply lemma 4.2.14 and theorem 4.3.1 to deduce that the connected components of $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, B\right)$ are in natural bijection with the homotopy classes of morphisms $A \to B$. ■

Corollary 4.8.11. *Let $\mathcal{M}$ be a model category. If $A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$, then:*

(i) *For each fibrant object B in $\mathcal{M}$, the left hom-complex $\mathcal{H}om_{\mathcal{M}}(A, B)$ is a Kan complex.*

(ii) *The left hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathcal{M} \to$ **sSet** sends weak equivalences between fibrant objects in $\mathcal{M}$ to homotopy equivalences of Kan complexes.*

(iii) *The total hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}\mathcal{M} \to$ **sSet** sends Reedy weak equivalences between* degreewise *fibrant simplicial objects in $\mathcal{M}$ to weak homotopy equivalences in* **sSet**.

Dually, if $B_\bullet$ is a simplicial resolution in $\mathcal{M}$, then:

(i′) *For each cofibrant object A in $\mathcal{M}$, the right hom-complex $\mathcal{H}om_{\mathcal{M}}(A, B)$ is a Kan complex.*

(ii′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(-, B) : \mathcal{M}^{\mathrm{op}} \to$ **sSet** sends weak equivalences between cofibrant objects in $\mathcal{M}$ to homotopy equivalences of Kan complexes.*

(iii′) *The total hom-complex functor $\mathcal{H}om_{\mathcal{M}}(-, B) : (\mathbf{c}\mathcal{M})^{\mathrm{op}} \to$ **sSet** sends Reedy weak equivalences between* degreewise *cofibrant cosimplicial objects in $\mathcal{M}$ to weak homotopy equivalences.*

Proof. (i) and (ii). Apply proposition 4.8.2 and Ken Brown's lemma (4.4.6) to the previous proposition.

(iii). First, consider the functor $\mathcal{M}(A^\bullet, -) : \mathbf{s}\mathcal{M} \to \mathbf{ssSet}$ that sends an object $B_\bullet$ in $\mathbf{s}\mathcal{M}$ to the bisimplicial set defined by the formula below:

$$\left(\mathcal{M}\left(A^\bullet, B_\bullet\right)\right)_m = \mathcal{H}om_{\mathcal{M}}\left(A, B_m\right)$$

Thus, by claim (ii), $\mathcal{M}(A^\bullet, -)$ sends Reedy weak equivalences between degreewise fibrant simplicial objects in $\mathcal{M}$ to Reedy weak equivalences in **ssSet**. We may then use lemma 1.5.7 and theorem 1.5.9, we may deduce that the total hom-complex functor has the required property. ■

Corollary 4.8.12. *Let $\mathcal{M}$ be a model category, let $\mathbf{c}_{\mathrm{r}}\mathcal{M}$ be the full subcategory of $\mathbf{c}\mathcal{M}$ spanned by the cosimplicial resolutions, and let $\mathbf{s}_{\mathrm{r}}\mathcal{M}$ be the full subcategory of $\mathbf{s}\mathcal{M}$ spanned by the simplicial resolutions. Then, the total hom-complex functor $\mathcal{H}om_{\mathcal{M}} : (\mathbf{c}_{\mathrm{r}}\mathcal{M})^{\mathrm{op}} \times \mathbf{s}_{\mathrm{r}}\mathcal{M} \to \mathbf{sSet}$ is a homotopical functor.*

Proof. Apply lemma 4.8.3 to corollary 4.8.11. ■

Proposition 4.8.13. *Let $\mathcal{M}$ be a model category.*

- *If $A^\bullet$ is a* degreewise *cofibrant weakly constant cosimplicial object in $\mathcal{M}$, then the functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}\mathcal{M} \to \mathbf{sSet}$ preserves weak equivalences between simplicial* resolutions.

- *If $B_\bullet$ is a* degreewise *fibrant weakly constant simplicial object in $\mathcal{M}$, then the functor $\mathcal{H}om_{\mathcal{M}}(-, B) : (\mathbf{c}\mathcal{M})^{\mathrm{op}} \to \mathbf{sSet}$ preserves weak equivalences between cosimplicial* resolutions.

Proof. The two claims are formally dual; we will prove the first version.

Let $A^\bullet$ be a degreewise cofibrant weakly constant cosimplicial object in $\mathcal{M}$ and let $p^\bullet : A^\bullet \to \operatorname{cosk}^0(A^0)$ be the component of the adjunction unit. By proposition 4.8.2, $p^\bullet$ is a Reedy weak equivalence. Let $f_\bullet : B_\bullet \to C_\bullet$ be a weak equivalence between simplicial resolutions. We then have the following commutative diagram in $\mathbf{sSet}$:

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{M}}(A, B) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(p,B)"] \arrow[d, "\mathcal{H}om_{\mathcal{M}}(A,f)"'] & \mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), B\right) \arrow[d, "\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right),f\right)"] \\
\mathcal{H}om_{\mathcal{M}}(A, C) \arrow[r, "\mathcal{H}om_{\mathcal{M}}(p,C)"'] & \mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), C\right)
\end{tikzcd}$$

Corollary 4.8.11 says that $\mathcal{H}om_{\mathcal{M}}(p, B)$ and $\mathcal{H}om_{\mathcal{M}}(p, C)$ are weak homotopy equivalences; but recalling lemma 4.8.3, we may then use remark 4.8.9 to deduce that $\mathcal{H}om_{\mathcal{M}}\left(\operatorname{cosk}^0\left(A^0\right), f\right)$ is a weak homotopy equivalence. Finally, we apply the 2-out-of-3 property of weak homotopy equivalences to conclude that $\mathcal{H}om_{\mathcal{M}}(A, f)$ itself is a weak homotopy equivalence. ■

Proposition 4.8.14. *Let $\mathcal{M}$ be a model category. If $A^\bullet$ is a cosimplicial resolution in $\mathcal{M}$ and $B_\bullet$ is a simplicial resolution in $\mathcal{M}$, then there is a natural diagram of weak homotopy equivalences in* **sSet** *of the form below,*

$$\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right) \longrightarrow \mathcal{H}om_{\mathcal{M}}(A, B) \longleftarrow \mathcal{H}om_{\mathcal{M}}\left(A^0, B\right)$$

where $\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right)$ is the left hom-complex, $\mathcal{H}om_{\mathcal{M}}\left(A^0, B\right)$ is the right hom-complex, $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the total hom-complex, the rightward arrow is the morphism induced by the adjunction counit component $i_\bullet : \operatorname{sk}_0\left(B_0\right) \to B_\bullet$, and the leftward arrow is the morphism induced by the adjunction unit component $p^\bullet : A^\bullet \to \operatorname{cosk}^0\left(A^0\right)$.

Proof. The two halves of the claim are formally dual; we will show that there is a natural weak homotopy equivalence $\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right) \to \mathcal{H}om_{\mathcal{M}}(A, B)$.

By proposition 4.8.2, B_0 is a fibrant object in $\mathcal{M}$ and $i_\bullet : \operatorname{sk}_0\left(B_0\right) \to B_\bullet$ is a Reedy weak equivalence. Lemma 4.8.3 says that each B_m is a fibrant object in $\mathcal{M}$, so $i_\bullet$ is moreover a Reedy weak equivalence between degreewise fibrant objects. Thus, $\mathcal{H}om_{\mathcal{M}}\left(A, \operatorname{sk}_0\left(B_0\right)\right) \to \mathcal{H}om_{\mathcal{M}}(A, B)$ is a weak homotopy equivalence, by corollary 4.8.11. Since the total hom-complex $\mathcal{H}om_{\mathcal{M}}\left(A, \operatorname{sk}_0\left(B_0\right)\right)$ is naturally isomorphic to the left hom-complex $\mathcal{H}om_{\mathcal{M}}\left(A, B_0\right)$ (by remark 4.8.9), this is the required natural weak homotopy equivalence. ∎

Definition 4.8.15. Let A and B be objects in a model category $\mathcal{M}$.

- A **left homotopy function complex** from A to B consists of the data $\left(\tilde{A}^\bullet, p^\bullet, \hat{B}, i, \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)\right)$, where $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial resolution of A, $\left(\hat{B}, i\right)$ is a fibrant replacement for B, and $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$ is the left hom-complex.

- A **right homotopy function complex** from A to B consists of the data $\left(\tilde{A}, p, \hat{B}_\bullet, i_\bullet, \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)\right)$, where (A, p) is a cofibrant replacement for A, $\left(\hat{B}_\bullet, i_\bullet\right)$ is a simplicial resolution of B, and $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$ is the right hom-complex.

- A **two-sided homotopy function complex** from A to B consists of the data $\left(\tilde{A}^\bullet, p^\bullet, \hat{B}_\bullet, i_\bullet, \mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)\right)$, where $\left(\tilde{A}^\bullet, p^\bullet\right)$ is a cosimplicial resolution of A, $\left(\hat{B}_\bullet, i\right)$ is a simplicial resolution of B, and $\mathcal{H}om_{\mathcal{M}}\left(\tilde{A}, \hat{B}\right)$ is the total hom-complex.

We will often abuse notation and say $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ is a (left, right, or two-sided) homotopy function complex from A to B, omitting mention of the other data.

Note that the weak homotopy type of $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, \hat{B})$ depends only on the isomorphism class of A and B in $\operatorname{Ho} \mathcal{M}$, by corollary 4.8.11.

Proposition 4.8.16. *Let $f : A \to B$ be a morphism in a model category $\mathcal{M}$.*

- *Let $(\hat{A}, i_A)$ and $(\hat{B}, i_B)$ be fibrant replacements for A and B, respectively, and let $\hat{f} : \hat{A} \to \hat{B}$ be any morphism in $\mathcal{M}$ making the diagram below commute:*

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "f"'] & \hat{A} \arrow[d, "\hat{f}"] \\ B \arrow[r, "i_B"'] & \hat{B} \end{tikzcd}$$

  The morphism $f : A \to B$ is a weak equivalence in $\mathcal{M}$ if and only if the induced morphism of left homotopy function complexes

$$\mathcal{H}om_{\mathcal{M}}(C, \hat{f}) : \mathcal{H}om_{\mathcal{M}}(C, \hat{A}) \to \mathcal{H}om_{\mathcal{M}}(C, \hat{B})$$

  is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$.

- *Let $(\tilde{A}, p_A)$ and $(\tilde{B}, p_B)$ be cofibrant replacements for A and B, respectively, and let $\tilde{f} : \tilde{A} \to \tilde{B}$ be any morphism in $\mathcal{M}$ making the diagram below commute:*

$$\begin{tikzcd} \tilde{A} \arrow[r, "p_A"] \arrow[d, "\tilde{f}"'] & A \arrow[d, "f"] \\ \tilde{B} \arrow[r, "p_B"'] & B \end{tikzcd}$$

  The morphism $f : A \to B$ is a weak equivalence in $\mathcal{M}$ if and only if the induced morphism of right homotopy function complexes

$$\mathcal{H}om_{\mathcal{M}}(\tilde{f}, C) : \mathcal{H}om_{\mathcal{M}}(\tilde{B}, C) \to \mathcal{H}om_{\mathcal{M}}(\tilde{A}, C)$$

  is a weak homotopy equivalence for all simplicial resolutions $C_\bullet$.

Proof. The two claims are formally dual; we will prove the first version.

First, suppose $f : A \to B$ is a weak equivalence in $\mathcal{M}$. Then, by axiom CM2, $\hat{f} : \hat{A} \to \hat{B}$ is also a weak equivalence, so we may use corollary 4.8.11 to deduce that $\mathcal{H}om_{\mathcal{M}}(C, \hat{f})$ is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$.

Conversely, suppose $\mathcal{H}om_{\mathcal{M}}(C, \hat{f})$ is a weak homotopy equivalence for all cosimplicial resolutions $C^\bullet$. Proposition 4.8.10 then implies that the hom-set map

$$\operatorname{Ho}\mathcal{M}(C^0, \hat{f}) : \operatorname{Ho}\mathcal{M}(C^0, \hat{A}) \to \operatorname{Ho}\mathcal{M}(C^0, \hat{B})$$

is a bijection for all cosimplicial resolutions $C^\bullet$; but theorem 4.7.20 implies every cofibrant object in $\mathcal{M}$ occurs as C^0 for some cosimplicial resolution $C^\bullet$, so using proposition 4.1.22, we deduce that $\hat{f} : \hat{A} \to \hat{B}$ is an isomorphism in $\operatorname{Ho}\mathcal{M}$. Theorem 4.3.1 then implies $\hat{f} : \hat{A} \to \hat{B}$ must be an weak equivalence in $\mathcal{M}$, and therefore (using axiom CM2) $f : A \to B$ itself is a weak equivalence in $\mathcal{M}$. ■

Definition 4.8.17. Let $\mathcal{M}$ be a model category.

- A **derived left hom-space functor** for an object B in $\mathcal{M}$ is a functor $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(-, B) : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \to \operatorname{Ho}\mathbf{sSet}$ equipped with natural isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A^0, B) \cong \mathcal{H}om_{\mathcal{M}}(A, \hat{B})$$

  in $\operatorname{Ho}\mathbf{sSet}$, where $A^\bullet$ varies over the cosimplicial resolutions in $\mathcal{M}$, $(\hat{B}, i)$ varies over the fibrant replacements for B, and $\mathcal{H}om_{\mathcal{M}}(A, \hat{B})$ is the left hom-complex.

- A **derived right hom-space functor** for an object A in $\mathcal{M}$ is a functor $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, -) : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ equipped with natural isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, B_0) \cong \mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$$

  in $\operatorname{Ho}\mathbf{sSet}$, where $(\tilde{A}, p)$ varies over the cofibrant replacements for A, $B_\bullet$ varies over the simplicial resolutions in $\mathcal{M}$, and $\mathcal{H}om_{\mathcal{M}}(\tilde{A}, B)$ is the right hom-complex.

- A **derived hom-space functor** for $\mathcal{M}$ is a functor $\mathbf{R}\mathrm{Hom}_{\mathcal{M}} : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \times \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ equipped with natural isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A^0, B_0) \cong \mathcal{H}om_{\mathcal{M}}(A, B)$$

  in $\operatorname{Ho}\mathbf{sSet}$, where $A^\bullet$ varies over the cosimplicial resolutions in $\mathcal{M}$, $B_\bullet$ varies over the simplicial resolutions in $\mathcal{M}$, and $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the total hom-complex.

We will often refer to the object $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}(A, B)$ as a **derived hom-space**, omitting mention of the other data.

The name 'derived hom-space' is justified by the following theorem.

Theorem 4.8.18. *Let $\mathcal{M}$ be a model category, let $\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}, Q^{\bullet}, p^{\bullet}\right)$ be a left Quillen deformation retract of $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, and let $\left(\mathbf{s}_{\mathrm{r}}\mathcal{M}, R_{\bullet}, i_{\bullet}\right)$ be a right Quillen deformation retract of $\mathbf{s}_{\mathrm{w}}\mathcal{M}$.*

(i) *$\left(\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{r}}\mathcal{M}, Q^{\bullet} \times R_{\bullet}, \left(p^{\bullet}, i_{\bullet}\right)\right)$ is a right deformation retract for the total hom-complex functor $\mathcal{Hom}_{\mathcal{M}} : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$.*

(ii) *$\mathcal{Hom}_{\mathcal{M}} : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \times \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$ has a total right derived functor; furthermore, if $\left(\mathbf{c}_{\mathrm{r}}\mathcal{M}, Q^{\bullet}, p^{\bullet}\right)$ and $\left(\mathbf{s}_{\mathrm{r}}\mathcal{M}, R_{\bullet}, i_{\bullet}\right)$ are functorial deformation retracts, then $\mathcal{Hom}_{\mathcal{M}}$ also has a homotopical right approximation.*

(iii) *The functor $\mathbf{R}\mathcal{Hom}_{\mathcal{M}}\left(\operatorname{cosk}^0(-), \operatorname{sk}_0(-)\right) : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \times \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ is a derived hom-space functor for $\mathcal{M}$.*

Proof. (i). Recall that proposition 4.8.2 says every cofibrant weakly constant cosimplicial object is a cosimplicial resolution, and every fibrant weakly constant simplicial object is a simplicial resolution. Thus, a cofibrant replacement for $\operatorname{cosk}^0(A)$ is a cosimplicial resolution of A, and a fibrant replacement for $\operatorname{sk}_0(B)$ is a simplicial resolution of B. The claim then follows from corollary 4.8.12.

(ii). Apply theorems 3.3.13 and 3.4.10.

(iii). This follows from claims (i) and (ii). ■

Theorem 4.8.19. *Let $\mathcal{M}$ be a model category. If B is a fibrant object in $\mathcal{M}$, then:*

(i) *The left hom-complex functor $\mathcal{Hom}_{\mathcal{M}}(-, B) : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \to \mathbf{sSet}$ sends trivial cofibrations in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ to weak homotopy equivalences in $\mathbf{sSet}$.*

(ii) *The left hom-complex functor $\mathcal{Hom}_{\mathcal{M}}(-, B) : \left(\mathbf{c}_{\mathrm{w}}\mathcal{M}\right)^{\mathrm{op}} \to \mathbf{sSet}$ admits a total right derived functor.*

(iii) *The functor $\mathbf{R}\mathcal{Hom}_{\mathcal{M}}\left(\operatorname{cosk}^0(-), B\right) : (\operatorname{Ho}\mathcal{M})^{\mathrm{op}} \to \operatorname{Ho}\mathbf{sSet}$ is a derived left hom-space functor.*

Dually, if A is a cofibrant object in $\mathcal{M}$, then:

(i′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$ sends trivial fibrations in $\mathbf{s}_{\mathrm{w}}\mathcal{M}$ to weak homotopy equivalences in $\mathbf{sSet}$.*

(ii′) *The right hom-complex functor $\mathcal{H}om_{\mathcal{M}}(A, -) : \mathbf{s}_{\mathrm{w}}\mathcal{M} \to \mathbf{sSet}$ admits a total right derived functor.*

(iii′) *The functor $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(A, \mathrm{sk}_0(-)\big) : \operatorname{Ho}\mathcal{M} \to \operatorname{Ho}\mathbf{sSet}$ is a derived right hom-space functor.*

Proof. (i). Let $f^\bullet : A^\bullet \to C^\bullet$ be a trivial cofibration in $\mathbf{c}_{\mathrm{w}}\mathcal{M}$, and choose a simplicial frame $\big(\hat{B}_\bullet, i_\bullet\big)$ on B. We then have a morphism of bisimplicial sets

$$\mathcal{M}\big(f^\bullet, \hat{B}_\bullet\big) : \mathcal{M}\big(C^\bullet, \hat{B}_\bullet\big) \to \mathcal{M}\big(A^\bullet, \hat{B}_\bullet\big)$$

and since each $f^n : A^n \to C^n$ is a trivial cofibration in $\mathcal{M}$ (by proposition 4.5.14), proposition 4.8.10 says that the components

$$\mathcal{M}\big(f^n, \hat{B}_\bullet\big) : \mathcal{M}\big(C^n, \hat{B}_\bullet\big) \to \mathcal{M}\big(A^n, \hat{B}_\bullet\big)$$

are trivial Kan fibrations, hence weak homotopy equivalences *a fortiori*. Thus, applying lemma 1.5.7 and theorem 1.5.9, we deduce that the morphism

$$\mathcal{H}om_{\mathcal{M}}\big(f, \hat{B}\big) : \mathcal{H}om_{\mathcal{M}}\big(C, \hat{B}\big) \to \mathcal{H}om_{\mathcal{M}}\big(A, \hat{B}\big)$$

is a weak homotopy equivalence. Finally, using proposition 4.8.14 and the 2-out-of-3 property of weak homotopy equivalences, we conclude that the morphism $\mathcal{H}om_{\mathcal{M}}(f, B) : \mathcal{H}om_{\mathcal{M}}(C, B) \to \mathcal{H}om_{\mathcal{M}}(A, B)$ is indeed a weak homotopy equivalence.

(ii). Remark 4.8.5 says $\mathbf{c}_{\mathrm{w}}\mathcal{M}$ is a derivable category, so theorem 4.4.11 yields the required total right derived functor.

(iii). The total derived functor theorem implies that $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(\mathrm{cosk}^0(A), B\big)$ is naturally isomorphic to the weak homotopy type of $\mathcal{H}om_{\mathcal{M}}\big(\tilde{A}, B\big)$ for any cofibrant replacement $\big(\tilde{A}^\bullet, p^\bullet\big)$ for $\mathrm{cosk}^0(A)$, and proposition 4.8.2 says any such $\big(\tilde{A}^\bullet, p^\bullet\big)$ is a cosimplicial resolution of A, so $\mathbf{R}\mathcal{H}om_{\mathcal{M}}\big(\mathrm{cosk}^0(-), B\big)$ is indeed a derived left hom-space functor. ■

Theorem 4.8.20. *Let $\mathcal{M}$ be a model category, let $\mathbf{c}_{\mathrm{r}}\mathcal{M}$ be the full subcategory of $\mathbf{c}\mathcal{M}$ spanned by the cosimplicial resolutions, and let $\mathbf{s}_{\mathrm{r}}\mathcal{M}$ be the full subcategory of $\mathbf{s}\mathcal{M}$ spanned by the simplicial resolutions. If $\hbar_\bullet : \mathcal{M} \to \left[(\mathbf{c}_{\mathrm{r}}\mathcal{M})^{\mathrm{op}}, \mathbf{sSet}\right]_{\mathrm{h}}$ is the functor defined by*

$$\hbar_B(A) = \mathcal{H}om_{\mathcal{M}}(A, B)$$

where $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the left hom-complex, then:

(i) *$\hbar_\bullet$ sends fibrations (resp. trivial fibrations) in $\mathcal{M}$ to componentwise Kan fibrations (resp. componentwise trivial Kan fibrations).*

(ii) *$\hbar_\bullet$ admits a total right derived functor.*

(iii) *For each cosimplicial resolution $A^\bullet$ in $\mathcal{M}$ and each object B in $\mathcal{M}$, $\mathbf{R}\hbar_B(A)$ is a derived hom-space $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A^0, B\right)$.*

Dually, if $\hbar^\bullet : \mathcal{M}^{\mathrm{op}} \to \left[\mathbf{s}_{\mathrm{r}}\mathcal{M}, \mathbf{sSet}\right]_{\mathrm{h}}$ is the functor defined by

$$\hbar^A(B) = \mathcal{H}om_{\mathcal{M}}(A, B)$$

where $\mathcal{H}om_{\mathcal{M}}(A, B)$ is the right hom-complex, then:

(i′) *$\hbar^\bullet$ sends cofibrations (resp. trivial cofibrations) in $\mathcal{M}$ to componentwise Kan fibrations (resp. componentwise trivial Kan fibrations).*

(ii′) *$\hbar^\bullet$ admits a total right derived functor.*

(iii′) *For each object A in $\mathcal{M}$ and each simplicial resolution $B_\bullet$ in $\mathcal{M}$, $\mathbf{R}\hbar_B(A)$ is a derived hom-space $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A, B_0\right)$.*

Proof. (i). This is proposition 4.8.10; note that corollary 4.8.11 implies that $\hbar_B : \left(\mathbf{c}_{\mathrm{r}}\mathcal{M}\right)^{\mathrm{op}} \to \mathbf{sSet}$ is indeed a homotopical functor.

(ii). Since the weak equivalences in $\left[(\mathbf{c}_{\mathrm{r}}\mathcal{M})^{\mathrm{op}}, \mathbf{sSet}\right]_{\mathrm{h}}$ are componentwise (by definition), we may apply theorem 4.4.11.

(iii). The total derived functor theorem implies that $\mathbf{R}\hbar_B(A)$ is isomorphic to the weak homotopy type of the left hom-complex $\mathcal{H}om_{\mathcal{M}}\left(A, \hat{B}\right)$, where $\left(\hat{B}, i\right)$ is any fibrant replacement for B, so $\mathbf{R}\hbar_B(A)$ is a derived hom-space $\mathbf{R}\mathrm{Hom}_{\mathcal{M}}\left(A^0, B\right)$. ■

— V —

Topics in model categories

5.1 Combinatorial model categories

Prerequisites. §§ 0.2, 0.3, 0.5, 4.1, A.3.

Definition 5.1.1. A **cofibrantly-generated model category** is a complete and cocomplete model category $\mathcal{M}$ such that there exist a set $\mathcal{I}$ of cofibrations and a set $\mathcal{I}'$ of trivial cofibrations satisfying these conditions:

- $(\mathcal{I}, \mathcal{M})$ admits the small object argument, and $\mathrm{cof}_{\mathcal{M}}\, \mathcal{I}$ is the class of all cofibrations in $\mathcal{M}$.
- $(\mathcal{I}', \mathcal{M})$ admits the small object argument, and $\mathrm{cof}_{\mathcal{M}}\, \mathcal{I}'$ is the class of all trivial cofibrations in $\mathcal{M}$.

Remark 5.1.2. By Quillen's small object argument (0.5.12), any cofibrantly-generated model category satisfies axiom CM5* and thus is a DHK model category.

Theorem 5.1.3 (Kan's recognition principle)**.** *Let $\mathcal{M}$ be a complete and cocomplete locally small category, let $\mathcal{W}$ be a subcategory of $\mathcal{M}$ containing all the objects, and let $\mathcal{I}$ and $\mathcal{I}'$ be subsets of* $\mathrm{mor}\, \mathcal{M}$*. Assume the following hypotheses:*

- *$\mathcal{W}$ is closed under retracts and has the 2-out-of-3 property in $\mathcal{M}$.*
- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ both admit the small object argument.*
- $\mathrm{inj}^{\mathcal{M}}\, \mathcal{I} \subseteq \mathcal{W} \cap \mathrm{inj}^{\mathcal{M}}\, \mathcal{I}'$.

- $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}' \subseteq \mathcal{W} \cap \mathrm{cof}_{\mathcal{M}}\,\mathcal{I}$.

If, in addition, either

- $\mathrm{inj}^{\mathcal{M}}\,\mathcal{I} = \mathcal{W} \cap \mathrm{inj}^{\mathcal{M}}\,\mathcal{I}'$*, or*
- $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}' = \mathcal{W} \cap \mathrm{cof}_{\mathcal{M}}\,\mathcal{I}$.

then there exists a unique model structure on $\mathcal{M}$ such that $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}$ is the class of cofibrations, $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}'$ is the class of trivial cofibrations, and $\mathcal{W}$ is the class of weak equivalences.

Proof. See Theorem 11.3.1 in [Hirschhorn, 2003]. □

Theorem 5.1.4 (Kan's lifting theorem)**.** *Let $\mathcal{M}$ be a complete and cocomplete locally small category, let $\mathcal{N}$ be a cofibrantly generated model category. Assume the following hypotheses:*

- *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is an adjunction of categories.*
- *$\mathcal{J}$ is a generating set of cofibrations in $\mathcal{N}$.*
- *$\mathcal{J}'$ is a generating set of trivial cofibrations in $\mathcal{N}$.*
- *$(\mathcal{I}, \mathcal{M})$ and $(\mathcal{I}', \mathcal{M})$ admit the small object argument, where $\mathcal{I}$ and $\mathcal{I}'$ are the following sets:*

$$\mathcal{I} = \{Ff \mid f \in \mathcal{J}\}$$
$$\mathcal{I}' = \{Ff \mid f \in \mathcal{J}'\}$$

- *G sends relative $\mathcal{I}'$-cell complexes in $\mathcal{M}$ to weak equivalences in $\mathcal{N}$.*

Then:

(i) *There is a unique model structure on $\mathcal{M}$ with $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}$ as the class of cofibrations and $\mathrm{cof}_{\mathcal{M}}\,\mathcal{I}'$ as the class of trivial cofibrations.*

(ii) *A morphism $g : A \to B$ in $\mathcal{M}$ is a weak equivalence in this model structure if and only if $Gg : GA \to GB$ is a weak equivalence in $\mathcal{N}$.*

(iii) *$F \dashv G : \mathcal{M} \to \mathcal{N}$ is a Quillen adjunction with respect to this model structure.*

Proof. See Theorem 11.3.2 in [Hirschhorn, 2003]. □

Theorem 5.1.5 (Existence of cofibrantly-generated projective model structures)**.** *Let $\mathcal{M}$ be a cofibrantly-generated model category. If $\mathbb{A}$ is a small category, then the projective model structure on* $[\mathbb{A}, \mathcal{M}]$ *exists and is cofibrantly generated.*

Proof. See Theorem 11.6.1 in [Hirschhorn, 2003]. □

Definition 5.1.6. A **combinatorial model category** is a cofibrantly-generated model category that is also a locally presentable category.

Remark 5.1.7. Since locally presentable categories are automatically complete and cocomplete,[1] in light of remark 0.5.9, to show that a locally presentable model category $\mathcal{M}$ is a combinatorial model category, it is enough to verify that there exist sets $\mathcal{I}$ and $\mathcal{I}'$ such that $\operatorname{cof}_{\mathcal{M}} \mathcal{I}$ is the class of all cofibrations in $\mathcal{M}$ and $\operatorname{cof}_{\mathcal{M}} \mathcal{I}'$ is the class of all trivial cofibrations in $\mathcal{M}$.

Theorem 5.1.8 (Existence of combinatorial injective model structures)**.** *Let $\mathcal{M}$ be a combinatorial model category. If $\mathbb{A}$ is a small category, then the injective model structure on* $[\mathbb{A}, \mathcal{M}]$ *exists and is combinatorial.*

Proof. This theorem is due to Lurie; see [HTT, Proposition A.2.8.2]. □

Definition 5.1.9. Let κ and λ be regular cardinals. A **strongly (κ, λ)-combinatorial model category** is a combinatorial model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.
- $\mathbf{K}_{\lambda}(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.
- Each hom-set in $\mathbf{K}_{\kappa}(\mathcal{M})$ is λ-small.
- There exist λ-small sets of morphisms in $\mathbf{K}_{\kappa}(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

Proposition 5.1.10. *For any combinatorial model category $\mathcal{M}$, there exist regular cardinals κ and λ and functorial factorisation systems making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category.*

Proof. Apply proposition 0.2.32, lemma 0.2.35, remark 0.3.4. ■

[1] See theorem 0.2.37.

Theorem 5.1.11. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category.*

(i) *There exist (trivial cofibration, fibration)- and (cofibration, trivial fibration)-factorisation functors that are κ-accessible and strongly λ-accessible.*

(ii) *Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be the full subcategory of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations (resp. trivial fibrations). Then $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.*

Proof. (i). Since the weak factorisation systems on $\mathcal{M}$ are cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$, and $\mathbf{K}_\kappa(\mathcal{M})$ is locally λ-small, we may apply the small object argument of either Quillen (theorem 0.5.12 and corollary 0.5.14) or Garner (proposition 0.5.23 and theorem 0.5.24) to obtain the required functorial weak factorisation systems.

(ii). This is corollary 0.5.27 says that $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$. ■

Theorem 5.1.12. *Let (L', R) and (L, R') be functorial weak factorisation systems on a locally presentable category $\mathcal{M}$ and let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the morphisms in the right class of of the weak factorisation systems induced by (L', R) and (L, R'), respectively. Suppose κ and λ are regular cardinals satisfying the following hypotheses:*

- *$\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.*
- *$\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.*
- *$R, R' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ preserve colimits for small κ-filtered diagrams and are strongly λ-accessible functors.*

Let $\mathcal{W}$ be the preimage of $\mathcal{F}'$ under the functor $R : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$. Then:

(i) *The functorial weak factorisation systems (L', R) and (L, R') restrict to functorial weak factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$.*

(ii) *The inclusions $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible functors.*

(iii) *$\mathcal{W}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

(iv) $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ *if and only if the same holds in* $\mathbf{K}_\lambda(\mathcal{M})$.

(v) $\mathcal{W}$ *(regarded as a class of morphisms in* $\mathcal{M}$*) has the 2-out-of-3 property in* $\mathcal{M}$ *if and only if the same is true in* $\mathbf{K}_\lambda(\mathcal{M})$.

(vi) *The weak factorisation systems induced by* (L', R) *and* (L, R') *underlie a model structure on* $\mathcal{M}$ *if and only if the restrictions to* $\mathbf{K}_\lambda(\mathcal{M})$ *underlie a model structure on* $\mathbf{K}_\lambda(\mathcal{M})$.

Proof. (i). It is clear that we can restrict (L', R) and (L, R') to obtain functorial factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$, and these are functorial *weak* factorisation systems by theorem A.3.29.

(ii). Since $R, R' : [\mathbb{2}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible, we may use proposition 0.5.28 to deduce that the inclusions $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible.

(iii). Since $\mathcal{F}'$ is a replete subcategory of $[\mathbb{2}, \mathcal{M}]$, we may useproposition 0.3.29 to deduce that $\mathcal{W}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$ and that the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.

(iv). Claims (ii) and (iii) and proposition 0.3.29 imply the inclusion $\mathcal{W} \cap \mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is strongly λ-accessible; but by propositions 0.2.44 and 0.3.28,

$$\mathbf{K}_\lambda(\mathcal{F}') = \mathcal{F}' \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right] \qquad \mathbf{K}_\lambda(\mathcal{W} \cap \mathcal{F}) = (\mathcal{W} \cap \mathcal{F}) \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$$

so $\mathcal{F}' = \mathcal{W} \cap \mathcal{F}$ if and only if $\mathcal{F}' \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right] = (\mathcal{W} \cap \mathcal{F}) \cap \left[\mathbb{2}, \mathbf{K}_\lambda(\mathcal{M})\right]$.

(v). Consider the three full subcategories $\Lambda^2_i(\mathcal{W})$ (where $i \in \{0, 1, 2\}$) of $[\mathbb{3}, \mathcal{M}]$ spanned (respectively) by the diagrams of the form below:

$$\begin{tikzcd} \bullet \arrow[r, "\in \mathcal{W}"] \arrow[dr, "\in \mathcal{W}"'] & \bullet \arrow[d] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, "\in \mathcal{W}"] \arrow[dr] & \bullet \arrow[d, "\in \mathcal{W}"] \\ & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[dr, "\in \mathcal{W}"'] & \bullet \arrow[d, "\in \mathcal{W}"] \\ & \bullet \end{tikzcd}$$

By proposition 0.3.15, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbb{3}, \mathcal{M}]$ a strongly λ-accessible inclusion of a full subcategory of $[\mathbb{2}, \mathcal{M}]^{\times 3}$ along the evident projection functor $[\mathbb{3}, \mathcal{M}] \to [\mathbb{2}, \mathcal{M}]^{\times 3}$; thus, each inclusion $\Lambda^2_i(\mathcal{W}) \hookrightarrow [\mathbb{3}, \mathcal{M}]$ is a strongly λ-accessible functor. We may then use proposition 0.3.28 as above to prove the claim.

(vi). Apply lemmas 4.1.10 and 4.1.11 and theorem 4.1.12. ■

Corollary 5.1.13. *Let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model category. Then the full subcategory $\mathcal{W}$ of $[\mathbb{2}, \mathcal{M}]$ spanned by the weak equivalences is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

Proof. Combine theorems 5.1.11 and 5.1.12. ■

Definition 5.1.14. Let κ and λ be regular cardinals. A (κ, λ)**-compact model category** is a model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a (κ, λ)-compactly generated category, and $\kappa \lhd \lambda$.
- $\mathcal{M}$ has limits for finite diagrams and colimits for λ-small diagrams.
- Each hom-set in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ is λ-small.
- There exist λ-small sets of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{M})$ that cofibrantly generate the model structure of $\mathcal{M}$.

Proposition 5.1.15. *If $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, then $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-compact model category (with the weak equivalences, cofibrations, and fibrations inherited from $\mathcal{M}$).*

Proof. By proposition 0.3.7, $\mathbf{K}_\lambda(\mathcal{M})$ is a (κ, λ)-compactly generated category, and lemma 0.2.15 implies it is closed under colimits for λ-small diagrams in $\mathcal{M}$. Now, choose a pair of functorial factorisation systems as in theorem 5.1.11, and recall that theorem A.3.29 says a morphism is in the left (resp. right) class of a functorial weak factorisation system if and only if it is a retract of the left (resp. right) half of its functorial factorisation. Since we chose factorisation functors that are strongly λ-accessible, it follows that the weak factorisation system on $\mathcal{M}$ restricts to a weak factorisation system on $\mathbf{K}_\lambda(\mathcal{M})$. It is then clear that $\mathbf{K}_\lambda(\mathcal{M})$ inherits a model structure from $\mathcal{M}$, and lemma 0.5.30 implies the model structure on $\mathbf{K}_\lambda(\mathcal{M})$ can be cofibrantly generated by λ-small sets of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$. The remaining axioms for a λ-compact model category are easily verified. ■

Proposition 5.1.16. *Let $\mathcal{K}$ be a (κ, λ)-compact model category and let $\mathcal{M}$ be the λ-ind-completion $\mathbf{Ind}^\lambda(\mathcal{K})$. Then there is a unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category such that the canonical embedding $\mathcal{K} \to \mathcal{M}$ preserves and reflects the model structure.*

Proof. We will regard $\mathcal{K}$ as a full subcategory of $\mathcal{M}$ via the canonical embedding $\mathcal{K} \to \mathcal{M}$. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be a λ-small set of morphisms in $\mathbf{K}_\kappa^\lambda(\mathcal{K})$ that generate the cofibrations (resp. trivial cofibrations) in $\mathcal{K}$. Let (L', R) and (L, R') be functorial weak factorisation systems cofibrantly generated by $\mathcal{I}'$ and $\mathcal{I}$ respectively; by corollary 0.5.14, we may assume $R, R' : [\mathbb{2}, \mathcal{K}] \to [\mathbb{2}, \mathcal{K}]$ are λ-accessible functors and preserve colimits for small κ-filtered diagrams.

Let $\mathcal{F}$ and $\mathcal{F}'$ be the full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the right class of the weak factorisation systems induced by (L', R) and (L, R'), respectively. It is not hard to see that any morphism in $\mathcal{K}$ is an object in $\mathcal{F}$ (resp. $\mathcal{F}'$) if and only if it is a fibration (resp. trivial fibration) in $\mathcal{K}$. Corollary 0.5.27 says $\mathcal{F}$ and $\mathcal{F}'$ are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, so we may now apply theorem 5.1.12 to deduce that $\mathcal{F}$ and $\mathcal{F}'$ induce a model structure on $\mathcal{M}$. It is clear that $\mathcal{M}$ equipped with this model structure is then a strongly (κ, λ)-combinatorial model category in a way that is compatible with the canonical embedding $\mathcal{K} \to \mathcal{M}$.

Finally, to see that the above construction is the unique way of making $\mathcal{M}$ into a strongly (κ, λ)-combinatorial model category satisfying the given conditions, we simply have to observe that the model structure of a strongly (κ, λ)-combinatorial model category is necessarily cofibrantly generated by the cofibrations and trivial cofibrations in (a small skeleton of) $\mathbf{K}_\kappa(\mathcal{M})$ (independently of the choice of $\mathcal{I}$ and $\mathcal{I}'$). ■

Remark 5.1.17. Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, let $\mathcal{M}$ be a strongly (κ, λ)-combinatorial model $\mathbf{U}$-category, and let $\mathcal{M} \hookrightarrow \mathcal{M}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-extension. By combining propositions 5.1.15 and 5.1.16, we may deduce that there is a unique way of making $\mathcal{M}^+$ into a strongly (κ, λ)-combinatorial model $\mathbf{U}^+$-category such that the embedding $\mathcal{M} \hookrightarrow \mathcal{M}^+$ preserves and reflects the model structure. In other words, combinatorial model categories are stable under universe enlargement.

5.2 Algebraic model categories

Prerequisites. §§ 0.2, 0.3, 0.5, 4.1, 5.1, A.3.

Though model categories equipped with functorial factorisations are better-behaved than general model categories, one can often extract a bit more structure by using Garner's small object argument (theorem 0.5.24). This leads to the notion of 'algebraic model structure', due to Riehl [2011a,b].

Definition 5.2.1. Let $\mathcal{M}$ be a category. An **algebraic model structure** on $\mathcal{M}$ consists of a pair of algebraic factorisation systems $(\mathsf{L}', \mathsf{R})$ and $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{M}$ and a morphism $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$ satisfying the following condition:

- There exists a model structure on $\mathcal{M}$ such that the cofibrations are the left class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$ and the fibrations are the right class of the weak factorisation system induced by $(\mathsf{L}', \mathsf{R})$.

An **algebraic model category** is a category with an algebraic model structure and limits and colimits for all finite diagrams.

The following lemma, originally part of Theorem 3.8 in [Riehl, 2011b], is useful in the construction of algebraic model structures:

Lemma 5.2.2. *Let $\mathcal{M}$ be a category with a model structure, let $(\mathsf{L}, \mathsf{R}')$ be an algebraic factorisation system on $\mathcal{M}$, and suppose $\mathcal{I}'$ is a generating set of trivial cofibrations in $\mathcal{M}$. If the left class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$ is the class of cofibrations, then there exists a subset $\tilde{\mathcal{I}}'$ with the following properties:*

- *$\tilde{\mathcal{I}}'$ has at most as many elements as $\mathcal{I}'$.*
- *The weak factorisation system on $\mathcal{M}$ cofibrantly generated by $\mathcal{I}'$ coincides with the one cofibrantly generated by $\mathcal{I}$.*
- *Each morphism in $\tilde{\mathcal{I}}'$ can be equipped with an L-coalgebra structure.*

In particular, if $(\mathsf{L}', \mathsf{R})$ is a free algebraic factorisation system cofibrantly generated by $\tilde{\mathcal{I}}'$, then there must exist a morphism $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$.

Proof. Let $\tilde{\mathcal{I}}' = \{ Le \,|\, e \in \mathcal{I}' \}$. Since L is a comonad, every morphism in $\tilde{\mathcal{I}}'$ admits an L-coalgebra structure. Consider the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} Z \arrow[r, "Le"] \arrow[d, "e"'] & W' \arrow[d, "R'e"] \\ W \arrow[r, "\mathrm{id}"'] & W \end{tikzcd}$$

Since $e : Z \to W$ is a trivial cofibration and $Re : W' \to W$ is a trivial fibration, there exists a morphism $i : W \to W'$ filling in the diagram. Hence, every morphism in $\mathcal{I}'$ is a retract of one in $\tilde{\mathcal{I}}'$, so by propositions A.3.3 and A.3.12, we have $\tilde{\mathcal{I}}'^{\boxslash} \subseteq \mathcal{I}'^{\boxslash}$. On the other hand, axiom CM2 implies $Le : Z \to W'$ is a trivial cofibration, and so $\tilde{\mathcal{I}}' \subseteq {}^{\boxslash}(\mathcal{I}'^{\boxslash})$. Thus, we have $\mathcal{I}'^{\boxslash} \subseteq \tilde{\mathcal{I}}'^{\boxslash}$ as well. ■

Proposition 5.2.3. *Let $\mathcal{M}$ be a combinatorial model category, let $\mathcal{I}$ be a set of generating cofibrations in $\mathcal{M}$, and let $\mathcal{I}'$ be a set of generating trivial cofibrations in $\mathcal{M}$.*

(i) *$\mathcal{I}$ cofibrantly generates an algebraically free algebraic factorisation system $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{M}$.*

(ii) *There exists a set $\tilde{\mathcal{I}}'$ of generating trivial cofibrations in $\mathcal{M}$ such that $\tilde{\mathcal{I}}'$ cofibrantly generates an algebraically free algebraic factorisation system $(\mathsf{L}', \mathsf{R})$ on $\mathcal{M}$ with a morphism $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$.*

In particular, $\mathcal{M}$ is the underlying model category of an algebraic model category.

Proof. (i). Apply Garner's small object argument (theorem 0.5.24).

(ii). Use lemma 5.2.2. ■

Definition 5.2.4. Let κ and λ be regular cardinals. A **strongly** (κ, λ)**-algebraic model category** is an algebraic model category $\mathcal{M}$ that satisfies these axioms:

- $\mathcal{M}$ is a locally κ-presentable category, and $\kappa \lhd \lambda$.
- $\mathbf{K}_\lambda(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.
- The underlying endofunctors of the two given algebraic factorisation systems on $\mathcal{M}$ preserve colimits for small κ-filtered diagrams and are strongly λ-accessible functors.
- The full subcategory $\mathcal{F}$ (resp. $\mathcal{F}'$) of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations (resp. trivial fibrations) in $\mathcal{M}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$.

Proposition 5.2.5. *If $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category, then there exist algebraic factorisation systems making $\mathcal{M}$ a strongly (κ, λ)-algebraic model category.*

Proof. Let $\mathcal{I}$ (resp. $\mathcal{I}'$) be a λ-small set of morphisms in $\mathbf{K}_\kappa(\mathcal{M})$ that generate the cofibrations (resp. trivial cofibrations) in $\mathcal{M}$. Replacing $\mathcal{I}$ with $\mathcal{I} \cup \mathcal{I}'$ if necessary, we may assume $\mathcal{I}' \subseteq \mathcal{I}$. Garner's small object argument (0.5.24) says that algebraically free algebraic factorisation systems cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$ exist and are free, and since $\mathcal{I}' \subseteq \mathcal{I}$, the universal property of free algebraic factorisation systems ensures we have the required morphism of algebraic factorisation systems. Lemma 0.3.32 and proposition 0.5.23 then say that the underlying endofunctors of the algebraic factorisation systems preserve colimits for small κ-filtered diagrams and are strongly λ-accessible. Finally, by corollary 0.5.27, the two full subcategories of $[\mathbb{2}, \mathcal{M}]$ spanned by the fibrations and trivial fibrations are closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$. ■

Theorem 5.2.6. *Let $\mathcal{M}$ be a strongly (κ, λ)-algebraic model category.*

(i) *The algebraic model structure on $\mathcal{M}$ restricts to an algebraic model structure on $\mathbf{K}_\lambda(\mathcal{M})$.*

(ii) *The inclusions $\mathcal{F} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ and $\mathcal{F}' \hookrightarrow [\mathbb{2}, \mathcal{M}]$ are strongly λ-accessible functors.*

(iii) *$\mathcal{W}$ is closed under colimits for small κ-filtered diagrams in $[\mathbb{2}, \mathcal{M}]$, and the inclusion $\mathcal{W} \hookrightarrow [\mathbb{2}, \mathcal{M}]$ is a strongly λ-accessible functor.*

Proof. (i). By definition, the underlying endofunctors of the given algebraic factorisation systems are strongly λ-accessible and so send morphisms in $\mathbf{K}_\lambda(\mathcal{M})$ back to $\mathbf{K}_\lambda(\mathcal{M})$. Thus, we obtain algebraic factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$, and it is clear that the given morphism of algebraic factorisation systems on $\mathcal{M}$ restricts to a morphism of algebraic factorisation systems on $\mathbf{K}_\lambda(\mathcal{M})$. Since $\mathbf{K}_\lambda(\mathcal{M})$ is a full subcategory of $\mathcal{M}$, it follows that the restricted data define an algebraic model structure on $\mathbf{K}_\lambda(\mathcal{M})$.

(ii) and (iii). Apply theorem 5.1.12. ■

5.3 Cisinski model categories

Prerequisites. § 0.5, 3.1, 3.5, 4.1, 5.1, A.3, A.4.

In this section we follow [Cisinski, 2002] and [Cisinski, 2006, Ch. 1].

Definition 5.3.1. A **Cisinski model category** is a combinatorial model category whose underlying category is a Grothendieck topos and whose cofibrations are the monomorphisms.

REMARK 5.3.2. Grothendieck toposes are always locally presentable categories, so we may replace ‘combinatorial’ with ‘cofibrantly-generated’ in the above definition.

Example 5.3.3. The Kan–Quillen model structure on **sSet** makes it into a Cisinski model category.

REMARK 5.3.4. In any topos, the unique morphism $0 \to X$ is always a monomorphism; thus, in a Cisinski model category, every object is cofibrant.

Proposition 5.3.5. *Let $f : X \to Y$ and $g : Z \to W$ be cofibrations in a Cisinski model category $\mathcal{M}$. Suppose the square in the diagram below is a pushout square in $\mathcal{M}$:*

$$\begin{tikzcd}
X \times Z \arrow[r, "\mathrm{id}_X \times g"] \arrow[d, "f \times \mathrm{id}_Z"'] & X \times W \arrow[d] \arrow[ddr, bend left, "f \times \mathrm{id}_W"] & \\
Y \times Z \arrow[r] \arrow[drr, bend right, "\mathrm{id}_Y \times g"'] & (X \times W) \cup^{X \times Z} (Y \times Z) \arrow[dr, dashed, "f \mathbin{\square} g"] & \\
& & Y \times W
\end{tikzcd}$$

(i) *The unique morphism $f \mathbin{\square} g$ making the diagram commute is a cofibration.*

(ii) *Assuming the class of trivial cofibrations in $\mathcal{M}$ is closed under binary products, if either f or g is a trivial cofibration, then $f \mathbin{\square} g$ is a trivial cofibration.*

Proof. (i). The claim is certainly true when $\mathcal{M}$ is a presheaf topos, and since the associated sheaf functor preserves colimits and finite limits, the claim holds for all sheaf toposes as well.

(ii). The two cases are symmetrical; we will assume $f : X \to Y$ is a trivial cofibration. Clearly, $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ and $f \times \mathrm{id}_W : X \times W \to Y \times W$ are monomorphisms, so the hypothesis implies they are trivial cofibrations. The class of trivial cofibrations is closed under pushouts (by proposition A.3.12), so the morphism $X \times W \to (X \times W) \cup^{X \times Z} (Y \times Z)$ is also a trivial cofibration. The 2-out-of-3 property of weak equivalences then implies $f \boxdot g$ must be a weak equivalence as well; hence, by claim (i), it is a trivial cofibration. ■

¶ 5.3.6. We will now see how to build Cisinski model structures. Throughout this section, $\mathcal{M}$ will be a Grothendieck topos, say $\mathcal{M} = \mathbf{Sh}(\mathbb{C}, J)$ for a small category $\mathbb{C}$ equipped with a Grothendieck topology J.

Definition 5.3.7. A **Cisinski cylinder functor** for $\mathcal{M}$ is a quadruple $\left(I, \iota^0, \iota^1, \rho\right)$ where $I : \mathcal{M} \to \mathcal{M}$ is a functor, $\iota^0, \iota^1 : \mathrm{id}_{\mathcal{M}} \Rightarrow I$ and $\rho : I \Rightarrow \mathrm{id}_{\mathcal{M}}$ are natural transformations, such that:

- $\rho \bullet \iota^0 = \rho \bullet \iota^1 = \mathrm{id}_{\mathrm{id}_{\mathcal{M}}}$.
- The induced morphism $\iota_X = \left(\!\left(\iota^0_X, \iota^1_X\right)\!\right) : X \amalg X \to IX$ is a monomorphism for every object X in $\mathcal{M}$.

We will often abuse notation and simply say that I is a cylinder functor, with the natural transformations ι^0, ι^1, and ρ understood.

Remark 5.3.8. By symmetry, $\left(I, \iota^0, \iota^1, \rho\right)$ is a Cisinski cylinder functor if and only if $\left(I, \iota^1, \iota^0, \rho\right)$ is a Cisinski cylinder functor.

Definition 5.3.9. Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor for $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$. An **I-homotopy** in $\mathcal{M}$ from f_0 to f_1 is a morphism $H : IX \to Y$ such that $H \circ \iota^0_X = f_0$ and $H \circ \iota^1_X = f_1$. We say f_0 and f_1 are **I-homotopic** if there is a zigzag of I-homotopies connecting f_0 to f_1.

Proposition 5.3.10. *Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor for $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$.*

(i) *For any morphism $g : Y \to Z$ in $\mathcal{M}$, if f_0 and f_1 are I-homotopic, then so are $g \circ f_0$ and $g \circ f_1$.*

(ii) *For any morphism $g : W \to X$ in $\mathcal{M}$, if f_0 and f_1 are I-homotopic, then so are $f_0 \circ g$ and $f_1 \circ g$.*

Proof. Obvious. ◆

Definition 5.3.11. Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor for $\mathcal{M}$. The **I-homotopy category** of $\mathcal{M}$ is the category $\mathrm{Ho}_I\, \mathcal{M}$ defined below:

- The objects of $\mathrm{Ho}_I\,\mathcal{M}$ are those of $\mathcal{M}$.
- The hom-set $\mathrm{Ho}_I\,\mathcal{M}(X,Y)$ is $\mathcal{M}(X,Y)$ modulo I-homotopy.
- Composition and identities are inherited from $\mathcal{M}$.

Proposition 5.3.12. *Let $\left(I, \iota^0, \iota^1, \rho\right)$ be a Cisinski cylinder functor and let $\gamma : \mathcal{M} \to \mathrm{Ho}_I\,\mathcal{M}$ be the functor that sends a morphism in $\mathcal{M}$ to its I-homotopy class.*

(i) *The functor $\gamma : \mathcal{M} \to \mathrm{Ho}_I\,\mathcal{M}$ is full.*

(ii) *Let $\mathcal{H}$ be the class of morphisms in $\mathcal{M}$ that γ sends to isomorphisms. If $\gamma\rho : \gamma I \Rightarrow \gamma$ is a natural isomorphism, then $\gamma : \mathcal{M} \to \mathrm{Ho}_I\,\mathcal{M}$ exhibits $\mathrm{Ho}_I\,\mathcal{M}$ as a localisation of $\mathcal{M}$ at $\mathcal{H}$.*

Proof. (i). Obvious.

(ii). Consider any functor $F : \mathcal{M} \to \mathcal{C}$ such that $F\rho : FI \Rightarrow F$ is a natural isomorphism. Then, we have $F\iota^0 = F\iota^1$, so F factors through $\gamma : \mathcal{M} \to \mathrm{Ho}_I\,\mathcal{M}$ in a unique way. In particular, if $\gamma\rho : \gamma I \Rightarrow \gamma$ itself is a natural isomorphism, then $\mathrm{Ho}_I\,\mathcal{M}$ has the universal property of a localisation of $\mathcal{M}$ at $\mathcal{H}$. ■

Definition 5.3.13. A **Cisinski trivial fibration** in $\mathcal{M}$ is a morphism that has the right lifting property with respect to all monomorphisms.

Proposition 5.3.14. *Let $p : X \to Y$ be a Cisinski trivial fibration in $\mathcal{M}$.*

(i) *There exists a morphism $s : Y \to X$ such that $p \circ s = \mathrm{id}_Y$.*

(ii) *For any such $s : Y \to X$ and any Cisinski cylinder functor $\left(I, \iota^0, \iota^1, \rho\right)$ for $\mathcal{M}$, there exists an I-homotopy from id_X to $s \circ p$.*

(iii) *The morphism $p : X \to Y$ becomes an isomorphism in $\mathrm{Ho}_I\,\mathcal{M}$.*

Proof. (i). The unique morphism $0 \to Y$ is a monomorphism in any topos, so the right lifting property of $p : X \to Y$ guarantees the existence of a section.

(ii). Consider the following commutative diagram in $\mathcal{M}$:

$$\begin{tikzcd} X \amalg X \arrow[r, "{(\mathrm{id}_X, s\circ p)}"] \arrow[d, "\iota_X"'] & X \arrow[d, "p"] \\ IX \arrow[r, "p\circ\rho_X"'] & Y \end{tikzcd}$$

By definition, $\iota_X : X \amalg X \to IX$ is a monomorphism, so the right lifting property of $p : X \to Y$ yields a morphism $H : IX \to X$ such that $H \circ \iota_X = (\!(\mathrm{id}_X, s \circ p)\!)$ and $p \circ H = p \circ \rho_X$; in particular, H is an I-homotopy from id_X to $s \circ p$.

(iii). Clearly, the morphisms $p : X \to Y$ and $s : Y \to X$ become mutual inverses in $\mathrm{Ho}_I\, \mathcal{M}$. ■

¶ 5.3.15. Let Ω be a subobject classifier for $\mathcal{M}$ and let $\top, \bot : 1 \to \Omega$ be the morphisms classifying the top and bottom subobjects of 1, respectively. Then the following diagram is a pullback square by definition,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[d, "\top"] \\ 1 \arrow[r, "\bot"'] & \Omega \end{tikzcd}$$

so the induced morphism $(\!(\top, \bot)\!) : 1 \amalg 1 \to \Omega$ is a monomorphism. Since monomorphisms are stable under pullback, the following definition is legitimate:

Definition 5.3.16. The **Lawvere cylinder functor** for $\mathcal{M}$ is the cylinder functor $(I, \iota^0, \iota^1, \rho)$ defined below:

- $I : \mathcal{M} \to \mathcal{M}$ is the functor $\Omega \times -$.
- The morphism $\iota^0_X : X \to \Omega \times X$ corresponds to $\top \times \mathrm{id}_X$.
- The morphism $\iota^1_X : X \to \Omega \times X$ corresponds to $\bot \times \mathrm{id}_X$.
- The morphism $\rho_X : \Omega \times X \to X$ is the product projection.

Proposition 5.3.17. *Let X be any object in $\mathcal{M}$ and let Ω be the subobject classifier for $\mathcal{M}$.*

(i) *The product projection $p_X : \Omega \times X \to X$ is a Cisinski trivial fibration.*

(ii) *For any Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$, there exists a commutative diagram of the following form:*

$$\begin{tikzcd} X \amalg X \arrow[r, "(\!(\top,\bot)\!) \times \mathrm{id}_X"] \arrow[d, "\iota_X"'] & \Omega \times X \arrow[d, "p_X"] \\ IX \arrow[r, "\rho_X"'] \arrow[ur, dashed] & X \end{tikzcd}$$

Proof. (i). Since the class of Cisinski trivial fibrations is closed under pullbacks (by proposition A.3.12), it suffices to show that the morphism $p_1 : \Omega \times 1 \to 1$ is a trivial fibration. However, Ω is canonically an injective object in $\mathcal{M}$ (with respect to the class of monomorphisms), i.e. the unique morphism $\Omega \to 1$ has the right lifting property with respect to all monomorphisms, so p_1 is indeed a Cisinski trivial fibration.

(ii). This follows from claim (i) and the requirement that $\iota_X : X \amalg X \to IX$ be a monomorphism. ■

Remark 5.3.18. Thus, any pair of morphisms that are homotopic with respect to the Lawvere cylinder functor must also be I-homotopic for any Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$.

Definition 5.3.19. An **elementary Cisinski homotopy structure** on $\mathcal{M}$ is a Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$ satisfying these axioms:

DH1. The functor $I : \mathcal{M} \to \mathcal{M}$ preserves monomorphisms and colimits for all small diagrams.

DH2. For all monomorphisms $g : Z \to W$ in $\mathcal{M}$, the following diagrams are pullback squares:

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\iota^0_Z"'] & W \arrow[d, "\iota^0_W"] \\ IZ \arrow[r, "Ig"'] & IW \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "\iota^1_Z"'] & W \arrow[d, "\iota^1_W"] \\ IZ \arrow[r, "Ig"'] & IW \end{tikzcd}$$

Proposition 5.3.20. *The Lawvere cylinder functor is an elementary Cisinski homotopy structure.*

Proof. The functor $A \times -$ always preserves monomorphisms, and toposes are cartesian closed, so for any object A in $\mathcal{M}$, the functor $A \times -$ preserves colimits. Thus the Lawvere cylinder functor satisfies axiom DH1. It is clear that axiom DH2 is also satisfied. ■

Definition 5.3.21. Let $(I, \iota^0, \iota^1, \rho)$ be an elementary Cisinski homotopy structure on $\mathcal{M}$. A **class of I-anodyne extensions** is a class $\mathcal{A}$ of morphisms in $\mathcal{M}$ satisfying these axioms:

An0. There exists a subset $\Lambda \subseteq \mathcal{A}$ such that the members of Λ are monomorphisms in $\mathcal{M}$ and $\mathcal{A} = {}^{\boxslash}\big(\Lambda^{\boxslash}\big)$. We say Λ is a **generating set** for $\mathcal{A}$.

An1. If $g : Z \to W$ is a monomorphism in $\mathcal{M}$ and $e \in \{0, 1\}$, then given a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\iota^e_Z"] \arrow[d, "g"'] & IZ \arrow[d] \arrow[ddr, bend left, "Ig"] & \\ W \arrow[r] \arrow[drr, bend right, "\iota^e_W"'] & V_e(g) \arrow[dr, dashed, "j_e"] & \\ & & IW \end{tikzcd}$$

where the top-left square is a pushout square, $j_e : V_e(g) \to IW$ is in $\mathcal{A}$.

An2. If $g : Z \to W$ is in $\mathcal{A}$, then given a commutative diagram

$$\begin{tikzcd} Z \amalg Z \arrow[r, "\iota_Z"] \arrow[d, "g"'] & IZ \arrow[d] \arrow[ddr, bend left, "Ig"] & \\ W \amalg W \arrow[r] \arrow[drr, bend right, "\iota_W"'] & U(g) \arrow[dr, dashed, "j"] & \\ & & IW \end{tikzcd}$$

where the top-left square is a pushout square, $j : U(g) \to IW$ is in $\mathcal{A}$.

Remark 5.3.22. Since I preserves colimits for all small diagrams, $I0$ must be an initial object in $\mathcal{M}$. Thus, by taking $Z = 0$, we see that the morphisms $\iota^0_W, \iota^1_W : W \to IW$ are always in any class of I-anodyne extensions.

Proposition 5.3.23. *Let $\big(I, \iota^0, \iota^1, \rho\big)$ be an elementary Cisinski homotopy structure on $\mathcal{M}$, let $\mathcal{A}$ be a class of I-anodyne extensions, and let Λ be a generating set for $\mathcal{A}$.*

(i) *There exists a functorial factorisation system on $\mathcal{M}$ with $\mathcal{A}$ as its left class.*

(ii) *$\mathcal{A}$ is the smallest class of morphisms containing Λ that is closed under pushouts, transfinite composition, and retracts.*

(iii) *Every morphism that is in $\mathcal{A}$ is a monomorphism.*

Proof. (i). Apply Quillen's small object argument (theorem 0.5.12).

(ii). This is corollary 0.5.13.

(iii). The class of monomorphisms in a Grothendieck topos is closed under pushouts, transfinite composition, and retracts because the class of injections in **Set** is closed under the same operations. Since Λ is a collection of monomorphisms, so too is $\mathcal{A}$. ■

Definition 5.3.24. A **Cisinski homotopy structure** on $\mathcal{M}$ is an elementary Cisinski homotopy structure on $\mathcal{M}$ together with a class of anodyne extensions.

Definition 5.3.25. Let $\mathcal{A}$ be the class of anodyne extensions of a Cisinski homotopy structure on $\mathcal{M}$. An **$\mathcal{A}$-fibrant object** in $\mathcal{M}$ is an object X such that the unique morphism $X \to 1$ has the right lifting property with respect to $\mathcal{A}$.

Definition 5.3.26. Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. A **weak equivalence** with respect to $(I, \mathcal{A})$ is a morphism $f : W \to Z$ in $\mathcal{M}$ such that, for every $\mathcal{A}$-fibrant object X, the induced map

$$\mathrm{Ho}_I\, \mathcal{M}(f, X) : \mathrm{Ho}_I\, \mathcal{M}(Z, X) \to \mathrm{Ho}_I\, \mathcal{M}(W, X)$$

is a bijection of sets.

Proposition 5.3.27. *$\mathcal{M}$ together with the class of weak equivalences with respect to a Cisinski homotopy structure $(I, \mathcal{A})$ constitute a saturated homotopical category.*

Proof. Obvious. ♦

Proposition 5.3.28. *Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. Then every morphism in $\mathcal{A}$ is a weak equivalence with respect to $(I, \mathcal{A})$.*

Proof. See Proposition 2.23 in [Cisinski, 2002]. □

Corollary 5.3.29. *Let $\mathcal{W}$ be the class of weak equivalences with respect to $(I, \mathcal{A})$ and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}$. If f_0 and f_1 are I-homotopic, then f_0 and f_1 become equal in* $\mathrm{Ho}(\mathcal{M}, \mathcal{W})$.

Proof. It suffices to verify the case where there is an I-homotopy $H : IX \to Y$ from f_0 to f_1. By remark 5.3.22, the morphisms $\iota_X^0, \iota_X^1 : X \to IX$ are anodyne extensions, and so are invertible in $\operatorname{Ho}(\mathcal{M}, \mathcal{W})$. We have $\rho_X \circ \iota_X^0 = \rho_X \circ \iota_X^1 = \mathrm{id}_X$ by definition, so ι_X^0 and ι_X^1 must be equal in $\operatorname{Ho}(\mathcal{M}, \mathcal{W})$; but $H \circ \iota_X^0 = f_0$ and $H \circ \iota_X^1 = f_1$, so f_0 and f_1 must be equal in $\operatorname{Ho}(\mathcal{M}, \mathcal{W})$. ■

Theorem 5.3.30. *Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. Then $\mathcal{M}$ is a combinatorial model category where*

- *the cofibrations are the monomorphisms in $\mathcal{M}$,*
- *the weak equivalences are the weak equivalences with respect to $(I, \mathcal{A})$, and*
- *the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.*

This is the **Cisinski model structure** *on $\mathcal{M}$ defined by $(I, \mathcal{A})$.*

Proof. See Théorème 2.13 in [Cisinski, 2002]. □

Definition 5.3.31. An **$\mathcal{M}$-localiser** is a class $\mathcal{W}$ of morphisms in $\mathcal{M}$ satisfying the following axioms:

L1. $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$.

L2. Every Cisinski trivial fibration is in $\mathcal{W}$.

L3. The class of monomorphisms that are in $\mathcal{W}$ is closed under pushout and transfinite composition.

A **generating set** for $\mathcal{W}$ is a set S such that $\mathcal{W}$ is the smallest $\mathcal{M}$-localiser containing S. An **accessible $\mathcal{M}$-localiser** is an $\mathcal{M}$-localiser that admits a generating set.

Proposition 5.3.32. *Let $\mathcal{W}$ be a class of morphisms in $\mathcal{M}$ satisfying the following axioms:*

FS1. *For any object X in $\mathcal{M}$, the morphism* $\mathrm{id} : X \to X$ *is in $\mathcal{W}$.*

FS2. *$\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{M}$.*

FS3. *$\mathcal{W}$ has the special 2-out-of-4 property in $\mathcal{M}$.*

Then the following are equivalent:

(i) *Every Cisinski trivial fibration is in $\mathcal{W}$.*

(ii) *Let $(I, \iota^0, \iota^1, \rho)$ be the Lawvere cylinder functor for $\mathcal{M}$. For all objects X in $\mathcal{M}$, the morphism $\rho_X : IX \to X$ is in $\mathcal{W}$.*

(iii) *There exists a Cisinski cylinder functor $(I, \iota^0, \iota^1, \rho)$ for $\mathcal{M}$ such that the morphism $\rho_X : IX \to X$ is in $\mathcal{W}$ for all objects X in $\mathcal{M}$.*

Proof. (i) ⇒ (ii). This was shown in proposition 5.3.17.

(ii) ⇒ (iii). Immediate.

(iii) ⇒ (i). Let $p : X \to Y$ be a Cisinski trivial fibration in $\mathcal{M}$. Proposition 5.3.14 then says that there exists a morphism $s : Y \to X$ and an I-homotopy from id_X to $s \circ p$, i.e. a morphism $H : IX \to X$ such that $H \circ \iota^0_X = \mathrm{id}_X$ and $H \circ \iota^1_X = s \circ p$. Since $\rho_X : IX \to X$ is in $\mathcal{W}$ and $\rho_X \circ \iota^0_X = \rho_X \circ \iota^1_X = \mathrm{id}_X$, axioms FS1 and FS2 imply that $\iota^0_X, \iota^1_X : X \to IX$ are in $\mathcal{W}$, and so $H : IX \to X$ is also in $\mathcal{W}$, and hence $s \circ p : X \to X$ is in $\mathcal{W}$ as well. We may now use axiom FS3 to deduce that $p : X \to Y$ is in $\mathcal{W}$. ■

Proposition 5.3.33. *Let $(I, \mathcal{A})$ be a Cisinski homotopy structure on $\mathcal{M}$. Then the class of weak equivalences with respect to $(I, \mathcal{A})$ is an accessible $\mathcal{M}$-localiser.*

Proof. See Proposition 3.8 in [Cisinski, 2002]. □

Theorem 5.3.34. *Let $\mathcal{W}$ be any accessible $\mathcal{M}$-localiser. Then $\mathcal{M}$ is a combinatorial model category where*

- *the cofibrations are the monomorphisms in $\mathcal{M}$,*
- *the weak equivalences are the morphisms that are in $\mathcal{W}$, and*
- *the fibrations are the morphisms that have the right lifting property with respect to the trivial cofibrations.*

This is the **Cisinski model structure** *on $\mathcal{M}$ associated with $\mathcal{W}$.*

Proof. See Théorème 3.9 in [Cisinski, 2002]. □

Corollary 5.3.35. *If $\mathcal{W}$ is any $\mathcal{M}$-localiser (not necessarily accessible), then $\mathcal{W}$ is closed under retracts.*

Proof. See Corollaire 3.10 in [Cisinski, 2002]. □

5.4 Monoidal model categories

Prerequisites. §§ 4.1, 4.2, 4.4, B.1, B.2.

Proposition 5.4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories with pullbacks, let $\mathcal{E}$ be a category with pushouts, and let $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, $\mathcal{J} \subseteq \operatorname{mor} \mathcal{D}$ and $\mathcal{K} \subseteq \operatorname{mor} \mathcal{E}$ be subensembles. Suppose we have the following functors*

$$\oslash : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$$
$$\pitchfork : \mathcal{D}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{C}$$
$$\multimapinv : \mathcal{E} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$$

and natural bijections:

$$\mathcal{E}(C \oslash D, E) \cong \mathcal{C}(C, D \pitchfork E)$$
$$\mathcal{E}(C \oslash D, E) \cong \mathcal{D}(D, E \multimapinv C)$$
$$\mathcal{C}(C, D \pitchfork E) \cong \mathcal{D}(D, E \multimapinv C)$$

Then the following are equivalent:

(i) *If $f : C \to C'$ is in $\mathcal{I}$, $g : D \to D'$ is in $\mathcal{J}$, and the square in the diagram below is a pushout square in $\mathcal{E}$,*

$$\begin{tikzcd}
C \oslash D \arrow[r, "\mathrm{id}_C \oslash g"] \arrow[d, "f \oslash \mathrm{id}_D"'] & C \oslash D' \arrow[d] \arrow[ddr, bend left, "f \oslash \mathrm{id}_{D'}"] & \\
C' \oslash D \arrow[r] \arrow[drr, bend right, "\mathrm{id}_{C'} \oslash g"'] & (C \oslash D') \cup^{C \oslash D} (C' \oslash D) \arrow[dr, dashed, "f \square g"] & \\
& & C' \oslash D'
\end{tikzcd}$$

then the unique morphism $f \square g$ making the diagram commute is in ${}^{\boxslash}\mathcal{K}$.

(ii) *If $g : D \to D'$ is in $\mathcal{J}$, $h : E \to E'$ is in $\mathcal{K}$, and the square in the diagram* *below is a pullback square in $\mathcal{C}$,*

$$\begin{tikzcd}
D' \pitchfork E \arrow[drr, bend left, "g \pitchfork \mathrm{id}_E"] \arrow[dr, dashed, "g \boxminus h"] \arrow[ddr, bend right, "\mathrm{id}_{D'} \pitchfork h"'] & & \\
& (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E) \arrow[r] \arrow[d] & D \pitchfork E \arrow[d, "\mathrm{id}_D \pitchfork h"] \\
& D' \pitchfork E' \arrow[r, "g \pitchfork \mathrm{id}_{E'}"'] & D \pitchfork E'
\end{tikzcd}$$

then the unique morphism $g \boxminus h$ making the diagram commute is in $\mathcal{I}^{\boxslash}$.

(iii) *If $h : E \to E'$ is in $\mathcal{K}$, $f : C \to C'$ is in $\mathcal{I}$ and the square in the diagram below is a pullback square in $\mathcal{D}$,*

$$\begin{tikzcd}
E \multimapdotinv C' \arrow[drr, bend left, "\mathrm{id}_E \multimapdotinv f"] \arrow[dr, dashed, "h \boxminus f"] \arrow[ddr, bend right, "h \multimapdotinv \mathrm{id}_C"'] & & \\
& (E' \multimapdotinv C') \times_{E' \multimapdotinv C} (E \multimapdotinv C) \arrow[r] \arrow[d] & E \multimapdotinv C \arrow[d, "h \multimapdotinv \mathrm{id}_C"] \\
& E' \multimapdotinv C' \arrow[r, "\mathrm{id}_{E'} \multimapdotinv f"'] & E' \multimapdotinv C
\end{tikzcd}$$

then the unique morphism $h \boxminus f$ making the diagram commute is in $\mathcal{J}^{\boxslash}$.

Proof. (i) $\Rightarrow$ (ii). Let $f : C \to C'$ be in $\mathcal{I}$, let $g : D \to D'$ be in $\mathcal{J}$, let $h : E \to E'$ be in $\mathcal{K}$, and suppose we have a commutative diagram of the following form:

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & D' \pitchfork E \arrow[d, "g \boxminus h"] \\
C' \arrow[r] \arrow[ur, dashed] & (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E)
\end{tikzcd}$$

By the universal property of pullbacks, this corresponds to a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & D' \pitchfork E \arrow[dr, "\mathrm{id}_{D'} \pitchfork h"] \arrow[dd, "g \pitchfork \mathrm{id}_E"] & \\
C' \arrow[ur, dashed] \arrow[rr, bend left=10] \arrow[dr] & & D' \pitchfork E' \arrow[d, "g \pitchfork \mathrm{id}_{E'}"] \\
& D \pitchfork E \arrow[r, "\mathrm{id}_D \pitchfork h"'] & D \pitchfork E'
\end{tikzcd}$$

and, by adjoint transposition, to a commutative diagram in $\mathcal{E}$ of the form

$$\begin{tikzcd}
C \oslash D \arrow[r, "\mathrm{id}_C \oslash g"] \arrow[d, "f \oslash \mathrm{id}_D"'] & C \oslash D' \arrow[dr] \arrow[dd, "f \otimes \mathrm{id}_{D'}"] & \\
C' \oslash D \arrow[rr, bend right=10] \arrow[dr, "\mathrm{id}_{C'} \oslash g"'] & & E \arrow[d, "h"] \\
& C' \oslash D' \arrow[ur, dashed] \arrow[r] & E'
\end{tikzcd}$$

whence, by the universal property of pushouts, commutative diagram in $\mathcal{E}$ of the following form:

$$\begin{tikzcd}
(C \oslash D') \cup^{C \oslash D} (C' \oslash D) \arrow[r] \arrow[d, "f \boxdot g"'] & E \arrow[d, "h"] \\
C' \oslash D' \arrow[ur, dashed] \arrow[r] & E'
\end{tikzcd}$$

But $(f \boxdot g) \boxslash h$, so we conclude that $f \boxslash (g \boxminus h)$.

(ii) $\Rightarrow$ (iii), (i) $\Rightarrow$ (ii). A similar argument works. ■

Definition 5.4.2. Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be three model categories. A **Quillen adjunction of two variables** consists of three functors $\oslash, \pitchfork, \multimapinv$ with natural bijections as in the proposition satisfying the following (equivalent) axioms:

(a) If $h : E \to E'$ is a fibration in $\mathcal{E}$ and $f : C \to C'$ is a cofibration in $\mathcal{C}$, then the morphism $h \boxminus f : E \multimapinv C' \to (E' \multimapinv C') \times_{E' \multimapinv C} (E \multimapinv C)$ is a fibration in $\mathcal{D}$, which is a weak equivalence if either h or f is.

(b) If $f : C \to C'$ is a cofibration in $\mathcal{C}$ and $g : D \to D'$ is a cofibration in $\mathcal{D}$, then the morphism $f \boxslash g : C \oslash D \to (C \oslash D') \cup^{C \oslash D} (C' \oslash D)$ is a cofibration in $\mathcal{E}$, which is a weak equivalence if either f or g is.

(c) If $g : D \to D'$ is a cofibration in $\mathcal{C}$ and $h : E \to E'$ is a fibration in $\mathcal{D}$, then the morphism $g \boxminus h : D' \pitchfork E \to (D' \pitchfork E') \times_{D \pitchfork E'} (D \pitchfork E)$ is a fibration in $\mathcal{C}$, which is a weak equivalence if either g or h is.

Proposition 5.4.3. *Let* $(\oslash, \pitchfork, \multimapinv)$ *be a Quillen adjunction of two variables as above.*

(i) *For each cofibrant object C in $\mathcal{C}$, the adjunction*

$$C \oslash (-) \dashv (-) \multimapinv C : \mathcal{E} \to \mathcal{D}$$

is a Quillen adjunction.

(ii) *For each cofibrant object D in $\mathcal{D}$, the adjunction*

$$(-) \oslash D \dashv D \pitchfork (-) : \mathcal{E} \to \mathcal{C}$$

is a Quillen adjunction.

(iii) *For each fibrant object E in $\mathcal{E}$, the adjunction*

$$E \multimapinv (-) \dashv (-) \pitchfork E : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}$$

is a Quillen adjunction.

Proof. Immediate from the definitions. ♦

Corollary 5.4.4.

(i) *For each object C in $\mathcal{C}$, $C \oslash (-)$ preserves weak equivalences between cofibrant objects, and $(-) \oslash C$ preserves weak equivalences between fibrant objects.*

(ii) *For each object D in $\mathcal{D}$, $(-) \oslash D$ preserves weak equivalences between cofibrant objects, and $D \pitchfork (-)$ preserves weak equivalences between fibrant objects.*

(iii) *For each object E in $\mathcal{E}$, $E \multimapinv (-)$ sends weak equivalences between cofibrant objects in $\mathcal{C}$ to weak equivalences between fibrant objects in $\mathcal{D}$, and $(-) \pitchfork E$ sends weak equivalences between cofibrant objects in $\mathcal{D}$ to weak equivalences between fibrant objects in $\mathcal{D}$.*

Proof. Apply Ken Brown's lemma (4.4.6). ■

Lemma 5.4.5. *Let $\mathcal{V}$ be a monoidal category, let $\mathcal{M}$ be a model category with fibrant and cofibrant replacement functors, and let $p : \tilde{I} \to I$ be a morphism in $\mathcal{V}$, where I is the monoidal unit of $\mathcal{V}$.*

If $\mathcal{M}$ has a left $\mathcal{V}$-action $\oslash$ and right adjoint right $\mathcal{V}^{\mathrm{op}}$-action $\multimapinv$ such that the adjunction

$$\tilde{I} \otimes (-) \dashv (-) \multimapinv \tilde{I} : \mathcal{M} \to \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

(i) *For all cofibrant objects X in $\mathcal{M}$, $p \oslash \mathrm{id}_X : \tilde{I} \oslash X \to I \oslash X$ is a weak equivalence.*

(ii) *For all fibrant objects Y in $\mathcal{M}$, $\mathrm{id}_Y \multimapinv p : Y \multimapinv I \to Y \multimapinv \tilde{I}$ is a weak equivalence.*

If $\mathcal{M}$ has a right $\mathcal{V}$-action $\owns$ and a right adjoint left $\mathcal{V}^{\mathrm{op}}$-action $\multimap$ such that the adjunction

$$(-) \varobslash \tilde{I} \dashv \tilde{I} \multimap (-) : \mathcal{M} \to \mathcal{M}$$

is a Quillen adjunction, then the following are equivalent:

(i′) *For all cofibrant objects X in $\mathcal{M}$, $\mathrm{id}_X \varobslash p : X \varobslash \tilde{I} \to X \varobslash I$ is a weak equivalence.*

(ii′) *For all fibrant objects Y in $\mathcal{M}$, $p \multimap \mathrm{id}_Y : I \multimap Y \to \tilde{I} \multimap Y$ is a weak equivalence.*

Proof. Since $\boldsymbol{\eta}_X : X \to I \oslash X$ is a natural isomorphism, the adjunction

$$I \oslash (-) \dashv (-) \multimapinv I : \mathcal{M} \to \mathcal{M}$$

is an adjoint equivalence of categories, and *a fortiori* a Quillen equivalence, and the natural transformations $p \oslash (-)$ and $(-) \multimapinv p$ constitute a conjugate pair. Theorem 3.3.20 says that the derived natural transformations for $p \oslash (-)$ and $(-) \multimapinv p$ constitute a conjugate pair of natural transformations between the derived adjunctions. Applying proposition 3.3.24 to theorem 4.4.12, we deduce that the following are equivalent:

- For all cofibrant objects X, $p \oslash \mathrm{id}_X$ is a weak equivalence.

- The left derived natural transformation for $p \oslash (-)$ is a natural isomorphism.
- The right derived natural transformation for $(-) \oslash p$ is a natural isomorphism.
- For all fibrant objects Y, $\mathrm{id}_Y \multimapinv p$ is a weak equivalence. ■

The following definition is due to Hovey [1999, § 4.2]:

Definition 5.4.6. A **monoidal model category** is a biclosed monoidal category $\mathcal{M}$ equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The right $\mathcal{M}$-hom system $(\otimes, \multimap, \multimapinv)$, where $\multimap$ (resp. $\multimapinv$) is the right (resp. left) internal hom functor of $\mathcal{M}$, is a Quillen adjunction of two variables.
- **Unit axiom.** For each cofibrant replacement $(\tilde{I}, p)$ of the monoidal unit I and each cofibrant object X in $\mathcal{M}$, the morphisms $p \otimes \mathrm{id}_X : \tilde{I} \otimes X \to I \otimes X$ and $\mathrm{id}_X \otimes p : X \otimes \tilde{I} \to X \otimes I$ are weak equivalences in $\mathcal{M}$.

Lemma 5.4.7. *Let $\mathcal{M}$ be a biclosed monoidal category equipped with a model structure satisfying the pushout–product axiom, and let X be any object in $\mathcal{M}$. The following are equivalent:*

(i) *There exists a cofibrant replacement $(\tilde{I}, p)$ of the monoidal unit I such that $p \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes p$ are weak equivalences in $\mathcal{M}$.*

(ii) *There exists a fibrant cofibrant replacement (QI, q) of the monoidal unit I such that $q \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes q$ are weak equivalences in $\mathcal{M}$.*

(iii) *For any cofibrant replacement $(\tilde{I}, p)$ of the monoidal unit I, both $p \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes p$ are weak equivalences in $\mathcal{M}$.*

Proof. (i) ⇒ (ii). Let (QI, q) be a fibrant cofibrant replacement of I; such exists by proposition 4.1.22. Since $\tilde{I}$ is cofibrant, axiom CM5 implies there is a morphism $w : \tilde{I} \to QI$ such that $q \circ w = p$, and the 2-out-of-3 property implies w is a weak equivalence. Corollary 5.4.4 says $w \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes w$ are weak equivalences, thus by the 2-out-of-3 property again $q \otimes \mathrm{id}_X$ and $\mathrm{id}_X \otimes q$ must be weak equivalences.

(ii) ⇒ (iii). A similar argument works.

(iii) ⇒ (i). Obvious, given the existence of cofibrant replacements. ■

Corollary 5.4.8. *Let $\mathcal{M}$ be a biclosed monoidal category equipped with a model structure. If the monoidal unit I is a cofibrant object in $\mathcal{M}$, then the following are equivalent:*

(i) *$\mathcal{M}$ is a monoidal model category.*

(ii) *$\mathcal{M}$ satisfies the pushout–product axiom.* ■

Proposition 5.4.9. *Let $\mathcal{M}$ be a monoidal model category, let I be the monoidal unit, and let $\multimap : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{M}$ be the right internal hom functor. If I is a cofibrant object and (J, i_0, i_1, p) is a cylinder object for I, then $(J \multimap X, i, p_0, p_1)$ is a path object for all fibrant X, where $i : X \to [J, X]$ is the morphism induced by $p : J \to I$, and $p_0, p_1 : [J, X] \to X$ are (respectively) the morphisms induced by $i_0, i_1 : I \to J$.*

TODO: State the version without the assumption that the unit is cofibrant.

Proof. Since I is a cofibrant object, $I + I$ is cofibrant (by proposition A.3.12), and hence J itself is cofibrant. Corollary 5.4.4 says the functor $(-) \multimap X : \mathcal{M}^{\mathrm{op}} \to \mathcal{M}$ sends weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences between fibrant objects in $\mathcal{M}$ when X is fibrant, so it follows that the morphism $i : X \to [J, X]$ is a weak equivalence. Similarly, since the morphism $I + I \to J$ induced by i_0 and i_1 is a cofibration, the morphism $[J, X] \to X \times X$ induced by p_0 and p_1 is a fibration, so $([J, X], i, p_0, p_1)$ is indeed a path object for X. ■

The following definition can be found in [Rezk, 2010, § 2] and [Simpson, 2012, § 7.7].

Definition 5.4.10. A **cartesian model category** is a cartesian closed category $\mathcal{M}$ equipped with a model structure satisfying the following additional axioms:

- **Pushout–product axiom.** The left $\mathcal{M}$-hom system $(\times, [-, -], [-, -])$ is a Quillen adjunction of two variables.
- **Cofibrant unit axiom.** Every terminal object in $\mathcal{M}$ is cofibrant.

Example 5.4.11. The Kan–Quillen model structure on **sSet** makes it a cartesian model category: **sSet** is a cartesian closed combinatorial model category (*a fortiori* a DHK model category), all simplicial sets are cofibrant, and the pushout–product axiom is just proposition 1.3.13.

Proposition 5.4.12. *Let $\mathcal{M}$ be a Cisinski model category.*[2] *The following are equivalent:*

(i) *$\mathcal{M}$ is a cartesian model category.*

(ii) *The class of weak equivalences in $\mathcal{M}$ is closed under binary products.*

(iii) *The class of trivial cofibrations in $\mathcal{M}$ is closed under binary products.*

Proof. (i) ⇒ (ii). Since all objects in $\mathcal{M}$ are cofibrant, corollary 5.4.4 implies that, for any object Y in $\mathcal{M}$, the functor $(-) \times Y : \mathcal{M} \to \mathcal{M}$ preserves weak equivalences. Thus, the class of weak equivalences in $\mathcal{M}$ is closed under binary products.

(ii) ⇒ (iii). The class of monomorphisms is always closed under binary products, so the class of trivial cofibrations (i.e. monic weak equivalences) is closed under binary products if the class of weak equivalences is.

(iii) ⇒ (i). This is the content of proposition 5.3.5. ∎

Theorem 5.4.13. *If $\mathcal{M}$ is a monoidal model category, then there is an induced monoidal biclosed structure on* $\operatorname{Ho}\mathcal{M}$ *where the monoidal product is the left derived functor of the monoidal product in $\mathcal{M}$ and the coherence data is inherited from $\mathcal{M}$.*

Proof. See Theorem 4.3.2 in [Hovey, 1999]. □

Proposition 5.4.14. *Let $\mathcal{M}$ be a cartesian model category and let $\mathcal{M}_{\mathrm{f}}$ be the full subcategory of fibrant objects.*

(i) *$\mathcal{M}_{\mathrm{f}}$ is closed under products of small families of objects in $\mathcal{M}$, and $[X, Y]$ is fibrant if X is cofibrant and Y is fibrant.*

(ii) *The localising functor $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho}\mathcal{M}$ preserves products of small families of objects; in particular,* $\operatorname{Ho}\mathcal{M}$ *has products for all small families of objects.*

(iii) $\operatorname{Ho}\mathcal{M}$ *is a cartesian closed category, and $\gamma[X, Y]$ is naturally isomorphic to $[\gamma X, \gamma Y]$ when X is cofibrant and Y is fibrant.*

[2] See definition 5.3.1.

(iv) *Let* $\Gamma : \mathcal{M} \to \mathbf{Set}$ *be the functor* $\mathcal{M}(1, -)$ *and let* $\tau_0 : \mathcal{M} \to \mathbf{Set}$ *be the functor* $\operatorname{Ho} \mathcal{M}(\gamma 1, \gamma -)$*. The functor* τ_0 *preserves small products in* $\mathcal{M}_{\mathrm{f}}$*, and the component* $\chi_Y : \Gamma Y \Rightarrow \tau_0 Y$ *of the natural transformation* $\chi : \Gamma \Rightarrow \tau_0$ *induced by the functor* γ *is surjective for all fibrant objects* Y *in* $\mathcal{M}$*.*

Proof. (i). That $\mathcal{M}_{\mathrm{f}}$ is closed in $\mathcal{M}$ under small products is a straightforward consequence of proposition A.3.12, and pushout–product axiom for cartesian model structures implies the other half of the claim.

(ii). Proposition 4.4.18 says $\operatorname{Ho}[I, \mathcal{M}] \to [I, \operatorname{Ho} \mathcal{M}]$ is an equivalence of categories for all sets I, so products in $\operatorname{Ho} \mathcal{M}$ coincide with homotopy products. Homotopy products in $\mathcal{M}_{\mathrm{f}}$ coincide with ordinary products, hence the localising functor $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho} \mathcal{M}$ preserves small products. Since every object in $\mathcal{M}$ is weakly equivalent to one in $\mathcal{M}_{\mathrm{f}}$, it follows that $\operatorname{Ho} \mathcal{M}$ has products for all small families of objects.

(iii). Apply theorem 5.4.13.

(iv). As a representable functor, $\operatorname{Ho} \mathcal{M}(\gamma 1, -) : \operatorname{Ho} \mathcal{M} \to \mathbf{Set}$ preserves small products, and by claim (ii), $\gamma : \mathcal{M}_{\mathrm{f}} \to \operatorname{Ho} \mathcal{M}$ preserves small products, so $\tau_0 : \mathcal{M}_{\mathrm{f}} \to \mathbf{Set}$ indeed preserves small products. Theorem 4.1.29 says that the localising functor induces hom-set maps $\mathcal{M}(X, Y) \to \operatorname{Ho} \mathcal{M}(\gamma X, \gamma Y)$ that are surjective when X is cofibrant and Y is fibrant; since 1 is cofibrant by hypothesis, it follows that the map $\chi_Y : \Gamma Y \to \tau_0 Y$ is surjective for all cofibrant objects Y. ■

Under stronger hypotheses, the homotopy category of a cartesian model category admits a description à la Hurewicz:

Proposition 5.4.15. *Let* $\mathcal{M}$ *be a cartesian model category, let* $\mathcal{M}_{\mathrm{f}}$ *be the full subcategory of fibrant objects, and let* $\operatorname{Ho} \mathcal{M}_{\mathrm{f}}$ *be the localisation of* $\mathcal{M}_{\mathrm{f}}$ *at the weak equivalences. If all fibrant objects in* $\mathcal{M}$ *are cofibrant, then:*

(i) $\mathcal{M}_{\mathrm{f}}$ *is a cartesian closed category.*

(ii) *The natural transformation* $\chi : \Gamma \Rightarrow \tau_0$ *induces a functor* $\mathcal{M}_{\mathrm{f}} \to \tau_0 \left[\underline{\mathcal{M}_{\mathrm{f}}} \right]$ *that is a bijection on objects, full, and preserves small products and exponential objects.*

(iii) *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in $\mathcal{M}_{\mathrm{f}}$. Then f_0 and f_1 are (right) homotopic if and only if they are sent to the same morphism in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$.*

(iv) *The canonical functor* $\operatorname{Ho} \mathcal{M}_{\mathrm{f}} \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$ *is an isomorphism of categories.*

Proof. (i). Since all fibrant objects are cofibrant, the exponential object $[X, Y]$ is fibrant for all X and Y in $\mathcal{M}_{\mathrm{f}}$; and by proposition 5.4.14, $\mathcal{M}_{\mathrm{f}}$ is closed under products of small objects in $\mathcal{M}$, so it follows that $\mathcal{M}_{\mathrm{f}}$ is a cartesian closed category.

(ii). This is a straightforward consequence of the fact that $\tau_0 : \mathcal{M}_{\mathrm{f}} \to \mathbf{Set}$ preserves small products, that we have a natural bijection $\Gamma[X, Y] \cong \mathcal{M}(X, Y)$ for all objects X and Y, and that $\chi_Z : \Gamma Z \to \tau_0 Z$ is a surjection for all fibrant objects Z.

(iii). Suppose $f_0, f_1 : X \to Y$ are related by a right homotopy, i.e. there exists a path object $\left(P, i, p_0, p_1\right)$ for Y and a morphism $f : X \to P$ such that $p_0 \circ f = f_0$ and $p_1 \circ f = f_1$. Since $p_0, p_1 : P \to Y$ are retractions of the weak equivalence $i : Y \to P$, the two maps $\tau_0[X, P] \to \tau_0[X, Y]$ induced by p_0 and p_1 must be equal. In particular, $\chi_{[X,Y]} : \Gamma[X, Y] \to \tau_0[X, Y]$ must map f_0 and f_1 to the same element.

Conversely, if f_0 and f_1 are sent to the same morphism in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$, then there must exist a cylinder object $\left(J, i_0, i_1, p\right)$ for 1 and a morphism $h : J \to [X, Y]$ such that $h \circ i_0$ (resp. $h \circ i_1$) is the exponential transpose of f_0 (resp. f_1). Taking exponential transposes again and using the fact that $[J, Y]$ is a path object for Y, we deduce that f_0 and f_1 are right homotopic.

(iv). The formal Whitehead theorem implies that weak equivalences in $\mathcal{M}_{\mathrm{f}}$ are mapped to isomorphisms in $\tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$, so the functor $\mathcal{M} \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$ induces a functor $\operatorname{Ho} \mathcal{M}_{\mathrm{f}} \to \tau_0\left[\underline{\mathcal{M}_{\mathrm{f}}}\right]$. A standard argument then shows that it is an isomorphism: see e.g. theorem 4.3.1. ■

Proposition 5.4.16. *Let $\mathcal{M}$ be a cartesian model category. If all objects in $\mathcal{M}$ are cofibrant, then:*

(i) *The functors $\gamma : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ and $\tau_0 : \mathcal{M} \to$* **Set** *both preserve finite products.*

(ii) *A morphism $f : X \to Y$ in $\mathcal{M}$ is a weak equivalence if and only if the induced maps*

$$\tau_0[f, Z] : \tau_0[Y, Z] \to \tau_0[X, Z]$$

are bijections for all fibrant objects Z in $\mathcal{M}$.

(iii) *The inclusion $\mathcal{M}_\mathrm{f} \hookrightarrow \mathcal{M}$ induces a fully faithful functor $\tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right] \to \tau_0[\underline{\mathcal{M}}]$ with a left adjoint.*

Proof. (i). It suffices to to show that $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ preserves finite products; that $\tau_0 : \mathcal{M} \to \mathbf{Set}$ preserves finite products will follow automatically. It is not hard to check that $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ preserves terminal objects for all model categories $\mathcal{M}$, and we will now show that γ preserves binary products.

The pushout–product axiom implies that, for all cofibrant objects Y, the functor $- \times Y : \mathcal{M} \to \mathcal{M}$ is a left Quillen functor. Since we are assuming that all objects are cofibrant, corollary 5.4.4 implies that $- \times Y$ preserves weak equivalences. We may then deduce that $- \times - : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ preserves all weak equivalences, and hence that it is a homotopical left approximation for itself. Thus, the localising functor $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ indeed preserves binary products.

(ii). If $f : X \to Y$ is a weak equivalence, then $[f, Z] : [Y, Z] \to [X, Z]$ is a weak equivalence for all fibrant objects Z, and hence $\tau_0[f, Z]$ must be a bijection. Conversely, suppose $\tau_0[f, Z]$ is a bijection for all fibrant objects Z. Let $R : \mathcal{M} \to \mathcal{M}$ be a fibrant replacement functor for $\mathcal{M}$. Then, the morphism $Rf : RX \to RY$ also induces bijections $\tau_0[Rf, Z]$ for all fibrant objects Z, and since RX and RY are in $\mathcal{M}_\mathrm{f}$, the Yoneda lemma implies that $Rf : RX \to RY$ is sent to an isomorphism in $\tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right]$, and hence must be a weak equivalence in $\mathcal{M}_\mathrm{f}$. The 2-out-of-3 property of weak equivalences then implies $f : X \to Y$ is a weak equivalence in $\mathcal{M}$.

(iii). It is clear that the induced functor $\tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right] \to \tau_0[\underline{\mathcal{M}}]$ is indeed fully faithful, and it is not hard to check that a fibrant replacement functor provides the required left adjoint $\tau_0[\underline{\mathcal{M}}] \to \tau_0\left[\underline{\mathcal{M}_\mathrm{f}}\right]$. ■

Definition 5.4.17. An **isocofibration** is a functor that is injective on objects. An **isofibration** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that, for every object C in $\mathcal{C}$ and every isomorphism $f : FC \to D$ in $\mathcal{D}$, there exists an isomorphism $\tilde{f} : C \to \tilde{D}$ in $\mathcal{C}$ such that $F\tilde{f} = f$.

Proposition 5.4.18. *Let* **Cat** *be the category of small categories. The following data constitute a model structure on* **Cat***:*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isocofibrations.*
- *The fibrations are the isofibrations.*

Moreover, the factorisations for axiom CM5 may be chosen functorially, so that **Cat** *becomes a DHK model category. This model structure is called the* **canonical model structure** *on* **Cat***.*

Proof. It is not hard to show that **Cat** has limits and colimits for all small diagrams, so axiom CM1* is satisfied. It is also clear that the announced class of weak equivalences has the 2-out-of-3 property, so by theorem 4.1.12, it is enough to show that we have a pair of compatible weak factorisation systems.

Let $I : \mathbb{A} \to \mathbb{B}$ be an isocofibration and $P : \mathbb{C} \to \mathbb{D}$ be an isofibration, and suppose we have a commutative diagram of the following form:

$$\begin{tikzcd} \mathbb{A} \arrow[r, "F"] \arrow[d, "I"'] & \mathbb{C} \arrow[d, "P"] \\ \mathbb{B} \arrow[r, "G"'] & \mathbb{D} \end{tikzcd}$$

First, suppose P is a weak equivalence. Then, P must be surjective on objects, so we may define a map $H : \operatorname{ob}\mathbb{B} \to \operatorname{ob}\mathbb{C}$ by taking $HB = FA$ if $B = IA$ for some A, and if B is not in the image of A, define HB to be any object in $\mathbb{C}$ such that $PHB = GB$; there is then a unique way of extending H to a functor $\mathbb{B} \to \mathbb{C}$ making the evident diagram commute.

Next, instead suppose I is a weak equivalence. Then, I may be regarded as the inclusion of a full subcategory that is essentially surjective on objects. For each object B in $\mathbb{B}$ that is not in the image of I, fix an object A in $\mathbb{A}$ and an isomorphism $IA \xrightarrow{\cong} B$. Since P is an isofibration, for each such B we may also choose an object C in $\mathbb{C}$ and an isomorphism $FA \xrightarrow{\cong} C$ whose image under P is $GIA \xrightarrow{\cong} GB$. There is then a unique functor $H : \mathbb{B} \to \mathbb{C}$ that makes the evident diagram commute and sends B to the chosen C and $IA \xrightarrow{\cong} B$ to $FA \xrightarrow{\cong} C$.

It remains to be shown that every functor can be factorised in the required manner. Let $F : \mathbb{C} \to \mathbb{D}$ be any functor. Consider the iso-comma category $(F \wr \mathbb{D})$:

- The objects are triples (C, D, α), where C is an object in $\mathbb{C}$, D is an object in $\mathbb{D}$, and $\alpha : FC \to D$ is an *isomorphism* in $\mathbb{D}$.
- The morphisms $(C, D, \alpha) \to (C', D', \alpha')$ is a morphism $f : C \to C'$ is in $\mathbb{C}$ together with a morphism $g : D \to D'$ in $\mathbb{D}$ such that $g \circ \alpha = \alpha' \circ F f$.[3]
- Composition and identities are inherited from $\mathbb{C}$ and $\mathbb{D}$.

There is an evident isocofibration $I : \mathbb{C} \to (F \wr \mathbb{D})$ sending an object C in $\mathbb{C}$ to the object $\left(C, FC, \mathrm{id}_{FC}\right)$, and it is easy to see that I is a weak equivalence. On the other hand, the projection $P : (F \wr \mathbb{D}) \to \mathbb{D}$ is an isofibration by construction, and obviously $F = PI$. Thus, we have factored F as a trivial isocofibration followed by an isofibration, and it is clear that this construction is functorial in F.

Now, consider instead the category $\mathbf{M}(F)$ defined below:

- $\operatorname{ob} \mathbf{M}(F) = \operatorname{ob} \mathbb{C} \amalg \operatorname{ob} \mathbb{D}$.
- If C and C' are objects in $\mathbb{C}$, while D and D' are objects in $\mathbb{D}$, then:

$$\begin{aligned} \operatorname{Hom}(C, C') &= \mathbb{D}(FC, FC') \\ \operatorname{Hom}(C, D') &= \mathbb{D}(FC, D') \\ \operatorname{Hom}(D, C') &= \mathbb{D}(D, FC') \\ \operatorname{Hom}(D, D') &= \mathbb{D}(D, D') \end{aligned}$$

- Composition and identities are inherited from $\mathbb{D}$.

There is an evident isocofibration $I : \mathbb{C} \to \mathbf{M}(F)$ that sends an object C in $\mathbb{C}$ to the corresponding object in $\mathbf{M}(F)$ and sends a morphism $f : C \to C'$ in $\mathbb{C}$ to the morphism in $\mathbf{M}(F)$ corresponding to $Ff : FC \to FC'$ in $\mathbb{D}$. On the other hand, there is an evident projection $P : \mathbf{M}(F) \to \mathbb{D}$ that is fully faithful and surjective on objects, i.e. P is a trivial isofibration. Of course, $F = PI$, so this is a factorisation of F as an isocofibration followed by a trivial isofibration, and it is clear that this construction is functorial in F. ■

[3] However, because α and α' are isomorphisms, f freely and uniquely determines g.

Theorem 5.4.19. *Let* **Cat** *be considered as a model category via the canonical model structure.*

(i) *Every object in* **Cat** *is both cofibrant and fibrant.*

(ii) **Cat** *is a combinatorial model category.*

(iii) **Cat** *is a cartesian model category.*

Proof. (i). The unique functor $\varnothing \to \mathbb{C}$ is vacuously an isocofibration, and the unique functor $\mathbb{C} \to \mathbb{1}$ is certainly an isofibration.

(ii). **Cat** is a locally finitely presentable category,[4] and it remains to be shown that the canonical model structure is a cofibrantly-generated model structure.

By the very definition of isofibration, the set $\{\{0\} \hookrightarrow \mathbf{I}\mathbb{2}\}$ is a generating set of trivial isocofibrations, where $\mathbf{I}\mathbb{2}$ is the groupoid containing only a pair of non-trivial isomorphisms. It is also straightforward to see that a functor is …

… surjective on objects if and only if it has the right lifting property with respect to the unique functor $\varnothing \to \mathbb{1}$;

… full if and only if it has the right lifting property with respect to the inclusion $\operatorname{disc} 2 \to \mathbb{2}$; and

… faithful if and only if it has the right lifting property with respect the surjective functor $\mathbb{E} \to \mathbb{2}$, where $\mathbb{E}$ is the category with a parallel pair of non-trivial morphisms.

However, a functor is a trivial isofibration if and only if it is fully faithful and surjective on objects, so $\{\varnothing \to \mathbb{1}, \operatorname{disc} 2 \to \mathbb{2}, \mathbb{E} \to \mathbb{2}\}$ is a set of generating isocofibrations.

[4] — because e.g. **Cat** is the category of models for a finite limit sketch; see Proposition 1.51 in [LPAC] or Proposition 5.6.4 in [Borceux, 1994b].

(iii). Let $F : \mathbb{C} \to \mathbb{C}'$ and $G : \mathbb{D} \to \mathbb{D}'$ be isocofibrations, and consider the functor $F \mathbin{\Box} F'$ defined by the diagram below:

$$\begin{tikzcd}
\mathbb{C}\times\mathbb{D} \arrow[r, "\mathrm{id}_{\mathbb{C}}\times G"] \arrow[d, "F\times\mathrm{id}_{\mathbb{D}}"'] & \mathbb{C}\times\mathbb{D}' \arrow[d] \arrow[ddr, bend left, "F\times\mathrm{id}_{\mathbb{D}'}"] & \\
\mathbb{C}'\otimes\mathbb{D} \arrow[r] \arrow[drr, bend right, "\mathrm{id}_{\mathbb{C}'}\times G"'] & (\mathbb{C}\times\mathbb{D}')\cup^{\mathbb{C}\times\mathbb{D}}(\mathbb{C}'\times\mathbb{D}) \arrow[dr, dashed, "F\mathbin{\Box}G"] & \\
& & \mathbb{C}'\times\mathbb{D}'
\end{tikzcd}$$

The functor ob : **Cat** $\to$ **Set** has both left and right adjoints, so it is easy to see that $F \mathbin{\Box} G$ is an isocofibration. Moreover, if $F : \mathbb{C} \to \mathbb{C}'$ is a trivial isocofibration, one may directly verify that $F\times\mathrm{id}_{\mathbb{D}} : \mathbb{C}\times\mathbb{D} \to \mathbb{C}'\times\mathbb{D}$ and $F\times\mathrm{id}_{\mathbb{D}'} : \mathbb{C}\times\mathbb{D}' \to \mathbb{C}'\times\mathbb{D}'$ are trivial isocofibrations; but trivial isocofibrations are closed under pushout, so applying the 2-out-of-3 property of weak equivalences, we conclude that $F \mathbin{\Box} G$ is a trivial isocofibration if F is. The symmetrical argument shows that $F \mathbin{\Box} G$ is a trivial isocofibration if G is.

Having shown that **Cat** satisfies the pushout–product axiom, we must now verify that **Cat** is cartesian closed and has a cofibrant unit; but the former is a very well-known fact, and the latter follows from claim (i). ■

Theorem 5.4.20. *Let* **Grpd** *be the category of small groupoids.*

(i) *The following data constitute a model structure on* **Grpd***:*

- *The weak equivalences are the functors that are fully faithful and essentially surjective on objects.*
- *The cofibrations are the isocofibrations.*
- *The fibrations are the isofibrations.*

This model structure is called the **canonical model structure** *on* **Grpd**.

(ii) *Every object in* **Grpd** *is both cofibrant and fibrant.*

(iii) **Grpd** *is a combinatorial model category.*

(iv) **Grpd** *is a cartesian model category.*

(v) *The inclusion* und : **Grpd** $\to$ **Cat** *preserves* and *reflects weak equivalences, isocofibrations, and isofibrations; moreover, it is both a left Quillen functor and a right Quillen functor.*

Proof. (i). The proof of proposition 5.4.18 goes through for **Grpd** without modifications.

(ii) – (iv). These can be proven in essentially the same way as proposition 5.4.18, though one should note that the generating isocofibrations and generating trivial isocofibrations for **Grpd** are different.

(v). It is clear that und : **Grpd** $\to$ **Cat** has the announced preservation and reflection properties. One may check that und has a left adjoint **I** : **Cat** $\to$ **Grpd** and a right adjoint iso : **Cat** $\to$ **Grpd**, so und is both a left Quillen functor and a right Quillen functor. ■

— VI —

Quasicategories

Quasicategories were first defined by Boardman and Vogt [BV] as simplicial classes that satisfy the “restricted Kan condition”. The modern name is due to Joyal [2002], who worked out much of the basic theory.

As the word itself suggests, a quasicategory is a structure that is like a category. More precisely, it is a model for an $(\infty, 1)$-category, i.e. a weak higher category with n-morphisms for all $n \geq 0$, such that every n-morphism with $n > 1$ is (weakly) invertible; alternatively, one may think of quasicategories as being homotopy-coherent categories, i.e. a structure which is like a category but only up to a specified, coherent system of homotopies.

6.1 Basics

Prerequisites. §§ 0.1, 1.1, 1.2, A.2.

In this section we use the explicit universe convention.

Definition 6.1.1. An **inner horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where $n \geq 2$ and $0 < k < n$, where Λ^n_k is the union of the faces of Δ^n that include the k-th vertex. (See also definition 1.3.1.)

Definition 6.1.2. A **quasicategory** is a simplicial set X such that the unique morphism $X \to 1$ has the right lifting property with respect to all inner horn inclusions.

¶ 6.1.3. Quasicategories are also called **∞-categories** (by e.g. Lurie [HTT]) or **weak Kan complexes** (by e.g. Cordier and Porter [1986]). We will usually use *bold upright* calligraphic letters to denote quasicategories, e.g. $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \ldots$. As with 'category', the word 'quasicategory' always means a quasicategory that is not necessarily small, even when we are using the implicit universe convention.

Proposition 6.1.4. *Let C be a category and let* $\mathrm{N}(C)$ *be its nerve. For $n \geq 2$ and $0 < k < n$, the unique morphism* $\mathrm{N}(C) \to 1$ *is* right orthogonal *to the inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$; in particular,* $\mathrm{N}(C)$ *is a quasicategory.*

Proof. This is a straightforward exercise using induction on n. ◊

¶ 6.1.5. We will often refer to vertices of a quasicategory as **objects**, and edges as **morphisms**. The **domain** of a morphism f in a quasicategory is the object $d_1(f)$, and the **codomain** of f is the object $d_0(f)$. An **identity morphism** is a degenerate edge; we write $f : x \to y$ to mean that x is the domain of f and y is the codomain of f. The **identity morphism** of an object x in a quasicategory is the degenerate edge $s_0(x)$, which we also denote by id_x. Note that all this terminology is compatible with the identification of categories C with their nerves $\mathrm{N}(C)$.

Also observe that there is an automorphism $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$ that sends coface maps $\delta^i_n : [n-1] \to [n]$ to $\delta^{n-i}_n : [n-1] \to [n]$ and codegeneracy maps $\sigma^i_n : [n] \to [n+1]$ to $\sigma^{n-i}_n : [n] \to [n+1]$ for all $n \geq 0$ and $0 \leq i \leq n$. This in turn induces an automorphism on the category of simplicial sets.

Definition 6.1.6. The **opposite** of a simplicial set X is the simplicial set X^{op} obtained by composing $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ with $(-)^{\mathrm{op}} : \mathbf{\Delta} \to \mathbf{\Delta}$.

Remark 6.1.7. It is not hard to check that a simplicial set X is a quasicategory if and only if the simplicial set X^{op} is a quasicategory.

Definition 6.1.8. Let f_0 and f_1 be a parallel pair of morphisms in a quasicategory.

- We say f_0 and f_1 are **left homotopic** if there exists a 2-simplex α such that $d_1(\alpha) = f_0$, $d_0(\alpha) = f_1$, and $d_2(\alpha) = s_0\big(d_0(f_0)\big)$.
- We say f_0 and f_1 are **right homotopic** if there exists a 2-simplex α such that $d_2(\alpha) = f_0$, $d_1(\alpha) = f_1$, and $d_0(\alpha) = s_0\big(d_0(f_0)\big)$.
- We say f_0 and f_1 are **homotopic** if they are both left and right homotopic, and we write $f_0 \sim f_1$ in this case.

Obviously, two edges are left homotopic in a quasicategory $\mathcal{C}$ if and only if they are right homotopic in $\mathcal{C}^{\mathrm{op}}$. In fact:

Lemma 6.1.9. *Let f_0 and f_1 be a parallel pair of morphisms in a quasicategory $\mathcal{C}$. The following are equivalent:*

(i) *f_0 and f_1 are left homotopic.*

(ii) *f_0 and f_1 are right homotopic.*

(iii) *f_0 and f_1 are homotopic.*

Proof. (i) ⇔ (ii). By duality, it suffices to show that (i) ⇒ (ii). Let α be a 2-simplex of $\mathcal{C}$ such that $d_1(\alpha) = f$, $d_0(\alpha) = f'$, and $d_2(\alpha) = s_0\big(d_0(f)\big)$. Using the right lifting property of $\mathcal{C} \to 1$ with respect to the inner horn inclusion $\Lambda^3_1 \to \mathcal{C}$, it is straightforward to obtain a 3-simplex ξ such that $d_2(\xi) = \alpha$, $d_3(\xi) = s_0\big(f_1\big)$, and $d_0(\xi) = s_1\big(f_1\big)$; thus the 2-simplex $d_1(\xi)$ is the required witness for the claim that f_0 and f_1 are right homotopic.

(i) and (ii) ⇔ (iii). This is by definition. ■

Lemma 6.1.10. *Let $\mathcal{C}$ be a quasicategory. The relation of homotopy is an equivalence relation on the set of edges of $\mathcal{C}$.*

Proof. See Proposition 1.2.3.5 in [HTT], or Lemma 4.11 in [BV]. □

Definition 6.1.11. The **homotopy category** of a quasicategory $\mathcal{C}$ is the category $\operatorname{Ho}\mathcal{C}$ defined below:

- The objects are the objects in $\mathcal{C}$.
- A morphism $x \to y$ is a homotopy class of morphisms $f : x \to y$ in $\mathcal{C}$.
- The identity morphism $x \to x$ is the homotopy class of the morphism id_x.
- Composition is induced by the existence of fillers for the inner horn Λ^2_1: if α is a 2-simplex of $\mathcal{C}$, then we have $d_0(\alpha) \circ d_2(\alpha) = d_1(\alpha)$.

Lemma 6.1.12. *The above construction is indeed a category.*

Proof. See Proposition 1.2.3.8 in [HTT]. □

Definition 6.1.13. Let $\mathbf{U}$ be a universe. A **$\mathbf{U}$-small quasicategory** is a quasicategory whose underlying simplicial set is $\mathbf{U}$-small.

Proposition 6.1.14. *Let $\mathbf{U}$ be a universe, let $\mathbf{sSet}$ be the category of simplicial $\mathbf{U}$-sets, and let $\mathbf{Cat}$ be the category of $\mathbf{U}$-small categories.*

(i) *The functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ that sends a $\mathbf{U}$-small category $\mathbb{C}$ to its nerve $\mathrm{N}(\mathbb{C})$ has a left adjoint $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ that sends a simplicial $\mathbf{U}$-set X to its fundamental category $\tau_1 X$.*

(ii) *The functor $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ preserves finite products.*

(iii) *For each quasicategory $\mathcal{C}$, there is a canonical isomorphism $\tau_1 \mathcal{C} \cong \operatorname{Ho} \mathcal{C}$.*

Proof. Claims (i) and (ii) were previously proven in proposition 1.2.1, and claim (iii) is essentially a consequence of the fact that $\tau_1 X$ can be presented explicitly in terms of generators and relations as in remark 1.2.3. □

¶ 6.1.15. Henceforth, we will regard all ordinary categories as quasicategories by implicitly identifying a category C with its nerve $\mathrm{N}(C)$. Continuing the terminological conventions in paragraph 6.1.5, we now define functors and natural transformations in the context of quasicategories.

Definition 6.1.16. A **functor** between quasicategories is a morphism of simplicial sets whose domain and codomain are quasicategories.

Recall that theorem A.2.22 implies that the category of simplicial $\mathbf{U}$-sets is cartesian closed for all universes $\mathbf{U}$. For brevity, we will identify morphisms $X \to Y$ with vertices of the exponential object $[X, Y]$; thus, a functor $\mathcal{C} \to \mathcal{D}$ will be both a morphism between simplicial sets and an vertex in $[\mathcal{C}, \mathcal{D}]$.

Definition 6.1.17. Let $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ be functors between quasicategories.

- A **natural transformation** $\alpha : f_0 \Rightarrow f_1$ is an edge $\alpha : f_0 \to f_1$ in the exponential object $[\mathcal{C}, \mathcal{D}]$.
- Two natural transformations are **homotopic** if they are *isomorphic* in the fundamental category $\tau_1[\mathcal{C}, \mathcal{D}]$.

Remark 6.1.18. It is a fact that the exponential object $[X, Y]$ is a quasicategory when Y is quasicategory: see corollary 6.2.12. Thus the fundamental category $\tau_1[\mathcal{C}, \mathcal{D}]$ can be computed using the homotopy category construction.

Definition 6.1.19. Let $\mathcal{C}$ be a quasicategory. An **equivalence** in $\mathcal{C}$ is a morphism f whose homotopy class is invertible in $\operatorname{Ho}\mathcal{C}$, and a **quasi-inverse** for f is a morphism in $\mathcal{C}$ whose homotopy class is an inverse for (the homotopy class of) f in $\operatorname{Ho}\mathcal{C}$.

One of the requirements for a model of the theory of $(\infty, 1)$-categories is that the groupoid-like instances should be models of ∞-groupoids. If by '∞-groupoid' one means a (weak) homotopy type of Kan complexes, then the following result is relevant:

Proposition 6.1.20. *Let $\mathcal{C}$ be a quasicategory. The following are equivalent:*

(i) $\mathcal{C}$ *(as a simplicial set) is a Kan complex.*

(ii) *Every morphism in $\mathcal{C}$ is an equivalence.*

(iii) $\operatorname{Ho}\mathcal{C}$ *is a groupoid.*

Proof. See Corollary 1.5 in [Joyal, 2002]. □

There is also a homotopy-coherent notion of equivalence. Let $\mathbf{I}\mathbb{2}$ be the groupoid obtained by freely inverting the arrows in the category $\mathbb{2}$ freely generated by a morphism $0 \to 1$.

Definition 6.1.21. A **homotopy-coherent equivalence** in a quasicategory $\mathcal{C}$ is a functor $\mathbf{I}[1] \to \mathcal{C}$.

Remark 6.1.22. More explicitly, a homotopy-coherent equivalence in $\mathcal{C}$ consists of the following data:

- A pair of objects in $\mathcal{C}$, say x and y.
- A pair of morphisms in $\mathcal{C}$, say $f : x \to y$ and $g : y \to x$.
- A pair of 2-simplices, say α and β, witnessing the fact that $\mathrm{id}_x \sim g \circ f$ and $f \circ g \sim \mathrm{id}_y$.
- A pair of 3-simplices witnessing the fact that α and β satisfy (versions of) the triangle identities for adjunctions.
- etc.

That is, for each natural number n, we have a pair of $(n+1)$-simplices witnessing a coherence axiom for the given pair of n-simplices. Note that the data for $n \leq 2$ already determine a mutually quasi-inverse pair of equivalences in $\mathcal{C}$; we will refer to $f : x \to y$ as the **underlying morphism** of the homotopy-coherent equivalence.

When $\mathcal{C}$ is an ordinary category, the 2-simplices are unique *if* they exist, and given the 2-simplices, the required n-simplices exist and are unique for $n \geq 2$.

In other words, every isomorphism in an ordinary category can be equipped with the structure of a homotopy-coherent equivalence. It turns out the same is true for quasicategories:

Proposition 6.1.23. *Let $\mathcal{C}$ be a quasicategory. If f is an equivalence in $\mathcal{C}$, then there is a homotopy-coherent equivalence whose underlying morphism is f.*

Proof. See Corollary 1.6 in [Joyal, 2002], or Theorem 4.14 in [TQA]. □

Definition 6.1.24. Let **U** be a universe. The **homotopy 2-category of U-small quasicategories** is the following 2-category $\mathfrak{Qcat}$:

- The objects are **U**-small quasicategories.
- The category of morphisms $\mathcal{C} \to \mathcal{D}$ is the fundamental category $\tau_1[\mathcal{C}, \mathcal{D}]$, which we also denote by $\mathbf{QFun}(\mathcal{C}, \mathcal{D})$.
- Composition and identity morphisms are induced by τ_1 from the cartesian closed structure of **sSet**.

The construction of the 2-category $\mathfrak{Qcat}$ enables us to apply definitions from formal category theory to the context of quasicategories.

Definition 6.1.25. Let $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ be functors between quasicategories. A **natural equivalence** is a natural transformation $\alpha : f_0 \Rightarrow f_1$ whose image in the fundamental category $\tau_1[\mathcal{C}, \mathcal{D}]$ is an isomorphism.

As with natural transformations of functors between ordinary categories, natural transformations of functors between quasicategories have components. It is a non-trivial fact that a natural transformation is a natural equivalence if and only if its components are equivalences:

Theorem 6.1.26. *Let $f_0, f_1 : \mathfrak{C} \to \mathfrak{D}$ be functors between quasicategories and let $\alpha : f_0 \Rightarrow f_1$ be a natural transformation. Then α is a natural equivalence if and only if, for every object c in $\mathfrak{C}$, the morphism $\alpha_c : f_0(x) \to f_1(x)$ is an equivalence in $\mathfrak{D}$.*

Proof. See Theorem 5.14 in [TQA], or Lemma 2.3.8 in [Riehl and Verity, 2013a] □

Definition 6.1.27. An **equivalence of quasicategories** is an equivalence in the 2-category $\mathfrak{Qcat}$, i.e. a tuple $(f, g, \eta, \varepsilon)$ where:

- $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{C}$ are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathfrak{C}} \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathfrak{D}}$ are natural equivalences.

We will often abuse notation and say that f is an equivalence of quasicategories, omitting mention of the other data.

Definition 6.1.28. An **adjunction of quasicategories** is an adjunction in the 2-category $\mathfrak{Qcat}$, i.e. a tuple $(f, g, \eta, \varepsilon)$ where:

- $f : \mathfrak{C} \to \mathfrak{D}$ and $g : \mathfrak{D} \to \mathfrak{C}$ are functors between quasicategories.
- $\eta : \mathrm{id}_{\mathfrak{C}} \Rightarrow g \circ f$ and $\varepsilon : f \circ g \Rightarrow \mathrm{id}_{\mathfrak{D}}$ are natural transformations.
- The triangle identities are satisfied:

$$\begin{aligned} \left(\varepsilon \circ \mathrm{id}_f\right) \bullet \left(\mathrm{id}_f \circ \eta\right) &= \mathrm{id}_f && \text{in } \mathbf{QFun}(\mathfrak{C}, \mathfrak{D}) \\ \left(\mathrm{id}_g \circ \varepsilon\right) \bullet \left(\eta \circ \mathrm{id}_g\right) &= \mathrm{id}_g && \text{in } \mathbf{QFun}(\mathfrak{D}, \mathfrak{C}) \end{aligned}$$

Remark 6.1.29. There also exist homotopy-coherent versions of the above definitions, but it is a theorem of Riehl and Verity [2013b] that every adjunction of quasicategories can be extended to a homotopy-coherent adjunction.

Lemma 6.1.30. *Let* **U** *be a universe, let* **Cat** *be the category of* **U***-small categories, and let* **Qcat** *be the category of* **U***-small quasicategories. The functor* $\mathrm{Ho} : \mathbf{Qcat} \to \mathbf{Cat}$ *is isomorphic to (the underlying functor of) a representable 2-functor* $\mathfrak{Qcat} \to \mathfrak{Cat}$.

Proof. This is an immediate consequence of the natural isomorphism $[\mathbb{1}, -] \cong \mathrm{id}_{\mathbf{Qcat}}$ and the fact that $\mathbf{QFun}(\mathbb{1}, -)$ is a 2-functor $\mathfrak{Qcat} \to \mathfrak{Cat}$. ■

6.2 The Joyal model structure

Prerequisites. §§ 0.5, 1.3, 4.1, 5.4, 6.1, A.2, A.4.

Just as there is a model structure on **Cat** whose homotopy category is the category of small categories modulo natural isomorphism of functors, there is a model structure on **sSet**, due to Joyal [TQ1], whose homotopy category is the category of small quasicategories modulo natural equivalence of functors.

¶ 6.2.1. Throughout this section, τ_0 denotes the functor $\mathbf{sSet} \to \mathbf{Set}$ that sends a simplicial set X to the set of isomorphism classes of objects in the fundamental category $\tau_1 X$. Note that it can be factored as $\pi_0 \circ \mathrm{iso} \circ \tau_1$, where $\mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$ is the *right* adjoint of the inclusion $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$.

Definition 6.2.2. A **weak categorical equivalence** is a morphism $f : W \to Z$ of simplicial sets such that the induced map

$$\tau_0[f, \mathcal{K}] : \tau_0[Z, \mathcal{K}] \to \tau_0[W, \mathcal{K}]$$

is a bijection for all small quasicategories $\mathcal{K}$.

Lemma 6.2.3. *Let $f : \mathfrak{C} \to \mathfrak{D}$ be a functor between small quasicategories. The following are equivalent:*

(i) *f is (part of) an equivalence of quasicategories.*

(ii) *f is a weak categorical equivalence.*

(iii) *For all small quasicategories $\mathcal{K}$, the induced map*

$$\tau_0[\mathcal{K}, f] : \tau_0[\mathcal{K}, \mathfrak{C}] \to \tau_0[\mathcal{K}, \mathfrak{D}]$$

is a bijection.

Proof. (i) ⇒ (ii). It is not hard to see that $f : \mathfrak{C} \to \mathfrak{D}$ is (part of) an equivalence of quasicategories if and only if the induced functor

$$\tau_1[f, \mathcal{K}] : \tau_1[\mathfrak{D}, \mathcal{K}] \to \tau_1[\mathfrak{C}, \mathcal{K}]$$

is (part of) an equivalence of categories for all small quasicategories $\mathcal{K}$. The functor $\pi_0 \circ \mathrm{iso} : \mathbf{Cat} \to \mathbf{Set}$ sends equivalences to bijections, so we may deduce that $f : \mathfrak{C} \to \mathfrak{D}$ is a weak categorical equivalence.

(i) ⇒ (iii). The proof is similar to that of (i) ⇒ (ii).

(ii) ⇒ (i), (iii) ⇒ (i). These are straightforward exercises in chasing identity morphisms. ■

Proposition 6.2.4. **sSet** *with the class of weak categorical equivalences constitute a saturated relative category; in particular, the class of weak categorical equivalences has the 2-out-of-6 property.*

Proof. The collection of functors $\tau_0[-, \mathcal{K}] : \mathbf{sSet} \to \mathbf{Set}$, as $\mathcal{K}$ varies over the small quasicategories, *jointly* reflect isomorphisms as weak categorical equivalences, so the class of weak categorical equivalences must be saturated. For the 2-out-of-6 property, see corollary A.4.15. ■

Definition 6.2.5. An **inner fibration** of simplicial sets is a morphism $f : X \to Y$ with the right lifting property with respect to the inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ for all $n \geq 2$ and $0 < i < n$.

Remark 6.2.6. It is clear that a simplicial set X is a quasicategory if and only if the unique morphism $X \to 1$ is an inner fibration. Unfortunately, these are *not* the fibrations in the Joyal model structure.

Definition 6.2.7. An **inner anodyne extension** is a morphism of simplicial sets with the left lifting property with respect to all inner fibrations.

Proposition 6.2.8. *There exist an $\aleph_0$-accessible functor $M : [\mathbb{2}, \mathbf{sSet}] \to \mathbf{sSet}$ and two natural transformations $i : \mathrm{dom} \Rightarrow M$ and $p : M \Rightarrow \mathrm{codom}$ such that, for all objects f in $[\mathbb{2}, \mathbf{sSet}]$:*

- $f = p_f \circ i_f$.
- *i_f is a relative $\mathcal{I}'$-cell complex, where $\mathcal{I}'$ is the set of inner horn inclusions.*
- *p_f is an inner fibration of simplicial sets.*

Proof. Using proposition 0.2.43, it is not hard to see that the inner horn inclusions are $\aleph_0$-compact as objects in $[\mathbb{2}, \mathbf{sSet}]$. We then apply corollary 0.5.14. ■

Corollary 6.2.9. *There exist an $\aleph_0$-accessible functor $R : \mathbf{sSet} \to \mathbf{sSet}$ and a natural transformation $i : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ such that, for all objects X in* **sSet***:*

- *RX is a small quasicategory.*
- *$i_X : X \to RX$ is an inner anodyne extension.* ■

Corollary 6.2.10. *Inner anodyne extensions are monomorphisms.*

Proof. The class of monomorphisms in **sSet** is closed under coproducts, pushouts, transfinite composition, and retracts, so we may apply corollary 0.5.13. ■

Theorem 6.2.11. *Let $i : Z \to W$ be a monomorphism in* **sSet** *and let $p : X \to Y$ be an inner fibration. Suppose we have a commutative diagram*

$$\begin{tikzcd}
{[W,X]} \arrow[dr, dashed, "q"] \arrow[drr, bend left, "{[i,X]}"] \arrow[ddr, bend right, "{[W,p]}"'] & & \\
& L(i,p) \arrow[r] \arrow[d] & {[Z,X]} \arrow[d, "{[Z,p]}"] \\
& {[W,Y]} \arrow[r, "{[i,Y]}"'] & {[Z,Y]}
\end{tikzcd}$$

where the square in the lower right is a pullback square.

(i) *The unique morphism $q : [W,X] \to L(i,p)$ making the diagram commute is an inner fibration.*

(ii) *If $i : Z \to W$ is an inner anodyne extension, then $q : [W,X] \to L(i,p)$ is a trivial Kan fibration.*

(iii) *If $p : Z \to W$ is a trivial Kan fibration, then so is $q : [W,X] \to L(i,p)$.*

Proof. (i) and (ii). See Theorem 2.18 in [TQA].

(iii). This is a special case of proposition 1.3.13. □

Corollary 6.2.12.

(i) *If $p : X \to Y$ is an inner fibration, then for all simplicial sets W, the morphism $[W,p] : [W,X] \to [W,Y]$ is also an inner fibration.*

(ii) *If $i : Z \to W$ is a monomorphism (resp. inner anodyne extension) and $\mathcal{K}$ is a small quasicategory, then the morphism $[i,\mathcal{K}] : [W,\mathcal{K}] \to [Z,\mathcal{K}]$ is an inner fibration (resp. trivial Kan fibration).*

(iii) *If W is any simplicial set and $\mathcal{K}$ is a small quasicategory, then $[W,\mathcal{K}]$ is also a small quasicategory.*

Proof. The proof is similar to that of corollary 1.3.14. ■

Corollary 6.2.13. **Qcat** *is an exponential ideal of* **sSet***; in particular,* **Qcat** *is a cartesian closed category.* ∎

Proposition 6.2.14. *Let $f : W \to Z$ be a morphism in* **sSet**. *The following are equivalent:*

(i) *For all small quasicategories $\mathcal{K}$, the induced functor*

$$[f, \mathcal{K}] : [Z, \mathcal{K}] \to [W, \mathcal{K}]$$

is (part of) an equivalence of quasicategories.

(ii) *For all small quasicategories $\mathcal{K}$, the induced functor*

$$\mathrm{Ho}\,[f, \mathcal{K}] : \mathrm{Ho}\,[Z, \mathcal{K}] \to \mathrm{Ho}\,[W, \mathcal{K}]$$

is (part of) an equivalence of categories.

(iii) *The morphism $f : W \to Z$ is a weak categorical equivalence.*

Proof. (i) ⇒ (ii). This is a corollary of lemma 6.1.30.

(ii) ⇒ (iii). Any equivalence of categories must induce a bijection on isomorphism classes of objects.

(iii) ⇒ (i). Suppose $f : W \to Z$ is a weak categorical equivalence, i.e. that the induced map

$$\tau_0[f, \mathcal{K}] : \tau_0[Z, \mathcal{K}] \to \tau_0[W, \mathcal{K}]$$

is a bijection of sets for all small quasicategories $\mathcal{K}$. Then, for all simplicial sets X and all small quasicategories $\mathcal{K}$, the induced map

$$\tau_0[f, [X, \mathcal{K}]] : \tau_0[Z, [X, \mathcal{K}]] \to \tau_0[W, [X, \mathcal{K}]]$$

is a bijection, because $[X, \mathcal{K}]$ is a quasicategory by corollary 6.2.13. Proposition A.2.11 then implies that the induced map

$$\tau_0[X, [f, \mathcal{K}]] : \tau_0[X, [Z, \mathcal{K}]] \to \tau_0[X, [W, \mathcal{K}]]$$

is a bijection for all simplicial sets X and all small quasicategories $\mathcal{K}$. Thus, by lemma 6.2.3, the induced functor $[f, \mathcal{K}] : [Z, \mathcal{K}] \to [W, \mathcal{K}]$ is an equivalence of quasicategories. ∎

Proposition 6.2.15. *The class of weak categorical equivalences is closed under binary products.*

Proof. Let $f : X \to Y$ and $g : Z \to W$ be weak categorical equivalences. Since $f \times g = (\mathrm{id}_Y \times g) \circ (f \times \mathrm{id}_Z)$, it suffices (by symmetry) to show that $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ is a weak categorical equivalence, i.e. that the induced map

$$\tau_0[f \times \mathrm{id}_Z, \mathcal{K}] : \tau_0[Y \times Z, \mathcal{K}] \to \tau_0[X \times Z, \mathcal{K}]$$

is a bijection for all small quasicategories $\mathcal{K}$. By proposition A.2.11, it is the same to show that

$$\tau_0[f, [Z, \mathcal{K}]] : \tau_0[Y, [Z, \mathcal{K}]] \to \tau_0[X, [Z, \mathcal{K}]]$$

is a bijection for all small quasicategories $\mathcal{K}$; but corollary 6.2.13 says that the exponential object $[Z, \mathcal{K}]$ is a small quasicategory and f is a weak categorical equivalence, so the maps are indeed bijections. ■

Proposition 6.2.16. *Inner anodyne extensions are weak categorical equivalences.*

Proof. See Corollary 2.29 in [TQA]. □

Remark 6.2.17. It is *a priori* not clear whether the notion of weak categorical equivalence is stable under universe enlargement, but in fact it is. First, notice that the notion of weak categorical equivalence *between quasicategories* is stable under universe enlargement, by lemma 6.2.3. Given any morphism $f : X \to Y$ in **sSet**, we may apply the functor R of corollary 6.2.9 to get a commutative diagram of the form below,

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & RX \arrow[d, "Rf"] \\ Y \arrow[r, "i_Y"'] & RY \end{tikzcd}$$

and proposition 6.2.4 implies that the class of weak categorical equivalences has the 2-out-of-3 property, so $f : X \to Y$ is a weak categorical equivalence if and only if $Rf : RX \to RY$ is an equivalence of quasicategories. Since R and i are stable under universe enlargement, it follows that the property of f being a weak categorical equivalence is also stable.

Definition 6.2.18. An **isofibration** of quasicategories is a functor $f : \mathcal{C} \to \mathcal{D}$ with the following properties:

- f (as a morphism of simplicial sets) is an inner fibration.
- f has the right lifting property with respect to the inclusion $\{0\} \hookrightarrow \mathbf{I}\mathbb{2}$.

Proposition 6.2.19. *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small quasicategories.*

(i) *If f (as a morphism of simplicial sets) has the right lifting property with respect to all monomorphisms in* **sSet***, then f is an isofibration.*

(ii) *If $\mathcal{D}$ is an ordinary category, then f is an inner fibration.*

(iii) *Assuming f (as a morphism of simplicial sets) is an inner fibration, $f : \mathcal{C} \to \mathcal{D}$ is an isofibration if and only if $\operatorname{Ho} f : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ is an isofibration.*

Proof. (i). This is an immediate consequence of the fact that isofibrations are morphisms that have the right lifting property with respect to certain monomorphisms in **sSet**.

(ii). See Proposition 2.2 in [TQA].

(iii). See Proposition 4.5 in [TQA]. □

Theorem 6.2.20 (Joyal)**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between small quasicategories. The following are equivalent:*

(i) *f is an isofibration of quasicategories.*

(ii) *f (as a morphism of simplicial sets) has the right lifting property with respect to all monic weak categorical equivalences in* **sSet***.*

Proof. See Theorem 6.11 in [TQA], or combine Proposition 2.2.5.8 and Corollary 2.4.6.5 in [HTT]. □

Theorem 6.2.21 (Joyal)**.** *The following data constitute a cofibrantly-generated model structure on* **sSet***:*

- *The weak equivalences are the weak categorical equivalences.*
- *The cofibrations are the monomorphisms.*

- *The fibrations are the morphisms that have the right lifting property with respect to monomorphisms that are weak categorical equivalences.*

This model structure is called the **Joyal model structure for quasicategories***, and the fibrant objects are the quasicategories.*

Proof. See Theorem 6.12 in [TQA], or combine Proposition 2.2.5.8 with Theorems 2.2.5.1 and 2.4.6.1 in [HTT]. □

Remark 6.2.22. Joyal's determination principle (proposition 4.3.5) implies the Joyal model structure is stable under universe enlargement. Indeed, the claim is obvious for the class of cofibrations, the class of fibrant objects, and lemma 6.2.3 implies that the class of weak equivalences between fibrant objects is stable under universe enlargement; but this is enough data to uniquely determine a model structure.

Proposition 6.2.23. *The Joyal model structure for quasicategories is cartesian.*

Proof. The Joyal model structure for quasicategories is a Cisinski model structure, so we may apply proposition 5.4.12 to proposition 6.2.15 to deduce the claim. ■

Proposition 6.2.24. *Let* $\mathbf{Cat}$ *be the category of small categories, let* $\mathbf{sSet}$ *be the category of small simplicial sets, let* $\mathbf{Qcat}$ *be the full subcategory spanned by the small quasicategories, and let* $\operatorname{Ho}\mathbf{Qcat}$ *be the localisation of* $\mathbf{Qcat}$ *at the weak categorical equivalences.*

(i) *The adjunction*
$$\tau_1 \dashv \mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$$
is a Quillen adjunction with respect to the canonical model structure on $\mathbf{Cat}$ *and the Joyal model structure on* $\mathbf{sSet}$*.*

(ii) *The functors* τ_1 *and* N *preserve weak equivalences, and the induced adjunction*
$$\operatorname{Ho}\tau_1 \dashv \operatorname{Ho}\mathrm{N} : \operatorname{Ho}\mathbf{Cat} \to \operatorname{Ho}\mathbf{Qcat}$$
exhibits $\operatorname{Ho}\mathbf{Cat}$ *as a reflective exponential ideal of* $\operatorname{Ho}\mathbf{Qcat}$*.*

Proof. (i). See Proposition 6.14 in [TQA].

(ii). Apply theorem 5.4.19, Ken Brown's lemma (4.4.6), propositions 5.4.14 and A.2.13, and the 2-functoriality of Ho (corollary A.4.20). ■

— VII —

Derivators

7.1 Basics

Prerequisites. §§ 3.1, 3.6, A.1, A.5.

The notion of derivator has a somewhat complicated history; the name and the original idea are due to Grothendieck [1983, 1991], but Heller [1988] studied essentially the same thing independently. The distinguishing characteristic of the theory of derivators is its agnosticism: a derivator is a way of studying homotopy-coherent diagrams and their limits/colimits without using any particular model for homotopical algebra.

In this section, we use the explicit universe convention, all 2-categories and 2-functors will be strict unless otherwise stated, and for simplicity, we say 'coproduct', 'product', 'pullback', etc. instead of '2-coproduct', '2-product', '2-pullback' etc., i.e. we tacitly assume that these have the relevant 2-dimensional universal property in addition to the usual 1-dimensional universal property.

Definition 7.1.1. A **derivator domain** is 2-category $\mathfrak{K}$ satisfying these axioms:

D0. $\mathfrak{K}$ has an initial object 0, a terminal object 1, and tensors with the category $\mathbb{2} = \{0 \to 1\}$.

D1. $\mathfrak{K}$ has finite coproducts and pullbacks.

D2. $\mathfrak{K}$ has comma objects of the form $(u \downarrow b)$ and $(b \downarrow u)$ for all morphisms $u : A \to B$ and $b : 1 \to B$.

A **subdomain** of a derivator domain is a 2-full 2-subcategory that is closed under constructions specified in the above axioms.

Definition 7.1.2. Let $\mathbf{U}$ be a universe. A **U-small prederivator** on $\mathfrak{K}$ is a 2-functor $\mathscr{D} : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$, where $\mathfrak{K}$ is a derivator domain and $\mathfrak{Cat}$ is the 2-category of $\mathbf{U}$-small categories. A **prederivator** is a 2-functor that is a $\mathbf{U}$-small prederivator for some universe $\mathbf{U}$.

We write $\mathscr{D}^A$ for the value of $\mathscr{D}$ at an object A in $\mathfrak{K}$, and we write either $\mathscr{D}^u$ or u^* for the functor $\mathscr{D}^B \to \mathscr{D}^A$ induced by a morphism $u : A \to B$ in $\mathfrak{K}$. If $f : x \to y$ is a morphism in $\mathscr{D}^B$, then we may sometimes write $f \upharpoonright u : x \upharpoonright u \to y \upharpoonright u$ instead of $u^*(f) : u^*(x) \to u^*(y)$. The **underlying category** of a prederivator $\mathscr{D}$ is the category $\mathscr{D}^1$, where 1 is any terminal object of $\mathfrak{K}$.

Remark 7.1.3. While it is true that $\mathfrak{K}$ is a derivator domain if and only if $\mathfrak{K}^{\mathrm{co}}$ is a derivator domain, the duality principle for general prederivators is somewhat subtle: because $(-)^{\mathrm{op}}$ is a 2-functor $\mathfrak{Cat}^{\mathrm{co}} \to \mathfrak{Cat}$, the opposite of a prederivator on $\mathfrak{K}$ is a prederivator on $\mathfrak{K}^{\mathrm{co}}$, which is in general not isomorphic or even equivalent to $\mathfrak{K}$.

One should be aware that some authors (e.g. Cisinski [2003]) define prederivators to be 2-functors $\mathfrak{K}^{\mathrm{coop}} \to \mathfrak{Cat}$; readers should take care to dualise results appropriately when translating between the two conventions.

Definition 7.1.4. A **semiderivator** on $\mathfrak{K}$ is prederivator $\mathscr{D} : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$ satisfying the following axioms:

- **Der1.** $\mathscr{D}$ sends coproducts of finite families of objects in $\mathfrak{K}$ to products in $\mathfrak{Cat}$.
- **Der2.** Let A be an object in $\mathfrak{K}$ and let $f : x \to y$ be a morphism in $\mathscr{D}^A$. Then, f is an isomorphism in $\mathscr{D}^A$ if and only if, for all morphisms $a : 1 \to A$ in $\mathfrak{K}$, the morphism $f \upharpoonright a : x \upharpoonright a \to y \upharpoonright a$ is an isomorphism in $\mathscr{D}^1$.

Example 7.1.5. If B is an object in $\mathfrak{K}$ and $\mathfrak{K}$ is a locally $\mathbf{U}$-small 2-category, then the 2-functor $\mathfrak{K}(-, B) : \mathfrak{K}^{\mathrm{op}} \to \mathfrak{Cat}$ is a prederivator. We say $\mathfrak{K}(-, B)$ is the **prederivator represented by** B.

Definition 7.1.6. Let C be a $\mathbf{U}$-small relative category. The **prederivator** of C, denoted by $\mathscr{D}(C)$, is the $\mathbf{U}$-small prederivator on $\mathfrak{RelCat}$ (or any subdomain thereof) defined by $\mathscr{D}(C)^{\mathcal{A}} = \mathrm{Ho}\,[\mathcal{A}, C]_{\mathrm{h}}$.

Proposition 7.1.7. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. If A is an object in $\mathfrak{K}$ and $\mathbb{C}$ is a category for which the tensor $\mathbb{C} \odot A$ exists, then there is a canonical comparison functor $\mathscr{D}^{\mathbb{C} \odot A} \to \left[\mathbb{C}, \mathscr{D}^A\right]$.*

Proof. By definition, the object $\mathbb{C} \odot A$ in $\mathfrak{K}$ induces isomorphisms

$$\mathfrak{K}(\mathbb{C} \odot A, B) \cong [\mathbb{C}, \mathfrak{K}(A, B)]$$

that are 2-natural in B. Since $\mathscr{D}$ is a prederivator on $\mathfrak{K}$, it induces a functor $\mathfrak{K}(A, B) \to \left[\mathscr{D}^B, \mathscr{D}^A\right]$ that is 2-natural in A and in B, so we obtain a 2-natural functor $\mathfrak{K}(\mathbb{C} \odot A, B) \to \left[\mathbb{C}, \left[\mathscr{D}^B, \mathscr{D}^A\right]\right]$ by composition; but we have 2-natural isomorphisms

$$\left[\mathbb{C}, \left[\mathscr{D}^B, \mathscr{D}^A\right]\right] \cong \left[\mathbb{C} \times \mathscr{D}^B, \mathscr{D}^A\right] \cong \left[\mathscr{D}^B, \left[\mathbb{C}, \mathscr{D}^A\right]\right]$$

so, taking $B = \mathbb{C} \odot A$, we obtain the required functor $\mathscr{D}^{\mathbb{C} \odot A} \to \left[\mathbb{C}, \mathscr{D}^A\right]$. ■

Definition 7.1.8. A **strong semiderivator** on $\mathfrak{K}$ is a semiderivator that satisfies the additional axiom below:

Der5. For any object A in $\mathfrak{K}$, the canonical functor $\mathscr{D}^{\mathbb{2} \odot A} \to \left[\mathbb{2}, \mathscr{D}^A\right]$ is full and essentially surjective on objects (but not necessarily faithful).

Remark 7.1.9. If $\mathscr{D}$ is the prederivator represented by an object in $\mathfrak{K}$, then $\mathscr{D}$ automatically satisfies axioms Der1 and Der5; and if $\mathfrak{K}$ is a 2-full 2-subcategory of $\mathfrak{Cat}$ with the same terminal object, then $\mathscr{D}$ will also satisfy axiom Der2.

Lemma 7.1.10. *If $\mathcal{C}$ is a uni-fractionable category, then the canonical comparison functor* $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \operatorname{Ho} \mathcal{C}]$ *is full and essentially surjective on objects.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be subcategories of $\operatorname{weq} \mathcal{C}$ such that $\mathcal{C}$ admits a three-arrow calculus with respect to $(\mathcal{U}, \mathcal{V})$, and let $\bar{f} : X \to Y$ be any morphism in $\operatorname{Ho} \mathcal{C}$. By the fundamental theorem of three-arrow calculi (3.6.9), there exist $u : Y \to \hat{Y}$ in $\mathcal{U}$, $v : \tilde{X} \to X$ in $\mathcal{V}$, and $f : \tilde{X} \to \hat{Y}$ such that $\bar{f} = u^{-1} \circ f \circ v^{-1}$ in $\operatorname{Ho} \mathcal{C}$, i.e. such that the following diagram in $\operatorname{Ho} \mathcal{C}$ commutes:

$$\begin{tikzcd} X \arrow[r, "v^{-1}"] \arrow[d, "\bar{f}"'] & \tilde{X} \arrow[d, "f"] \\ Y \arrow[r, "u"'] & \hat{Y} \end{tikzcd}$$

It immediately follows that $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is essentially surjective on objects.

It remains to be shown that $\mathrm{Ho}\,[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is a full functor. Let $x : X_1 \to X_2$ and $y : Y_1 \to Y_2$ be morphisms in $\mathcal{C}$, let $\bar{f}_1 : X_1 \to Y_1$ and $\bar{f}_2 : X_2 \to Y_2$ be morphisms in $\mathrm{Ho}\,\mathcal{C}$, and suppose we have $\bar{f}_2 \circ x = y \circ \bar{f}_1$; note this constitutes a morphism in $[\mathbb{2}, \mathcal{C}]$ between objects in the image of the functor $\mathrm{Ho}\,[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$. As before, we may choose $u_1 : Y_1 \to \hat{Y}_1$ and $u_2 : Y_2 \to \hat{Y}_2$ in $\mathcal{U}$, $v_1 : \tilde{X}_1 \to X_1$ and $v_2 : \tilde{X}_2 \to X_2$ in $\mathcal{V}$, and $f_1 : \tilde{X}_1 \to \hat{Y}_1$ and $f_2 : \tilde{X}_2 \to \hat{Y}_2$ in $\mathcal{C}$ such that the equations below hold in $\mathrm{Ho}\,\mathcal{C}$:

$$\bar{f}_1 = u_1^{-1} \circ f_1 \circ v_1^{-1} \qquad\qquad \bar{f}_2 = u_2^{-1} \circ f_2 \circ v_2^{-1}$$

Using axioms A2 and A3, there exist $u_2' : Y_2 \to Z$ in $\mathcal{U}$, $v_1' : W \to X_1$ in $\mathcal{V}$, and $z : \hat{Y}_1 \to Z$ and $w : W \to \tilde{X}_2$ making the following diagrams in $\mathcal{C}$ commute,

$$\begin{tikzcd}
X_1 \arrow[d, equal] & \tilde{X}_1 \arrow[l, "v_1"'] \arrow[r, "f_1"] \arrow[d, equal] & \hat{Y}_1 \arrow[d, dashed, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_1 & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "z \circ f_1"'] & Z & Y_2 \arrow[l, dashed, "u_2'"]
\end{tikzcd}$$

$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & W \arrow[l, dashed, "v_1'"'] \arrow[r, "f_2 \circ w"] \arrow[d, dashed, "w"] & \hat{Y}_2 \arrow[d, equal] & Y_2 \arrow[l, "u_2"'] \arrow[d, equal] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & Y_2 \arrow[l, "u_2"]
\end{tikzcd}$$

and since $\bar{f}_2 \circ x = y \circ \bar{f}_1$, the fundamental theorem says there exist a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
X_1 \arrow[d, equal] & W \arrow[l, "v_1'"'] \arrow[r, "f_2 \circ w"] & \hat{Y}_2 & Y_2 \arrow[l, "u_2"'] \arrow[d, equal] \\
X_1 \arrow[d, equal] & \bullet \arrow[l, "v_5"'] \arrow[r, "f_5"] \arrow[u, "v_3"] \arrow[d, "w_3"'] & \bullet \arrow[u, "v_4"'] \arrow[d, "w_4"] & Y_2 \arrow[l, "u_6"'] \arrow[d, equal] \\
X_1 \arrow[d, equal] & \bullet \arrow[l, "v_6"] \arrow[r, "f_6"'] & \bullet & Y_2 \arrow[l, "u_6"] \arrow[d, equal] \\
X_1 & \tilde{X}_1 \arrow[l, "v_1"] \arrow[r, "z \circ f_1"'] \arrow[u, "u_3"] & Z \arrow[u, "u_4"'] & Y_2 \arrow[l, "u_2'"]
\end{tikzcd}$$

where u_3, u_4, u_5, u_6 are in $\mathcal{U}$, v_3, v_4, v_5, v_6 are in $\mathcal{V}$, and w_3, w_4 are weak equivalences in $\mathcal{C}$.

It is easy to verify that the following diagram in $\mathcal{C}$ commutes,

$$\begin{tikzcd}
X_1 \arrow[d, "x"'] & W \arrow[l, "v_1'"'] \arrow[r, equal] \arrow[d, "w"'] & W \arrow[d, "f_2 \circ w"'] & \bullet \arrow[l, "v_3"'] \arrow[r, "w_3"] \arrow[d, "f_5"'] & \bullet \arrow[d, "f_6"] & \tilde{X}_1 \arrow[l, "u_3"'] \arrow[r, "f_1"] \arrow[d, "z \circ f_1"] & \hat{Y}_1 \arrow[d, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & \bullet \arrow[l, "v_4"] \arrow[r, "w_4"'] & \bullet & Z \arrow[l, "u_4"] \arrow[r, equal] & Z & Y_2 \arrow[l, "u_2'"]
\end{tikzcd}$$

and this is the required lift of $\left(\bar{f}_1, \bar{f}_2\right)$ to $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}}$, because the diagram in $\mathcal{C}$ shown below commutes:

$$\begin{tikzcd}
X_1 \arrow[r, equal] \arrow[d, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 \arrow[r, equal] & X_1 & \tilde{X}_1 \arrow[l, "v_1"'] \arrow[r, "f_1"] \arrow[d, equal] & \hat{Y}_1 \arrow[d, equal] & Y_1 \arrow[l, "u_1"'] \arrow[d, equal] \\
X_1 \arrow[d, "x"'] & W \arrow[l, "v_1'"'] \arrow[r, equal] \arrow[u, "v_1'"] \arrow[d, "w"'] & W \arrow[u, "v_1'"] \arrow[d, "f_2 \circ w"'] & \bullet \arrow[l, "v_3"'] \arrow[r, "w_3"] \arrow[u, "v_5"] \arrow[d, "f_5"'] & \bullet \arrow[u, "v_6"'] \arrow[d, "f_6"] & \tilde{X}_1 \arrow[l, "u_3"'] \arrow[r, "f_1"] \arrow[d, "z \circ f_1"] & \hat{Y}_1 \arrow[d, "z"] & Y_1 \arrow[l, "u_1"'] \arrow[d, "y"] \\
X_2 \arrow[d, equal] & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] \arrow[d, equal] & \hat{Y}_2 \arrow[d, equal] & \bullet \arrow[l, "v_4"] \arrow[r, "w_4"'] & \bullet & Z \arrow[l, "u_4"] \arrow[r, equal] & Z & Y_2 \arrow[l, "u_2'"] \arrow[d, equal] \\
X_2 & \tilde{X}_2 \arrow[l, "v_2"] \arrow[r, "f_2"'] & \hat{Y}_2 & Y_2 \arrow[l, "u_2"] \arrow[r, equal] \arrow[u, "u_5"] & Y_2 \arrow[r, equal] \arrow[u, "u_6"'] & Y_2 \arrow[r, equal] \arrow[u, "u_2'"'] & Y_2 \arrow[r, equal] \arrow[u, "u_2'"'] & Y_2
\end{tikzcd}$$

We may therefore conclude that $\operatorname{Ho}[\min \mathbb{2}, \mathcal{C}]_{\mathrm{h}} \to [\mathbb{2}, \mathcal{C}]$ is indeed full. ■

Proposition 7.1.11. *Let $\mathscr{D}$ be the prederivator of a* $\mathbf{U}$*-small relative category $\mathcal{M}$.*

(i) *$\mathscr{D}$ satisfies axiom Der1.*

(ii) *Moreover, if $\mathcal{M}$ is a (necessarily saturated) homotopical category and each homotopical functor category $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ admits a three-arrow calculus, then $\mathscr{D}$ is a strong semiderivator.*

Proof. (i). Proposition A.4.19 implies $\mathscr{D}$ sends finite coproducts in $\mathfrak{RelCat}$ to products in $\mathfrak{Cat}^{+}$, so axiom Der1 is satisfied.

(ii). Suppose $f : X \to Y$ is a morphism in $\operatorname{Ho}[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ such that all its components are isomorphisms in $\operatorname{Ho}\mathcal{M}$. The fundamental theorem of three-arrow calculi (3.6.9) says $f : X \to Y$ may be represented by a zigzag in $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ of the form below,

$$X \xleftarrow{\psi} \tilde{X} \xrightarrow{\theta} \hat{Y} \xleftarrow{\varphi} Y$$

where ψ and φ are natural weak equivalences. Thus, if A is an object in $\mathcal{A}$, then the following zigzag represents an isomorphism in $\operatorname{Ho} \mathcal{M}$:

$$XA \xleftarrow{\psi_A} \tilde{X}A \xrightarrow{\theta_A} \hat{Y}A \xleftarrow{\varphi_A} YA$$

However, proposition 3.6.10 says $\mathcal{M}$ is a saturated homotopical category, so θ_A must be a weak equivalence in $\mathcal{M}$ as well; hence, $f : X \to Y$ is an isomorphism in $\operatorname{Ho} [\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$. This shows that $\mathscr{D}$ satisfies axiom Der2.

Finally, observe that $[\min \mathbb{2} \times \mathcal{A}, \mathcal{M}]_{\mathrm{h}} \cong \left[\min \mathbb{2}, [\mathcal{A}, \mathcal{M}]_{\mathrm{h}}\right]_{\mathrm{h}}$, and the hypothesis says $[\mathcal{A}, \mathcal{M}]_{\mathrm{h}}$ admits a three-arrow calculus, so we apply lemma 7.1.10 to deduce that axiom Der5 is satisfied. ■

Definition 7.1.12. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, let $u : A \to B$ be a morphism in $\mathfrak{K}$, and let X be an object in $\mathscr{D}^A$.

- A **left $\mathscr{D}$-extension** of X along u is an initial object in the comma category $(X \downarrow u^*)$.
- A **right $\mathscr{D}$-extension** of X along u is a terminal object in the comma category $(u^* \downarrow X)$.
- We say $\mathscr{D}$ has **left extensions** along u if the functor $u^* : \mathscr{D}^B \to \mathscr{D}^A$ has a left adjoint, which we denote by $u_! : \mathscr{D}^A \to \mathscr{D}^B$.
- We say $\mathscr{D}$ has **right extensions** along u if the functor $u^* : \mathscr{D}^B \to \mathscr{D}^A$ has a right adjoint, which we denote by $u_* : \mathscr{D}^A \to \mathscr{D}^B$.

We may refer to left and right $\mathscr{D}$-extensions generically as **homotopy Kan extensions** in $\mathscr{D}$.

Remark 7.1.13. It is straightforward to check that $\mathscr{D}$ has left (resp. right) extensions along u if and only if, for every object X in $\mathscr{D}^A$, there exists a left (resp. right) $\mathscr{D}$-extension of X along u.

Example 7.1.14. If $\mathfrak{K}$ is a 2-full 2-subcategory of $\mathfrak{Cat}$ and $\mathscr{D}$ is the prederivator represented by an object in $\mathfrak{K}$, then $\mathscr{D}$-extensions are exactly the same thing as Kan extensions in the usual sense.

As we saw in theorem A.5.15, pointwise left (resp. right) Kan extensions can be computed as colimits (resp. limits) of certain diagrams whose shapes are comma categories. We shall shortly see that more is true.

Definition 7.1.15. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$ and suppose we have a diagram in $\mathfrak{K}$ of the following form:

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

- We say the square is a **left $\mathscr{D}$-exact square** if $\mathscr{D}$ has left extensions along $u : A \to C$ and $q : D \to B$ and the induced diagram shown below satisfies the left Beck–Chevalley condition:

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\theta^*}{\nearrow}"] & \mathscr{D}^B \arrow[d, "q^*"] \\ \mathscr{D}^A \arrow[r, "p^*"'] & \mathscr{D}^D \end{tikzcd}$$

- We say the square is a **right $\mathscr{D}$-exact square** if $\mathscr{D}$ has right extensions along $v : B \to C$ and $p : D \to A$ and the induced diagram shown below satisfies the right Beck–Chevalley condition:

$$\begin{tikzcd} \mathscr{D}^C \arrow[r, "u^*"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\underset{\theta^*}{\swarrow}"] & \mathscr{D}^A \arrow[d, "p^*"] \\ \mathscr{D}^B \arrow[r, "q^*"'] & \mathscr{D}^D \end{tikzcd}$$

- A **$\mathscr{D}$-exact square** in $\mathfrak{K}$ is a diagram in $\mathfrak{K}$ that is both left $\mathscr{D}$-exact and right $\mathscr{D}$-exact.

Proposition 7.1.16. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. Given the following diagram in $\mathfrak{K}$,*

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] & C \end{tikzcd}$$

if $\mathscr{D}$ has left extensions along $u : A \to C$ and $q : D \to B$, and $\mathscr{D}$ has right extensions along $v : B \to C$ and $p : D \to A$, then the following are equivalent:

(i) *The diagram is a $\mathscr{D}$-exact square.*

(ii) *The diagram is a left $\mathscr{D}$-exact square.*

(iii) *The diagram is a right $\mathscr{D}$-exact square.*

Proof. Statement (i) is just the conjunction of statements (ii) and (iii), and when the required left and right adjoints exist, proposition A.1.11 implies that statements (ii) and (iii) are equivalent. ■

Lemma 7.1.17 (Pasting exact squares)**.** *Let $\mathscr{D}$ be a prederivator, and consider pasting diagrams of the following forms in $\mathfrak{K}$:*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

In either diagram, if both squares are left (resp. right) $\mathscr{D}$-exact squares, then the rectangle obtained by pasting the two squares is also a left (resp. right) $\mathscr{D}$-exact square.

Proof. Apply lemma A.1.10. ■

Lemma 7.1.18. *Let* **Set** *be the category of* **U***-sets. If $\mathscr{D}$ is the prederivator of* **Set** *restricted to the subdomain $\mathfrak{Cat}$, then every comma square in $\mathfrak{Cat}$ is a right $\mathscr{D}$-exact square.*

Proof. Suppose we have the following comma square in $\mathfrak{Cat}$:

$$\begin{tikzcd} (u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\theta \nearrow"] & \mathbb{B} \arrow[d, "v"] \\ \mathbb{A} \arrow[r, "u"'] & \mathbb{C} \end{tikzcd}$$

Let $Y : \mathbb{B} \to \mathbf{Set}$ be a functor and let (Z, ε) be a right Kan extension of Y along v, i.e. a terminal object in the comma category $(v^* \downarrow Y)$. In view of lemma A.1.9, to deduce the claim, it is enough to show that $\left(u^*(Z), q^*(\varepsilon) \bullet \theta^*_Z\right)$ is a terminal object in the comma category $(p^* \downarrow q^*(Y))$, i.e. a right Kan extension of Yq along p; but this was done in lemma A.5.8. ■

Proposition 7.1.19. *Let $\mathcal{M}$ be a locally* **U**-*small category, and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted to $\mathfrak{Cat}$.*

- *If $\mathcal{M}$ has colimits for all* **U**-*small diagrams, then every comma square in $\mathfrak{Cat}$ is a left $\mathscr{D}$-exact square.*
- *If $\mathcal{M}$ has limits for all* **U**-*small diagrams, then every comma square in $\mathfrak{Cat}$ is a right $\mathscr{D}$-exact square.*

Proof. The two claims are formally dual; we will prove the first version.

Consider a comma square in $\mathfrak{Cat}$:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & \mathbb{B} \arrow[d, "v"] \\ \mathbb{A} \arrow[r, "u"'] & \mathbb{C} \end{tikzcd}$$

If $\mathcal{M}$ has colimits for all **U**-small diagrams, then theorem A.5.15 implies that, for any functor $X : \mathbb{A} \to \mathcal{M}$, the left Kan extension of X along u exists and is pointwise, and same is true for the left Kan extension of $p^*(X)$ along q. Thus, for any object M in $\mathcal{M}$, if $\hbar_M : \mathcal{M}^{\mathrm{op}} \to \mathbf{Set}^+$ is the representable functor $\mathcal{M}(-, M)$, we may use lemma A.1.9 to deduce that the following (commutative!) diagrams satisfy the *right* Beck–Chevalley condition:

$$\begin{tikzcd} {[\mathbb{C}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{C}^{\mathrm{op}}, \hbar_M]}"] \arrow[d, "(u^{\mathrm{op}})^*"'] & {[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[d, "(u^{\mathrm{op}})^*"] \\ {[\mathbb{A}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{A}^{\mathrm{op}}, \hbar_M]}"'] & {[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}^+]} \end{tikzcd} \qquad \begin{tikzcd} {[\mathbb{B}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{A}^{\mathrm{op}}, \hbar_M]}"] \arrow[d, "(q^{\mathrm{op}})^*"'] & {[\mathbb{B}^{\mathrm{op}}, \mathbf{Set}^+]} \arrow[d, "(q^{\mathrm{op}})^*"] \\ {[\mathbb{D}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]} \arrow[r, "{[\mathbb{B}^{\mathrm{op}}, \hbar_M]}"'] & {[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}^+]} \end{tikzcd}$$

On the other hand, lemma 7.1.18 says the diagram below satisfies the right Beck–Chevalley condition,

$$\begin{tikzcd}
[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}^{+}] \arrow[r, "(v^{\mathrm{op}})^*"] \arrow[d, "(u^{\mathrm{op}})^*"'] & [\mathbb{B}^{\mathrm{op}}, \mathbf{Set}^{+}] \arrow[d, "(q^{\mathrm{op}})^*"] \arrow[dl, phantom, "\Swarrow (\theta^{\mathrm{op}})^*"] \\
[\mathbb{A}^{\mathrm{op}}, \mathbf{Set}^{+}] \arrow[r, "(p^{\mathrm{op}})^*"'] & [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}^{+}]
\end{tikzcd}$$

and the family $\left\{ \hbar_M : \mathcal{M}^{\mathrm{op}} \to \mathbf{Set}^{+} \;\middle|\; M \in \mathrm{ob}\,\mathcal{M} \right\}$ is jointly conservative, so we deduce that the right Beck–Chevalley condition for the following diagram is satisfied,

$$\begin{tikzcd}
[\mathbb{C}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}] \arrow[r, "(v^{\mathrm{op}})^*"] \arrow[d, "(u^{\mathrm{op}})^*"'] & [\mathbb{B}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}] \arrow[d, "(q^{\mathrm{op}})^*"] \arrow[dl, phantom, "\Swarrow (\theta^{\mathrm{op}})^*"] \\
[\mathbb{A}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}] \arrow[r, "(p^{\mathrm{op}})^*"'] & [\mathbb{D}^{\mathrm{op}}, \mathcal{M}^{\mathrm{op}}]
\end{tikzcd}$$

and therefore this diagram satisfies the *left* Beck–Chevalley condition:

$$\begin{tikzcd}
[\mathbb{C}, \mathcal{M}] \arrow[r, "v^*"] \arrow[d, "u^*"'] & [\mathbb{B}, \mathcal{M}] \arrow[d, "q^*"] \\
[\mathbb{A}, \mathcal{M}] \arrow[r, "p^*"'] \arrow[ur, phantom, "\theta^* \Nearrow"] & [\mathbb{D}, \mathcal{M}]
\end{tikzcd}$$

We then conclude that every comma square in $\mathfrak{Cat}$ is a left $\mathscr{D}$-exact square. ■

Definition 7.1.20. A **$\mathfrak{K}$-cocomplete semiderivator** is a semiderivator $\mathscr{D}$ on $\mathfrak{K}$ satisfying these additional axioms:

Der3L. $\mathscr{D}$ has left extensions along every morphism $u : A \to B$ in $\mathfrak{K}$.

Der4L. Every comma square in $\mathfrak{K}$ of the form below is a left $\mathscr{D}$-exact square:

$$\begin{tikzcd}
(u \downarrow c) \arrow[r] \arrow[d] & 1 \arrow[d, "c"] \\
A \arrow[r, "u"'] \arrow[ur, phantom, "\Nearrow"] & C
\end{tikzcd}$$

Dually, a **$\mathfrak{K}$-complete U-semiderivator** is one satisfying these axioms:

Der3R. $\mathscr{D}$ has right extensions along every morphism $u : A \to B$ in $\mathfrak{K}$.

Der4R. Every comma square in $\mathfrak{K}$ of the form below is a right $\mathscr{D}$-exact square:

$$\begin{tikzcd}
(c \downarrow v) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, "v"] \\
1 \arrow[r, "c"'] & C
\end{tikzcd}$$

Theorem 7.1.21. *Let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$, let $\mathcal{M}$ be a $\mathbf{U}^+$-small category, and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted to $\mathfrak{Cat}$.*

(i) *$\mathscr{D}$ is a strong semiderivator.*

(ii) *$\mathscr{D}$ is $\mathfrak{Cat}$-cocomplete (resp. $\mathfrak{Cat}$-complete) if and only if $\mathcal{M}$ is $\mathbf{U}$-complete (resp. $\mathbf{U}$-complete).*

Proof. (i). This can be shown using the same arguments as remark 7.1.9.

(ii). This is the content of proposition 7.1.19. ■

Finally, we come to the definition of the subject of this chapter:

Definition 7.1.22. A **derivator** on $\mathfrak{K}$ is a semiderivator that is $\mathfrak{K}$-cocomplete and $\mathfrak{K}$-complete, and a **strong derivator** is one that satisfies axiom Der5.

Remark 7.1.23. The definition of 'subdomain' ensures that the restriction of any derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator) on $\mathfrak{K}$ to any subdomain of $\mathfrak{K}$ is again a derivator (resp. semiderivator, complete semiderivator, cocomplete semiderivator).

Proposition 7.1.24. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, and let $u \dashv v : B \to A$ be an adjunction in $\mathfrak{K}$, with unit $\eta : \mathrm{id}_A \Rightarrow v \circ u$ and counit $\varepsilon : u \circ v \Rightarrow \mathrm{id}_B$.*

(i) *We have an adjunction $v^* \dashv u^* : \mathscr{D}^B \to \mathscr{D}^A$, with unit $\eta^* : \mathrm{id}_{\mathscr{D}^A} \Rightarrow u^* \circ v^*$ and counit $\varepsilon^* : v^* \circ u^* \Rightarrow \mathrm{id}_{\mathscr{D}^B}$; in particular, $\mathscr{D}$ has left extensions along $u : A \to B$ and right extensions along $v : B \to A$.*

(ii) *Consider the following commutative diagrams in $\mathfrak{K}$:*

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "p"'] & B \arrow[d, "q"] \\
1 \arrow[r, "\mathrm{id}"'] & 1
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \arrow[r, "q"] \arrow[d, "v"'] & 1 \arrow[d, "\mathrm{id}"] \\
A \arrow[r, "p"'] & 1
\end{tikzcd}$$

The diagram on the left is a left $\mathscr{D}$-exact square, and the diagram on the right is a right $\mathscr{D}$-exact square.

(iii) *Moreover, if $\mathscr{D}$ has left extensions along $p : A \to 1$ and $q : A \to 1$, then the diagram on the right is a left $\mathscr{D}$-exact square; and if $\mathscr{D}$ has right extensions along $p : A \to 1$ and $q : A \to 1$, then the diagram on the left is a right $\mathscr{D}$-exact square.*

Proof. (i). Since $\mathscr{D}$ is a 2-functor, it preserves the triangle identities; thus $v^* \dashv u^*$ is indeed an adjunction. (The left and right adjoints are exchanged because $\mathscr{D}$ is contravariant.)

(ii). The two halves of the claim are formally dual; we will prove the first version. By claim (i), we may take $u_! = v^*$; but the left Beck–Chevalley transformation

$$u_! p^* \Rightarrow u_! p^* \mathrm{id}^* \mathrm{id}_! \Rightarrow u_! u^* q^* \mathrm{id}_! \Rightarrow q^* \mathrm{id}_!$$

is then equal to $\varepsilon^* q^* : v^* u^* q^* \Rightarrow q^*$, and $\varepsilon^* q^* = (q\varepsilon)^* = \mathrm{id}$, because 1 is a terminal object in $\mathfrak{K}$. Thus the left Beck–Chevalley condition is satisfied.

(iii). This is a special case of proposition 7.1.16. ■

Theorem 7.1.25. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$ that satisfies axioms Der3L and Der3R, and let 1 be a terminal object in $\mathfrak{K}$. The following are equivalent:*

(i) *$\mathscr{D}$ is a derivator.*

(ii) *$\mathscr{D}$ satisfies axiom Der4L.*

(iii) *Every comma square in $\mathfrak{K}$ is left $\mathscr{D}$-exact.*

(iv) *$\mathscr{D}$ satisfies axiom Der4R.*

(v) *Every comma square in $\mathfrak{K}$ is right $\mathscr{D}$-exact.*

Proof. Obviously, statement (i) implies statements (ii)–(v), and the conjunction of statements (iii) and (v) implies statement (i). We are assuming that $\mathscr{D}$ has left and right extensions along all morphisms in $\mathfrak{K}$, so the equivalence of statements (iii) and (v) is just proposition 7.1.16. It remains to be shown that (ii) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (v), but the two implications are formally dual, so it is enough to prove just one; we prove the former.

Consider a general comma square in $\mathfrak{K}$:

$$\begin{tikzcd}
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}$$

Let $b : 1 \to B$ be a morphism in $\mathfrak{K}$, and let $c = v \circ b$, and consider the following pasting diagrams,

$$\begin{tikzcd}
(q \downarrow b) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\tau}{\nearrow}"] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
(u \downarrow c) \arrow[r, "r"] \arrow[d, "j"'] \arrow[dr, phantom, "/\!\!/"] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & B \arrow[d, "v"] \\
A \arrow[r, "u"'] & C
\end{tikzcd}$$

where the upper square of the diagram on the left is a comma square, and the upper square of the diagram on the right is a 2-pullback square; note that the pasting lemma for comma squares implies that the outer rectangle of the diagram on the right is also a comma square.

Let $\pi = p \circ j$ and let $\lambda = \theta \circ \mathrm{id}_j$. By the universal property of comma objects, there is a unique morphism $f : (q \downarrow b) \to (u \downarrow c)$ such that $\pi \circ f = p \circ s$, $r \circ f = t$, and $\lambda \circ \mathrm{id}_f = \left(\mathrm{id}_v \circ \tau\right) \bullet \left(\theta \circ \mathrm{id}_s\right)$; and similarly there is a unique morphism $g : (u \downarrow c) \to (q \downarrow b)$ such that $s \circ g = j$, $t \circ g = r$, and $\tau \circ \mathrm{id}_g = \mathrm{id}_{q \circ j} = \mathrm{id}_{b \circ r}$. Then,

$$\pi \circ (f \circ g) = p \circ s \circ g = p \circ j = \pi \qquad\qquad r \circ (f \circ g) = r$$

so $f \circ g = \mathrm{id}_{(u \downarrow c)}$; and since $p \circ s = p \circ s \circ g \circ f$, we may think of $\mathrm{id}_{p \circ s}$ as a 2-cell $\beta : p \circ s \Rightarrow p \circ s \circ g \circ f$, whereas $b \circ t = q \circ s \circ g \circ f$, so $\tau : q \circ s \Rightarrow b \circ t$ is also a 2-cell $\gamma : q \circ s \Rightarrow q \circ s \circ g \circ f$, but then

$$\left(\theta \circ \mathrm{id}_{s \circ g \circ f}\right) \bullet \left(\mathrm{id}_u \circ \beta\right) = \theta \circ \mathrm{id}_j \circ \mathrm{id}_f = \lambda \circ \mathrm{id}_f = \left(\mathrm{id}_v \circ \gamma\right) \bullet \left(\theta \circ \mathrm{id}_s\right)$$

so by the 2-universal property of $(u \downarrow v)$, there is a unique 2-cell $\alpha : s \Rightarrow s \circ g \circ f$ such that $\mathrm{id}_p \circ \alpha = \beta$ and $\mathrm{id}_q \circ \alpha = \gamma$; and furthermore,

TODO: Justify this more carefully...

$$\left(\tau \circ \mathrm{id}_{g \circ f}\right) \bullet \left(\mathrm{id}_q \circ \alpha\right) = \left(\mathrm{id}_b \circ \mathrm{id}_{t \circ g \circ f}\right) \bullet \tau$$

therefore there is a unique 2-cell $\eta : \mathrm{id}_{(q\downarrow b)} \Rightarrow g \circ f$ such that $\mathrm{id}_s \circ \eta = \alpha$ and $\mathrm{id}_t \circ \eta = \mathrm{id}_{t\circ g\circ f}$.

We will now show that we have an adjunction $f \dashv g : (u \downarrow c) \to (q \downarrow b)$ in $\mathfrak{K}$; since $f \circ g = \mathrm{id}_{(u\downarrow c)}$, it is enough to check that $\mathrm{id}_f \circ \eta = \mathrm{id}_f$ and $\eta \circ \mathrm{id}_g = \mathrm{id}_g$. By construction, $\mathrm{id}_\pi \circ \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_p \circ \mathrm{id}_s \circ \eta = \mathrm{id}_{p\circ s}$, and $\mathrm{id}_r \circ \left(\mathrm{id}_f \circ \eta\right) = \mathrm{id}_t$, so indeed $\mathrm{id}_f \circ \eta = \mathrm{id}_f$; and $\mathrm{id}_s \circ \left(\eta \circ \mathrm{id}_g\right) = \mathrm{id}_s$ and $\mathrm{id}_t \circ \left(\eta \circ \mathrm{id}_g\right) = \mathrm{id}_t$, so $\eta \circ \mathrm{id}_g = \mathrm{id}_g$ as well. Thus, by proposition 7.1.24, the commutative diagram in $\mathfrak{K}$ shown below on the left is a left $\mathscr{D}$-exact square,

$$\begin{tikzcd}
(u \downarrow c) \arrow[r, "r"] \arrow[d, "g"'] & 1 \arrow[d, "\mathrm{id}"] \\
(q \downarrow b) \arrow[r] & 1
\end{tikzcd}
\qquad
\begin{tikzcd}
(q \downarrow b) \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\overset{\tau}{\nearrow}"] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"'] & B
\end{tikzcd}$$

and the diagram on the right is a left $\mathscr{D}$-exact square by hypothesis, so by the pasting lemma (7.1.17), the following commutative diagram is also a left $\mathscr{D}$-exact square:

$$\begin{tikzcd}
(u \downarrow c) \arrow[r] \arrow[d, "j"'] & 1 \arrow[d, "b"] \\
(u \downarrow v) \arrow[r, "q"'] & B
\end{tikzcd}$$

The hypothesis also implies that this diagram satisfies the left Beck–Chevalley condition,

$$\begin{tikzcd}
\mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\overset{\lambda^*}{\nearrow}"] & \mathscr{D}^1 \arrow[d, "r^*"] \\
\mathscr{D}^A \arrow[r, "\pi^*"'] & \mathscr{D}^{(u\downarrow c)}
\end{tikzcd}$$

but the pasting lemma (A.1.10) says that the left Beck–Chevalley transformation $r_! \pi^* \Rightarrow c^* u_!$ is obtained by pasting together the left Beck–Chevalley transformations of the squares in the diagram below,

$$\begin{tikzcd}
\mathscr{D}^C \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\underset{\theta^*}{\swarrow}"] & \mathscr{D}^B \arrow[r, "b^*"] \arrow[d, "q^*"] \arrow[dr, phantom, "="] & \mathscr{D}^1 \arrow[d, "r^*"] \\
\mathscr{D}^A \arrow[r, "p^*"'] & \mathscr{D}^{(u\downarrow v)} \arrow[r, "j^*"'] & \mathscr{D}^{(u\downarrow c)}
\end{tikzcd}$$

and so, allowing $b : 1 \to B$ to vary, we deduce that every component of the left Beck–Chevalley transformation $v^* u_! \Rightarrow q_! p^*$ is an isomorphism in $\mathscr{D}^1$. We may then apply axiom Der2 to conclude that the comma square we started with is a left $\mathscr{D}$-exact square. ■

7.2 Homotopy limits and colimits

Prerequisites. §§ 3.3, 4.1, 4.6, 7.1.

¶ 7.2.1. In this section, we use the **two-universe convention**: we assume that there are two universes $\mathbf{U}$ and $\mathbf{U}^+$, with $\mathbf{U} \in \mathbf{U}^+$. We refer to $\mathbf{U}$-sets, $\mathbf{U}$-small categories, etc. as 'small', and we refer to $\mathbf{U}^+$-sets, $\mathbf{U}^+$-small categories, etc. as 'moderate'.

Definition 7.2.2. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$, let A be an object in $\mathfrak{K}$, let 1 be a terminal object in $\mathfrak{K}$, let $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ be the functor induced by the unique morphism $A \to 1$ in $\mathfrak{K}$, and let X be an object in $\mathscr{D}^A$.

- A **$\mathscr{D}$-colimit** for X is an initial object in the comma category $(X \downarrow \Delta_A)$.
- A **$\mathscr{D}$-limit** for X is a terminal object in the comma category $(\Delta_A \downarrow X)$.
- We say $\mathscr{D}$ **has colimits for diagrams of shape** A if $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ has a left adjoint, which we denote by $\operatorname{ho}\varinjlim_A : \mathscr{D}^A \to \mathscr{D}^1$.
- We say $\mathscr{D}$ **has limits for diagrams of shape** A if $\Delta_A : \mathscr{D}^1 \to \mathscr{D}^A$ has a right adjoint, which we denote by $\operatorname{ho}\varprojlim_A : \mathscr{D}^A \to \mathscr{D}^1$.

We may refer to $\mathscr{D}$-colimits (resp. $\mathscr{D}$-limits) generically as **homotopy colimits** (resp. homotopy limits) in $\mathscr{D}$.

Remark 7.2.3. Of course, homotopy colimits (resp. homotopy limits) in $\mathscr{D}$ are a special case of homotopy left (resp. right) Kan extensions in $\mathscr{D}$; in particular, $\mathscr{D}$ has colimits (resp. limits) for diagrams of shape A if and only if, for every object X in $\mathscr{D}^A$, there exists a $\mathscr{D}$-colimit (resp. $\mathscr{D}$-limit) for X.

Proposition 7.2.4. *Let $\mathcal{M}$ be a moderate model category and let $\mathscr{D}$ be the prederivator of $\mathcal{M}$ restricted along* $\min : \mathfrak{Cat} \to \mathfrak{RelCat}$.

(i) *$\mathscr{D}$ satisfies axiom Der1.*

(ii) *$\mathscr{D}$ satisfies axiom Der5 at the terminal category $\mathbb{1}$, i.e. the canonical comparison functor $\mathscr{D}^{\mathbb{2}} \to \left[\mathbb{2}, \mathscr{D}^{\mathbb{1}}\right]$ is full and essentially surjective on objects.*

(iii) *Moreover, if $\mathcal{M}$ satisfies axiom CM5*, then $\mathscr{D}$ is a strong semiderivator.*

Proof. (i). Proposition A.4.19 implies $\mathscr{D}$ sends finite coproducts in $\mathfrak{RelCat}$ to products in $\mathfrak{Cat}^{+}$, and the embedding min : $\mathfrak{Cat} \to \mathfrak{RelCat}$ preserves finite coproducts, so axiom Der1 is satisfied.

(ii). By theorem 4.1.29, $\mathcal{M}$ admits a three-arrow calculus, so the claim follows from lemma 7.1.10.

(iii). Moreover, if $\mathcal{M}$ satisfies axiom CM5*, then $\mathcal{M}$ admits a *functorial* three-arrow calculus, so by proposition 3.6.8, each $[\mathbb{A}, \mathcal{M}]_{\mathrm{h}}$ admits a componentwise three-arrow calculus. Theorem 4.3.1 implies $\mathcal{M}$ is a saturated homotopical category, so we deduce that $\mathscr{D}$ is a strong semiderivator using proposition 7.1.11. ∎

Theorem 7.2.5. *If $\mathcal{M}$ is a locally small DHK model category, then the restriction of $\mathscr{D}(\mathcal{M})$ to $\mathfrak{Cat}$ is a strong derivator.*

Proof. Let $\mathscr{D}$ be the restriction of $\mathscr{D}(\mathcal{M})$ to $\mathfrak{Cat}$. We have already shown in proposition 7.2.4 that $\mathscr{D}$ is a strong semiderivator, so it remains to be proven that $\mathscr{D}$ is cocomplete and complete. Cocompleteness and completeness are formally dual, so it suffices to demonstrate just one half of the claim; we will show that $\mathscr{D}$ is cocomplete.

By theorem 4.6.16, for every functor $u : \mathbb{A} \to \mathbb{B}$ between small categories, the functor $\mathrm{Lan}_u : [\mathbb{A}, \mathcal{M}] \to [\mathbb{B}, \mathcal{M}]$ is left deformable, so theorem 3.3.20 implies the functor $\mathrm{Ho}\, u^* : \mathrm{Ho}\, [\mathbb{B}, \mathcal{M}] \to \mathrm{Ho}\, [\mathbb{A}, \mathcal{M}]$ has a left adjoint, namely the total left derived functor $\mathbf{L}\big(\mathrm{Lan}_u\big) : \mathrm{Ho}\, [\mathbb{A}, \mathcal{M}] \to \mathrm{Ho}\, [\mathbb{B}, \mathcal{M}]$. Thus $\mathscr{D}$ satisfies axiom Der3L.

Finally, to conclude, we note that proposition 4.6.18 is precisely the statement that axiom Der4L is satisfied. This completes the proof that $\mathscr{D}$ is cocomplete. ∎

Theorem 7.2.6 (Cisinski)**.** *Let $\mathcal{M}$ be a locally small model category and let $\mathscr{D}(\mathcal{M})$ be its associated prederivator. If $\mathcal{M}$ has colimits and limits for all small diagrams, then the restriction of $\mathscr{D}(\mathcal{M})$ to the 2-category of small categories is a derivator.*

Proof. See Theorem 6.11 in [Cisinski, 2003]. □

Definition 7.2.7. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$.

- A **$\mathscr{D}$-cofinal morphism** is a morphism $v : B \to A$ in $\mathfrak{K}$ such that the diagram below is a left $\mathscr{D}$-exact square,

$$\begin{tikzcd} B \arrow[r] \arrow[d, "v"'] & 1 \arrow[d, "\mathrm{id}"] \\ A \arrow[r] & 1 \end{tikzcd}$$

  i.e. such that the left Beck–Chevalley transformation

$$\underrightarrow{\mathrm{holim}}_B \circ v^* \Rightarrow \underrightarrow{\mathrm{holim}}_A$$

  is a natural isomorphism.

- A **$\mathscr{D}$-coinitial morphism** is a morphism $u : A \to B$ in $\mathfrak{K}$ such that the diagram below is a right $\mathscr{D}$-exact square,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d] & B \arrow[d] \\ 1 \arrow[r, "\mathrm{id}"'] & 1 \end{tikzcd}$$

  i.e. such that the right Beck–Chevalley transformation

$$\underleftarrow{\mathrm{holim}}_B \Rightarrow \underleftarrow{\mathrm{holim}}_A \circ u^*$$

  is a natural isomorphism.

Example 7.2.8. For any derivator $\mathscr{D}$ on $\mathfrak{K}$, every right adjoint (resp. left adjoint) in $\mathfrak{K}$ is a $\mathscr{D}$-cofinal (resp. $\mathscr{D}$-coinitial) morphism: this is the content of proposition 7.1.24.

Example 7.2.9. A category $\mathbb{A}$ has a terminal object if and only if the unique functor $\mathbb{A} \to \mathbb{1}$ has a right adjoint $t : \mathbb{1} \to \mathbb{A}$; thus, for any derivator on $\mathfrak{Cat}$, if $\mathbb{A}$ is a small category with a terminal object, then the left Beck–Chevalley transformation $t^* \Rightarrow \underrightarrow{\mathrm{holim}}_{\mathbb{A}}$ is a natural isomorphism.

Definition 7.2.10. Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$. A $\mathscr{D}$**-equivalence** is a morphism $u : A \to B$ in $\mathfrak{K}$ satisfying the following condition:

- For all X and Y in $\mathscr{D}^1$, the map $\mathscr{D}^B\big(\Delta_B X, \Delta_B Y\big) \to \mathscr{D}^A\big(\Delta_A X, \Delta_A Y\big)$ induced by $u^* : \mathscr{D}^B \to \mathscr{D}^A$ is a bijection.

Proposition 7.2.11. *Let $\mathscr{D}$ be a prederivator on $\mathfrak{K}$ and let $u : A \to B$ be a morphism in $\mathfrak{K}$. If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then the following are equivalent:*

(i) *The morphism $u : A \to B$ is a $\mathscr{D}$-equivalence.*

(ii) *For η^B the unit of $\operatorname{ho}\varinjlim_B \dashv \Delta_B$ and ε^A the counit of $\operatorname{ho}\varinjlim_A \dashv \Delta_A$, the natural transformation*

$$\left(\varepsilon^A \circ \operatorname{ho}\varinjlim_B \circ \Delta_B\right) \bullet \left(\operatorname{ho}\varinjlim_A \circ u^* \circ \eta^B \circ \Delta_B\right) : \operatorname{ho}\varinjlim_A \circ \Delta_A \Rightarrow \operatorname{ho}\varinjlim_B \circ \Delta_B$$

is a natural isomorphism.

(iii) *For ε^u the counit of $u_! \dashv u^*$, the natural transformation*

$$\operatorname{ho}\varinjlim_B \circ \varepsilon^u \circ \Delta_B : \operatorname{ho}\varinjlim_A \circ \Delta_A \Rightarrow \operatorname{ho}\varinjlim_B \circ \Delta_B$$

is a natural isomorphism.

Dually, if $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then the following are equivalent:

(i′) *The morphism $u : A \to B$ is a $\mathscr{D}$-equivalence.*

(ii′) *For η^A the unit of $\Delta_A \dashv \operatorname{ho}\varprojlim_A$ and ε^B the counit of $\Delta_B \dashv \operatorname{ho}\varprojlim_B$, the natural transformation*

$$\left(\operatorname{ho}\varprojlim_A \circ u^* \circ \varepsilon^B \circ \Delta_B\right) \bullet \left(\eta^A \circ \operatorname{ho}\varprojlim_B \circ \Delta_B\right) : \operatorname{ho}\varprojlim_B \circ \Delta_B \Rightarrow \operatorname{ho}\varprojlim_A \circ \Delta_A$$

is a natural isomorphism.

(iii′) *For η^u the unit of $u^* \dashv u_*$, the natural transformation*

$$\operatorname{ho}\varprojlim_B \circ \eta^u \circ \Delta_B : \operatorname{ho}\varprojlim_B \circ \Delta_B \Rightarrow \operatorname{ho}\varprojlim_A \circ \Delta_A$$

is a natural isomorphism.

Proof. The two sets of claims are formally dual; we will prove the first version.

Observe that every morphism $u : A \to B$ in $\mathfrak{K}$ induces a commutative diagram of the following form:

$$\begin{tikzcd}
\mathscr{D}^1\left(\underrightarrow{\operatorname{holim}}_B \Delta_B X, Y\right) \arrow[r, "\cong"] \arrow[d] & \mathscr{D}^B(\Delta_B X, \Delta_B Y) \arrow[d] \\
\mathscr{D}^1\left(\underrightarrow{\operatorname{holim}}_A \Delta_A X, Y\right) \arrow[r, "\cong"'] & \mathscr{D}^A(\Delta_A X, \Delta_A Y)
\end{tikzcd}$$

Thus, a morphism $u : A \to B$ in $\mathfrak{K}$ satisfies condition (ii) if and only if it is a $\mathscr{D}$-equivalence. By factoring the counit $\varepsilon^A : \underrightarrow{\operatorname{holim}}_A \Delta_A \Rightarrow \mathrm{id}_{\mathscr{D}^1}$ in terms of the counit $\varepsilon^u : u_! u^* \Rightarrow \mathrm{id}_{\mathscr{D}^B}$ and using the left triangle identity, we deduce that

$$\left(\varepsilon^A \circ \underrightarrow{\operatorname{holim}}_B \circ \Delta_B\right) \bullet \left(\underrightarrow{\operatorname{holim}}_A \circ u^* \circ \eta^B \circ \Delta_B\right) = \underrightarrow{\operatorname{holim}}_B \circ \varepsilon^u \circ \Delta_B$$

and so condition (ii) is satisfied if and only if condition (iii) is satisfied. ■

Corollary 7.2.12.

- *If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then every $\mathscr{D}$-cofinal morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.*
- *If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then every $\mathscr{D}$-coinitial morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.* ■

REMARK 7.2.13. In particular:

- If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then every right adjoint morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.
- If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then every left adjoint morphism in $\mathfrak{K}$ is a $\mathscr{D}$-equivalence.

Proposition 7.2.14. *Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{K}$ be the underlying 1-category of $\mathfrak{K}$. For any prederivator $\mathscr{D}$ on $\mathfrak{K}$, the category $\mathcal{K}$ with the class of $\mathscr{D}$-equivalences in $\mathfrak{K}$ constitute a saturated homotopical category.*

Proof. We will assume that, for every object A in $\mathfrak{K}$, the category $\mathscr{D}^A$ is locally small, but there is no loss of generality in doing so because we may always enlarge the universe.

Observe that, for all objects X and Y in $\mathscr{D}^1$, the functor $\mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ defined by $C \mapsto \mathscr{D}^C(\Delta_C X, \Delta_C Y)$ sends every $\mathscr{D}$-equivalence in $\mathfrak{K}$ to a bijection. Thus, if $u : A \to B$ is a morphism in $\mathfrak{K}$ that becomes invertible in the localisation of $\mathcal{K}$ at $\mathscr{D}$-equivalences, then for all objects X and Y in $\mathscr{D}^1$, the map $\mathscr{D}^B(\Delta_B X, \Delta_B Y) \to \mathscr{D}^A(\Delta_A X, \Delta_A Y)$ induced by u must be a bijection, so u must be a $\mathscr{D}$-equivalence. ■

Proposition 7.2.15. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$.*

- *Given a commutative triangle in $\mathfrak{K}$ as below,*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

  if $\mathscr{D}$ is $\mathfrak{K}$-cocomplete and, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ induced by $u : A \to B$ is a $\mathscr{D}$-equivalence, then $u : A \to B$ is itself a $\mathscr{D}$-equivalence.

- *Given a commutative triangle in $\mathfrak{K}$ as below,*

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

  if $\mathscr{D}$ is $\mathfrak{K}$-complete and, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism ${}^c u : (c \downarrow p) \to (c \downarrow q)$ induced by $u : A \to B$ is a $\mathscr{D}$-equivalence, then $u : A \to B$ is itself a $\mathscr{D}$-equivalence.

Proof. We will use the characterisation of $\mathscr{D}$-equivalences afforded by proposition 7.2.11. We wish to show that the natural transformation defined by the following pasting diagram is a natural isomorphism:

(1)
$$\begin{tikzcd} \mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "\operatorname{ho}\varinjlim"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "="] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "/\!/"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\Nearrow"'] & \\ & \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "\operatorname{ho}\varinjlim"'] & \mathscr{D}^1 \end{tikzcd}$$

By factoring $A \to 1$ and $B \to 1$ through $C \to 1$ and applying the left triangle identity, we see that it is enough to show that the natural transformation defined below is a natural isomorphism:

(2)
$$\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"'] \arrow[d, phantom, "="] & \mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\nearrow"] & \mathscr{D}^C \arrow[d, "p^*"] \arrow[dr, "\mathrm{id}"] \arrow[dr, phantom, "\nearrow"'] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "p_!"'] & \mathscr{D}^C
\end{tikzcd}$$

Axiom Der4L says that the following comma square in $\mathfrak{K}$ is left $\mathscr{D}$-exact,

$$\begin{tikzcd}
(p \downarrow c) \arrow[r] \arrow[d, "r"'] \arrow[dr, phantom, "\nearrow"] & 1 \arrow[d, "c"] \\
A \arrow[r, "p"'] & C
\end{tikzcd}$$

i.e. the left Beck–Chevalley transformation it induces is a natural isomorphism:

(3)
$$\begin{tikzcd}
\mathscr{D}^A \arrow[r, "p_!"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^A \arrow[r, "r^*"'] & \mathscr{D}^{(p\downarrow c)} \arrow[r, "\operatorname{ho}\varinjlim"'] \arrow[ur, phantom, "\nearrow"] & \mathscr{D}^1
\end{tikzcd}$$

Similarly, the comma square in $\mathfrak{K}$ shown below

$$\begin{tikzcd}
(q \downarrow c) \arrow[r] \arrow[d, "s"'] \arrow[dr, phantom, "\nearrow"] & 1 \arrow[d, "c"] \\
B \arrow[r, "q"'] & C
\end{tikzcd}$$

induces a left Beck–Chevalley transformation that is a natural isomorphism:

(4)
$$\begin{tikzcd}
\mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "q^*"'] \arrow[dl, phantom, "\nearrow"] \arrow[dr, phantom, "\nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^B \arrow[r, "s^*"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "\operatorname{ho}\varinjlim"'] \arrow[ur, phantom, "\nearrow"] & \mathscr{D}^1
\end{tikzcd}$$

Our hypothesis is that unique morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ making the following diagram commute is a $\mathscr{D}$-equivalence,

$$\begin{tikzcd}
(p \downarrow c) \arrow[r, "u_c"] \arrow[d, "r"'] & (q \downarrow c) \arrow[d, "s"] \\
A \arrow[r, "u"'] & B
\end{tikzcd}$$

i.e. the natural transformation defined below is a natural isomorphism:

$$\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "\operatorname{ho}\varinjlim"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "="] & \mathscr{D}^1 \arrow[d, "\Delta"] \arrow[dr, "\mathrm{id}"] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "(u_c)^*"'] & \mathscr{D}^{(p\downarrow c)} \arrow[r, "\operatorname{ho}\varinjlim"'] \arrow[u, phantom, "\Nearrow"', shift right=4] & \mathscr{D}^1
\end{tikzcd} \tag{5}$$

However, the natural transformations defined by the following pasting diagrams are equal,

$$\begin{tikzcd}
\mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"'] \arrow[dr, phantom, "="] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[d, "\Delta"] \\
\mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "r^*"'] & \mathscr{D}^{(p\downarrow c)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "q^*"'] \arrow[dr, phantom, "\Nearrow"] & \mathscr{D}^1 \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "="] & \mathscr{D}^1 \arrow[d, "\Delta"] \\
\mathscr{D}^B \arrow[r, "s^*"'] & \mathscr{D}^{(q\downarrow c)} \arrow[r, "(u_c)^*"'] & \mathscr{D}^{(p\downarrow c)}
\end{tikzcd}$$

so, the natural transformation obtained by pasting together (2) and (3) is equal to the natural transformation obtained by pasting together (4) and (5); but the latter is a natural isomorphism, so we deduce that the former is a natural isomorphism as well. Thus,

$$\begin{tikzcd}
\mathscr{D}^1 \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \mathscr{D}^B \arrow[r, "q_!"] \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \mathscr{D}^C \arrow[r, "\mathrm{id}"] \arrow[d, "q^*"] \arrow[dl, phantom, "\Nearrow"] \arrow[dr, phantom, "="] & \mathscr{D}^C \arrow[r, "c^*"] \arrow[d, "p^*"'] \arrow[dr, "\mathrm{id}"] & \mathscr{D}^1 \arrow[dr, "\mathrm{id}"] \arrow[d, phantom, "="] & \\
& \mathscr{D}^1 \arrow[r, "\Delta"'] & \mathscr{D}^B \arrow[r, "u^*"'] & \mathscr{D}^A \arrow[r, "p_!"'] \arrow[u, phantom, "\Nearrow"', shift right=4] & \mathscr{D}^C \arrow[r, "c^*"'] & \mathscr{D}^1
\end{tikzcd}$$

defines a natural isomorphism. Since $c : 1 \to C$ was arbitrary, we may use axiom Der2 to deduce that (1) itself defines a natural isomorphism, as claimed. ∎

Lemma 7.2.16. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$. Consider a diagram of the following form in $\mathfrak{K}$:*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & 1 \arrow[d, "b"] \\ 1 \arrow[r, "a"'] \arrow[ur, phantom, "\overset{\theta}{\nearrow}"] & C \end{tikzcd}$$

If $\mathscr{D}$ is $\mathfrak{K}$-cocomplete (resp. $\mathfrak{K}$-cocomplete), then the following are equivalent:

(i) *The diagram above is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).*

(ii) *The morphism $w : E \to (a \downarrow b)$ induced by the universal property of $(a \downarrow b)$ is a $\mathscr{D}$-equivalence.*

Proof. The two claims are formally dual; we will prove the first version.

By definition, the diagram above is a left $\mathscr{D}$-exact square if and only if the left Beck–Chevalley transformation

$$\underrightarrow{\operatorname{holim}}_E \Delta_E \Rightarrow \underrightarrow{\operatorname{holim}}_E \Delta_E a^* a_! \Rightarrow \underrightarrow{\operatorname{holim}}_E \Delta_E b^* a_! \Rightarrow b^* a_!$$

is a natural isomorphism. However, $\Delta_E = w^* \Delta_{(a\downarrow b)}$, and axiom Der4L says the left Beck–Chevalley transformation

$$\underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} \Rightarrow \underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} a^* a_! \Rightarrow \underrightarrow{\operatorname{holim}}_{(a\downarrow b)} \Delta_{(a\downarrow b)} b^* a_! \Rightarrow b^* a_!$$

is a natural isomorphism, so using the counit of the adjunction $w_! \dashv w^*$, proposition 7.2.11, and the 2-out-of-3 property of natural isomorphisms, we may deduce that conditions (i) and (ii) are equivalent. ■

Theorem 7.2.17. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$. Consider the following diagram in $\mathfrak{K}$:*

$$(\square) \qquad \begin{tikzcd} D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "v"] \\ A \arrow[r, "u"'] \arrow[ur, phantom, "\overset{\theta}{\nearrow}"] & C \end{tikzcd}$$

If $\mathscr{D}$ is $\mathfrak{K}$-cocomplete (resp. $\mathfrak{K}$-cocomplete), then the following are equivalent:

(i) *Diagram ($\square$) is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).*

(ii) *For all morphisms $a : 1 \to A$ and $b : 1 \to B$, for all diagrams of the form below in $\mathfrak{K}$,*

$$(*) \qquad \begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{p.b.}"] & (q \downarrow b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\beta"] & 1 \arrow[d, "b"] \\ (a \downarrow p) \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow\alpha"] & D \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\nearrow\theta"] & B \arrow[d, "v"] \\ 1 \arrow[r, "a"'] & A \arrow[r, "u"'] & C \end{tikzcd}$$

where the top-left square is a pullback square and the squares inhabited by α and β are comma squares, the outer square is a left $\mathscr{D}$-exact square (resp. right $\mathscr{D}$-exact square).

(iii) *For all diagrams of the form $(*)$ in $\mathfrak{K}$, the morphism $E \to (u \circ a \downarrow v \circ b)$ induced by the universal property of $(u \circ a \downarrow v \circ b)$ is a $\mathscr{D}$-equivalence.*

Proof. (i) $\Rightarrow$ (ii). The pasting lemma for comma diagrams implies the left rectangle of $(*)$ is a comma diagram, and we may apply lemma 7.1.17 and theorem 7.1.25 to deduce that the outer square of $(*)$ is a left $\mathscr{D}$-exact square.

(ii) $\Leftrightarrow$ (iii). This is a special case of the previous lemma.

(ii) $\Rightarrow$ (i). Using axioms Der2 and Der4L as well as the 2-out-of-3 property for natural isomorphisms, we may deduce that diagram $(\square)$ is left $\mathscr{D}$-exact if every diagram of the form $(*)$ is left $\mathscr{D}$-exact. ∎

Corollary 7.2.18. *Let $\mathscr{D}$ be a semiderivator on $\mathfrak{K}$. If $\mathscr{D}$ is $\mathfrak{K}$-cocomplete, then the following are equivalent for a morphism $v : B \to A$ in $\mathfrak{K}$:*

(i) *The morphism $v : B \to A$ is a $\mathscr{D}$-cofinal morphism.*

(ii) *For every morphism $a : 1 \to A$ in $\mathfrak{K}$, the unique morphism $(a \downarrow v) \to 1$ is a $\mathscr{D}$-equivalence.*

Dually, if $\mathscr{D}$ is $\mathfrak{K}$-complete, then the following are equivalent for a morphism $u : A \to B$ in $\mathfrak{K}$:

(i) *The morphism $u : A \to B$ is a $\mathscr{D}$-coinitial morphism.*

(ii) *For every morphism $b : 1 \to B$ in $\mathfrak{K}$, the unique morphism $(u \downarrow b) \to 1$ is a $\mathscr{D}$-equivalence.* ∎

7.3 Basic localisers

Prerequisites. §§ 3.1, 7.1.

Definition 7.3.1. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{K}$ be its underlying 1-category. A **basic right localiser** (resp. **basic left localiser**) for $\mathfrak{K}$ is a subcategory $\mathcal{W}$ of $\mathcal{K}$ satisfying these axioms:

LF1. Every identity morphism in $\mathcal{K}$ is also in $\mathcal{W}$, $\mathcal{W}$ has the 2-out-of-3 property in $\mathcal{K}$, and $\mathcal{W}$ is closed under retracts in $\mathcal{K}$.

LF2. For any object A in $\mathfrak{K}$, if the unique morphism $A \to 1$ has a right adjoint (resp. left adjoint), then $A \to 1$ is in $\mathcal{W}$.

LF3. Given a commutative triangle in $\mathfrak{K}$,

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "p"'] & & B \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

if, for every morphism $c : 1 \to C$ in $\mathfrak{K}$, the morphism $u_c : (p \downarrow c) \to (q \downarrow c)$ (resp. ${}^c u : (c \downarrow p) \to (c \downarrow q)$) induced by $u : A \to B$ is in $\mathcal{W}$, then $u : A \to B$ itself is in $\mathcal{W}$.

A **basic localiser** for $\mathfrak{K}$ is a subcategory of $\mathcal{K}$ that is both a basic left localiser and a basic right localiser.

Definition 7.3.2. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$. A $\mathcal{W}$**-equivalence** is a morphism that is in $\mathcal{W}$. A $\mathcal{W}$**-aspherical object** is an object A in $\mathfrak{K}$ such that the unique morphism $A \to 1$ is a $\mathcal{W}$-equivalence.

¶ 7.3.3. The above terminology is non-standard: it is more conventional to refer to basic right localisers as 'basic localisers' and ignore basic left localisers; cf. [Cisinski, 2004]. However, this is unproblematic in the case where $\mathfrak{K} = \mathfrak{Cat}$: one can show that all three notions coincide then. The chirality of the above terminology is chosen to agree with the chirality of the induced asphericity structures (cf. [Maltsiniotis, 2005]).

Proposition 7.3.4. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathscr{D}$ is a $\mathfrak{K}$-cocomplete semiderivator, then the class of $\mathscr{D}$-equivalences is a basic right localiser for $\mathfrak{K}$.*
- *If $\mathscr{D}$ is a $\mathfrak{K}$-complete semiderivator, then the class of $\mathscr{D}$-equivalences is a basic left localiser for $\mathfrak{K}$.*

Proof. The two claims are formally dual; we will prove the first version.

Proposition 7.2.14 implies that the class of $\mathscr{D}$-equivalences satisfies axiom LF1, and proposition 7.2.15 says that axiom LF3 is satisfied. Axiom LF2 remains to be verified, so suppose A is an object in $\mathfrak{K}$ such that the unique morphism $p : A \to 1$ has a right adjoint, say $t : 1 \to A$. By remark 7.2.13, t is a $\mathscr{D}$-equivalence; but $p \circ t = \mathrm{id}_1$ since 1 is a terminal object in $\mathfrak{K}$, so we may deduce that $p : A \to 1$ is also a $\mathscr{D}$-equivalence by using axiom LF1. ■

Corollary 7.3.5. *If $\mathscr{D}$ is a derivator on $\mathfrak{K}$, then the class of $\mathscr{D}$-equivalences is a basic localiser for $\mathfrak{K}$.* ■

Example 7.3.6. Let $\mathscr{D}$ be the prederivator of **Set** (restricted to $\mathfrak{Cat}$). By theorem 7.1.21, $\mathscr{D}$ is a derivator, and it is straightforward to verify that the $\mathscr{D}$-equivalences are precisely the functors $u : \mathbb{A} \to \mathbb{B}$ that induce bijections $\pi_0 u : \pi_0 \mathbb{A} \to \pi_0 \mathbb{B}$, where $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ is the connected components functor.[1]

REMARK 7.3.7. It is not hard to see that the intersection of any family of basic localisers (resp. basic left localisers, basic right localisers) for a derivator domain $\mathfrak{K}$ is automatically a basic localiser (resp. basic left localiser, basic right localiser) for $\mathfrak{K}$; thus, there is a unique minimal basic localiser (resp. basic left localiser, basic right localiser) for $\mathfrak{K}$.

Definition 7.3.8. Let $\mathfrak{K}$ be a derivator domain and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$.

- A **right $\mathcal{W}$-aspherical morphism** is a morphism $u : A \to B$ in $\mathfrak{K}$ such that, for all morphisms $b : 1 \to B$ in $\mathfrak{K}$, the unique morphism $(u \downarrow b) \to 1$ is a $\mathcal{W}$-equivalence.
- A **left $\mathcal{W}$-aspherical morphism** is a morphism $v : B \to A$ in $\mathfrak{K}$ such that, for all morphisms $a : 1 \to A$ in $\mathfrak{K}$, the unique morphism $(a \downarrow v) \to 1$ is a $\mathcal{W}$-equivalence.

[1] Recall proposition A.2.15.

Remark 7.3.9. In view of corollary 7.2.18, one might also call right (resp. left) $\mathcal{W}$-aspherical morphisms **$\mathcal{W}$-coinitial** (resp. **$\mathcal{W}$-cofinal**).

Lemma 7.3.10. *Let $\mathfrak{K}$ be a derivator domain.*

- *If a morphism $u : A \to B$ in $\mathfrak{K}$ has a right adjoint, then for any morphism $b : 1 \to B$, the unique morphism $(u \downarrow b) \to 1$ has a right adjoint.*
- *If a morphism $v : B \to A$ in $\mathfrak{K}$ has a left adjoint, then for any morphism $a : 1 \to A$, the unique morphism $(a \downarrow v) \to 1$ has a left adjoint.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose the following diagram is a comma square in $\mathfrak{K}$:

$$\begin{tikzcd} (u \downarrow b) \arrow[r, "q"] \arrow[d, "p"'] \arrow[dr, phantom, "\overset{\theta}{\nearrow}"] & 1 \arrow[d, "b"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

Let $v : B \to A$ be a right adjoint of $u : A \to B$, say with counit $\varepsilon : u \circ v \Rightarrow \mathrm{id}_B$. Consider the morphism $t : 1 \to (u \downarrow b)$ induced by the diagram in $\mathfrak{K}$ shown below:

$$\begin{tikzcd} 1 \arrow[r, "v \circ b"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\overset{\varepsilon \circ b}{\nearrow}"] & 1 \arrow[d, "b"] \\ A \arrow[r, "u"'] & B \end{tikzcd}$$

Via the 2-dimensional universal property of $(u \downarrow b)$, $\theta : u \circ p \Rightarrow b \circ q$ induces a 2-cell $\eta : \mathrm{id}_{(u \downarrow b)} \Rightarrow t \circ q$, and using the 2-dimensional Yoneda lemma, it is straightforward to check that η is the unit of an adjunction $q \dashv t : 1 \to (u \downarrow b)$. Thus, the unique morphism $(u \downarrow b) \to 1$ indeed has a right adjoint. ∎

Corollary 7.3.11. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every morphism in $\mathfrak{K}$ that has a right adjoint is a right $\mathcal{W}$-aspherical morphism.*
- *If $\mathcal{W}$ is a basic left localiser for $\mathfrak{K}$, then every morphism in $\mathfrak{K}$ that has a left adjoint is a left $\mathcal{W}$-aspherical morphism.* ∎

Proposition 7.3.12. *Let $\mathfrak{K}$ be a derivator domain.*

- *If $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every right $\mathcal{W}$-aspherical morphism is a $\mathcal{W}$-equivalence; in particular every morphism in $\mathfrak{K}$ that has a right adjoint is a $\mathcal{W}$-equivalence.*

- *If $\mathcal{W}$ is a basic left localiser for $\mathfrak{K}$, then every left $\mathcal{W}$-aspherical morphism is a $\mathcal{W}$-equivalence; in particular every morphism in $\mathfrak{K}$ that has a left adjoint is a $\mathcal{W}$-equivalence.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $u : A \to B$ is a right $\mathcal{W}$-aspherical morphism. Consider the following commutative triangle in $\mathfrak{K}$:

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dr, "u"'] & & B \arrow[dl, "\mathrm{id}"] \\ & B & \end{tikzcd}$$

Let $b : 1 \to B$ be a morphism in $\mathfrak{K}$. Since the unique morphism $(u \downarrow b) \to 1$ is a $\mathcal{W}$-equivalence, axioms LF1 and LF2 and lemma 7.3.10 imply the induced morphism $u_b : (u \downarrow b) \to \left(\mathrm{id}_B \downarrow b\right)$ is also a $\mathcal{W}$-equivalence. We may then apply axiom LF3 to deduce that $u : A \to B$ itself is a $\mathcal{W}$-equivalence. ■

Lemma 7.3.13. *Let A be an object in a derivator domain $\mathfrak{K}$. If $\mathcal{W}$ is a basic left or right localiser for $\mathfrak{K}$, then the morphism $p : \mathbb{2} \odot A \to \mathbb{1} \odot A \cong A$ induced by the unique functor $\mathbb{2} \to \mathbb{1}$ is a $\mathcal{W}$-equivalence.*

Proof. The unique functor $\mathbb{2} \to \mathbb{1}$ has both a left adjoint and a right adjoint, so the induced morphism $p : \mathbb{2} \odot A \to A$ has both a left adjoint and a right adjoint. Proposition 7.3.12 then implies that it is a $\mathcal{W}$-equivalence. ■

Proposition 7.3.14. *Let $u_0, u_1 : A \to B$ be a parallel pair of morphisms in a derivator domain $\mathfrak{K}$ and let $\mathcal{W}$ be either a basic left localiser or a basic right localiser for $\mathfrak{K}$. If there exists a 2-cell $\alpha : u_0 \Rightarrow u_1$, then the following are equivalent:*

(i) *The morphism $u_0 : A \to B$ is a $\mathcal{W}$-equivalence.*

(ii) *The morphism $u_1 : A \to B$ is a $\mathcal{W}$-equivalence.*

Proof. Let $i_0, i_1 : A \to \mathbb{2} \odot A$ be the morphisms induced by the left and right adjoints of the unique functor $\mathbb{2} \to \mathbb{1}$; note that functoriality yields $p \circ i_0 = \mathrm{id}_A = p \circ i_1$. The previous lemma says that p is a $\mathcal{W}$-equivalence, so we may then use axiom LF1 to deduce that i_0 and i_1 are both $\mathcal{W}$-equivalences.

By definition, there is a bijection

$$\mathcal{K}(\mathbb{2} \odot A, B) \cong \mathrm{Fun}(\mathbb{2}, \mathfrak{K}(A, B))$$

that is natural in B; thus, the 2-cell $\alpha : u_0 \Rightarrow u_1$ corresponds to a morphism $h : \mathbb{2} \odot A \to B$ such that $h \circ i_0 = u_0$ and $h \circ i_1 = u_1$. Axiom LF1 then implies that u_0 is a $\mathcal{W}$-equivalence if and only if u_1 is a $\mathcal{W}$-equivalence. ■

Corollary 7.3.15. *If $\mathcal{W}$ is a basic left or right localiser for a derivator domain $\mathfrak{K}$, then every left or right adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence.*

Proof. One half of the claim was proved in proposition 7.3.12; it now suffices to show that, if $\mathcal{W}$ is a basic right localiser for $\mathfrak{K}$, then every right adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence. We already know that every left adjoint in $\mathfrak{K}$ is a $\mathcal{W}$-equivalence, so axiom LF1 and the above proposition together imply that right adjoints are also $\mathcal{W}$-equivalences. ■

7.4 The minimal basic localiser

Prerequisites. §§ 1.1, 1.2, 1.3, 1.4, 1.7, 7.1, 7.2, 7.3.

In this section, we follow [Cisinski, 2004, § 2.2].

Proposition 7.4.1. *Let $\mathcal{W}$ be a basic left or right localiser for $\mathfrak{Cat}$. For any functor $u : \mathbb{A} \to \mathbb{B}$, the following are equivalent:*

(i) *The functor $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}$-equivalence.*

(ii) *The functor $u^{\mathrm{op}} : \mathbb{A}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is a $\mathcal{W}$-equivalence.*

Proof. See Proposition 1.2.6 in [Cisinski, 2004]. □

Corollary 7.4.2. *Let $\mathcal{W}$ be a subcategory of* **Cat**. *The following are equivalent:*

(i) *$\mathcal{W}$ is a basic localiser for $\mathfrak{Cat}$.*

(ii) *$\mathcal{W}$ is a basic right localiser for $\mathfrak{Cat}$.*

(iii) $\mathcal{W}$ *is a basic left localiser for* $\mathfrak{Cat}$. ■

¶ 7.4.3. Throughout this section, let $\mathcal{W}$ be any basic localiser for $\mathfrak{Cat}$.

Definition 7.4.4. A **weak homotopy equivalence of categories** is a functor $f : \mathbb{A} \to \mathbb{B}$ such that the induced morphism $\mathrm{N}(f) : \mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathbb{B})$ is a weak homotopy equivalence of simplicial sets. We write $\mathcal{W}_\infty$ for the class of weak homotopy equivalences of categories.

¶ 7.4.5. Weak homotopy equivalences of categories are also called **∞-equivalences**, but we should avoid this term as it conflicts with the terminology of higher category theory.

Definition 7.4.6. The **category of simplices** of a simplicial set X is the category $\mathbf{\Delta}(X)$ defined below:

- The objects are simplices of X.
- For $x \in X_n$ and $x' \in X_{n'}$, the morphisms $x \to x'$ are simplicial operators $f : [n] \to [n']$ such that $f(x') = x$.
- Composition and identities are the obvious ones.

We write $\pi_\Delta : \mathbf{\Delta}(X) \to \mathbf{\Delta}$ for the evident projection functor that sends an n-simplex of X to the object $[n]$ in $\mathbf{\Delta}$.

¶ 7.4.7. For brevity, if $\mathbb{A}$ is a small category, then we write $\mathbf{\Delta}(\mathbb{A})$ instead of $\mathbf{\Delta}(\mathrm{N}(\mathbb{A}))$. This is consistent with the notation of § 4.6. We will also use the left and right projection functors defined in definition 4.6.9.

Lemma 7.4.8. *If* $\mathbb{A}$ *is a small category with a terminal object, then the category* $\mathbf{\Delta}(\mathbb{A})$ *is* $\mathcal{W}$*-aspherical.*

Proof. Straightforward. (This is Lemme 2.2.2 in [Cisinski, 2004].) □

Proposition 7.4.9 (Grothendieck). *For all small categories* $\mathbb{A}$*, the right projection* $\pi_\mathrm{R} : \mathbf{\Delta}(\mathbb{A}) \to \mathbb{A}$ *is right* $\mathcal{W}$*-aspherical; in particular, it is a* $\mathcal{W}$*-equivalence.*

Proof. Let a be an object in $\mathbb{A}$. Lemma 4.6.12 says that the canonical comparison functor $\mathbf{\Delta}(\mathbb{A}_{/a}) \to (\pi_\mathrm{R} \downarrow a)$ is an isomorphism, and lemma 7.4.8 implies $\mathbf{\Delta}(\mathbb{A}_{/a})$ is $\mathcal{W}$-aspherical, so the induced functor $(\pi_\mathrm{R} \downarrow a) \to \mathbb{A}_{/a}$ is a $\mathcal{W}$-equivalence. Thus, $\pi_\mathrm{R} : \mathbf{\Delta}(\mathbb{A}) \to \mathbb{A}$ is right $\mathcal{W}$-aspherical. ■

Corollary 7.4.10. *A functor $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}$-equivalence if and only if the functor $\mathbf{\Delta}(u) : \mathbf{\Delta}(\mathbb{A}) \to \mathbf{\Delta}(\mathbb{B})$ is a $\mathcal{W}$-equivalence.*

Proof. Use the naturality of π_{R} and axiom LF1. ∎

¶ 7.4.11. Now, let $\mathcal{W}_{\mathbf{\Delta}}$ be the subcategory of **sSet** consisting of those morphisms $f : X \to Y$ such that $\mathbf{\Delta}(f) : \mathbf{\Delta}(X) \to \mathbf{\Delta}(Y)$ are $\mathcal{W}$-equivalences.

Proposition 7.4.12. *For all simplicial sets X and all natural numbers n, the projection $\pi : X \times \Delta^n \to X$ is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. Since $\Delta^m \times \Delta^n \cong \mathrm{N}([m] \times [n])$, lemma 7.4.8 implies $\mathbf{\Delta}(\Delta^m \times \Delta^n)$ is $\mathcal{W}$-aspherical. Now, let x be an m-simplex of X, and consider the comma category $(\mathbf{\Delta}(\pi) \downarrow x)$. It is not hard to see that $(\mathbf{\Delta}(\pi) \downarrow x)$ is isomorphic to $\mathbf{\Delta}(\Delta^m \times \Delta^n)$, and so the induced functor $(\mathbf{\Delta}(\pi) \downarrow x) \to \mathbf{\Delta}(X)_{/x}$ is a $\mathcal{W}$-equivalence. Thus, $\mathbf{\Delta}(\pi) : \mathbf{\Delta}(X \times \Delta^n) \to \mathbf{\Delta}(X)$ is right $\mathcal{W}$-aspherical, and in particular $\pi : X \times \Delta^n \to X$ is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence. ∎

Corollary 7.4.13. *Every trivial Kan fibration is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. Apply proposition 1.4.6. ∎

Proposition 7.4.14. *Every trivial cofibration in* **sSet** *is a $\mathcal{W}_{\mathbf{\Delta}}$-equivalence.*

Proof. See Proposition 2.2.9 in [Cisinski, 2004]. □

Theorem 7.4.15 (Cisinski)**.** *Any $\mathcal{W}_\infty$-equivalence is also a $\mathcal{W}$-equivalence.*

Proof. Propositions 1.3.8 and 1.3.16 together imply that every weak homotopy equivalence in **sSet** can be factored as a trivial cofibration followed by a trivial Kan fibration, so applying corollaries 7.4.10 and 7.4.13 and proposition 7.4.14, we deduce that every $\mathcal{W}_\infty$-equivalence is a $\mathcal{W}$-equivalence. ∎

Proposition 7.4.16. *Let $\mathscr{D}$ be the prederivator of* **sSet** *(with the Kan–Quillen model structure) restricted to $\mathfrak{Cat}$. Then a functor $u : \mathbb{A} \to \mathbb{B}$ is a $\mathscr{D}$-equivalence if and only if $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}_\infty$-equivalence.*

Proof. Theorem 7.2.5 says $\mathscr{D}$ is a derivator, so we may use the characterisation of $\mathscr{D}$-equivalences afforded by proposition 7.2.11. Let X be any simplicial set. By theorem 1.7.9, $\operatorname{holim}_{\longrightarrow \mathbb{A}} \Delta_{\mathbb{A}} X$ can be computed using the bar construction $\mathrm{B}(\Delta 1, \mathbb{C}, X)$, which is readily seen to be isomorphic to the simplicial set

$\mathrm{N}(\mathbb{C}) \times X$. Using the explicit description of the derived unit and counit in the proof of theorem 3.3.20, it can be shown that the diagram shown below commutes,

$$\begin{tikzcd}
\underrightarrow{\operatorname{holim}}_{\mathbb{A}} \Delta_{\mathbb{A}} X \arrow[r] \arrow[d, "\cong"'] & \underrightarrow{\operatorname{holim}}_{\mathbb{B}} \Delta_{\mathbb{B}} X \arrow[d, "\cong"] \\
\mathrm{N}(\mathbb{A}) \times X \arrow[r, "\mathrm{N}(u)\times\mathrm{id}_X"'] & \mathrm{N}(\mathbb{B}) \times X
\end{tikzcd}$$

but by proposition 1.3.19, $\mathrm{N}(u) \times \mathrm{id}_X$ is a weak homotopy equivalence for all X if and only if $\mathrm{N}(u) : \mathrm{N}(\mathbb{A}) \to \mathrm{N}(\mathbb{B})$ is a weak homotopy equivalence, i.e. if and only if $u : \mathbb{A} \to \mathbb{B}$ is a $\mathcal{W}_\infty$-equivalence. ■

We thus obtain a proof of Grothendieck's conjecture ([1983, § 81]):

Corollary 7.4.17. *The minimal basic localiser for $\mathfrak{Cat}$ is $\mathcal{W}_\infty$.* ■

— A —

Generalities

A.1 Adjoints and mates

We begin by recalling a standard definition:

Definition A.1.1. An **adjunction of categories** consists of the following data:

- A functor $F : \mathcal{C} \to \mathcal{D}$, called the **left adjoint**.
- A functor $G : \mathcal{D} \to \mathcal{C}$, called the **right adjoint**.
- A natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, called the **unit**.
- A natural transformation $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, called the **counit**.

These are moreover required to satisfy the **triangle identities**:

$$\varepsilon F \bullet F\eta = \mathrm{id}_F \qquad\qquad G\varepsilon \bullet \eta G = \mathrm{id}_G$$

If such data exist, we write

$$F \dashv G : \mathcal{D} \to \mathcal{C}$$

and say that F is a left adjoint of G, and G is a right adjoint of F.

Proposition A.1.2. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ε. The following are equivalent:*

(i) *The left adjoint $F : \mathcal{C} \to \mathcal{D}$ is fully faithful.*

(ii) *The adjunction unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is a natural isomorphism.*

(iii) *The natural transformation* $F\eta G : FG \Rightarrow FGFG$ *is a natural isomorphism,* $F : \mathcal{C} \to \mathcal{D}$ *is conservative, and* $G : \mathcal{D} \to \mathcal{C}$ *is essentially surjective on objects.*

Dually, the following are equivalent:

(i′) *The right adjoint* $G : \mathcal{D} \to \mathcal{C}$ *is fully faithful.*

(ii′) *The adjunction counit* $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ *is a natural isomorphism.*

(iii′) *The natural transformation* $G\varepsilon F : GFGF \Rightarrow GF$ *is a natural isomorphism,* $G : \mathcal{D} \to \mathcal{C}$ *is conservative, and* $F : \mathcal{C} \to \mathcal{D}$ *is essentially surjective on objects.*

Proof. (i) $\Leftrightarrow$ (ii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. By naturality, we have $\eta_Y \circ f = GFf \circ \eta_X$; but the triangle identities imply the hom-set map $\mathcal{D}(FX, B) \to \mathcal{C}(X, GB)$ given by $g \mapsto Gg \circ \eta_X$ is also a bijection, so we deduce that the hom-set map $\mathcal{C}(X, Y) \to \mathcal{C}(X, GFY)$ given by $f \mapsto \eta_Y \circ f$ is a bijection if and only if the hom-set map $\mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$ given by $f \mapsto Ff$ is a bijection because F is fully faithful. We may then deduce that η is a natural isomorphism if and only if F is fully faithful.

(i) $\Rightarrow$ (iii). We have already shown that $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is a natural isomorphism, so in particular $F\eta G : FG \Rightarrow FGFG$ is a natural isomorphism. Fully faithful functors are conservative, so F is conservative. On the other hand, since η is a natural isomorphism, G is essentially surjective on objects.

(iii) $\Rightarrow$ (ii). If F is conservative and $F\eta G$ is a natural isomorphism, then ηG is also a natural isomorphism. Since every object in $\mathcal{C}$ is isomorphic to one in the image of G, it follows that η is a natural isomorphism. ■

Proposition A.1.3. *Let* $F \dashv G : \mathcal{D} \to \mathcal{C}$ *be an adjunction.*

- $G : \mathcal{D} \to \mathcal{C}$ *is fully faithful if and only if, for all categories* $\mathcal{E}$*, the induced functor* $F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ *is fully faithful.*

- $F : \mathcal{C} \to \mathcal{D}$ *is fully faithful if and only if, for all categories* $\mathcal{E}$*, the induced functor* $G^* : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$ *is fully faithful.*

Proof. The two claims are formally dual; we will prove the first version.

Suppose $G : \mathcal{D} \to \mathcal{C}$ is fully faithful. By proposition A.1.2, the adjunction counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ must be a natural isomorphism. On the other hand, we have an induced adjunction $G^* \dashv F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ with counit induced by ε, so the same proposition implies F^* must be fully faithful.

Conversely, suppose $F^* : [\mathcal{C}, \mathcal{E}] \to [\mathcal{D}, \mathcal{E}]$ is a fully faithful functor for all categories $\mathcal{E}$. Then the induced adjunction counit $\varepsilon^* : G^* F^* \Rightarrow \mathrm{id}_{[\mathcal{D},\mathcal{E}]}$ is a natural isomorphism. In particular, this is true when $\mathcal{E} = \mathcal{D}$, so by considering the component of ε^* at $\mathrm{id}_{\mathcal{D}}$, we see that $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ itself is a natural isomorphism. Thus $G : \mathcal{D} \to \mathcal{C}$ must be fully faithful. ■

Proposition A.1.4. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be adjunctions, let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\eta' : \mathrm{id}_{\mathcal{C}'} \Rightarrow G'F'$ be the respective units, and let $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\varepsilon' : F'G' \Rightarrow \mathrm{id}_{\mathcal{D}'}$ be the respective counits. Let $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ be functors, and let φ and ψ be natural transformations as in the diagrams below:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swArrow\varphi"] & \mathcal{C}' \arrow[d, "F'"] \\
\mathcal{D} \arrow[r, "K"'] & \mathcal{D}'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "\psi\neArrow"] & \mathcal{D}' \arrow[d, "G'"] \\
\mathcal{C} \arrow[r, "H"'] & \mathcal{C}'
\end{tikzcd}$$

Then, the following are equivalent:

(i) $\varepsilon' KF \bullet F'\psi F \bullet F'H\eta = \varphi$.

(ii) $\psi F \bullet H\eta = G'\varphi \bullet \eta' H$.

(iii) $\psi = G'K\varepsilon \bullet G'\varphi G \bullet \eta' HG$.

(iv) $\varepsilon' K \bullet F'\psi = K\varepsilon \bullet \varphi G$.

Proof.

(i) ⇒ (ii).
$$\begin{aligned}
G'\varphi \bullet \eta' H &= G'\varepsilon' KF \bullet G'F'\psi F \bullet G'F'H\eta \bullet \eta' H \\
&= G'\varepsilon' KF \bullet \eta' G'KF \bullet \psi F \bullet H\eta \\
&= \psi F \bullet H\eta
\end{aligned}$$

(ii) ⇒ (iii).
$$\begin{aligned} G'K\varepsilon \bullet G'\varphi G \bullet \eta' HG &= G'K\varepsilon \bullet \psi FG \bullet H\eta G \\ &= \psi \bullet HG\varepsilon \bullet H\eta G \\ &= \psi \end{aligned}$$

(iii) ⇒ (iv).
$$\begin{aligned} \varepsilon' K \bullet F'\psi &= \varepsilon' K \bullet F'G'K\varepsilon \bullet F'G'\varphi G \bullet F'\eta' HG \\ &= K\varepsilon \bullet \varphi G \bullet \varepsilon' HG \bullet F'\eta' HG \\ &= K\varepsilon \bullet \varphi G \end{aligned}$$

(iv) ⇒ (i).
$$\begin{aligned} \varepsilon' KF \bullet F'\psi F \bullet F'H\eta &= K\varepsilon F \bullet \varphi GF \bullet F'H\eta \\ &= K\varepsilon F \bullet KF\eta \bullet \varphi \\ &= \varphi \end{aligned}$$
■

Definition A.1.5. A **conjugate pair of natural transformations** is a pair (φ, ψ) satisfying the equivalent conditions of the above proposition. Given such, we say φ is the **left mate** of ψ, and ψ is the **right mate** of φ.

Definition A.1.6. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ be functors, and let φ and ψ be a conjugate pair of natural transformations as in the diagrams below:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C}' \arrow[d, "F'"] \\ \mathcal{D} \arrow[r, "K"'] & \mathcal{D}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "\psi\nearrow"] & \mathcal{D}' \arrow[d, "G'"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \end{tikzcd}$$

We say the diagram on the right satisfies the **left Beck–Chevalley condition** if the left mate φ is a natural isomorphism, and we say the diagram on the left satisfies the **right Beck–Chevalley condition** if the right mate ψ is a natural isomorphism. More generally, the **local left Beck–Chevalley condition** is satisfied at an object C in $\mathcal{C}$ if the component $\varphi_C : F'HC \to KFC$ is an isomorphism, and the **local right Beck–Chevalley condition** is satisfied at an object D in $\mathcal{D}$ if the component $\psi_D : HGD \to G'KD$ is an isomorphism.

Remark A.1.7. Unfortunately, the Beck–Chevalley conditions are not vacuous. For example, consider the following (strictly!) commutative diagram of forgetful functors:

$$\begin{tikzcd} \mathbf{CRing} \arrow[r] \arrow[d] & \mathbf{Ab} \arrow[d] \\ \mathbf{Set} \arrow[r, "\mathrm{id}"'] & \mathbf{Set} \end{tikzcd}$$

The left mate of the trivial natural transformation in the above diagram is the group homomorphism $\mathbb{Z}X \to \mathbb{Z}[X]$ that sends a generator in $\mathbb{Z}X$ to the corresponding generator in $\mathbb{Z}[X]$; clearly, this is never an isomorphism. However, this is unsurprising: we do not expect the free abelian group generated by X to be naturally isomorphic to the additive group of free commutative ring generated by X.

Example A.1.8. Let $\mathcal{C}$ be a category with pullbacks, and suppose the following diagram is a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

Let $\Sigma_f : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ etc. be the functor that sends an object $p : E \to X$ in $\mathcal{C}_{/X}$ to the object $f \circ p : E \to Y$ in $\mathcal{C}_{/Y}$, and consider the induced (strictly!) commutative diagram of functors:

$$\begin{tikzcd} \mathcal{C}_{/Z} \arrow[r, "\Sigma_z"] \arrow[d, "\Sigma_g"'] & \mathcal{C}_{/X} \arrow[d, "\Sigma_f"] \\ \mathcal{C}_{/W} \arrow[r, "\Sigma_w"'] & \mathcal{C}_{/Y} \end{tikzcd}$$

Since $\mathcal{C}$ has pullbacks, Σ_g and Σ_f have right adjoints,[1] and the pullback pasting lemma then implies that the above square satisfies the right Beck–Chevalley condition.

[1] See lemma A.2.17.

Lemma A.1.9. *Given a diagram of functors and natural transformations of the form below,*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\psi}{\nearrow}"] & \mathcal{D}' \arrow[d, "G'"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \end{tikzcd}$$

where $\psi : HG \Rightarrow G'K$ is a natural isomorphism, $F \dashv G$, and $F' \dashv G'$, for each object C in $\mathcal{C}$, the following are equivalent:

(i) *The diagram satisfies the local left Beck–Chevalley condition at C.*

(ii) *The functor $(C \downarrow G) \to (HC \downarrow G')$ sending an object (D, f) in the comma category $(C \downarrow G)$ to the object $\left(KD, \psi_D \circ Hf\right)$ in $(HC \downarrow G')$ preserves initial objects.*

Proof. We know $\left(FC, \eta_C\right)$ is an initial object of $(C \downarrow G)$ and $\left(F'HC, \eta'_{HC}\right)$ is an initial object of $(HC \downarrow G')$, so there is a unique morphism $\varphi_C : F'HC \to KFC$ such that $G'\varphi_C \circ \eta'_{HC} = \psi_{FC} \circ H\eta_C$. However, we observe that

$$\begin{aligned} \varphi_C &= \varphi_C \circ \varepsilon'_{F'HC} \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'G'\varphi_C \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'\psi_{FC} \circ F'H\eta_C \end{aligned}$$

so φ_C is precisely the component at C of the left mate of ψ. ■

Lemma A.1.10 (Pasting conjugate pairs)**.**

(i) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$, $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$, and $F'' \dashv G'' : \mathcal{D}'' \to \mathcal{C}''$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$, $H' : \mathcal{C}' \to \mathcal{C}''$, $K : \mathcal{D} \to \mathcal{D}'$, and $K' : \mathcal{D}' \to \mathcal{D}''$ be functors, and let $\varphi, \varphi', \psi, \psi'$ be natural transformations as in the following pasting diagrams:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F"'] \arrow[dr, phantom, "\swarrow\varphi"] & \mathcal{C}' \arrow[r, "H'"] \arrow[d, "F'"] \arrow[dr, phantom, "\swarrow\varphi'"] & \mathcal{C}'' \arrow[d, "F''"] \\ \mathcal{D} \arrow[r, "K"'] & \mathcal{D}' \arrow[r, "K'"'] & \mathcal{D}'' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[r, "K"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\psi}{\nearrow}"] & \mathcal{D}' \arrow[r, "K'"] \arrow[d, "G'"] \arrow[dr, phantom, "\overset{\psi'}{\nearrow}"] & \mathcal{D}'' \arrow[d, "G''"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \arrow[r, "H'"'] & \mathcal{C}'' \end{tikzcd}$$

Let $\bar{\varphi} = K'\varphi \bullet \varphi'H$ and $\bar{\psi} = \psi'K \bullet H'\psi$. If (φ, ψ) and (φ', ψ') are conjugate pairs, then $(\bar{\varphi}, \bar{\psi})$ is also a conjugate pair.

(ii) *Let $F_1 \dashv G_1 : \mathcal{D} \to \mathcal{C}$, $F_2 \dashv G_2 : \mathcal{E} \to \mathcal{D}$, $F_1' \dashv G_1' : \mathcal{D}' \to \mathcal{C}'$, and $F_2' \dashv G_2' : \mathcal{E}' \to \mathcal{D}'$ be adjunctions, let $H : \mathcal{C} \to \mathcal{C}'$, $K : \mathcal{D} \to \mathcal{D}'$, and $L : \mathcal{E} \to \mathcal{E}'$ be functors, and let $\varphi_1, \varphi_2, \psi_1, \psi_2$ be natural transformations as in the following pasting diagrams:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[d, "F_1"'] \arrow[dr, phantom, "\swarrow\varphi_1"] & \mathcal{C}' \arrow[d, "F_1'"] \\ \mathcal{D} \arrow[r, "K"] \arrow[d, "F_2"'] \arrow[dr, phantom, "\swarrow\varphi_2"] & \mathcal{D}' \arrow[d, "F_2'"] \\ \mathcal{E} \arrow[r, "L"'] & \mathcal{E}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "L"] \arrow[d, "G_2"'] \arrow[dr, phantom, "\psi_2\nearrow"] & \mathcal{E}' \arrow[d, "G_2'"] \\ \mathcal{D} \arrow[r, "K"] \arrow[d, "G_1"'] \arrow[dr, phantom, "\psi_1\nearrow"] & \mathcal{D}' \arrow[d, "G_1'"] \\ \mathcal{C} \arrow[r, "H"'] & \mathcal{C}' \end{tikzcd}$$

Let $\varphi = \varphi_2 F_1 \bullet F_2' \varphi_1$ and $\psi = G_1' \psi_2 \bullet \psi_1 G_2$. If (φ_1, ψ_1) and (φ_2, ψ_2) are conjugate pairs, then (φ, ψ) is also a conjugate pair.

Proof. These are straightforward exercises in using the triangle identities. ◆

Proposition A.1.11. *Let $u_! \dashv u^* : \mathcal{C} \to \mathcal{A}$, $q_! \dashv q^* : \mathcal{B} \to \mathcal{D}$, $v^* \dashv v_* : \mathcal{B} \to \mathcal{C}$, and $p^* \dashv p_* : \mathcal{D} \to \mathcal{A}$ be adjunctions, and let $\theta : u^* p^* \Rightarrow v^* q^*$ be a natural transformation.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "v^*"] \arrow[d, "u^*"'] \arrow[dr, phantom, "\theta\nearrow"] & \mathcal{B} \arrow[d, "q^*"] \\ \mathcal{A} \arrow[r, "p^*"'] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, "u^*"] \arrow[d, "v^*"'] \arrow[dr, phantom, "\swarrow\theta"] & \mathcal{A} \arrow[d, "p^*"] \\ \mathcal{B} \arrow[r, "q^*"'] & \mathcal{D} \end{tikzcd}$$

The following are equivalent:

(i) *The diagram on the left satisfies the left Beck–Chevalley condition.*

(ii) *The diagram on the right satisfies the right Beck–Chevalley condition.*

Proof. Let $\varphi : q_! p^* \Rightarrow v^* u_!$ be the left mate of θ, and let $\psi : u^* v_* \Rightarrow p_* q^*$ be the right mate of θ. Then, by proposition A.1.4,

$$\theta u_! \bullet p^* \eta^u = q^* \varphi \bullet \eta^q p^* \qquad \varepsilon^q v^* \bullet q_! \theta = v^* \varepsilon^u \bullet \varphi u^*$$
$$\psi v^* \bullet u^* \eta^v = p_* \theta \bullet \eta^p u^* \qquad \varepsilon^p q^* \bullet p^* \psi = q^* \varepsilon^v \bullet \theta v_*$$

where the η denote the various adjunction units and the ε denote the various adjunction counits, thus:

$$\begin{aligned}
\psi v^* u_! \bullet \left(u^* \eta^v u_! \bullet \eta^u\right) &= p_* \theta u_! \bullet \eta^p u^* u_! \bullet \eta^u \\
p_* \theta u_! \bullet p_* p^* \eta^u \bullet \eta^p &= p_* q^* \varphi \bullet \left(p_* \eta^q p^* \bullet \eta^p\right) \\
\left(\varepsilon^q \bullet q_! \varepsilon^p q^*\right) \bullet q_! p^* \psi &= \varepsilon^q \bullet q_! q^* \varepsilon^v \bullet q_! \theta v_* \\
\varepsilon^v \bullet \varepsilon^q v^* v_* \bullet q_! \theta v_* &= \left(\varepsilon^v \bullet v^* \varepsilon^u v_*\right) \bullet \varphi u^* v_*
\end{aligned}$$

Thus, (φ, ψ) is a conjugate pair of natural transformations between the adjunctions $v^* u_! \dashv u^* v_*$ and $q_! p^* \dashv p_* q^*$. It follows (by lemma A.1.10) that φ is a natural isomorphism if and only if ψ is a natural isomorphism. ■

A.2 Cartesian closed categories

Definition A.2.1. Let C be a category with binary products, and let Y and Z be objects in C. An **exponential object** for Y and Z is an object $[Y, Z]_C$ in C and a morphism $\mathrm{ev}_{Y,Z} : [Y, Z]_C \times Y \to Z$ with the following universal property:

- For all morphisms $f : X \times Y \to Z$ in C, there exists a unique morphism $\bar{f} : X \to [Y, Z]_C$ such that $\mathrm{ev}_{Y,Z} \circ \left(\bar{f} \times \mathrm{id}_Y\right) = f$.

An **exponentiable object** in C is an object Y such that, for all objects Z in C, the exponential object $[Y, Z]_C$ exists. We may write $[Y, Z]$ or Z^Y instead of $[Y, Z]_C$ if there is no risk of confusion.

Lemma A.2.2. *Let Y be an object in a category C with binary products. The following are equivalent:*

(i) *Y is an exponentiable object in C.*

(ii) *The functor $- \times Y : C \to C$ has a right adjoint $[Y, -]_C : C \to C$, and the counit of this adjunction is $\mathrm{ev}_{Y,-}$.*

Proof. Immediate from the definitions. ◆

Definition A.2.3. A **cartesian closed category** is a category with finite products, in which every object is exponentiable. A **locally cartesian closed category** is a category C such that, for every object I, the slice category $C_{/I}$ is a cartesian closed category.

Example A.2.4. **Set** is cartesian closed category; in fact, it is even a locally cartesian closed category.

Proposition A.2.5. *Let $\mathcal{C}$ be a cartesian closed category.*

(i) *The assignment $(Y, Z) \mapsto [Y, Z]_{\mathcal{C}}$ extends to a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.*

(ii) *For each object Z, the functor $[-, Z]_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is a contravariant right adjoint for itself.*

Proof. (i). This is an instance of the parametrised adjunction theorem.[2]

(ii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}(X, [Y, Z]) &\cong \mathcal{C}(X \times Y, Z) \\ &\cong \mathcal{C}(Y \times X, Z) \\ &\cong \mathcal{C}(Y, [X, Z]) \end{aligned}$$ ■

Lemma A.2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories. If $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves binary products, then:*

(i) *For any two objects X and Y in $\mathcal{C}$, there is a unique morphism $\varphi_{Y,Z} : F[X, Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ such that the following diagram commutes:*

$$\begin{tikzcd} F[X, Y]_{\mathcal{C}} \times FX \arrow[r, "\cong"] \arrow[d, "\varphi_{X,Y} \times \mathrm{id}"'] & F\big([X, Y]_{\mathcal{C}} \times X\big) \arrow[d, "F\mathrm{ev}_{X,Y}"] \\ {[FX, FY]_{\mathcal{D}}} \times FX \arrow[r, "\mathrm{ev}_{FX,FY}"'] & FY \end{tikzcd}$$

(ii) *The morphism $\varphi_{Y,Z}$ is natural in both Y and Z.*

Proof. The existence and uniqueness of $\varphi_{X,Y}$ follows from the universal property of $[FX, FY]_{\mathcal{D}}$ as an exponential object, and a standard argument proves naturality. ◆

Definition A.2.7. A **cartesian closed functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ between cartesian closed categories such that the canonical comparison morphisms $\varphi_{X,Y} : F[X, Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ described above are isomorphisms.

[2] See Theorem 3 in [CWM, Ch. IV, § 7].

Proposition A.2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories, and let Y be an object in $\mathcal{C}$ and let Z be an object in $\mathcal{D}$. Suppose we have an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$; then:*

(i) *If $\psi_{FY,Z} : G[FY,Z]_{\mathcal{D}} \to [GFY, GZ]_{\mathcal{C}}$ is the canonical comparison morphism, then $\theta_{Y,Z} = \left[\eta_Y, GZ\right]_{\mathcal{C}} \circ \psi_{FY,Z}$ is the unique morphism in $\mathcal{C}$ making the following diagram commute:*

$$
\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\mathrm{id}\times\eta_Y"] \arrow[d, "\theta_{Y,Z}\times\mathrm{id}"'] & G[FY,Z]_{\mathcal{D}} \times GFY \arrow[d, "\cong"] \\
{[Y,GZ]_{\mathcal{C}}} \times Y \arrow[d, "\mathrm{ev}_{Y,GZ}"'] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[d, "G\mathrm{ev}_{FY,Z}"] \\
GZ \arrow[r, equal] & GZ
\end{tikzcd}
$$

(ii) *If the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism for all objects X in $\mathcal{C}$, and $\varphi_{Y,GZ} : F[Y,GZ]_{\mathcal{C}} \to [FY, FGZ]_{\mathcal{D}}$ is the canonical comparison morphism, then $\chi_{Y,Z} = \left[FY, \varepsilon_Z\right]_{\mathcal{D}} \circ \varphi_{Y,GZ}$ is the unique morphism in $\mathcal{D}$ making the following diagram commute:*

$$
\begin{tikzcd}
F[Y,GZ]_{\mathcal{C}} \times FY \arrow[r, "\cong"] \arrow[d, "\chi_{Y,Z}\times\mathrm{id}"'] & F\big([Y,GZ]_{\mathcal{C}} \times Y\big) \arrow[d, "\mathrm{ev}_{Y,GZ}"] \\
{[FY,Z]_{\mathcal{D}}} \times FY \arrow[d, "\mathrm{ev}_{FY,Z}"'] & FGZ \arrow[d, "\varepsilon_Z"] \\
Z \arrow[r, equal] & Z
\end{tikzcd}
$$

Moreover, under this hypothesis, $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$ is a two-sided inverse for $\theta_{Y,Z}$.

(iii) *If $\theta_{Y,Z}$ is an isomorphism for all objects Z in $\mathcal{D}$, then for all objects X in $\mathcal{C}$, the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism.*

Proof. (i). The claim is proven by the commutativity of the following diagram:

$$\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\mathrm{id}\times\eta_Y"] \arrow[d, "\psi_{FY,Z}\times\mathrm{id}"'] & G[FY,Z]_{\mathcal{D}} \times GFY \arrow[r, "\cong"] \arrow[d, "\psi_{FY,Z}\times\mathrm{id}"] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[dd, "G\mathrm{ev}_{FY,Z}"] \\
{[GFY,GZ]_{\mathcal{C}} \times Y} \arrow[r, "\mathrm{id}\times\eta_Y"'] \arrow[d, "{[\eta_Y,Z]_{\mathcal{C}}}"'] & {[GFY,GZ]_{\mathcal{C}} \times GFY} \arrow[dr, "\mathrm{ev}_{GFY,GZ}"] & \\
{[Y,GZ]_{\mathcal{C}}} \arrow[rr, "\mathrm{ev}_{Y,GZ}"'] & & GZ
\end{tikzcd}$$

(ii). To show that $\chi_{Y,Z}$ makes the diagram commute, one uses the fact that $\mathrm{ev}_{FY,Z} : [FY,Z]_{\mathcal{D}} \times FY \to Z$ is natural in Z. Since F preserves products with Y, we have the following natural bijections:

$$\begin{aligned}
\mathcal{C}\big(X, G[FY,Z]_{\mathcal{D}}\big) &\cong \mathcal{D}\big(FX, [FY,Z]_{\mathcal{D}}\big) \cong \mathcal{D}(FX \times FY, Z) \\
&\cong \mathcal{D}(F(X \times Y), Z) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{C}\big(X, [Y,GZ]_{\mathcal{C}}\big)
\end{aligned}$$

One obtains explicit isomorphisms by chasing id_X in both directions. Taking $X = [Y,GZ]_{\mathcal{C}}$, we find that the isomorphism $[Y,GZ]_{\mathcal{C}} \to G[FY,Z]_{\mathcal{D}}$ is precisely $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$, and taking $X = G[FY,Z]_{\mathcal{D}}$, we find that the inverse is the right exponential transpose of

$$G\big(\mathrm{ev}_{FY,Z} \circ \big(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_Y\big)\big) \circ \eta_{G[FY,Z]_{\mathcal{D}}\times Y}$$

where we have suppressed the comparison isomorphism $F\big(G[FY,Z]_{\mathcal{D}} \times Y\big) \cong FG[FY,Z]_{\mathcal{D}} \times FY$; but naturality of the comparison morphisms for binary products gives us the commutative diagram below,

$$\begin{tikzcd}
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\eta"] \arrow[dd, equal] & GF\big(G[FY,Z]_{\mathcal{D}} \times Y\big) \arrow[d, "\cong"'] & \\
& G\big(FG[FY,Z]_{\mathcal{D}} \times FY\big) \arrow[r, "G(\varepsilon\times\mathrm{id})"] \arrow[d, "\cong"'] & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \arrow[d, "\cong"] \\
G[FY,Z]_{\mathcal{D}} \times Y \arrow[r, "\eta\times\eta"] \arrow[rr, bend right, "\mathrm{id}\times\eta"'] & GFG[FY,Z]_{\mathcal{D}} \times GFY \arrow[r, "G\varepsilon\times\mathrm{id}"] & G[FY,Z]_{\mathcal{D}} \times GFY
\end{tikzcd}$$

so, suppressing the comparison isomorphisms, we obtain the following equation:

$$G\big(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_{FY}\big) \circ \eta_{G[FY,Z]_{\mathcal{D}} \times Y} = \mathrm{id}_{G[FY,Z]_{\mathcal{D}}} \times \eta_Y$$

Thus, the isomorphism $G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is indeed $\theta_{Y,Z}$, as claimed.

(iii). Now, suppose $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is an isomorphism for all Z. Then, we have the natural bijections

$$\begin{aligned} \mathcal{D}(FX \times FY, Z) &\cong \mathcal{D}\big(FX, [FY, Z]_{\mathcal{D}}\big) \cong \mathcal{C}\big(X, G[FY, Z]_{\mathcal{D}}\big) \\ &\cong \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{D}(F(X \times Y), Z) \end{aligned}$$

and by chasing id_Z for $Z = FX \times FY$, we conclude that the *canonical* comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism. ■

Definition A.2.9. A **Frobenius adjunction of cartesian closed categories** is an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories, such that the natural morphisms $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ described above are isomorphisms, or equivalently, such that the left adjoint $F : \mathcal{C} \to \mathcal{D}$ preserves binary products.

Remark A.2.10. If $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories and $G : \mathcal{D} \to \mathcal{C}$ is any functor that preserves finite products, then G induces a $\mathcal{C}$-enrichment of $\mathcal{D}$ from the cartesian closed structure of $\mathcal{D}$, and the exponential comparison morphisms $\psi_{Y,Z} : G[Y, Z]_{\mathcal{D}} \to [GY, GZ]_{\mathcal{C}}$ makes $G : \mathcal{D} \to \mathcal{C}$ into a $\mathcal{C}$-enriched functor.

Now, suppose G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$. The adjunction $F \dashv G$ is a Frobenius adjunction precisely when it is compatible with the $\mathcal{C}$-enrichments of $\mathcal{C}$ and $\mathcal{D}$. (Of course, this means F is also a $\mathcal{C}$-enriched functor.)

However, not all enriched adjunctions between cartesian closed categories are of the above form.

Proposition A.2.11. *Let X, Y, and Z be any three objects in a cartesian closed category $\mathcal{C}$.*

(i) *There is a unique morphism $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ making* *the following diagram commute:*

$$\begin{tikzcd}
([X \times Y, Z] \times X) \times Y \arrow[r, "\cong"] \arrow[d, "(\lambda_{X,Y,Z} \times \mathrm{id}_X) \times \mathrm{id}_Y"'] & {[X \times Y, Z] \times (X \times Y)} \arrow[dd, "\mathrm{ev}_{X \times Y, Z}"] \\
([X, [Y, Z]] \times X) \times Y \arrow[d, "\mathrm{ev}_{X,[Y,Z]} \times \mathrm{id}_X"'] & \\
{[Y, Z] \times Y} \arrow[r, "\mathrm{ev}_{Y,Z}"'] & Z
\end{tikzcd}$$

(ii) *The morphisms* $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ *constitute a natural isomorphism.*

Proof. The existence and uniqueness of $\lambda_{X,Y,Z}$ follows from the universal property of $[X, [Y, Z]]$ and $[Y, Z]$ as exponential objects, and a standard argument shows that $\lambda_{X,Y,Z}$ is natural in X, Y, and Z. By the associativity of cartesian products, we have the following natural bijections:

$$\begin{aligned}
\mathcal{C}(T, [X \times Y, Z]) &\cong \mathcal{C}(T \times (X \times Y), Z) \\
&\cong \mathcal{C}((T \times X) \times Y, Z) \cong \mathcal{C}(T \times X, [Y, Z]) \cong \mathcal{C}(T, [X, [Y, Z]])
\end{aligned}$$

Chasing id_T for $T = [X \times Y, Z]$, we find that $\lambda_{X,Y,Z}$ is an isomorphism. ∎

Definition A.2.12. Let $\mathcal{C}$ be a cartesian closed category. An **exponential ideal** of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that, for all objects Y in $\mathcal{C}$, if Z is in $\mathcal{D}$, then the exponential object $[Y, Z]_{\mathcal{C}}$ is (isomorphic to) an object in $\mathcal{D}$. A **reflective exponential ideal** of $\mathcal{C}$ is an exponential ideal $\mathcal{D}$ such that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint.

Proposition A.2.13. *Let* $\mathcal{C}$ *be a cartesian closed category, let* $G : \mathcal{D} \to \mathcal{C}$ *be the inclusion of a full subcategory, and suppose* G *has a left adjoint* $F : \mathcal{C} \to \mathcal{D}$. *The following are equivalent:*

(i) F *preserves finite products.*

(ii) F *preserves binary products.*

(iii) $\mathcal{D}$ *is a reflective exponential ideal of* $\mathcal{C}$.

(iv) $\mathcal{D}$ *is a cartesian closed category,* $G : \mathcal{D} \to \mathcal{C}$ *is a cartesian closed functor, and the canonical morphisms* $G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ *are isomorphisms.*

Proof. (i) ⇒ (ii). Immediate.

(ii) ⇒ (iii). Under our hypotheses, the product of two objects X and Y in $\mathcal{D}$ can be computed as $F(GX \times GY)$. Let $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ be the unit of the adjunction. We have the following natural bijections:

$$\begin{aligned} \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big) &\cong \mathcal{C}(X \times Y, GZ) \\ &\cong \mathcal{D}(FX \times FY, Z) \\ &\cong \mathcal{D}(FGFX \times FY, Z) \\ &\cong \mathcal{C}(GFX \times Y, GZ) \\ &\cong \mathcal{C}\big(GFX, [Y, GZ]_{\mathcal{C}}\big) \end{aligned}$$

By chasing these maps explicitly, we find that every morphism $X \to [Y, GZ]_{\mathcal{C}}$ factors through $\eta_X : X \to GFX$ in a unique way. In particular, we have

$$\mathrm{id}_{[Y,GZ]_{\mathcal{C}}} = r_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$$

for a unique $r_{Y,Z} : GF[Y, GZ]_{\mathcal{C}} \to [Y, GZ]_{\mathcal{C}}$. The triangle identity then implies $Fr_{Y,Z} = \varepsilon_{F[Y,GZ]_{\mathcal{C}}}$, thus,

$$\eta_{[Y,GZ]_{\mathcal{C}}} \circ r_{Y,Z} = GFr_{Y,Z} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = G\varepsilon_{F[Y,GZ]_{\mathcal{C}}} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = \mathrm{id}_{GF[Y,GZ]_{\mathcal{C}}}$$

and therefore $r_{Y,Z}$ is an isomorphism.

(iii) ⇒ (iv). It is a standard fact that a reflective subcategory is closed under all limits that exist in $\mathcal{C}$, so $\mathcal{D}$ must have finite products and $G : \mathcal{D} \to \mathcal{C}$ preserves them. If $\mathcal{D}$ is an exponential ideal, then $\eta_{[Y,GZ]_{\mathcal{C}}} : [Y, GZ]_{\mathcal{C}} \to GF[Y, GZ]_{\mathcal{C}}$ must be an isomorphism, so we obtain natural bijections

$$\begin{aligned} \mathcal{D}(X \times Y, Z) &\cong \mathcal{C}(GX \times GY, GZ) \\ &\cong \mathcal{C}\big(GX, [GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{C}\big(GX, GF[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(FGX, F[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(X, F[GY, GZ]_{\mathcal{C}}\big) \end{aligned}$$

and therefore we may take $[Y,Z]_{\mathcal{D}} = F[GY,GZ]_{\mathcal{C}}$. Obviously, this makes $G : \mathcal{D} \to \mathcal{C}$ into a cartesian closed functor. We also have

$$\begin{aligned}
\mathcal{C}\big(X, G[FY,Z]_{\mathcal{D}}\big) &= \mathcal{C}\big(X, GF[GFY,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(X, [GFY,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, [X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, GF[X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, GF[X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, [X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(X, [Y,GZ]_{\mathcal{C}}\big)
\end{aligned}$$

and so the canonical morphism $G[FY,Z]_{\mathcal{D}} \to [Y,GZ]_{\mathcal{C}}$ is an isomorphism.

(iv) ⇒ (i). Since $\mathcal{D}$ has a terminal object and $G : \mathcal{D} \to \mathcal{C}$ preserves it, $F1$ must be a terminal object in $\mathcal{D}$. Now apply proposition A.2.8. ■

Corollary A.2.14. *If $\mathcal{E}$ is a reflective exponential ideal of $\mathcal{D}$, and $\mathcal{D}$ is a reflective exponential ideal of $\mathcal{C}$, then $\mathcal{E}$ is also a reflective exponential ideal of $\mathcal{C}$.* ■

Proposition A.2.15. *Let* **Cat** *be the category of small categories, and let* **Grpd** *be the full subcategory of groupoids.*

(i) *There exist adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Cat}$$

where $\mathrm{ob}\,\mathbb{C}$ *is the set of objects in a category* $\mathbb{C}$*,* $\mathrm{disc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and all arrows trivial, and* $\mathrm{codisc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and a unique arrow between any two objects.*

(ii) *The functor* $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ *is fully faithful and exhibits* **Set** *as a reflective exponential ideal of* **Cat**.

(iii) *The functor* $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ *preserves finite products.*

(iv) *There exist adjunctions*

$$\mathbf{I} \dashv \mathrm{und} \dashv \mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$$

where $\mathrm{und} : \mathbf{Grpd} \to \mathbf{Cat}$ *is the inclusion and* $\mathrm{iso}\,\mathbb{C}$ *is the maximal subgroupoid of a category* $\mathbb{C}$*.*

(v) $\mathbf{Grpd}$ *is a reflective exponential ideal of* $\mathbf{Cat}$.

(vi) *The functor* $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$ *preserves finite products.*

(vii) *The adjunctions in (i) factor through* $\mathbf{Grpd}$, *yielding adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Grpd}$$

where $\pi_0 : \mathbf{Grpd} \to \mathbf{Set}$ *again preserves finite products.*

(viii) *The functor* $\mathbf{Cat} \to \mathbf{Set}$ *that sends a category* $\mathbb{C}$ *to the set of isomorphism classes in* $\mathbb{C}$ *preserves finite products.*

Proof. (i). The functor $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ obviously satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$; the existence of the other adjunctions is obvious.

(ii). It is clear that $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ is fully faithful, and direct computation shows that $[\mathbb{C}, \mathrm{disc}\, X]$ is a discrete category for any $\mathbb{C}$, so $\mathbf{Set}$ is indeed a reflective exponential ideal of $\mathbf{Cat}$.

(iii). Thus, by proposition A.2.13, $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ must preserve finite products.

(iv). It is not hard to check that the inclusion $\mathbf{Grpd} \to \mathbf{Cat}$ satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$; the fact that $\mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$ is right adjoint to the inclusion is obvious.

(v). Direct computation shows that $[\mathbb{C}, \mathbb{G}]$ is a groupoid whenever $\mathbb{G}$ is, so $\mathbf{Grpd}$ is indeed a reflective exponential ideal of $\mathbf{Cat}$.

(vi). Thus, $\mathbf{I} : \mathbf{Cat} \to \mathbf{Grpd}$ must preserve finite products.

(vii). Clearly, $\mathrm{disc}\, X$ and $\mathrm{codisc}\, X$ are already groupoids, so the adjunctions do indeed factor through $\mathbf{Grpd}$.

(viii). The set of isomorphism classes of objects in $\mathbb{C}$ is precisely $\pi_0\, \mathrm{iso}\, \mathbb{C}$. ■

Definition A.2.16. The **dependent sum** of an object $p : X \to I$ in $\mathcal{C}_{/I}$ along a morphism $j : I \to J$ in $\mathcal{C}$ is the object $j \circ p : X \to J$ in $\mathcal{C}_{/J}$, and we write $\Sigma_j : \mathcal{C}_{/I} \to \mathcal{C}_{/J}$ for the functor sending an object to its dependent sum along j.

Lemma A.2.17. *Let $j : I \to J$ be a morphism in a category C. The following are equivalent:*

(i) *C has pullbacks along j.*

(ii) *There exists an adjunction*

$$\Sigma_j \dashv j^* : C_{/J} \to C_{/I}$$

where Σ_j is the dependent sum functor, and the right adjoint $j^* : C_{/J} \to C_{/I}$ is the pullback functor.

Proof. This is just a matter of unwinding the definitions. ◆

Definition A.2.18. Let C be a category with pullbacks. A **dependent product** of an object $p : X \to I$ in $C_{/I}$ along a morphism $j : I \to J$ in C is an object $\Pi_j p$ in $C_{/J}$ and a morphism $\mathrm{ev}_{j,p} : j^*\Pi_j p \to p$ in $C_{/I}$ with the following universal property:

- For all morphisms $f : j^*q \to p$ in $C_{/I}$, there exists a unique morphism $\bar{f} : q \to \Pi_j p$ in $C_{/J}$ such that $\mathrm{ev}_{j,p} \circ j^*\bar{f} = f$.

A **ΣΠ-category** is a category C with finite limits such that, for every morphism $j : I \to J$ in C, dependent products along j exist.

Lemma A.2.19. *Let $j : I \to J$ be a morphism in a category C with pullbacks. The following are equivalent:*

(i) *For all objects $p : X \to I$ in C, a dependent product of p along j exists.*

(ii) *The pullback functor $j^* : C_{/J} \to C_{/I}$ has a right adjoint $\Pi_j : C_{/I} \to C_{/J}$ that sends an object to its dependent product along j, and the counit of this adjunction is $\mathrm{ev}_{j,-}$.*

Proof. This is just a matter of unwinding the definitions. ◆

Corollary A.2.20. *If $j : I \to J$ is a morphism in a ΣΠ-category C, then the pullback functor $j^* : C_{/J} \to C_{/I}$ preserves all limits and colimits.* ■

Proposition A.2.21. *Let C be a category with a terminal object. The following are equivalent:*

(i) *C is a ΣΠ-category.*

(ii) *$\mathcal{C}$ is a locally cartesian closed category.*

Proof. See Proposition 9.20 in [Awodey, 2010]. □

Theorem A.2.22. *Let $\mathbb{D}$ be a small category, and let $\mathcal{C} = [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$. Then:*

(i) *$\mathcal{C}$ has limits and colimits for all small diagrams, and these can be constructed componentwise in* **Set***: a cone (resp. cocone) in $\mathcal{C}$ over (resp. under) a diagram in $\mathcal{C}$ is a limiting cone (resp. colimiting cocone) if and only if it is so in every component.*

(ii) *Every internal equivalence relation in $\mathcal{C}$ is the kernel pair of its coequaliser.*

(iii) *For all morphisms $j : I \to J$ in $\mathcal{C}$, the pullback functor $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$ preserves all limits and colimits.*

(iv) *The Yoneda embedding $\hbar_\bullet : \mathbb{D} \to \mathcal{C}$ is a dense functor, i.e. for every presheaf $X : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$, the tautological cocone*[3] *from the canonical diagram $(\hbar_\bullet \downarrow X) \to \mathcal{C}$ to X is a colimiting cocone.*

(v) *$\mathcal{C}$ is a locally finitely presentable category.*

(vi) *$\mathcal{C}$ is a $\Sigma\Pi$-category.*

Proof. (i). This is a standard fact about presheaf categories.

(ii) and (iii). The claims are true for **Set**, and hence for $\mathcal{C}$ by claim (i).

(iv). See proposition A.5.25.

(v). See theorem 0.2.37.

(vi). Apply theorem 0.2.47 to construct a right adjoint for $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$. ■

Remark A.2.23. The Yoneda lemma gives us an explicit description of the exponential objects in $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$: given two presheaves $Y, Z : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$, if Z^Y is their exponential object, then we must have

$$Z^Y(d) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d, Z^Y\big) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d \times Y, Z\big)$$

and so we may *define* Y^Z by $Y^Z(d) = [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d \times Y, Z\big)$.

[3] See definition A.5.7.

Definition A.2.24. Let Y and Z be topological spaces, and let $[Y, Z]$ be the set of all *continuous* maps $Y \to Z$. The **compact–open topology** on $[Y, Z]$ is the coarsest topology such that the subsets

$$V(K, U) = \left\{ f \in [Y, Z] \,\middle|\, K \subseteq f^{-1}U \right\}$$

are open in $[Y, Z]$ for all compact subsets $K \subseteq X$ and all open subsets $U \subseteq Y$.

Remark A.2.25. If Y is a discrete space, then the compact–open topology on $[Y, Z]$ coincides with the product topology on Z^Y.

Definition A.2.26. A **compactly-generated Hausdorff space** is a Hausdorff topological space X such that a subset $U \subseteq X$ is open if and only if, for every continuous map $f : K \to X$ where K is a compact Hausdorff space, $f^{-1}U$ is an open subset of K. We write **CGHaus** for the category of compactly-generated Hausdorff spaces and continuous maps.

Proposition A.2.27.

(i) *If Y is a locally compact Hausdorff space, then for all topological spaces Z, the set of all continuous maps $Y \to Z$, equipped with the compact–open topology, is an exponential object* $[Y, Z]$ *in* **Top**.

(ii) **Top** *is* not *a cartesian closed category.*

(iii) **CGHaus** *is a cartesian closed category.*

Proof. Claim (i) follows from Theorems 46.10 and 46.11 in [Munkres, 2000], and claim (ii) is Proposition 7.1.2 in [Borceux, 1994a], and claim (iii) is proved in [GZ, Ch. III, § 2]. □

A.3 Factorisation systems

Definition A.3.1. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a category $\mathcal{C}$. Given a commutative square in $\mathcal{C}$,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

a **lift** is a morphism $h : W \to X$ such that $f \circ h = w$ and $h \circ g = z$.

We say g has the **left lifting property** with respect to f and f has the **right lifting property** with respect to g, and we write $g \boxslash f$, if every commutative square in $\mathcal{C}$ of the form above has a lift. We say f is **left orthogonal** to g and g is **right orthogonal** to f, and we write $g \perp f$ if lifts exist *and* are unique.

Given $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, we define the following subensembles of $\operatorname{mor} \mathcal{C}$:

$$\begin{aligned}
{}^{\boxslash}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\, f \boxslash g \} \\
\mathcal{I}^{\boxslash} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\, f \boxslash g \} \\
{}^{\perp}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\, f \perp g \} \\
\mathcal{I}^{\perp} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\, f \perp g \}
\end{aligned}$$

Lemma A.3.2. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a locally small category $\mathcal{C}$. Consider the commutative diagram in* **Set** *shown below,*

$$\begin{tikzcd}
\mathcal{C}(W,X) \arrow[dr, dashed] \arrow[drr, bend left, "f_*"] \arrow[ddr, bend right, "g^*"'] & & \\
& \bullet \arrow[r] \arrow[d] & \mathcal{C}(W,Y) \arrow[d, "g^*"] \\
& \mathcal{C}(Z,X) \arrow[r, "f_*"'] & \mathcal{C}(Z,Y)
\end{tikzcd}$$

where the inner square is a pullback diagram.

(i) *The dashed arrow is a surjection if and only if $g \boxslash f$.*

(ii) *The dashed arrow is a bijection if and only if $g \perp f$.*

Proof. This is just a restatement of the definition. ■

Proposition A.3.3. *Let $\mathcal{C}$ be a category.*

(i) *If $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\perp}\mathcal{R} \subseteq {}^{\boxslash}\mathcal{R}$.*

(ii) *If $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\boxslash}\mathcal{R}' \supseteq {}^{\boxslash}\mathcal{R}$.*

(iii) *If $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, then ${}^{\perp}\mathcal{R}' \supseteq {}^{\perp}\mathcal{R}$.*

Dually:

(i′) *If* $\mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash}$.

(ii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}'^{\boxslash} \supseteq \mathcal{L}^{\boxslash}$.

(iii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$*, then* $\mathcal{L}'^{\perp} \supseteq \mathcal{L}^{\perp}$.

Moreover, we have the following antitone Galois connections:

$$\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\boxslash}$$
$$\mathcal{L} \subseteq {}^{\perp}\mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\perp}$$

Proof. Obvious. ♦

Corollary A.3.4. *We have the following identities:*

$$\begin{aligned} {}^{\boxslash}\big(({}^{\boxslash}\mathcal{R})^{\boxslash}\big) &= {}^{\boxslash}\mathcal{R} & \qquad {}^{\perp}\big(({}^{\perp}\mathcal{R})^{\perp}\big) &= {}^{\perp}\mathcal{R} \\ \big({}^{\boxslash}\big(\mathcal{L}^{\boxslash}\big)\big)^{\boxslash} &= \mathcal{L}^{\boxslash} & \qquad \big({}^{\perp}\big(\mathcal{L}^{\perp}\big)\big)^{\perp} &= \mathcal{L}^{\perp} \end{aligned}$$

Proof. This is a standard fact about (antitone) Galois connections. ■

Lemma A.3.5. *Let* $f : X \to Y$ *be a morphism in a category* $\mathcal{C}$*. The following are equivalent:*

(i) f *is an isomorphism.*

(ii) f *is right orthogonal to any morphism in* $\mathcal{C}$.

(iii) f *has the right lifting property with respect to any morphism in* $\mathcal{C}$.

(iv) f *has the right lifting property with respect to itself.*

Dually, the following are equivalent:

(i′) f *is an isomorphism.*

(ii′) f *is left orthogonal to any morphism in* $\mathcal{C}$.

(iii′) f *has the left lifting property with respect to any morphism in* $\mathcal{C}$.

(iv′) f *has the left lifting property with respect to itself.*

Proof. (i) ⇒ (ii). Suppose $r : Y \to X$ is a morphism such that $r \circ f = \mathrm{id}_X$. Then, for any commutative square as below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

we have $(r \circ w) \circ g = r \circ f \circ z = z$; but if $f \circ r = \mathrm{id}_Y$ as well, then $f \circ (r \circ w) = w$; thus $r \circ w : W \to X$ is the required lift. It is clearly unique, as f is monic.

(ii) ⇒ (iii), (iii) ⇒ (iv). Obvious.

(iv) ⇒ (i). Consider the following commutative square:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

Since f has the right lifting property with respect to itself, there exists a morphism $h : Y \to X$ such that $h \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. ■

Definition A.3.6. A **weak factorisation system** for a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of subclasses of $\operatorname{mor} \mathcal{C}$ satisfying these conditions:

- For each morphism f in $\mathcal{C}$ there exists a pair (g, h) with $g \in \mathcal{L}$ and $h \in \mathcal{R}$ such that $f = h \circ g$. Such a pair is a $(\mathcal{L}, \mathcal{R})$**-factorisation** of f.
- A morphism is in $\mathcal{L}$ if and only if it has the left lifting property with respect to every morphism in $\mathcal{R}$, i.e. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$.
- A morphism is in $\mathcal{R}$ if and only if it has the right lifting property with respect to every morphism in $\mathcal{L}$, i.e. $\mathcal{R} = \mathcal{L}^{\boxslash}$.

An **orthogonal factorisation system** is defined like a weak factorisation system, except for replacing '… has the left/right lifting property with respect to …' with '… is left/right orthogonal to …'.

Remark A.3.7. Obviously, $(\mathcal{L}, \mathcal{R})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}$ if and only if $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}^{\mathrm{op}}$.

Proposition A.3.8. *Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on $\mathcal{C}$. If either*

- *every morphism in $\mathcal{R}$ is a monomorphism in $\mathcal{C}$, or*
- *every morphism in $\mathcal{L}$ is an epimorphism in $\mathcal{C}$,*

then $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system.

Proof. The two hypotheses are formally dual, so it is enough to check the first case. Observe that, given a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] \arrow[ur, dashed, "h"] & Y \end{tikzcd}$$

where $f : X \to Y$ is a monomorphism, for any $h' : W \to X$ such that $f \circ h' = w$, we must have $h = h'$. Thus, for any monomorphism $f : X \to Y$, $g \boxslash f$ if and only if $g \perp f$. Hence, $\mathcal{L} = {}^{\boxslash}\mathcal{R} = {}^{\perp}\mathcal{R}$. On the other hand, $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash} = \mathcal{R}$, so $\mathcal{R} = \mathcal{L}^{\perp}$ as well. ∎

Definition A.3.9. A **proper factorisation system** on a category $\mathcal{C}$ is an orthogonal factorisation system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ such that every morphism in $\mathcal{E}$ is an epimorphism *and* every morphism in $\mathcal{M}$ is a monomorphism.

Example A.3.10. In **Set**, if $\mathcal{E}$ is the class of surjective maps and $\mathcal{M}$ is the class of injective maps, then $(\mathcal{E}, \mathcal{M})$ is a proper factorisation system.

Lemma A.3.11. *Let A be an object in a category $\mathcal{C}$ with a weak (resp. orthgonal) factorisation system $(\mathcal{L}, \mathcal{R})$. Then the slice category $\mathcal{C}_{/A}$ has a weak (resp. orthogonal) factorisation system where a morphism is in the left or right class if and only if it is so in $\mathcal{C}$.*

Proof. The projection $\mathcal{C}_{/A} \to \mathcal{C}$ induces a bijection between solutions for lifting problems in $\mathcal{C}_{/A}$ and solutions for the corresponding lifting problems in $\mathcal{C}$. ∎

Proposition A.3.12 (Closure properties)**.** *Let $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$ and suppose either $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ or $\mathcal{L} = {}^{\perp}\mathcal{R}$.*

(i) *Given a pushout diagram in $\mathcal{C}$ as below,*

$$\begin{tikzcd} Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[d, "g"] \\ W' \arrow[r, "i_W"'] & W \end{tikzcd}$$

if the morphism g' is in $\mathcal{L}$, then g is also in $\mathcal{L}$.

(ii) *Let I be a set. If $g_i : Z_i \to W_i$ is a morphism in $\mathcal{L}$ for all i in I and the coproduct $\coprod_i g_i : \coprod_i Z_i \to \coprod_i W_i$ exists in $\mathcal{C}$, then $\coprod_i g_i$ is also in $\mathcal{L}$.*

(iii) *Given a commutative diagram of the form*

$$\begin{tikzcd} Z' \arrow[r, "i_Z"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "g'"'] & Z \arrow[r, "r_Z"'] \arrow[d, "g"'] & Z' \arrow[d, "g'"] \\ W' \arrow[r, "i_W"] \arrow[rr, bend right, "\mathrm{id}"'] & W \arrow[r, "r_W"] & W' \end{tikzcd}$$

if g is in $\mathcal{L}$, then so is g'; in other words, $\mathcal{L}$ is closed under retracts.

(iv) *$\mathcal{L}$ is closed under composition.*

(v) *Let γ be an ordinal and let $Z : \gamma \to \mathcal{C}$ be a colimit-preserving functor. We write Z_α for $Z(\alpha)$, where $\alpha < \gamma$, and $g_{\alpha,\beta} : Z_\alpha \to Z_\beta$ for the morphism $Z(\alpha \to \beta)$, where $\alpha < \beta < \gamma$. If λ is a colimiting cocone from Z to W and each $g_{\alpha,\beta}$ is in $\mathcal{L}$, then each component $\lambda_\alpha : Z_\alpha \to W$ is also in $\mathcal{L}$.*

Proof. (i). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{tikzcd} Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[r, "z"] \arrow[d, "g"] & X \arrow[d, "f"] \\ W' \arrow[r, "i_W"'] & W \arrow[r, "w"'] & Y \end{tikzcd}$$

There exists $h' : W' \to X$ such that $h' \circ g' = z \circ i_Z$ and $f \circ h' = w \circ i_W$. In particular, there exists a unique morphism $h : W \to X$ such that $h \circ g = z$ and

$h \circ i_W = h'$, by the universal property of pullbacks. Thus $f \circ h \circ i_W = f \circ h' = w \circ i_W$ and $f \circ h \circ g = f \circ z = w \circ g$, but i_W and g are jointly epic, so $f \circ h = w$. This shows h is the required lift, and h is unique if h' is.

(ii). We may construct the required lift componentwise.

(iii). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{tikzcd}
Z' \arrow[r, "i_Z"] \arrow[d, "g'"'] & Z \arrow[r, "r_Z"] \arrow[d, "g"'] & Z' \arrow[r, "z"] \arrow[d, "g'"'] & X \arrow[d, "f"] \\
W' \arrow[r, "i_W"'] & W \arrow[r, "r_W"'] & W' \arrow[r, "w"'] & Y
\end{tikzcd}$$

There exists $h : W \to X$ such that $h \circ g = z \circ r_Z$ and $f \circ h = w \circ r_W$, and so for $h' = h \circ i_W$:

$$f \circ h' = f \circ h \circ i_W = w \circ r_W \circ i_W = w$$
$$h' \circ g' = h \circ i_W \circ g' = h \circ g \circ i_Z = z \circ r_Z \circ i_Z = z$$

Thus $h' : W' \to X$ is the required lift, and h' is unique if h is (because r_W is split epic).

(iv). Suppose $g' : Z' \to Z$ and $g : Z \to W$ are in $\mathcal{L}$ and $f : X \to Y$ is in $\mathcal{R}$. Consider the following commutative diagram:

$$\begin{tikzcd}
Z' \arrow[r, "z'"] \arrow[d, "g'"'] & X \arrow[dd, "f"] \\
Z \arrow[d, "g"'] & \\
W \arrow[r, "w"'] & Y
\end{tikzcd}$$

There must exist a morphism $z : Z \to X$ such that $z \circ g' = z'$ and $f \circ z' = w \circ g$, and hence a morphism $h : W \to X$ such that $h \circ g = z$ and $f \circ h = w$. Obviously, $h \circ (g' \circ g) = z'$, so h is the required lift. Moreover, h unique if $\mathcal{L} = {}^{\perp}\mathcal{R}$.

(v). We may assume without loss of generality that $\alpha = 0$, since any non-empty terminal segment of γ is cofinal in γ. Suppose $f : X \to Y$ is in $\mathcal{R}$ and consider the following commutative diagram:

$$\begin{tikzcd} Z_0 \arrow[r, "z_0"] \arrow[d, "\lambda_0"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

For each $\alpha < \gamma$, given z_α making the following diagram commute,

$$\begin{tikzcd} Z_\alpha \arrow[rr, "z_\alpha"] \arrow[d, "g_{\alpha,\alpha+1}"'] & & X \arrow[d, "f"] \\ Z_{\alpha+1} \arrow[r, "\lambda_{\alpha+1}"'] & W \arrow[r, "w"'] & Y \end{tikzcd}$$

choose a lift $z_{\alpha+1} : Z_{\alpha+1} \to X$; for each limit ordinal $\beta < \gamma$, let $z_\beta : Z_\beta \to X$ be the unique morphism such that $z_\beta \circ g_{\alpha,\beta} = z_\alpha$ for all $\alpha < \beta$. (Such z_β exist and are unique because $Z_\beta = \varinjlim_{\alpha<\beta} Z_\alpha$.) Note that the universal property of W then guarantees that $w \circ \lambda_\beta = f \circ z_\beta$.

Having constructed morphisms $z_\alpha : Z_\alpha \to X$ for all $\alpha < \gamma$ as above, we may now obtain $h : W \to X$ as the unique morphism such that $h \circ \lambda_\alpha = z_\alpha$ for all $\alpha < \gamma$, and again we automatically have $f \circ h = w$. It is also clear that h is unique if $\mathcal{L} = {}^\perp\mathcal{R}$. ■

Proposition A.3.13 (Cancellation properties)**.** *Let $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$.*

(i) *Let $\mathcal{L}$ be either ${}^\boxslash\mathcal{R}$ or ${}^\perp\mathcal{R}$, let $e : A \to Z$ be an epimorphism in $\mathcal{C}$, and let $g : Z \to W$ be a morphism in $\mathcal{C}$. If $g \circ e$ is in $\mathcal{L}$, then so is g.*

(ii) *Suppose $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{R}$, and let $e : A \to Z$ be in $\mathcal{L}$. Then, a morphism $g : Z \to W$ is in $\mathcal{L}$ if and only $g \circ e$ is in $\mathcal{L}$.*

Dually, let $\mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$.

(i′) *Let $\mathcal{R}$ be either $\mathcal{L}^\boxslash$ or $\mathcal{L}^\perp$, let $m : Y \to B$ be an monomorphism in $\mathcal{C}$, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If $m \circ f$ is in $\mathcal{R}$, then so is f.*

(ii′) *Suppose $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{R}$, and let $m : Y \to B$ be in $\mathcal{L}$. Then, a morphism $f : X \to Y$ is in $\mathcal{L}$ if and only $g \circ e$ is in $\mathcal{L}$.*

Proof. (i). The epimorphism $e : A \to Z$ induces a bijection between solutions of lifting problems in $\mathcal{C}$ of the form

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and lifting problems of the form

$$\begin{tikzcd} A \arrow[r, "z \circ e"] \arrow[d, "g \circ e"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

so $g \boxslash f$ (resp. $g \perp f$) if and only if $g \circ e \boxslash f$ (resp. $g \circ e \perp f$).

(ii). By proposition A.3.12, we know $g \circ e$ is in $\mathcal{L}$ if both g and e are in $\mathcal{L}$; the converse remains to be shown. Let $r \circ l$ be an $(\mathcal{L}, \mathcal{R})$-factorisation of g. If $g \circ e$ is in $\mathcal{L}$, then there exists a unique s making the diagram below commute,

$$\begin{tikzcd} A \arrow[r, "l \circ e"] \arrow[d, "g \circ e"'] & M \arrow[d, "r"] \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "s"] & W \end{tikzcd}$$

so $r \circ s = \mathrm{id}_W$, but then we also have

$$r \circ (s \circ r) = r$$
$$(s \circ r) \circ (l \circ e) = s \circ (g \circ e) = l \circ e$$

and $l \circ e \perp r$, so we must have $s \circ r = \mathrm{id}_M$. Hence, g is also in $\mathcal{L}$. ■

Proposition A.3.14. *Let $\mathcal{C}$ be a category and let $(\mathcal{L}, \mathcal{R})$ be a pair of subclasses of* mor $\mathcal{C}$ *such that $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$. If every morphism in $\mathcal{C}$ admits an $(\mathcal{L}, \mathcal{R})$-factorisation, then the following are equivalent:*

(i) *$(\mathcal{L}, \mathcal{R})$ is a weak factorisation system.*

(ii) *$\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts in $\mathcal{C}$.*

Proof. (i) ⇒ (ii). This is a special case of proposition A.3.12.

(ii) ⇒ (i). Suppose $f : X \to Y$ is in $\mathcal{L}^{\boxslash}$. Let $p \circ i$ be an $(\mathcal{L}, \mathcal{R})$-factorisation of f. Then, there must exist a morphism r such that $r \circ i = \mathrm{id}_X$ and $f \circ r = p$, as in the diagram below:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ Z \arrow[r, "p"'] \arrow[ur, dashed, "r"] & Y \end{tikzcd}$$

Hence, we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & Z \arrow[r, "r"'] \arrow[d, "p"] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"] & Y \arrow[r, "\mathrm{id}"] & Y \end{tikzcd}$$

Since $\mathcal{R}$ is closed under retracts, we deduce that f is in $\mathcal{R}$. Thus, $\mathcal{L}^{\boxslash} \subseteq \mathcal{R}$. The dual argument proves that ${}^{\boxslash}\mathcal{R} \subseteq \mathcal{L}$, so $(\mathcal{L}, \mathcal{R})$ is indeed a weak factorisation system. ■

Corollary A.3.15. *Every orthogonal factorisation system is also a weak factorisation system.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorisation system on a category $\mathcal{C}$. Proposition A.3.3 implies $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$, and proposition A.3.12 says $\mathcal{L}$ and $\mathcal{R}$ are both closed under retracts, so we may use the earlier proposition to deduce that $(\mathcal{L}, \mathcal{R})$ is a weak factorisation system. ■

Definition A.3.16. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ is **cofibrantly generated** by a subensemble $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{R} = \mathcal{I}^{\boxslash}$. Dually, $(\mathcal{L}, \mathcal{R})$ is **fibrantly generated** by a subensemble $\mathcal{F} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{L} = {}^{\boxslash}\mathcal{F}$.

Remark A.3.17. Of course, $(\mathcal{L}, \mathcal{R})$ is always cofibrantly generated by $\mathcal{L}$. The condition is most useful when $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by a (small) subset of $\mathcal{L}$, but it is convenient to have the more general definition available.

Definition A.3.18. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on a category $\mathcal{C}$. An **extension** of $(\mathcal{L}, \mathcal{R})$ along a functor $i : \mathcal{C} \to \mathcal{C}^+$ is a weak factorisation system $(\mathcal{L}^+, \mathcal{R}^+)$ on $\mathcal{C}^+$ with the following properties:

- A morphism $f : X \to Y$ in $\mathcal{C}$ is in $\mathcal{R}$ if and only if $if : iX \to iY$ is in $\mathcal{R}^+$.
- A morphism $g : Z \to W$ in $\mathcal{C}$ is in $\mathcal{L}$ if and only if $ig : iZ \to iW$ is in $\mathcal{L}^+$.

Proposition A.3.19. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on $\mathcal{C}$, and let $(\mathcal{L}^+, \mathcal{R}^+)$ be a weak factorisation system on $\mathcal{C}^+$.*

(i) *If $\mathcal{L} \subseteq \mathcal{L}^+$, then $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both cofibrantly generated by the same ensemble $\mathcal{I}$, then $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

Dually:

(i′) *If $\mathcal{R} \subseteq \mathcal{R}^+$, then $\mathcal{L} \supseteq \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii′) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both fibrantly generated by the same ensemble $\mathcal{F}$, then $\mathcal{L} = \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

Moreover:

(iii) *If $\mathcal{L} \subseteq \mathcal{L}^+$ and $\mathcal{R} \subseteq \mathcal{R}^+$, then $(\mathcal{L}^+, \mathcal{R}^+)$ is an extension of $(\mathcal{L}, \mathcal{R})$.*

Proof. Since $\mathcal{C}$ is a full subcategory of $\mathcal{C}^+$, if $g : Z \to W$ and $f : X \to Y$ are morphisms in $\mathcal{C}$, then any lifting problem of the following form in $\mathcal{C}^+$ is already in $\mathcal{C}$,

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r] & Y \end{tikzcd}$$

and moreover any solution to the above lifting problem in $\mathcal{C}^+$ is also a solution in $\mathcal{C}$. Thus, $g \boxslash f$ in $\mathcal{C}$ if and only if $g \boxslash f$ in $\mathcal{C}^+$.

(i). Suppose f is in $\mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$. Then f has the right lifting property in $\mathcal{C}^+$ with respect to every morphism in $\mathcal{L}^+$, and in particular, f has the right lifting property in $\mathcal{C}$ with respect to every morphism in $\mathcal{L}$; hence f is in $\mathcal{R}$, and therefore $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(ii). A morphism is in $\mathcal{R}$ (resp. $\mathcal{R}^+$) if and only if it has the right lifting property in $\mathcal{C}$ (resp. $\mathcal{C}^+$) with respect to every morphism in $\mathcal{I}$, so by our initial observation, we must have $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(iii). Immediately follows from claims (i) and (i′). ■

Proposition A.3.20. *Let $(\mathcal{L}, \mathcal{R})$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}$, and let $(\mathcal{L}', \mathcal{R}')$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}'$. Given an adjunction*

$$F \dashv U : \mathcal{C}' \to \mathcal{C}$$

the following are equivalent:

(i) *F sends morphisms in $\mathcal{L}$ to morphisms in $\mathcal{L}'$.*

(ii) *U sends morphisms in $\mathcal{R}'$ to morphisms in $\mathcal{R}$.*

Proof. The adjunction induces a bijection between solutions to the two lifting problems shown below:

$$\begin{tikzcd} FZ \arrow[r] \arrow[d, "Fg"'] & X \arrow[d, "f"] \\ FW \arrow[r] \arrow[ur, dashed, "?"] & Y \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & UX \arrow[d, "Uf"] \\ W \arrow[r] \arrow[ur, dashed, "?"] & UY \end{tikzcd}$$

Thus, $Fg \boxslash f$ (resp. $Fg \perp f$) if and only if $g \boxslash Uf$ (resp. $g \perp Uf$). ■

¶ A.3.21. Let $\mathbb{2}$ be the category $\{0 \to 1\}$, and let $\mathbb{3}$ be $\{0 \to 1 \to 2\}$. Thus, given a category $\mathcal{C}$, the functor category $[\mathbb{2}, \mathcal{C}]$ is the category of arrows and commutative squares in $\mathcal{C}$. There are three embeddings $d^0, d^1, d^2 : \mathbb{2} \to \mathbb{3}$:

$$\begin{array}{lll} d^0(0) = 1 & d^1(0) = 0 & d^2(0) = 0 \\ d^0(1) = 2 & d^1(1) = 2 & d^2(1) = 1 \end{array}$$

These then induce (by precomposition) three functors $d_0, d_1, d_2 : [\mathbb{3}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$.

Definition A.3.22. A **functorial factorisation system** on a category $\mathcal{C}$ is a pair of functors $L, R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ for which there exists a (necessarily unique) functor $F : [\mathbb{2}, \mathcal{C}] \to [\mathbb{3}, \mathcal{C}]$ satisfying the following equations:

$$d_2 F = L \qquad d_1 F = \mathrm{id}_{[\mathbb{2}, \mathcal{C}]} \qquad d_0 F = R$$

A **functorial weak** (resp. **orthogonal**) **factorisation system** on $\mathcal{C}$ is a weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$ together with a functorial factorisation system (L, R) such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$. We will often abuse notation and refer to the functorial factorisation system (L, R) as a functorial weak (resp. orthogonal) factorisation system, omitting mention of the weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$.

Lemma A.3.23. *Let A be an object in a category $\mathcal{C}$ and let $\Sigma_A : \mathcal{C}_{/A} \to \mathcal{C}$ be the projection from the slice category.*

(i) *For each functorial factorisation system (L, R) on $\mathcal{C}$, there exists a unique functorial factorisation system $\left(L_A, R_A\right)$ on $\mathcal{C}_{/A}$ such that*
$$\left[\mathbb{2}, \Sigma_A\right] \circ L_A = L \circ \left[\mathbb{2}, \Sigma_A\right] \qquad \left[\mathbb{2}, \Sigma_A\right] \circ R_A = R \circ \left[\mathbb{2}, \Sigma_A\right]$$
where $\left[\mathbb{2}, \Sigma_A\right] : \left[\mathbb{2}, \mathcal{C}_{/A}\right] \to [\mathbb{2}, \mathcal{C}]$ is the evident induced functor.

(ii) *If (L, R) is part of a functorial weak or orthogonal factorisation system on $\mathcal{C}$, then $\left(L_A, R_A\right)$ is compatible with the induced weak or orthogonal factorisation system on $\mathcal{C}_{/A}$ as well.*

Proof. Obvious. ♦

Proposition A.3.24. *Any orthogonal factorisation system can be extended to a functorial one.*

Proof. For each morphism f in a category $\mathcal{C}$ with an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$, choose a factorisation $f = Rf \circ Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$. Given a commutative square in $\mathcal{C}$, say

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

the lifting property ensures that the dashed arrow in the diagram below exists,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[d, "Lf"] \\ \bullet \arrow[r, dashed] \arrow[d, "Rg"'] & \bullet \arrow[d, "Rf"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

and orthogonality ensures uniqueness and hence functoriality. ■

Corollary A.3.25. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on a category $\mathcal{C}$, then, for any category $\mathcal{J}$, there exists an orthogonal factorisation system on the functor category $[\mathcal{J}, \mathcal{C}]$ where a natural transformation is in the left (resp. right) class if and only if all its components are in $\mathcal{L}$ (resp. $\mathcal{R}$).*

Proof. Obviously, every morphism in $[\mathcal{J}, \mathcal{C}]$ admits such a factorisation, since $(\mathcal{L}, \mathcal{R})$-factorisations in $\mathcal{C}$ are functorial. By considering a commutative diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd}
Z' \arrow[rr] \arrow[dd, "g'"'] \arrow[dr] & & X' \arrow[dd, "f'" near start] \arrow[dr] & \\
& Z \arrow[rr] \arrow[dd, "g" near start] & & X \arrow[dd, "f"] \\
W' \arrow[rr] \arrow[dr] & & Y' \arrow[dr] & \\
& W \arrow[rr] & & Y
\end{tikzcd}$$

where f and f' are in $\mathcal{R}$ while g and g' are in $\mathcal{L}$, using the fact that $(\mathcal{E}, \mathcal{M})$ is an *orthogonal* factorisation system, one may show that lifting problems in $[\mathcal{J}, \mathcal{C}]$ admit unique solutions, and that these solutions are moreover constructed componentwise. Thus, $(\mathcal{L}, \mathcal{R})$ induces an orthogonal factorisation system on $[\mathcal{J}, \mathcal{C}]$. ■

The following characterisation of functorial orthogonal factorisation systems is due to Grandis and Tholen [2006]:

Theorem A.3.26. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *L is the underlying endofunctor of an idempotent comonad on $[\mathbb{2}, \mathcal{C}]$ with counit given by $\varepsilon_k = (\mathrm{id}_{\operatorname{dom} k}, Rk)$, and R is the underlying endofunctor of an idempotent monad on $[\mathbb{2}, \mathcal{C}]$ with unit given by $\eta_h = (h, \mathrm{id}_{\operatorname{codom} h})$.*

(ii) *For all morphisms h in $\mathcal{C}$, RLh and LRh are isomorphisms in $\mathcal{C}$.*

(iii) *For any two morphisms in $\mathcal{C}$, say h and k, we have $Lk \perp Rh$.*

(iv) $(\mathcal{L}, \mathcal{R})$ *is an orthogonal factorisation system on* $\mathcal{C}$ *extending* (L, R)*, where:*

$$\mathcal{L} = \{g \in \operatorname{mor} \mathcal{C} \mid Rg \text{ is an isomorphism in } \mathcal{C}\}$$
$$\mathcal{R} = \{f \in \operatorname{mor} \mathcal{C} \mid Lf \text{ is an isomorphism in } \mathcal{C}\}$$

(v) *There exists an orthogonal factorisation system* $(\mathcal{L}, \mathcal{R})$ *extending* (L, R).

Proof. (i) $\Leftrightarrow$ (ii). This is a standard fact about idempotent (co)monads.

(ii) $\Rightarrow$ (iii). Now, consider the following lifting problem:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

Since (L, R) is a functorial factorisation system, we get a commutative diagram of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[d, "Lf"] \\ W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[d, "Rf"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

but Rg and Lf are isomorphisms, so $(Lf)^{-1} \circ t \circ (Rg)^{-1}$ is the required lift $W \to X$. On the other hand, if $s : W \to X$ is any morphism such that $f \circ s = w$ and $s \circ g = z$, then by taking (L, R)-factorisations of the vertical arrows in the following diagram,

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "g"'] & W \arrow[r, "s"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & X \arrow[d, "f"] \\ W \arrow[r, "\mathrm{id}"'] & W \arrow[r, "s"'] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

we find it must be the case that $Lf \circ s \circ Rg = t$, so we indeed have $g \perp f$.

(iii) ⇒ (iv). In particular, $g \perp Rg$ and $Lf \perp f$, so there must exist morphisms i and r making the diagrams below commute:

$$\begin{tikzcd} Z \arrow[r, "Lg"] \arrow[d, "g"'] & W' \arrow[d, "Rg"] \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "i"] & W \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"'] & X \arrow[d, "f"] \\ X' \arrow[r, "Rf"'] \arrow[ur, dashed, "r"'] & Y \end{tikzcd}$$

We then obtain the following equations,

$$\begin{aligned} (i \circ Rg) \circ Lg &= Lg & \qquad (Lf \circ r) \circ Lf &= Lf \\ Rg \circ (i \circ Rg) &= Rg & \qquad Rf \circ (Lf \circ r) &= Rf \end{aligned}$$

and since $Lg \perp Rg$ and $Lf \perp Rf$, we must have $i \circ Rg = \mathrm{id}_{W'}$ and $Lf \circ r = \mathrm{id}_{X'}$. Thus, $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and the same argument now shows that ${}^{\perp}\mathcal{R} \subseteq \mathcal{L}$ and $\mathcal{L}^{\perp} \subseteq \mathcal{R}$.

It remains to be shown that $\mathcal{L} \subseteq {}^{\perp}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\perp}$. First, suppose $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and consider the following lifting problem:

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

With r and i as in the previous paragraph, we obtain a commutative diagram of the form below,

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"] & X \arrow[dd, "f"] \\ & W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[d, "Rf"] \arrow[ur, "r"'] & \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, "i"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

where the arrow t is obtained by the functoriality of (L, R)-factorisations. Thus, $r \circ t \circ i$ is the required lift $W \to X$, and it is unique, since Rg and Lf are isomorphisms. (Recall the proof of (ii) ⇒ (iii).) We conclude that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.

(iv) ⇒ (v). Immediate.

(v) ⇒ (iii). If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{C}$ such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$, then we must have $Lk \perp Rh$ for all h and k in mor $\mathcal{C}$, as required.

(iv) ⇒ (ii). Immediate. ∎

REMARK A.3.27. It is clear that a functorial factorisation system is associated with *at most one* orthogonal factorisation system: indeed, if $(\mathcal{L}', \mathcal{R}')$ is any orthogonal factorisation system extending a functorial factorisation system (L, R), and $(\mathcal{L}, \mathcal{R})$ is the induced orthogonal factorisation system as in the theorem, then each morphism in $\mathcal{L}$ (resp. $\mathcal{R}$) is a retract of some morphism in in $\mathcal{L}'$ (resp. $\mathcal{R}'$); but by proposition A.3.12, this implies $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{R} \subseteq \mathcal{R}'$, and applying proposition A.3.3, we also get $\mathcal{L} \supseteq \mathcal{L}'$ and $\mathcal{R} \supseteq \mathcal{R}'$.

Corollary A.3.28. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on a category $\mathcal{C}$, then:*

(i) *$\mathcal{L}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is replete and coreflective.*

(ii) *$\mathcal{L}$ is closed under all colimits in $[\mathbb{2}, \mathcal{C}]$.*

(iii) *If a diagram in $\mathcal{L}$ has a limit in $[\mathbb{2}, \mathcal{C}]$, then it also has a limit in $\mathcal{L}$.*

Dually:

(i′) *$\mathcal{R}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is replete and reflective.*

(ii′) *$\mathcal{R}$ is closed under all limits in $[\mathbb{2}, \mathcal{C}]$.*

(iii′) *If a diagram in $\mathcal{R}$ has a colimit in $[\mathbb{2}, \mathcal{C}]$, then it also has a colimit in $\mathcal{R}$.*

Proof. Using proposition A.3.24 and theorem A.3.26, the above claims amount to standard facts about the Eilenberg–Moore category for idempotent (co)monads. ∎

There is a similar characterisation of functorial weak factorisation systems:

Theorem A.3.29. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *For any two morphisms in $\mathcal{C}$, say h and k, $Lk \boxslash Rh$.*

(ii) $(\mathcal{L}, \mathcal{R})$ *is an weak factorisation system on C extending (L, R), where:*

$$\mathcal{L} = \left\{ g \in \operatorname{mor} C \,\middle|\, \exists i \in \operatorname{mor} C .\, i \circ g = Lg \wedge Rg \circ i = \mathrm{id}_{\operatorname{codom} g} \right\}$$
$$\mathcal{R} = \left\{ f \in \operatorname{mor} C \,\middle|\, \exists r \in \operatorname{mor} C .\, f \circ r = Rf \wedge r \circ Lf = \mathrm{id}_{\operatorname{dom} f} \right\}$$

(iii) *There exists a weak factorisation system $(\mathcal{L}, \mathcal{R})$ extending (L, R).*

Proof. The proof is essentially the same as that of theorem A.3.26. ∎

Remark A.3.30. As with orthogonal factorisation systems, there is *at most one* weak factorisation system extending any functorial factorisation system.

Proposition A.3.31. *Let (L, R) be a functorial factorisation system on C and let $\lambda : \mathrm{id}_{[2,C]} \Rightarrow R$ and $\rho : L \Rightarrow \mathrm{id}_{[2,C]}$ be the natural transformations whose component at an object f in $[2, C]$ correspond to the following commutative squares in C:*

$$\begin{tikzcd} \bullet \arrow[r, "Lf"] \arrow[d, "f"'] & \bullet \arrow[d, "Rf"] \\ \bullet \arrow[r, equal] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "Lf"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "Rf"'] & \bullet \end{tikzcd}$$

Suppose (L, R) extends to a functorial weak factorisation system. Then the following are equivalent for a morphism $g : Z \to W$ in C:

(i) *The morphism g is in the left class of the induced weak factorisation system.*

(ii) *There exists a morphism i in C such that the diagram below commutes:*

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, "g"'] & Z \arrow[r, equal] \arrow[d, "Lg"'] & Z \arrow[d, "g"] \\ W \arrow[r, "i"] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "Rg"] & W \end{tikzcd}$$

(iii) *The object g in $[2, C]$ admits a coalgebra structure for the copointed endofunctor (L, ρ).*

Dually, the following are equivalent for a morphism $f : X \to Y$ in $\mathcal{C}$:

(i′) *The morphism f is in the right class of the induced weak factorisation system.*

(ii′) *There exists a morphism r in $\mathcal{C}$ such that the diagram below commutes:*

$$\begin{tikzcd} X \arrow[r, "Lf"'] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] & \bullet \arrow[r, "r"'] \arrow[d, "Rf"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd}$$

(iii′) *The object f in $[\mathbb{2}, \mathcal{C}]$ admits an algebra structure for the pointed endofunctor (R, λ).*

Proof. (i) ⇒ (ii). Consider the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} Z \arrow[r, "Lg"] \arrow[d, "g"'] & \bullet \arrow[d, "Rg"] \\ W \arrow[r, "\mathrm{id}"'] & W \end{tikzcd}$$

Thus, a morphism i of the required form exists in $\mathcal{C}$ as soon as $g \boxslash Rg$.

(ii) ⇔ (iii). This is simply the definition of (L, ρ)-coalgebra.

(ii) ⇒ (i). By definition, the morphism Lf is in the left class of the induced weak factorisation system; but the given diagram exhibits f as a retract of Lf, so we may apply proposition A.3.12 to deduce that f is also in the left class. ■

The results above motivate the following definition:

Definition A.3.32. A **natural weak factorisation system**[4] on a category $\mathcal{C}$ is a pair $(\mathbf{L}, \mathbf{R})$ satisfying the following conditions:

- $\mathbf{L} = (L, \varepsilon, \delta)$ is a comonad on $[\mathbb{2}, \mathcal{C}]$, where $\varepsilon_k = \big(\mathrm{id}_{\operatorname{dom} k}, Rk\big)$.
- $\mathbf{R} = (R, \eta, \mu)$ is a monad on $[\mathbb{2}, \mathcal{C}]$, where $\eta_h = \big(Lh, \mathrm{id}_{\operatorname{codom} h}\big)$.

[4] — in the sense of Grandis and Tholen [2006], *not* Garner [2009].

- (L, R) constitute a functorial factorisation system on $\mathcal{C}$.

Given natural weak factorisation systems $(\mathsf{L}', \mathsf{R})$ and $(\mathsf{L}, \mathsf{R}')$ on $\mathcal{C}$, a morphism $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$ is a pair $\left(\theta^L, \theta^R\right)$, where $\theta^L : L' \Rightarrow L$ and $\theta^R : R \Rightarrow R'$ are natural transformations such that the equations below hold,

$$\varepsilon \bullet \theta^L = \varepsilon' \qquad \left(\theta^L \circ \theta^L\right) \bullet \delta' = \delta \bullet \theta^L$$
$$\theta^R \bullet \eta = \eta' \qquad \mu' \bullet \left(\theta^R \circ \theta^R\right) = \theta^R \bullet \mu$$

and furthermore we require $d_0 \theta^L = d_1 \theta^R$.

Remark A.3.33. In other words, a morphism of natural weak factorisation systems is a natural transformation of functors $[\mathbb{2}, \mathcal{C}] \to [\mathbb{3}, \mathcal{C}]$ such that the left half is a morphism of comonads and the right half is a morphism of monads. In particular, we must have $d_1 \theta^L = \mathrm{id}$ and $d_0 \theta^R = \mathrm{id}$; so for every object f in $[\mathbb{2}, \mathcal{C}]$, we obtain a commutative diagram in $\mathcal{C}$ of the form below:

$$\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, "L' f"'] & \bullet \arrow[d, "L f"] \\
\bullet \arrow[r, "\theta_f"] \arrow[d, "R f"'] & \bullet \arrow[d, "R' f"] \\
\bullet \arrow[r, equal] & \bullet
\end{tikzcd}$$

Proposition A.3.34. *Any functorial orthogonal factorisation system extends to a natural weak factorisation system in a unique way; conversely, a natural weak factorisation system induces an orthogonal factorisation system if and only if the underlying comonad and monad are both idempotent.*

Proof. This follows from the definition above and theorem A.3.26. ■

Proposition A.3.35. *Let (L, R) be an natural weak factorisation system on a category $\mathcal{C}$.*

(i) *Let $f : X \to Y$ and $g : Z \to W$ be objects in $[\mathbb{2}, \mathcal{C}]$. If $\alpha : Rf \to f$ is a R-algebra structure and $\beta : g \to Lg$ is a L-coalgebra structure, then $\alpha_1 : Y \to Y$ and $\beta_0 : Z \to Z$ are identity morphisms, and we have the following identities:*

$$\alpha_0 \circ Lf = \mathrm{id}_X \qquad Rg \circ \beta_1 = \mathrm{id}_W$$
$$f \circ \alpha_0 = Rf \qquad \beta_1 \circ g = Lg$$

(ii) *If f admits a* L*-coalgebra structure and g admits an* R*-algebra structure, then $f \boxslash g$.*

(iii) *There exists a (unique) weak factorisation system $(\mathcal{L}, \mathcal{R})$ on C such that $Lk \in \mathcal{L}$ and $Rh \in \mathcal{R}$ for all h and k in* mor C.

Proof. (i). The claim follows from the L-coalgebra counitality axiom and the R-algebra unitality axiom:

$$\alpha \circ \eta_f = \mathrm{id}_f \qquad\qquad \varepsilon_g \circ \beta = \mathrm{id}_g$$

(ii). It then follows that the diagram below commutes,

$$\begin{tikzcd}
Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"] & X \arrow[dd, "f"] \\
& W' \arrow[r, dashed, "t"] \arrow[d, "Rg"'] & X' \arrow[ur, "\alpha_0"'] \arrow[d, "Rf"] & \\
W \arrow[r, "\mathrm{id}"'] \arrow[ur, "\beta_1"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y
\end{tikzcd}$$

where the arrow t is obtained by the functoriality of (L, R)-factorisations; clearly, $\alpha_0 \circ t \circ \beta_1$ is the required lift.

(iii). Finally, for any two morphisms in C, say h and k, we simply note that $\delta_k : Lk \to LLk$ is an L-coalgebra structure and $\mu_h : RRh \to Rh$ is an R-algebra structure, so we may apply theorem A.3.29 to obtain the conclusion. ■

Proposition A.3.36. *Let $(\mathsf{L}', \mathsf{R})$ and $(\mathsf{L}, \mathsf{R}')$ be natural weak factorisation systems on a category C. If there exists a morphism $(\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$, then:*

- *Every morphism in the left class of the weak factorisation system induced by $(\mathsf{L}', \mathsf{R})$ is also in the left class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$.*

- *Every morphism in the right class of the weak factorisation system induced by $(\mathsf{L}, \mathsf{R}')$ is also in the right class of the weak factorisation system induced by $(\mathsf{L}', \mathsf{R})$.*

Proof. The two claims are formally dual; we will prove the first version.

Let L (resp. L') be the underlying endofunctor of L (resp. L') and let ε (resp. ε') be the counit of L (resp. L'). Suppose we have a morphism $\theta : (\mathsf{L}', \mathsf{R}) \to (\mathsf{L}, \mathsf{R}')$. By proposition A.3.31, it suffices to show that every morphism that admits a (L', ε')-coalgebra structure also admits a (L, ε)-coalgebra structure. But if i is a (L', ε')-coalgebra structure on g, then $\theta^L_g \circ i$ is a (L, ε)-coalgebra structure on g, because $\varepsilon_g \circ \theta^L_g = \varepsilon'_g$. ■

Remark A.3.37. Let (L, R) be a natural weak factorisation system. Then, for each morphism $f : X \to Y$, we have a commutative diagram of the following form in $\mathcal{C}$,

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "Lf"'] & X \arrow[d, "LLf"] \\
\bullet \arrow[r, "d_0(\delta_f)"] \arrow[d, "LRf"'] \arrow[dr, "\mathrm{id}"] & \bullet \arrow[d, "RLf"] \\
\bullet \arrow[r, "d_1(\mu_f)"'] \arrow[d, "RRf"'] & \bullet \arrow[d, "Rf"] \\
Y \arrow[r, equal] & Y
\end{tikzcd}$$

where the upper square corresponds to $\delta_f : Lf \to LLf$ and the lower square corresponds to $\mu_f : RRf \to Rf$; note that the middle square commutes because $(\eta \circ \mathrm{id}_L) \bullet \delta = \mathrm{id}_L$ and $\mu \bullet (\varepsilon \circ \mathrm{id}_R) = \mathrm{id}_R$. Thus, we obtain a canonical natural transformation $\xi : LR \Rightarrow RL$.

The following definition is due to Garner [2009]:

Definition A.3.38. Let $\mathcal{C}$ be a category. An **algebraic factorisation system** on $\mathcal{C}$ is a pair (L, R) satisfying the following conditions:

- (L, R) is a natural weak factorisation system; in particular, $\mathsf{L} = (L, \varepsilon, \delta)$ is a comonad on $[\mathbb{2}, \mathcal{C}]$ and $\mathsf{R} = (R, \eta, \mu)$ is a monad on $[\mathbb{2}, \mathcal{C}]$.

- The canonical natural transformation $\xi : LR \Rightarrow RL$ is a **distributive law**, i.e.

  $$(\mathrm{id}_{d_0} \circ \delta) \bullet (\mathrm{id}_{d_1} \circ \mu) = (\mathrm{id}_{d_1} \circ \mu \circ \mathrm{id}_L) \bullet (\mathrm{id}_M \circ \xi) \bullet (\mathrm{id}_{d_0} \circ \delta \circ \mathrm{id}_R)$$

  where $M = d_0 L = d_1 R$.

¶ A.3.39. Let $\mathcal{C}$ be a category and let $U : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ be a functor. We define a category $\mathbf{RLP}_{\mathcal{C}}(U)$ over $[\mathbb{2}, \mathcal{C}]$ as follows:

- The objects in $\mathbf{RLP}_{\mathcal{C}}(U)$ are morphisms in $\mathcal{C}$ equipped with a coherent choice of liftings, i.e. a pair (f, Φ) where f is a morphism in $\mathcal{C}$ equipped with a chosen morphism $\Phi(e, h) : d_0(Ue) \to d_1(f)$ in $\mathcal{C}$ for each morphism $h : Ue \to f$ in $[\mathbb{2}, \mathcal{C}]$ such that the following diagram in $\mathcal{C}$ commutes,

$$\begin{tikzcd} \bullet \arrow[r, "d_1(h)"] \arrow[d, "Ue"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(h)"'] \arrow[ur, "\Phi(e,h)"] & \bullet \end{tikzcd}$$

  and furthermore, for each morphism $k : e' \to e$ in $\mathcal{I}$, we require that the following diagram commute:

$$\begin{tikzcd} \bullet \arrow[r, "d_1(Uk)"] \arrow[d, "Ue'"'] & \bullet \arrow[r, "d_1(h)"] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(Uk)"'] \arrow[urr, "\Phi(e', h \circ Uk)"] & \bullet \arrow[r, "d_0(h)"'] \arrow[ur, "\Phi(e,h)"'] & \bullet \end{tikzcd}$$

- The morphisms in $\mathbf{RLP}_{\mathcal{C}}(U)$ are commutative squares in $\mathcal{C}$ that are compatible with the chosen liftings, i.e. a morphism $l : (f', \Phi') \to (f, \Phi)$ is a morphism $l : f' \to f$ in $[\mathbb{2}, \mathcal{C}]$ such that, for all morphisms $h' : Ue \to f'$ in $[\mathbb{2}, \mathcal{C}]$, the following diagram commutes:

$$\begin{tikzcd} \bullet \arrow[r, "d_1(h')"] \arrow[d, "Ue"'] & \bullet \arrow[r, "d_1(l)"] \arrow[d] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "d_0(h')"'] \arrow[ur, "\Phi(e,h')"] \arrow[urr, "\Phi(e, l \circ h')"'] & \bullet \arrow[r, "d_0(l)"'] & \bullet \end{tikzcd}$$

- Composition and identities are inherited from $[\mathbb{2}, \mathcal{C}]$.

- The structure functor $\mathbf{RLP}_{\mathcal{C}}(U) \to [\mathbb{2}, \mathcal{C}]$ is the evident forgetful functor sending (f, Φ) to f.

Note that the construction of $\mathbf{RLP}_{\mathcal{C}}(U)$ is contravariantly functorial in U.

Proposition A.3.40. *Let C be a category, let (L, R) be a natural weak factorisation system on C, let $\mathcal{L}$ be the category of L-coalgebras, and let $\mathcal{R}$ be the category of R-algebras in $[\mathbb{2}, C]$. Then there is a natural functor $T : \mathcal{R} \to \mathbf{RLP}_C(U_{\mathsf{L}})$ making the diagram below commute,*

$$\begin{tikzcd} \mathcal{R} \arrow[rr, dashed, "T"] \arrow[dr, "U^{\mathsf{R}}"'] & & \mathbf{RLP}_C(U_{\mathsf{L}}) \arrow[dl, "U"] \\ & {[\mathbb{2}, C]} & \end{tikzcd}$$

where $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, C]$, $U^{\mathsf{R}} : \mathcal{R} \to [\mathbb{2}, C]$ and $U : \mathbf{RLP}_C(U_{\mathsf{L}}) \to [\mathbb{2}, C]$ are the respective forgetful functors.

Proof. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in C, let $(r, \mathrm{id}) : Rf \to f$ be an R-algebra structure on f, and let $(\mathrm{id}, i) : g \to Lf$ be an L-coalgebra structure on g. Given a commutative square in C of the form below,

$$\begin{tikzcd} Z \arrow[r, "z"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "w"'] & Y \end{tikzcd}$$

we choose the lifting $W \to X$ defined by the following commutative diagram,

$$\begin{tikzcd} Z \arrow[r, "\mathrm{id}"] \arrow[dd, "g"'] & Z \arrow[r, "z"] \arrow[d, "Lg"'] & X \arrow[r, "\mathrm{id}"] \arrow[d, "Lf"'] & X \arrow[dd, "f"] \\ & W' \arrow[r] \arrow[d, "Rg"'] & X' \arrow[ur, "r"'] \arrow[d, "Rf"] & \\ W \arrow[r, "\mathrm{id}"'] \arrow[ur, "i"] & W \arrow[r, "w"'] & Y \arrow[r, "\mathrm{id}"'] & Y \end{tikzcd}$$

where the morphism $W' \to X'$ is the one given by the functorial factorisation. It is not hard to see that this choice of liftings is compatible with the morphisms in $\mathcal{L}$, so we have an object in $\mathcal{L}^{\boxslash}$. Similarly, one may verify that the liftings are compatible with the morphisms in $\mathcal{R}$. Thus, we have the required functor $T : \mathcal{R} \to \mathbf{RLP}_C(U_{\mathsf{L}})$ compatible with the forgetful functors, and it is clearly natural in (L, R). ■

Definition A.3.41. Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. An **algebraically free natural weak factorisation system** on $\mathcal{C}$ cofibrantly generated by U is a natural weak factorisation system (L, R) on $\mathcal{C}$ equipped with a functor $E : \mathcal{I} \to \mathcal{L}$ making the following diagram commute,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, dashed, "E"] \arrow[dr, "U"'] & & \mathcal{L} \arrow[dl, "U_{\mathsf{L}}"] \\ & {[\mathbb{2}, \mathcal{C}]} & \end{tikzcd}$$

where $\mathcal{L}$ is the category of L-algebras in $[\mathbb{2}, \mathcal{C}]$ and $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ is the forgetful functor, such that the composite functor shown below is an isomorphism,

$$\mathcal{R} \xrightarrow{T} \mathbf{RLP}_{\mathcal{C}}\left(U_{\mathsf{L}}\right) \xrightarrow{E^*} \mathbf{RLP}_{\mathcal{C}}(U)$$

where $\mathcal{R}$ is the category of R-algebras and $T : \mathcal{R} \to \mathbf{LLP}\left(U_{\mathsf{L}}\right)$ is the canonical functor given in proposition A.3.40.

Remark A.3.42. If $\mathcal{C}$ admits an algebraically free natural weak factorisation system (L, R) cofibrantly generated by $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$, then the forgetful functor $\mathbf{RLP}_{\mathcal{C}}(U) \to [\mathbb{2}, \mathcal{C}]$ is monadic, and the induced monad is isomorphic to R. Garner's small object argument (0.5.24) gives sufficient conditions for the existence of algebraically free natural weak factorisation systems; note that natural weak factorisation systems so constructed also satisfy the distributive law and are therefore algebraic factorisation systems.

Proposition A.3.43. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subensemble of* $\operatorname{mor} \mathcal{C}$*, and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be the evident embedding. If* (L, R) *is an algebraically free natural weak factorisation system cofibrantly generated by U, then the underlying weak factorisation system of* (L, R) *is cofibrantly generated by $\mathcal{I}$.*

Proof. This follows from the definitions and proposition A.3.35. ■

Definition A.3.44. Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. A **free algebraic factorisation system** on $\mathcal{C}$ cofibrantly generated by U is an algebraic factorisation system (L, R) equipped with a functor $E : \mathcal{I} \to \mathcal{L}$ making the following diagram commute,

$$\begin{tikzcd} \mathcal{I} \arrow[rr, dashed, "E"] \arrow[dr, "U"'] & & \mathcal{L} \arrow[dl, "U_{\mathsf{L}}"] \\ & {[\mathbb{2}, \mathcal{C}]} & \end{tikzcd}$$

where $\mathcal{L}$ is the category of L-algebras in $[\mathbb{2}, \mathcal{C}]$ and $U_{\mathsf{L}} : \mathcal{L} \to [\mathbb{2}, \mathcal{C}]$ is the forgetful functor, such that (L, R) and E have the following universal property:

- For all algebraic factorisation systems $(\mathsf{L}', \mathsf{R}')$ and all functors $E' : \mathcal{I} \to \mathcal{L}'$ where $\mathcal{L}'$ is the category of L'-coalgebras and E' is compatible with the forgetful functors, there exists a unique morphism $\theta : (\mathsf{L}, \mathsf{R}) \to (\mathsf{L}', \mathsf{R}')$ such that $E' = \theta^L_* E$, where $\theta^L_* : \mathcal{L} \to \mathcal{L}'$ is the functor induced by the comonad morphism $\theta^L : \mathsf{L} \to \mathsf{L}'$.

Theorem A.3.45. *Let $\mathcal{C}$ be a category and let $U : \mathcal{I} \to [\mathbb{2}, \mathcal{C}]$ be a functor. If (L, R) is an algebraic factorisation system on $\mathcal{C}$ and also an algebraically free natural weak factorisation system cofibrantly generated by U, then (L, R) is a free algebraic factorisation system cofibrantly generated by U.*

Proof. See Theorem A.1 in [Garner, 2009]. □

Remark A.3.46. The cited proof of the theorem above uses the distributive law for algebraic factorisation systems.

A.4 Relative categories

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.4.1. A **relative category** $\mathcal{C}$ consists of a category $\operatorname{und} \mathcal{C}$ and a subcategory $\operatorname{weq} \mathcal{C}$ such that $\operatorname{ob} \operatorname{und} \mathcal{C} = \operatorname{ob} \operatorname{weq} \mathcal{C}$. We say $\operatorname{und} \mathcal{C}$ is the **underlying category** of $\mathcal{C}$, and that the morphisms in $\operatorname{weq} \mathcal{C}$ are the **weak equivalences** in $\mathcal{C}$. A **relative subcategory** of a relative category $\mathcal{C}$ is a relative category $\mathcal{C}'$ such that $\operatorname{und} \mathcal{C}'$ is a subcategory of $\operatorname{und} \mathcal{C}$, and we further demand that $\operatorname{weq} \mathcal{C}' = \operatorname{weq} \mathcal{C} \cap \operatorname{und} \mathcal{C}'$.

REMARK A.4.2. The subcategory $\operatorname{weq} \mathcal{C}$ is entirely determined by $\operatorname{mor} \operatorname{weq} \mathcal{C}$, so a relative category may equivalently be defined as a category equipped with a distinguished subset of morphisms closed under composition and containing all the identity morphisms.

For brevity, we will write $\operatorname{ob} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, $\operatorname{mor} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, and we may occasionally abuse notation and write $\operatorname{weq} \mathcal{C}$ instead of $\operatorname{mor} \operatorname{weq} \mathcal{C}$.

REMARK A.4.3. Every category $\mathcal{C}$ can be endowed with the structure of a relative category in two ways: we can make it into a **minimal relative category** $\min \mathcal{C}$ by taking $\operatorname{weq} \min \mathcal{C}$ to be the set of identity morphisms in $\mathcal{C}$; or we could make it into a **maximal relative category** $\max \mathcal{C}$ by taking $\operatorname{weq} \max \mathcal{C} = \operatorname{mor} \mathcal{C}$. We may also define the **minimal saturated relative category** $\min^+ \mathcal{C}$ by taking $\operatorname{weq} \min^+ \mathcal{C}$ to be the set of all isomorphisms in $\mathcal{C}$.

Definition A.4.4. Given a relative category $\mathcal{C}$, the **opposite relative category** $\mathcal{C}^{\mathrm{op}}$ is defined by $\operatorname{und} \mathcal{C}^{\mathrm{op}} = (\operatorname{und} \mathcal{C})^{\mathrm{op}}$ and $\operatorname{weq} \mathcal{C}^{\mathrm{op}} = (\operatorname{weq} \mathcal{C})^{\mathrm{op}}$.

Definition A.4.5. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **relative functor** $\mathcal{C} \to \mathcal{D}$ is a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ that sends weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. The **relative functor category** $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is the full subcategory of $[\operatorname{und} \mathcal{C}, \operatorname{und} \mathcal{D}]$ spanned by the relative functors, and the weak equivalences in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ are defined to be the natural transformations that are componentwise weak equivalences in $\mathcal{D}$.

Definition A.4.6. Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$. A **localisation of $\mathcal{C}$ at $\mathcal{W}$** is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ with the following universal property:

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that $F f$ is an isomorphism for all f in $\mathcal{W}$, there exists a unique functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $\overline{F} \gamma = F$.

The functor $\gamma : \mathcal{C} \to \operatorname{Ho} \mathcal{C}$ is called the **localising functor**.

REMARK A.4.7. The universal property in the above definition is strict; as such, $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to unique isomorphism. Nonetheless, $\mathcal{C}[\mathcal{W}^{-1}]$ automatically has a 2-universal property: if $F, G : \mathcal{C} \to \mathcal{D}$ both factor through $\mathcal{C}[\mathcal{W}^{-1}]$, then so do all natural transformations $F \Rightarrow G$.

Proposition A.4.8. *If $\mathcal{C}$ is a* **U**-*small category, then there exists a* **U**-*small category with the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$.*

Proof. Use the general adjoint functor theorem. □

Definition A.4.9. The **homotopy category** of a relative category $\mathcal{C}$ is a localisation of $\operatorname{und}\mathcal{C}$ at $\operatorname{weq}\mathcal{C}$ and is denoted $\operatorname{Ho}\mathcal{C}$.

Definition A.4.10.

- A **semi-saturated relative category** is a relative category in which every isomorphism is a weak equivalence.

- A **saturated relative category** is a relative category $\mathcal{C}$ such that the weak equivalences in $\mathcal{C}$ are precisely the ones that become isomorphisms in $\operatorname{Ho}\mathcal{C}$.

REMARK A.4.11. Obviously, there is no loss of generality in considering semi-saturated relative categories and their homotopy categories instead of localisations $\mathcal{C}[\mathcal{W}^{-1}]$ for arbitrary subsets $\mathcal{W} \subseteq \operatorname{mor}\mathcal{C}$.

REMARK A.4.12. Clearly, every saturated relative category is semi-saturated, and a minimal saturated relative category is indeed saturated in the sense above.

Definition A.4.13. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subset of $\operatorname{mor}\mathcal{C}$. The **2-out-of-3 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, if any two of f, g, or $g \circ f$ are in $\mathcal{W}$, then all of them are.

The **2-out-of-6 property** for $\mathcal{W}$ says:

- Given any three morphisms $f : X \to Y$, $g : Y \to Z$, $h : Y \to Z$ in $\mathcal{C}$, if both $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so too are f, g, h, and $h \circ g \circ f$.

Lemma A.4.14. *Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor}\mathcal{C}$.*

(i) *If $\mathcal{W}$ has the 2-out-of-6 property, then it also has the 2-out-of-3 property.*

(ii) *The set of all isomorphisms in $\mathcal{C}$ has the 2-out-of-6 property.*

(iii) *If $F : \mathcal{C}' \to \mathcal{C}$ is a functor and $\mathcal{W}$ has either the 2-out-of-3 property or the 2-out-of-6 property, then $F^{-1}\mathcal{W}$ has the same property.*

Proof. (i). Consider the three cases $f = \mathrm{id}$, $g = \mathrm{id}$, $h = \mathrm{id}$ in turn.

(ii). If $h \circ g$ and $g \circ f$ are isomorphisms, then g must be split epic and split monic; thus g itself is an isomorphism, hence so too are f and h.

(iii). Obvious. ■

Corollary A.4.15. *If $\mathcal{C}$ is a saturated relative category, then* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-6 property.* ■

Definition A.4.16. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subset of $\operatorname{mor} \mathcal{C}$. The **2-out-of-4 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to X$ in $\mathcal{C}$, if $f \circ g$ and $g \circ f$ are in $\mathcal{W}$, then both f and g are in $\mathcal{W}$.

The **special 2-out-of-4 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to X$ in $\mathcal{C}$, if $f \circ g$ is in $\mathcal{W}$ and $g \circ f = \mathrm{id}_X$, then both f and g are in $\mathcal{W}$.

Lemma A.4.17. *Let $\mathcal{C}$ be a relative category.*

(i) *If* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-4 property, then* $\operatorname{weq} \mathcal{C}$ *has the special 2-out-of-4 property.*

(ii) *If* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-6 property, then* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-4 property.*

(iii) *If* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-3 property and is closed under retracts, then* $\operatorname{weq} \mathcal{C}$ *has the special 2-out-of-4 property.*

Proof. (i) and (ii). Obvious.

(iii). Let $f : X \to Y$ and $g : Y \to X$ be morphisms in $\mathcal{C}$ such that $f \circ g$ is a weak equivalence and $g \circ f = \mathrm{id}_X$. Consider the following diagram:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{id}"] \arrow[d, "g"'] & Y \arrow[r, "\mathrm{id}"] \arrow[d, "f \circ g"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & X \end{tikzcd}$$

Since $g \circ f = \mathrm{id}_X$, the diagram commutes, so we see that $g : Y \to X$ is a retract of $f \circ g : Y \to Y$. We deduce that g is a weak equivalence in $\mathcal{C}$ using the fact that $\operatorname{weq} \mathcal{C}$ is closed under retracts, and then we deduce that f is a weak equivalence using the the 2-out-of-3 property of $\operatorname{weq} \mathcal{C}$. ∎

Proposition A.4.18. *Let* **RelCat** *be the category of* **U***-small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories, and let* **Cat** *be the category of* **U***-small categories and functors.*

(i) **RelCat** *is a cartesian closed category, where the product of $\mathcal{C}$ and $\mathcal{D}$ is the cartesian product $\mathcal{C} \times \mathcal{D}$ with weak equivalences taken componentwise, and the exponential of $\mathcal{E}$ by $\mathcal{D}$ is the relative functor category $[\mathcal{D}, \mathcal{E}]_{\mathrm{h}}$.*

(ii) **RelCat** *is a locally finitely presentable* **U***-category,*[5] *and the two functors* $\operatorname{und}, \operatorname{weq} : \mathbf{RelCat} \to \mathbf{Cat}$ *are $\aleph_0$-accessible*[6] *and jointly conservative.*

(iii) **SsRelCat** *is a locally finitely presentable* **U***-category, and the inclusion* $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$ *is $\aleph_0$-accessible and has a left adjoint.*

(iv) **SsRelCat** *is an exponential ideal in* **RelCat***.*

(v) *The full subcategory spanned by the minimal relative categories is an exponential ideal in* **RelCat***.*

(vi) *The full subcategory spanned by the minimal saturated relative categories is an exponential ideal in* **SsRelCat***.*

Proof. (i). This is straightforward from the definitions.

(ii). Obviously, a relative functor $F : \mathcal{C} \to \mathcal{D}$ such that $\operatorname{und} F : \operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ and $\operatorname{weq} F : \operatorname{weq} \mathcal{C} \to \operatorname{weq} \mathcal{D}$ are both isomorphisms is itself an isomorphism, so $\operatorname{und}, \operatorname{weq} : \mathbf{RelCat} \to \mathbf{Cat}$ are indeed jointly conservative.

It is also not hard to check that limits for all **U**-small diagrams and colimits for **U**-small filtered diagrams in **RelCat** exist and can be computed componentwise in **Cat**, so (by theorem 0.2.37) it is enough to show that **RelCat** is a $\aleph_0$-accessible **U**-category. Clearly, a relative category $\mathcal{C}$ such that $\operatorname{und} \mathcal{C}$ is finitely presentable in **Cat** and $\operatorname{weq} \mathcal{C}$ is a finitely-generated subcategory of $\operatorname{und} \mathcal{C}$ is itself finitely presentable in **RelCat**, so **RelCat** is indeed $\aleph_0$-accessible.

[5] See definition 0.2.33.
[6] See definition 0.2.24.

(Alternatively, one may appeal to the sketchability theorem[7] and the fact that a relative category is manifestly a model for a certain finite-limit sketch.)

(iii). It is clear that **SsRelCat** is closed in **RelCat** under limits for all **U**-small diagrams and colimits for all **U**-small filtered diagrams, and we know that **RelCat** is a locally finitely presentable category, so (by proposition 0.2.28) it is enough to construct a left adjoint for the inclusion $\mathbf{SsRelCat} \hookrightarrow \mathbf{RelCat}$. This may be done using the general adjoint functor theorem.

(iv) – (vi). All straightforward. ■

Proposition A.4.19. *Let* **RelCat** *be the category of* **U***-small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories and relative functors, and let* **Cat** *be the category of* **U***-small categories and functors. We have the following strings of adjoint functors:*

$$\min \dashv \mathrm{und} \dashv \max \dashv \mathrm{weq} : \mathbf{RelCat} \to \mathbf{Cat}$$
$$\mathrm{Ho} \dashv \min^+ \dashv \mathrm{und} \dashv \max \dashv \mathrm{weq} : \mathbf{SsRelCat} \to \mathbf{Cat}$$

The functors $\min$, $\min^+$, *and* $\max$ *are moreover fully faithful, and* Ho *preserves finite products.*

Proof. All but the last of the above claims are obvious; for the preservation of finite products under Ho, we refer to proposition A.2.13. ■

Corollary A.4.20. $\mathrm{Ho} : \mathbf{SsRelCat} \to \mathbf{Cat}$ *is 2-functorial.*

Proof. Apply remark A.2.10. ■

Proposition A.4.21. *Let* $\mathcal{C}$ *be a relative category and let* $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ *be the localising functor.*

(i) *For all categories* $\mathcal{D}$*, the induced functor* $\gamma^* : [\mathrm{Ho}\,\mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$ *is fully faithful and injective on objects.*

(ii) *Any left or right adjoint for* $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$ *is a fully faithful functor.*

[7] See Proposition 1.51 in [LPAC] or Proposition 5.6.4 in [Borceux, 1994b].

Proof. (i). It is an immediate consequence of the universal property of $\operatorname{Ho} \mathcal{C}$ that $\gamma^* : [\operatorname{Ho} \mathcal{C}, \mathcal{D}] \to [\mathcal{C}, \mathcal{D}]$ is injective on objects. It is moreover fully faithful because we have the following natural isomorphism,

$$[\operatorname{Ho} \mathcal{C}, \mathcal{D}] \cong \operatorname{und} \left[\mathcal{C}, \min^+ \mathcal{D}\right]_{\mathrm{h}}$$

and $\operatorname{und} \left[\mathcal{C}, \min^+ \mathcal{D}\right]_{\mathrm{h}}$ is manifestly a full subcategory of $[\mathcal{C}, \mathcal{D}]$.

(ii). Apply proposition A.1.3. ■

Definition A.4.22. A **zigzag type** is a relative category T where $\operatorname{und} T$ is the free category on an inhabited finite planar graph of the form

$$\bullet \;\text{———}\; \bullet \;\text{———}\; \cdots \;\text{———}\; \bullet \;\text{———}\; \bullet$$

where the edges are arrows that point either leftwards or rightwards, and $\operatorname{weq} T$ is generated by the leftward-pointing arrows. A **morphism of zigzag types** is a relative functor that maps the leftmost object to the leftmost object and the rightmost object to the rightmost object. We write $\mathbf{T}$ for the category of zigzag types.[8]

A **zigzag** of type T in a relative category $\mathcal{C}$ is a relative functor $T \to \mathcal{C}$. Given objects X and Y in $\mathcal{C}$, we denote by $\mathcal{C}^T(X, Y)$ the category whose objects are the zigzags starting at X and ending at Y and whose morphisms are commutative diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd}
X \arrow[r, dash] \arrow[d, equal] & \bullet \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] \arrow[d] & Y \arrow[d, equal] \\
X \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots \arrow[r, dash] & \bullet \arrow[r, dash] & Y
\end{tikzcd}$$

where the rows are zigzags of type T and the unmarked vertical arrows are weak equivalences.

Example A.4.23. If $f : X \to Y$ is a weak equivalence in a relative category $\mathcal{C}$, then we have commutative diagrams

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, equal] \\
X \arrow[r, "f"'] & Y & X \arrow[l, "f"]
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[d, equal] & X \arrow[l, "f"'] \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, equal] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

and these correspond to morphisms of zigzags in $\mathcal{C}$.

[8] **Warning.** This is the *opposite* of the category $\mathbf{T}$ defined in [DHKS, § 34].

REMARK A.4.24. It is clear that $C^T(X, Y)$ is a subcategory of the relative functor category $[T, C]_{\mathrm{h}}$. Thus, if C is a $\mathbf{U}$-small relative category, precomposition makes the assignment $T \mapsto C^T(X, Y)$ into a functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$, which we denote by $C^*(X, Y)$. A Grothendieck construction applied to this functor yields the following $\mathbf{U}$-small category $C^{(\mathbf{T})}(X, Y)$:

- Its objects are pairs (T, f), where T is a zigzag type and f is a zigzag of type T in C.
- A morphism $(T', f') \to (T, f)$ is a pair (α, β) where $\alpha : T \to T'$ is a morphism in $\mathbf{T}$ and $\beta : \alpha^* f' \to f$ is a morphism in $C^T(X, Y)$.
- The composite of a pair of morphisms $(\alpha', \beta') : (T'', f'') \to (T', f')$ and $(\alpha, \beta) : (T', f') \to (T, f)$ is given by $(\alpha' \circ \alpha, \beta \circ \alpha^* \beta')$.

There is an evident projection functor $C^{(\mathbf{T})}(X, Y) \to \mathbf{T}^{\mathrm{op}}$, and by construction it is a Grothendieck opfibration with a canonical splitting.

Lemma A.4.25. *Given a commutative diagram of the form below in a relative category C,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "a"'] & Y \arrow[d, "b"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

if a and b are weak equivalences in C, then we obtain the following morphisms of zigzags:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "b"] \arrow[d, "b"] & Y' \arrow[d, equal] \\ X' \arrow[r, "a"'] & X' \arrow[r, "f'"'] & Y' \arrow[r, equal] & Y' \end{tikzcd}$$

$$\begin{tikzcd} X' \arrow[d, equal] & X \arrow[l, "a"'] \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, equal] \arrow[d, "b"] & Y \arrow[d, equal] \\ X' \arrow[r, equal] & X' \arrow[r, "f'"'] & Y' & Y \arrow[l, "b"] \end{tikzcd}$$

In particular, $X \xrightarrow{f} Y \xrightarrow{b} Y'$ *and* $X \xrightarrow{a} X' \xrightarrow{f'} Y'$ *are in the same connected component of* $\mathcal{C}^{(\mathbf{T})}(X, Y')$*; and* $X' \xleftarrow{a} X \xrightarrow{f} Y$ *and* $X' \xrightarrow{f'} Y' \xleftarrow{b} Y$ *are in the same connected component of* $\mathcal{C}^{(\mathbf{T})}(X', Y)$. ■

Theorem A.4.26. *Let* X *and* Y *be objects in a relative category* $\mathcal{C}$.

(i) *For each zigzag type* T*, the map that sends an object in* $\mathcal{C}^{T}(X, Y)$ *to the corresponding composite in* $\operatorname{Ho}\mathcal{C}(X, Y)$ *is a functor when the latter is regarded as a discrete category.*

(ii) *The functors described above constitute a jointly surjective cocone from the diagram* $\mathcal{C}^{*}(X, Y)$ *to* $\operatorname{Ho}\mathcal{C}(X, Y)$.

(iii) *The induced functor* $\mathcal{C}^{(\mathbf{T})}(X, Y) \to \operatorname{Ho}\mathcal{C}(X, Y)$ *is surjective, and moreover two objects in* $\mathcal{C}^{(\mathbf{T})}(X, Y)$ *become equal in* $\operatorname{Ho}\mathcal{C}$ *if and only if they are in the same connected component.*

Proof. All obvious except for the last part of claim (iii), for which we refer to paragraphs 33.8 and 33.10 in [DHKS]. ■

A.5 Kan extensions

Prerequisites. §§ 0.1, A.1.

In this section we use the explicit universe convention.

Definition A.5.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors. A **left Kan extension** (resp. **right Kan extension**) of G along F is an initial (resp. terminal) object of the category $(G \downarrow F^{*})$ (resp. $(F^{*} \downarrow G)$) described below:

- The objects are pairs (H, α) where H is a functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).

Remark A.5.2. Clearly, Kan extensions are unique up to unique isomorphism if they exist. We write $(\operatorname{Lan}_F G, \eta)$ for the left Kan extension of G along F and say η is the **unit** of $\operatorname{Lan}_F G$; dually, we write $(\operatorname{Ran}_F G, \varepsilon)$ for the right Kan extension of G along F and say ε is the **counit** of $\operatorname{Ran}_F G$.

Lemma A.5.3. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. Let* $\mathcal{B}$ *be a* $\mathbf{U}$*-small category and let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category. Given functors* $F : \mathcal{B} \to \mathcal{C}$ *and* $G : \mathcal{B} \to \mathbf{Set}$*, if* $H : \mathcal{C} \to \mathbf{Set}$ *is the functor defined by the formula below,*

$$H(C) = [\mathcal{B}, \mathbf{Set}](\mathcal{C}(C, F-), G-)$$

and $\varepsilon_B : H(FB) \to G(B)$ *is defined by evaluation at* id_{FB}*, then* (H, ε) *is the right Kan extension of* G *along* F*.*

Proof. Note that $H(C)$ so defined is indeed a $\mathbf{U}$-set, because $\mathcal{B}$ is $\mathbf{U}$-small and $\mathcal{C}$ is locally $\mathbf{U}$-small. The claim amounts to saying that (H, ε) is a terminal object in the comma category $(F^* \downarrow G)$, so that is what we must show.

Let $\varphi : (X, \alpha) \to (H, \varepsilon)$ be a morphism in $(F^* \downarrow G)$, i.e. a natural transformation $\varphi : X \Rightarrow H$ such that $\varepsilon \bullet \varphi F = \alpha$. Let C be an object in $\mathcal{C}$, let x be an element of $X(C)$, and consider the element $\varphi_C(x)$ of $H(C)$. By definition, this is a natural transformation $\mathcal{C}(C, F) \Rightarrow G$, so we may consider its component at an object B in $\mathcal{B}$, which will be a map $\mathcal{C}(C, FB) \to G(B)$. Let $f : C \to FB$ be an arrow in $\mathcal{C}$. By hypothesis,

$$\alpha_C(x) = \varepsilon_C\big(\varphi_C(x)_B \circ \mathcal{C}(f, FB)\big) = \varphi_C(x)_B(f)$$

thus the action of φ is entirely determined by α. Conversely, given any object (X, α) in the comma category $(F^* \downarrow G)$, it is easily verified that the above equation defines a morphism $\varphi : (X, \alpha) \to (H, \varepsilon)$, so (H, ε) is indeed a terminal object in $(F^* \downarrow G)$. ■

Corollary A.5.4. *For any two functors* $F : \mathcal{B} \to \mathcal{C}$ *and* $G : \mathcal{B} \to \mathbf{Set}$*, if* $\mathcal{B}$ *is* $\mathbf{U}$*-small and* $\mathcal{C}$ *is locally* $\mathbf{U}$*-small, then the following are equivalent:*

(i) $\big(\mathrm{Ran}_F\, G, \varepsilon\big)$ *is a right Kan extension of* G *along* F*.*

(ii) *The maps* $\big(\mathrm{Ran}_F\, G\big)(C) \to [\mathcal{B}, \mathbf{Set}](\mathcal{C}(C, F), G)$ *defined by* $x \mapsto \varepsilon \bullet \theta_x F$*, where* $\theta_x : \mathcal{C}(C, -) \Rightarrow G$ *is the unique natural transformation such that* $\big(\theta_x\big)_C\big(\mathrm{id}_C\big) = x$*, are bijections that are natural in* C*.* ■

Definition A.5.5. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors.

- A functor $L : \mathcal{E} \to \mathcal{F}$ **preserves** left Kan extensions of G along F if, given any left Kan extension (H, α) of G along F, $(LH, L\alpha)$ is a left Kan extension of LG along F.

- A functor $R : \mathcal{E} \to \mathcal{F}$ **preserves** right Kan extensions of G along F if, given any right Kan extension (H, α) of G along F, $(RH, R\alpha)$ is a right Kan extension of LG along F.

If a Kan extension is preserved by *all* functors, then it is said to be **absolute**.

Definition A.5.6. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-small sets, let $\mathcal{E}$ be a locally $\mathbf{U}$-small category, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors.

- A **pointwise left Kan extension** of G along F is one that is preserved by all functors of the form $\mathcal{E}(-, E) : \mathcal{E} \to \mathbf{Set}^{\mathrm{op}}$.

- A **pointwise right Kan extension** of G along F is one that is preserved by all functors of the form $\mathcal{E}(E, -) : \mathcal{E} \to \mathbf{Set}$.

Definition A.5.7. Let $F : \mathcal{B} \to \mathcal{C}$ be a functor and let C be an object in $\mathcal{C}$.

- The **tautological cocone** to C induced by F is the cocone $\theta : F P_C \Rightarrow \Delta C$, where $P_C : (F \downarrow C) \to \mathcal{B}$ is the projection functor sending an object (B, f) in the comma category $(F \downarrow C)$ to the object B in $\mathcal{B}$, and $\theta_{(B,f)} = f$.

- The **tautological cone** from C induced by F is the cone $\theta : \Delta C \Rightarrow F P^C$, where $P^C : (C \downarrow F) \to \mathcal{C}$ is the projection functor sending an object (B, f) in the comma category $(C \downarrow F)$ to the object B in $\mathcal{B}$, and $\theta_{(B,f)} = f$.

Lemma A.5.8. *Let $\mathcal{A}$ be any category, let $\mathcal{B}$ be a $\mathbf{U}$-small category, let $\mathcal{C}$ be locally $\mathbf{U}$-small category, and let $U : \mathcal{A} \to \mathcal{C}$, $V : \mathcal{B} \to \mathcal{C}$, and $Y : \mathcal{B} \to \mathbf{Set}$ be functors. Consider the following diagram of functors and natural transformations,*

$$\begin{tikzcd} (U \downarrow V) \arrow[r, "Q"] \arrow[d, "P"'] & \mathcal{B} \arrow[d, "V"] \\ \mathcal{A} \arrow[r, "U"'] \arrow[ur, phantom, "\overset{\theta}{\nearrow}"] & \mathcal{C} \end{tikzcd}$$

where $(U \downarrow V)$ is the comma category, $P : (U \downarrow V) \to \mathcal{A}$ and $Q : (U \downarrow V) \to \mathcal{B}$ are the two projections, and $\theta : UP \Rightarrow VQ$ is the tautological natural transformation defined by $\theta_{(A,B,f)} = f$. If (Z, ε) is a right Kan extension of Y along V, then $(ZU, \varepsilon Q \bullet Z\theta)$ is a right Kan extension of YQ along P.

Proof. By lemma A.5.3, we may take $Z : \mathcal{C} \to \mathbf{Set}$ to be the functor defined by the formula below,

$$Z(C) = [\mathbb{B}, \mathbf{Set}](\mathcal{C}(C, F-), Y-)$$

with $\varepsilon : V^*(Z) \Rightarrow Y$ being the natural transformation obtained by evaluating elements of $Z(VB)$ at id_{VB}.

Let $\varphi : (X, \alpha) \to (ZU, \varepsilon Q \bullet Z\theta)$ be a morphism in $(P^* \downarrow YQ)$, i.e. a natural transformation $\varphi : X \Rightarrow ZU$ such that $\varepsilon Q \bullet Z\theta \bullet \varphi P = \alpha$. Let A be an object in $\mathcal{A}$, let x be an element of $X(A)$, and consider the element $\varphi_A(x)$ of $Z(UA)$. By definition, this is a natural transformation $\mathrm{N}^V(C) \Rightarrow Y$, so we may consider its component at an object B in $\mathcal{B}$, which will be a map $\mathcal{C}(UA, VB) \to Y(B)$. Let $f : UA \to VB$ be an arrow in $\mathcal{C}$; then (A, B, f) is an object in the comma category $(U \downarrow V)$, and $\theta_{(A,B,f)} = f$ by definition. By hypothesis,

$$\alpha_{(A,B,f)}(x) = \varepsilon_B\big(\varphi_A(x)_B \circ \mathcal{C}(f, VB)\big) = \varphi_A(x)_B(f)$$

thus the action of φ is entirely determined by α. Conversely, given any object (X, α) in the comma category $(P^* \downarrow YQ)$, it is easily verified that the above equation defines a morphism $\varphi : (X, \alpha) \to (ZU, \varepsilon Q \bullet Z\theta)$, so $(ZU, \varepsilon Q \bullet Z\theta)$ is indeed a terminal object in $(P^* \downarrow YQ)$. ■

Corollary A.5.9. *Let $\mathcal{B}$ be a* **U***-small category and let $\mathcal{C}$ be a locally* **U***-small category. Given functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{B} \to$* **Set***, if (H, ε) is a right Kan extension of G along F, then, for each object C in $\mathcal{C}$, the image under H of the tautological cone from C induced by F is a limiting cone in* **Set**.

Proof. In the lemma, take $\mathcal{A}$ to be the terminal category $\mathbb{1}$, take $U : \mathbb{1} \to \mathcal{C}$ to be the functor sending the unique object in $\mathbb{1}$ to C, and take $V = F$; then $(HU, \varepsilon Q \bullet H\theta)$ is a right Kan extension of $GQ : (C \downarrow F) \to \mathbf{Set}$ along the unique functor $P : (C \downarrow F) \to \mathbb{1}$, but it is clear that a right Kan extension of GQ along P amounts to a limit for the diagram GQ in **Set**. ■

It is convenient at this juncture to introduce a concept borrowed from enriched category theory. The notation below follows [Kelly, 2005, § 3.1].

Definition A.5.10. Let **U** be a pre-universe, let **Set** be the category of **U**-sets, and let $\mathcal{C}$ be a locally **U**-small category. Given functors $W : \mathcal{J} \to \mathbf{Set}$ and

$A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted limit** of A is an object $\{W, A\}^{\mathcal{J}}$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) \cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A))$$

that are natural in C. We may also write $\varprojlim^{Wj}_{j:\mathcal{J}} Aj$ instead of $\{W, A\}^{\mathcal{J}}$, if we wish to use an explicit variable j.

Dually, given functors $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ and $A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted colimit** of A is an object $W \star_{\mathcal{J}} A$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(W \star_{\mathcal{J}} A, C\big) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}](W, \mathcal{C}(A, C))$$

that are natural in C. We may also write $\varinjlim^{Wj}_{j:\mathcal{J}} Aj$ instead of $W \star_{\mathcal{J}} A$, if we wish to use an explicit variable j.

Remark A.5.11. Clearly, weighted limits and colimits are unique up to unique isomorphism if they exist.

It is also not hard to spell out the above definition in elementary terms; for example, one notes that to give a natural transformation $W \Rightarrow \mathcal{C}(C, A)$, one must give a morphism $\lambda_{j,x} : C \to Aj$ for each object j in $\mathcal{J}$ *and* each element x of Wj, and these are required to make various diagrams commute. This is a **W-weighted cone** from C to A, and $\{W, A\}^{\mathcal{J}}$ is an object equipped with a universal W-weighted cone to A. Similarly, one may define the notion of a **W-weighted cocone** from A to C, and then $W \star_{\mathcal{J}} A$ is an object equipped with a universal W-weighted cocone from A. In particular, if $Wj = 1$ for all j, then W-weighted limits and colimits reduce to ordinary limits and colimits.

The above discussion also shows that the concept of a weighted limit or colimit (within a fixed category!) does not depend on $\mathbf{U}$ in any essential way.

Lemma A.5.12. *Let $\mathcal{J}$ be a $\mathbf{U}$-small category. Given functors $F, G : \mathcal{J} \to \mathbf{Set}$, the F-weighted limit of G exists in $\mathbf{Set}$, and we have bijections*

$$\{F, G\}^{\mathcal{J}} \cong [\mathcal{J}, \mathbf{Set}](F, G)$$

that are natural in F and G.

Proof. One simply has to check that this works. ⧫

Proposition A.5.13. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $F : \mathcal{C} \to \mathcal{D}$ be any functor where $\mathcal{C}$ and $\mathcal{D}$ are locally $\mathbf{U}$-small categories.*

(i) *For each weight* $W : \mathcal{J} \to \mathbf{Set}$ *and each diagram* $A : \mathcal{J} \to \mathcal{C}$*, if the weighted limits* $\{W, A\}^{\mathcal{J}}$ *and* $\{W, FA\}^{\mathcal{J}}$ *both exist, then there is a canonical comparison morphism*

$$F\{W, A\}^{\mathcal{J}} \to \{W, FA\}^{\mathcal{J}}$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FC, FA))$$

induced by the functor F.

(ii) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ *preserves* all *weighted limits.*

(iii) *The functors* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ jointly *reflect weighted limits.*

(iv) *If* F *has a left adjoint, then* F *preserves weighted limits.*

Dually:

(i′) *For each weight* $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ *and each diagram* $A : \mathcal{J} \to \mathcal{C}$*, if the weighted colimits* $W \star_{\mathcal{J}} A$ *and* $W \star_{\mathcal{J}} FA$ *both exist, then there is a canonical comparison morphism*

$$W \star_{\mathcal{J}} FA \to F\big(W \star_{\mathcal{J}} A\big)$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(A, C)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FA, FC))$$

induced by the functor F.

(ii′) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *sends any weighted colimit in* $\mathcal{C}$ *to the corresponding weighted limit in* $\mathbf{Set}$.

(iii′) *The functors* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ jointly *reflect weighted colimits.*

(iv′) *If* F *has a right adjoint, then* F *preserves weighted colimits.*

Proof. All straightforward. ⧫

Definition A.5.14. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{D}$ be a locally $\mathbf{U}$-small category. Given a functor $F : \mathcal{C} \to \mathcal{D}$, the **$F$-nerve functor** $\mathrm{N}^F : \mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is defined by

$$\mathrm{N}^F(D)(C) = \mathcal{D}(FC, D)$$

i.e. $\mathrm{N}^F = F^* \hbar_\bullet$, where $\hbar_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is the usual Yoneda embedding.

Theorem A.5.15. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be locally $\mathbf{U}$-small categories. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the following are equivalent:*

(i) *(H, α) is a pointwise right Kan extension of G along F.*

(ii) *For each object d in $\mathcal{D}$, the weighted limit $\left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$ exists in $\mathcal{E}$, and there are isomorphisms*

$$Hd \cong \left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$$

natural in d, with $\alpha_c : HFc \to Gc$ corresponding to the element id_{Fc} of $\mathrm{N}^{F^{\mathrm{op}}}(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii) *(Assuming $\mathcal{C}$ is $\mathbf{U}$-small.) For each object d in $\mathcal{D}$, if $P^d : (d \downarrow F) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(d \downarrow F)$ to c, and $\varphi : \Delta d \Rightarrow FP^d$ is the tautological cone in $\mathcal{D}$, then the cone $\alpha P^d \bullet H\varphi : \Delta Hd \Rightarrow GP^d$ is limiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(d' \downarrow F) \to (d \downarrow F)$ sending (c', f') to $(c', f' \circ g)$. In particular, $\alpha_c : HFc \to Gc$ must be (equal to) the component of the limiting cone $\Delta Fc \Rightarrow GP^d$ at the object $\left(c, \mathrm{id}_{Fc}\right)$ of $(Fc \downarrow F)$.*

In particular, if $\mathcal{C}$ is a $\mathbf{U}$-small category and $\mathcal{E}$ is $\mathbf{U}$-complete, then the right Kan extension of G along F exists and is pointwise.

Dually, the following are equivalent:

(i′) *(H, α) is a pointwise left Kan extension of G along F.*

(ii′) *For each object d in $\mathcal{D}$, the weighted colimit $\mathrm{N}^F(d) \star_{\mathcal{C}} G$ exists in $\mathcal{E}$, and there are isomorphisms*

$$Hd \cong \mathrm{N}^F(d) \star_{\mathcal{C}} G$$

natural in d, with $\alpha_c : Gc \to HFc$ corresponding to the element id_{Fc} of $\mathrm{N}^F(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii′) *(Assuming $\mathcal{C}$ is* $\mathbf{U}$*-small.) For each object d in $\mathcal{D}$, if $P_d : (F \downarrow d) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(F \downarrow d)$ to c, and $\varphi : F P_d \Rightarrow \Delta d$ is the tautological cocone in $\mathcal{D}$, then the cocone $H\varphi \bullet \alpha P_d : G P_d \Rightarrow \Delta H d$ is colimiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(F \downarrow d) \to (F \downarrow d')$ sending (c, f) to $(c, g \circ f)$. In particular, $\alpha_c : Gc \to HFc$ must be (equal to) the component of the colimiting cocone $G P_d \Rightarrow \Delta F c$ at the object $\left(c, \mathrm{id}_{Fc}\right)$ of $(F \downarrow Fc)$.*

In particular, if $\mathcal{C}$ is a $\mathbf{U}$*-small category and $\mathcal{E}$ is* $\mathbf{U}$*-cocomplete, then the left Kan extension of G along F exists and is pointwise.*

Proof. (i) ⇔ (ii). This is just a matter of unwinding the definitions.

(i) ⇔ (iii). Corollary A.5.9 implies that the construction in (iii) does indeed define a right Kan extension in the special case $\mathcal{E} = \mathbf{Set}$, so we deduce that statements (i) and (iii) are equivalent by applying the Yoneda lemma; see also [CWM, Ch. X, §§ 3 and 5]. □

Remark A.5.16. It is possible to extract an elementary characterisation of pointwise Kan extensions from the results above, thereby showing that the property of being pointwise does not depend on the choice of universe $\mathbf{U}$.

Corollary A.5.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $\mathcal{C}$ is* $\mathbf{U}$*-small and $\mathcal{D}$ is locally* $\mathbf{U}$*-small, then the functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ has both a left adjoint* Lan_F *and a right adjoint* Ran_F. ■

Corollary A.5.18. *If (H, α) is a pointwise right Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$, and $R : \mathcal{E} \to \mathcal{F}$ is a functor, then $(RH, R\alpha)$ is a pointwise right Kan extension of RG along F, provided either:*

(i) *R preserves all weighted limits, or*

(ii) *R preserves limits for* $\mathbf{U}$*-small diagrams and $\mathcal{C}$ is* $\mathbf{U}$*-small.*

If (H, α) is a pointwise left Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along $F : \mathcal{C} \to \mathcal{D}$, and $L : \mathcal{E} \to \mathcal{F}$ is a functor, then $(LH, L\alpha)$ is a pointwise left Kan extension of LG along F, provided either:

(i′) *L preserves all weighted colimits, or* (ii′) *L preserves colimits for* $\mathbf{U}$*-small diagrams and* $\mathcal{C}$ *is* $\mathbf{U}$*-small.* ∎

Corollary A.5.19. *If (H, α) is a pointwise right (resp. left) Kan extension of $G : \mathcal{C} \to \mathcal{E}$ along a fully faithful functor $F : \mathcal{C} \to \mathcal{D}$, then $\alpha : HF \Rightarrow G$ (resp. $\alpha : G \Rightarrow HF$) is a natural isomorphism.*

Proof. If F is fully faithful, then the comma category $(Fc \downarrow F)$ (resp. $(F \downarrow Fc)$) has an initial (resp. terminal) object, namely (c, id_{Fc}), so the component $\alpha_c : HFc \to Gc$ (resp. $\alpha_c : Gc \to HFc$) must be an isomorphism. ∎

Theorem A.5.20. *Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ be functors, and let $i : \mathcal{C} \to \mathcal{C}^+$ and $j : \mathcal{D} \to \mathcal{D}^+$ be fully faithful functors. Consider the following (not necessarily commutative) diagram:*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] \arrow[d, "F"'] & \mathcal{D} \arrow[r, "j"] & \mathcal{D}^+ \\
\mathcal{C} \arrow[ur, "H"'] \arrow[d, "i"'] & & \\
\mathcal{C}^+ \arrow[uurr, "H^+"'] & &
\end{tikzcd}$$

(i) *If H^+ is a pointwise right Kan extension of jG along iF, and $H^+ i \cong jH$, then H is a pointwise right Kan extension of G along F.*

(ii) *Suppose jH is a pointwise right Kan extension of jG along F. If H^+ is a pointwise right Kan extension of jH along i, then the counit $H^+ i \Rightarrow jH$ is a natural isomorphism, and H^+ is also a pointwise right Kan extension of jG along iF; conversely, if H^+ is a pointwise right Kan extension of jG along iF, then it is also a pointwise right Kan extension of jH along i.*

(iii) *If* $\mathbf{U}$ *is a pre-universe such that $\mathcal{A}$ is* $\mathbf{U}$*-small and j preserves limits for all* $\mathbf{U}$*-small diagrams, and H is a pointwise right Kan extension of G along F, then a pointwise right Kan extension of jG along iF can be computed as a pointwise right Kan extension of jH along i (if either one exists).*

Dually:

(i′) *If H^+ is a pointwise left Kan extension of jG along iF, and $H^+ i \cong jH$, then H is a pointwise left Kan extension of G along F.*

(ii′) *Suppose jH is a pointwise left Kan extension of jG along F. If H^+ is a pointwise right Kan extension of jH along i, then the unit $jH \Rightarrow H^+ i$ is a natural isomorphism, and H^+ is also a pointwise left Kan extension of jG along iF; conversely, if H^+ is a pointwise left Kan extension of jG along iF, then it is also a pointwise left Kan extension of jH along i.*

(iii′) *If $\mathbf{U}$ is a pre-universe such that $\mathcal{A}$ is $\mathbf{U}$-small and j preserves colimits for all $\mathbf{U}$-small diagrams, and H is a pointwise left Kan extension of G along F, then a pointwise left Kan extension of jG along iF can be computed as a pointwise left Kan extension of jH along i (if either one exists).*

Proof. (i). Theorem A.5.15 gives an explicit description of $H^+ : \mathcal{C}^+ \to \mathcal{D}^+$ as a weighted limit:

$$H^+(C') \cong \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}$$

Since i is fully faithful, the weights $\mathcal{C}(C, F)$ and $\mathcal{C}^+(iC, iF)$ are naturally isomorphic, hence,

$$jH(C) \cong H^+(iC) \cong \{\mathcal{C}^+(iC, iF), jG\}^{\mathcal{A}} \cong \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}$$

but, since j is fully faithful, j reflects *all* weighted limits, therefore H must be a pointwise right Kan extension of G along F.

(ii). Let $\mathbf{U}^+$ be a pre-universe such that $\mathcal{A}$ and $\mathcal{C}$ are $\mathbf{U}^+$-small categories and $\mathcal{D}, \mathcal{C}^+, \mathcal{D}^+$ are locally $\mathbf{U}^+$-small categories, and let $\mathbf{Set}^+$ be the category of $\mathbf{U}^+$-sets. Using the interchange law (theorem A.6.13) and propositions A.6.7 and A.6.14, we obtain the following natural bijections:

$$\begin{aligned}
\mathcal{D}^+(D', H^+(C')) &\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', i), jH\}^{\mathcal{C}}\big) \\
&\cong \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jHC)) \\
&\cong \int_{C:\mathcal{C}} \mathbf{Set}^+\big(\mathcal{C}^+(C', iC), \mathcal{D}^+\big(D', \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}\big)\big) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathbf{Set}^+(\mathcal{C}(C, FA), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{A:\mathcal{A}} \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA)))
\end{aligned}$$

$$\begin{aligned}&\cong \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iFA), \mathcal{D}^+(D', jGA)) \\ &\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}\big)\end{aligned}$$

Thus, H^+ is a pointwise right Kan extension of jG along iF if and only if H^+ is a pointwise right Kan extension of jH along i. The fact that the counit $H^+ i \Rightarrow jH$ is a natural isomorphism is just corollary A.5.19.

(iii). Apply corollary A.5.18 to claim (ii). ■

Proposition A.5.21. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be any two functors. The following are equivalent:*

(i) *$F \dashv G$, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

(ii) *(F, ε) is an absolute right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G.*

(iii) *(F, ε) is a right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G that is preserved by F.*

(iv) *(G, η) is an absolute left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F.*

(v) *(G, η) is a left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F that is preserved by G.*

Proof. See [CWM, Ch. X, § 7]. □

Proposition A.5.22.

- *Left adjoints preserve all left Kan extensions.*
- *Right adjoints preserve all right Kan extensions.*

Proof. See Theorem 1 in [CWM, Ch. X, § 5]. □

Definition A.5.23. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. A **dense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the F-nerve functor $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. A **dense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a dense functor.

Dually, a **codense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the opposite functor $F^{\mathrm{op}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is dense, and a **codense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a codense functor.

Example A.5.24. The Yoneda lemma implies $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a dense and codense functor.

One may extract an elementary definition for '(co)dense functor' from the following proposition.

Proposition A.5.25. *With notation as in definition A.5.23, the following are equivalent:*

(i) $F : \mathcal{B} \to \mathcal{C}$ *is a dense functor.*

(ii) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C, C') \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]\big(\mathrm{N}^F(C), \mathcal{C}(F, C')\big)$$

induced by $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted colimit $\mathrm{N}^F(C) \star_{\mathcal{B}} F$ in $\mathcal{C}$.

(iii) *For each object C in $\mathcal{C}$, the tautological cocone to C induced by F is a colimiting cocone.*

(iv) $\big(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F\big)$ *is a pointwise left Kan extension of F along F.*

Dually, the following are equivalent:

(i′) $F : \mathcal{B} \to \mathcal{C}$ *is a codense functor.*

(ii′) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C', C) \to [\mathcal{B}, \mathbf{Set}]\big(\mathrm{N}^{F^{\mathrm{op}}}(C), \mathcal{C}(C', F)\big)$$

induced by $\mathrm{N}^{F^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to [\mathcal{B}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted limit $\big\{\mathrm{N}^{F^{\mathrm{op}}}(C), F\big\}^{\mathcal{B}}$ in $\mathcal{C}$.

(iii′) *For each object C in $\mathcal{C}$, the tautological cone from C induced by F is a limiting cone.*

(iv′) $\big(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F\big)$ *is a pointwise right Kan extension of F along F.*

Proof. (i) ⇔ (ii). The indicated maps are bijections for all C and C' if and only if N^F is fully faithful, by definition.

(ii) ⇔ (iii) ⇔ (iv). This is an application of theorem A.5.15. ■

Definition A.5.26. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. A **densely-defined partial left adjoint** for G is a triple (F, i, η), where $F : \mathcal{B} \to \mathcal{D}$ is a functor, $i : \mathcal{B} \to \mathcal{C}$ is a dense functor, and $\eta : i \Rightarrow GF$ is a natural transformation such that the maps

$$\begin{aligned} \mathcal{D}(FB, D) &\to \mathcal{C}(iB, GD) \\ g &\mapsto Gg \circ \eta_B \end{aligned}$$

are bijections that are natural in B and D.

Dually, given a functor $F : \mathcal{C} \to \mathcal{D}$, a **codensely-defined partial right adjoint** for F is a triple (G, j, ε), where $G : \mathcal{B} \to \mathcal{C}$ is a functor, $j : \mathcal{B} \to \mathcal{C}$ is a codense functor, and $\varepsilon : FG \Rightarrow j$ is a natural transformation such that the maps

$$\begin{aligned} \mathcal{C}(C, GB) &\to \mathcal{D}(FC, jB) \\ f &\mapsto \varepsilon_B \circ Ff \end{aligned}$$

are bijections that are natural in B and C.

Example A.5.27. The Yoneda embedding $\hbar_\bullet : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ has a densely-defined partial left adjoint, namely $\left(\mathrm{id}_{\mathcal{B}}, \hbar_\bullet, \mathrm{id}_{\hbar_\bullet}\right)$.

Remark A.5.28. $\left(F, \mathrm{id}_{\mathcal{C}}, \eta\right)$ is a densely-defined partial left adjoint for G if and only if F is a left adjoint for G in the usual sense, with η being the adjunction unit.

Proposition A.5.29. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathcal{C}$ *and* $\mathcal{D}$ *be locally* $\mathbf{U}$*-small categories. Given functors* $G : \mathcal{D} \to \mathcal{C}$, $F : \mathcal{B} \to \mathcal{D}$, *and* $i : \mathcal{B} \to \mathcal{C}$, *the following are equivalent:*

(i) (F, i, η) *is a densely-defined partial left adjoint for* G.

(ii) *The functor* $i : \mathcal{B} \to \mathcal{C}$ *is dense, and there exists a diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\hbar_\bullet"] \arrow[d, "G"'] \arrow[dr, phantom, "\swarrow\alpha"] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\ \mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

where α *factors through* $\eta^* : \mathrm{N}^{GF} \Rightarrow \mathrm{N}^i$ *and is a natural isomorphism.*

(iii) *The functor $i : \mathcal{B} \to \mathcal{C}$ is dense, and the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\text{よ}_\bullet"] \arrow[d, "G"'] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\ \mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

commutes up to natural isomorphism.

Dually, given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{B} \to \mathcal{C}$, and $j : \mathcal{B} \to \mathcal{D}$, the following are equivalent:

(i′) *(G, j, ε) is a codensely-defined partial right adjoint for F.*

(ii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and there exists a diagram*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\text{よ}^\bullet"] \arrow[d, "F^{\mathrm{op}}"'] \arrow[dr, phantom, "\swArrow\beta"] & {[\mathcal{C}, \mathbf{Set}]} \arrow[d, "G^*"] \\ \mathcal{D}^{\mathrm{op}} \arrow[r, "\mathrm{N}^{j^{\mathrm{op}}}"'] & {[\mathcal{B}, \mathbf{Set}]} \end{tikzcd}$$

where β factors through $(\varepsilon^{\mathrm{op}})^* : \mathrm{N}^{F^{\mathrm{op}}G^{\mathrm{op}}} \Rightarrow \mathrm{N}^{j^{\mathrm{op}}}$ and is a natural isomorphism.

(iii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and the diagram*

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "\text{よ}^\bullet"] \arrow[d, "F^{\mathrm{op}}"'] & {[\mathcal{C}, \mathbf{Set}]} \arrow[d, "G^*"] \\ \mathcal{D}^{\mathrm{op}} \arrow[r, "\mathrm{N}^{j^{\mathrm{op}}}"'] & {[\mathcal{B}, \mathbf{Set}]} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof. (i) ⇒ (ii). This immediately follows from the definition.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). The displayed diagram commutes up to natural isomorphism precisely when there are bijections

$$\alpha_{B,D} : \mathcal{D}(FB, D) \to \mathcal{C}(iB, GD)$$

that are natural in both B and D. Taking $D = FB$, let $\eta_B : iB \to GFB$ be the morphism corresponding to $\mathrm{id}_{FB} : FB \to FB$. Applying the Yoneda lemma, we see that the natural bijection $\alpha_{B,D}$ must be the map $g \mapsto Gg \circ \eta_B$. ■

Corollary A.5.30. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a densely-defined partial left adjoint, then G preserves:*

(i) *limits for all diagrams in $\mathcal{D}$,*

(ii) *weighted limits, and*

(iii) *pointwise right Kan extensions.*

Dually, if a functor $F : \mathcal{C} \to \mathcal{D}$ has a codensely-defined partial right adjoint, then F preserves:

(i′) *colimits for all diagrams in $\mathcal{C}$,*

(ii′) *weighted colimts, and*

(iii′) *pointwise left Kan extensions.*

Proof. Choose a universe $\mathbf{U}$ such that the domain of $i : \mathcal{B} \to \mathcal{C}$ is $\mathbf{U}$-small and both $\mathcal{C}$ and $\mathcal{D}$ are locally $\mathbf{U}$-small, and consider the following diagram:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\text{\usefont{U}{min}{m}{n}\symbol{'210}}_\bullet"] \arrow[d, "G"'] & {[\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]} \arrow[d, "(F^{\mathrm{op}})^*"] \\ \mathcal{C} \arrow[r, "\mathrm{N}^i"'] & {[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]} \end{tikzcd}$$

Since i is dense, the i-nerve functor $\mathrm{N}^i : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. Corollary A.5.17 implies $(F^{\mathrm{op}})^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is a right adjoint, and the Yoneda embedding $\text{\usefont{U}{min}{m}{n}\symbol{'210}}_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ preserves all limits and weighted limits (see proposition A.5.13), so we use the fact that N^i reflects limits and weighted limits to conclude that G preserves them. We then apply corollary A.5.18. ■

Definition A.5.31. A **cofinal functor** (resp. **coinitial functor**) is a functor $F : \mathcal{C} \to \mathcal{D}$ such that, for each object D in $\mathcal{D}$, the comma category $(D \downarrow F)$ (resp. $(F \downarrow D)$) is connected.

Theorem A.5.32. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between* $\mathbf{U}$*-small categories. The following are equivalent:*

(i) $F : \mathcal{C} \to \mathcal{D}$ *is a coinitial functor.*

(ii) *The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[d, "F^*"] \\ \mathbf{Set} \arrow[r, "\Delta"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iii) *The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & \mathbf{Set} \arrow[d, "\Delta"] \\ {[\mathcal{D}, \mathbf{Set}]} \arrow[r, "F^*"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iv) *For all locally small categories* $\mathcal{E}$ *and all diagrams* $G : \mathcal{D} \to \mathcal{E}$, $\varprojlim_{\mathcal{C}} GF$ *exists if and only if* $\varprojlim_{\mathcal{D}} G$ *exists, in which case the canonical comparison morphism* $\varprojlim_{\mathcal{D}} G \to \varprojlim_{\mathcal{C}} GF$ *is an isomorphism.*

Dually, the following are equivalent:

(i′) $F : \mathcal{C} \to \mathcal{D}$ *is a cofinal functor.*

(ii′) *The commutative diagram of functors shown below satisfies the right Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\Delta"] \arrow[d, "\mathrm{id}"'] & {[\mathcal{D}, \mathbf{Set}]} \arrow[d, "F^*"] \\ \mathbf{Set} \arrow[r, "\Delta"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iii′) *The commutative diagram of functors shown below satisfies the left Beck–Chevalley condition:*

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "\mathrm{id}"] \arrow[d, "\Delta"'] & \mathbf{Set} \arrow[d, "\Delta"] \\ {[\mathcal{D}, \mathbf{Set}]} \arrow[r, "F^*"'] & {[\mathcal{C}, \mathbf{Set}]} \end{tikzcd}$$

(iv′) *For all locally small categories $\mathcal{E}$ and all diagrams $G : \mathcal{D} \to \mathcal{E}$, $\varinjlim_{\mathcal{C}} GF$ exists if and only if $\varinjlim_{\mathcal{D}} G$ exists, in which case the canonical comparison morphism $\varinjlim_{\mathcal{C}} GF \to \varinjlim_{\mathcal{D}} G$ is an isomorphism.*

Proof. (i) ⇔ (ii). Using the colimit formula for $\mathrm{Lan}_F : [\mathcal{C}, \mathbf{Set}] \to [\mathcal{D}, \mathbf{Set}]$ indicated in theorem A.5.15, it is clear that the comma categories $(F \downarrow D)$ is connected if and only if the left Beck–Chevalley transformation $\Delta \Rightarrow \mathrm{Ran}_F(\Delta -)$ is a natural isomorphism.

(ii) ⇔ (iii). Apply proposition A.1.11.

(iii) ⇒ (iv). We have the following natural bijections:

$$\begin{aligned} [\mathcal{C}, \mathcal{E}](\Delta E, GF) &\cong \varinjlim_{\mathcal{C}} \mathcal{E}(E, GF) \\ &\cong \varinjlim_{\mathcal{D}} \mathcal{E}(E, G) \\ &\cong [\mathcal{D}, \mathcal{E}](\Delta E, G) \end{aligned}$$

Thus, there is a natural bijection between cones from E to GF and cones from E to G; this implies that limits for GF exist in $\mathcal{E}$ if and only if limits for G exist in $\mathcal{E}$ and that they are canonically isomorphic.

(iv) ⇒ (iii). Obvious. ∎

A.6 Ends and coends

Prerequisites. §§ 0.1, A.5

In this section we use the explicit universe convention.

Definition A.6.1. Let $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A **dinatural transformation** $\alpha : F \overset{\diamond}{\to} G$ is a family $\big(\alpha_C : F(C,C) \to G(C,C) \,\big|\, C \in \operatorname{ob} \mathcal{C}\big)$ such that the diagram

$$\begin{tikzcd}
 & F(C,C) \arrow[r, "\alpha_C"] & G(C,C) \arrow[dr, "G(f{,}\mathrm{id}_C)"] & \\
F(C,C') \arrow[ur, "F(\mathrm{id}_C{,}f)"] \arrow[dr, "F(f{,}\mathrm{id}_{C'})"'] & & & G(C',C) \\
 & F(C',C') \arrow[r, "\alpha_{C'}"'] & G(C',C') \arrow[ur, "G(\mathrm{id}_{C'}{,}f)"'] &
\end{tikzcd}$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$.

Example A.6.2. Let $\mathbf{U}$ be a pre-universe, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets. Consider the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ that sends a pair of objects in $\mathcal{C}$ to their hom-set. For each natural number n, we have an dinatural transformation $\mathrm{Hom}_{\mathcal{C}} \overset{\diamond}{\to} \mathrm{Hom}_{\mathcal{C}}$ defined by $e \mapsto e^n$, where e^n denotes the n-fold iterate of the endomorphism e.

Definition A.6.3. A **wedge** from an object D in $\mathcal{D}$ to a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a dinatural transformation $\Delta D \overset{\diamond}{\to} G$, where $\Delta D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is the constant functor with value D; dually, a **cowedge** from a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to an object D in $\mathcal{D}$ is a dinatural transformation $F \overset{\diamond}{\to} \Delta D$.

Definition A.6.4. An **end** for a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a wedge $\lambda : \Delta E \overset{\diamond}{\to} G$ with the following universal property:

- For each wedge $\varphi : \Delta D \overset{\diamond}{\to} G$, there is a unique morphism $f : D \to E$ in $\mathcal{D}$ such that $\varphi_C = \lambda_C \circ f$ for all objects C in $\mathcal{C}$.

We write the following formula to mean that E is an end for G:

$$E = \int_{C:\mathcal{C}} G(C,C)$$

Dually, a **coend** for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a cowedge $\lambda : F \overset{\diamond}{\to} \Delta E$ with the following universal property:

- For each cowedge $\varphi : F \overset{\diamond}{\to} \Delta D$, there is a unique morphism $f : E \to D$ in $\mathcal{D}$ such that $\varphi_C = f \circ \lambda_C$ for all objects C in $\mathcal{C}$.

We write the following formula to mean that E is a coend for F:

$$E = \int^{C:\mathcal{C}} F(C, C)$$

Remark A.6.5. Let $\mathbf{U}$ be a pre-universe, let $\mathbb{D}$ be a $\mathbf{U}$-small category, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Then, for all functors $F, G : \mathbb{D} \to \mathcal{C}$, we have a bijection

$$[\mathbb{D}, \mathcal{C}](F, G) \cong \int_{d:\mathbb{D}} \mathcal{C}(Fd, Gd)$$

and this is natural in both F and G. (The size restriction ensures that the LHS is a $\mathbf{U}$-set.) See also lemma A.5.12.

Proposition A.6.6. *Let $\mathbf{U}$ be a pre-universe and let $\mathbb{D}$ be a $\mathbf{U}$-small category. If $\mathcal{C}$ is a $\mathbf{U}$-complete category, then $\mathcal{C}$ has ends for all functors $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$. Dually, if $\mathcal{C}$ is a $\mathbf{U}$-cocomplete category, then $\mathcal{C}$ has coends for all functors $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$.*

Proof. It is clear from the definition that an end is a special kind of limit, and a coend is a special kind of colimit. To make this precise, one can use Mac Lane's subdivision category $\mathcal{C}^{\S}$: see [CWM, Ch. IX, § 5]. □

Proposition A.6.7. *Let $\mathbf{U}$ be a pre-universe, let* **Set** *be the category of $\mathbf{U}$-sets, and let $F : \mathcal{C} \to \mathcal{D}$ be any functor where $\mathcal{C}$ and $\mathcal{D}$ are locally $\mathbf{U}$-small categories.*

(i) *For any functor $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, if the ends $\int_{\mathcal{J}} A$ and $\int_{\mathcal{J}} FA$ both exist, with λ being the universal wedge in $\mathcal{C}$, then there is a canonical comparison morphism*

$$F \int_{\mathcal{J}} A \to \int_{\mathcal{J}} FA$$

induced by the wedge $F\lambda$.

(ii) *For any object C in $\mathcal{C}$, the functor $\mathcal{C}(C, -) : \mathcal{C} \to$* **Set** *preserves* all *ends.*

(iii) *The functors $\mathcal{C}(C, -)$* jointly *reflect ends.*

(iv) *If F has a left adjoint, then F preserves ends.*

Dually:

(i′) *For any functor* $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$*, if the coends* $\int^{\mathcal{J}} A$ *and* $\int^{\mathcal{J}} FA$ *both exist, with* λ *being the universal cowedge in* $\mathcal{C}$*, then there is a canonical comparison morphism*

$$\int^{\mathcal{J}} FA \to F \int^{\mathcal{J}} A$$

induced by the cowedge $F\lambda$*.*

(ii′) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}$ *sends any coend in* $\mathcal{C}$ *to the corresponding end in* **Set***.*

(iii′) *The functors* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ jointly *reflect coends.*

(iv′) *If* F *has a right adjoint, then* F *preserves coends.*

Proof. All straightforward. ⧫

Definition A.6.8. Let $\mathbf{U}$ be a pre-universe, let **Set** be the category of $\mathbf{U}$-sets, and let $\mathbb{1}$ be the trivial category with $*$ as its only object. A **tensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted colimit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a tensored $\mathbf{U}$-category, then we write $X \odot C$ for the weighted colimit $W \star_{\mathbb{1}} A$, where $X = W(*)$ and $C = A(*)$.

Dually, a **cotensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted limit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a cotensored $\mathbf{U}$-category, then we write $X \pitchfork C$ for the weighted limit $\{W, A\}^{\mathbb{1}}$, where $X = W(*)$ and $C = A(*)$.

Proposition A.6.9 (Tensor–hom–cotensor adjunction)**.** *Let* $\mathbf{U}$ *be a pre-universe, let* **Set** *be the category of* $\mathbf{U}$*-sets, let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category.*

(i) *If* $\mathcal{C}$ *is a tensored* $\mathbf{U}$*-category, then the assignment* $(X, C) \mapsto X \odot C$ *can be extended to a functor* $\mathbf{Set} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object* C*, we have the following adjunction:*

$$- \odot C \dashv \mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$$

(ii) *If $\mathcal{C}$ is a cotensored* $\mathbf{U}$*-category, then the assignment* $(X, C) \mapsto X \pitchfork C$ *can be extended to a functor* $\mathbf{Set}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object* C*, the functors* $- \pitchfork C : \mathbf{Set}^{\mathrm{op}} \to \mathcal{C}$ *and* $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *are contravariantly adjoint on the right.*

(iii) *If* $\mathcal{C}$ *is a tensored and cotensored* $\mathbf{U}$*-category, then for each set* X*, we have the following adjunction:*

$$X \odot - \dashv X \pitchfork - : \mathcal{C} \to \mathcal{C}$$

Proof. Claims (i) and (ii) are formally dual and are straightforward applications of the parametrised adjunction theorem.[9] For claim (iii), simply observe that we have bijections

$$\mathcal{C}(X \odot A, B) \cong \mathbf{Set}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, X \pitchfork B)$$

and these are natural in A, B, and X. ■

Theorem A.6.10. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathcal{C}$ *be a locally* $\mathbf{U}$*-small category. The following are equivalent:*

(i) $\mathcal{C}$ *is a* $\mathbf{U}$*-complete category.*

(ii) $\mathcal{C}$ *is a cotensored* $\mathbf{U}$*-category and, for all* $\mathbf{U}$*-small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, an end for* A *exists in* $\mathcal{C}$*.*

(iii) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted limit for* A*, provided* $\mathbb{D}$ *is a* $\mathbf{U}$*-small category.*

Dually, the following are equivalent:

(i′) $\mathcal{C}$ *is a* $\mathbf{U}$*-cocomplete category.*

(ii′) $\mathcal{C}$ *is a tensored* $\mathbf{U}$*-category and, for all* $\mathbf{U}$*-small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, a coend for* A *exists in* $\mathcal{C}$*.*

(iii′) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted colimit for* A*, provided* $\mathbb{D}$ *is a* $\mathbf{U}$*-small category.*

[9] See Theorem 3 in [CWM, Ch. IV, § 7].

Proof. (i) ⇒ (ii). It is clear that $X \pitchfork C$ is nothing more than an X-fold product of copies of C, so C is certainly $\mathbf{U}$-cotensored if it is $\mathbf{U}$-complete, and proposition A.6.6 says C also has the required ends in that case.

(ii) ⇒ (iii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}\big(C, \{W, A\}^{\mathbb{D}}\big) &\cong [\mathbb{D}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\ &\cong \int_{d:\mathbb{D}} \mathbf{Set}(Wd, \mathcal{C}(C, Ad)) \\ &\cong \int_{d:\mathbb{D}} \mathcal{C}(C, Wd \pitchfork Ad) \\ &\cong \mathcal{C}\left(C, \int_{d:\mathbb{D}} Wd \pitchfork Ad\right) \end{aligned}$$

Thus, using the Yoneda lemma and *assuming $\mathcal{C}$ is a cotensored $\mathbf{U}$-category*, the weighted limit $\{W, A\}^{\mathbb{D}}$ exists if and only if the end $\int_{d:\mathbb{D}} Wd \pitchfork Ad$ exists.

(iii) ⇒ (i). Ordinary limits are a special case of weighted limits, as remarked in A.5.11. ∎

Proposition A.6.11. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, and let $\mathcal{J}$ be any category. If $\mathcal{C}$ is a tensored $\mathbf{U}$-category and has weighted limits for all weights $W : \mathcal{J} \to \mathbf{Set}$ and diagrams $A : \mathcal{J} \to \mathcal{C}$, then:*

(i) $(W, A) \mapsto \{W, A\}^{\mathcal{J}}$ *extends to a functor* $[\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.

(ii) *For each diagram $A : \mathcal{J} \to \mathcal{C}$, the functors $\{-, A\}^{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \to \mathcal{C}$ and $\mathcal{C}(-, A) : \mathcal{C}^{\mathrm{op}} \to [\mathcal{J}, \mathbf{Set}]$ are contravariantly adjoint on the right.*

(iii) *For each weight $W : \mathcal{J} \to \mathbf{Set}$, we have the following adjunction:*

$$W \odot - \dashv \{W, -\}^{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$$

Here, $W \odot C : \mathcal{J} \to \mathcal{C}$ is the diagram $j \mapsto Wj \odot C$.

Dually, if $\mathcal{C}$ is a cotensored $\mathbf{U}$-category and has weighted colimits for all weights $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ and diagrams $A : \mathcal{J} \to \mathcal{C}$, then:

(i′) $(W, A) \mapsto W \star_{\mathcal{J}} A$ *extends to a functor* $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}] \times \mathcal{C} \to \mathcal{C}$.

(ii′) *For each diagram* $A : \mathcal{J} \to \mathcal{C}$*, we have the following adjunction:*

$$- \star_{\mathcal{J}} A \dashv \mathcal{C}(A, -) : \mathcal{C} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$$

(iii′) *For each weight* $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$*, we have the following adjunction:*

$$W \star_{\mathcal{J}} - \dashv W \pitchfork - : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$$

Here, $W \pitchfork C : \mathcal{J} \to \mathcal{C}$ *is the diagram* $j \mapsto Wj \pitchfork C$*.*

Proof. Claim (i) is straightforward, and for claims (ii) and (iii), observe that we have bijections

$$\begin{aligned}
\mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) &\cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\
&\cong \int_{j:\mathcal{J}} \mathbf{Set}(Wj, \mathcal{C}(C, Aj)) \\
&\cong \int_{j:\mathcal{J}} \mathcal{C}(Wj \odot C, Aj) \\
&\cong [\mathcal{J}, \mathcal{C}](W \odot C, A)
\end{aligned}$$

and these are natural in W, A, and C. ∎

Lemma A.6.12. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $\mathbb{I}$ *and* $\mathbb{J}$ *be* $\mathbf{U}$*-small categories. For all functors* $A : \mathbb{I}^{\mathrm{op}} \times \mathbb{J}^{\mathrm{op}} \times \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$*:*

(i) *The assignment* $(i', i) \mapsto \int_{j:\mathbb{J}} A(i', j, i, j)$ *extends to a functor* $\mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathbf{Set}$*.*

(ii) *There is a unique morphism* θ *making the diagram below commute for all* i *and* j*,*

$$\begin{tikzcd}
\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') \arrow[r] \arrow[d, dashed, "\theta"'] & \int_{j':\mathbb{J}} A(i, j', i, j') \arrow[d] \\
\int_{(i',j'):\mathbb{I}\times\mathbb{J}} A(i', j', i', j') \arrow[r] & A(i, j, i, j)
\end{tikzcd}$$

where the unlabelled arrows are the components of the respective universal wedges, and θ *is moreover an isomorphism.*

(iii) *There is a unique morphism σ making the diagram below commute for all i and j,*

$$\begin{tikzcd}
\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i',j',i',j') \arrow[r] \arrow[d, dashed, "\sigma"'] & \int_{j':\mathbb{J}} A(i,j',i,j') \arrow[dr] & \\
 & & A(i,j,i,j) \\
\int_{j':\mathbb{J}} \int_{i':\mathbb{I}} A(i',j',i',j') \arrow[r] & \int_{i':\mathbb{I}} A(i,j',i,j') \arrow[ur] &
\end{tikzcd}$$

where the unmarked arrows are the components of the respective universal wedges, and σ is moreover an isomorphism.

Proof. See [CWM, Ch. IX, § 8]. □

Theorem A.6.13 (Interchange law for ends and coends)**.** *Let $\mathcal{C}$ be any category and let $A : \mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \times \mathcal{I} \times \mathcal{J} \to$ **Set** be any functor. If the end $\int_{i:\mathcal{I}} A(i,j',i,j)$ exists in $\mathcal{C}$ for all j' and j in $\mathcal{J}$, and the end $\int_{j:\mathcal{J}} A(i',j,i,j)$ exists in $\mathcal{C}$ for all i' and i in $\mathcal{I}$, then the following are equivalent:*

(i) *The end $\int_{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(ii) *The iterated end $\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(iii) *The iterated end $\int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

In this case, we have a canonical isomorphism in $\mathcal{C}$:

$$\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j) \cong \int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$$

Dually, if the coend $\int^{i:\mathcal{I}} A(i,j',i,j)$ exists in $\mathcal{C}$ for all j' and j in $\mathcal{J}$, and the coend $\int^{j:\mathcal{J}} A(i',j,i,j)$ exists in $\mathcal{C}$ for all i' and i in $\mathcal{I}$, then the following are equivalent:

(i′) *The coend $\int^{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(ii′) *The iterated coend $\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(iii′) *The iterated coend $\int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

In this case, we have a canonical isomorphism in $\mathcal{C}$:

$$\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j) \cong \int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$$

Proof. Choose a pre-universe $\mathbf{U}$ such that $\mathcal{I}$ and $\mathcal{J}$ are $\mathbf{U}$-small categories and $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and use the Yoneda lemma to reduce the claims to the previous lemma. ■

Proposition A.6.14. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ and $\mathcal{J}$ be locally $\mathbf{U}$-small categories.*

(i) *For all j in $\mathcal{J}$ and all functors $A : \mathcal{J} \to \mathcal{C}$, the Yoneda bijection*

$$\mathcal{C}(C, Aj) \cong [\mathcal{J}, \mathbf{Set}]\big(\hbar^j, \mathcal{C}(C, A)\big)$$

exhibits Aj as the weighted limit $\left\{\hbar^j, A\right\}^{\mathcal{J}}$ in $\mathcal{C}$.

(ii) *If $\mathcal{C}$ is a cotensored $\mathbf{U}$-category, then the end $\int_{j':\mathcal{J}} \mathcal{J}(j,j') \pitchfork Aj'$ exists in $\mathcal{C}$ and can be canonically identified with Aj.*

(iii) *For all functors $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, the weighted limit $\left\{\mathrm{Hom}_{\mathcal{J}}, H\right\}^{\mathcal{J}^{\mathrm{op}} \times \mathcal{J}}$ exists in $\mathcal{C}$ if and only if the end $\int_{j:\mathcal{J}} H(j,j)$ exists in $\mathcal{C}$, and there is a canonical identification of the two.*

Dually:

(i′) *For all j in $\mathcal{J}$ and all functors $A : \mathcal{J} \to \mathcal{C}$, the Yoneda bijection*

$$\mathcal{C}(Aj, C) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_j, \mathcal{C}(A, C)\big)$$

exhibits Aj as the weighted colimit $\hbar_j \star_{\mathcal{J}} A$ in $\mathcal{C}$.

(ii′) *If $\mathcal{C}$ is a tensored $\mathbf{U}$-category, then the coend $\int^{j':\mathcal{J}} \mathcal{J}(j',j) \odot Aj'$ exists in $\mathcal{C}$ and can be canonically identified with Aj.*

(iii′) *For all functors $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, the weighted colimit $\mathrm{Hom}_{\mathcal{J}^{\mathrm{op}}} \star_{\mathcal{J}^{\mathrm{op}} \times \mathcal{J}} H$ exists in $\mathcal{C}$ if and only if the coend $\int^{j:\mathcal{J}} H(j,j)$ exists in $\mathcal{C}$, and there is a canonical identification of the two.*

Proof. (i). This is an immediate consequence of the Yoneda lemma and the definition of weighted limit.

(ii). Use the identification constructed in the proof of theorem A.6.10.

(iii). For all objects C in $\mathcal{C}$, using claim (ii) and the interchange law for ends (theorem A.6.13), there are bijections

$$\begin{aligned}[\mathcal{J}^{\mathrm{op}} \times \mathcal{J}, \mathbf{Set}]\big(\mathrm{Hom}_{\mathcal{J}}, \mathcal{C}(C, H)\big) &\cong \int_{(j',j):\mathcal{J}^{\mathrm{op}}\times\mathcal{J}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(H(j', j))) \\ &\cong \int_{j:\mathcal{J}} \int_{j':\mathcal{J}^{\mathrm{op}}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(H(j', j))) \\ &\cong \int_{j:\mathcal{J}} \mathcal{C}(C, H(j, j))\end{aligned}$$

and these are natural in C; now apply propositions A.5.13 and A.6.7. ■

— B —

Higher generalities

B.1 Monoidal categories

Standard references for monoidal categories include [CWM, Ch. VII and Ch. XI] and [Kelly, 2005, Ch. 1]. To fix notation, we will quickly review the main definitions in the theory of monoidal categories.

Definition B.1.1. A **strict monoidal category** is a category $\mathcal{C}$ together with an object I and a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ satisfying the following axioms:

- (Left unit). $I \otimes (-) = \mathrm{id}_{\mathcal{C}}$.
- (Right unit). $(-) \otimes I = \mathrm{id}_{\mathcal{C}}$.
- (Associativity). For all objects X, Y, and Z in $\mathcal{C}$,

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$$

  and similarly for morphisms in $\mathcal{C}$.

I is called the **monoidal unit**, and $\otimes$ is called the **monoidal product**.

In short, a strict monoidal category is an internal monoid in the metacategory of all categories.

Example B.1.2. For any category $\mathcal{C}$, the endofunctor category $[\mathcal{C}, \mathcal{C}]$ is a strict monoidal category with $\mathrm{id}_{\mathcal{C}}$ as the monoidal unit and endofunctor composition as the monoidal product.

Despite the above example, strict monoidal categories turn out to be less useful than one might hope: not even **Set** equipped with the usual cartesian product is a strict monoidal category.[1] The problem is in the *equations* we have imposed in the axioms above: in naturally-occurring examples, we do not get *identities* but only natural isomorphisms. This observation led Bénabou [1963] to propose the following notion instead:

Definition B.1.3. A **monoidal category** is a category $\mathcal{C}$ together with an object I, a functor $(-) \otimes (-) : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and three natural isomorphisms $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$,[2] of type

$$\boldsymbol{\lambda}_X : I \otimes X \xrightarrow{\cong} X$$
$$\boldsymbol{\rho}_X : X \otimes I \xrightarrow{\cong} X$$
$$\boldsymbol{\alpha}_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$$

such that the following diagrams commute for all choices of objects in $\mathcal{C}$:

$$\begin{tikzcd}
 & (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\boldsymbol{\alpha}_{W,X \otimes Y,Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\boldsymbol{\alpha}_{W,X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "\boldsymbol{\alpha}_{W \otimes X,Y,Z}"'] & & (W \otimes ((X \otimes Y) \otimes Z)) \arrow[d, "\mathrm{id}_W \otimes \boldsymbol{\alpha}_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y \otimes Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\boldsymbol{\alpha}_{X,I,Y}"] \arrow[dr, "\boldsymbol{\rho}_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\boldsymbol{\lambda}_X \otimes \mathrm{id}_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

The natural isomorphisms $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ are called, respectively, the **left unitor**, **right unitor**, and **associator** of the monoidal category $\mathcal{C}$.

[1] In fact, even if we identify all isomorphic objects, there is still a problem: see the closing remarks in [CWM, Ch. VII, § 1].

[2] Beware: Mac Lane [CWM, Ch. VII] uses the opposite convention for $\boldsymbol{\alpha}$.

Remark B.1.4. Since $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ are natural *isomorphisms*, a monoidal structure on $\mathcal{C}$ induces a monoidal structure on $\mathcal{C}^{\mathrm{op}}$. Less obviously, we can define a monoidal category $\mathcal{C}^{\mathrm{rev}}$ whose underlying category is the same as $\mathcal{C}$, but $X \otimes^{\mathrm{rev}} Y = Y \otimes X$, $\boldsymbol{\lambda}^{\mathrm{rev}} = \boldsymbol{\rho}$, $\boldsymbol{\rho}^{\mathrm{rev}} = \boldsymbol{\lambda}$, and $\boldsymbol{\alpha}^{\mathrm{rev}} = \boldsymbol{\alpha}^{-1}$.

¶ B.1.5. A fairly non-trivial theorem of Mac Lane [1963] and Kelly [1964] essentially states that these two axioms are enough to prove that "all diagrams involving only $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\alpha}$ commute". For example, using the pentagon axiom and the triangle axiom, we may derive

$$\begin{tikzcd} (I \otimes X) \otimes Y \arrow[rr, "\boldsymbol{\alpha}_{I,X,Y}"] \arrow[dr, "\boldsymbol{\lambda}_X \otimes \mathrm{id}_Y"'] & & I \otimes (X \otimes Y) \arrow[dl, "\boldsymbol{\lambda}_{X \otimes Y}"] \\ & X \otimes Y & \end{tikzcd}$$

from which the equation (!) below can be obtained:

$$\boldsymbol{\lambda}_I = \boldsymbol{\rho}_I$$

Definition B.1.6. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A **lax monoidal functor** $\mathcal{C} \to \mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ of the underlying categories, together with a morphism $\boldsymbol{\eta} : I_{\mathcal{D}} \to F I_{\mathcal{C}}$ in $\mathcal{D}$ and a natural transformation $\boldsymbol{\mu}$ of type $F(-) \otimes_{\mathcal{D}} F(-) \to F\big(- \otimes_{\mathcal{C}} -\big)$ making these diagrams commute:

$$\begin{tikzcd} I_{\mathcal{D}} \otimes_{\mathcal{D}} FX \arrow[r, "\boldsymbol{\eta} \otimes_{\mathcal{D}} \mathrm{id}_{FX}"] \arrow[d, "\boldsymbol{\lambda}_{FX}"'] & FI_{\mathcal{C}} \otimes_{\mathcal{D}} FX \arrow[d, "\boldsymbol{\mu}_{I_{\mathcal{C}},X}"] \\ FX & F\big(I_{\mathcal{C}} \otimes_{\mathcal{C}} X\big) \arrow[l, "F\boldsymbol{\lambda}_X"] \end{tikzcd} \qquad \begin{tikzcd} FX \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \boldsymbol{\eta}"] \arrow[d, "\boldsymbol{\rho}_{FX}"'] & FX \otimes_{\mathcal{D}} FI_{\mathcal{C}} \arrow[d, "\boldsymbol{\mu}_{X,I_{\mathcal{C}}}"] \\ FX & F\big(X \otimes_{\mathcal{C}} I_{\mathcal{C}}\big) \arrow[l, "F\boldsymbol{\rho}_X"] \end{tikzcd}$$

$$\begin{tikzcd} \big(FX \otimes_{\mathcal{D}} FY\big) \otimes_{\mathcal{D}} FZ \arrow[r, "\boldsymbol{\alpha}_{FX,FY,FZ}"] \arrow[d, "\boldsymbol{\mu}_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{FZ}"'] & FX \otimes_{\mathcal{D}} \big(FY \otimes_{\mathcal{D}} FZ\big) \arrow[d, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \boldsymbol{\mu}_{Y,Z}"] \\ F\big(X \otimes_{\mathcal{C}} Y\big) \otimes_{\mathcal{D}} FZ \arrow[d, "\boldsymbol{\mu}_{X \otimes_{\mathcal{C}} Y, Z}"'] & FX \otimes_{\mathcal{D}} F\big(Y \otimes_{\mathcal{C}} Z\big) \arrow[d, "\boldsymbol{\mu}_{X, Y \otimes_{\mathcal{C}} Z}"] \\ F\big(\big(X \otimes_{\mathcal{C}} Y\big) \otimes_{\mathcal{C}} Z\big) \arrow[r, "F\boldsymbol{\alpha}_{X,Y,Z}"'] & F\big(X \otimes_{\mathcal{C}} \big(Y \otimes_{\mathcal{C}} Z\big)\big) \end{tikzcd}$$

An **oplax monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax monoidal functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. A **strong monoidal functor** is a lax monoidal functor such that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are isomorphisms. A **strict monoidal functor** is a lax monoidal functor such that $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ are *identities*.

Definition B.1.7. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and let $F, F' : \mathcal{C} \to \mathcal{D}$ be lax monoidal functors. A **monoidal natural transformation** $\varphi : F \Rightarrow F'$ is a natural transformation making the following diagrams commute:

$$\begin{tikzcd} I_{\mathcal{D}} \arrow[r, "\boldsymbol{\eta}"] \arrow[d, "\mathrm{id}"'] & F I_{\mathcal{C}} \arrow[d, "\varphi_{I_{\mathcal{C}}}"] \\ I_{\mathcal{D}} \arrow[r, "\boldsymbol{\eta}'"'] & F' I_{\mathcal{C}} \end{tikzcd} \qquad \begin{tikzcd} FX \otimes_{\mathcal{D}} FY \arrow[r, "\boldsymbol{\mu}_{X,Y}"] \arrow[d, "\varphi_X \otimes_{\mathcal{D}} \varphi_Y"'] & F\left(X \otimes_{\mathcal{C}} Y\right) \arrow[d, "\varphi_{X \otimes_{\mathcal{C}} Y}"] \\ F'X \otimes_{\mathcal{D}} F'Y \arrow[r, "\boldsymbol{\mu}'_{X,Y}"'] & F'\left(X \otimes_{\mathcal{C}} Y\right) \end{tikzcd}$$

Remark B.1.8. Note that if $\mathcal{C}$ and $\mathcal{D}$ are both strict monoidal categories, then the diagrams above simplify to more familiar ones:

$$\begin{tikzcd} FX \arrow[r, "\boldsymbol{\eta} \otimes_{\mathcal{D}} \mathrm{id}_{FX}"] \arrow[dr, "\mathrm{id}_{FX}"'] & F I_{\mathcal{C}} \otimes_{\mathcal{D}} FX \arrow[d, "\boldsymbol{\mu}_{I_{\mathcal{C}},X}"] \\ & FX \end{tikzcd} \qquad \begin{tikzcd} FX \arrow[r, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \boldsymbol{\eta}"] \arrow[dr, "\mathrm{id}_{FX}"'] & FX \otimes_{\mathcal{D}} F I_{\mathcal{C}} \arrow[d, "\boldsymbol{\mu}_{X,I_{\mathcal{C}}}"] \\ & FX \end{tikzcd}$$

$$\begin{tikzcd} & FX \otimes_{\mathcal{D}} FY \otimes_{\mathcal{D}} FZ \arrow[dl, "\boldsymbol{\mu}_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{FZ}"'] \arrow[dr, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \boldsymbol{\mu}_{Y,Z}"] & \\ F\left(X \otimes_{\mathcal{C}} Y\right) \otimes_{\mathcal{D}} FZ \arrow[dr, "\boldsymbol{\mu}_{X \otimes_{\mathcal{C}} Y, Z}"'] & & FX \otimes_{\mathcal{D}} F\left(Y \otimes_{\mathcal{C}} Z\right) \arrow[dl, "\boldsymbol{\mu}_{X, Y \otimes_{\mathcal{C}} Z}"] \\ & F\left(X \otimes_{\mathcal{C}} Y \otimes_{\mathcal{C}} Z\right) & \end{tikzcd}$$

Thus, we see one reason for defining lax monoidal functors as we have done: if $\mathbb{1}$ is the terminal category, then a lax monoidal functor $\mathbb{1} \to \mathcal{D}$ is the same thing as an internal monoid[3] in $\mathcal{D}$, and a monoidal natural transformation of such lax monoidal functors is the same thing as a homomorphism of internal monoids.

Many natural examples of monoidal categories have a "commutative" monoidal product. For example, the cartesian product in any category satisfies $X \times Y \cong Y \times X$. As usual, to do anything useful, we must demand not only the

[3] — in the monoidal category sense, of course.

existence of such isomorphisms but also that they be natural and coherent in the following sense:

Definition B.1.9. A **braided monoidal category** is a monoidal category C together with a natural isomorphism $\boldsymbol{\gamma}$ of type

$$\boldsymbol{\gamma}_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X$$

such that the following diagrams commute for all choices of objects in C:

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[dr, "\boldsymbol{\gamma}_{X,Y\otimes Z}"] & \\
(X \otimes Y) \otimes Z \arrow[ur, "\boldsymbol{\alpha}_{X,Y,Z}"] \arrow[d, "\boldsymbol{\gamma}_{X,Y}\otimes \mathrm{id}_Z"'] & & (Y \otimes Z) \otimes X \arrow[d, "\boldsymbol{\alpha}_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[dr, "\boldsymbol{\alpha}_{Y,X,Z}"'] & & Y \otimes (Z \otimes X) \\
& Y \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}_Y \otimes \boldsymbol{\gamma}_{X,Z}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
& (X \otimes Y) \otimes Z \arrow[dr, "\boldsymbol{\gamma}_{X\otimes Y,Z}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\boldsymbol{\alpha}^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X \otimes \boldsymbol{\gamma}_{Y,Z}"'] & & Z \otimes (X \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{Y,Z,X}"] \\
X \otimes (Z \otimes Y) \arrow[dr, "\boldsymbol{\alpha}^{-1}_{Y,X,Z}"'] & & (Z \otimes X) \otimes Y \\
& (X \otimes Z) \otimes Y \arrow[ur, "\boldsymbol{\gamma}_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes X \arrow[rr, "\boldsymbol{\gamma}_{I,X}"] \arrow[dr, "\boldsymbol{\lambda}_X"'] & & X \otimes I \arrow[dl, "\boldsymbol{\rho}_X"] \\
& X &
\end{tikzcd}$$

The natural isomorphism $\boldsymbol{\gamma}$ is called the **braiding** of C. A **symmetric monoidal category** is a braided monoidal category C satisfying the following additional axiom:

$$\gamma \bullet \gamma = \mathrm{id}_{\mathcal{C}}$$

A **braided / symmetric strict monoidal category** is a braided / symmetric monoidal category that is strict as a monoidal category.

There is a coherence theorem for braided and symmetric monoidal categories as well, but in the braided case it is somewhat subtle compared to the coherence theorem for monoidal categories – we cannot be so cavalier as to say that "all diagrams commute" in a braided monoidal category. Instead, just as before, every braided / symmetric monoidal category is equivalent to a strict one via functors respecting the various structural isomorphisms.

Definition B.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories. A **lax / oplax / strong / strict braided monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax / oplax / strong / strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ making the diagram below commute:

$$\begin{tikzcd}
FX \otimes_{\mathcal{D}} FY \arrow[r, "\mu_{X,Y}"] \arrow[d, "\gamma_{FX,FY}"'] & F\big(X \otimes_{\mathcal{C}} Y\big) \arrow[d, "F\gamma_{X,Y}"] \\
FY \otimes_{\mathcal{D}} FX \arrow[r, "\mu_{Y,X}"'] & F\big(Y \otimes_{\mathcal{C}} X\big)
\end{tikzcd}$$

Remark B.1.11. The appropriate notion of natural transformation for lax braided monoidal functors is precisely that of a monoidal natural transformation: we need not impose any extra conditions.

Here is an example of an equation that does *not* necessarily hold in a braided monoidal category, even though they have the same domain and codomain:

$$\gamma_{X,Y} \stackrel{?}{=} \gamma_{Y,X}^{-1}$$

Indeed, if it were true, then every braided monoidal category would be a symmetric monoidal category! On the other hand, in a symmetric strict monoidal category, it is true that any two composites of braiding operations with the same domain and codomain are equal – provided each object is identified with a different letter, so that we do not get absurdities like this:

$$\gamma_{X,X} \stackrel{?}{=} \mathrm{id}_{X \otimes X}$$

A similar restriction applies to our claim that "all diagrams commute" in a monoidal category, so it is not unreasonable to say the same is true in a symmetric monoidal category.

We pause briefly to indicate an important special case of a symmetric monoidal category.

Definition B.1.12. A **cartesian monoidal category** is a category with products for all finite families of objects, and a **cartesian monoidal functor** is a functor between cartesian monoidal categories that preserves all finite products.

Proposition B.1.13.

(i) *A category with all finite products is automatically a symmetric monoidal category, with the terminal object* 1 *as its monoidal unit and the cartesian product* $\times$ *as the monoidal product.*

(ii) *If* $\mathcal{C}$ *and* $\mathcal{D}$ *are two categories with finite products regarded as symmetric monoidal categories, then every functor* $\mathcal{C} \to \mathcal{D}$ *can be equipped with a canonical oplax braided monoidal functor structure.*

(iii) *A cartesian monoidal functor is canonically equipped with the structure of a strong braided monoidal functor.*

Proof. (i). The verification of the axioms is straightforward and left to the reader as an exercise.

(ii). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The universal property of the terminal object gives a unique morphism $\boldsymbol{\varepsilon} : F1 \to 1$ in $\mathcal{D}$, and the universal property of binary products gives a canonical morphism $\boldsymbol{\delta}_{X,Y} : F(X \times Y) \to FX \times FY$. It can be shown that the diagrams below commute,

$$\begin{tikzcd}
F\big(1_{\mathcal{C}} \times_{\mathcal{C}} X\big) \arrow[r, "\boldsymbol{\delta}_{1_{\mathcal{C}},X}"] \arrow[d, "F\boldsymbol{\lambda}_X"'] & F1_{\mathcal{C}} \times_{\mathcal{D}} FX \arrow[d, "\boldsymbol{\varepsilon}\times_{\mathcal{D}}\mathrm{id}_{FX}"] \\
FX & 1_{\mathcal{D}} \times_{\mathcal{D}} FX \arrow[l, "\boldsymbol{\lambda}_{FX}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F\big(X \times_{\mathcal{C}} 1_{\mathcal{C}}\big) \arrow[r, "\boldsymbol{\delta}_{X,1_{\mathcal{C}}}"] \arrow[d, "F\boldsymbol{\rho}_X"'] & FX \times_{\mathcal{D}} F1_{\mathcal{C}} \arrow[d, "\mathrm{id}_{FX}\times_{\mathcal{D}}\boldsymbol{\varepsilon}"] \\
FX & FX \times_{\mathcal{D}} 1_{\mathcal{D}} \arrow[l, "\boldsymbol{\rho}_{FX}"]
\end{tikzcd}$$

$$\begin{tikzcd}
F\big((X \times_C Y) \times_C Z\big) \arrow[r, "F\boldsymbol{\alpha}_{X,Y,Z}"] \arrow[d, "\boldsymbol{\delta}_{X\times_C Y,Z}"'] & F\big(X \times_C (Y \times_C Z)\big) \arrow[d, "\boldsymbol{\delta}_{X,Y\times_C Z}"] \\
F\big(X \times_C Y\big) \times_D FZ \arrow[d, "\boldsymbol{\delta}_{X,Y}\times_D \mathrm{id}_{FZ}"'] & FX \times_D F\big(Y \times_C Z\big) \arrow[d, "\mathrm{id}_{FX}\times_D \boldsymbol{\delta}_{Y,Z}"] \\
\big(FX \times_D FY\big) \times_D FZ \arrow[r, "\boldsymbol{\alpha}_{FX,FY,FZ}"'] & FX \times_D \big(FY \times_D FZ\big)
\end{tikzcd}$$

$$\begin{tikzcd}
F\big(X \times_C Y\big) \arrow[r, "\boldsymbol{\delta}_{X,Y}"] \arrow[d, "F\boldsymbol{\gamma}_{X,Y}"'] & FX \times_D FY \arrow[d, "\boldsymbol{\gamma}_{FX,FY}"] \\
F\big(Y \otimes_C X\big) \arrow[r, "\boldsymbol{\delta}_{Y,X}"'] & FY \otimes_D FX
\end{tikzcd}$$

making F into an oplax braided monoidal functor $\mathcal{C} \to \mathcal{D}$.

(iii). A functor is cartesian monoidal precisely if $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}$ as defined above are isomorphisms. $\lozenge$

Definition B.1.14. Let Y and Z be objects in a monoidal category $\mathcal{C}$.

- A **right internal hom object** for Y and Z is an object $\mathcal{H}om(Y,Z)$ in $\mathcal{C}$ together with a morphism $\mathrm{ev}_{Y,Z} : \mathcal{H}om(Y,Z) \otimes Y \to Z$ having the following universal property: for all morphisms $f : X \otimes Y \to Z$ in $\mathcal{C}$, there is a unique morphism $\tilde{f} : X \to \mathcal{H}om(Y,Z)$ in $\mathcal{C}$ such that $\mathrm{ev}_{Y,Z} \circ \big(\tilde{f} \otimes \mathrm{id}_Y\big) = f$; equivalently, $\mathcal{H}om(Y,Z)$ is an object in $\mathcal{C}$ equipped with bijections

  $$\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, \mathcal{H}om(Y,Z))$$

  that are natural for each object X in $\mathcal{C}$. We may also write $[Y,Z]$ or $Y \multimap Z$ for a right internal hom object for Y and Z.

- A **left internal hom object** for Y and Z is a right internal hom object $Y \pitchfork Z$ in the reverse monoidal structure on $\mathcal{C}$; equivalently, $Y \pitchfork Z$ is an object equipped with bijections

  $$\mathcal{C}(Y \otimes X, Z) \cong \mathcal{C}(X, Y \pitchfork Z)$$

  that are natural for each object X in $\mathcal{C}$. We may also write Z^Y or $Z \multimapinv Y$ for a left internal hom object for Y and Z.

- A **right-closed monoidal category** is a monoidal category that has right internal hom object for all pairs of objects.
- A **left-closed monoidal category** is a monoidal category that has left internal hom objects for all pairs of objects.
- A **biclosed monoidal category** is a monoidal category that is both left-closed and right-closed.

Note that in a symmetric monoidal category, $Y \pitchfork Z$ and $\mathcal{H}om(Y, Z)$ are naturally isomorphic if they exist; a **closed symmetric monoidal category** is a symmetric monoidal category that is biclosed.

Proposition B.1.15. *Let C be a right-closed monoidal category.*

(i) *The assignment $(Y, Z) \mapsto \mathcal{H}om(Y, Z)$ extends to a functor $C^{\mathrm{op}} \times C \to C$ making the bijection*

$$C(X \otimes Y, Z) \cong C(X, \mathcal{H}om(Y, Z))$$

natural in X, Y, and Z.

(ii) *For each object Y, we have an adjunction*

$$(-) \otimes Y \dashv \mathcal{H}om(Y, -) : C \to C$$

whose counit is $\mathrm{ev}_{Y,-} : \mathcal{H}om(Y, -) \otimes Y \Rightarrow \mathrm{id}_C$.

(iii) *If I is the monoidal unit of C, then there is a bijection*

$$C(Y, Z) \cong C(I, \mathcal{H}om(Y, Z))$$

that is natural in Y and Z.

Proof. (i). This is a straightforward example of an adjunction with a parameter.[4]

(ii). This is clear from the definition of $\mathcal{H}om(Y, Z)$ and $\mathrm{ev}_{Y,-}$.

(iii). The left unitor $\boldsymbol{\lambda}_Y : Y \xrightarrow{\cong} I \otimes Y$ induces the required bijection. ■

REMARK B.1.16. A cartesian monoidal category is a closed symmetric monoidal category if and only if it is a cartesian closed category (definition A.2.3).

[4] See [CWM, Ch. IV, § 7].

B.2 Categories with actions

Prerequisites. § B.1.

Definition B.2.1. Let $\mathcal{V}$ be a monoidal category. A **left $\mathcal{V}$-action** on a category $\mathcal{C}$ is a strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]$, where $[\mathcal{C}, \mathcal{C}]$ is regarded as a strict monoidal category under composition. Similarly, a **right $\mathcal{V}$-action** on $\mathcal{C}$ is a strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]^{\mathrm{rev}}$.

Remark B.2.2. We can unfold the above definition somewhat by taking the left exponential transpose of the strong monoidal functor $\mathcal{V} \to [\mathcal{C}, \mathcal{C}]$: let $\oslash$ be the corresponding functor $\mathcal{V} \times \mathcal{C} \to \mathcal{C}$. Since the original functor was strong monoidal, we get a natural isomorphism $\boldsymbol{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow I \oslash (-)$ and a natural isomorphism $\boldsymbol{\mu}_{X,Y} : X \oslash (Y \oslash (-)) \Rightarrow (X \otimes Y) \oslash (-)$ for each pair of objects X and Y in $\mathcal{V}$; these moreover satisfy the following coherence laws:

$$
\begin{tikzcd}
X \oslash (Y \oslash (-)) \arrow[r, "\boldsymbol{\mu}_{X,Y}"] \arrow[d, "f \oslash (g \oslash \mathrm{id})"'] & (X \otimes Y) \oslash (-) \arrow[d, "(f \otimes g) \oslash \mathrm{id}"] \\
X' \oslash (Y' \oslash (-)) \arrow[r, "\boldsymbol{\mu}_{X',Y'}"'] & (X \otimes Y) \oslash (-)
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \oslash (-) \arrow[r, "\boldsymbol{\eta} \bullet (X \oslash (-))"] \arrow[d, "\mathrm{id}"'] & I \oslash (X \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{I,X}"] \\
X \oslash (-) & (I \otimes X) \oslash (-) \arrow[l, "\boldsymbol{\lambda}_X \oslash \mathrm{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \oslash (-) \arrow[r, "(X \oslash (-)) \bullet \boldsymbol{\eta}"] \arrow[d, "\mathrm{id}"'] & X \oslash (I \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{X,I}"] \\
X \oslash (-) & (X \otimes I) \oslash (-) \arrow[l, "\boldsymbol{\rho}_X \oslash \mathrm{id}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
& W \oslash (X \oslash (Y \oslash (-))) \arrow[dl, "\boldsymbol{\mu}_{W,X} \bullet (Y \oslash (-))"'] \arrow[dr, "\mathrm{id}_W \oslash \boldsymbol{\mu}_{X,Y}"] & \\
(W \otimes X) \oslash (Y \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{W \otimes X, Y}"'] & & W \oslash ((X \otimes Y) \oslash (-)) \arrow[d, "\boldsymbol{\mu}_{W, X \otimes Y}"] \\
((W \otimes X) \otimes Y) \oslash (-) \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y} \oslash \mathrm{id}"'] & & (W \otimes (X \otimes Y)) \oslash (-)
\end{tikzcd}
$$

Conversely, any functor $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ equipped with such a collection of natural isomorphisms defines a left $\mathcal{V}$-action on $\mathcal{C}$.

Proposition B.2.3 (Bénabou)**.** *For any monoidal category $\mathcal{C}$, there is a faithful strong monoidal functor $F : \mathcal{C} \to [\mathcal{C}, \mathcal{C}]$ defined by the following data:*

$$FX = X \otimes (-)$$
$$\boldsymbol{\eta} = \boldsymbol{\lambda}^{-1}$$
$$\left(\boldsymbol{\mu}_{X,Y}\right)_Z = \boldsymbol{\alpha}^{-1}_{X,Y,Z}$$

In particular, this defines a left $\mathcal{C}$-action on $\mathcal{C}$, called the **left regular representation** *of $\mathcal{C}$.*

Proof. F is clearly a faithful functor. In this case, the strong monoidal functor axioms become the following diagrams:

$$\begin{tikzcd} X \otimes Y \arrow[r, "\boldsymbol{\lambda}^{-1}_{X\otimes Y}"] \arrow[d, "\mathrm{id}"'] & I \otimes (X \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{I,X,Y}"] \\ X \otimes Y & (I \otimes X) \otimes Y \arrow[l, "\boldsymbol{\lambda}_X \otimes \mathrm{id}_Y"] \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y \arrow[r, "\mathrm{id}_X \otimes \boldsymbol{\lambda}^{-1}_Y"] \arrow[d, "\mathrm{id}"'] & X \otimes (I \otimes Y) \arrow[d, "\boldsymbol{\alpha}^{-1}_{X,I,Y}"] \\ X \otimes Y & (X \otimes I) \otimes Y \arrow[l, "\boldsymbol{\rho}_X \otimes \mathrm{id}_Y"] \end{tikzcd}$$

$$\begin{tikzcd} & W \otimes (X \otimes (Y \otimes Z)) \arrow[dl, "\boldsymbol{\alpha}^{-1}_{W,X,Y\otimes Z}"'] \arrow[dr, "\mathrm{id}_W \otimes \boldsymbol{\alpha}^{-1}_{X,Y,Z}"] & \\ (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\boldsymbol{\alpha}^{-1}_{W\otimes X,Y,Z}"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[d, "\boldsymbol{\alpha}^{-1}_{W,X\otimes Y,Z}"] \\ ((W \otimes X) \otimes Y) \otimes Z \arrow[rr, "\boldsymbol{\alpha}_{W,X,Y} \otimes \mathrm{id}_Z"'] & & (W \otimes (X \otimes Y)) \otimes Z \end{tikzcd}$$

The left square commutes by the coherence theorem, while the right square and the pentagon are seen to be immediate consequences of the triangle and pentagon axioms, respectively. ■

Proposition B.2.4. *Let $\mathcal{V}$ be a monoidal category and let $\mathcal{C}$ be a category.*

- *If $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $X \oslash (-)$ has a right adjoint $(-) \multimapinv X$, then the functor $\multimapinv : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ defines a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$.*

- *If $\obslash : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ defines a right $\mathcal{V}$-action on $\mathcal{C}$ such that, for each object X in $\mathcal{V}$, the endofunctor $(-) \obslash X$ has a right adjoint $X \multimap (-)$, then the functor $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ defines a left $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$.*

- *If* $\multimap\!\!-$: $\mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ *defines a right* $\mathcal{V}^{\mathrm{op}}$*-action on* $\mathcal{C}$ *such that, for each object* X *in* $\mathcal{V}$*, the endofunctor* $X \multimap\!\!- (-)$ *has a left adjoint* $X \oslash (-)$*, then the functor* $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$ *defines a left* $\mathcal{V}$*-action on* $\mathcal{C}$*.*
- *If* $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *defines a left* $\mathcal{V}^{\mathrm{op}}$*-action on* $\mathcal{C}$ *such that, for each object* X *in* $\mathcal{V}$*, the endofunctor* $X \multimap (-)$ *has a left adjoint* $(-) \oslash X$*, then the functor* $\oslash : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$ *defines a right* $\mathcal{V}$*-action on* $\mathcal{C}$*.*

Proof. The four statements are related by applying $(-)^{\mathrm{op}}$ and $(-)^{\mathrm{rev}}$ at the appropriate points, so it suffices to prove the first claim.

First, note that $\multimap\!\!-$ is indeed a functor $\mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$, by the parameter theorem for adjunctions.[5] Let $\mathrm{ev}_{X,A} : X \oslash (A \multimap\!\!- X) \to A$ denote the component of the counit of the adjunction $X \oslash (-) \dashv (-) \multimap\!\!- X$ at an object A in $\mathcal{C}$. For each pair of objects X and Y in $\mathcal{V}$ and each object A in $\mathcal{C}$, we define the morphism $\left(\boldsymbol{\delta}_{X,Y}\right)_A : A \multimap\!\!- (X \otimes Y) \to (A \multimap\!\!- X) \multimap\!\!- Y$ to be the right adjoint transpose of $\mathrm{ev}_{X \otimes Y, A} \circ \left(\boldsymbol{\mu}_{X,Y}\right)_{(A \multimap\!\!- X) \multimap\!\!- Y}$, and for each A, we define $\boldsymbol{\varepsilon}_A : A \multimap\!\!- I \to A$ to be the composite $\mathrm{ev}_{I,A} \circ \boldsymbol{\eta}_{A \multimap\!\!- I}$. These are clearly natural in A, and it is straightforward to check that $\boldsymbol{\delta}_{X,Y}$ is also natural in X and Y. One may then use the calculus of mates to show that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\delta}_{X,Y}$ are natural isomorphisms and that they satisfy the axioms for making the right exponential transpose of $\multimap\!\!- : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{C}$ into a strong monoidal functor $\mathcal{V}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{C}]^{\mathrm{rev}}$, i.e. a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$. ∎

Example B.2.5. $\mathcal{V}$ is a left-closed (resp. right-closed) monoidal category if and only if the left (resp. right) self-action of $\mathcal{V}$ has a parametrised right adjoint as in the proposition, and the right adjoint right (resp. left) $\mathcal{V}^{\mathrm{op}}$-action so obtained is precisely a left (resp. right) internal hom functor.

Definition B.2.6. Let $\mathcal{V}$ be a monoidal category and let $\mathcal{C}$ be a category.

- A **right $\mathcal{V}$-hom system** for $\mathcal{C}$ consists of a left $\mathcal{V}$-action $\oslash : \mathcal{V} \times \mathcal{C} \to \mathcal{C}$, a functor $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, and a right $\mathcal{V}^{\mathrm{op}}$-action $\multimap\!\!- : \mathcal{C} \times \mathcal{V}^{\mathrm{op}} \to \mathcal{V}$ together with natural bijections of the types below,

$$\mathcal{V}(X, \underline{\mathcal{C}}(A, B)) \cong \mathcal{C}(A, B \multimap\!\!- X)$$
$$\mathcal{C}(X \oslash A, B) \cong \mathcal{C}(A, B \multimap\!\!- X)$$
$$\mathcal{C}(X \oslash A, B) \cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B))$$

  where X varies over the objects in $\mathcal{V}$, and A and B vary over the objects in $\mathcal{C}$, such that the cyclic composition of the three bijections is the identity.

[5] See [CWM, Ch. IV, § 7].

- A **left $\mathcal{V}$-hom system** for $\mathcal{C}$ consists of a right $\mathcal{V}$-action $\oslash : \mathcal{C} \times \mathcal{V} \to \mathcal{C}$, a functor $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, and a left $\mathcal{V}^{\mathrm{op}}$-action $\multimap : \mathcal{V}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$, together with natural bijections of the types below,

$$\begin{aligned} \mathcal{V}(X, \underline{\mathcal{C}}(A, B)) &\cong \mathcal{C}(A, X \multimap B) \\ \mathcal{C}(A \oslash X, B) &\cong \mathcal{C}(A, X \multimap B) \\ \mathcal{C}(A \oslash X, B) &\cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B)) \end{aligned}$$

  where X varies over the objects in $\mathcal{V}$, and A and B vary over the objects in $\mathcal{C}$, such that the cyclic composition of the three bijections is the identity.

Example B.2.7. If $\mathcal{V}$ is a biclosed monoidal category with right internal hom functor $\mathcal{H}om$ and left internal hom functor $\pitchfork$, then $(\otimes, \pitchfork, \mathcal{H}om)$ is a left $\mathcal{V}$-hom system for $\mathcal{V}$:

$$\begin{aligned} \mathcal{V}(Y, X \pitchfork Z) &\cong \mathcal{V}(X, \mathcal{H}om(Y, Z)) \\ \mathcal{V}(X \otimes Y, Z) &\cong \mathcal{V}(X, \mathcal{H}om(Y, Z)) \\ \mathcal{V}(X \otimes Y, Z) &\cong \mathcal{V}(Y, X \pitchfork Z) \end{aligned}$$

Example B.2.8. If $\mathcal{C}$ is a locally small category that has products and coproducts for all small families of objects, then $\mathcal{C}$ admits a left **Set**-action and a right $\mathbf{Set}^{\mathrm{op}}$-action that are related by the following adjunctions:

$$\begin{aligned} \mathbf{Set}(X, \mathcal{C}(A, B)) &\cong \mathcal{C}(A, B \multimapinv X) \\ \mathcal{C}(X \oslash A, B) &\cong \mathcal{C}(A, B \multimapinv X) \\ \mathcal{C}(X \oslash A, B) &\cong \mathbf{Set}(X, \mathcal{C}(A, B)) \end{aligned}$$

(The adjointness claim was checked in proposition A.6.9, and the coherence laws are straightforwardly verified.) Thus, $(\oslash, \mathcal{C}, \multimapinv)$ is a right **Set**-hom system for $\mathcal{C}$.

Theorem B.2.9. *Let $\mathcal{V}$ be a monoidal category and let $\mathcal{C}$ be a category.*

(i) *If $\oslash$ is a left $\mathcal{V}$-action on $\mathcal{C}$ and $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is a functor with natural bijections of the form below,*

$$\mathcal{C}(X \oslash A, B) \cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B))$$

then $\underline{\mathcal{C}}$ is the hom functor of a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ whose underlying ordinary category is isomorphic to $\mathcal{C}$.

(ii) *If $\multimapinv$ is a right $\mathcal{V}^{\mathrm{op}}$-action on $\mathcal{C}$ and $\underline{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ is a functor with natural bijections of the form below,*

$$\mathcal{C}(A, B \multimapinv X) \cong \mathcal{V}(X, \underline{\mathcal{C}}(A, B))$$

then $\underline{\mathcal{C}}$ is the hom functor of a $\mathcal{V}$-enriched category $\underline{\mathcal{C}}$ whose underlying ordinary category is isomorphic to $\mathcal{C}$.

Proof. (i). The natural isomorphism $A \cong I \oslash A$ induces a family of bijections

$$\mathcal{C}(A, B) \cong \mathcal{V}(I, \underline{\mathcal{C}}(A, B))$$

natural in A and B, so we have a morphism $e_A : I \to \underline{\mathcal{C}}(A, A)$ in $\mathcal{V}$ for every object A in $\mathcal{C}$ corresponding to $\mathrm{id}_A : A \to A$ in $\mathcal{C}$. Let $\mathrm{ev}_{A,B} : \underline{\mathcal{C}}(A, B) \oslash A \to B$ be the component at B of the counit of the adjunction $(-) \oslash A \dashv \underline{\mathcal{C}}(A, -)$, and define $c_{A,B,C} : \underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B) \to \underline{\mathcal{C}}(A, C)$ to be the right adjoint transpose of the following morphism in $\mathcal{C}$:

$$\mathrm{ev}_{B,C} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(B,C)} \oslash \mathrm{ev}_{A,B}\right) \circ \left(\boldsymbol{\mu}_{\underline{\mathcal{C}}(B,C),\underline{\mathcal{C}}(A,B)}\right)_A^{-1} : (\underline{\mathcal{C}}(B, C) \otimes \underline{\mathcal{C}}(A, B)) \oslash A \to C$$

By definition, the left adjoint transpose of e_B is $\boldsymbol{\eta}_B^{-1}$, so the left and right unit axioms are satisfied:

$$c_{A,B,B} \circ \left(e_B \otimes \mathrm{id}_{\underline{\mathcal{C}}(A,B)}\right) = \boldsymbol{\lambda}_{\underline{\mathcal{C}}(A,B)}$$
$$c_{B,B,C} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(B,C)} \otimes e_B\right) = \boldsymbol{\rho}_{\underline{\mathcal{C}}(B,C)}$$

One may similarly verify the associativity axiom:

$$c_{A,B,D} \circ \left(c_{B,C,D} \otimes \mathrm{id}_{\underline{\mathcal{C}}(A,B)}\right) = c_{A,C,D} \circ \left(\mathrm{id}_{\underline{\mathcal{C}}(C,D)} \otimes c_{A,B,C}\right) \circ \boldsymbol{\alpha}_{\underline{\mathcal{C}}(C,D),\underline{\mathcal{C}}(B,C),\underline{\mathcal{C}}(A,B)}$$

(ii). By duality and symmetry, $\multimapinv$ induces a left $\mathcal{V}^{\mathrm{rev}}$-action on $\mathcal{C}^{\mathrm{op}}$, so we may construct a $\mathcal{V}^{\mathrm{rev}}$-enriched category $\underline{\mathcal{C}^{\mathrm{op}}}$ using claim (i) and thence a $\mathcal{V}$-enriched category $\underline{\mathcal{C}} = \left(\underline{\mathcal{C}^{\mathrm{op}}}\right)^{\mathrm{op}}$. ■

Definition B.2.10. Let $\mathcal{V}$ be a monoidal category, and let $\mathcal{C}$ and $\mathcal{D}$ be categories with left $\mathcal{V}$-actions. A **$\mathcal{V}$-strength** for a functor $F : \mathcal{C} \to \mathcal{D}$ is a natural transformation $\sigma : (-) \oslash F(-) \Rightarrow F(- \oslash -)$ making these diagrams commute:

$$\begin{tikzcd}
 & FA \arrow[dl, "\eta_{FA}"'] \arrow[dr, "F\eta_A"] & \\
I \oslash FA \arrow[rr, "\sigma_{I,A}"'] & & F(I \oslash A)
\end{tikzcd}$$

$$\begin{tikzcd}
 & X \oslash (Y \oslash FA) \arrow[dl, "\mathrm{id}_X \oslash \sigma_{Y,A}"'] \arrow[dr, "(\mu_{X,Y})_A"] & \\
X \oslash F(Y \oslash A) \arrow[d, "\sigma_{X,Y\oslash A}"'] & & (X \otimes Y) \oslash FA \arrow[d, "\sigma_{X\otimes Y,A}"] \\
F(X \oslash (Y \oslash A)) \arrow[rr, "F(\mu_{X,Y})_A"'] & & F((X \otimes Y) \oslash A)
\end{tikzcd}$$

A $\mathcal{V}$**-strong functor** is a functor equipped with a $\mathcal{V}$-strength.

Definition B.2.11. Let $\mathcal{V}$ be a monoidal category, let C and $\mathcal{D}$ be categories with left $\mathcal{V}$-actions, and let $F, F' : C \to \mathcal{D}$ be functors with $\mathcal{V}$-strengths σ and σ' respectively. A $\mathcal{V}$**-strong natural transformation** $\varphi : F \Rightarrow F'$ is a natural transformation making the following diagram commute:

$$\begin{tikzcd}
X \oslash FA \arrow[r, "\sigma_{X,A}"] \arrow[d, "\mathrm{id}_X \oslash \varphi_A"'] & F(X \oslash A) \arrow[d, "\varphi_{X\oslash A}"] \\
X \oslash F'A \arrow[r, "\sigma'_{X,A}"'] & F'(X \oslash A)
\end{tikzcd}$$

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