Notes on homotopical algebra

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Preface

These notes are intended as a kind of annotated index to the various standard references in homotopical algebra: the focus is on definitions and statements of results, *not* proofs.

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FOUNDATIONS

0.1 Set theory

In category theory it is often convenient to invoke a certain set-theoretic device commonly known as a 'Grothendieck universe', but we shall say simply 'universe', so as to simplify exposition and proofs by eliminating various circumlocutions involving cardinal bounds, proper classes etc.

Definition 0.1.1. A **pre-universe** is a set $\mathbf{U}$ satisfying these axioms:

1. If $x \in y$ and $y \in \mathbf{U}$, then $x \in \mathbf{U}$.
2. If $x \in \mathbf{U}$ and $y \in \mathbf{U}$ (but not necessarily distinct), then $\{x, y\} \in \mathbf{U}$.
3. If $x \in \mathbf{U}$, then $\mathscr{P}(x) \in \mathbf{U}$, where $\mathscr{P}(x)$ denotes the set of all subsets of x.
4. If $x \in \mathbf{U}$ and $f : x \to \mathbf{U}$ is a map, then $\bigcup_{i \in x} f(i) \in \mathbf{U}$.

A **universe** is a pre-universe $\mathbf{U}$ with this additional property:

5. $\omega \in \mathbf{U}$, where ω is the set of all finite (von Neumann) ordinals.

Example 0.1.2. The empty set is a pre-universe, and with very mild assumptions, so is the set **HF** of all hereditarily finite sets.

¶ **0.1.3.** The notion of universe makes sense in any material set theory, but their existence must be postulated. We adopt the following:

- **Grothendieck–Verdier universe axiom.** For each set x, there exists a universe $\mathbf{U}$ with $x \in \mathbf{U}$.

For definiteness, we may take our base theory to be Mac Lane set theory, which is a weak subsystem of Zermelo–Fraenkel set theory with choice (ZFC). Readers interested in the details of Mac Lane set theory are referred to [Mathias, 2001], but in practice as long as one is working at all times *inside some universe*, one may as well be working in ZFC. Indeed:

Proposition 0.1.4. *With the assumptions of Mac Lane set theory, any universe is a transitive model of ZFC.*

Proof. Let $\mathbf{U}$ be a universe. By definition, $\mathbf{U}$ is a transitive set containing pairs, power sets, unions, and ω, so the axioms of extensionality, empty set, pairs, power sets, unions, choice, and infinity are all automatically satisfied. We must show that the axiom schemas of separation and replacement are also satisfied, and in fact it is enough to check that replacement is valid; but this is straightforward using axioms 2 and 4. ■

Definition 0.1.5. Let $\mathbf{U}$ be a pre-universe. A **U-set** is a member of $\mathbf{U}$, a **U-class** is a subset of $\mathbf{U}$, and a **proper U-class** is a $\mathbf{U}$-class that is not a $\mathbf{U}$-set.

Lemma 0.1.6. *A* $\mathbf{U}$*-class* X *is a* $\mathbf{U}$*-set if and only if there exists a* $\mathbf{U}$*-class* Y *such that* $X \in Y$. ■

Proposition 0.1.7. *If* $\mathbf{U}$ *is a universe, then the collection of* $\mathbf{U}$*-classes is a transitive model of Morse–Kelley class–set theory (MK), and so is a transitive model of von Neumann–Bernays–Gödel class–set theory (NBG) in particular.* ■

Definition 0.1.8. A **U-small category** is a category $\mathbb{C}$ such that $\operatorname{ob}\mathbb{C}$ and $\operatorname{mor}\mathbb{C}$ are $\mathbf{U}$-sets. A **locally U-small category** is a category $\mathcal{D}$ satisfying these conditions:

- $\operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{D}$ are $\mathbf{U}$-classes, and
- for all objects x and y in $\mathcal{D}$, the hom-set $\mathcal{D}(x, y)$ is a $\mathbf{U}$-set.

An **essentially U-small category** is a category $\mathcal{D}$ for which there exist a $\mathbf{U}$-small category $\mathbb{C}$ and a functor $\mathbb{C} \to \mathcal{D}$ that is fully faithful and essentially surjective on objects.

Proposition 0.1.9. *If* $\mathbb{C}$ *is a* $\mathbf{U}$*-small category and* $\mathcal{D}$ *is a locally* $\mathbf{U}$*-small category, then the functor category* $[\mathbb{C}, \mathcal{D}]$ *is locally* $\mathbf{U}$*-small.*

Proof. Strictly speaking, this depends on the set-theoretic implementation of ordered pairs, categories, functors, etc., but at the very least $[\mathbb{C}, \mathcal{D}]$ should be isomorphic to a locally $\mathbf{U}$-small category.

In the context of $[\mathbb{C}, \mathcal{D}]$, we may regard functors $\mathbb{C} \to \mathcal{D}$ as being the pair consisting of the *graph* of the object map $\operatorname{ob} \mathbb{C} \to \operatorname{ob} \mathcal{D}$ and the *graph* of the morphism map $\operatorname{mor} \mathbb{C} \to \operatorname{mor} \mathcal{D}$, and these are $\mathbf{U}$-sets by the $\mathbf{U}$-replacement axiom. Similarly, if F and G are objects in $[\mathbb{C}, \mathcal{D}]$, then we may regard a natural transformation $\alpha : F \Rightarrow G$ as being the triple (F, G, A), where A is the set of all pairs $\left(c, \alpha_c\right)$. ■

One complication introduced by having multiple universes concerns the existence of (co)limits.

Theorem 0.1.10 (Freyd)**.** *Let $\mathcal{C}$ be a category and let κ be a cardinal such that $|\operatorname{mor} \mathcal{C}| \le \kappa$. If $\mathcal{C}$ has products for families of size κ, then any two parallel morphisms in $\mathcal{C}$ must be equal.*

Proof. Suppose, for a contradiction, that $f, g : X \to Y$ are distinct morphisms in $\mathcal{C}$. Let Z be the product of κ-many copies of Y in $\mathcal{C}$. The universal property of products implies there are at least 2^κ-many distinct morphisms $X \to Z$; but $\mathcal{C}(X, Z) \subseteq \operatorname{mor} \mathcal{C}$, so this is an absurdity. ■

Definition 0.1.11. Let $\mathbf{U}$ be a pre-universe. A **$\mathbf{U}$-complete** (resp. **$\mathbf{U}$-cocomplete**) **category** is a category $\mathcal{C}$ with the following property:

- For all $\mathbf{U}$-small categories $\mathbb{D}$ and all diagrams $A : \mathbb{D} \to \mathcal{C}$, a limit (resp. colimit) of A exists in $\mathcal{C}$.

We may instead say $\mathcal{C}$ has all **finite limits** (resp. **finite colimits**) in the special case $\mathbf{U} = \mathbf{HF}$.

Proposition 0.1.12. *Let $\mathcal{C}$ be a category and let $\mathbf{U}$ be a non-empty pre-universe. The following are equivalent:*

(i) *$\mathcal{C}$ is $\mathbf{U}$-complete.*

(ii) *$\mathcal{C}$ has all finite limits and products for all families of objects indexed by a $\mathbf{U}$-set.*

(iii) *For each* **U**-*small category* $\mathbb{D}$*, there exists an adjunction*

$$\Delta \dashv \varprojlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$$

where ΔX *is the constant functor with value* X*.*

Dually, the following are equivalent:

(i′) $\mathcal{C}$ *is* **U**-*cocomplete.*

(ii′) $\mathcal{C}$ *has all finite colimits and coproducts for all families of objects indexed by a* **U**-*set.*

(iii′) *For each* **U**-*small category* $\mathbb{D}$*, there exists an adjunction*

$$\varinjlim_{\mathbb{D}} \dashv \Delta : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$$

where ΔX *is the constant functor with value* X*.*

Proof. This is a standard result; but we remark that we do require a sufficiently powerful form of the axiom of choice to pass from (ii) to (iii). □

¶ **0.1.13.** In the **explicit universe convention**, the words ‘set’, ‘class’, etc. have their usual meanings, and in the **implicit universe convention**, these instead abbreviate ‘**U**-set’, ‘**U**-class’, etc. for a fixed (but arbitrary) universe **U**. However, the word ‘category’ always refers to a category that is contained in *some* universe, which may or may not be locally **U**-small, and we shall use the word ‘ensemble’ to refer to sets which may or may not be in **U**. In subsequent chapters, the implicit universe convention should be assumed *unless otherwise stated.*

We now recall some definitions and results about ordinal and cardinal numbers. Readers familiar with axiomatic set theory may wish to skip ahead.

Definition 0.1.14. A **von Neumann ordinal** is a set α with the following properties:

- If $x \in y$ and $y \in \alpha$, then $x \in \alpha$.
- The binary relation $\in$ is strict total ordering of α.
- If S is a subset of α such that

- $\varnothing \in S$,
- If $\beta \in S$ and $\beta \cup \{\beta\} \in \alpha$, then $\beta \cup \{\beta\} \in S$.
- If $T \subseteq S$, then $\bigcup T \in S$.

then $S = \alpha$.

We identify 0 with the von Neumann ordinal $\varnothing$, and by induction, we identify the natural number $n + 1$ with the von Neumann ordinal $\{0, \dots, n\}$.

Proposition 0.1.15.

(i) *If α is a von Neumann ordinal, then every member of α is an initial segment of α and is in particular a von Neumann ordinal.*

(ii) *If α is a von Neumann ordinal, so is $\alpha \cup \{\alpha\}$. (This is usually denoted by $\alpha + 1$ and called the* **successor** *of α.)*

(iii) *The union of a set S of von Neumann ordinals is another von Neumann ordinal. (This is usually denoted by* $\sup S$ *and called the* **supremum** *of S.)*

(iv) *If $\mathbf{U}$ is a pre-universe and $\kappa(\mathbf{U})$ is the set of von Neumann ordinals in $\mathbf{U}$, then $\kappa(\mathbf{U})$ a von Neumann ordinal, but $\kappa(\mathbf{U}) \notin \mathbf{U}$.*

Proof. Claims (i) – (iii) are all easy, and claim (iv) is Burali-Forti's paradox. □

Theorem 0.1.16 (Classification of well-orderings)**.**

(i) *In Zermelo–Fraenkel set theory, every well-ordered set is isomorphic to a unique von Neumann ordinal.*

(ii) *In Mac Lane set theory, if $\mathbf{U}$ is a pre-universe and X is a well-ordered set in $\mathbf{U}$, then X is isomorphic to a unique von Neumann ordinal in $\mathbf{U}$.*

Proof. Claim (i) is a standard result in axiomatic set theory, and claim (ii) is an obvious corollary. □

Definition 0.1.17. A **transitive set** is a set T such that, given $x \in y$, if $y \in T$, then $x \in T$ as well. The **transitive closure** of a set X is a set $\operatorname{tcl}(X)$ such that, for all transitive sets T with $X \subseteq T$, we have $\operatorname{tcl}(X) \subseteq T$ as well.

Lemma 0.1.18. *In Mac Lane set theory, every set has a unique transitive closure.*

Proof. One of the axioms of Mac Lane set theory states that every set X is a member of some transitive set T, and so $X \subseteq T$. Clearly, the intersection of any family of transitive sets containing X is again a transitive set containing X, so $\operatorname{tcl}(X)$ exists and is unique so long as there is at least one transitive set containing X. ■

Definition 0.1.19. A **partial rank function** from a transitive set T to a well-ordered set W is a partial function $\rho : T \to W$ with these properties:

- If $\emptyset \in T$, then $\rho(\emptyset)$ is the least element of W.
- If $y \in T$ and $\rho(x)$ is defined for all $x \in y$, then

  $$\rho(y) = \min \{w \in W \mid \forall x \in y.\, \rho(x) < w\}$$

  provided the RHS is defined.
- Otherwise $\rho(y)$ is undefined.

A **total rank function** is a partial rank function that is defined on its entire domain. The **rank** of a set X, if it exists, the least von Neumann ordinal $\operatorname{rank}(X)$ for which there exists a total rank function $\operatorname{tcl}(X) \to \operatorname{rank}(X)$.

Proposition 0.1.20. *In Mac Lane set theory:*

(i) *If T is a transitive set and W is a well-ordered set, then there is a unique partial rank function $\rho : T \to W$.*

(ii) *If $\mathbf{U}$ is a pre-universe and $x \in \mathbf{U}$, then* $\operatorname{rank}(x)$ *can be defined by a Δ_0-formula with $\mathbf{U}$ as a parameter, and for each von Neumann ordinal α in $\mathbf{U}$, the set*

$$\mathbf{V}_\alpha = \{x \in \mathbf{U} \mid \operatorname{rank}(x) < \alpha\}$$

is a $\mathbf{U}$-set.

(iii) *Assuming the Grothendieck–Verdier universe axiom,* $\operatorname{rank}(x)$ *is defined for all x.*

Proof. (i). This is a straightforward application of well-founded induction.

(ii). $\mathbf{U}$ is a transitive set and the set $\kappa(\mathbf{U})$ of all von Neumann ordinals in $\mathbf{U}$ is well-ordered by inclusion, so by claim (i) there is a partial rank function ρ :

$\mathbf{U} \rightharpoonup \kappa(\mathbf{U})$. ZFC proves that every set has a rank, so ρ must in fact be a total rank function; hence, for any $x \in \mathbf{U}$, $\operatorname{rank}(x)$ is defined. It is clear that ρ can be defined by a Δ_0-formula with only $\mathbf{U}$ as a parameter, and the rest of the claim follows.

(iii). Obvious, assuming claim (ii). □

Definition 0.1.21. Two sets are **equinumerous** if there exists a bijection between them. A **cardinality class** in a pre-universe $\mathbf{U}$ is an equivalence class under the relation of equinumerosity.

Definition 0.1.22. An **ℵ-number** is an infinite von Neumann ordinal κ such that, for any von Neumann ordinal λ such that κ and λ are equinumerous, we have $\kappa \subseteq \lambda$.

Example 0.1.23. The first infinite von Neumann ordinal, i.e. $\omega = \{0, 1, 2, \ldots\}$, is the ℵ-number $\aleph_0$.

Theorem 0.1.24 (Classification of cardinalities)**.**

(i) *In Zermelo–Fraenkel set theory, for every well-ordered infinite set X, there exists a unique ℵ-number κ such that X and κ are equinumerous.*

(ii) *In Zermelo–Fraenkel set theory with the axiom of choice, the same is true for any infinite set whatsoever.*

(iii) *In Mac Lane set theory, if $\mathbf{U}$ is a universe and X is an infinite set in $\mathbf{U}$, then there exists a unique ℵ-number κ in the cardinality class of X.*

(iv) *In Mac Lane set theory with the Grothendieck–Verdier universe axiom, if $\mathbf{U}$ is a pre-universe and κ is an ℵ-number not in $\mathbf{U}$, then the cardinality of $\mathbf{U}$ is at most κ.*

Proof. Claim (i) is a standard fact, whence claims (ii) and (iii), by the well-ordering theorem. Claim (iv) can be proven using axiom 4 for pre-universes. □

¶ **0.1.25.** Henceforth, we identify the cardinality class of a finite set with the unique von Neumann ordinal contained in that class, and similarly we identify the cardinality class of an infinite set with the unique ℵ-number in that class. These are the **cardinal numbers**.

Definition 0.1.26. A **cofinal subset** of a partially-ordered set X is a subset $Y \subseteq X$ such that, for all x in X, there exists some y in Y such that $x \leq y$. A **regular cardinal number** is an $\aleph$-number κ such that any cofinal subset of κ has cardinality equal to κ. A **singular cardinal number** is an $\aleph$-number that is not regular.

The following helps to motivate the definition of regular cardinal numbers.

Definition 0.1.27. Let $\mathbf{U}$ be a pre-universe. An **arity class** in $\mathbf{U}$ is a $\mathbf{U}$-class K of cardinal numbers satisfying the following conditions:

- $1 \in K$.
- If $\kappa \in K$ and $\lambda : \kappa \to K$ is a function, then the cardinal sum $\sum_{\alpha \in \kappa} \lambda(\alpha)$ is also in K.
- If $\kappa \in K$ and $\lambda : \kappa \to \mathbf{U}$ is a function such that each $\lambda(\alpha)$ is a cardinal number and $\sum_{\alpha \in \kappa} \lambda(\alpha) \in K$, then $\lambda(\alpha) \in K$ as well.

Theorem 0.1.28 (Classification of arity classes)**.** *In Mac Lane set theory, if K is an arity class in a pre-universe $\mathbf{U}$, then K must be either*

- $\{1\}$, *or*
- $\{0, 1\}$, *or*
- *of the form $\{\lambda \in \mathbf{U} \mid \lambda$ is a cardinal number and $\lambda < \kappa\}$ for some regular cardinal number κ (possibly not in $\mathbf{U}$).*

Proof. The notion of arity class and this result are due to Shulman [2012]. □

Definition 0.1.29. Let κ be a regular cardinal number. A **κ-small category** is a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ has cardinality *less than* κ. A **finite category** is an $\aleph_0$-small category, i.e. a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ is finite. A **finite diagram** (resp. **κ-small diagram**, **U-small diagram**) in a category $\mathcal{C}$ is a functor $\mathbb{D} \to \mathcal{C}$ where $\mathbb{D}$ is a finite (resp. κ-small, $\mathbf{U}$-small) category.

Theorem 0.1.30. *Let $\mathbf{U}$ be a pre-universe, and let $\mathbf{U}^+$ be a universe with $\mathbf{U} \in \mathbf{U}^+$. Let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathbf{Set}^+$ be the category of $\mathbf{U}^+$-sets.*

(i) *If $X : \mathbb{D} \to \mathbf{Set}$ is a $\mathbf{U}$-small diagram, then there exist a limit and a colimit for X in $\mathbf{Set}$.*

(ii) *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *is fully faithful and preserves limits and colimits for all* $\mathbf{U}$*-small diagrams.*

Proof. One can construct products, equalisers, coproducts, coequalisers, and hom-sets in a completely explicit way, making the preservation properties obvious. □

Corollary 0.1.31. *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Corollary 0.1.32. *For any* $\mathbf{U}$*-small category* $\mathbb{C}$*:*

(i) *The functor category* $[\mathbb{C}, \mathbf{Set}]$ *is* $\mathbf{U}$*-complete and* $\mathbf{U}$*-cocomplete, with limits and colimits for* $\mathbf{U}$*-small diagrams computed componentwise in* $\mathbf{Set}$*.*

(ii) *The inclusion* $[\mathbb{C}, \mathbf{Set}] \hookrightarrow [\mathbb{C}, \mathbf{Set}^+]$ *is fully faithful and both preserves and reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Definition 0.1.33. An **strongly inaccessible cardinal number** is a regular cardinal number κ such that, for all sets X of cardinality less than κ, the power set $\mathscr{P}(X)$ is also of cardinality less than κ.

Example 0.1.34. $\aleph_0$ is a strongly inaccessible cardinal number and is the only one that can be proven to exist in ZFC. It is more conventional to exclude $\aleph_0$ from the definition of strongly inaccessible cardinal number by demanding that they be uncountable.

Proposition 0.1.35. *In Mac Lane set theory:*

(i) *If* $\mathbf{U}$ *is a non-empty pre-universe, then there exists a strongly inaccessible cardinal number* κ *such that the members of* $\mathbf{U}$ *are all the sets of rank less than* κ*. Moreover, this* κ *is the rank and the cardinality of* $\mathbf{U}$*.*

(ii) *If* $\mathbf{U}$ *is a universe and* κ *is a strongly inaccessible cardinal number such that* $\kappa \in \mathbf{U}$*, then there exists a* $\mathbf{U}$*-set* $\mathbf{V}_\kappa$ *whose members are all the sets of rank less than* κ*, and* $\mathbf{V}_\kappa$ *is a pre-universe.*

(iii) *If* $\mathbf{U}$ *and* $\mathbf{U}'$ *are pre-universes, then either* $\mathbf{U} \subseteq \mathbf{U}'$ *or* $\mathbf{U}' \subseteq \mathbf{U}$*; and if* $\mathbf{U} \subsetneqq \mathbf{U}'$*, then* $\mathbf{U} \in \mathbf{U}'$*.*

Proof. (i). Let κ be the set of all von Neumann ordinals in $\mathbf{U}$; this exists by Δ_0-separation applied to $\mathbf{U}$. Since $\mathbf{U}$ is closed under power sets and internally-indexed unions, κ must be a strongly inaccessible cardinal.

We can construct the set all of $\mathbf{U}$-sets of rank less than κ using transfinite recursion on κ as follows: starting with $\mathbf{V}_0 = \emptyset$, for each von Neumann ordinal α less than κ, we set $\mathbf{V}_{\alpha+1} = \mathscr{P}(\mathbf{V}_\alpha)$, and for each ordinal λ that is not a successor, we set $\mathbf{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha$. The well-foundedness of $\in$ (restricted to $\mathbf{U}$) implies that in fact this must be all of $\mathbf{U}$.

Clearly, every set of rank less than κ is in fact a $\mathbf{U}$-set, and $\mathbf{U}$ is itself a set of rank κ. The cardinality of $\mathbf{U}$ is also κ, since κ is a regular cardinal number and any cardinal number less than κ is a member of $\mathbf{U}$.

(ii). We may construct $\mathbf{V}_\kappa$ using the same method as in (i). By construction $\mathbf{V}_\kappa$ satisfies axiom 1; since κ is infinite, $\mathbf{V}_\kappa$ satisfies axioms 2 and 3; and since κ is strongly inaccessible, $\mathbf{V}_\kappa$ satisfies axiom 4. Thus $\mathbf{V}_\kappa$ is a pre-universe.

(iii). Again, let κ be the rank of $\mathbf{U}$. If $\kappa \in \mathbf{U}'$ then we can show by transfinite induction that $\mathbf{V}_\kappa \in \mathbf{U}'$ and so $\mathbf{U} \subsetneqq \mathbf{U}'$; else we must have $\mathbf{U}' \subseteq \mathbf{V}_\kappa = \mathbf{U}$. ■

0.2 Accessibility and ind-completions

Prerequisites. § 0.1.

A classical technology for controlling size problems in category theory, due to Gabriel and Ulmer [1971], is the notion of accessibility. Though we make use of universes, accessibility remains important and is a crucial tool in verifying the stability of various universal constructions when one passes from one universe to a larger one.

Definition 0.2.1. Let κ be a regular cardinal. A **κ-filtered category** is a category $\mathcal{J}$ satisfying these conditions:

- $\mathcal{J}$ is **inhabited**, i.e. there exists an object in $\mathcal{J}$.
- If λ is a cardinal number strictly less than κ and S is a subset of $\operatorname{ob}\mathcal{J}$ of cardinality λ, then there exist an object j and arrows $f_i : i \to j$ for each object i in S.
- If $f, g : i \to j$ are a pair of parallel arrows in $\mathcal{J}$, then there exist an object k and an arrow $h : j \to k$ such that $h \circ f = h \circ g$.

A κ**-directed preorder** is a preordered set that is κ-filtered when considered as a category; note that the third condition is then vacuous. A κ**-filtered diagram** (resp. κ**-directed diagram**) in a category C is a functor $\mathbb{D} \to C$ such that $\mathbb{D}$ is a κ-filtered category (resp. κ-directed preorder). It is conventional to omit mention of κ when $\kappa = \aleph_0$.

Example 0.2.2. The category with one object $*$ and only one non-trivial arrow f is filtered if and only if $f = f \circ f$.

Example 0.2.3. Let X be any set. The set of all finite subsets of X, partially ordered by inclusion, is a directed preorder. More generally, if κ is any regular cardinal, then the set of all subsets of X with cardinality strictly less than κ is a κ-directed preorder.

Theorem 0.2.4. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* κ *be any regular cardinal. Given a* $\mathbf{U}$*-small category* $\mathbb{D}$*, the following are equivalent:*

(i) $\mathbb{D}$ *is a* κ*-filtered category.*

(ii) *The functor* $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathbf{Set}] \to \mathbf{Set}$ *preserves limits for all diagrams that are both* κ*-small and* $\mathbf{U}$*-small.*

Proof. The claim (i) $\Rightarrow$ (ii) is very well known, and the converse is an exercise in using the Yoneda lemma and manipulating limits and colimits for diagrams of representable functors. □

Definition 0.2.5. Let κ be a regular cardinal in a universe $\mathbf{U}^+$ and let $\mathbf{U}$ be a pre-universe with $\mathbf{U} \subseteq \mathbf{U}^+$. A $(\kappa, \mathbf{U})$**-compact object** in a locally $\mathbf{U}^+$-small category C is an object A such that the representable functor $C(A, -) : C \to \mathbf{Set}^+$ preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams. A κ**-compact object** is one that is $(\kappa, \mathbf{U})$-compact for all pre-universes $\mathbf{U}$.

Though the above definition is stated using a pre-universe $\mathbf{U}$ contained in a universe $\mathbf{U}^+$, the following lemma shows there is no dependence on $\mathbf{U}^+$.

Lemma 0.2.6. *Let* A *be an object in a locally* $\mathbf{U}^+$*-small category* C*. The following are equivalent:*

(i) A *is a* $(\kappa, \mathbf{U})$*-compact object in* C*.*

(ii) *For all* $\mathbf{U}$*-small* κ*-filtered diagrams* $B : \mathbb{D} \to \mathcal{C}$*, if* $\lambda : B \Rightarrow \Delta C$ *is a colimiting cocone, then for any morphism* $f : A \to C$*, there exist an object* i *in* $\mathbb{D}$ *and a morphism* $f' : A \to Bi$ *in* $\mathcal{C}$ *such that* $f = \lambda_i \circ f'$*; and moreover if* $f = \lambda_j \circ f''$ *for some morphism* $f'' : A \to Bj$ *in* $\mathcal{C}$*, then there exists an object* k *and a pair of arrows* $g : i \to k$*,* $h : i \to k$ *in* $\mathbb{D}$ *such that* $Bg \circ f' = Bh \circ f''$*.*

Proof. Use the explicit description of $\varinjlim_{\mathbb{D}} \mathcal{C}(A, B)$ as a filtered colimit of sets; see Definition 1.1 in [LPAC], or Proposition 5.1.3 in [Borceux, 1994b]. □

Corollary 0.2.7. *Let* $B : \mathbb{D} \to \mathcal{C}$ *be a* $\mathbf{U}$*-small* κ*-filtered diagram, and let* $\lambda : B \Rightarrow \Delta C$ *be a colimiting cocone in* $\mathcal{C}$*. If* C *is a* $(\kappa, \mathbf{U})$*-compact object in* $\mathcal{C}$*, then for some object* i *in* $\mathbb{D}$*,* $\lambda_i : Bi \to C$ *is a* split *epimorphism.* ■

Lemma 0.2.8. *Let* A *be an object in a category* $\mathcal{C}$*.*

(i) *If* $\mathbf{U}$ *is a pre-universe contained in a universe* $\mathbf{U}^+$ *and* κ *is a regular cardinal such that* A *is* $(\kappa, \mathbf{U}^+)$*-compact, then* A *is* $(\kappa, \mathbf{U})$*-compact as well.*

(ii) *If* κ *is a regular cardinal such that* A *is* $(\kappa, \mathbf{U})$*-compact and* λ *is any regular cardinal such that* $\kappa \le \lambda$*, then* A *is also* $(\lambda, \mathbf{U})$*-compact.*

Proof. Obvious. ■

Lemma 0.2.9. *Let* λ *be a regular cardinal in a universe* $\mathbf{U}^+$*, and let* $\mathbf{U}$ *be a pre-universe with* $\mathbf{U} \subseteq \mathbf{U}^+$*. If* $B : \mathbb{D} \to \mathcal{C}$ *is a* λ*-small diagram of* $(\lambda, \mathbf{U})$*-compact objects in a locally* $\mathbf{U}^+$*-small category, then the colimit* $\varinjlim_{\mathbb{D}} B$*, if it exists, is a* $(\lambda, \mathbf{U})$*-compact object in* $\mathcal{C}$*.*

Proof. Use theorem 0.2.4 and the fact that $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^+$ maps colimits in $\mathcal{C}$ to limits in $\mathbf{Set}^+$. ■

Corollary 0.2.10. *A retract of a* $(\lambda, \mathbf{U})$*-compact object is also a* $(\kappa, \mathbf{U})$*-compact object.*

Proof. Suppose $r : A \to B$ and $s : B \to A$ are morphisms in $\mathcal{C}$ such that $r \circ s = \mathrm{id}_B$. Then $e = s \circ r$ is an idempotent morphism and the diagram below

$$A \underset{e}{\overset{\mathrm{id}_A}{\rightrightarrows}} A \xrightarrow{r} B$$

is a (split) coequaliser diagram in $\mathcal{C}$, so B is $(\lambda, \mathbf{U})$-compact if A is. ■

Proposition 0.2.11. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. For any* $\mathbf{U}$*-set* A*, the following are equivalent:*

(i) *A has cardinality less than κ.*

(ii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iii) *The representable functor* $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ *preserves colimits for all* $\mathbf{U}$*-small* κ*-directed diagrams.*

Proof. The claim (i) $\Rightarrow$ (ii) follows from the theorem, and (ii) $\Rightarrow$ (iii) is obvious. To see (iii) $\Rightarrow$ (i), we may use corollary 0.2.7 and the fact that every set is the directed union of its subsets of cardinality at most κ. ■

Corollary 0.2.12. *A set is κ-compact if and only if its cardinality is $< \kappa$.* ■

Definition 0.2.13. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ satisfying the following conditions:

- $\mathcal{C}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams.
- There exists a $\mathbf{U}$-set $\mathcal{G}$ whose members are $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ such that, for every object B in $\mathcal{C}$, there exists a $\mathbf{U}$-small κ-filtered diagram of objects in $\mathcal{G}$ with B as its colimit in $\mathcal{C}$.

We write $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the $(\kappa, \mathbf{U})$-compact objects.

Example 0.2.14. The category of $\mathbf{U}$-sets is a κ-accessible $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Remark 0.2.15. Lemma 0.2.9 implies that, for each object A in an accessible $\mathbf{U}$-category, there exists a regular cardinal λ in $\mathbf{U}$ such that A is $(\lambda, \mathbf{U})$-compact.

Theorem 0.2.16. *Let $\mathcal{C}$ be a locally* $\mathbf{U}$*-small category, and let κ be a regular cardinal in* $\mathbf{U}$*. There exist a locally* $\mathbf{U}$*-small category* $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *and a functor* $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *with the following properties:*

(i) *The objects of* $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ *are* $\mathbf{U}$*-small κ-filtered diagrams* $B : \mathbb{D} \to \mathcal{C}$*, and γ sends an object C in $\mathcal{C}$ to the corresponding trivial diagram* $\mathbb{1} \to \mathcal{C}$ *with value C.*

(ii) *The functor* $\gamma : \mathcal{C} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is fully faithful, injective on objects, preserves all limits that exist in* $\mathcal{C}$*, and preserves all* κ*-small colimits that exist in* $\mathcal{C}$*.*

(iii) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *has colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams.*

(iv) *For every object* C *in* $\mathcal{C}$*, the object* γC *is* $(\kappa, \mathbf{U})$*-compact in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$*, and for each* $\mathbf{U}$*-small* κ*-filtered diagram* $B : \mathbb{D} \to \mathcal{C}$*, there is a canonical colimiting cocone* $\gamma B \Rightarrow \Delta B$ *in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$*.*

(v) *If* $\mathcal{D}$ *is a category with colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams, then for each functor* $F : \mathcal{C} \to \mathcal{D}$*, there exists a functor* $\bar{F} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *that preserves colimits for all* $\mathbf{U}$*-small* κ*-filtered diagrams in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *such that* $\gamma\bar{F} = F$*, and given any functor* $\bar{G} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *whatsoever, the induced map* $\mathrm{Nat}(\bar{F}, \bar{G}) \to \mathrm{Nat}(F, \gamma\bar{G})$ *is a bijection.*

The category $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is called the* **free** $(\kappa, \mathbf{U})$**-ind-completion** *of* $\mathcal{C}$*, or the* **category of** $(\kappa, \mathbf{U})$**-ind-objects** *in* $\mathcal{C}$*.*

Proof. If $B : \mathbb{D} \to \mathcal{C}$ and $B' : \mathbb{D}' \to \mathcal{C}$ are two $\mathbf{U}$-small κ-filtered diagrams, then properties (ii) and (iii) together imply that

$$\mathrm{Hom}(B', B) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and so, taking the RHS as the *definition* of the LHS, we need only find a suitable notion of composition to make $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ into a locally $\mathbf{U}$-small category. However, we observe that, if $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, then

$$\mathrm{Hom}\left(\varinjlim_{\mathbb{D}'} \mathrm{N}B', \varinjlim_{\mathbb{D}} \mathrm{N}B\right) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and, assuming property (v), the Yoneda embedding $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ must extend along γ to a functor $\bar{\mathrm{N}} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that preserves colimits for $\mathbf{U}$-small κ-filtered diagram, so, in consideration of properties (i) and (iv), we may as well *define* the composition in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ so that $\bar{\mathrm{N}}$ becomes fully faithful. This completes the definition of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ as a category.

It remains to be shown that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ actually has properties (ii), (iii), (iv), and (v); see Corollary 6.4.14 in [Borceux, 1994a] and Theorem 2.26 in [LPAC]. Note that the fact that γ preserves colimits for κ-small diagrams essentially follows from theorem 0.2.4. □

Proposition 0.2.17. *Let $\mathbb{B}$ be a* $\mathbf{U}$*-small category and let κ be a regular cardinal in* $\mathbf{U}$.

(i) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ *is a κ-accessible* $\mathbf{U}$*-category.*

(ii) *Every $(\kappa, \mathbf{U})$-compact object in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is a retract of an object of the form γB, where $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is the canonical embedding.*

(iii) $\mathbf{K}^{\mathbf{U}}_{\kappa}\left(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right)$ *is an essentially* $\mathbf{U}$*-small category.*

Proof. (i). This claim more-or-less follows from the properties of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ explained in the previous theorem.

(ii). Use corollary 0.2.10.

(iii). Since $\mathbb{B}$ is $\mathbf{U}$-small and $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is locally $\mathbf{U}$-small, claim (ii) implies that $\mathbf{K}^{\mathbf{U}}_{\kappa}\left(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right)$ must be essentially $\mathbf{U}$-small. ■

Definition 0.2.18. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **$(\kappa, \mathbf{U})$-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and
- F preserves all colimits for $\mathbf{U}$-small κ-filtered diagrams.

We write $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D})$ for the full subcategory of the functor category $[\mathcal{C}, \mathcal{D}]$ spanned by the $(\kappa, \mathbf{U})$-accessible functors. An **accessible functor** is a functor that is $(\kappa, \mathbf{U})$-accessible functor for some regular cardinal κ in some universe $\mathbf{U}$.

Theorem 0.2.19 (Classification of accessible categories)**.** *Let κ be a regular cardinal in a universe* $\mathbf{U}$*, and let $\mathcal{C}$ be a locally* $\mathbf{U}$*-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a κ-accessible* $\mathbf{U}$*-category.*

(ii) *The inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ extends along the embedding $\gamma : \mathcal{C} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ to a $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}\left(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})\right) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *There exist a* $\mathbf{U}$*-small category $\mathbb{B}$ and a functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

Proof. See Theorem 2.26 in [LPAC], or Theorem 5.35 in [Borceux, 1994b]. □

Corollary 0.2.20. *If $\mathcal{C}$ is a κ-accessible* **U***-category and $\mathcal{D}$ is any category, then:*

(i) *The restriction $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \to \left[\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C}), \mathcal{D}\right]$ is fully faithful and surjective on objects.*

(ii) *In particular, if $\mathcal{D}$ is also locally* **U***-small, then $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D})$ is equivalent to a locally* **U***-small category.*

(iii) *If $\mathcal{D}$ has colimits for all* **U***-small κ-filtered diagrams, then the inclusion $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \hookrightarrow [\mathcal{C}, \mathcal{D}]$ has a left adjoint.* ■

Proposition 0.2.21. *Let $\mathcal{C}$ be a κ-accessible* **U***-category and let $\mathcal{D}$ be a locally* **U***-small category. Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, if G is fully faithful and preserves colimits for all* **U***-small κ-filtered diagrams, then $\mathcal{D}$ is also a κ-accessible* **U***-category.*

Proof. Under our hypotheses, given any **U**-small κ-filtered diagram $A : \mathbb{J} \to \mathcal{D}$, we may take $F \varinjlim_{\mathbb{J}} GA$ as its colimit in $\mathcal{D}$. Our hypotheses also imply that F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\kappa, \mathbf{U})$-compact objects in $\mathcal{D}$; thus if $\mathcal{G}$ is a **U**-small set of objects that generates $\mathcal{C}$ under **U**-small κ-filtered colimits, then $\{FX \mid X \in \mathcal{G}\}$ is a **U**-small set of objects that generates $\mathcal{D}$ in the same sense. ■

Definition 0.2.22. Let κ be a regular cardinal in a universe **U**. A **locally κ-presentable U-category** is a κ-accessible **U**-category that is also **U**-cocomplete. A **locally presentable U-category** is one that is a locally κ-presentable **U**-category for some regular cardinal κ in **U**, and we often say 'locally finitely presentable' instead of 'locally $\aleph_0$-presentable'.

Example 0.2.23. The category of **U**-sets is a locally κ-presentable **U**-category for any regular cardinal κ in **U**.

Lemma 0.2.24. *Let $\mathcal{C}$ be a locally κ-presentable* **U***-category.*

(i) *For any regular cardinal λ in* **U***, if $\kappa \leq \lambda$, then $\mathcal{C}$ is a locally λ-presentable* **U***-category.*

(ii) *With λ as above, if $F : \mathcal{C} \to \mathcal{D}$ is a $(\kappa, \mathbf{U})$-accessible functor, then it is also a $(\lambda, \mathbf{U})$-accessible functor.*

(iii) *If* $\mathbf{U}^+$ *is any universe with* $\mathbf{U} \in \mathbf{U}^+$, *and* C *is a locally* κ*-presentable* $\mathbf{U}^+$*-category, then* C *must be a preorder.*

Proof. (i). See the remark after Theorem 1.20 in [LPAC], or Propositions 5.3.2 and 5.2.3 in [Borceux, 1994b].

(ii). A λ-filtered diagram is certainly κ-filtered, so if F preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in C, it must also preserve colimits for all $\mathbf{U}$-small λ-filtered diagrams.

(iii). This is a corollary of theorem 0.1.10. ■

Corollary 0.2.25. *A category* C *is a locally presentable* $\mathbf{U}$*-category for* at most *one universe* $\mathbf{U}$, *provided* C *is not a preorder.*

Proof. Use proposition 0.1.35 together with the above lemma. ■

Theorem 0.2.26 (Classification of locally presentable categories)**.** *Let* κ *be a regular cardinal in a universe* $\mathbf{U}$, *let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* C *be a locally* $\mathbf{U}$*-small category. The following are equivalent:*

(i) C *is a locally* κ*-presentable* $\mathbf{U}$*-category.*

(ii) *There exist a* $\mathbf{U}$*-small category* $\mathbb{B}$ *that has colimits for* κ*-small diagrams and a functor* $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}) \to C$ *that is fully faithful and essentially surjective on objects.*

(iii) *The restricted Yoneda embedding* $C \to \left[\mathbf{K}_{\kappa}^{\mathbf{U}}(C)^{\mathrm{op}}, \mathbf{Set}\right]$ *is fully faithful,* $(\kappa, \mathbf{U})$*-accessible, and has a left adjoint.*

(iv) *There exists a* $\mathbf{U}$*-small category* $\mathbb{A}$ *and a fully faithful* $(\kappa, \mathbf{U})$*-accessible functor* $R : C \to [\mathbb{A}, \mathbf{Set}]$ *such that* $\mathbb{A}$ *has limits for all* κ*-small diagrams,* R *has a left adjoint, and* R *is essentially surjective onto the full subcategory of functors* $\mathbb{A} \to \mathbf{Set}$ *that preserve finite limits.*

(v) *There exists a* $\mathbf{U}$*-small category* $\mathbb{A}$ *and a fully faithful* $(\kappa, \mathbf{U})$*-accessible functor* $R : C \to [\mathbb{A}, \mathbf{Set}]$ *such that* R *has a left adjoint.*

(vi) C *is a* κ*-accessible* $\mathbf{U}$*-category and is* $\mathbf{U}$*-complete.*

Proof. See Proposition 1.27, Corollary 1.28, Theorem 1.46, and Corollary 2.47 in [LPAC], or Theorems 5.2.7 and 5.5.8 in [Borceux, 1994b]. □

Remark 0.2.27. If $\mathcal{C}$ is equivalent to $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ for some $\mathbf{U}$-small category $\mathbb{B}$ that has limits for all κ-small diagrams, then $\mathbb{B}$ must be equivalent to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ by proposition 0.2.17. In other words, every locally κ-presentable $\mathbf{U}$-category is, up to equivalence, the $(\kappa, \mathbf{U})$-ind-completion of an essentially unique $\mathbf{U}$-small κ-complete category.

Example 0.2.28. Obviously, for any $\mathbf{U}$-small category $\mathbb{A}$, the functor category $[\mathbb{A}, \mathbf{Set}]$ is locally finitely presentable. More generally, one may show that for any κ-ary algebraic theory $\mathbb{T}$, possibly many-sorted, the category of $\mathbb{T}$-algebras in $\mathbf{U}$ is a locally κ-presentable $\mathbf{U}$-category. The above theorem can also be used to show that $\mathbf{Cat}$, the category of $\mathbf{U}$-small categories, is a locally finitely presentable $\mathbf{U}$-small category.

Corollary 0.2.29. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category. For any $\mathbf{U}$-small κ-filtered diagram $\mathbb{D}$, $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ preserves κ-small limits.*

Proof. The claim is certainly true when $\mathcal{C} = [\mathbb{A}, \mathbf{Set}]$, by theorem 0.2.4. In general, choose a $(\kappa, \mathbf{U})$-accessible fully faithful functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ with a left adjoint, and simply note that R creates limits for all $\mathbf{U}$-small diagrams as well as colimits for all $\mathbf{U}$-small κ-filtered diagrams. ■

Proposition 0.2.30. *If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category and $\mathbb{D}$ is any $\mathbf{U}$-small category, then the functor category $[\mathbb{D}, \mathcal{C}]$ is also a locally κ-presentable category.*

Proof. This can be proven using the classification theorem by noting that the 2-functor $[\mathbb{D}, -]$ preserves reflective subcategories, but see also Corollary 1.54 in [LPAC]. □

It is commonplace to say 'λ-presentable object' instead of 'λ-compact object', especially in algebraic contexts. The following proposition justifies the alternative terminology:

Proposition 0.2.31. *Let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category, and let λ be a regular cardinal in $\mathbf{U}$ with $\lambda \geq \kappa$. If $\mathcal{H}$ is a small full subcategory of $\mathcal{C}$ such that*

- *every $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$ is isomorphic to an object in $\mathcal{H}$, and*
- *$\mathcal{H}$ is closed in $\mathcal{C}$ under colimits for λ-small diagrams,*

then every $(\lambda, \mathbf{U})$*-compact object in* $\mathcal{C}$ *is isomorphic to an object in* $\mathcal{H}$*. In particular,* $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ *is the smallest replete full subcategory of* $\mathcal{C}$ *containing* $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ *and closed in* $\mathcal{C}$ *under colimits for* λ*-small diagrams.*

TODO: Simplify this argument.

Proof. Let C be any $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$. Clearly, the comma category $(\mathcal{H} \downarrow C)$ is a $\mathbf{U}$-small λ-filtered category. Let $\mathcal{G} = \mathcal{H} \cap \mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$. One can show that $(\mathcal{G} \downarrow C)$ is a cofinal subcategory in $(\mathcal{H} \downarrow C)$, and the classification theorem (0.2.26) plus proposition A.4.20 implies that the tautological cocone on the diagram $(\mathcal{G} \downarrow C) \to \mathcal{C}$ is colimiting, so the tautological cocone on the diagram $(\mathcal{H} \downarrow C) \to \mathcal{C}$ is also colimiting. Now, by corollary 0.2.7, C is a retract of an object in $\mathcal{H}$, and hence C must be isomorphic to an object in $\mathcal{H}$, because $\mathcal{H}$ is closed under coequalisers.

For the final claim, note that $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ is certainly a replete full subcategory of $\mathcal{C}$ and contained in any replete full subcategory containing $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and closed in $\mathcal{C}$ under colimits for λ-small diagrams, so we just have to show that $\mathbf{K}_\lambda^{\mathbf{U}}(\mathcal{C})$ is also closed in $\mathcal{C}$ under colimits for λ-small diagrams; for this, we simply appeal to lemma 0.2.9. ■

Proposition 0.2.32. *Let* $\mathcal{C}$ *be a locally* κ*-presentable* $\mathbf{U}$*-category and let* $\mathbb{D}$ *be a* μ*-small category in* $\mathbf{U}$*. The* $(\lambda, \mathbf{U})$*-compact objects in* $[\mathbb{D}, \mathcal{C}]$ *are precisely the diagrams* $\mathbb{D} \to \mathcal{C}$ *that are componentwise* $(\lambda, \mathbf{U})$*-compact, so long as* $\lambda \geq \max\{\kappa, \mu\}$.

Proof. First, note that Mac Lane's subdivision category[1] $\mathbb{D}^\S$ is also μ-small, so $[\mathbb{D}, \mathcal{C}](A, B)$ is computed as the limit of a μ-small diagram of hom-sets. More precisely, using end notation,[2]

$$[\mathbb{D}, \mathcal{C}](A, B) \cong \int_{d:\mathbb{D}} \mathcal{C}(Ad, Bd)$$

and so if A is componentwise $(\lambda, \mathbf{U})$-compact, then $[\mathbb{D}, \mathcal{C}](A, -)$ preserves colimits for $\mathbf{U}$-small λ-filtered diagrams, hence A is itself $(\lambda, \mathbf{U})$-compact.

Now, suppose A is a $(\lambda, \mathbf{U})$-compact object in $[\mathbb{D}, \mathcal{C}]$. Let d be an object in $\mathbb{D}$, let $d^* : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ be evaluation at d, and let $d_* : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$ be the right adjoint, which is explicitly given by

$$\left(d_* C\right)(d') = \mathbb{D}(d', d) \pitchfork C$$

[1] See [CWM, Ch. IX, § 5].
[2] See § A.5.

where $\pitchfork$ is defined by following adjunction:

$$\mathbf{Set}(X, \mathcal{C}(C, C')) \cong \mathcal{C}(C, X \pitchfork C')$$

The unit $\eta_A : A \to d_* d^* A$ is constructed using the universal property of $\pitchfork$ in the obvious way, and the counit $\epsilon_C : d^* d_* C \to C$ is the projection $\mathbb{D}(d, d) \pitchfork C \to C$ corresponding to $\mathrm{id}_d \in \mathbb{D}(d, d)$. Since $\mathcal{C}$ is a locally λ-presentable $\mathbf{U}$-category, there exist a $\mathbf{U}$-small λ-filtered diagram $B : \mathbb{J} \to \mathcal{C}$ consisting of $(\lambda, \mathbf{U})$-compact objects in $\mathcal{C}$ and a colimiting cocone $\alpha : B \Rightarrow \Delta d^* A$, and since each $\mathbb{D}(d', d)$ has cardinality less than μ, the cocone $d_* \alpha : d_* B \Rightarrow \Delta d_* d^* A$ is also colimiting, by corollary 0.2.29. Lemma 0.2.6 then implies $\eta_A : A \to d_* d^* A$ factors through $d_* \alpha_j : d_*(Bj) \to d_* d^* A$ for some j in $\mathbb{J}$, say

$$\eta_A = d_* \alpha_j \circ \sigma$$

for some $\sigma : A \to d_* Bj$. But then, by the triangle identity,

$$\mathrm{id}_{Ad} = \epsilon_{Ad} \circ d^* \eta_A = \epsilon_{Ad} \circ d^* d_* \alpha_j \circ d^* \sigma = \alpha_j \circ \epsilon_{Bj} \circ d^* \sigma$$

and so $\alpha_j : Bj \to Ad$ is a split epimorphism, hence Ad is a $(\lambda, \mathbf{U})$-compact object, by corollary 0.2.10. ■

Remark 0.2.33. The claim in the above proposition can fail if $\mu > \lambda \geq \kappa$. For example, we could take $\mathcal{C} = \mathbf{Set}$, with $\mathbb{D}$ being the set ω considered as a discrete category; then the terminal object in $[\mathbb{D}, \mathbf{Set}]$ is componentwise finite, but is not itself an $\aleph_0$-compact object in $\mathbf{Set}$.

Lemma 0.2.34. *Let κ and λ be regular cardinals in a universe $\mathbf{U}$, with $\kappa \leq \lambda$.*

(i) *If $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and $G : \mathcal{D} \to \mathcal{C}$ is a $(\lambda, \mathbf{U})$-accessible functor that preserves limits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, then, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object.*

(ii) *If $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, $\mathcal{D}$ is a locally $\mathbf{U}$-small category, and $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, then, for any object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object.*

Proof. (i). Let $\mathcal{F}$ be the full subcategory of $(C \downarrow G)$ spanned by those (D, g) where D is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$. G preserves colimits for all $\mathbf{U}$-small λ-filtered diagrams, so, by lemma 0.2.6, $\mathcal{F}$ must be a weakly initial family in $(C \downarrow G)$. Proposition 0.2.17 implies $\mathcal{F}$ is an essentially $\mathbf{U}$-small category, and since $\mathcal{D}$ has limits for all $\mathbf{U}$-small diagrams and G preserves them, $(C \downarrow G)$ is also $\mathbf{U}$-complete. Thus, the inclusion $\mathcal{F} \hookrightarrow (C \downarrow G)$ has a limit, and it can be shown that this is an initial object in $(C \downarrow G)$.[3]

(ii). Let $\mathcal{G}$ be the full subcategory of $(F \downarrow D)$ spanned by those (C, f) where C is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$; note that proposition 0.2.17 implies $\mathcal{G}$ is an essentially $\mathbf{U}$-small category. Since $\mathcal{C}$ has colimits for all $\mathbf{U}$-small diagrams and F preserves them, $(F \downarrow D)$ is also $\mathbf{U}$-cocomplete.[4] Let (C, f) be a colimit for the inclusion $\mathcal{G} \hookrightarrow (F \downarrow D)$. It is not hard to check that (C, f) is a weakly terminal object in $(F \downarrow D)$, so the formal dual of Freyd's initial object lemma[5] gives us a terminal object in $(F \downarrow D)$; explicitly, it may be constructed as the joint coequaliser of all the endomorphisms of (C, f). ■

Theorem 0.2.35 (Accessible adjoint functor theorem)**.** *Let κ and λ be regular cardinals in a universe* $\mathbf{U}$*, with $\kappa \le \lambda$, let $\mathcal{C}$ be a locally κ-presentable* $\mathbf{U}$*-category, and let $\mathcal{D}$ be a locally λ-presentable* $\mathbf{U}$*-category.*

Given a functor $F : \mathcal{C} \to \mathcal{D}$, the following are equivalent:

(i) *F has a right adjoint $G : \mathcal{D} \to \mathcal{C}$, and G is a $(\lambda, \mathbf{U})$-accessible functor.*

(ii) *F preserves colimits for all* $\mathbf{U}$*-small diagrams and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(iii) *F has a right adjoint and sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

On the other hand, given a functor $G : \mathcal{D} \to \mathcal{C}$, the following are equivalent:

(iv) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(v) *G is a $(\lambda, \mathbf{U})$-accessible functor and preserves limits for all* $\mathbf{U}$*-small diagrams.*

[3] See Theorem 1 in [CWM, Ch. X, § 2].
[4] See the Lemma in [CWM, Ch. V, § 6].
[5] See Theorem 1 in [CWM, Ch. V, § 6].

(vi) *G is a $(\lambda, \mathbf{U})$-accessible functor and there exist a functor $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ and hom-set bijections*

$$\mathcal{C}(C, GD) \cong \mathcal{D}\big(F_0 C, D\big)$$

natural in D for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, where D varies in $\mathcal{D}$.

Proof. We will need to refer back to the details of the proof of this theorem later, so here is a sketch of the constructions involved.

(i) ⇒ (ii). If F is a left adjoint, then F certainly preserves colimits for all $\mathbf{U}$-small diagrams. Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathbb{J} \to \mathcal{D}$, observe that

$$\mathcal{D}\left(FC, \varinjlim_{\mathbb{J}} B\right) \cong \mathcal{C}\left(C, G \varinjlim_{\mathbb{J}} B\right) \cong \mathcal{C}\left(C, \varinjlim_{\mathbb{J}} GB\right) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(C, GB) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(FC, B)$$

and thus FC is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$.

(ii) ⇒ (iii). It is enough to show that, for each object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object $\big(GD, \varepsilon_D\big)$;[6] but this was done in the previous lemma.

(iii) ⇒ (i). Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathbb{J} \to \mathcal{D}$, observe that

$$\mathcal{C}\left(C, G \varinjlim_{\mathbb{J}} B\right) \cong \mathcal{D}\left(FC, \varinjlim_{\mathbb{J}} B\right) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(FC, B) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(C, GB) \cong \mathcal{C}\left(C, \varinjlim_{\mathbb{J}} GB\right)$$

because FC is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$; but theorem 0.2.26 says the restricted Yoneda embedding $\mathcal{C} \to \big[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\big]$ is fully faithful, so this is enough to conclude that G preserves colimits for $\mathbf{U}$-small λ-filtered diagrams.

(iv) ⇒ (v). If G is a right adjoint, then G certainly preserves colimits for all $\mathbf{U}$-small diagrams; the rest of the claim is subsumed by (iii) ⇒ (i).

[6] See Theorem 2 in [CWM, Ch. IV, § 1].

(v) ⇒ (vi). It is enough to show that, for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object $(F_0 C, \eta_C)$; but this was done in the previous lemma. It is clear how to make F_0 into a functor $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$.

(vi) ⇒ (iv). We use theorems 0.2.16 and 0.2.26 to extend $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ along the inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ to get $(\kappa, \mathbf{U})$-accessible functor $F : \mathcal{C} \to \mathcal{D}$. We then observe that, for any $\mathbf{U}$-small κ-filtered diagram $A : \mathbb{I} \to \mathcal{C}$ of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}\Big(\varinjlim_{\mathbb{I}} A, GD\Big) &\cong \varprojlim_{\mathbb{I}} \mathcal{C}(A, GD) \cong \varprojlim_{\mathbb{I}} \mathcal{C}\big(F_0 A, D\big) \\ &\cong \mathcal{C}\Big(\varinjlim_{\mathbb{I}} FA, D\Big) \cong \mathcal{C}\Big(F \varinjlim_{\mathbb{I}} A, D\Big) \end{aligned}$$

is a series of bijections natural in D, where D varies in $\mathcal{D}$; but $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, so this is enough to show that F is a left adjoint of G. The remainder of the claim is a corollary of (i) ⇒ (ii). ■

Corollary 0.2.36. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable* $\mathbf{U}$*-categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a left adjoint, then there exists a regular cardinal μ in* $\mathbf{U}$ *such that G is a $(\mu, \mathbf{U})$-accessible functor.*

Proof. Suppose $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, and $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint for G. Since $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ is an essentially $\mathbf{U}$-small category, recalling lemma 0.2.8, there certainly exists a regular cardinal μ in $\mathbf{U}$ such that $\mu \geq \lambda$ and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U})$-compact objects in $\mathcal{D}$. The above theorem, plus lemma 0.2.24, implies G is an $(\mu, \mathbf{U})$-accessible functor. ■

0.3 Change of universe

Prerequisites. §§ 0.1, 0.2, A.4.

Having introduced universes into our ontology, it becomes necessary to ask whether an object with some universal property retains that property when we enlarge the universe. Though it sounds inconceivable, there do exist examples of badly-behaved constructions that are not stable under change-of-universe; for example, Waterhouse [1975] defined a functor $F : \mathbf{CRing} \to \mathbf{Set}^+$, where $\mathbf{CRing}$

is the category of commutative rings in a universe $\mathbf{U}$ and $\mathbf{Set}^{+}$ is the category of $\mathbf{U}^{+}$-sets for some universe $\mathbf{U}^{+}$ with $\mathbf{U} \in \mathbf{U}^{+}$, such that the value of F at any given commutative ring in $\mathbf{U}$ does not depend on $\mathbf{U}$, and yet the value of the fpqc sheaf associated with F at the field $\mathbb{Q}$ depends on the size of $\mathbf{U}$.

Many of the universal properties of interest concern adjunctions, so that is where we begin.

Definition 0.3.1. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ and $F' \dashv G' : \mathcal{D}' \to \mathcal{C}'$ be adjunctions, and let $H : \mathcal{C} \to \mathcal{C}'$ and $K : \mathcal{D} \to \mathcal{D}'$ be functors. The **mate** of a natural transformation $\alpha : HG \Rightarrow G'K$ is the natural transformation

$$\varepsilon' KF \bullet F' \alpha F \bullet F' H \eta : F' H \Rightarrow KF$$

where $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ is the unit of $F \dashv G$ and $\varepsilon : F'G' \Rightarrow \mathrm{id}_{\mathcal{D}}$ is the counit of $F' \dashv G'$; dually, the **mate** of a natural transformation $\beta : F'H \Rightarrow KF$ is the natural transformation

$$G' K \varepsilon \bullet G' \beta G \bullet \eta' HG : HG \Rightarrow G'K$$

where $\eta' : \mathrm{id}_{\mathcal{C}'} \Rightarrow G'F'$ is the unit of $F' \dashv G'$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ is the counit of $F \dashv G$.

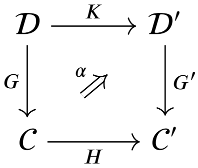
Lemma 0.3.2. *In the above notation, the two mates constructions constitute a mutually inverse pair of bijections*

$$\mathrm{Nat}(F'H, KF) \cong \mathrm{Nat}(HG, G'K)$$

and moreover, given a further adjunction $F'' \dashv G'' : \mathcal{C}'' \to \mathcal{D}''$ and functors $H' : \mathcal{C}' \to \mathcal{C}''$ and $K' : \mathcal{D}' \to \mathcal{D}''$, if $\alpha : HG \Rightarrow G'K$ and $\alpha' : H'G' \Rightarrow G''K'$ have mates $\beta : F'H \Rightarrow KF$ and $\beta' : F''H' \Rightarrow K'F'$ respectively, then the composite natural transformation $\alpha' K \bullet H' \alpha : H'HG \Rightarrow H''K'K$ has mate $K'\beta \bullet \beta' H : F''H'H \Rightarrow K'KF$.

Proof. This is an exercise in using the triangle identities for adjunctions. ☐

Definition 0.3.3. Given a diagram of the form



where $\alpha : HG \Rightarrow G'K$ is a natural isomorphism, $F \dashv G$ and $F' \dashv G'$, we say the diagram satisfies the **left Beck–Chevalley condition** if the mate of α is also a natural isomorphism. Dually, given a diagram of the form

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ {\scriptstyle F}\downarrow & \swarrow_{\beta} & \downarrow{\scriptstyle F'} \\ \mathcal{D} & \xrightarrow[K]{} & \mathcal{D}' \end{array}$$

where $\beta : F'H \Rightarrow KF$ is a natural isomorphism, $F \dashv G$ and $F' \dashv G'$, we say the diagram satisfies the **right Beck–Chevalley condition** if the mate of β is also a natural isomorphism.

Remark 0.3.4. Unfortunately, the Beck–Chevalley conditions are not vacuous. For example, consider the following (strictly!) commutative diagram of forgetful functors:

$$\begin{array}{ccc} \mathbf{CRing} & \longrightarrow & \mathbf{Ab} \\ \downarrow & & \downarrow \\ \mathbf{Set} & \xrightarrow[\mathrm{id}]{} & \mathbf{Set} \end{array}$$

The mate of the trivial natural transformation in the above diagram is the group homomorphism $\mathbb{Z}X \to \mathbb{Z}[X]$ that sends a generator in $\mathbb{Z}X$ to the corresponding generator in $\mathbb{Z}[X]$; clearly, this is never an isomorphism. However, this is unsurprising: we do not expect the additive group of free commutative ring generated by X to be naturally isomorphic to the free abelian group generated by X.

Example 0.3.5. Let $\mathcal{C}$ be a category with pullbacks, and suppose

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

is a pullback square in $\mathcal{C}$. Let $\Sigma_f : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ etc. be the functor that sends an object $p : E \to X$ in $\mathcal{C}_{/X}$ to the object $f \circ p : E \to Y$ in $\mathcal{C}_{/Y}$, and consider the induced (strictly!) commutative diagram of functors:

$$\begin{array}{ccc} \mathcal{C}_{/Z} & \xrightarrow{\Sigma_z} & \mathcal{C}_{/X} \\ {\scriptstyle \Sigma_g}\downarrow & & \downarrow{\scriptstyle \Sigma_f} \\ \mathcal{C}_{/W} & \xrightarrow[\Sigma_w]{} & \mathcal{C}_{/Y} \end{array}$$

Since $\mathcal{C}$ has pullbacks, Σ_g and Σ_f have right adjoints,[1] and the pullback pasting lemma then implies that the above square satisfies the right Beck–Chevalley condition.

Lemma 0.3.6. *Given a diagram of the form*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ {\scriptstyle G}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle G'} \\ \mathcal{C} & \xrightarrow[H]{} & \mathcal{C}' \end{array}$$

where $\alpha : HG \Rightarrow G'K$ is a natural isomorphism, $F \dashv G$ and $F' \dashv G'$, the diagram satisfies the left Beck–Chevalley condition if and only if, for every object C in $\mathcal{C}$, the functor $(C \downarrow G) \to (HC \downarrow G')$ sending an object (D, f) in the comma category $(C \downarrow G)$ to the object $\left(KD, \alpha_D \circ Hf\right)$ in $(HC \downarrow G')$ preserves initial objects.

Proof. We know $\left(FC, \eta_C\right)$ is an initial object of $(C \downarrow G)$ and $\left(F'HC, \eta'_{HC}\right)$ is an initial object of $(HC \downarrow G')$, so there is a unique morphism $\beta_C : F'HC \to KFC$ such that $G'\beta_C \circ \eta'_{HC} = \alpha_{FC} \circ H\eta_C$. However, we observe that

$$\begin{aligned} \beta_C &= \beta_C \circ \varepsilon'_{F'HC} \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'G'\beta_C \circ F'\eta'_{HC} \\ &= \varepsilon'_{KFC} \circ F'\alpha_{FC} \circ F'H\eta_C \end{aligned}$$

so β_C is precisely the component at C of the mate of α. Thus β_C is an isomorphism for all C if and only if the Beck–Chevalley condition holds. ■

Definition 0.3.7. Let κ be a regular cardinal in a universe $\mathbf{U}$, and let $\mathbf{U}^+$ be a universe with $\mathbf{U} \subseteq \mathbf{U}^+$. A $(\kappa, \mathbf{U}, \mathbf{U}^+)$**-accessible extension** is a $(\kappa, \mathbf{U})$-accessible functor $i : \mathcal{C} \to \mathcal{C}^+$ such that

[1] See lemma A.1.17.

- C is a κ-accessible $\mathbf{U}$-category,
- C^+ is a κ-accessible $\mathbf{U}^+$-category,
- i sends $(\kappa, \mathbf{U})$-compact objects in C to $(\kappa, \mathbf{U}^+)$-compact objects in C^+, and
- the functor $\mathbf{K}_\kappa^{\mathbf{U}}(C) \to \mathbf{K}_\kappa^{\mathbf{U}^+}(C^+)$ so induced by i is fully faithful and essentially surjective on objects.

Remark 0.3.8. Let $\mathbb{B}$ be a $\mathbf{U}$-small category in which idempotents split. Then the $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B}) \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ obtained by extending the embedding $\gamma^+ : \mathbb{B} \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ along $\gamma : \mathbb{B} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-extension, by proposition 0.2.17. The classification theorem (0.2.19) implies all examples of $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions are essentially of this form.

Proposition 0.3.9. *Let $i : C \to C^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

(i) *C is a locally κ-presentable $\mathbf{U}$-category if and only if C^+ is a locally κ-presentable $\mathbf{U}^+$-category.*

(ii) *The functor $i : C \to C^+$ is fully faithful.*

(iii) *If $B : \mathbb{J} \to C$ is any diagram (not necessarily $\mathbf{U}$-small) and C has a limit for B, then i preserves this limit.*

Proof. (i). If C is a locally κ-presentable $\mathbf{U}$-category, then $\mathbf{K}_\kappa^{\mathbf{U}}(C)$ has colimits for all κ-small diagrams, so $\mathbf{K}_\kappa^{\mathbf{U}^+}(C^+)$ also has colimits for all κ-small diagrams. The classification theorem (0.2.19) then implies C^+ is a locally κ-presentable $\mathbf{U}^+$-category. Reversing this argument proves the converse.

(ii). Let $A : \mathbb{I} \to C$ and $B : \mathbb{J} \to C$ be two $\mathbf{U}$-small κ-filtered diagrams of $(\kappa, \mathbf{U})$-compact objects in C. Then,

$$\begin{aligned} C\left(\varinjlim_{\mathbb{I}} A, \varinjlim_{\mathbb{J}} B\right) &\cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} C(A, B) \cong \varprojlim_{\mathbb{I}} \varinjlim_{\mathbb{J}} C^+(iA, iB) \\ &\cong C^+\left(\varinjlim_{\mathbb{I}} iA, \varinjlim_{\mathbb{J}} iB\right) \cong C^+\left(i \varinjlim_{\mathbb{I}} A, i \varinjlim_{\mathbb{J}} B\right) \end{aligned}$$

because i is $(\kappa, \mathbf{U})$-accessible and is fully faithful on the subcategory $\mathbf{K}_\kappa^{\mathbf{U}}(C)$, and therefore $i : C \to C^+$ itself is fully faithful. Note that this hinges crucially on theorem 0.1.30.

(iii). Let $B : \mathbb{J} \to \mathcal{C}$ be any diagram. We observe that, for any $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$,

$$\begin{aligned} \mathcal{C}^+\Big(iC, i \varprojlim_{\mathbb{J}} B\Big) &\cong \mathcal{C}\Big(C, \varprojlim_{\mathbb{J}} B\Big) && \text{because } i \text{ is fully faithful} \\ &\cong \varprojlim_{\mathbb{J}} \mathcal{C}(C, B) && \text{by definition of limit} \\ &\cong \varprojlim_{\mathbb{J}} \mathcal{C}^+(iC, iB) && \text{because } i \text{ is fully faithful} \end{aligned}$$

but we know the restricted Yoneda embedding $\mathcal{C}^+ \to \left[\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}^+\right]$ is fully faithful, so this is enough to conclude that $i \varprojlim_{\mathbb{J}} B$ is the limit of iB in $\mathcal{C}^+$. ■

Remark 0.3.10. Similar methods show that any fully faithful functor $\mathcal{C} \to \mathcal{C}^+$ satisfying the four bulleted conditions in the definition above is necessarily $(\kappa, \mathbf{U})$-accessible.

Lemma 0.3.11. *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$*, and let* κ *be a regular cardinal in* $\mathbf{U}$*. Suppose:*

- *$\mathcal{C}$ and $\mathcal{D}$ are locally κ-presentable $\mathbf{U}$-categories.*
- *$\mathcal{C}^+$ and $\mathcal{D}^+$ are locally κ-presentable $\mathbf{U}^+$-categories.*
- *$i : \mathcal{C} \to \mathcal{C}^+$ and $j : \mathcal{D} \to \mathcal{D}^+$ are $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions.*

Given a strictly commutative diagram of the form below,

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\;j\;} & \mathcal{D}^+ \\ {\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G^+} \\ \mathcal{C} & \xrightarrow[\;i\;]{} & \mathcal{C}^+ \end{array}$$

where G is $(\kappa, \mathbf{U})$-accessible, G^+ is $(\kappa, \mathbf{U}^+)$-accessible, if both have left adjoints, then the diagram satisfies the left Beck–Chevalley condition.

Proof. Let C be a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$. Inspecting the proof of theorem 0.2.35, we see that the functor $(C \downarrow G) \to (iC \downarrow G^+)$ induced by j preserves initial objects. As in the proof of lemma 0.3.6, this implies the component at C of the left Beck–Chevalley natural transformation $F^+ i \Rightarrow jF$ is an isomorphism; but $\mathcal{C}$ is generated by $\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})$ and the functors F, F^+, i, j all preserve colimits for $\mathbf{U}$-small κ-filtered diagrams, so in fact $F^+ i \Rightarrow jF$ is a natural isomorphism. ■

Proposition 0.3.12. *If $i : \mathcal{C} \to \mathcal{C}^+$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, then i preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$.*

Proof. It is well-known that a functor preserves colimits for all $\mathbf{U}$-small diagrams if and only if it preserves coequalisers for all parallel pairs and coproducts for all $\mathbf{U}$-small families, but coproducts for $\mathbf{U}$-small families can be constructed in a uniform way using coproducts for κ-small families and colimits for $\mathbf{U}$-small κ-filtered diagrams. It is therefore enough to show that $i : \mathcal{C} \to \mathcal{C}^+$ preserves all colimits for κ-small diagrams, since i is already $(\kappa, \mathbf{U})$-accessible.

Let $\mathbb{D}$ be a κ-small category. Recalling proposition 0.1.12, our problem amounts to showing that the diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\;i\;} & \mathcal{C}^+ \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta^+} \\ [\mathbb{D}, \mathcal{C}] & \xrightarrow[i_*]{} & [\mathbb{D}, \mathcal{C}^+] \end{array}$$

satisfies the left Beck–Chevalley condition. It is clear that i_* is fully faithful. Colimits for $\mathbf{U}$-small diagrams in $[\mathbb{D}, \mathcal{C}]$ and in $[\mathbb{D}, \mathcal{C}^+]$ are computed componentwise, so Δ and i_* are certainly $(\kappa, \mathbf{U})$-accessible, and Δ^+ is $(\kappa, \mathbf{U}^+)$-accessible. Using proposition 0.2.32, we see that i_* is also a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, so we apply the lemma above to conclude that the left Beck–Chevalley condition is satisfied. ■

Theorem 0.3.13 (Stability of accessible adjoint functors)**.** *Let $\mathbf{U}$ and $\mathbf{U}^+$ be universes, with $\mathbf{U} \in \mathbf{U}^+$, and let κ and λ be regular cardinals in $\mathbf{U}$, with $\kappa \le \lambda$. Suppose:*

- *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category.*
- *$\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.*
- *$\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*
- *$\mathcal{D}^+$ is a locally λ-presentable $\mathbf{U}^+$-category.*

Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and let $j : \mathcal{D} \to \mathcal{D}^+$ be a fully faithful functor.

(i) *Given a strictly commutative diagram of the form below,*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{j} & \mathcal{D}^{+} \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G^{+}} \\ \mathcal{C} & \xrightarrow[i]{} & \mathcal{C}^{+} \end{array}$$

where G is $(\lambda, \mathbf{U})$-accessible and G^{+} is $(\lambda, \mathbf{U}^{+})$-accessible, if both have left adjoints and j is a $(\lambda, \mathbf{U}, \mathbf{U}^{+})$-accessible extension, then the diagram satisfies the left Beck–Chevalley condition.

(ii) *Given a strictly commutative diagram of the form below,*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{C}^{+} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^{+}} \\ \mathcal{D} & \xrightarrow[j]{} & \mathcal{D}^{+} \end{array}$$

if both F and F^{+} have right adjoints, then the diagram satisfies the right Beck–Chevalley condition.

Proof. (i). The proof is essentially the same as lemma 0.3.11, though we have to use proposition 0.3.12 to ensure that j preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams in $\mathcal{C}$.

(ii). Let D be any object in $\mathcal{D}$. Inspecting the proof of theorem 0.2.35, we see that our hypotheses, plus the fact that i preserves colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$, imply that the functor $(F \downarrow D) \to (F^{+} \downarrow jD)$ induced by i preserves terminal objects. Thus lemma 0.3.6 implies that the diagram satisfies the right Beck–Chevalley condition. ■

Theorem 0.3.14. *If $i : \mathcal{C} \to \mathcal{C}^{+}$ is a $(\kappa, \mathbf{U}, \mathbf{U}^{+})$-accessible extension and $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, then:*

(i) *If λ is a regular cardinal and $\kappa \le \lambda \in \mathbf{U}$, then $i : \mathcal{C} \to \mathcal{C}^{+}$ is also a $(\lambda, \mathbf{U}, \mathbf{U}^{+})$-accessible extension.*

(ii) *If μ is the cardinality of $\mathbf{U}$, then $i : \mathcal{C} \to \mathcal{C}^{+}$ factors through the inclusion $\mathbf{K}_{\mu}^{\mathbf{U}^{+}}(\mathcal{C}^{+}) \hookrightarrow \mathcal{C}^{+}$ as functor $\mathcal{C} \to \mathbf{K}_{\mu}^{\mathbf{U}^{+}}(\mathcal{C}^{+})$ that is (fully faithful and) essentially surjective on objects.*

(iii) *The* $(\mu, \mathbf{U}^{+})$*-accessible functor* $\mathbf{Ind}^{\mu}_{\mathbf{U}^{+}}(\mathcal{C}) \to \mathcal{C}^{+}$ *induced by* $i : \mathcal{C} \to \mathcal{C}^{+}$ *is fully faithful and essentially surjective on objects.*

Proof. (i). Since $i : \mathcal{C} \to \mathcal{C}^{+}$ is a $(\kappa, \mathbf{U})$-accessible functor, it is certainly also $(\lambda, \mathbf{U})$-accessible, by lemma 0.2.24. It is therefore enough to show that i restricts to a functor $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathbf{K}^{\mathbf{U}^{+}}_{\kappa}(\mathcal{C}^{+})$ that is (fully faithful and) essentially surjective on objects.

Proposition 0.2.31 says $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ is the smallest replete full subcategory of $\mathcal{C}$ that contains $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ and is closed in $\mathcal{C}$ under colimits for λ-small diagrams, therefore the replete closure of the image of $\mathbf{K}^{\mathbf{U}}_{\lambda}(\mathcal{C})$ must be the smallest replete full subcategory of $\mathcal{C}^{+}$ that contains $\mathbf{K}^{\mathbf{U}^{+}}_{\kappa}(\mathcal{C}^{+})$ and is closed in $\mathcal{C}^{+}$ under colimits for λ-small diagrams, since i is fully faithful and preserves colimits for all $\mathbf{U}$-small diagrams. This proves the claim.

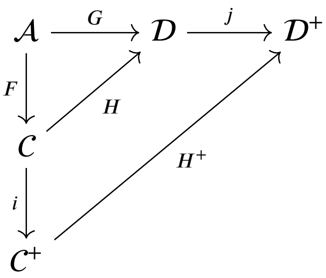
(ii). Since every object in $\mathcal{C}$ is $(\lambda, \mathbf{U})$-compact for some regular cardinal $\lambda < \mu$, claim (i) implies that the image of $i : \mathcal{C} \to \mathcal{C}^{+}$ is contained in $\mathbf{K}^{\mathbf{U}^{+}}_{\mu}(\mathcal{C})$. To show i is essentially surjective onto $\mathbf{K}^{\mathbf{U}^{+}}_{\mu}(\mathcal{C})$, we simply have to observe that the inaccessibility of μ (proposition 0.1.35) and proposition 0.2.31 imply that, for C' any $(\mu, \mathbf{U}^{+})$-compact object in $\mathcal{C}^{+}$, there exists a regular cardinal $\lambda < \mu$ such that C' is also a $(\lambda, \mathbf{U}^{+})$-compact object, which reduces the question to claim (i).

(iii). This is an immediate corollary of claim (ii) and the classification theorem (0.2.19) applied to $\mathcal{C}^{+}$, considered as a $(\mu, \mathbf{U}^{+})$-accessible category. ■

Remark 0.3.15. Although the fact $i : \mathcal{C} \to \mathcal{C}^{+}$ that preserves limits and colimits for all $\mathbf{U}$-small diagrams in $\mathcal{C}$ is a formal consequence of the theorem above (via e.g. corollary A.4.25), it is not clear whether the theorem can be proven without already knowing this.

Corollary 0.3.16. *If* $\mathbb{B}$ *is a* $\mathbf{U}$*-small category and has colimits for all* κ*-small diagrams, and* μ *is the cardinality of* $\mathbf{U}$*, then the canonical* $(\mu, \mathbf{U}^{+})$*-accessible functor* $\mathbf{Ind}^{\mu}_{\mathbf{U}^{+}}\left(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\right) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}^{+}}(\mathbb{B})$ *is fully faithful and essentially surjective on objects.* ■

Theorem 0.3.17 (Stability of pointwise Kan extensions)**.** *Let* $F : \mathcal{A} \to \mathcal{C}$ *and* $G : \mathcal{A} \to \mathcal{D}$ *be functors, and let* $i : \mathcal{C} \to \mathcal{C}^{+}$ *and* $j : \mathcal{D} \to \mathcal{D}^{+}$ *be fully faithful* *functors. Consider the following (not necessarily commutative) diagram:*



(i) *If H^+ is a pointwise right Kan extension of jG along iF, and $H^+ i \cong jH$, then H is a pointwise right Kan extension of G along F.*

(ii) *Suppose jH is a pointwise right Kan extension of jG along F. If H^+ is a pointwise right Kan extension of jH along i, then the counit $H^+ i \Rightarrow jH$ is a natural isomorphism, and H^+ is also a pointwise right Kan extension of jG along iF; conversely, if H^+ is a pointwise right Kan extension of jG along iF, then it is also a pointwise right Kan extension of jH along i.*

(iii) *If* **U** *is a pre-universe such that $\mathcal{A}$ is* **U***-small and j preserves limits for all* **U***-small diagrams, and H is a pointwise right Kan extension of G along F, then a pointwise right Kan extension of jG along iF can be computed as a pointwise right Kan extension of jH along i (if either one exists).*

Dually:

(i′) *If H^+ is a pointwise left Kan extension of jG along iF, and $H^+ i \cong jH$, then H is a pointwise left Kan extension of G along F.*

(ii′) *If jH is a pointwise left Kan extension of jG along F, and H^+ is a pointwise left Kan extension of jH along i, then the unit $jH \Rightarrow H^+ i$ is a natural isomorphism, and H^+ is also a pointwise left Kan extension of jG along iF.*

(iii′) *If* **U** *is a pre-universe such that $\mathcal{A}$ is* **U***-small and j preserves limits for all* **U***-small diagrams, and H is a pointwise right Kan extension of G along F, then a pointwise right Kan extension of jG along iF can be computed as a pointwise right Kan extension of jH along i (if either one exists).*

Proof. (i). Theorem A.4.11 gives an explicit description of $H^+ : C^+ \to D^+$ as a weighted limit:

$$H^+(C') \cong \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}$$

Since i is fully faithful, the weights $\mathcal{C}(C, F)$ and $\mathcal{C}^+(iC, iF)$ are naturally isomorphic, hence,

$$jH(C) \cong H^+(iC) \cong \{\mathcal{C}^+(iC, iF), jG\}^{\mathcal{A}} \cong \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}$$

but, since j is fully faithful, j reflects *all* weighted limits, therefore H must be a pointwise right Kan extension of G along F.

(ii). Let $\mathbf{U}^+$ be a pre-universe such that $\mathcal{A}, \mathcal{C}, \mathcal{D}, \mathcal{C}^+, \mathcal{D}^+$ are all locally $\mathbf{U}^+$-small categories, and let $\mathbf{Set}^+$ be the category of $\mathbf{U}^+$-sets. Using the interchange law (theorem A.5.13) and propositions A.5.7 and A.5.14, we obtain the following natural bijections:

$$\begin{aligned}
\mathcal{D}^+(D', H^+(C')) &\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', i), jH\}^{\mathcal{C}}\big) \\
&\cong \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jHC)) \\
&\cong \int_{C:\mathcal{C}} \mathbf{Set}^+\big(\mathcal{C}^+(C', iC), \mathcal{D}^+\big(D', \{\mathcal{C}(C, F), jG\}^{\mathcal{A}}\big)\big) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathbf{Set}^+(\mathcal{C}(C, FA), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{C:\mathcal{C}} \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{A:\mathcal{A}} \int_{C:\mathcal{C}} \mathbf{Set}^+(\mathcal{C}(C, FA), \mathbf{Set}^+(\mathcal{C}^+(C', iC), \mathcal{D}^+(D', jGA))) \\
&\cong \int_{A:\mathcal{A}} \mathbf{Set}^+(\mathcal{C}^+(C', iFA), \mathcal{D}^+(D', jGA)) \\
&\cong \mathcal{D}^+\big(D', \{\mathcal{C}^+(C', iF), jG\}^{\mathcal{A}}\big)
\end{aligned}$$

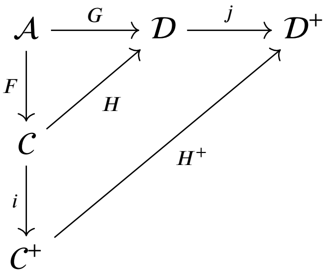
Thus, H^+ is a pointwise right Kan extension of jG along iF if and only if H^+ is a pointwise right Kan extension of jH along i. The fact that the counit $H^+ i \Rightarrow jH$ is a natural isomorphism is just corollary A.4.15.

(iii). Apply corollary A.4.14 to claim (ii). ∎

Corollary 0.3.18. *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$, *and let* κ *and* λ *be regular cardinals in* $\mathbf{U}$. *Suppose:*

- *$\mathcal{C}$ is a locally κ-presentable* **U**-*category.*
- *$\mathcal{D}$ is a locally λ-presentable* **U**-*category.*
- *$\mathcal{C}^+$ is a locally κ-presentable* $\mathbf{U}^+$-*category.*
- *$\mathcal{D}^+$ is a locally λ-presentable* $\mathbf{U}^+$-*category.*

Let $F : \mathcal{A} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{D}$ be functors, let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, and let $j : \mathcal{D} \to \mathcal{D}^+$ be a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension. Consider the following (not necessarily commutative) diagram:



(i) *If H is a pointwise right Kan extension of G along F, then jH is a pointwise right Kan extension of jG along F, and if H^+ is a pointwise right Kan extension of jH along i, then H^+ is also a pointwise right Kan extension of jG along iF.*

(ii) *Assuming $\mathcal{A}$ is* **U**-*small, if H is a pointwise left Kan extension of G along F, then jH is a pointwise left Kan extension of jG along F, and if H^+ is a pointwise left Kan extension of jH along i, then H^+ is also a pointwise left Kan extension of jG along iF.*

Proof. Use the theorem and the fact that i and j preserve limits for *all* diagrams and colimits for **U**-small diagrams. ∎

0.4 Small object arguments

Prerequisites. §§ 0.1, 0.2, 0.3, A.2.

The small object argument is a recurring construction in homotopical algebra, originally due to Quillen [1967, Ch. II, § 3] but refined by many authors since—notably by Garner [2009]. Roughly speaking, the small object argument shows that, under certain hypotheses, starting from a small set $\mathcal{I}$ of morphisms in a cocomplete category $\mathcal{C}$, one can define the notions of 'relative $\mathcal{I}$-cell complex' and '$\mathcal{I}$-fibration' so that every morphism in $\mathcal{C}$ factors as a relative $\mathcal{I}$-cell complex followed by an $\mathcal{I}$-fibration.

In this section, we will study the small object argument with a view toward questions of stability under change-of-universe.

Definition 0.4.1. Let $\mathcal{C}$ be a category, and let $\mathcal{I}$ be a subset of $\operatorname{mor}\mathcal{C}$. A **presentation for a relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ consists of the following data:

- An ordinal α. (We say the presentation is **indexed over** α.)

- A colimit-preserving functor $X_\bullet : [\alpha] \to \mathcal{C}$, where $[\alpha]$ is the well-ordered set $\{0, \ldots, \alpha\}$ considered as a preorder category.

- For each ordinal $\beta < \alpha$, a (possibly empty) indexing set T_β; and for each element j of T_β, a commutative diagram of the form below,

$$\begin{array}{ccc} U_{\beta,j} & \xrightarrow{u_{\beta,j}} & X_\beta \\ {\scriptstyle e_{\beta,j}}\downarrow & & \downarrow{\scriptstyle X_{\beta\to\beta+1}} \\ V_{\beta,j} & \xrightarrow[v_{\beta,j}]{} & X_{\beta+1} \end{array}$$

  where $e_{\beta,j} : U_{\beta,j} \to V_{\beta,j}$ is a morphism in $\mathcal{I}$.

These data are moreover required to satisfy the following condition:

- For each ordinal $\beta < \gamma$, the coproducts $\coprod_{j\in T_\beta} S_{\beta,j}$ and $\coprod_{j\in T_\beta} D_{\beta,j}$ exist in $\mathcal{C}$, and the induced diagram

$$\begin{array}{ccc} \coprod_{j\in T_\beta} U_{\beta,j} & \xrightarrow{u_\beta} & X_\beta \\ {\scriptstyle \coprod_{j\in T_\beta} e_{\beta,j}}\downarrow & & \downarrow{\scriptstyle X_{\beta\to\beta+1}} \\ \coprod_{j\in T_\beta} V_{\beta,j} & \xrightarrow[v_\beta]{} & X_{\beta+1} \end{array}$$

  is a pushout square in $\mathcal{C}$.

The presentation is said to be **U-small** (resp. **κ-small** for a regular cardinal κ) if α is an ordinal in **U** (resp. $|\alpha| < \kappa$) and the disjoint union $\coprod_{\beta<\alpha} T_\beta$ is in **U** (resp. has cardinality less than κ). A **sequential presentation** is one where each T_β is a singleton, in which case we suppress the index j in $e_{\beta,j}$, $u_{\beta,j}$, and $v_{\beta,j}$.

A **relative $\mathcal{I}$-cell complex** in $\mathcal{C}$ is a morphism $f : X \to Y$ in $\mathcal{C}$ for which there exists a presentation as above with f equal to $X_0 \to X_\alpha$. Given an initial object 0 in $\mathcal{C}$, an **$\mathcal{I}$-cell complex** in $\mathcal{C}$ is an object Y for which the unique morphism $0 \to Y$ is a relative $\mathcal{I}$-cell complex.

Remark 0.4.2. For any object X in $\mathcal{C}$ and any subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, the morphism $\mathrm{id} : X \to X$ is a relative $\mathcal{I}$-cell complex in $\mathcal{C}$, with the obvious presentation indexed over 0). More generally, every isomorphism in $\mathcal{C}$ is a relative $\mathcal{I}$-cell complex, with a presentation indexed over 1 (and $T_0 = \varnothing$); but in order to get a *sequential* presentation, one must assume that there is an isomorphism in $\mathcal{I}$.

Proposition 0.4.3. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$, *let κ be a regular cardinal, and let* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes in $\mathcal{C}$ that admit a κ-small presentation.*

(i) *Every morphism in $\mathcal{I}$ is also in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$.

(ii) *For each object X in $\mathcal{C}$, the morphism* $\mathrm{id} : X \to X$ *is in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$.

(iii) *If $f : X \to Y$ and $g : Y \to Z$ are both in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$, *then so is $g \circ f$.*

(iv) *Let α be an ordinal and let $X_\bullet : \alpha \to \mathcal{C}$ be a colimit-preserving functor. If $|\alpha| < \kappa$ and λ is a colimiting cocone from $X_\bullet$ to Y and, for $\beta \le \gamma < \alpha$, the morphism $X_{\beta\to\gamma} : X_\beta \to X_\gamma$ is in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$, *then each component $\lambda_\beta : X_\beta \to Y$ is also in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$.

(v) *Given a pushout diagram of the form below in $\mathcal{C}$,*

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

if g is in $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$ *and $\mathcal{C}$ has colimits for all κ-small diagrams, then f is also in* $\mathrm{cell}_{\mathcal{I},\kappa}\, \mathcal{C}$.

Proof. (i). Given any morphism $e : U \to V$ in $\mathcal{I}$, we have the following pushout diagram:

$$\begin{array}{ccc} U & \xrightarrow{\mathrm{id}} & U \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e} \\ V & \xrightarrow[\mathrm{id}]{} & V \end{array}$$

Thus $e : U \to V$ is in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$.

(ii). See remark 0.4.2.

(iii). It is clear that appending any κ-small presentation for g to any κ-small presentation for f yields a κ-small presentation of $g \circ f$.

(iv). The case $\alpha = 0$ falls under claim (ii). If $\alpha = \gamma + 1$, then the component $\lambda_\gamma : X_\gamma \to Y$ must be an isomorphism, and thus $\lambda_\beta = \lambda_\gamma \circ X_{\beta \to \gamma}$ is also in $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$; and if α is a positive limit ordinal, since every terminal segment of α is cofinal in α, it is clear that concatenating κ-small presentations for $X_{\gamma \to \gamma+1}$ for $\beta \le \gamma < \alpha$ yields a κ-small presentation for $\lambda_\beta : X_\beta \to Y$.

(v). Fix a κ-small presentation of $g : Z \to W$. By the pushout pasting lemma, given a commutative diagram of the form below,

$$\begin{array}{ccccc} \coprod_{j \in T_\beta} U_{\beta,j} & \xrightarrow{u_\beta} & Z_\beta & \longrightarrow & X_\beta \\ {\scriptstyle \coprod_{j \in T_\beta} e_{\beta,j}}\downarrow & & \downarrow{\scriptstyle z_{\beta \to \beta+1}} & & \downarrow{\scriptstyle x_{\beta \to \beta+1}} \\ \coprod_{j \in T_\beta} V_{\beta,j} & \xrightarrow[v_\beta]{} & Z_{\beta+1} & \longrightarrow & X_{\beta+1} \end{array}$$

if both squares are pushout diagrams, then the outer rectangle is a pushout diagram as well. Since pushout along $z : Z \to X$ is the left adjoint of the evident functor $z^* : {}^{X/}\mathcal{C} \to {}^{Z/}\mathcal{C}$, it preserves all colimits, and thus we obtain a κ-small presentation of $f : X \to Y$. ■

Definition 0.4.4. Let $\mathcal{C}$ be a category and let $\mathcal{I}$ be a subset of $\operatorname{mor} \mathcal{C}$. An **$\mathcal{I}$-injective morphism** in $\mathcal{C}$ is a morphism that has the right lifting property with respect to every morphism in $\mathcal{I}$.[1] An **$\mathcal{I}$-cofibration** in $\mathcal{C}$ is a morphism that has the left lifting property with respect to every $\mathcal{I}$-injective morphism.

[1] Equivalently, it is a morphism $f : X \to Y$ in $\mathcal{C}$ that is an $\mathcal{I}$-injective object in the slice category $\mathcal{C}_{/Y}$.

Proposition 0.4.5. *Let $\mathcal{C}$ be a category, let $\mathcal{I}$ be a subset of* $\operatorname{mor} \mathcal{C}$, *and let* $\operatorname{cell}_{\mathcal{I}} \mathcal{C}$, $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$, *and* $\operatorname{cof}_{\mathcal{I}} \mathcal{C}$ *be the set of relative $\mathcal{I}$-cell complexes, $\mathcal{I}$-injections, and $\mathcal{I}$-cofibrations in $\mathcal{C}$, respectively.*

(i) *We have* $\mathcal{I} \subseteq \operatorname{cell}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}$.

(ii) *A morphism is in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ *if and only if it has the right lifting property with respect to every $\mathcal{I}$-cofibration.*

(iii) *In particular, a morphism is in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$ *if and only if it has the right lifting property with respect to every relative $\mathcal{I}$-cell complex.*

Proof. (i). Follows immediately from the definition of 'relative $\mathcal{I}$-cell complex' and proposition A.2.9.

(ii) and (iii). See proposition A.2.3. ■

Some authors define 'relative $\mathcal{I}$-cell complex' so that every such morphism admits a *sequential* presentation. The following lemma and its corollary show that there is no loss of generality in doing so.

Lemma 0.4.6. *Let κ be a regular cardinal, let $\mathcal{C}$ be a category with colimits for all κ-small diagrams, and let α be an ordinal of cardinality less than κ. For each ordinal $\beta < \alpha$, let $e_\beta : U_\beta \to V_\beta$ be a morphism in $\mathcal{C}$, and for each ordinal $\beta \le \alpha$, let*

$$C_\beta = \left(\coprod_{\gamma < \beta} V_\gamma \right) \amalg \left(\coprod_{\beta \le \gamma < \alpha} U_\gamma \right)$$

be a coproduct in $\mathcal{C}$ with coproduct insertions $u_{\gamma,\beta} : U_\gamma \to C_\beta$ (for $\beta \le \gamma < \alpha$) and $v_{\gamma,\beta} : V_\gamma \to C_\beta$ (for $\gamma < \beta$).

Given ordinals $\beta < \beta' \le \alpha$, there is a unique morphism $C_\beta \to C_{\beta'}$ such that, for $\zeta < \beta \le \zeta' < \beta' \le \zeta''$, the following diagrams commute:

$$\begin{array}{ccc} V_\zeta & \xrightarrow{v_{\zeta,\beta}} & C_\beta \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ V_\zeta & \xrightarrow[v_{\zeta,\beta'}]{} & C_{\beta'} \end{array} \qquad \begin{array}{ccc} U_{\zeta'} & \xrightarrow{u_{\zeta',\beta}} & C_\beta \\ {\scriptstyle e_{\zeta'}}\downarrow & & \downarrow \\ V_{\zeta'} & \xrightarrow[v_{\zeta',\beta'}]{} & C_{\beta'} \end{array} \qquad \begin{array}{ccc} U_{\zeta''} & \xrightarrow{u_{\zeta'',\beta}} & C_\beta \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ U_{\zeta''} & \xrightarrow[u_{\zeta'',\beta'}]{} & C_{\beta'} \end{array}$$

This yields a functor $C_\bullet : [\alpha] \to C$, and it preserves colimits. Moreover, the diagrams below are pushout squares for all ordinals $\beta < \alpha$:

$$\begin{array}{ccc} U_\beta & \xrightarrow{u_{\beta,\beta}} & C_\beta \\ {\scriptstyle e_\beta}\downarrow & & \downarrow \\ V_\beta & \xrightarrow[v_{\beta,\beta+1}]{} & C_{\beta+1} \end{array}$$

Proof. This is a straightforward exercise. See Proposition 10.2.7 in [Hirschhorn, 2003]. □

Corollary 0.4.7. *Let κ be a regular cardinal, let C be a category with colimits for κ-small diagrams, and let $\mathcal{I}$ be a subset of* $\operatorname{mor} C$. *If $f : X \to Y$ is a relative $\mathcal{I}$-cell complex in C that admits a κ-small presentation, and either*

- *$X = Y$ and $f = \mathrm{id}_X$, or*
- *f is an isomorphism and $\mathcal{I}$ contains an isomorphism, or*
- *f is not an isomorphism,*

then f also admits a κ-small sequential presentation.

Proof. We have already commented on the first two cases in remark 0.4.2. The third case is proven by transfinite induction, where in the induction step we may assume that f is presented by just one pushout diagram:

$$\begin{array}{ccc} \coprod_{j \in T} U_j & \xrightarrow{u} & X \\ {\scriptstyle \coprod_{j\in T} e_j}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_{j \in T} V_j & \xrightarrow[v]{} & Y \end{array}$$

By decomposing the morphism $\coprod_{j \in T} e_j : \coprod_{j \in T} U_j \to \coprod_{j \in T} V_j$ as in the earlier lemma and applying the pushout pasting lemma, we obtain a sequential presentation of f, which is κ-small precisely if $|T| < \kappa$. ■

Definition 0.4.8. Let $\mathbf{U}$ be a universe, let C be a category, let $\mathcal{I}$ be a subset of $\operatorname{mor} C$, and let $\operatorname{cell}_{\mathcal{I},\mathbf{U}} C$ be the set of relative $\mathcal{I}$-cell complexes in C that have a $\mathbf{U}$-small presentation. We say $(\mathcal{I}, C)$ is **admissible for the U-small object argument** when the following conditions are satisfied:

- $\mathcal{I}$ is a **U**-set.
- $\mathcal{C}$ be a locally **U**-small category with colimits for all **U**-small diagrams.
- There is a regular cardinal κ in **U** such that, for every morphism $e : U \to V$ in $\mathcal{I}$, every ordinal α in **U**, and every functor $X_\bullet : \alpha \to \mathcal{C}$, if $|\alpha| \geq \kappa$, and the morphism $X_{\beta\to\gamma} : X_\beta \to X_\gamma$ is in $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$ for all ordinals $\beta \leq \gamma < \alpha$, then the canonical comparison map $\varinjlim_{\beta<\alpha} \mathcal{C}(U, X_\beta) \to \mathcal{C}\left(U, \varinjlim_{\beta<\alpha} X_\beta\right)$ is a bijection.

The **sequential U-rank** of $\mathcal{I}$ in $\mathcal{C}$ is the least cardinal κ with the above property.

Remark 0.4.9. Notice that, if $|\alpha| \geq \kappa$, then α is a κ-directed preorder. Thus, for any locally presentable **U**-category $\mathcal{C}$ and any **U**-subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ whatsoever, $(\mathcal{I}, \mathcal{C})$ is admissible for the **U**-small object argument.

Definition 0.4.10. Let **U** be a universe. A **U-cofibrantly-generated factorisation system** on a category $\mathcal{C}$ on is a weak factorisation system on $\mathcal{C}$ that is cofibrantly generated by some **U**-subset of $\operatorname{mor} \mathcal{C}$.

Theorem 0.4.11 (Quillen's small object argument)**.** *Let* **U** *be a universe, let* $\mathcal{C}$ *be a locally* **U**-*small category with colimits for all* **U**-*small diagrams, and let* $\mathcal{I}$ *be a* **U**-*subset of* $\operatorname{mor} \mathcal{C}$.

(i) *There exist a functor* $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ *and two natural transformations* $i : \operatorname{dom} \Rightarrow M$, $p : M \Rightarrow \operatorname{codom}$ *such that, for all morphisms* $f : X \to Y$ *in* $\mathcal{C}$, *the morphism* $i_f : X \to M(f)$ *is in* $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}$, *and we have* $f = p_f \circ i_f$.

(ii) *If* $(\mathcal{I}, \mathcal{C})$ *is moreover admissible for the* **U**-*small object argument, then we may choose* M, i, *and* p *so that, for all morphisms* $f : X \to Y$ *in* $\mathcal{C}$, *the morphism* $p_f : M(f) \to Y$ *in* $\operatorname{inj}^{\mathcal{I}} \mathcal{C}$.

(iii) *In particular, if* $(\mathcal{I}, \mathcal{C})$ *is admissible, then* $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ *is a* **U**-*cofibrantly-generated factorisation system on* $\mathcal{C}$ *and extends to a functorial weak factorisation system.*

Proof. (i). Let κ be any regular cardinal, and let α be the least ordinal of cardinality κ.[2] For each morphism $f : X \to Y$ in $\mathcal{C}$, we construct by transfinite

[2] We could also take $\kappa = 0$, but then the factorisation so obtained is trivial.

recursion a colimit-preserving functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ and a cocone $p_{f;\bullet} : M_\bullet(f) \to Y$ satisfying the following conditions:

- $M_0(f) = X$, $p_{f;0} = p$.
- For each ordinal $\beta < \alpha$, if $T_\beta(f)$ is the set of all commutative diagrams in $\mathcal{C}$ of the form below,

$$\begin{array}{ccc} U_{\beta,j} & \xrightarrow{u_{\beta,j}} & M_\beta(f) \\ {\scriptstyle e_{\beta,j}}\downarrow & & \downarrow{\scriptstyle p_{f;\beta}} \\ V_{\beta,j} & \xrightarrow[v_{\beta,j}]{} & Y \end{array}$$

  is in $\mathcal{I}$ and $u_{\beta,j} : U_{\beta,j} \to X_\beta$ is in $\mathcal{C}$, then $T_\beta(f)$ is a **U**-set (because $\mathcal{I}$ is a **U**-set and $\mathcal{C}$ is a locally **U**-small category), and we have a pushout square of the following form,

$$\begin{array}{ccc} \coprod_{j \in T_\beta(f)} U_{\beta,j} & \xrightarrow{u_\beta} & M_\beta(f) \\ {\scriptstyle \coprod_{j \in T_\beta(f)} e_{\beta,j}}\downarrow & & \downarrow{\scriptstyle X_{\beta\to\beta+1}} \\ \coprod_{j \in T_\beta(f)} V_{\beta,j} & \xrightarrow[\bar{v}_\beta]{} & M_{\beta+1}(f) \end{array}$$

  where $u_\beta : \coprod_{j \in T_\beta(f)} U_{\beta,j} \to X_\beta$ is the evident morphism induced by the universal property of coproducts. Observe that there is then a unique morphism $p_{f;\beta+1} : M_{\beta+1}(f) \to Y$ such that

$$p_{f;\beta+1} \circ M_{\beta\to\beta+1}(f) = p_\beta$$

  and

$$p_{f;\beta+1} \circ \bar{v}_{\beta,j} = v_{\beta,j}$$

  for all j in $T_\beta(f)$, where $\bar{v}_{\beta,j} : V_{\beta,j} \to M_{\beta+1}(f)$ is the evident component of $\bar{v}_\beta : \coprod_{j \in T_\beta(f)} V_{\beta,j} \to M_{\beta+1}(f)$.
- For limit ordinals $\gamma \le \alpha$, $M_\gamma(f) = \varinjlim_{\beta<\gamma} M_\beta(f)$, and $p_\gamma : M_\gamma(f) \to Y$ is defined by the universal property of X_γ.

It is not hard to see that the functor $M_\bullet(f) : [\alpha] \to \mathcal{C}$ so defined is itself functorial in f; in particular, defining $M(f) = M_\alpha(f)$, $i_f = M_{0\to\alpha}(f)$, $p_f = p_{f;\alpha}$, we obtain a functor $M : [\mathbb{2}, \mathcal{C}] \to \mathcal{C}$ with two natural transformations $i : M \Rightarrow \mathrm{dom}$

and $p : M \Rightarrow \mathrm{codom}$; by construction, we have $f = p_f \circ i_f$, and $i_f : X \to M(f)$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\, \mathcal{C}$.

(ii). Now, take κ to be a regular cardinal as in definition 0.4.8. We wish to show that the morphism p_f constructed above has the right lifting property with respect to all morphisms in $\mathcal{I}$. Consider a lifting problem of the form below,

$$\begin{array}{ccc} U & \xrightarrow{u} & M(f) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle p_f} \\ V & \xrightarrow[v]{} & Y \end{array}$$

where $e : U \to V$ is in $\mathcal{I}$. Since $\mathcal{I}$ is admissible, there must exist an ordinal $\beta < \alpha$ and a morphism $u' : U \to M_\beta(f)$ such that $u = M_{\beta\to\alpha}(f) \circ u'$. We then obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{u'} & M_\beta(f) \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle p_{f;\beta}} \\ V & \xrightarrow[v]{} & Y \end{array}$$

Since this is one of the diagrams in the set $T_\beta(f)$, it must embed in a commutative diagram of the form below,

$$\begin{array}{ccccc} U & \xrightarrow{u'} & M_\beta(f) & \longrightarrow & M_\alpha(f) \\ {\scriptstyle e}\downarrow & & \downarrow & \nearrow & \\ V & \longrightarrow & M_{\beta+1}(f) & & \downarrow{\scriptstyle p_{f;\alpha}} \\ \| & & & & \\ V & & \xrightarrow[v]{} & & Y \end{array}$$

and thus we have the required lift $V \to M(f)$.

(iii). Finally, apply proposition 0.4.5 and theorem A.2.23. ■

Corollary 0.4.12. *With other notation in the theorem, a morphism $g : Z \to W$ is in $\mathrm{cof}_{\mathcal{I}}\, \mathcal{C}$ if and only if there exists a commutative diagram of the following* *form in $\mathcal{C}$,*

$$\begin{array}{ccc} Z & \xrightarrow{i} & W' \\ {\scriptstyle g}\downarrow & \nearrow & \downarrow \\ W & \xrightarrow[\mathrm{id}]{} & W \end{array}$$

where $i : Z \to W'$ is in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$.

Proof. (i). If $g : Z \to W$ is in $\mathrm{cof}_{\mathcal{I}}\,\mathcal{C}$, then g has the left lifting property with respect to $p_g : M(g) \to W$, and so there exists a commutative diagram of the required form. Conversely, suppose we have $g = p \circ i$, $i = j \circ g$, and $\mathrm{id}_W = p \circ j$ for some $i : Z \to W'$ in $\mathrm{cell}_{\mathcal{I},\mathbf{U}}\,\mathcal{C}$ and some $j : W \to W'$ in $\mathcal{C}$. Then g is a retract of i,

$$\begin{array}{ccccc} Z & \xrightarrow{\mathrm{id}} & Z & \xrightarrow{\mathrm{id}} & Z \\ {\scriptstyle g}\downarrow & & {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{j} & W' & \xrightarrow{p} & W \end{array} \qquad p \circ j = \mathrm{id}$$

but proposition 0.4.5 says i is in $\mathrm{cof}_{\mathcal{I}}\,\mathcal{C}$, so by proposition A.2.9, g is also in $\mathrm{cof}_{\mathcal{I}}\,\mathcal{C}$. ■

Lemma 0.4.13. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $\mathcal{I}$ be a subset of* $\mathrm{mor}\,\mathcal{C}$, *and let κ be a regular cardinal. If $\mathcal{C}$ is closed in $\mathcal{C}^+$ under colimits for all κ-small diagrams, then* $\mathrm{cell}_{\mathcal{I},\kappa}\,\mathcal{C} = \mathrm{cell}_{\mathcal{I},\kappa}\,\mathcal{C}^+ \cap \mathrm{mor}\,\mathcal{C}$.

Proof. Obvious. □

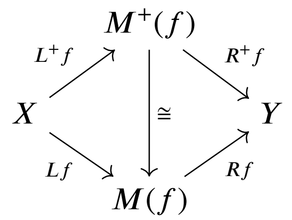
Theorem 0.4.14 (Stability of cofibrantly-generated factorisation systems)**.** *Let* $\mathbf{U}$ *and* $\mathbf{U}^+$ *be universes, with* $\mathbf{U} \in \mathbf{U}^+$. *Suppose:*

- *$\mathcal{C}$ is a locally* $\mathbf{U}$*-small and* $\mathbf{U}$*-cocomplete category.*
- *$\mathcal{C}^+$ is a locally* $\mathbf{U}^+$*-small and* $\mathbf{U}^+$*-cocomplete category.*
- *The inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ preserves colimits for all* $\mathbf{U}$*-small diagrams.*
- *$\mathcal{I}$ is a* $\mathbf{U}$*-subset of* $\mathrm{mor}\,\mathcal{C}$.
- *$(\mathcal{I}, \mathcal{C})$ is admissible for the* $\mathbf{U}$*-small object argument, and (L, R) is the functorial factorisation system on $\mathcal{C}$ constructed by Quillen's small object argument argument.*

- $(\mathcal{I}, \mathcal{C}^+)$ *is admissible for the* $\mathbf{U}^+$*-small object argument, and* (L^+, R^+) *is the functorial factorisation system on* $\mathcal{C}^+$ *constructed by Quillen's small object argument argument.*

Under these hypotheses, if the sequential $\mathbf{U}$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}$ *is equal to the sequential* $\mathbf{U}^+$*-rank of* $\mathcal{I}$ *in* $\mathcal{C}^+$*, then:*

(i) *For each morphism* $f : X \to Y$ *in* $\mathcal{C}$*, we have a commutative diagram of the following form in* $\mathcal{C}^+$*,*



and the isomorphism $M^+(f) \to M(f)$ *is moreover canonical and natural in* f*.*

(ii) *We have* $\operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C} \subseteq \operatorname{cell}_{\mathcal{I},\mathbf{U}} \mathcal{C}^+ \subseteq \operatorname{cell}_{\mathcal{I},\mathbf{U}^+} \mathcal{C}^+$.

(iii) $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}^+, \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+\right)$ *is an extension of* $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$.

Proof. (i). This can be seen by examining the explicit construction in the proof of theorem 0.4.11.

(ii). This is implied by the lemma.

(iii). Since $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}, \operatorname{inj}^{\mathcal{I}} \mathcal{C}\right)$ and $\left(\operatorname{cof}_{\mathcal{I}} \mathcal{C}^+, \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+\right)$ are both cofibrantly generated by $\mathcal{I}$, by proposition A.2.14, we have $\operatorname{inj}^{\mathcal{I}} \mathcal{C} \subseteq \operatorname{inj}^{\mathcal{I}} \mathcal{C}^+$ and so $\operatorname{cof}_{\mathcal{I}} \mathcal{C} \supseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}^+ \cap \operatorname{mor} \mathcal{C}$. It remains to be shown that $\operatorname{cof}_{\mathcal{I}} \mathcal{C} \subseteq \operatorname{cof}_{\mathcal{I}} \mathcal{C}^+$, but this is implied by corollary 0.4.12 applied to claim (ii). ■

Remark 0.4.15. Let κ be a regular cardinal in $\mathbf{U}$, let $\mathcal{B}$ be a $\mathbf{U}$-small category with colimits for all κ-small diagrams, let $\mathcal{C} = \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{B})$, and let $\mathcal{C}^+ = \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathcal{B})$. Then $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}^+$ is an accessible $(\kappa, \mathbf{U}, \mathbf{U}^+)$ extension, and any $\mathbf{U}$-subset $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ whatsoever will satisfy the hypotheses of the theorem.

Theorem 0.4.16 (Garner's small object argument)**.** *Let $\mathcal{C}$ be a locally presentable* $\mathbf{U}$*-category and let $\mathcal{I}$ be any* $\mathbf{U}$*-subset of* $\operatorname{mor} \mathcal{C}$*. There then exists an algebraic factorisation system* (L, R) *on $\mathcal{C}$ such that the induced weak factorisation system is cofibrantly generated by $\mathcal{I}$.*

Proof. See Theorem 4.4 in [Garner, 2009]. □

— I —

Simplicial sets

Simplicial sets, like simplicial complexes, are combinatorial models for spaces built up by gluing standard n-simplices together; unlike simplicial complexes, an n-simplex in a simplicial set need not be uniquely determined by its vertices. It is for this reason that simplicial sets were once known by the unwieldy name 'complete semi-simplicial (c.s.s.) complex'.

In the 1960s, it was discovered that one can mimic the definitions and constructions of classical homotopy theory by combinatorial means using simplicial sets, and that the resulting theory is moreover equivalent to the classical theory in a natural, functorial way. More recently, it has been shown that the homotopy theory of simplicial sets is *universal* in a precise sense,[1] so it seems fitting that we begin here.

1.1 Basics

Definition 1.1.1. The **simplex category** is the category $\mathbf{\Delta}$ whose objects are the positive finite ordinals and whose morphisms are the monotone maps. We use the geometer's convention: $[n]$ denotes the ordinal $\{0, 1, \ldots, n\}$.

Definition 1.1.2. A **simplicial object** in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and a **morphism of simplicial objects** in $\mathcal{C}$ is a natural transformation of such functors. The **category of simplicial objects** in $\mathcal{C}$ is the functor category $[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}]$ and is denoted by $\mathbf{s}\mathcal{C}$.

[1] See [Dugger, 2001].

Definition 1.1.3. The **coface maps** in $\mathbf{\Delta}$ are the morphisms $\delta^i_n : [n-1] \to [n]$, where δ^i_n is the unique injective monotone map that misses i; and the **codegeneracy maps** in $\mathbf{\Delta}$ are the morphisms $\sigma^i_n : [n+1] \to [n]$, where σ^i_n is the unique surjective monotone map with $\sigma^i_n(i) = \sigma^i_n(i+1) = i$.

Theorem 1.1.4 (Cosimplicial identities)**.** *The following equations hold in* $\mathbf{\Delta}$*:*

$$
\begin{aligned}
\delta^{j+1}_{n+1} \circ \delta^i_n &= \delta^i_{n+1} \circ \delta^j_n && \textit{if } 0 \le i \le j \le n \\
\sigma^j_n \circ \sigma^i_{n+1} &= \sigma^i_n \circ \sigma^{j+1}_{n+1} && \textit{if } 0 \le i \le j \le n \\
\sigma^{j+1}_{n+1} \circ \delta^i_{n+1} &= \delta^i_n \circ \sigma^j_n && \textit{if } 0 \le i \le j \le n \\
\delta^{j+1}_n \circ \sigma^i_n &= \sigma^i_{n+1} \circ \delta^{j+2}_{n+1} && \textit{if } 0 \le i < j < n \\
\sigma^i_n \circ \delta^i_n &= \mathrm{id} && \textit{if } 0 \le i \le n \\
\sigma^{i+1}_n \circ \delta^i_n &= \mathrm{id} && \textit{if } 0 \le i < n
\end{aligned}
$$

Equivalently, the following diagrams commute:

$$
\begin{array}{ccc}
[n-1] & \xrightarrow{\delta^i} & [n] \\
{\scriptstyle \delta^j}\big\downarrow & & \big\downarrow{\scriptstyle \delta^{j+1}} \\
[n] & \xrightarrow[\delta^i]{} & [n+1]
\end{array}
\qquad \textit{for } 0 \le i \le j \le n
$$

$$
\begin{array}{ccc}
[n+1] & \xrightarrow{\sigma^i} & [n] \\
{\scriptstyle \sigma^{j+1}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma^{j}} \\
[n] & \xrightarrow[\sigma^i]{} & [n-1]
\end{array}
\qquad \textit{for } 0 \le i \le j \le n
$$

$$
\begin{array}{ccc}
[n] & \xrightarrow{\delta^i} & [n+1] \\
{\scriptstyle \sigma^{j}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma^{j+1}} \\
[n-1] & \xrightarrow[\delta^i]{} & [n]
\end{array}
\qquad \textit{for } 0 \le i \le j \le n
$$

$$
\begin{array}{ccc}
[n] & \xrightarrow{\sigma^i} & [n-1] \\
{\scriptstyle \delta^{j+2}}\big\downarrow & & \big\downarrow{\scriptstyle \delta^{j+1}} \\
[n+1] & \xrightarrow[\sigma^i]{} & [n]
\end{array}
\qquad \textit{for } 0 \le i < j < n
$$

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta^i} & [n] \\ {\scriptstyle \delta^i}\downarrow & \overset{\text{id}}{\searrow} & \downarrow{\scriptstyle \sigma^{i+1}} \\ [n] & \xrightarrow[\sigma^i]{} & [n-1] \end{array} \quad \textit{for } 0 \le i \le n$$

Moreover, every morphism $[n] \to [m]$ *in* $\mathbf{\Delta}$ *is* uniquely *a composite of the form*

$$\delta_m^{j_1} \circ \cdots \circ \delta_k^{j_{m-k}} \circ \sigma_k^{i_{n-k}} \circ \cdots \circ \sigma_n^{i_1}$$

where $k \le \min\{n, m\}$*, and*

$$0 \le i_{n-k} \le \cdots \le i_1 \le n$$
$$0 \le j_{m-k} \le \cdots \le j_1 \le m$$

The category $\mathbf{\Delta}$ *is uniquely characterised by these properties.*

Proof. See [May, 1967, § 2], [GZ, Ch. II, § 2], or [Weibel, 1994, § 8.1]. □

Definition 1.1.5. Let A be a simplicial object in a category $\mathcal{C}$. A **face operator** for A is a morphism of the form $A(\delta_n^i) : A([n]) \to A([n-1])$, and a **degeneracy operator** for A is a morphism of the form $A(\sigma_n^i) : A([n]) \to A([n+1])$. For brevity, we will usually write A_n instead of $A([n])$, d_i^n instead of $A(\delta_n^i)$, and s_i^n instead of $A(\sigma_n^i)$.

Corollary 1.1.6 (Simplicial identities)**.** *The face and degeneracy operators of a simplicial object satisfy the formal duals of the equations in theorem 1.1.4.* ■

Corollary 1.1.7. *A simplicial object A is uniquely determined by the sequence of objects* $A_0, A_1, A_2, \ldots$ *together with the face and degeneracy operators. Conversely, any sequence of objects equipped with face and degeneracy operators satisfying the simplicial identities defined a simplicial object.* ■

Definition 1.1.8. A **simplicial set** is a simplicial object in **Set**, and the **category of simplicial sets** is denoted by **sSet**.

Lemma 1.1.9.

(i) *Limits (resp. colimits) in* **sSet** *are constructed degreewise: a cone (resp. cocone) in* **sSet** *over a diagram is limiting (resp. colimiting) if and only if it is so in every degree.*

(ii) *A morphism of* **sSet** *is monic (resp. epic) if and only if it is degreewise injective (resp. surjective).*

Proof. These are standard facts about functor categories. □

Definition 1.1.10. The **standard** ***n*****-simplex** in **sSet**, denoted by Δ^n, is the representable presheaf $\mathbf{\Delta}(-, [n])$.

Theorem 1.1.11. *Let* $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ *be the functor* $[n] \mapsto \Delta^n$.

(i) *For any simplicial set* X*, the map* $\mathbf{sSet}(\Delta^n, X) \to X_n$ *defined by* $f \mapsto f_n(\mathrm{id}_{[n]})$ *is a bijection and is moreover natural in* $[n]$ *and* X.

(ii) **sSet** *has limits and colimits for all small diagrams, every epimorphism is effective, and for all morphisms* $f : X \to Y$ *in* **sSet***, the pullback functor* $f^* : \mathbf{sSet}_{/Y} \to \mathbf{sSet}_{/X}$ *preserves colimits.*

(iii) $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ *is a dense functor, i.e. for any simplicial set* X*, the tautological cocone*[1] *from the canonical diagram* $(\Delta^\bullet \downarrow X) \to \mathbf{sSet}$ *to* X *is colimiting.*

(iv) *Let* $\mathcal{E}$ *be a locally small category with colimits for all small diagrams. If* $F : \mathbf{sSet} \to \mathcal{E}$ *is a functor that preserves small colimits, then it is left adjoint to the functor* $\mathcal{E} \to \mathbf{sSet}$ *defined by* $E \mapsto \mathcal{E}(F\Delta^\bullet, E)$.

(v) *With* $\mathcal{E}$ *as above, the functor* $F \mapsto F\Delta^\bullet$ *from the category of colimit-preserving functors* $\mathbf{sSet} \to \mathcal{E}$ *to the category of all functors* $\mathbf{\Delta} \to \mathcal{E}$ *is fully faithful and essentially surjective on objects.*

Proof. Claim (i) is just the Yoneda lemma, claim (ii) follows from the lemma above, and claims (iii)–(v) are just facts about dense functors, pointwise left Kan extensions, weighted colimits: see proposition A.4.20, theorem A.4.11, and proposition A.5.11. ■

Definition 1.1.12. An element of X_n is called an ***n*****-simplex** of X; in particular, an element of X_0 is a **vertex** of X and an element of X_1 is an **edge** of X. This is justified by statement (i) in the above theorem. Given an edge f of X, the **source** of f is the vertex $d_1(f)$, and the **target** of f is the vertex $d_0(f)$; we write $f : x \to y$ to mean $d_1(f) = x$ and $d_2(f) = y$.

[1] See definition A.4.10.

Definition 1.1.13. The **standard n-simplex** in **Top**, denoted by $|\Delta^n|$, is the topological space

$$|\Delta^n| = \left\{ \left(x_0, \dots, x_n\right) \in [0,1]^{n+1} \;\middle|\; x_0 + \cdots + x_n = 1 \right\}$$

where $[0,1]$ is the closed unit interval with the standard metric. The functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ sends $[n]$ to $|\Delta^n|$ and is defined on morphisms by linearly interpolating the obvious map of vertices.

Corollary 1.1.14. *There exists an adjunction*

$$|{-}| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$$

extending the functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ defined above, and this adjunction is unique up to unique isomorphism. Explicitly, we may take

$$\mathrm{S}(Y)_n = \mathbf{Top}\left(|\Delta^n|, Y\right)$$

with the evident face and degeneracy operators induced by the coface and codegeneracy maps in $\mathbf{\Delta}$. ■

Definition 1.1.15. The **geometric realisation** of a simplicial set X is the topological space $|X|$, and the **singular set** of a topological space Y is the simplicial set $\mathrm{S}(Y)$.

Remark 1.1.16. The geometric realisation $|X|$ is stable under universe enlargement, by theorem 0.3.17.

Theorem 1.1.17. *Let* **CGHaus** *be the category of compactly-generated Hausdorff spaces*[2] *and continuous maps.*

(i) *The topological standard n-simplex $|\Delta^n|$ is a compact Hausdorff space.*

(ii) *For any simplicial set X, the geometric realisation $|X|$ is a compactly-generated Hausdorff space.*

(iii) *The previously-constructed adjunction $|{-}| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$ restricts to an adjunction between* **CGHaus** *and* **sSet***, and moreover the functor $|{-}| : \mathbf{sSet} \to \mathbf{CGHaus}$ preserves finite limits and reflects isomorphisms.*

Proof. Claim (i) is a standard fact, while claims (ii) and (iii) are proven in [GZ, Ch. III, § 3]. □

[2] See definition A.1.26.

1.2 Nerves, skeletons, and coskeletons

Prerequisites. § 1.1, A.1.

Proposition 1.2.1. *Let* $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{C})_n = \mathrm{Fun}([n], \mathbb{C})$$

where $[n]$ *here denotes the preorder category* $\{0 \to \cdots \to n\}$.

(i) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *has a left adjoint* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *such that* $\tau_1 \Delta^n = [n]$.

(ii) *The functor* N *is fully faithful and exhibits* **Cat** *as a reflective subcategory of* **sSet**.

(iii) $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(iv) *The functor* τ_1 *preserves finite products.*

Proof. (i). Apply theorem 1.1.11.

(ii). A functor is entirely determined by its action on objects, arrows, and composable strings of arrows, so N is fully faithful.

(iii). N preserves binary products, so we have the following natural bijections:

$$\begin{aligned}
\mathbf{sSet}(\Delta^n, \mathrm{N}([\mathbb{C}, \mathbb{D}])) &\cong \mathrm{Fun}([n], [\mathbb{C}, \mathbb{D}]) \\
&\cong \mathrm{Fun}([n] \times \mathbb{C}, \mathbb{D}) \\
&\cong \mathbf{sSet}(\mathrm{N}([n] \times \mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]) \times \mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})) \\
&\cong \mathbf{sSet}(\mathrm{N}([n]), [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]) \\
&\cong \mathbf{sSet}(\Delta^n, [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})])
\end{aligned}$$

Thus, by the Yoneda lemma, the canonical morphism $\mathrm{N}([\mathbb{C}, \mathbb{D}]) \to [\mathrm{N}(\mathbb{C}), \mathrm{N}(\mathbb{D})]$ is an isomorphism.

(iv). It is clear that τ_1 preserves terminal objects. Let X and Y be simplicial sets. We wish to show that the canonical morphism $\tau_1(X \times Y) \to \tau_1 X \times \tau_1 Y$ is an isomorphism; but since τ_1 is a left adjoint and both **sSet** and **Cat** are cartesian closed, it is enough to check the claim for $Y = \Delta^n$, because **sSet** is generated under colimits by $\{\Delta^n \mid n \in \mathbb{N}\}$. We have the following natural bijections:

$$\begin{aligned}
\mathrm{Fun}\big(\tau_1(X \times \Delta^n), \mathbb{C}\big) &\cong \mathbf{sSet}(X \times \Delta^n, \mathrm{N}(\mathbb{C})) \\
&\cong \mathbf{sSet}\big(X, \mathrm{N}(\mathbb{C})^{\Delta^n}\big) \\
&\cong \mathbf{sSet}(X, \mathrm{N}([[n], \mathbb{C}])) \\
&\cong \mathrm{Fun}\big(\tau_1 X, [[n], \mathbb{C}]\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times [n], \mathbb{C}\big) \\
&\cong \mathrm{Fun}\big(\tau_1 X \times \tau_1 \Delta^n, \mathbb{C}\big)
\end{aligned}$$

The claim follows by the Yoneda lemma. ■

Definition 1.2.2. The **fundamental category** of a simplicial set X is the small category $\tau_1 X$, and the **nerve** of a small category $\mathbb{C}$ is the simplicial set $\mathrm{N}(\mathbb{C})$.

Remark 1.2.3. Given a simplicial set X, the fundamental category $\tau_1 X$ admits the following presentation by generators and relations: the objects are the vertices of X, and the arrows are generated by the edges of X, modulo the relation $d_0(\alpha) \circ d_2(\alpha) = d_1(\alpha)$ for all 2-simplices α in X. This shows that $\tau_1 X$ is stable under universe enlargement.

Proposition 1.2.4. *Let* disc : **Set** → **sSet** *be the functor defined by the formula*

$$(\operatorname{disc} Y)_n = Y$$

with id_Y *for all the face and degeneracy maps.*

(i) disc : **Set** → **sSet** *has a left adjoint* π_0 : **sSet** → **Set** *such that* $\pi_0 \Delta^n = 1$.

(ii) *The functor* disc *is fully faithful and exhibits* **Set** *as a reflective subcategory of* **sSet**.

(iii) N : **Set** → **sSet** *is a cartesian closed functor.*

(iv) *The functor* π_0 *preserves products.*

Proof. (i). We could apply theorem 1.1.11, but it is also fairly straightforward to check that this explicit construction works: for each simplicial set X, we define $\pi_0 X$ by the coequaliser diagram in **Set** shown below,

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0 \longrightarrow \pi_0 X$$

and for each morphism $f : X \to Y$ in **sSet**, we define $\pi_0 f$ to be the unique morphism making the evident diagram commute.

(ii). It is clear that disc is fully faithful.

(iii). By proposition A.1.15, we have an analogous adjunction $\pi_0 \dashv \operatorname{disc} : \mathbf{Set} \to \mathbf{Cat}$. It is clear that we have a natural isomorphism $\mathrm{N}(\operatorname{disc} Y) \cong \operatorname{disc} Y$ for every set Y, and we know $\operatorname{disc} : \mathbf{Set} \to \mathbf{Cat}$ and $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ are cartesian closed functors, so $\operatorname{disc} : \mathbf{Set} \to \mathbf{sSet}$ must also be cartesian closed.

(iv). Similarly, for any simplicial set X, we have a natural isomorphism $\pi_0 X \cong \pi_0 \tau_1 X$; but we know that $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ preserves finite products, and $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ preserves finite products by proposition 1.2.1, so $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ must also preserve finite products. ■

Definition 1.2.5. The **set of connected components** of a simplicial set X is the set $\pi_0 X$, and a **discrete simplicial set** is one that is isomorphic to $\operatorname{disc} Y$ for some set Y.

¶ **1.2.6.** We will usually not distinguish between Y and $\operatorname{disc} Y$ notationally.

Proposition 1.2.7. *Let* $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *be the functor defined by the formula*

$$\mathrm{N}(\mathbb{G})_n = \mathrm{Fun}(\mathbf{I}[n], \mathbb{G})$$

where $\mathbf{I}[n]$ *here denotes the groupoid obtained by freely inverting the arrows in the preorder category* $[n]$.

(i) *For any groupoid* $\mathbb{G}$*, the nerve* $\mathrm{N}(\mathbb{G})$ *is the same (up to isomorphism) whether computed for* $\mathbb{G}$ *as a groupoid or* $\mathbb{G}$ *as a category.*

(ii) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *has a left adjoint* $\pi_1 : \mathbf{sSet} \to \mathbf{Grpd}$ *such that* $\pi_1 \Delta^n = \mathbf{I}[n]$.

(iii) *The functor* N *is fully faithful and exhibits* **Grpd** *as a reflective subcategory of* **sSet**.

(iv) $\mathrm{N} : \mathbf{Grpd} \to \mathbf{sSet}$ *is a cartesian closed functor.*

(v) *The functor* π_1 *preserves finite products.*

Proof. (i). By the universal property of $\mathbf{I}[n]$, there is a natural bijection

$$\mathrm{Fun}(\mathbf{I}[n], \mathbb{G}) \cong \mathrm{Fun}([n], \mathbb{G})$$

for all groupoids $\mathbb{G}$, so the two nerve constructions do indeed agree.

(ii) and (iii). These are proven in exactly the same way as in proposition 1.2.1.

(iv) and (v). These are proven in exactly the same way as in proposition 1.2.4. ∎

Definition 1.2.8. The **fundamental groupoid** of a simplicial set X is the small groupoid $\pi_1 X$.

Remark 1.2.9. Given a simplicial set X, the fundamental groupoid $\pi_1 X$ admits a presentation of the same kind as the fundamental category $\tau_1 X$, and in fact $\pi_1 X$ is isomorphic to the groupoid obtained by freely inverting the arrows in $\tau_1 X$:

$$\mathrm{Fun}\big(\pi_1 X, \mathbb{G}\big) \cong \mathbf{sSet}(X, \mathrm{N}(\mathbb{G})) \cong \mathrm{Fun}\big(\tau_1 X, \mathbb{G}\big)$$

This shows that $\pi_1 X$ is stable under universe enlargement.

Definition 1.2.10. Let n be a natural number, and let $\mathbf{\Delta}_{\leq n}$ be the full subcategory of $\mathbf{\Delta}$ spanned by the objects $[0], \ldots, [n]$. An ***n*-truncated simplicial set** is a functor ${\mathbf{\Delta}_{\leq n}}^{\mathrm{op}} \to \mathbf{Set}$, and we write $\mathbf{sSet}_{\leq n}$ for the category of n-truncated simplicial sets. The **brutal *n*-truncation** of a simplicial set X is the n-truncated simplicial set $X_{\leq n}$ defined by the evident reduct:

$$X_{\leq n}([m]) = X([m])$$

Proposition 1.2.11. *Let n be a natural number, and let $j : \mathbf{\Delta}_{\leq n} \to \mathbf{\Delta}$ be the inclusion.*

(i) *The functor $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$ has a left adjoint $\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$.*

(ii) *The unit $\mathrm{id} \Rightarrow j^* \mathrm{Lan}_j$ is a natural isomorphism.*

(iii) *$\mathrm{Lan}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$ is a fully faithful functor.*

(i′) *The functor $j^* : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$ has a right adjoint $\mathrm{Ran}_j : \mathbf{sSet}_{\leq n} \to \mathbf{sSet}$.*

(ii′) *The counit $j^* \mathrm{Ran}_j \Rightarrow \mathrm{id}$ is a natural isomorphism.*

(iii′) $\operatorname{Ran}_j : \mathbf{sSet}_{\le n} \to \mathbf{sSet}$ *is a fully faithful functor.*

Proof. (i) and (i′). Use theorem A.4.11.

(ii) and (ii′). The inclusion $j : \mathbf{\Delta}_{\le n} \to \mathbf{\Delta}$ is fully faithful, so the unit $\mathrm{id} \Rightarrow j^* \operatorname{Lan}_j$ and the counit $j^* \operatorname{Ran}_j \Rightarrow \mathrm{id}$ are natural isomorphisms, by corollary A.4.15.

(iii) and (iii′). It is a well-known fact that the unit (resp. counit) of an adjunction is a natural isomorphism if and only if the left (resp. right) adjoint is fully faithful.[1] ■

Definition 1.2.12. For each natural number n, with notation as above, let $\mathrm{sk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Lan}_j j^*$, and let $\mathrm{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$ be the composite $\operatorname{Ran}_j j^*$. The ***n*-skeleton** of a simplicial set X is the simplicial set $\mathrm{sk}_n(X)$, and the ***n*-coskeleton** of a simplicial set is the simplicial set $\mathrm{cosk}_n(X)$. A ***n*-skeletal simplicial set** is one that is isomorphic to the n-skeleton of some simplicial set, and an ***n*-coskeletal simplicial set** is one that is isomorphic to the n-coskeleton of some simplicial set.

Remark 1.2.13. In the special case $n = 0$, Lan_j may be identified with the functor $\mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$ defined in proposition 1.2.4. Thus, 0-skeletal simplicial sets are precisely the discrete simplicial sets. On the other hand, given a set X, $\operatorname{Ran}_j X$ can be identified with the simplicial set whose m-simplices are $(m + 1)$-tuples of elements of X, with face and degeneracy maps induced by the appropriate projections.

Proposition 1.2.14. *Let n be a natural number.*

(i) *The full subcategory of n-skeletal simplicial sets is a coreflective subcategory of* **sSet**, *with coreflector* sk_n.

(ii) sk_n *is the underlying endofunctor of an idempotent comonad on* **sSet**.

(iii) *A simplicial set X is n-skeletal if and only if the counit $\mathrm{sk}_n(X) \to X$ is an isomorphism.*

(iv) *If $m \ge n$, then any n-skeletal simplicial set is also m-skeletal.*

[1] See e.g. [CWM, Ch. IV, § 3].

(i′) *The full subcategory of n-coskeletal simplicial sets is a reflective subcategory of* **sSet**, *with reflector* cosk_n.

(ii′) cosk_n *is the underlying endofunctor of an idempotent monad on* **sSet**.

(iii′) *A simplicial set* X *is n-coskeletal if and only if the unit* $X \to \operatorname{cosk}_n(X)$ *is an isomorphism.*

(iv′) *If* $m \geq n$*, then any n-coskeletal simplicial set is also m-coskeletal.*

Proof. All straightforward from the definitions. □

Proposition 1.2.15. *Let n be a natural number, and let X be a simplicial set.*

(i) *We have the following adjunction:*

$$\operatorname{sk}_n \dashv \operatorname{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$$

(ii) *The counit* $\operatorname{sk}_n(X) \to X$ *is a monomorphism, and* X *is n-skeletal if and only if all m-simplices of* X *are degenerate for* $m > n$.

(iii) X *is n-coskeletal if and only if, for all natural numbers m, the map*

$$X_m \cong \mathbf{sSet}(\Delta^m, X) \to \mathbf{sSet}\big(\operatorname{sk}_n(\Delta^m), X\big)$$

induced by the counit $\operatorname{sk}_n(\Delta^m) \to \Delta^m$ *is a bijection.*

Proof. (i). Immediate from the definition of sk_n and cosk_n.

(ii). The most straightforward way of seeing this is to construct $\operatorname{sk}_n(X)$ explicitly as the smallest simplicial subset of X containing all of its n-simplices.

(iii). Apply the Yoneda lemma in conjunction with claim (i). ■

Example 1.2.16. For any small category $\mathbb{C}$, the nerve $\mathrm{N}(\mathbb{C})$ is a 2-coskeletal simplicial set: by definition, an m-simplex of $\mathrm{N}(\mathbb{C})$ is just a functor $[m] \to \mathbb{C}$, but the property of being a functor can be detected by only inspecting the vertices, edges, and 2-cells.

Proposition 1.2.17. *The following full subcategories are exponential ideals of* **sSet***:*

(i) *Discrete simplicial sets.*

(ii) *Simplicial sets isomorphic to the nerve of some category.*

(iii) *Simplicial sets isomorphic to the nerve of some groupoid.*

(iv) *n-coskeletal simplicial sets for some natural number n.*

Proof. Apply proposition A.1.13 to propositions 1.2.4, 1.2.1, 1.2.7, and 1.2.14. ■

1.3 The Kan–Quillen model structure

Prerequisites. §§ 1.1, A.2.

In [1967], Quillen constructed an axiomatic framework for doing homotopy theory in abstract categories, which he called 'closed model categories', and showed that **sSet** can be endowed with a model structure such that the resulting homotopy theory is equivalent in a strong sense to the homotopy theory of topological spaces.

Definition 1.3.1. A **horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where Λ^n_k is the union of the images of $\delta^0_n, \dots, \delta^{k-1}_n, \delta^{k+1}_n, \dots, \delta^n_n : \Delta^{n-1} \to \Delta^n$ in **sSet**. In other words, Λ^n_k is the union of all the faces of Δ^n that include the k-th vertex. The **boundary** of Δ^n is the simplicial subset $\partial\Delta^n \subseteq \Delta^n$ generated by the images of $\delta^0_n, \dots, \delta^n_n : \Delta^{n-1} \to \Delta^n$.

Remark 1.3.2. The boundary $\partial\Delta^n$ may be identified with $\mathrm{sk}_{n-1}\Delta^n$.

Definition 1.3.3. A **cofibration** in **sSet** is a monomorphism. A **Kan fibration** is a morphism $f : X \to Y$ in **sSet** that has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, where $n \geq 1$ and $0 \leq k \leq n$. A **Kan complex** is a simplicial set X such that the unique morphism $X \to 1$ is a Kan fibration.

Remark 1.3.4. In other words, a Kan complex is a simplicial set X satisfying the **Kan condition**: every horn $\alpha' : \Lambda^n_k \to X$ has a **filler**, i.e. a morphism $\alpha : \Delta^n \to X$ (equivalently, an n-simplex of X) such that α' is the restriction along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$.

Lemma 1.3.5. *If X is a Kan complex, then the fundamental category $\tau_1 X$ is a groupoid, and the unit $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is an epimorphism.*

Proof. Let x, y, and z be vertices in X, and let $f : x \to y$ and $g : y \to z$ be edges in X.[1] Then the pair (f, g) defines a horn $\Lambda^2_1 \to X$, and so by the Kan condition, there exists a 2-simplex α of X such that $d_2(\alpha) = f$ and $d_0(\alpha) = g$. By remark remark 1.2.3, the composite $g \circ f$ defined in $\tau_1 X$ must correspond to the edge $d_1(\alpha)$. Since the arrows in $\tau_1 X$ are generated by the edges of X, we conclude by induction that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on vertices and edges.

Similarly, given an edge $f : x \to y$, the Kan condition ensures that there exist two 2-simplices β and γ such that

$$\begin{aligned} d_2(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_x \\ d_0(\alpha) &= f & \qquad d_1(\alpha) &= \mathrm{id}_y \end{aligned}$$

where $\mathrm{id}_x : x \to x$ is the edge $s_0(x)$, and $\mathrm{id}_y : y \to y$ is the edge $s_0(y)$. Together with the argument in the previous paragraph, this shows that $\tau_1 X$ is a groupoid.

Finally, to show that $\eta_X : X \to \mathrm{N}(\tau_1 X)$ is a surjection on n-simplices for $n \geq 2$, we simply observe that an n-simplex of $\mathrm{N}(\tau_1 X)$ is just a string of n composable edges of X, so we may appeal to the Kan condition again to obtain the corresponding n-simplex of X. ■

Corollary 1.3.6. *If X is a Kan complex, then the unit $\eta_X : X \to \mathrm{N}(\pi_1 X)$ is an epimorphism.*

Proof. Since $\tau_1 X$ is already a groupoid, the canonical functor $\tau_1 X \to \pi_1 X$ must be an isomorphism. (See remark 1.2.9.) ■

Proposition 1.3.7.

(i) *There exists a functorial weak factorisation system on* **sSet** *such that the right class is the class of all Kan fibrations, and every morphism in the left class is a monomorphism (but not vice versa).*

(ii) *There exists a functorial weak factorisation system on* **sSet** *such that the left class is the class of all monomorphisms, and every morphism in the right class is a Kan fibration (but not vice versa).*

Proof. Each claim can be proven by a suitable small object argument. See Theorems 3.1.1 and 3.1.2 in [Joyal and Tierney, 2008] as well as Theorem 2.1.14 in [Hovey, 1999]. □

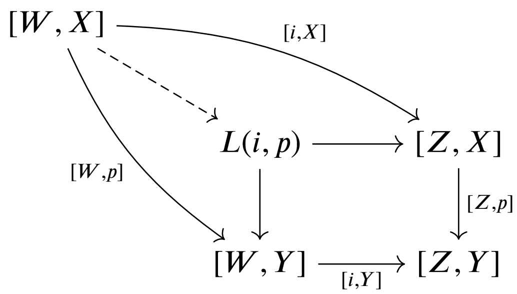
[1] Recall definition 1.1.12.

Definition 1.3.8. An **anodyne extension**, or **trivial cofibration** in **sSet**, is a cofibration that has the left lifting property with respect to all Kan fibrations. A **trivial Kan fibration** is a Kan fibration that has the right lifting property with respect to all cofibrations.

Proposition 1.3.9. *A morphism $f : X \to Y$ is a trivial Kan fibration if and only if it has the right lifting property with respect to the boundary inclusions $\partial\Delta^n \hookrightarrow \Delta^n$, for all $n \geq 0$.*

Proof. See the remarks at the beginning of [Joyal and Tierney, 2008, § 3.1], or Proposition 1 in [Quillen, 1967, Ch. II, § 2]. ■

Proposition 1.3.10. *Let $i : Z \to W$ be a cofibration in* **sSet** *and let $p : X \to Y$ be a Kan fibration. Suppose we have a commutative diagram*



where the square in the lower right is a pullback square.

(i) *The unique morphism $[W, X] \to L(i, p)$ making the diagram commute is a Kan fibration.*

(ii) *If $i : Z \to W$ is an anodyne extension, then $[W, X] \to L(i, p)$ is a trivial Kan fibration.*

(iii) *If $p : Z \to W$ is a trivial Kan fibration, then so is $[W, X] \to L(i, p)$.*

Proof. (i). See Theorem 3.3.1 in [Hovey, 1999], or Proposition 5.2 in [GJ, Ch. I].

(ii) and (iii). See Proposition 11.5 in [GJ, Ch. I]; for a purely combinatorial proof, see Theorem 3.2.1 in [Joyal and Tierney, 2008]. □

Corollary 1.3.11.

(i) *If $p : X \to Y$ is a Kan fibration (resp. trivial Kan fibration), then for all simplicial sets W, the morphism $[W, p] : [W, X] \to [W, Y]$ is also a Kan fibration (resp. trivial Kan fibration).*

(ii) *If $i : Z \to W$ is a cofibration (resp. anodyne extension) and X is a Kan complex, then the morphism $[i, X] : [W, X] \to [Z, X]$ is a Kan fibration (resp. trivial Kan fibration).*

(iii) *If W is any simplicial set and X is a Kan complex, then $[W, X]$ is also a Kan complex.*

Proof. (i). Take $Z = \varnothing$; noting that the canonical morphism $\varnothing \to W$ is a cofibration, and that $[\varnothing, p] : [\varnothing, X] \to [\varnothing, Y]$ is an isomorphism, the proposition above then implies $[W, p] : [W, X] \to [W, Y]$ is a Kan fibration (resp. trivial Kan fibration).

(ii). Take $Y = 1$; since $[W, 1] \to [Z, 1]$ is an isomorphism, the proposition above implies $[i, X] : [W, X] \to [Z, X]$ is a Kan fibration (resp. trivial Kan fibration).

(iii). Noting that $[\varnothing, X]$ is a terminal object in **sSet**, we apply claim (ii) to the case $Z = \varnothing$ to obtain the desired conclusion. ■

The following combinatorial definition of weak homotopy equivalence is due to Joyal and Tierney [2008]. Recalling the definition of $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ from proposition 1.2.4 as the functor sending a simplicial set X to the set π_0 of its connected components,

Definition 1.3.12. A **weak homotopy equivalence of simplicial sets** is a morphism $f : W \to Z$ such that, for every Kan complex K, the induced map

$$\pi_0[f, K] : \pi_0[Z, K] \to \pi_0[W, K]$$

is a bijection of sets.

Proposition 1.3.13.

(i) *A Kan fibration $p : X \to Y$ is trivial if and only if it is a weak homotopy equivalence.*

(ii) *A cofibration $i : Z \to W$ is an anodyne extension if and only if it is a weak homotopy equivalence.*

Proof. See Propositions 3.4.1 and 3.4.2 in [Joyal and Tierney, 2008]. □

In summary, we have:

Theorem 1.3.14. **sSet**, *regarded as a* **sSet**-*enriched category via its cartesian closed structure, is a simplicial model category where*

- *the cofibrations are the monomorphisms in* **sSet**,
- *the fibrations are the Kan fibrations, and*
- *the weak equivalences are the weak homotopy equivalences.*

This is the **Kan–Quillen model structure on simplicial sets**.

Proof. We know **sSet** has limits and colimits for all small diagrams and is a cartesian closed category, so it satisfies axioms CM1 and SM0. Using the definition of weak homotopy equivalence given above, the class of weak homotopy equivalences has the 2-out-of-6 property by lemma A.3.13, hence axiom CM2 is satisfied. Proposition 1.3.7 plus theorem 3.1.4 then shows that the announced cofibrations, fibrations, and weak equivalences do indeed constitute a closed model structure on **sSet**.

Finally, we note that proposition 1.3.10 is precisely the condition required by axiom SM7. ■

Proposition 1.3.15. *There exist a functor* $R : \mathbf{sSet} \to \mathbf{sSet}$ *and a natural transformation* $\eta : \mathrm{id}_{\mathbf{sSet}} \Rightarrow R$ *such that, for all simplicial sets* X, RX *is a Kan complex and* $i_X : X \to RX$ *is an anodyne extension.*

Proof. By proposition 1.3.7, for each X, there is a factorisation of the unique morphism $X \to 1$ as an anodyne extension $i_X : X \to RX$ followed by a Kan fibration $RX \to 1$, and this is moreover functorial in X. ■

1.4 Intrinsic homotopy

Prerequisites. § 1.3.

Definition 1.4.1. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in **sSet**. An **intrinsic homotopy** $\alpha : f_0 \Rightarrow f_1$ is an edge of the exponential object $[X, Y]$ such that $d_1(\alpha) = f_0$ and $d_0(\alpha) = f_1$. (Note the indices!) We say f_0 and f_1 are **intrisically homotopic** if there is a zigzag of intrinsic homotopies connecting f_0 and f_1, or equivalently, if f_0 and f_1 are in the same connected component of $[X, Y]$.

Remark 1.4.2. By the Yoneda lemma,

$$[X, Y]_1 \cong \mathbf{sSet}\big(\Delta^1, [X, Y]\big) \cong \mathbf{sSet}\big(\Delta^1 \times X, Y\big)$$

so an intrinsic homotopy $\alpha : f_0 \Rightarrow f_1$ is essentially the same thing as a morphism $\tilde{\alpha} : \Delta^1 \times X \to Y$ such that $\tilde{\alpha} \circ \big(\delta^1 \times \mathrm{id}_Y\big) = f_0$ and $\tilde{\alpha} \circ \big(\delta^0 \times \mathrm{id}_Y\big) = f_1$, just as in classical homotopy theory. Also,

$$\mathbf{sSet}\big(\Delta^1 \times X, Y\big) \cong \mathbf{sSet}\big(X, \big[\Delta^1, Y\big]\big)$$

so we can also formulate intrinsic homotopies as morphisms $X \to \big[\Delta^1, Y\big]$.

The notion of intrinsic homotopy is not well-behaved for general simplicial sets Y. For example, the existence of an intrinsic homotopy $f_0 \Rightarrow f_1$ does not guarantee the existence of an "inverse" intrinsic homotopy $f_1 \Rightarrow f_0$, and even if we have intrinsic homotopies $f_0 \Rightarrow f_1$ and $f_1 \Rightarrow f_2$, there need not be an intrinsic homotopy $f_0 \Rightarrow f_2$. However:

Proposition 1.4.3. *For any simplicial set X and any Kan complex Y, the relation $\sim$ on* $\mathbf{sSet}(X, Y)$ *defined by*

$$f_0 \sim f_1 \textit{ if and only if there exists an intrinsic homotopy } f_0 \Rightarrow f_1$$

is an equivalence relation.

Proof. The relation $\sim$ is certainly reflexive whether or not Y is a Kan complex. Recalling lemma 1.3.5, the transitivity of $\sim$ may be deduced from the fact that the unit $\eta_X : X \to \mathrm{N}\big(\tau_1 X\big)$ is an epimorphism, and the symmetry of $\sim$ corresponds to the fact that $\tau_1 X$ is a groupoid. ■

¶ **1.4.4.** Let **Kan** be the full subcategory of **sSet** spanned by the Kan complexes. For each category C with finite products and each functor $F : \mathbf{sSet} \to C$ that preserves finite products, let $F[\mathbf{Kan}]$ denote the following C-enriched category:

- $\mathrm{ob}\, F[\mathbf{Kan}] = \mathrm{ob}\, \mathbf{Kan}$.
- For each pair of Kan complexes X and Y, the hom-object is $F[X, Y]$, where $[X, Y]$ is the exponential object in **sSet**.
- Composition and identity morphisms are induced by F from the cartesian closed structure of **sSet**.

The next definition is a prime example of the above construction.

Definition 1.4.5. The **homotopy category of Kan complexes** is the category $\mathbf{H} = \pi_0[\mathbf{Kan}]$. A **homotopy type** is an isomorphism class of objects in $\mathbf{H}$.

Proposition 1.4.6. *For each simplicial set Z, let $\eta_Z : Z_0 \to \pi_0 Z$ be the map of vertices induced by the adjunction unit $\mathrm{id}_{\mathbf{sSet}} \Rightarrow \mathrm{disc}\, \pi_0$.*

(i) *There is a (unique) functor $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ that acts as the identity on objects and as $\eta_{[X,Y]} : [X,Y]_0 \to \pi_0[X,Y]$ on morphisms.*

(ii) *The functor $\boldsymbol{\pi}$ is full, surjective on objects, and preserves finite products.*

(iii) ***Kan** is closed under products for all small families in **sSet**, and **H** has products for finite families.*

(iv) ***Kan** and **H** are cartesian closed categories, and $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ is a cartesian closed functor.*

(v) *A morphism $f : X \to Y$ in **Kan** is a weak homotopy equivalence if and only if $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ is an isomorphism in **H**.*

Proof. (i). The construction of $\mathbf{H}$ as $\pi_0[\mathbf{Kan}]$ ensures that $\boldsymbol{\pi}$ is indeed a functor.

(ii). It is clear from the construction of $\pi_0 Z$ as a coequaliser that $\eta_Z : Z_0 \to \pi_0 Z$ is a surjection; thus $\boldsymbol{\pi}$ is a full functor. It is obviously surjective on objects, and it preserves finite products because π_0 does.

(iii). By proposition A.2.9, the class of Kan fibrations is closed under products for small families, so **Kan** is as well. By claim (ii), **H** inherits finite products from **Kan**.

(iv). By proposition 1.3.10, $[Y, K]$ is a Kan complex whenever K is, which combined with claim (iii) implies **Kan** is cartesian closed. Proposition A.1.11 says we have natural isomorphisms $[X \times Y, K] \cong [X, [Y, K]]$, so it follows that we have natural bijections

$$\pi_0[X \times Y, K] \cong \pi_0[X, [Y, K]]$$

for all X, Y, and K in **Kan**, and this descends along $\boldsymbol{\pi}$ to make **H** cartesian closed.

(v). The Joyal–Tierney definition says $f : X \to Y$ is a weak equivalence if and only if $\pi_0[f, K] : \pi_0[Y, K] \to \pi_0[X, K]$ is a bijection for all Kan complexes K; but this is natural in K, so the Yoneda lemma implies this happens if and only if $\boldsymbol{\pi} f : \boldsymbol{\pi} X \to \boldsymbol{\pi} Y$ is an isomorphism in $\mathbf{H}$. ∎

Proposition 1.4.7.

(i) *For each simplicial set X, there exists a Kan complex RX such that the functors $\pi_0[X, -], \pi_0[RX, -] : \mathbf{Kan} \to \mathbf{Set}$ are isomorphic.*

(ii) *For each simplicial set X, the functor $\pi_0[X, -] : \mathbf{Kan} \to \mathbf{Set}$ factors through $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ as a representable functor on $\mathbf{H}$.*

(iii) *The functor $\boldsymbol{\pi} : \mathbf{Kan} \to \mathbf{H}$ extends to a functor $\mathbf{sSet} \to \mathbf{H}$ that sends weak homotopy equivalences to isomorphisms, and this extension is unique up (not necessarily unique) isomorphism.*

Proof. (i). By proposition 1.3.15, there is an anodyne extension $i : X \to RX$ where RX is a Kan complex; but proposition 1.3.13 says that anodyne extensions are weak homotopy equivalences, so $\pi_0[i, K] : \pi_0[RX, K] \to \pi_0[X, K]$ is a bijection natural in K, as required.

(ii). The claim is certainly true if X were a Kan complex, and by claim (i), $\pi_0[X, -]$ is always isomorphic to $\pi_0[RX, -]$ for some Kan complex RX.

(iii). Formally, what we seek is a functor $F : \mathbf{sSet} \to \mathbf{H}$ such that, for all Kan complexes Y and K,

$$\mathbf{H}(FY, \boldsymbol{\pi} K) = \pi_0[Y, K]$$

and, for all weak homotopy equivalences $f : X \to Y$ in $\mathbf{sSet}$, the induced hom-set map $\mathbf{H}(Ff, \boldsymbol{\pi} K) : \mathbf{H}(FY, \boldsymbol{\pi} K) \to \mathbf{H}(FX, \boldsymbol{\pi} K)$ is a bijection for all Kan complexes K. Clearly, for any such F and any simplicial set X, there must be bijections

$$\mathbf{H}(FX, \boldsymbol{\pi} K) \cong \pi_0[X, K]$$

that are natural in K, but by claim (ii), this is representable as a functor $\mathbf{H} \to \mathbf{Set}$ for each X, so we can certainly construct such a functor F, and it is unique up to isomorphism. ∎

— II —

Homotopical categories

2.1 Basics

Prerequisites. § A.3.

Definition 2.1.1. A relative category $\mathcal{C}$ is a **category with weak equivalences** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property, and it is a **homotopical category** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property. A **homotopical functor** is a relative functor between homotopical categories.

Example 2.1.2. Any saturated relative category is automatically a homotopical category, by corollary A.3.14. In particular, any minimal saturated relative category is a homotopical category. On the other hand, any maximal relative category is obviously a homotopical category.

Remark 2.1.3. A relative category $\mathcal{C}$ is a category with weak equivalences or a homotopical category if and only if the opposite relative category $\mathcal{C}^{\mathrm{op}}$ is.

Lemma 2.1.4. *Let A be an object in a homotopical category (resp. category with weak equivalences) $\mathcal{C}$. Then the slice category $\mathcal{C}_{/A}$ is also a homotopical category (resp. category with weak equivalences) if we declare a morphism in $\mathcal{C}_{/A}$ to be a weak equivalence if and only if it is a weak equivalence in $\mathcal{C}$.*

Proof. Use lemma A.3.13 on the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$. ■

Definition 2.1.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two (not necessarily relative) functors between relative categories. A **natural weak equivalence** $\alpha : F \Rightarrow G$ is a natural transformation such that $\alpha_C : FC \to GC$ is a weak equivalence in $\mathcal{D}$ for all objects C in $\mathcal{C}$, and we say F and G are **naturally weakly equivalent** if they can be connected by a *zigzag* of natural weak equivalences.

Remark 2.1.6. If F and G are relative functors, then this is precisely the notion of weak equivalence in the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Although the definition above applies to all functors, if $H : \mathcal{D} \to \mathcal{E}$ is a functor, then the natural transformation $H\alpha : HF \Rightarrow HG$ is only guaranteed to be a natural weak equivalence if we assume H is a relative functor.

Definition 2.1.7. A **homotopical equivalence** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ for which there exists a relative functor $G : \mathcal{D} \to \mathcal{C}$ such that GF is naturally weakly equivalent to $\mathrm{id}_{\mathcal{C}}$ and FG is naturally weakly equivalent to $\mathrm{id}_{\mathcal{D}}$. Such a G is said to be a **homotopical inverse** of F.

Proposition 2.1.8. *If $F : \mathcal{C} \to \mathcal{D}$ is a homotopical equivalence of relative categories with homotopical inverse $G : \mathcal{D} \to \mathcal{C}$, then* $\operatorname{Ho} F : \operatorname{Ho} \mathcal{C} \to \operatorname{Ho} \mathcal{D}$ *is an equivalence of categories, with quasi-inverse* $\operatorname{Ho} G : \operatorname{Ho} \mathcal{D} \to \operatorname{Ho} \mathcal{C}$. ■

2.2 Homotopical Kan extensions

Prerequisites. § 2.1.

Definition 2.2.1. Let $\mathcal{C}$ be a homotopical category. A **homotopically initial object** in $\mathcal{C}$ is an object A for which there exists a zigzag of natural transformations of the form

$$\Delta A \;\text{———}\; F \xrightarrow{\alpha} G \;\text{———}\; \mathrm{id}_{\mathcal{C}}$$

where $\Delta A : \mathcal{C} \to \mathcal{C}$ is the constant functor with value A, $\alpha_A : FA \to GA$ is a weak equivalence in $\mathcal{C}$, and the unmarked lines denote (possibly trivial) zigzags of natural weak equivalences. Dually, a **homotopically terminal object** in $\mathcal{C}$ is a homotopically initial object in $\mathcal{C}^{\mathrm{op}}$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a homotopical category. If A is a homotopically initial (resp. homotopically terminal) object in $\mathcal{C}$, then:*

(i) *Any object in $\mathcal{C}$ weakly equivalent to A is also a homotopically initial (resp. homotopically terminal) object in $\mathcal{C}$.*

(ii) *A is an initial (resp. terminal) object in* $\operatorname{Ho} \mathcal{C}$.

(iii) *If C is a minimal homotopical category, then A is an initial (resp. terminal) object in C as well.*

Conversely, any initial (resp. terminal) object in C is also homotopically initial (resp. homotopically terminal).

Proof. Obvious. (This is Proposition 38.3 in [DHKS].) ■

Definition 2.2.3. A **homotopically contractible category** is a homotopical category C such that the unique (homotopical) functor $C \to \mathbb{1}$ is a homotopical equivalence, where $\mathbb{1}$ is the trivial category with only one object.

Proposition 2.2.4. *Let C be a homotopical category. The following are equivalent:*

(i) *C is homotopically contractible.*

(ii) *C is inhabited, and for every object A in C, the constant functor ΔA is naturally weakly equivalent to id_C.*

(iii) *There exists an object A in C such that ΔA and id_C are naturally weakly equivalent.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ■

Proposition 2.2.5. *Let C be a homotopically contractible category.*

(i) *Every morphism in C is a weak equivalence.*

(ii) *The unique functor $\mathrm{Ho}\, C \to \mathbb{1}$ is an equivalence of categories.*

(iii) *If C is a minimal homotopical category, then $C \to \mathbb{1}$ is also an equivalence of categories.*

(iv) *The opposite homotopical category C^{op} and the homotopical functor category $[\mathcal{D}, C]_{\mathrm{h}}$ (for any homotopical category $\mathcal{D}$) are also homotopically contractible.*

(v) *Every object in C is both homotopically initial and homotopically terminal.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ■

Proposition 2.2.6. *Let $\mathcal{C}$ be a homotopical category. If $\mathcal{D}$ is the full homotopical subcategory of $\mathcal{C}$ spanned by the homotopically initial (or homotopically terminal) objects, then $\mathcal{D}$ is homotopically contractible.*

Proof. See paragraph 38.5 in [DHKS]. □

Definition 2.2.7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two (not necessarily homotopical) functors between homotopical categories. A **homotopical left Kan extension** (resp. **homotopical right Kan extension**) of G along F is a homotopically initial (resp. homotopically terminal) object of the relative category $(G \downarrow F^*)_{\mathrm{h}}$ (resp. $(F^* \downarrow G)_{\mathrm{h}}$) described below:

- The objects are pairs (H, α) where H is a homotopical functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).
- The weak equivalences are the natural weak equivalences.

— III —

Model categories

3.1 Basics

Prerequisites. §§ 2.1, A.2.

In [1967], Quillen introduced the notion of a 'closed model category' (but we shall say simply 'model category') for homotopy theory, so as to formalise the similarities between the homotopy theory of spaces and homological algebra. The idea was that, to do homotopy theory, one really only needed to know which morphisms are cofibrations, which are weak equivalences, and which are fibrations.

Definition 3.1.1. A **model category** is a locally small category $\mathcal{M}$ equipped with three subclasses $\mathcal{C}, \mathcal{W}, \mathcal{F}$ of $\operatorname{mor} \mathcal{M}$ satisfying the following axioms:[1]

- **CM1.** $\mathcal{M}$ has finite limits and finite colimits.
- **CM2.** $\mathcal{W}$ has the 2-out-of-3 property.
- **CM3.** $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{F}$ are closed under retracts.
- **CM4.** Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

[1] This presentation is due to Quillen [1969].

where i is in $\mathcal{C}$ and p is in $\mathcal{F}$, if at least one of i or p is also in $\mathcal{W}$, then there exists a morphism $B \to X$ making the evident triangles commute.

- **CM5.** Any morphism f in $\mathcal{M}$ may be factored in two ways:
  - $f = p \circ i$ where i is in $\mathcal{C} \cap \mathcal{W}$ and p is in $\mathcal{F}$, and
  - $f = q \circ j$, where j is in $\mathcal{C}$ and q is in $\mathcal{W} \cap \mathcal{F}$.

The triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is said to be a **model structure** on $\mathcal{M}$. Given such a model structure on $\mathcal{M}$,

- a **cofibration** is a morphism in $\mathcal{C}$,
- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **fibration** is a morphism in $\mathcal{F}$,
- a **trivial cofibration** (or **acyclic cofibration**) is a morphism in $\mathcal{C} \cap \mathcal{W}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$;
- a **cofibrant object** is an object X such that the unique morphism $0 \to X$ is a cofibration, and
- a **fibrant object** is an object X such that the unique morphism $X \to 1$ is a fibration.
- a **cofibrant–fibrant object** is an object that is both cofibrant and fibrant.

Remark 3.1.2. The above presentation of the axioms is due to Quillen [1969], and is the one used in [DS] and [GJ]; however, [DHKS], [Hirschhorn, 2003], and [Hovey, 1999] use a variant definition that replaces axioms CM1 and CM5 with stronger ones:

- **CM1*.** $\mathcal{M}$ is complete and cocomplete.
- **CM5*.** The $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations can be chosen *functorially* in the sense of definition A.2.17.

Note also that Hovey [1999] considers the functorial factorisations to be a *structure* rather than a property.

Remark 3.1.3. Let $\mathcal{M}$ be a category with finite limits and finite colimits. Then, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{M}$ if and only if $(\mathcal{F}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ is a model structure on $\mathcal{M}^{\mathrm{op}}$.

Theorem 3.1.4. *Let $\mathcal{M}$ be a locally small category and let $\mathcal{C}, \mathcal{W}, \mathcal{F}$ be subclasses of* $\operatorname{mor} \mathcal{M}$*. Assuming $\mathcal{M}$ has finite limits and finite colimits, the following are equivalent:*

(i) *$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure for $\mathcal{M}$.*

(ii) *$\mathcal{M}$ is a saturated homotopical category with* $\operatorname{weq} \mathcal{M} = \mathcal{W}$*, and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems for $\mathcal{M}$.*

(iii) *$(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences (as in definition 2.1.1), and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems for $\mathcal{M}$.*

Proof. (i) ⇒ (ii). The fact that we have two weak factorisation systems follows from Lemma 1.1 in [GJ, Ch. II] or Proposition 7.2.3 in [Hirschhorn, 2003]; and the saturation property follows from Theorems 1.10 and 1.11 in [GJ, Ch. II], or Theorem 8.3.10 in [Hirschhorn, 2003].

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). Use proposition A.2.9. □

Lemma 3.1.5. *Let A be an object in a model category $\mathcal{M}$. Then the slice category $\mathcal{M}_{/A}$ has the* **slice model structure***, where a morphism in $\mathcal{M}_{/A}$ is a cofibration, weak equivalence, or fibration if it is so in $\mathcal{M}$.*

Proof. Use lemmas 2.1.4 and A.2.8, plus the fact that $\mathcal{M}_{/A}$ has finite limits and finite colimits if $\mathcal{M}$ does. ■

Definition 3.1.6. A **left Quillen functor** is a functor $F : \mathcal{M} \to \mathcal{N}$ between model categories that has a right adjoint and preserves cofibrations and trivial cofibrations; dually, a **right Quillen functor** is a functor $G : \mathcal{N} \to \mathcal{M}$ between model categories that has a left adjoint and preserves fibrations and trivial fibrations. A **Quillen adjunction** is an adjunction

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

where $\mathcal{M}$ and $\mathcal{N}$ are model categories, such that F is a left Quillen functor and G is a right Quillen functor. A **Quillen equivalence** is a Quillen adjunction as above satisfying this additional condition:

- Given a cofibrant object A in $\mathcal{N}$ and fibrant object X in $\mathcal{M}$, a morphism $FA \to X$ is a weak equivalence in $\mathcal{M}$ if and only if its adjoint transpose $A \to GX$ is a weak equivalence in $\mathcal{N}$.

Proposition 3.1.7. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be an adjunction between model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F is a left Quillen functor.*

(iii) *G is a right Quillen functor.*

(iv) *F preserves cofibrations and G preserves fibrations.*

(v) *F preserves trivial cofibrations and G preserves trivial fibrations.*

Proof. Use proposition A.2.15. ■

Lemma 3.1.8 (Kenneth S. Brown)**.** *Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a category with weak equivalences. If $F : \mathcal{M} \to \mathcal{C}$ sends trivial cofibrations (resp. trivial fibrations) in $\mathcal{M}$ to weak equivalences in $\mathcal{C}$, then F preserves all weak equivalences between cofibrant (resp. fibrant) objects.*

Proof. See Lemma 9.9 in [DS], Lemma 7.7.1 in [Hirschhorn, 2003], or Lemma 14.5 in [DHKS]. □

Corollary 3.1.9. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

(i) *If A and B are cofibrant objects in $\mathcal{N}$ and $f : A \to B$ is a weak equivalence in $\mathcal{N}$, then Ff is a weak equivalence in $\mathcal{M}$.*

(ii) *If X and Y are fibrant objects in $\mathcal{M}$ and $g : X \to Y$ is a weak equivalence in $\mathcal{M}$, then Gg is a weak equivalence in $\mathcal{N}$.* ■

Proposition 3.1.10 (Dugger)**.** *Let $F \dashv G$ be an adjunction between [strong???] model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(iii) *G preserves fibrations between fibrant objects and all trivial fibrations.*

Proof. This is Proposition 8.5.4 in [Hirschhorn, 2003]. □

Definition 3.1.11. Let X be an object in a model category $\mathcal{M}$.

- A **cofibrant replacement** for X is a pair $\left(\tilde{X}, p\right)$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$ and p is a weak equivalence $\tilde{X} \to X$.

- A **fibrant replacement** for X is a pair $\left(\hat{X}, i\right)$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$ and i is a weak equivalence $X \to \hat{X}$.

- A **fibrant cofibrant replacement** for X is a cofibrant replacement $\left(\tilde{X}, p\right)$ where $p : \tilde{X} \to X$ is a trivial fibration.

- A **cofibrant fibrant replacement** for X is a fibrant replacement $\left(\hat{X}, i\right)$ where $i : X \to \hat{X}$ is a trivial cofibration.

Remark 3.1.12. Note that a fibrant cofibrant replacement for X is precisely a cofibrant replacement for X that is fibrant as an object in $\mathcal{M}_{/X}$, and a cofibrant fibrant replacement for X is precisely a fibrant replacement for X that is cofibrant as an object in ${}^{X/}\mathcal{M}$.

Moreover, if X is fibrant and $\left(\tilde{X}, p\right)$ is a fibrant cofibrant replacement for X, then $\tilde{X}$ is both fibrant and cofibrant in $\mathcal{M}$, and if X is cofibrant and $\left(\hat{X}, i\right)$ is a cofibrant fibrant replacement for X, then $\hat{X}$ is both cofibrant and fibrant in $\mathcal{M}$.

Proposition 3.1.13. *Any object in a model category has both a fibrant cofibrant replacement and a cofibrant fibrant replacement.*

Proof. Use axiom CM5. ■

3.2 Left and right homotopy

Prerequisites. § 3.1.

Definition 3.2.1. Let X be an object in a model category $\mathcal{M}$. A **cylinder object** for X is a quadruple $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$, where $\mathrm{Cyl}(X)$ is an object in $\mathcal{M}$, $p : \mathrm{Cyl}(X) \to X$ is a weak equivalence, and $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are sections of p such that the morphism $\big[i_0, i_1\big] : X + X \to \mathrm{Cyl}(X)$ is a cofibration. Dually, a **path object** for X is a quadruple $\big(\mathrm{Path}(X), i, p_0, p_1\big)$, where $\mathrm{Path}(X)$ is an object in $\mathcal{M}$, $i : X \to \mathrm{Path}(X)$ is a weak equivalence, and $p_0, p_1 : \mathrm{Path}(X) \to X$ are retractions of i such that the morphism $\langle p_0, p_1 \rangle : \mathrm{Path}(X) \to X \times X$ is a fibration.

Proposition 3.2.2. *Let X be an object in a model category $\mathcal{M}$.*

(i) *There exists a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, where the morphism $p : \mathrm{Cyl}(X) \to X$ is a trivial fibration.*

(ii) *There exists a path object $\big(\mathrm{Path}(X), i, p_0, p_1\big)$ for X, where the morphism $i : X \to \mathrm{Path}(X)$ is a trivial cofibration.*

Proof. Use axioms CM1 and CM5. ■

Definition 3.2.3. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$. A **left homotopy** from f_0 to f_1 with respect to a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ is a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Dually, a **right homotopy** from f_0 to f_1 with respect to a path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ is a morphism $H : X \to \mathrm{Path}(Y)$ such that $p_0 \circ H = f_0$ and $p_1 \circ H = f_1$. We say f_0 and f_1 are **left homotopic** if there exists a left homotopy from f_0 to f_1 with respect to some cylinder object for X, and we say f_0 and f_1 are **right homotopic** if there exists a right homotopy from f_0 to f_1 with respect to some path object for Y.

Remark 3.2.4. If f_0 and f_1 are either left homotopic or right homotopic, then they must represent the same morphism in $\mathrm{Ho}\,\mathcal{M}$. For definiteness, let us write $\gamma : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ for the universal functor, and suppose $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1. Since i_0 and i_1 are both sections of the weak equivalence $p : \mathrm{Cyl}(X) \to X$, we must have $\gamma i_0 = (\gamma p)^{-1} = \gamma i_1$; but $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$, so indeed $\gamma f_0 = \gamma f_1$. This is one of the reasons for calling $\mathrm{Ho}\,\mathcal{M}$ the *homotopy* category of $\mathcal{M}$.

However, it is not quite true that $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are either left homotopic or right homotopic; this only happens in special cases. In general, being left/right homotopic fails to even be an equivalence relation.

Lemma 3.2.5. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *Given any cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, $f_0 \circ p : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to itself.*

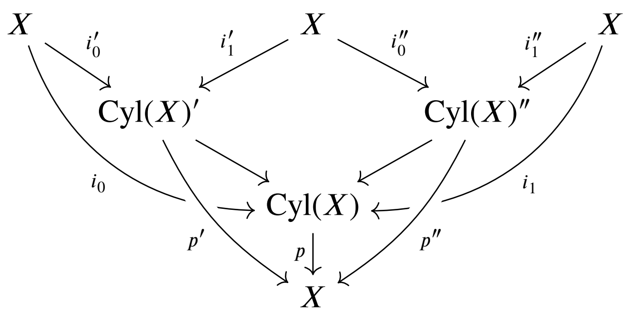
(ii) *Given any path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, $i \circ f_0 : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to itself.*

(iii) *If $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1 with respect to a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, then the same H is a left homotopy from f_1 to f_0 for the cylinder object $\big(\mathrm{Cyl}(X), i_1, i_0, p\big)$.*

(iv) *If $H : X \to \mathrm{Path}(Y)$ is a right homotopy from f_0 to f_1 with respect to a path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, then the same H is a right homotopy from f_1 to f_0 for the path object $\big(\mathrm{Path}(Y), i, p_1, p_0\big)$.*

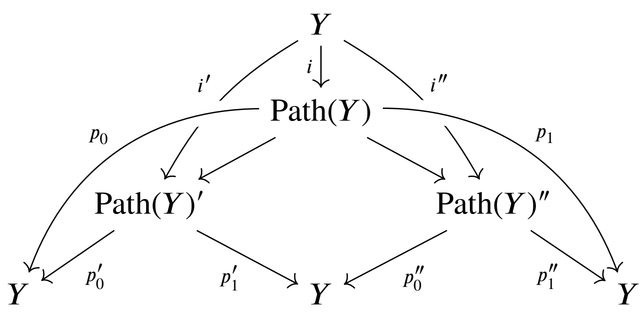
Proof. Obvious. ■

Lemma 3.2.6. *Let X be a cofibrant object in a model category $\mathcal{M}$. Given two cylinder objects for X, say $\big(\mathrm{Cyl}(X)', i'_0, i'_1, p'\big)$ and $\big(\mathrm{Cyl}(X)'', i''_0, i''_1, p''\big)$, there exists a third cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ such that the diagram below commutes,*



and the diamond is a pushout diagram.

Dually, if Y is a fibrant object in $\mathcal{M}$, and we have two path objects for Y, say $\big(\mathrm{Path}(Y)', i', p'_0, p'_1\big)$ and $\big(\mathrm{Path}(Y)'', i'', p''_0, p''_1\big)$, then there exists a third path *object* $(\mathrm{Path}(Y), i, p_0, p_1)$ *such that the diagram below commutes,*



and the diamond is a pullback diagram.

Proof. See Lemma 1.5 in [GJ, Ch. II], or Lemma 7.4.2 in [Hirschhorn, 2003]. □

Corollary 3.2.7. *Let* $f_0, f_1, f_2 : X \to Y$ *be three parallel morphisms in a model category* $\mathcal{M}$*.*

(i) *If* f_0 *and* f_1 *are left homotopic, and* f_1 *and* f_2 *are left homotopic, then* f_0 *and* f_2 *are also left homotopic.*

(ii) *If* f_0 *and* f_1 *are right homotopic, and* f_1 *and* f_2 *are right homotopic, then* f_0 *and* f_2 *are also right homotopic.* ■

Lemma 3.2.8. *Let* $f_0, f_1 : X \to Y$ *be a parallel pair of morphisms in a model category* $\mathcal{M}$*.*

(i) *If* X *is cofibrant, and* f_0 *and* f_1 *are left homotopic, given any path object* $(\mathrm{Path}(Y), i, p_0, p_1)$ *for* Y*, there is a right homotopy* $H : X \to \mathrm{Path}(Y)$ *from* f_0 *to* f_1*.*

(ii) *If* Y *is fibrant, and* f_0 *and* f_1 *are right homotopic, given any cylinder object* $(\mathrm{Cyl}(X), i_0, i_1, p)$ *for* X*, there is a left homotopy* $H : \mathrm{Cyl}(X) \to Y$ *from* f_0 *to* f_1*.*

Proof. See Proposition 1.8 in [GJ, Ch. II], or Proposition 7.4.7 in [Hirschhorn, 2003]. □

Proposition 3.2.9. *Let* X *and* Y *be objects in a model category* $\mathcal{M}$*.*

(i) *If X is cofibrant, then being left homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(ii) *If Y is fibrant, then being right homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(iii) *If X is cofibrant and Y is fibrant, then these two equivalence relations on $\mathcal{M}(X, Y)$ coincide.*

Proof. Use the preceding lemmas. ∎

Lemma 3.2.10. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *If f_0 and f_1 are right homotopic and $g : W \to X$ is any morphism in $\mathcal{M}$, then $f_0 \circ g$ and $f_1 \circ g$ are also right homotopic.*

(ii) *If f_0 and f_1 are left homotopic and $g : Y \to Z$ is any morphism in $\mathcal{M}$, then $g \circ f_0$ and $g \circ f_1$ are also left homotopic.*

Proof. Obvious. ∎

Corollary 3.2.11. *Let $\mathcal{M}$ be a model category, and let $\mathcal{M}_{\mathrm{cf}}$ be the full subcategory spanned by the cofibrant–fibrant objects. Then the equivalence relation induced by homotopy is a congruence on $\mathcal{M}_{\mathrm{cf}}$; in particular, there exist a locally small category $\mathcal{M}'$ and a full functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ with these properties:*

- *The objects of $\mathcal{M}'$ are those of $\mathcal{M}_{\mathrm{cf}}$.*
- *The hom-set $\mathcal{M}'(X, Y)$ is $\mathcal{M}(X, Y)$ modulo homotopy.*
- *The functor $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ sends each morphism in $\mathcal{M}'$ to its homotopy class.* ∎

The next result is a version of Whitehead's theorem; however, this is a purely formal consequence of the model category axioms and has no real content, unlike the original theorem.

Proposition 3.2.12. *Let X and Y be cofibrant–fibrant objects in a model category $\mathcal{M}$. If $f : X \to Y$ is a weak equivalence, then f has a* **homotopy inverse** *in $\mathcal{M}$, i.e. a morphism $g : Y \to X$ such that $g \circ f$ and id_X are homotopic, and $f \circ g$ and id_Y are homotopic.*

Proof. See Theorem 1.10 in [GJ, Ch. II], or Theorem 7.5.10 in [Hirschhorn, 2003]. □

Corollary 3.2.13. *Let W, X, Y, Z be cofibrant–fibrant objects in a model category $\mathcal{M}$, and let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms.*

(i) *If $g : W \to X$ is a weak equivalence such that $f_0 \circ g$ and $f_1 \circ g$ are homotopic, then f_0 and f_1 are homotopic.*

(ii) *If $g : Y \to Z$ is a weak equivalence such that $g \circ f_0$ and $g \circ f_1$ are homotopic, then f_0 and f_1 are homotopic.*

Proof. Use a homotopy inverse to cancel g. ■

3.3 The homotopy category

Prerequisites. §§ 3.1, 3.2, A.3.

Definition 3.3.1. The **Quillen homotopy category** (or, more simply, **homotopy category**) of a model category $\mathcal{M}$ is the category $\operatorname{Ho} \mathcal{M}$ obtained by freely inverting the weak equivalences in $\mathcal{M}$, as in definition A.3.9.

Theorem 3.3.2. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ be the universal functor.*

(i) *$\operatorname{Ho} \mathcal{M}$ is equivalent to the locally small category $\mathcal{M}'$ defined in corollary 3.2.11, and $\mathcal{M}$ is a saturated homotopical category.*

(ii) *If X and Y are cofibrant–fibrant objects in $\mathcal{M}$, then the hom-ensemble map $\mathcal{M}(X, Y) \to \operatorname{Ho} \mathcal{M}(X, Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

(iii) *For any two objects X and Y in $\mathcal{M}$, every morphism $X \to Y$ in $\operatorname{Ho} \mathcal{M}$ can be represented as a zigzag of the form*

$$X \xleftarrow{p} \tilde{X} \longrightarrow \hat{Y} \xleftarrow{i} Y$$

where $(\tilde{X}, p)$ is any cofibrant replacement for X and $(\hat{Y}, i)$ is any fibrant replacement for Y.

Proof. (i). This is Theorem 1.11 in [GJ, Ch. II], or Proposition 5.8 in [DS].

(ii). Implied by claim (i).

(iii). Using claim (ii), every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{M}$ can be represented as a zigzag of the form

$$\begin{array}{ccccccc} X & \xleftarrow{p} & \tilde{X} & & \hat{Y} & \xleftarrow{i} & Y \\ & & \downarrow{\scriptstyle i'} & & \uparrow{\scriptstyle p'} & & \\ & & R\tilde{X} & \xrightarrow{f'} & Q\hat{Y} & & \end{array}$$

where $\left(R\tilde{X}, i'\right)$ is a cofibrant fibrant replacement for $\tilde{X}$ and $\left(Q\hat{Y}, p'\right)$ is a fibrant cofibrant replacement for $\hat{Y}$; but such a zigzag is manifestly equivalent to the zigzag

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} \hat{Y} \xleftarrow{i} Y$$

where $f = p' \circ f' \circ i'$. □

Corollary 3.3.3. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the universal functor. If X is a cofibrant object in $\mathcal{M}$ and Y is a fibrant object in $\mathcal{M}$, then the hom-class map $\mathcal{M}(X, Y) \to \operatorname{Ho}\mathcal{M}(X, Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

Proof. As noted in remark 3.2.4, if $f_0, f_1 : X \to Y$ are homotopic, then we must have $\gamma f_0 = \gamma f_1$. Conversely, suppose $\gamma f_0 = \gamma f_1$ with X cofibrant and Y fibrant. Let (RX, i') be a cofibrant fibrant replacement for X and (QY, p') be a fibrant cofibrant replacement for Y. Then, there exists morphisms $f_0', f_1' : RX \to QY$ such that $f_0 = p' \circ f_0' \circ i'$ and $f_1 = p' \circ f_1' \circ i'$. Since $i' : X \to RX$ and $p' : QY \to Y$ are weak equivalences, we must have $\gamma f_0' = \gamma f_1'$ in $\operatorname{Ho}\mathcal{M}$. The theorem then implies f_0' and f_1' are homotopic; thus f_0 and f_1 are also homotopic, by lemmas 3.2.8 and 3.2.10. ■

— A —

Generalities

A.1 Cartesian closed categories

Definition A.1.1. Let C be a category with binary products, and let Y and Z be objects in C. An **exponential object** for Y and Z is an object $[Y, Z]$ in C and a morphism $\mathrm{ev}_{Y,Z} : [Y, Z]_C \times Y \to Z$ with the following universal property:

- For all morphisms $f : X \times Y \to Z$ in C, there exists a unique morphism $\bar{f} : X \to [Y, Z]_C$ such that $\mathrm{ev}_{Y,Z} \circ \left(\bar{f} \times \mathrm{id}_Y\right) = f$.

An **exponentiable object** in C is an object Y such that, for all objects Z in C, the exponential object $[Y, Z]_C$ exists. We may write $[Y, Z]$ or Z^Y instead of $[Y, Z]_C$ if there is no risk of confusion.

Lemma A.1.2. *Let Y be an object in a category C with binary products. The following are equivalent:*

(i) *Y is an exponentiable object in C.*

(ii) *The functor $- \times Y : C \to C$ has a right adjoint $[Y, -]_C : C \to C$, and the counit of this adjunction is $\mathrm{ev}_{Y,-}$.*

Proof. Immediate from the definitions. □

Definition A.1.3. A **cartesian closed category** is a category with finite products, in which every object is exponentiable. A **locally cartesian closed category** is a category C such that, for every object I, the slice category $C_{/I}$ is a cartesian closed category.

Example A.1.4. **Set** is cartesian closed category; in fact, it is even a locally cartesian closed category.

Proposition A.1.5. *Let $\mathcal{C}$ be a cartesian closed category.*

(i) *The assignment $(Y, Z) \mapsto [Y, Z]_{\mathcal{C}}$ extends to a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.*

(ii) *For each object Z, the functor $[-, Z]_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ is a contravariant right adjoint for itself.*

Proof. (i). This is an instance of the parametrised adjunction theorem.[1]

(ii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}(X, [Y, Z]) &\cong \mathcal{C}(X \times Y, Z) \\ &\cong \mathcal{C}(Y \times X, Z) \\ &\cong \mathcal{C}(Y, [X, Z]) \end{aligned}$$

■

Lemma A.1.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories. If $F : \mathcal{C} \to \mathcal{D}$ is a functor that preserves binary products, then:*

(i) *For any two objects X and Y in $\mathcal{C}$, there is a unique morphism $\varphi_{Y,Z}$: $F[X, Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} F[X,Y]_{\mathcal{C}} \times FX & \xrightarrow{\cong} & F\big([X,Y]_{\mathcal{C}} \times X\big) \\ {\scriptstyle \varphi_{X,Y} \times \mathrm{id}} \downarrow & & \downarrow {\scriptstyle F\mathrm{ev}_{X,Y}} \\ [FX, FY]_{\mathcal{D}} \times FX & \xrightarrow[\mathrm{ev}_{FX,FY}]{} & FY \end{array}$$

(ii) *The morphism $\varphi_{Y,Z}$ is natural in both Y and Z.*

Proof. The existence and uniqueness of $\varphi_{X,Y}$ follows from the universal property of $[FX, FY]_{\mathcal{D}}$ as an exponential object, and a standard argument proves naturality. □

[1] See Theorem 3 in [CWM, Ch. IV, § 7].

Definition A.1.7. A **cartesian closed functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ between cartesian closed categories such that the canonical comparison morphisms $\varphi_{X,Y} : F[X,Y]_{\mathcal{C}} \to [FX, FY]_{\mathcal{D}}$ described above are isomorphisms.

Proposition A.1.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be cartesian closed categories, and let Y be an object in $\mathcal{C}$ and let Z be an object in $\mathcal{D}$. Suppose we have an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : \mathrm{id}_{\mathcal{C}} \Rightarrow FG$; then:*

(i) *If $\psi_{FY,Z} : G[FY,Z]_{\mathcal{D}} \to [GFY, GZ]_{\mathcal{C}}$ is the canonical comparison morphism, then $\theta_{Y,Z} = \left[\eta_Y, GZ\right]_{\mathcal{C}} \circ \psi_{FY,Z}$ is the unique morphism in $\mathcal{C}$ making the following diagram commute:*

$$\begin{array}{ccc} G[FY,Z]_{\mathcal{D}} \times Y & \xrightarrow{\mathrm{id}\times\eta_Y} & G[FY,Z]_{\mathcal{D}} \times GFY \\ {\scriptstyle \theta_{Y,Z}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \cong} \\ [Y,GZ]_{\mathcal{C}} \times Y & & G\big([FY,Z]_{\mathcal{D}} \times FY\big) \\ {\scriptstyle \mathrm{ev}_{Y,GZ}}\downarrow & & \downarrow{\scriptstyle G\mathrm{ev}_{FY,Z}} \\ GZ & = & GZ \end{array}$$

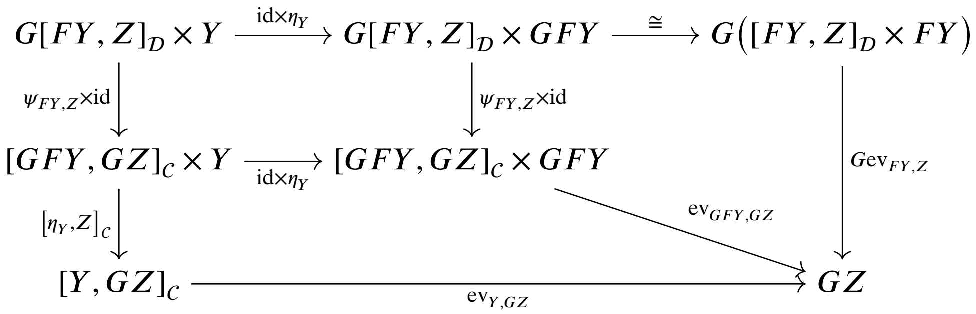
(ii) *If the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism for all objects X in $\mathcal{C}$, and $\varphi_{Y,GZ} : F[Y,GZ]_{\mathcal{C}} \to [FY, FGZ]_{\mathcal{D}}$ is the canonical comparison morphism, then $\chi_{Y,Z} = \left[FY, \varepsilon_Z\right]_{\mathcal{D}} \circ \varphi_{Y,GZ}$ is the unique morphism in $\mathcal{D}$ making the following diagram commute:*

$$\begin{array}{ccc} F[Y,GZ]_{\mathcal{C}} \times FY & \xrightarrow{\cong} & F\big([Y,GZ]_{\mathcal{C}} \times Y\big) \\ {\scriptstyle \chi_{Y,Z}\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{Y,GZ}} \\ [FY,Z]_{\mathcal{D}} \times FY & & FGZ \\ {\scriptstyle \mathrm{ev}_{FY,Z}}\downarrow & & \downarrow{\scriptstyle \varepsilon_Z} \\ Z & = & Z \end{array}$$

Moreover, under this hypothesis, $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$ is a two-sided inverse for $\theta_{Y,Z}$.

(iii) *If $\theta_{Y,Z}$ is an isomorphism for all objects Z in $\mathcal{D}$, then for all objects X in $\mathcal{C}$, the canonical comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism.*

Proof. (i). The claim is proven by the commutativity of the following diagram:



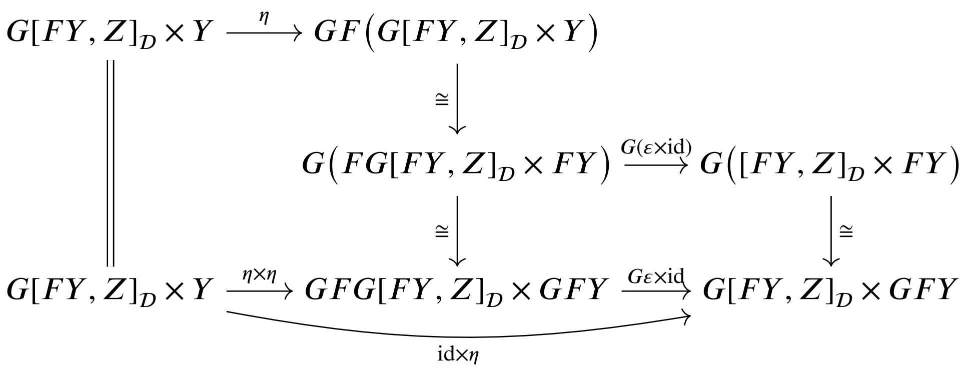
(ii). To show that $\chi_{Y,Z}$ makes the diagram commute, one uses the fact that $\mathrm{ev}_{FY,Z} : [FY,Z]_{\mathcal{D}} \times FY \to Z$ is natural in Z. Since F preserves products with Y, we have the following natural bijections:

$$\begin{aligned}\mathcal{C}\big(X, G[FY,Z]_{\mathcal{D}}\big) &\cong \mathcal{D}\big(FX, [FY,Z]_{\mathcal{D}}\big) \cong \mathcal{D}(FX \times FY, Z)\\ &\cong \mathcal{D}(F(X\times Y), Z) \cong \mathcal{C}(X\times Y, GZ) \cong \mathcal{C}\big(X, [Y,GZ]_{\mathcal{C}}\big)\end{aligned}$$

One obtains explicit isomorphisms by chasing id_X in both directions. Taking $X = [Y,GZ]_{\mathcal{C}}$, we find that the isomorphism $[Y,GZ]_{\mathcal{C}} \to G[FY,Z]_{\mathcal{D}}$ is precisely $G\chi_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$, and taking $X = G[FY,Z]_{\mathcal{D}}$, we find that the inverse is the right exponential transpose of

$$G\big(\mathrm{ev}_{FY,Z} \circ \big(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_Y\big)\big) \circ \eta_{G[FY,Z]_{\mathcal{D}}\times Y}$$

where we have suppressed the comparison isomorphism $F\big(G[FY,Z]_{\mathcal{D}} \times Y\big) \cong FG[FY,Z]_{\mathcal{D}} \times FY$; but naturality of the comparison morphisms for binary products gives us the commutative diagram below,



so, suppressing the comparison isomorphisms, we obtain the following equation:

$$G(\varepsilon_{[FY,Z]_{\mathcal{D}}} \times \mathrm{id}_{FY}) \circ \eta_{G[FY,Z]_{\mathcal{D}} \times Y} = \mathrm{id}_{G[FY,Z]_{\mathcal{D}}} \times \eta_Y$$

Thus, the isomorphism $G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is indeed $\theta_{Y,Z}$, as claimed.

(iii). Now, suppose $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [GY, Z]_{\mathcal{C}}$ is an isomorphism for all Z. Then, we have the natural bijections

$$\begin{aligned} \mathcal{D}(FX \times FY, Z) &\cong \mathcal{D}(FX, [FY, Z]_{\mathcal{D}}) \cong \mathcal{C}(X, G[FY, Z]_{\mathcal{D}}) \\ &\cong \mathcal{C}(X, [Y, GZ]_{\mathcal{C}}) \cong \mathcal{C}(X \times Y, GZ) \cong \mathcal{D}(F(X \times Y), Z) \end{aligned}$$

and by chasing id_Z for $Z = FX \times FY$, we conclude that the *canonical* comparison morphism $F(X \times Y) \to FX \times FY$ is an isomorphism. ■

Definition A.1.9. A **Frobenius adjunction of cartesian closed categories** is an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ where $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories, such that the natural morphisms $\theta_{Y,Z} : G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ described above are isomorphisms, or equivalently, such that the left adjoint $F : \mathcal{C} \to \mathcal{D}$ preserves binary products.

Remark A.1.10. If $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed categories and $G : \mathcal{D} \to \mathcal{C}$ is any functor that preserves finite products, then G induces a $\mathcal{D}$-enrichment of $\mathcal{C}$ from the cartesian closed structure of $\mathcal{C}$, and the exponential comparison morphisms $\psi_{Y,Z} : G[Y, Z]_{\mathcal{C}} \to [GY, GZ]_{\mathcal{D}}$ makes $G : \mathcal{D} \to \mathcal{C}$ into a $\mathcal{D}$-enriched functor.

Now, suppose G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$. The adjunction $F \dashv G$ is a Frobenius adjunction precisely when it is compatible with the $\mathcal{D}$-enrichments of $\mathcal{C}$ and $\mathcal{D}$. (Of course, this means F is also a $\mathcal{D}$-enriched functor.)

However, not all enriched adjunctions between cartesian closed categories are of the above form.

Proposition A.1.11. *Let X, Y, and Z be any three objects in a cartesian closed category $\mathcal{C}$.*

(i) *There is a unique morphism* $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ *making the following diagram commute:*

$$\begin{array}{ccc} ([X\times Y,Z]\times X)\times Y & \xrightarrow{\cong} & [X\times Y,Z]\times(X\times Y) \\ {\scriptstyle (\lambda_{X,Y,Z}\times \mathrm{id}_X)\times\mathrm{id}_Y}\downarrow & & \\ ([X,[Y,Z]]\times X)\times Y & & \downarrow{\scriptstyle \mathrm{ev}_{X\times Y,Z}} \\ {\scriptstyle \mathrm{ev}_{X,[Y,Z]}\times \mathrm{id}_X}\downarrow & & \\ [Y,Z]\times Y & \xrightarrow[\mathrm{ev}_{Y,Z}]{} & Z \end{array}$$

(ii) *The morphisms* $\lambda_{X,Y,Z} : [X \times Y, Z] \to [X, [Y, Z]]$ *constitute a natural isomorphism.*

Proof. The existence and uniqueness of $\lambda_{X,Y,Z}$ follows from the universal property of $[X, [Y, Z]]$ and $[Y, Z]$ as exponential objects, and a standard argument shows that $\lambda_{X,Y,Z}$ is natural in X, Y, and Z. By the associativity of cartesian products, we have the following natural bijections:

$$\begin{aligned} \mathcal{C}(T, [X \times Y, Z]) &\cong \mathcal{C}(T \times (X \times Y), Z) \\ &\cong \mathcal{C}((T \times X) \times Y, Z) \cong \mathcal{C}(T \times X, [Y, Z]) \cong \mathcal{C}(T, [X, [Y, Z]]) \end{aligned}$$

Chasing id_T for $T = [X \times Y, Z]$, we find that $\lambda_{X,Y,Z}$ is an isomorphism. ■

Definition A.1.12. Let $\mathcal{C}$ be a cartesian closed category. An **exponential ideal** of $\mathcal{C}$ is a full subcategory $\mathcal{D} \subseteq \mathcal{C}$ such that, for all objects Y in $\mathcal{C}$, if Z is in $\mathcal{D}$, then the exponential object $[Y, Z]_{\mathcal{C}}$ is (isomorphic to) an object in $\mathcal{D}$. A **reflective exponential ideal** of $\mathcal{C}$ is an exponential ideal $\mathcal{D}$ such that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint.

Proposition A.1.13. *Let* $\mathcal{C}$ *be a cartesian closed category, let* $G : \mathcal{D} \to \mathcal{C}$ *be the inclusion of a full subcategory, and suppose* G *has a left adjoint* $F : \mathcal{C} \to \mathcal{D}$*. The following are equivalent:*

(i) F *preserves finite products.*

(ii) F *preserves binary products.*

(iii) $\mathcal{D}$ *is a reflective exponential ideal of* $\mathcal{C}$*.*

(iv) *$\mathcal{D}$ is a cartesian closed category, $G : \mathcal{D} \to \mathcal{C}$ is a cartesian closed functor, and the canonical morphisms $G[FY, Z]_{\mathcal{D}} \to [Y, GZ]_{\mathcal{C}}$ are isomorphisms.*

Proof. (i) ⇒ (ii). Immediate.

(ii) ⇒ (iii). Under our hypotheses, the product of two objects X and Y in $\mathcal{D}$ can be computed as $F(GX \times GY)$. Let $\eta : \mathrm{id}_{\mathcal{C}} \to GF$ be the unit of the adjunction. We have the following natural bijections:

$$\begin{aligned} \mathcal{C}\big(X, [Y, GZ]_{\mathcal{C}}\big) &\cong \mathcal{C}(X \times Y, GZ) \\ &\cong \mathcal{D}(FX \times FY, Z) \\ &\cong \mathcal{D}(FGFX \times FY, Z) \\ &\cong \mathcal{C}(GFX \times Y, GZ) \\ &\cong \mathcal{C}\big(GFX, [Y, GZ]_{\mathcal{C}}\big) \end{aligned}$$

By chasing these maps explicitly, we find that every morphism $X \to [Y, GZ]_{\mathcal{C}}$ factors through $\eta_X : X \to GFX$ in a unique way. In particular, we have

$$\mathrm{id}_{[Y,GZ]_{\mathcal{C}}} = r_{Y,Z} \circ \eta_{[Y,GZ]_{\mathcal{C}}}$$

for a unique $r_{Y,Z} : GF[Y, GZ]_{\mathcal{C}} \to [Y, GZ]_{\mathcal{C}}$. The triangle identity then implies $Fr_{Y,Z} = \varepsilon_{F[Y,GZ]_{\mathcal{C}}}$, thus,

$$\eta_{[Y,GZ]_{\mathcal{C}}} \circ r_{Y,Z} = GFr_{Y,Z} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = G\varepsilon_{F[Y,GZ]_{\mathcal{C}}} \circ \eta_{GF[Y,GZ]_{\mathcal{C}}} = \mathrm{id}_{GF[Y,GZ]_{\mathcal{C}}}$$

and therefore $r_{Y,Z}$ is an isomorphism.

(iii) ⇒ (iv). It is a standard fact that a reflective subcategory is closed under all limits that exist in $\mathcal{C}$, so $\mathcal{D}$ must have finite products and $G : \mathcal{D} \to \mathcal{C}$ preserves them. If $\mathcal{D}$ is an exponential ideal, then $\eta_{[Y,GZ]_{\mathcal{C}}} : [Y, GZ]_{\mathcal{C}} \to GF[Y, GZ]_{\mathcal{C}}$ must be an isomorphism, so we obtain natural bijections

$$\begin{aligned} \mathcal{D}(X \times Y, Z) &\cong \mathcal{C}(GX \times GY, GZ) \\ &\cong \mathcal{C}\big(GX, [GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{C}\big(GX, GF[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(FGX, F[GY, GZ]_{\mathcal{C}}\big) \\ &\cong \mathcal{D}\big(X, F[GY, GZ]_{\mathcal{C}}\big) \end{aligned}$$

and therefore we may take $[Y,Z]_{\mathcal{D}} = F[GY,GZ]_{\mathcal{C}}$. Obviously, this makes $G : \mathcal{D} \to \mathcal{C}$ into a cartesian closed functor. We also have

$$\begin{aligned}
\mathcal{C}\big(X, G[FY,Z]_{\mathcal{D}}\big) &= \mathcal{C}\big(X, GF[GFY,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(X, [GFY,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, [X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(GFY, GF[X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, GF[X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(Y, [X,GZ]_{\mathcal{C}}\big) \\
&\cong \mathcal{C}\big(X, [Y,GZ]_{\mathcal{C}}\big)
\end{aligned}$$

and so the canonical morphism $G[FY,Z]_{\mathcal{D}} \to [Y,GZ]_{\mathcal{C}}$ is an isomorphism.

(iv) ⇒ (i). It is not hard to show that $\eta_1 : 1 \to GF1$ is an isomorphism for any adjunction whatsoever; but G is fully faithful, so this implies $F1$ is a terminal object in $\mathcal{D}$. Now apply proposition A.1.8. ■

Corollary A.1.14. *If $\mathcal{E}$ is a reflective exponential ideal of $\mathcal{D}$, and $\mathcal{D}$ is a reflective exponential ideal of $\mathcal{C}$, then $\mathcal{E}$ is also a reflective exponential ideal of $\mathcal{C}$.* ■

Proposition A.1.15. *Let* **Cat** *be the category of small categories, and let* **Grpd** *be the full subcategory of groupoids.*

(i) *There exist adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Cat}$$

where $\mathrm{ob}\,\mathbb{C}$ *is the set of objects in a category* $\mathbb{C}$*,* $\mathrm{disc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and all arrows trivial, and* $\mathrm{codisc}\,X$ *is the category with* $\mathrm{ob}\,\mathrm{disc}\,X = X$ *and a unique arrow between any two objects.*

(ii) *The functor* $\mathrm{disc} : \mathbf{Set} \to \mathbf{Cat}$ *is fully faithful and exhibits* **Set** *as a reflective exponential ideal of* **Cat**.

(iii) *The functor* $\pi_0 : \mathbf{Cat} \to \mathbf{Set}$ *preserves finite products.*

(iv) *There exist adjunctions*

$$\mathbf{I} \dashv \mathrm{und} \dashv \mathrm{iso} : \mathbf{Cat} \to \mathbf{Grpd}$$

where $\mathrm{und} : \mathbf{Grpd} \to \mathbf{Cat}$ *is the inclusion and* $\mathrm{iso}\,\mathbb{C}$ *is the maximal subgroupoid of a category* $\mathbb{C}$.

(v) **Grpd** *is a reflective exponential ideal of* **Cat**.

(vi) *The functor* **I** : **Cat** → **Grpd** *preserves finite products.*

(vii) *The adjunctions in (i) factor through* **Grpd**, *yielding adjunctions*

$$\pi_0 \dashv \mathrm{disc} \dashv \mathrm{ob} \dashv \mathrm{codisc} : \mathbf{Set} \to \mathbf{Grpd}$$

where π_0 : **Grpd** → **Set** *again preserves finite products.*

(viii) *The functor* **Cat** → **Set** *that sends a category* $\mathbb{C}$ *to the set of isomorphism classes in* $\mathbb{C}$ *preserves finite products.*

Proof. (i). The functor disc : **Set** → **Cat** obviously satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint π_0 : **Cat** → **Set**; the existence of the other adjunctions is obvious.

(ii). It is clear that disc : **Set** → **Cat** is fully faithful, and direct computation shows that $[\mathbb{C}, \mathrm{disc}\, X]$ is a discrete category for any $\mathbb{C}$, so **Set** is indeed a reflective exponential ideal of **Cat**.

(iii). Thus, by proposition A.1.13, π_0 : **Cat** → **Set** must preserve finite products.

(iv). It is not hard to check that the inclusion **Grpd** → **Cat** satisfies the solution set condition, so the general adjoint functor theorem gives us a left adjoint **I** : **Cat** → **Grpd**; the fact that iso : **Cat** → **Grpd** is right adjoint to the inclusion is obvious.

(v). Direct computation shows that $[\mathbb{C}, \mathbb{G}]$ is a groupoid whenever $\mathbb{G}$ is, so **Grpd** is indeed a reflective exponential ideal of **Cat**.

(vi). Thus, **I** : **Cat** → **Grpd** must preserve finite products.

(vii). Clearly, disc X and codisc X are already groupoids, so the adjunctions do indeed factor through **Grpd**.

(viii). The set of isomorphism classes of objects in $\mathbb{C}$ is precisely $\pi_0\, \mathrm{iso}\, \mathbb{C}$. ■

Definition A.1.16. The **dependent sum** of an object $p : X \to I$ in $\mathcal{C}_{/I}$ along a morphism $j : I \to J$ in $\mathcal{C}$ is the object $j \circ p : X \to J$ in $\mathcal{C}_{/J}$, and we write $\Sigma_j : \mathcal{C}_{/I} \to \mathcal{C}_{/J}$ for the functor sending an object to its dependent sum along j.

Lemma A.1.17. *Let $j : I \to J$ be a morphism in a category C. The following are equivalent:*

(i) *C has pullbacks along j.*

(ii) *There exists an adjunction*

$$\Sigma_j \dashv j^* : C_{/J} \to C_{/I}$$

where Σ_j is the dependent sum functor, and the right adjoint $j^* : C_{/J} \to C_{/I}$ is the pullback functor.

Proof. This is just a matter of unwinding the definitions. □

Definition A.1.18. Let C be a category with pullbacks. A **dependent product** of an object $p : X \to I$ in $C_{/I}$ along a morphism $j : I \to J$ in C is an object $\Pi_j p$ in $C_{/J}$ and a morphism $\mathrm{ev}_{j,p} : j^*\Pi_j p \to p$ in $C_{/I}$ with the following universal property:

- For all morphisms $f : j^*q \to p$ in $C_{/I}$, there exists a unique morphism $\bar{f} : q \to \Pi_j p$ in $C_{/J}$ such that $\mathrm{ev}_{j,p} \circ j^*\bar{f} = f$.

A **ΣΠ-category** is a category C with finite limits such that, for every morphism $j : I \to J$ in C, dependent products along j exist.

Lemma A.1.19. *Let $j : I \to J$ be a morphism in a category C with pullbacks. The following are equivalent:*

(i) *For all objects $p : X \to I$ in C, a dependent product of p along j exists.*

(ii) *The pullback functor $j^* : C_{/J} \to C_{/I}$ has a right adjoint $\Pi_j : C_{/I} \to C_{/J}$ that sends an object to its dependent product along j, and the counit of this adjunction is $\mathrm{ev}_{j,-}$.*

Proof. This is just a matter of unwinding the definitions. □

Corollary A.1.20. *If $j : I \to J$ is a morphism in a ΣΠ-category C, then the pullback functor $j^* : C_{/J} \to C_{/I}$ preserves all limits and colimits.* ■

Proposition A.1.21. *Let C be a category with a terminal object. The following are equivalent:*

(i) *$\mathcal{C}$ is a ΣΠ-category.*

(ii) *$\mathcal{C}$ is a locally cartesian closed category.*

Proof. See Proposition 9.20 in [Awodey, 2010]. □

Theorem A.1.22. *Let $\mathbb{D}$ be a small category, and let $\mathcal{C} = [\mathbb{D}^{\text{op}}, \mathbf{Set}]$. Then:*

(i) *$\mathcal{C}$ has limits and colimits for all small diagrams, and these can be constructed componentwise in* **Set***: a cone (resp. cocone) in $\mathcal{C}$ over (resp. under) a diagram in $\mathcal{C}$ is a limiting cone (resp. colimiting cocone) if and only if it is so in every component.*

(ii) *Every internal equivalence relation in $\mathcal{C}$ is the kernel pair of its coequaliser.*

(iii) *For all morphisms $j : I \to J$ in $\mathcal{C}$, the pullback functor $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$ preserves all limits and colimits.*

(iv) *The Yoneda embedding $\hbar_\bullet : \mathbb{D} \to \mathcal{C}$ is a dense functor, i.e. for every presheaf $X : \mathbb{D}^{\text{op}} \to \mathbf{Set}$, the tautological cocone*[2] *from the canonical diagram $(\hbar_\bullet \downarrow X) \to \mathcal{C}$ to X is a colimiting cocone.*

(v) *$\mathcal{C}$ is a locally finitely presentable category.*

(vi) *$\mathcal{C}$ is a ΣΠ-category.*

Proof. (i). This is a standard fact about presheaf categories.

(ii) and (iii). The claims are true for **Set**, and hence for $\mathcal{C}$ by claim (i).

(iv). See proposition A.4.20.

(v). See theorem 0.2.26.

(vi). Apply theorem 0.2.35 to construct a right adjoint for $j^* : \mathcal{C}_{/J} \to \mathcal{C}_{/I}$. ■

[2] See definition A.4.10.

Remark A.1.23. The Yoneda lemma gives us an explicit description of the exponential objects in $[\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]$: given two presheaves $Y, Z : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$, if Z^Y is their exponential object, then we must have

$$Z^Y(d) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d, Z^Y\big) \cong [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d \times Y, Z\big)$$

and so we may *define* Y^Z by $Y^Z(d) = [\mathbb{D}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_d \times Y, Z\big)$.

Definition A.1.24. Let Y and Z be topological spaces, and let $[Y, Z]$ be the set of all *continuous* maps $Y \to Z$. The **compact–open topology** on $[Y, Z]$ is the coarsest topology such that the subsets

$$V(K, U) = \left\{ f \in [Y, Z] \,\middle|\, K \subseteq f^{-1} U \right\}$$

are open in $[Y, Z]$ for all compact subsets $K \subseteq X$ and all open subsets $U \subseteq Y$.

Remark A.1.25. If Y is a discrete space, then the compact–open topology on $[Y, Z]$ coincides with the product topology on Z^Y.

Definition A.1.26. A **compactly-generated Hausdorff space** is a Hausdorff topological space X such that a subset $U \subseteq X$ is open if and only if, for every continuous map $f : K \to X$ where K is a compact Hausdorff space, $f^{-1} U$ is an open subset of K. We write **CGHaus** for the category of compactly-generated Hausdorff spaces and continuous maps.

Proposition A.1.27.

(i) *If Y is a locally compact Hausdorff space, then for all topological spaces Z, the set of all continuous maps $Y \to Z$, equipped with the compact–open topology, is an exponential object* $[Y, Z]$ *in* **Top**.

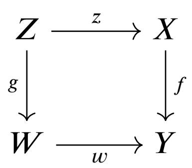
(ii) **Top** *is* not *a cartesian closed category.*

(iii) **CGHaus** *is a cartesian closed category.*

Proof. Claim (i) follows from Theorems 46.10 and 46.11 in [Munkres, 2000], and claim (ii) is Proposition 7.1.2 in [Borceux, 1994a], and claim (iii) is proved in [GZ, Ch. III, § 2]. □

A.2 Factorisation systems

Definition A.2.1. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a category $\mathcal{C}$. Given a commutative square in $\mathcal{C}$,



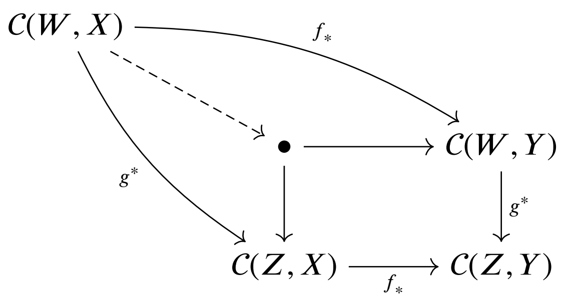
a **lift** is a morphism $h : W \to X$ such that $f \circ h = w$ and $h \circ g = z$.

We say g has the **left lifting property** with respect to f and f has the **right lifting property** with respect to g, and we write $g \boxslash f$, if every commutative square in $\mathcal{C}$ of the form above has a lift. We say f is **left orthogonal** to g and g is **right orthogonal** to f, and we write $g \perp f$ if lifts exist *and* are unique.

Given $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$, we define the following subensembles of $\operatorname{mor} \mathcal{C}$:

$$
\begin{aligned}
{}^{\boxslash}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\, f \boxslash g \} \\
\mathcal{I}^{\boxslash} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\, f \boxslash g \} \\
{}^{\perp}\mathcal{I} &= \{ f \in \operatorname{mor} \mathcal{C} \mid \forall g \in \mathcal{I}.\, f \perp g \} \\
\mathcal{I}^{\perp} &= \{ g \in \operatorname{mor} \mathcal{C} \mid \forall f \in \mathcal{I}.\, f \perp g \}
\end{aligned}
$$

Lemma A.2.2. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a locally small category $\mathcal{C}$. Consider the commutative diagram in* **Set** *shown below,*



where the inner square is a pullback diagram.

(i) *The dashed arrow is a surjection if and only if $g \boxslash f$.*

(ii) *The dashed arrow is a bijection if and only if $g \perp f$.*

Proof. This is just a restatement of the definition. ∎

Proposition A.2.3. *Let $\mathcal{C}$ be a category.*

(i) *If* $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, *then* ${}^{\perp}\mathcal{R} \subseteq {}^{\boxslash}\mathcal{R}$.

(ii) *If* $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, *then* ${}^{\boxslash}\mathcal{R}' \supseteq {}^{\boxslash}\mathcal{R}$.

(iii) *If* $\mathcal{R}' \subseteq \mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$, *then* ${}^{\perp}\mathcal{R}' \supseteq {}^{\perp}\mathcal{R}$.

Dually:

(i′) *If* $\mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$, *then* $\mathcal{L}^{\perp} \subseteq \mathcal{L}^{\boxslash}$.

(ii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$, *then* $\mathcal{L}'^{\boxslash} \supseteq \mathcal{L}^{\boxslash}$.

(iii′) *If* $\mathcal{L}' \subseteq \mathcal{L} \subseteq \operatorname{mor} \mathcal{C}$, *then* $\mathcal{L}'^{\perp} \supseteq \mathcal{L}^{\perp}$.

Moreover, we have the following antitone Galois connections:

$$\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\boxslash}$$
$$\mathcal{L} \subseteq {}^{\perp}\mathcal{R} \text{ if and only if } \mathcal{R} \subseteq \mathcal{L}^{\perp}$$

Proof. Obvious. □

Corollary A.2.4. *We have the following identities:*

$$ {}^{\boxslash}\big(({}^{\boxslash}\mathcal{R})^{\boxslash}\big) = {}^{\boxslash}\mathcal{R} \qquad {}^{\perp}\big(({}^{\perp}\mathcal{R})^{\perp}\big) = {}^{\perp}\mathcal{R}$$
$$ \big({}^{\boxslash}(\mathcal{L}^{\boxslash})\big)^{\boxslash} = \mathcal{L}^{\boxslash} \qquad \big({}^{\perp}(\mathcal{L}^{\perp})\big)^{\perp} = \mathcal{L}^{\perp}$$

Proof. This is a standard fact about (antitone) Galois connections. ∎

Lemma A.2.5. *Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. The following are equivalent:*

(i) *f is an isomorphism.*

(ii) *f is right orthogonal to any morphism in $\mathcal{C}$.*

(iii) *f has the right lifting property with respect to any morphism in $\mathcal{C}$.*

(iv) *f has the right lifting property with respect to itself.*

Dually, the following are equivalent:

(i′) *f is an isomorphism.*

(ii′) *f is left orthogonal to any morphism in $\mathcal{C}$.*

(iii′) *f has the left lifting property with respect to any morphism in $\mathcal{C}$.*

(iv′) *f has the left lifting property with respect to itself.*

Proof. (i) ⇒ (ii). Suppose $r : Y \to X$ is a morphism such that $r \circ f = \mathrm{id}_X$. Then, for any commutative square as below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

we have $(r \circ w) \circ g = r \circ f \circ z = z$; but if $f \circ r = \mathrm{id}_Y$ as well, then $f \circ (r \circ w) = w$; thus $r \circ w : W \to X$ is the required lift. It is clearly unique, as f is monic.

(ii) ⇒ (iii), (iii) ⇒ (iv). Obvious.

(iv) ⇒ (i). Consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

Since f has the right lifting property with respect to itself, there exists a morphism $h : Y \to X$ such that $h \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. ■

Definition A.2.6. A **weak factorisation system** for a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of subclasses of $\operatorname{mor} \mathcal{C}$ satisfying these conditions:

- For each morphism f in $\mathcal{C}$ there exists a pair (g, h) with $g \in \mathcal{L}$ and $h \in \mathcal{R}$ such that $f = h \circ g$. Such a pair is a $(\mathcal{L}, \mathcal{R})$-**factorisation** of f.
- A morphism is in $\mathcal{L}$ if and only if it has the left lifting property with respect to every morphism in $\mathcal{R}$, i.e. $\mathcal{L} = {}^{\boxslash}\mathcal{R}$.
- A morphism is in $\mathcal{R}$ if and only if it has the right lifting property with respect to every morphism in $\mathcal{L}$, i.e. $\mathcal{R} = \mathcal{L}^{\boxslash}$.

An **orthogonal factorisation system** is defined like a weak factorisation system, except for replacing '… has the left/right lifting property with respect to …' with '… is left/right orthogonal to …'.

Remark A.2.7. Obviously, $(\mathcal{L}, \mathcal{R})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}$ if and only if $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}^{\mathrm{op}}$.

Lemma A.2.8. *Let A be an object in a category $\mathcal{C}$ with a weak (resp. orthgonal) factorisation system $(\mathcal{L}, \mathcal{R})$. Then the slice category $\mathcal{C}_{/A}$ has a weak (resp. orthogonal) factorisation system where a morphism is in the left or right class if and only if it is so in $\mathcal{C}$.*

Proof. The projection $\mathcal{C}_{/A} \to \mathcal{C}$ induces a bijection between solutions for lifting problems in $\mathcal{C}_{/A}$ and solutions for the corresponding lifting problems in $\mathcal{C}$. ■

Proposition A.2.9. *Let $\mathcal{R} \subseteq \operatorname{mor} \mathcal{C}$ and suppose either $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ or $\mathcal{L} = {}^{\perp}\mathcal{R}$.*

(i) *Given a pushout diagram in $\mathcal{C}$ as below,*

$$\begin{array}{ccc} Z' & \xrightarrow{i_Z} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ W' & \xrightarrow[i_W]{} & W \end{array}$$

if the morphism g' is in $\mathcal{L}$, then g is also in $\mathcal{L}$.

(ii) *Let I be a set. If $g_i : Z_i \to W_i$ is a morphism in $\mathcal{L}$ for all i in I and the coproduct $\coprod_i g_i : \coprod_i Z_i \to \coprod_i W_i$ exists in $\mathcal{C}$, then $\coprod_i g_i$ is also in $\mathcal{L}$.*

(iii) *Given a commutative diagram of the form*

$$\begin{array}{ccccc} Z & \xrightarrow[i_Z]{} & Z' & \xrightarrow[r_Z]{} & Z \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ W' & \xrightarrow{i_W} & W & \xrightarrow{r_W} & W' \end{array} \qquad r_Z \circ i_Z = \mathrm{id}, \quad r_W \circ i_W = \mathrm{id}$$

if g is in $\mathcal{L}$, then so is g'; in other words, $\mathcal{L}$ is closed under retracts.

(iv) *$\mathcal{L}$ is closed under composition.*

(v) *Let γ be an ordinal and let $Z : \gamma \to \mathcal{C}$ be a colimit-preserving functor. We write Z_α for $Z(\alpha)$, where $\alpha < \gamma$, and $g_{\alpha,\beta} : Z_\alpha \to Z_\beta$ for the morphism $Z(\alpha \to \beta)$, where $\alpha < \beta < \gamma$. If λ is a colimiting cocone from Z to W and each $g_{\alpha,\beta}$ is in $\mathcal{L}$, then each component $\lambda_\alpha : Z_\alpha \to W$ is also in $\mathcal{L}$.*

Proof. (i). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{array}{ccccc} Z' & \xrightarrow{i_Z} & Z & \xrightarrow{z} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow[i_W]{} & W & \xrightarrow[w]{} & Y \end{array}$$

There exists $h' : W' \to X$ such that $h' \circ g' = z \circ i_Z$ and $f \circ h' = w \circ i_W$. In particular, there exists a unique morphism $h : W \to X$ such that $h \circ g = z$ and $h \circ i_W = h'$, by the universal property of pullbacks. Thus $f \circ h \circ i_W = f \circ h' = w \circ i_W$ and $f \circ h \circ g = f \circ z = w \circ g$, but i_W and g are jointly epic, so $f \circ h = w$. This shows h is the required lift, and h is unique if h' is.

(ii). We may construct the required lift componentwise.

(iii). Suppose f is in $\mathcal{R}$, and consider the following commutative diagram:

$$\begin{array}{ccccccc} Z & \xrightarrow{i_Z} & Z' & \xrightarrow{r_Z} & Z & \xrightarrow{z} & X \\ {\scriptstyle g'}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ W' & \xrightarrow[i_W]{} & W & \xrightarrow[r_W]{} & W' & \xrightarrow[w]{} & Y \end{array}$$

There exists $h : W \to X$ such that $h \circ g = z \circ r_Z$ and $f \circ h = w \circ r_W$, and so for $h' = h \circ i_W$:

$$f \circ h' = f \circ h \circ i_W = w \circ r_W \circ i_W = w$$
$$h' \circ g' = h \circ i_W \circ g' = h \circ g \circ i_Z = z \circ r_Z \circ i_Z = z$$

Thus $h' : W' \to X$ is the required lift, and h' is unique if h is (because r_W is split epic).

(iv). Suppose $g' : Z' \to Z$ and $g : Z \to W$ are in $\mathcal{L}$ and $f : X \to Y$ is in $\mathcal{R}$. Consider the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{z'} & X \\ {\scriptstyle g'}\big\downarrow & & \\ Z & & \big\downarrow{\scriptstyle f} \\ {\scriptstyle g}\big\downarrow & & \\ W & \xrightarrow[w]{} & Y \end{array}$$

There must exist a morphism $z : Z \to X$ such that $z \circ g' = z'$ and $f \circ z' = w \circ g$, and hence a morphism $h : W \to X$ such that $h \circ g = z$ and $f \circ h = w$. Obviously, $h \circ (g' \circ g) = z'$, so h is the required lift. Moreover, h unique if $\mathcal{L} = {}^{\perp}\mathcal{R}$.

(v). We may assume without loss of generality that $\alpha = 0$, since any non-empty terminal segment of γ is cofinal in γ. Suppose $f : X \to Y$ is in $\mathcal{R}$ and consider the following commutative diagram:

$$\begin{array}{ccc} Z_0 & \xrightarrow{z_0} & X \\ {\scriptstyle \lambda_0}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

For each $\alpha < \gamma$, given z_α making the following diagram commute,

$$\begin{array}{ccccc} Z_\alpha & & \xrightarrow{\quad z_\alpha \quad} & & X \\ {\scriptstyle g_{\alpha,\alpha+1}}\big\downarrow & & & & \big\downarrow{\scriptstyle f} \\ Z_{\alpha+1} & \xrightarrow[\lambda_{\alpha+1}]{} & W & \xrightarrow[w]{} & Y \end{array}$$

choose a lift $z_{\alpha+1} : Z_{\alpha+1} \to X$; for each limit ordinal $\beta < \gamma$, let $z_\beta : Z_\beta \to X$ be the unique morphism such that $z_\beta \circ g_{\alpha,\beta} = z_\alpha$ for all $\alpha < \beta$. (Such z_β exist and are unique because $Z_\beta = \varinjlim_{\alpha<\beta} Z_\alpha$.) Note that the universal property of W then guarantees that $w \circ \lambda_\beta = f \circ z_\beta$.

Having constructed morphisms $z_\alpha : Z_\alpha \to X$ for all $\alpha < \gamma$ as above, we may now obtain $h : W \to X$ as the unique morphism such that $h \circ \lambda_\alpha = z_\alpha$ for all $\alpha < \gamma$, and again we automatically have $f \circ h = w$. It is also clear that h is unique if $\mathcal{L} = {}^{\perp}\mathcal{R}$. ■

Proposition A.2.10. *Every orthogonal factorisation system is also a weak factorisation system.*

Proof. Let $(\mathcal{L}, \mathcal{R})$ be an orthogonal factorisation system on a category $\mathcal{C}$. Proposition A.2.3 implies $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$, so by duality it is enough to check that $\mathcal{L} \supseteq {}^{\boxslash}\mathcal{R}$.

Suppose $g : Z \to W$ is in ${}^{\boxslash}\mathcal{R}$, with $(\mathcal{L}, \mathcal{R})$-factorisation $g = r \circ g'$. Then the diagram below commutes,

$$\begin{array}{ccc} Z & \xrightarrow{g'} & W' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} \\ W & \xrightarrow[\mathrm{id}]{} & W \end{array}$$

so there must exist $i : W \to W'$ such that $r \circ i = \mathrm{id}_W$ and $i \circ g = g'$, and hence we have the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{\mathrm{id}} & Z & \xrightarrow{\mathrm{id}} & Z \\ {\scriptstyle g}\downarrow & & {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g'} \\ W & \xrightarrow{i} & W' & \xrightarrow{r} & W \end{array} \qquad r \circ i = \mathrm{id}$$

It follows from proposition A.2.9 that g is also in $\mathcal{L}$, so $\mathcal{L} \supseteq {}^{\boxslash}\mathcal{R}$ as required. ■

Definition A.2.11. A weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ is **cofibrantly generated** by a subensemble $\mathcal{I} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{R} = \mathcal{I}^{\boxslash}$. Dually, $(\mathcal{L}, \mathcal{R})$ is **fibrantly generated** by a subensemble $\mathcal{F} \subseteq \operatorname{mor} \mathcal{C}$ if $\mathcal{L} = {}^{\boxslash}\mathcal{F}$.

Remark A.2.12. Of course, $(\mathcal{L}, \mathcal{R})$ is always cofibrantly generated by $\mathcal{L}$. The condition is most useful when $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated by a (small) subset of $\mathcal{L}$, but it is convenient to have the more general definition available.

Definition A.2.13. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on a category $\mathcal{C}$. An **extension** of $(\mathcal{L}, \mathcal{R})$ along a functor $i : \mathcal{C} \to \mathcal{C}^+$ is a weak factorisation system $(\mathcal{L}^+, \mathcal{R}^+)$ on $\mathcal{C}^+$ with the following properties:

- A morphism $f : X \to Y$ in $\mathcal{C}$ is in $\mathcal{R}$ if and only if $if : iX \to iY$ is in $\mathcal{R}^+$.
- A morphism $g : Z \to W$ in $\mathcal{C}$ is in $\mathcal{L}$ if and only if $ig : iZ \to iW$ is in $\mathcal{L}^+$.

Proposition A.2.14. *Let $\mathcal{C}$ be a full subcategory of a category $\mathcal{C}^+$, let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system on $\mathcal{C}$, and let $(\mathcal{L}^+, \mathcal{R}^+)$ be a weak factorisation system on $\mathcal{C}^+$.*

(i) *If $\mathcal{L} \subseteq \mathcal{L}^+$, then $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both cofibrantly generated by the same ensemble $\mathcal{I}$, then $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.*

Dually:

(i′) *If $\mathcal{R} \subseteq \mathcal{R}^+$, then $\mathcal{L} \supseteq \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

(ii′) *If $(\mathcal{L}, \mathcal{R})$ and $(\mathcal{L}^+, \mathcal{R}^+)$ are both fibrantly generated by the same ensemble $\mathcal{F}$, then $\mathcal{L} = \mathcal{L}^+ \cap \operatorname{mor} \mathcal{C}$.*

Moreover:

(iii) *If $\mathcal{L} \subseteq \mathcal{L}^+$ and $\mathcal{R} \subseteq \mathcal{R}^+$, then $(\mathcal{L}^+, \mathcal{R}^+)$ is an extension of $(\mathcal{L}, \mathcal{R})$.*

Proof. Since $\mathcal{C}$ is a full subcategory of $\mathcal{C}^+$, if $g : Z \to W$ and $f : X \to Y$ are morphisms in $\mathcal{C}$, then any lifting problem of the following form in $\mathcal{C}^+$ is already in $\mathcal{C}$,

$$\begin{array}{ccc} Z & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \longrightarrow & Y \end{array}$$

and moreover any solution to the above lifting problem in $\mathcal{C}^+$ is also a solution in $\mathcal{C}$. Thus, $g \boxslash f$ in $\mathcal{C}$ if and only if $g \boxslash f$ in $\mathcal{C}^+$.

(i). Suppose f is in $\mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$. Then f has the right lifting property in $\mathcal{C}^+$ with respect to every morphism in $\mathcal{L}^+$, and in particular, f has the right lifting property in $\mathcal{C}$ with respect to every morphism in $\mathcal{L}$; hence f is in $\mathcal{R}$, and therefore $\mathcal{R} \supseteq \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(ii). A morphism is in $\mathcal{R}$ (resp. $\mathcal{R}^+$) if and only if it has the right lifting property in $\mathcal{C}$ (resp. $\mathcal{C}^+$) with respect to every morphism in $\mathcal{I}$, so by our initial observation, we must have $\mathcal{R} = \mathcal{R}^+ \cap \operatorname{mor} \mathcal{C}$.

(iii). Immediately follows from claims (i) and (i′). ■

Proposition A.2.15. *Let $(\mathcal{L}, \mathcal{R})$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}$, and let $(\mathcal{L}', \mathcal{R}')$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}'$. Given an adjunction*

$$F \dashv U : \mathcal{C}' \to \mathcal{C}$$

the following are equivalent:

(i) *F sends morphisms in $\mathcal{L}$ to morphisms in $\mathcal{L}'$.*

(ii) *U sends morphisms in $\mathcal{R}'$ to morphisms in $\mathcal{R}$.*

Proof. The adjunction induces a bijection between solutions to the two lifting problems shown below:

$$\begin{array}{ccc} FZ & \longrightarrow & X \\ {\scriptstyle Fg}\downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle f} \\ FW & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} Z & \longrightarrow & UX \\ {\scriptstyle g}\downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle Uf} \\ W & \longrightarrow & UY \end{array}$$

Thus, $Fg \boxslash f$ (resp. $Fg \perp f$) if and only if $g \boxslash Uf$ (resp. $g \perp Uf$). ■

¶ **A.2.16.** Let $\mathbb{2}$ be the category $\{0 \to 1\}$, and let $\mathbb{3}$ be $\{0 \to 1 \to 2\}$. Thus, given a category $\mathcal{C}$, the functor category $[\mathbb{2}, \mathcal{C}]$ is the category of arrows and commutative squares in $\mathcal{C}$. There are three embeddings $d^0, d^1, d^2 : \mathbb{2} \to \mathbb{3}$:

$$\begin{array}{lll} d^0(0) = 1 & d^1(0) = 0 & d^2(0) = 0 \\ d^0(1) = 2 & d^1(1) = 2 & d^2(1) = 1 \end{array}$$

These then induce (by precomposition) three functors $d_0, d_1, d_2 : [\mathbb{3}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$.

Definition A.2.17. A **functorial factorisation system** on a category $\mathcal{C}$ is a pair of functors $L, R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ for which there exists a (necessarily unique) functor $F : [\mathbb{2}, \mathcal{C}] \to [\mathbb{3}, \mathcal{C}]$ satisfying the following equations:

$$d_2 F = L \qquad d_1 F = \mathrm{id}_{[\mathbb{2}, \mathcal{C}]} \qquad d_0 F = R$$

A **functorial weak** (resp. **orthogonal**) **factorisation system** on $\mathcal{C}$ is a weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$ together with a functorial factorisation system (L, R) such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$.

Lemma A.2.18. *Let A be an object in a category $\mathcal{C}$ and let $\Sigma_A : \mathcal{C}_{/A} \to \mathcal{C}$ be the projection from the slice category.*

(i) *For each functorial factorisation system (L, R) on $\mathcal{C}$, there exists a unique functorial factorisation system (L_A, R_A) on $\mathcal{C}_{/A}$ such that*

$$[\mathbb{2}, \Sigma_A] \circ L_A = L \circ [\mathbb{2}, \Sigma_A] \qquad [\mathbb{2}, \Sigma_A] \circ R_A = R \circ [\mathbb{2}, \Sigma_A]$$

where $[\mathbb{2}, \Sigma_A] : [\mathbb{2}, \mathcal{C}_{/A}] \to [\mathbb{2}, \mathcal{C}]$ is the evident induced functor.

(ii) *If (L, R) is part of a functorial weak or orthogonal factorisation system on $\mathcal{C}$, then (L_A, R_A) is compatible with the induced weak or orthogonal factorisation system on $\mathcal{C}_{/A}$ as well.*

Proof. Obvious. □

Proposition A.2.19. *Any orthogonal factorisation system can be extended to a functorial one.*

Proof. For each morphism f in a category $\mathcal{C}$ with an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$, choose a factorisation $f = Rf \circ Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$. Given a commutative square in $\mathcal{C}$, say

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

the lifting property ensures that the dashed arrow in the diagram below exists,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle Lg}\downarrow & & \downarrow{\scriptstyle Lf} \\ \bullet & \dashrightarrow & \bullet \\ {\scriptstyle Rg}\downarrow & & \downarrow{\scriptstyle Rf} \\ W & \xrightarrow[w]{} & Y \end{array}$$

and orthogonality ensures uniqueness and hence functoriality. ■

The following characterisation of functorial orthogonal factorisation systems is due to Grandis and Tholen [2006]:

Theorem A.2.20. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *L is the underlying endofunctor of an idempotent comonad on $[\mathbb{2}, \mathcal{C}]$ with counit given by $\varepsilon_k = \left(\mathrm{id}_{\mathrm{dom}\, k}, Rk\right)$, and R is the underlying endofunctor of an idempotent monad on $[\mathbb{2}, \mathcal{C}]$ with unit given by $\eta_h = \left(h, \mathrm{id}_{\mathrm{codom}\, h}\right)$.*

(ii) *For all morphisms h in $\mathcal{C}$, RLh and LRh are isomorphisms in $\mathcal{C}$.*

(iii) *For any two morphisms in $\mathcal{C}$, say h and k, we have $Lk \perp Rh$.*

(iv) *$(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{C}$ extending (L, R), where:*

$$\mathcal{L} = \{g \in \operatorname{mor} \mathcal{C} \mid Rg \text{ is an isomorphism in } \mathcal{C}\}$$
$$\mathcal{R} = \{f \in \operatorname{mor} \mathcal{C} \mid Lf \text{ is an isomorphism in } \mathcal{C}\}$$

(v) *There exists an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$ extending (L, R).*

Proof. (i) ⇔ (ii). This is a standard fact about idempotent (co)monads.

(ii) ⇒ (iii). Now, consider the following lifting problem:

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

Since (L, R) is a functorial factorisation system, we get a commutative diagram of the form below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle Lg}\big\downarrow & & \big\downarrow{\scriptstyle Lf} \\ W' & \overset{t}{\dashrightarrow} & X' \\ {\scriptstyle Rg}\big\downarrow & & \big\downarrow{\scriptstyle Rf} \\ W & \xrightarrow[w]{} & Y \end{array}$$

but Rg and Lf are isomorphisms, so $(Lf)^{-1} \circ t \circ (Rg)^{-1}$ is the required lift $W \to X$. On the other hand, if $s : W \to X$ is any morphism such that $f \circ s = w$

and $s \circ g = z$, then by taking (L, R)-factorisations of the vertical arrows in the following diagram,

$$\begin{array}{ccccccc}
Z & \xrightarrow{g} & W & \xrightarrow{s} & X & \xrightarrow{\mathrm{id}} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\
W & \xrightarrow[\mathrm{id}]{} & W & \xrightarrow[s]{} & X & \xrightarrow[f]{} & Y
\end{array}$$

we find it must be the case that $Lf \circ s \circ Rg = t$, so we indeed have $g \perp f$.

(iii) ⇒ (iv). In particular, $g \perp Rg$ and $Lf \perp f$, so there must exist morphisms i and r making the diagrams below commute:

$$\begin{array}{ccc}
Z & \xrightarrow{Lg} & W' \\
{\scriptstyle g}\downarrow & \overset{i}{\nearrow} & \downarrow{\scriptstyle Rg} \\
W & \xrightarrow[\mathrm{id}]{} & W
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{\mathrm{id}} & X \\
{\scriptstyle Lf}\downarrow & \underset{r}{\nearrow} & \downarrow{\scriptstyle f} \\
X' & \xrightarrow[Rf]{} & Y
\end{array}$$

We then obtain the following equations,

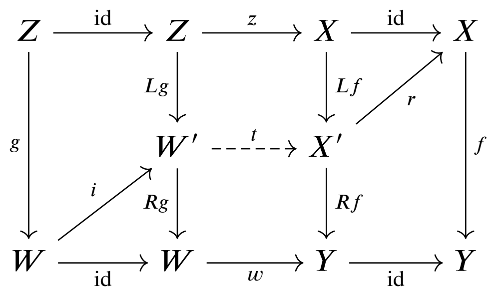
$$\begin{aligned}
(i \circ Rg) \circ Lg &= Lg & \qquad (Lf \circ r) \circ Lf &= Lf \\
Rg \circ (i \circ Rg) &= Rg & \qquad Rf \circ (Lf \circ r) &= Rf
\end{aligned}$$

and since $Lg \perp Rg$ and $Lf \perp Rf$, we must have $i \circ Rg = \mathrm{id}_{W'}$ and $Lf \circ r = \mathrm{id}_{X'}$. Thus, $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and the same argument now shows that ${}^{\perp}\mathcal{R} \subseteq \mathcal{L}$ and $\mathcal{L}^{\perp} \subseteq \mathcal{R}$.

It remains to be shown that $\mathcal{L} \subseteq {}^{\perp}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\perp}$. First, suppose $g \in \mathcal{L}$ and $f \in \mathcal{R}$, and consider the following lifting problem:

$$\begin{array}{ccc}
Z & \xrightarrow{z} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
W & \xrightarrow[w]{} & Y
\end{array}$$

With r and i as in the previous paragraph, we obtain a commutative diagram of the form below,



where the arrow t is obtained by the functoriality of (L, R)-factorisations. Thus, $r \circ t \circ i$ is the required lift $W \to X$, and it is unique, since Rg and Lf are isomorphisms. (Recall the proof of (ii) ⇒ (iii).) We conclude that $\mathcal{L} = {}^{\perp}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\perp}$.

(iv) ⇒ (v). Immediate.

(v) ⇒ (iii). If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on $\mathcal{C}$ such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$, then we must have $Lk \perp Rh$ for all h and k in $\operatorname{mor} \mathcal{C}$, as required.

(iv) ⇒ (ii). Immediate. ■

Remark A.2.21. It is clear that a functorial factorisation system is associated with *at most one* orthogonal factorisation system: indeed, if $(\mathcal{L}', \mathcal{R}')$ is any orthogonal factorisation system extending a functorial factorisation system (L, R), and $(\mathcal{L}, \mathcal{R})$ is the induced orthogonal factorisation system as in the theorem, then each morphism in $\mathcal{L}$ (resp. $\mathcal{R}$) is a retract of some morphism in in $\mathcal{L}'$ (resp. $\mathcal{R}'$); but by proposition A.2.9, this implies $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathcal{R} \subseteq \mathcal{R}'$, and applying proposition A.2.3, we also get $\mathcal{L} \supseteq \mathcal{L}'$ and $\mathcal{R} \supseteq \mathcal{R}'$.

Corollary A.2.22. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system on a category $\mathcal{C}$, then:*

(i) *$\mathcal{L}$, considered as a full subcategory of $[\mathbb{2}, \mathcal{C}]$, is replete and coreflective.*

(ii) *$\mathcal{L}$ is closed under all colimits in $[\mathbb{2}, \mathcal{C}]$.*

(iii) *If a diagram in $\mathcal{L}$ has a limit in $[\mathbb{2}, \mathcal{C}]$, then it also has a limit in $\mathcal{L}$.*

Dually:

(i′) *$\mathcal{R}$, considered as a full subcategory of* $[\mathbb{2}, \mathcal{C}]$*, is replete and reflective.*

(ii′) *$\mathcal{R}$ is closed under all limits in* $[\mathbb{2}, \mathcal{C}]$.

(iii′) *If a diagram in $\mathcal{R}$ has a colimit in* $[\mathbb{2}, \mathcal{C}]$*, then it also has a colimit in $\mathcal{R}$.*

Proof. Using proposition A.2.19 and theorem A.2.20, the above claims amount to standard facts about the Eilenberg–Moore category for idempotent (co)monads. ∎

There is a similar characterisation of functorial weak factorisation systems:

Theorem A.2.23. *Let (L, R) be a functorial factorisation system on a category $\mathcal{C}$. The following are equivalent:*

(i) *For any two morphisms in $\mathcal{C}$, say h and k, $Lk \boxslash Rh$.*

(ii) *$(\mathcal{L}, \mathcal{R})$ is an weak factorisation system on $\mathcal{C}$ extending (L, R), where:*

$$\mathcal{L} = \left\{ g \in \operatorname{mor} \mathcal{C} \,\middle|\, \exists i \in \operatorname{mor} \mathcal{C} .\, i \circ g = Lg \wedge Rg \circ i = \mathrm{id}_{\operatorname{codom} g} \right\}$$
$$\mathcal{R} = \left\{ f \in \operatorname{mor} \mathcal{C} \,\middle|\, \exists r \in \operatorname{mor} \mathcal{C} .\, f \circ r = Rf \wedge r \circ Lf = \mathrm{id}_{\operatorname{dom} f} \right\}$$

(iii) *There exists a weak factorisation system $(\mathcal{L}, \mathcal{R})$ extending (L, R).*

Proof. The proof is essentially the same as that of theorem A.2.20. ∎

Remark A.2.24. As with orthogonal factorisation systems, there is *at most one* weak factorisation system extending any functorial factorisation system.

The two theorems above motivate the following definition:

Definition A.2.25. A **algebraic factorisation system**[1] on a category $\mathcal{C}$ is a pair $(\mathbf{L}, \mathbf{R})$ satisfying the following conditions:

- $\mathbf{L} = (L, \varepsilon, \delta)$ is a comonad on $[\mathbb{2}, \mathcal{C}]$, where $\varepsilon_k = \left(\mathrm{id}_{\operatorname{dom} k}, Rk \right)$.
- $\mathbf{R} = (R, \eta, \mu)$ is a monad on $[\mathbb{2}, \mathcal{C}]$, where $\eta_h = \left(Lh, \mathrm{id}_{\operatorname{codom} h} \right)$.
- (L, R) constitute a functorial factorisation system on $\mathcal{C}$.

[1] — or **natural weak factorisation system** in the sense of Grandis and Tholen [2006] and Garner [2009].

Corollary A.2.26. *Any functorial orthogonal factorisation system extends to an algebraic factorisation system in a unique way; conversely, an algebraic factorisation system induces an orthogonal factorisation system if and only if the underlying comonad and monad are both idempotent.*

Proof. This follows from the definition above and theorem A.2.20. ■

Proposition A.2.27. *Let* (L, R) *be an algebraic factorisation system on a category* $\mathcal{C}$.

(i) *Let* $f : X \to Y$ *and* $g : Z \to W$ *be objects in* $[\mathbb{2}, \mathcal{C}]$. *If* $\alpha : Rf \to f$ *is a* R*-algebra structure and* $\beta : g \to Lg$ *is a* L*-coalgebra structure, then* $\alpha_1 : Y \to Y$ *and* $\beta_0 : Z \to Z$ *are identity morphisms, and we have the following identities:*

$$\alpha_0 \circ Lf = \mathrm{id}_X \qquad\qquad Rg \circ \beta_1 = \mathrm{id}_W$$
$$f \circ \alpha_0 = Rf \qquad\qquad \beta_1 \circ g = Lg$$

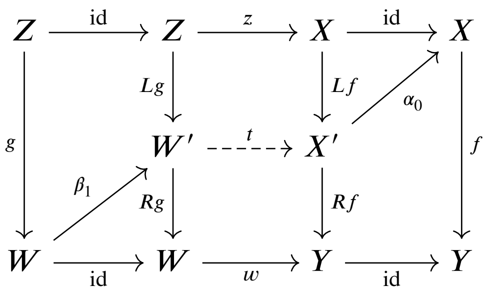
(ii) *If* f *admits a* L*-coalgebra structure and* g *admits an* R*-algebra structure, then* $f \boxslash g$.

(iii) *There exists a (unique) weak factorisation system* $(\mathcal{L}, \mathcal{R})$ *on* $\mathcal{C}$ *such that* $Lk \in \mathcal{L}$ *and* $Rh \in \mathcal{R}$ *for all* h *and* k *in* $\operatorname{mor} \mathcal{C}$.

Proof. (i). The claim follows from the L-coalgebra counitality axiom and the R-algebra unitality axiom:

$$\alpha \circ \eta_f = \mathrm{id}_f \qquad\qquad \varepsilon_g \circ \beta = \mathrm{id}_g$$

(ii). It then follows that the diagram below commutes,



where the arrow t is obtained by the functoriality of (L, R)-factorisations; clearly, $\alpha_0 \circ t \circ \beta_1$ is the required lift.

(iii). Finally, for any two morphisms in $\mathcal{C}$, say h and k, we simply note that $\delta_k : Lk \to LLk$ is an $\mathbf{L}$-coalgebra structure and $\mu_h : RRh \to Rh$ is an $\mathbf{R}$-algebra structure, so we may apply theorem A.2.23 to obtain the conclusion. ■

A.3 Relative categories

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.3.1. A **relative category** $\mathcal{C}$ consists of a category $\operatorname{und} \mathcal{C}$ and a subcategory $\operatorname{weq} \mathcal{C}$ such that $\operatorname{ob} \operatorname{und} \mathcal{C} = \operatorname{ob} \operatorname{weq} \mathcal{C}$. We say $\operatorname{und} \mathcal{C}$ is the **underlying category** of $\mathcal{C}$, and that the morphisms in $\operatorname{weq} \mathcal{C}$ are the **weak equivalences** in $\mathcal{C}$. A **relative subcategory** of a relative category $\mathcal{C}$ is a relative category $\mathcal{C}'$ such that $\operatorname{und} \mathcal{C}'$ is a subcategory of $\operatorname{und} \mathcal{C}$, and we further demand that $\operatorname{weq} \mathcal{C}' = \operatorname{weq} \mathcal{C} \cap \operatorname{und} \mathcal{C}'$.

Remark A.3.2. The subcategory $\operatorname{weq} \mathcal{C}$ is entirely determined by $\operatorname{mor} \operatorname{weq} \mathcal{C}$, so a relative category may equivalently be defined as a category equipped with a distinguished subset of morphisms closed under composition and containing all the identity morphisms.

For brevity, we will write $\operatorname{ob} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, $\operatorname{mor} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, and we may occasionally abuse notation and write $\operatorname{weq} \mathcal{C}$ instead of $\operatorname{mor} \operatorname{weq} \mathcal{C}$.

Remark A.3.3. Every category $\mathcal{C}$ can be endowed with the structure of a relative category in two ways: we can make it into a **minimal relative category** $\min \mathcal{C}$ by taking $\operatorname{weq} \min \mathcal{C}$ to be the set of identity morphisms in $\mathcal{C}$; or we could make it into a **maximal relative category** $\max \mathcal{C}$ by taking $\operatorname{weq} \max \mathcal{C} = \operatorname{mor} \mathcal{C}$. We may also define the **minimal saturated relative category** $\min^+ \mathcal{C}$ by taking $\operatorname{weq} \min^+ \mathcal{C}$ to be the set of all isomorphisms in $\mathcal{C}$.

Definition A.3.4. Given a relative category $\mathcal{C}$, the **opposite relative category** $\mathcal{C}^{\mathrm{op}}$ is defined by $\operatorname{und} \mathcal{C}^{\mathrm{op}} = (\operatorname{und} \mathcal{C})^{\mathrm{op}}$ and $\operatorname{weq} \mathcal{C}^{\mathrm{op}} = (\operatorname{weq} \mathcal{C})^{\mathrm{op}}$.

Definition A.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **relative functor** $\mathcal{C} \to \mathcal{D}$ is a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ that sends weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. The **relative functor category** $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is the full subcategory of $[\mathrm{und}\,\mathcal{C}, \mathrm{und}\,\mathcal{D}]$ spanned by the relative functors, and the weak equivalences in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ are defined to be the natural transformations that are componentwise weak equivalences in $\mathcal{D}$.

Definition A.3.6. Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \mathrm{mor}\,\mathcal{C}$. A **localisation of** $\mathcal{C}$ **away from** $\mathcal{W}$ is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ with the following universal property:

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that Ff is an isomorphism for all f in $\mathcal{W}$, there exists a unique functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $\overline{F}\gamma = F$.

Remark A.3.7. The universal property in the above definition is strict; as such, $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to unique isomorphism. Nonetheless, $\mathcal{C}[\mathcal{W}^{-1}]$ automatically has a 2-universal property: if $F, G : \mathcal{C} \to \mathcal{D}$ both factor through $\mathcal{C}[\mathcal{W}^{-1}]$, then so do all natural transformations $F \Rightarrow G$.

Proposition A.3.8. *If $\mathcal{C}$ is a* $\mathbf{U}$*-small category, then there exists a* $\mathbf{U}$*-small category with the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$.*

Proof. Use the general adjoint functor theorem. □

Definition A.3.9. The **homotopy category** of a relative category $\mathcal{C}$ is a localisation of $\mathrm{und}\,\mathcal{C}$ away from $\mathrm{weq}\,\mathcal{C}$ and is denoted $\mathrm{Ho}\,\mathcal{C}$. A **semi-saturated relative category** is a relative category in which every isomorphism is a weak equivalence. A **saturated relative category** is a relative category $\mathcal{C}$ such that the weak equivalences in $\mathcal{C}$ are precisely the ones that become isomorphisms in $\mathrm{Ho}\,\mathcal{C}$.

Remark A.3.10. Obviously, there is no loss of generality in considering semi-saturated relative categories and their homotopy categories instead of localisations $\mathcal{C}[\mathcal{W}^{-1}]$ for arbitrary subsets $\mathcal{W} \subseteq \mathrm{mor}\,\mathcal{C}$.

Remark A.3.11. Clearly, every saturated relative category is semi-saturated, and a minimal saturated relative category is indeed saturated in the sense above.

Definition A.3.12. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subset of $\mathrm{mor}\,\mathcal{C}$. The **2-out-of-3 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, if any two of f, g, or $g \circ f$ are in $\mathcal{W}$, then all of them are.

The **2-out-of-6 property** for $\mathcal{W}$ says:

- Given any three morphisms $f : X \to Y$, $g : Y \to Z$, $h : Y \to Z$ in $\mathcal{C}$, if both $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so too are f, g, h, and $h \circ g \circ f$.

Lemma A.3.13. *Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$.*

(i) *If $\mathcal{W}$ has the 2-out-of-6 property, then it also has the 2-out-of-3 property.*

(ii) *The set of all isomorphisms in $\mathcal{C}$ has the 2-out-of-6 property.*

(iii) *If $F : \mathcal{C}' \to \mathcal{C}$ is a functor and $\mathcal{W}$ has either the 2-out-of-3 property or the 2-out-of-6 property, then $F^{-1}\mathcal{W}$ has the same property.*

Proof. (i). Consider the three cases $f = \mathrm{id}$, $g = \mathrm{id}$, $h = \mathrm{id}$ in turn.

(ii). If $h \circ g$ and $g \circ f$ are isomorphisms, then g must be split epic and split monic; thus g itself is an isomorphism, hence so too are f and h.

(iii). Obvious. ■

Corollary A.3.14. *If $\mathcal{C}$ is a saturated relative category, then* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-6 property.* ■

Proposition A.3.15. *Let* **RelCat** *be the category of* **U***-small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories, and let* **Cat** *be the category of* **U***-small categories and functors.*

(i) **RelCat** *is a cartesian closed category, where the product of $\mathcal{C}$ and $\mathcal{D}$ is the cartesian product $\mathcal{C} \times \mathcal{D}$ with weak equivalences taken componentwise, and the exponential of $\mathcal{E}$ by $\mathcal{D}$ is the relative functor category $[\mathcal{D}, \mathcal{E}]_{\mathrm{h}}$.*

(ii) **RelCat** *is a locally finitely presentable* **U***-category,*[1] *and the two functors* $\mathrm{und}, \mathrm{weq} :$ **RelCat** $\to$ **Cat** *are $\aleph_0$-accessible*[2] *and jointly conservative.*

(iii) **SsRelCat** *is a locally finitely presentable* **U***-category, and the inclusion* **SsRelCat** $\hookrightarrow$ **RelCat** *is $\aleph_0$-accessible and has a left adjoint.*

(iv) **SsRelCat** *is an exponential ideal in* **RelCat**.

(v) *The full subcategory spanned by the minimal relative categories is an exponential ideal in* **RelCat**.

[1] See definition 0.2.22.
[2] See definition 0.2.18.

(vi) *The full subcategory spanned by the minimal saturated relative categories is an exponential ideal in* **SsRelCat**.

Proof. (i). This is straightforward from the definitions.

(ii). Obviously, a relative functor $F : \mathcal{C} \to \mathcal{D}$ such that $\operatorname{und} F : \operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ and $\operatorname{weq} F : \operatorname{weq} \mathcal{C} \to \operatorname{weq} \mathcal{D}$ are both isomorphisms is itself an isomorphism, so und, weq : **RelCat** → **Cat** are indeed jointly conservative.

It is also not hard to check that limits for all **U**-small diagrams and colimits for **U**-small filtered diagrams in **RelCat** exist and can be computed componentwise in **Cat**, so (by theorem 0.2.26) it is enough to show that **RelCat** is a $\aleph_0$-accessible **U**-category. Clearly, a relative category $\mathcal{C}$ such that $\operatorname{und} \mathcal{C}$ is finitely presentable in **Cat** and $\operatorname{weq} \mathcal{C}$ is a finitely-generated subcategory of $\operatorname{und} \mathcal{C}$ is itself finitely presentable in **RelCat**, so **RelCat** is indeed $\aleph_0$-accessible.

(Alternatively, one may appeal to the sketchability theorem[3] and the fact that a relative category is manifestly a model for a certain finite-limit sketch.)

(iii). It is clear that **SsRelCat** is closed in **RelCat** under limits for all **U**-small diagrams and colimits for all **U**-small filtered diagrams, and we know that **RelCat** is a locally finitely presentable category, so (by proposition 0.2.21) it is enough to construct a left adjoint for the inclusion **SsRelCat** ↪ **RelCat**. This may be done using the general adjoint functor theorem.

(iv) – (vi). All straightforward. ■

Proposition A.3.16. *Let* **RelCat** *be the category of* **U**-*small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories and relative functors, and let* **Cat** *be the category of* **U**-*small categories and functors. We have the following strings of adjoint functors:*

$$\min \dashv \operatorname{und} \dashv \max \dashv \operatorname{weq} : \mathbf{RelCat} \to \mathbf{Cat}$$

$$\operatorname{Ho} \dashv \min^{+} \dashv \operatorname{und} \dashv \max \dashv \operatorname{weq} : \mathbf{SsRelCat} \to \mathbf{Cat}$$

The functors min, $\min^{+}$, *and* max *are moreover fully faithful, and* Ho *preserves finite products.*

Proof. All but the last of the above claims are obvious; for the preservation of finite products under Ho, we refer to proposition A.1.13. ■

[3] See Proposition 1.51 in [LPAC] or Proposition 5.6.4 in [Borceux, 1994b]

Definition A.3.17. A **zigzag type** is a relative category T where $\operatorname{und} T$ is the free category on an inhabited finite planar graph of the form

$$\bullet \;\text{———}\; \bullet \;\text{———}\; \cdots \;\text{———}\; \bullet \;\text{———}\; \bullet$$

where the edges are arrows that point either left or right, and $\operatorname{weq} T$ consists of all identities and all composites of left-pointing arrows. A **morphism of zigzag types** is a relative functor that maps the leftmost object to the leftmost object and the rightmost object to the rightmost object. We write $\mathbf{T}$ for the category of zigzag types.[4]

A **zigzag** of type T in a relative category C is a relative functor $T \to C$. Given objects X and Y in C, we denote by $C^T(X, Y)$ the category whose objects are the zigzags starting at X and ending at Y and whose morphisms are commutative diagrams in C of the form

$$\begin{array}{ccccccccc}
X & \text{———} & \bullet & \text{———} & \cdots & \text{———} & \bullet & \text{———} & Y \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \text{———} & \bullet & \text{———} & \cdots & \text{———} & \bullet & \text{———} & Y
\end{array}$$

where the rows are zigzags of type T and the unmarked columns are weak equivalences.

Example A.3.18. If $f : X \to Y$ is a weak equivalence in a relative category C, then we have commutative diagrams

$$\begin{array}{ccccc}
X & \xrightarrow{\mathrm{id}} & X & \xleftarrow{\mathrm{id}} & X \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \xrightarrow[f]{} & Y & \xleftarrow[f]{} & X
\end{array}
\qquad
\begin{array}{ccccc}
Y & \xleftarrow{f} & X & \xrightarrow{f} & Y \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\
Y & \xleftarrow[\mathrm{id}]{} & Y & \xrightarrow[\mathrm{id}]{} & Y
\end{array}$$

and these correspond to morphisms of zigzags in C.

Remark A.3.19. It is clear that $C^T(X, Y)$ is a subcategory of the relative functor category $[T, C]_{\mathrm{h}}$. Thus, if C is a $\mathbf{U}$-small relative category, precomposition makes the assignment $T \mapsto C^T(X, Y)$ into a functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$, which we denote by $C^*(X, Y)$. A Grothendieck construction applied to this functor yields the following $\mathbf{U}$-small category $C^{(\mathbf{T})}(X, Y)$:

[4] Warning: This is the *opposite* of the category $\mathbf{T}$ defined in [DHKS, § 34].

- Its objects are pairs (T, f), where T is a zigzag type and f is a zigzag of type T in $\mathcal{C}$.
- A morphism $(T', f') \to (T, f)$ is a pair (α, β) where $\alpha : T \to T'$ is a morphism in $\mathbf{T}$ and $\beta : \alpha^* f' \to f$ is a morphism in $\mathcal{C}^T(X, Y)$.
- The composite of a pair of morphisms $(\alpha', \beta') : (T'', f'') \to (T', f')$ and $(\alpha, \beta) : (T', f') \to (T, f)$ is given by $(\alpha' \circ \alpha, \beta \circ \alpha^* \beta')$.

There is an evident projection functor $\mathcal{C}^{(\mathbf{T})}(X, Y) \to \mathbf{T}^{\mathrm{op}}$, and by construction it is a Grothendieck opfibration with a canonical splitting.

Theorem A.3.20. *Let X and Y be objects in a relative category $\mathcal{C}$.*

(i) *For each zigzag type T, the map that sends an object in $\mathcal{C}^T(X, Y)$ to the corresponding composite in* $\operatorname{Ho}\mathcal{C}(X, Y)$ *is a functor when the latter is regarded as a discrete category.*

(ii) *The functors described above constitute a jointly surjective cocone from the diagram $\mathcal{C}^*(X, Y)$ to* $\operatorname{Ho}\mathcal{C}(X, Y)$.

(iii) *The induced functor $\mathcal{C}^{(\mathbf{T})}(X, Y) \to$* $\operatorname{Ho}\mathcal{C}(X, Y)$ *is surjective, and moreover two objects in $\mathcal{C}^{(\mathbf{T})}(X, Y)$ become equal in* $\operatorname{Ho}\mathcal{C}$ *if and only if they are in the same connected component.*

Proof. All obvious except for the last part of claim (iii), for which we refer to paragraphs 33.8 and 33.10 in [DHKS]. ∎

Definition A.3.21. Two objects in a relative category are **weakly equivalent** if they can be connected by a *zigzag* of weak equivalences.

Remark A.3.22. If X and Y are weakly equivalent in a relative category $\mathcal{C}$, then they are isomorphic in $\operatorname{Ho}\mathcal{C}$.

A.4 Kan extensions

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.4.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors. A **left Kan extension** (resp. **right Kan extension**) of G along F is an initial (resp. terminal) object of the category $(G \downarrow F^*)$ (resp. $(F^* \downarrow G)$) described below:

- The objects are pairs (H, α) where H is a functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).

Remark A.4.2. Clearly, Kan extensions are unique up to unique isomorphism if they exist. We write $\left(\operatorname{Lan}_F G, \eta\right)$ for the left Kan extension of G along F and say η is the **unit** of $\operatorname{Lan}_F G$; dually, we write $\left(\operatorname{Ran}_F G, \varepsilon\right)$ for the right Kan extension of G along F and say ε is the **counit** of $\operatorname{Ran}_F G$.

Proposition A.4.3. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets. For any two functors* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{C} \to \mathbf{Set}$*, if* $\mathcal{D}$ *is locally* $\mathbf{U}$*-small, then the following are equivalent:*

(i) $\left(\operatorname{Ran}_F G, \varepsilon\right)$ *is a right Kan extension of* G *along* F.

(ii) *The maps* $\left(\operatorname{Ran}_F G\right)(D) \to [\mathcal{C}, \mathbf{Set}](\mathcal{D}(D, F), G)$ *defined by* $x \mapsto \varepsilon \bullet F^* \theta_x$*, where* $\theta_x : \mathcal{D}(D, -) \Rightarrow G$ *is the unique natural transformation such that* $\left(\theta_x\right)_D\left(\mathrm{id}_D\right) = x$*, are bijections that are natural in* D.

Proof. This is a straightforward exercise in applying the Yoneda lemma to the definition of right Kan extensions. □

Definition A.4.4. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{E}$, and $L : \mathcal{E} \to \mathcal{F}$ be three functors. We say L **preserves** a left (resp. right) Kan extension (H, α) of G along F if $(LH, L\alpha)$ is a left (resp. right) Kan extension of LF along G.

Let $\mathbf{Set}$ be the category of $\mathbf{U}$-small sets, and suppose $\mathcal{E}$ is locally $\mathbf{U}$-small. We say a left Kan extension $\left(\operatorname{Lan}_G F, \eta\right)$ is **pointwise** if it is preserved by all functors of the form $\mathcal{E}(-, E) : \mathcal{E} \to \mathbf{Set}^{\mathrm{op}}$.

Dually, we say a right Kan extension $\left(\operatorname{Ran}_G F, \varepsilon\right)$ is **pointwise** if it is preserved by all functors of the form $\mathcal{E}(E, -) : \mathcal{E} \to \mathbf{Set}$.

If a Kan extension is preserved by *all* functors, then it is said to be **absolute**.

It is convenient at this juncture to introduce a concept borrowed from enriched category theory. The notation below follows [Kelly, 2005, § 3.1].

Definition A.4.5. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Given functors $W : \mathcal{J} \to \mathbf{Set}$ and $A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted limit** of A is an object $\{W, A\}^{\mathcal{J}}$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) \cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A))$$

that are natural in C. We may also write $\varprojlim_{j:\mathcal{J}}^{Wj} Aj$ instead of $\{W, A\}^{\mathcal{J}}$, if we wish to use an explicit variable j.

Dually, given functors $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ and $A : \mathcal{J} \to \mathcal{C}$, a **$W$-weighted colimit** of A is an object $W \star_{\mathcal{J}} A$ in $\mathcal{C}$ together with bijections

$$\mathcal{C}\big(W \star_{\mathcal{J}} A, C\big) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}](W, \mathcal{C}(A, C))$$

that are natural in C. We may also write $\varinjlim_{j:\mathcal{J}}^{Wj} Aj$ instead of $W \star_{\mathcal{J}} A$, if we wish to use an explicit variable j.

Remark A.4.6. Clearly, weighted limits and colimits are unique up to unique isomorphism if they exist.

It is also not hard to spell out the above definition in elementary terms; for example, one notes that to give a natural transformation $W \Rightarrow \mathcal{C}(C, A)$, one must give a morphism $\lambda_{j,x} : C \to Aj$ for each object j in $\mathcal{J}$ *and* each element x of Wj, and these are required to make various diagrams commute. This is a **W-weighted cone** from C to A, and $\{W, A\}^{\mathcal{J}}$ is an object equipped with a universal W-weighted cone to A. Similarly, one may define the notion of a **W-weighted cocone** from A to C, and then $W \star_{\mathcal{J}} A$ is an object equipped with a universal W-weighted cocone from A. In particular, if $Wj = 1$ for all j, then W-weighted limits and colimits reduce to ordinary limits and colimits.

The above discussion also shows that the concept of a weighted limit or colimit (within a fixed category!) does not depend on $\mathbf{U}$ in any essential way.

Lemma A.4.7. *Let $\mathcal{J}$ be a $\mathbf{U}$-small category. Given functors $F, G : \mathcal{J} \to \mathbf{Set}$, the F-weighted limit of G exists in $\mathbf{Set}$, and we have bijections*

$$\{F, G\}^{\mathcal{J}} \cong [\mathcal{J}, \mathbf{Set}](F, G)$$

that are natural in F and G.

Proof. One simply has to check that this works. □

Proposition A.4.8. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $F : \mathcal{C} \to \mathcal{D}$ *be any functor where* $\mathcal{C}$ *and* $\mathcal{D}$ *are locally* $\mathbf{U}$*-small categories.*

(i) *For each weight* $W : \mathcal{J} \to \mathbf{Set}$ *and each diagram* $A : \mathcal{J} \to \mathcal{C}$*, if the weighted limits* $\{W, A\}^{\mathcal{J}}$ *and* $\{W, FA\}^{\mathcal{J}}$ *both exist, then there is a canonical comparison morphism*

$$F\{W, A\}^{\mathcal{J}} \to \{W, FA\}^{\mathcal{J}}$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FC, FA))$$

induced by the functor F.

(ii) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ *preserves* all *weighted limits.*

(iii) *The functors* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ jointly *reflect weighted limits.*

(iv) *If* F *has a left adjoint, then* F *preserves weighted limits.*

Dually:

(i′) *For each weight* $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$ *and each diagram* $A : \mathcal{J} \to \mathcal{C}$*, if the weighted colimits* $W \star_{\mathcal{J}} A$ *and* $W \star_{\mathcal{J}} FA$ *both exist, then there is a canonical comparison morphism*

$$W \star_{\mathcal{J}} FA \to F\big(W \star_{\mathcal{J}} A\big)$$

corresponding to the natural maps

$$[\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(A, C)) \to [\mathcal{J}, \mathbf{Set}](W, \mathcal{D}(FA, FC))$$

induced by the functor F.

(ii′) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *sends any weighted colimit in* $\mathcal{C}$ *to the corresponding weighted limit in* $\mathbf{Set}$.

(iii′) *The functors* $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$ jointly *reflect weighted colimits.*

(iv′) *If* F *has a right adjoint, then* F *preserves weighted colimits.*

Proof. All straightforward. ☐

Definition A.4.9. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{D}$ be a locally $\mathbf{U}$-small category. Given a functor $F : \mathcal{C} \to \mathcal{D}$, the ***F*-nerve functor** $\mathrm{N}^F : \mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is defined by

$$\mathrm{N}^F(D)(C) = \mathcal{D}(FC, D)$$

i.e. $\mathrm{N}^F = F^* \hbar_\bullet$, where $\hbar_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ is the usual Yoneda embedding.

Definition A.4.10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and let D be an object in $\mathcal{D}$. The **tautological cocone** to D induced by F is the cocone $\varphi : F P_D \Rightarrow \Delta D$, where $P_D : (F \downarrow D) \to \mathcal{C}$ is the projection functor sending an object (C, f) in the comma category $(F \downarrow D)$ to the object C in $\mathcal{C}$, and $\varphi_{(C,f)} = f$.

Dually, the **tautological cone** from D induced by F is the cone $\varphi : \Delta D \Rightarrow F P^D$, where $P^D : (D \downarrow F) \to \mathcal{C}$ is the projection functor sending an object (C, f) in the comma category $(D \downarrow F)$ to the object C in $\mathcal{C}$, and $\varphi_{(C,f)} = f$.

Theorem A.4.11. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be locally* $\mathbf{U}$*-small categories. Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$, the following are equivalent:*

(i) *(H, α) is a pointwise right Kan extension of G along F.*

(ii) *For each object d in $\mathcal{D}$, the weighted limit $\left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$ exists in $\mathcal{E}$, and there are isomorphisms*

$$H d \cong \left\{\mathrm{N}^{F^{\mathrm{op}}}(d), G\right\}^{\mathcal{C}}$$

natural in d, with $\alpha_c : HFc \to Gc$ corresponding to the element id_{Fc} *of* $\mathrm{N}^{F^{\mathrm{op}}}(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii) *(Assuming $\mathcal{C}$ is* $\mathbf{U}$*-small.) For each object d in $\mathcal{D}$, if $P^d : (d \downarrow F) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(d \downarrow F)$ to c, and $\varphi : \Delta d \Rightarrow F P^d$ is the tautological cone in $\mathcal{D}$, then the cone $\alpha P^d \bullet H\varphi : \Delta H d \Rightarrow G P^d$ is limiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(d' \downarrow F) \to (d \downarrow F)$ sending (c', f') to $(c', f' \circ g)$. In particular, $\alpha_c : HFc \to Gc$ must be (equal to) the component of the limiting cone $\Delta Fc \Rightarrow G P^d$ at the object $\left(c, \mathrm{id}_{Fc}\right)$ of $(Fc \downarrow F)$.*

In particular, if $\mathcal{C}$ is a **U**-*small category and $\mathcal{E}$ is* **U**-*complete, then the right Kan extension of G along F exists and is pointwise.*

Dually, the following are equivalent:

(i′) (H, α) *is a pointwise left Kan extension of G along F.*

(ii′) *For each object d in $\mathcal{D}$, the weighted colimit $\mathrm{N}^F(d) \star_{\mathcal{C}} G$ exists in $\mathcal{E}$, and there are isomorphisms*

$$Hd \cong \mathrm{N}^F(d) \star_{\mathcal{C}} G$$

natural in d, with $\alpha_c : Gc \to HFc$ corresponding to the element id_{Fc} of $\mathrm{N}^F(Fc)(c) = \mathcal{D}(Fc, Fc)$.

(iii′) *(Assuming $\mathcal{C}$ is* **U**-*small.) For each object d in $\mathcal{D}$, if $P_d : (F \downarrow d) \to \mathcal{C}$ is the projection sending (c, f) in the comma category $(F \downarrow d)$ to c, and $\varphi : FP_d \Rightarrow \Delta d$ is the tautological cocone in $\mathcal{D}$, then the cocone $H\varphi \bullet \alpha P_d : GP_d \Rightarrow \Delta Hd$ is colimiting; and for each $g : d \to d'$ in $\mathcal{D}$, the morphism $Hg : Hd \to Hd'$ is the one induced by the functor $(F \downarrow d) \to (F \downarrow d')$ sending (c, f) to $(c, g \circ f)$. In particular, $\alpha_c : Gc \to HFc$ must be (equal to) the component of the colimiting cocone $GP_d \Rightarrow \Delta Fc$ at the object $\left(c, \mathrm{id}_{Fc}\right)$ of $(F \downarrow Fc)$.*

In particular, if $\mathcal{C}$ is a **U**-*small category and $\mathcal{E}$ is* **U**-*cocomplete, then the left Kan extension of G along F exists and is pointwise.*

Proof. (i) ⇔ (ii). This is just a matter of unwinding the definitions.

(i) ⇔ (iii). One first proves that the construction in (iii) does indeed define a right Kan extension in the special case $\mathcal{E} = \mathbf{Set}$; once this is done, showing that (i) and (iii) are equivalent is simply a matter of applying the Yoneda lemma. See [CWM, Ch. X, §§ 3 and 5]. □

Remark A.4.12. It is possible to extract an elementary characterisation of pointwise Kan extensions from the results above, thereby showing that the property of being pointwise does not depend on the choice of universe **U**.

Corollary A.4.13. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $\mathcal{C}$ is* **U**-*small and $\mathcal{D}$ is locally* **U**-*small, then the functor $F^* : [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ has both a left adjoint Lan_F and a right adjoint Ran_F.* ■

Corollary A.4.14. *Let $L : \mathcal{E} \to \mathcal{F}$ be a functor. With other notation as in the theorem, if (H, α) is a pointwise right Kan extension of G along F, then $(LH, L\alpha)$ is a pointwise right Kan extension of LG along F, provided either:*

(i) *L preserves all weighted limits, or*

(ii) *L preserves limits for $\mathbf{U}$-small diagrams and $\mathcal{C}$ is $\mathbf{U}$-small.*

Dually, if (H, α) is a pointwise left Kan extension of G along F, then $(LH, L\alpha)$ is a pointwise left Kan extension of LG along F, provided either:

(i′) *L preserves all weighted colimits, or*

(ii′) *L preserves colimits for $\mathbf{U}$-small diagrams and $\mathcal{C}$ is $\mathbf{U}$-small.* ■

Corollary A.4.15. *With notation as in the theorem, if F is fully faithful and (H, α) is a pointwise right (resp. left) Kan extension of G along F, then $\alpha : HF \Rightarrow G$ (resp. $\alpha : G \Rightarrow HF$) is a natural isomorphism.*

Proof. If F is fully faithful, then the comma category $(Fc \downarrow F)$ (resp. $(F \downarrow Fc)$) has an initial (resp. terminal) object, namely $\left(c, \mathrm{id}_{Fc}\right)$, so the component $\alpha_c : HFc \to Gc$ (resp. $\alpha_c : Gc \to HFc$) must be an isomorphism. ■

Proposition A.4.16. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be any two functors. The following are equivalent:*

(i) *$F \dashv G$, with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and counit $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.*

(ii) *(F, ε) is an absolute right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G.*

(iii) *(F, ε) is a right Kan extension of $\mathrm{id}_{\mathcal{D}}$ along G that is preserved by F.*

(iv) *(G, η) is an absolute left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F.*

(v) *(G, η) is a left Kan extension of $\mathrm{id}_{\mathcal{C}}$ along F that is preserved by G.*

Proof. See [CWM, Ch. X, § 7]. □

Proposition A.4.17.

(i) *Right adjoints preserve all right Kan extensions.*

(ii) *Left adjoints preserve all left Kan extensions.*

Proof. See Theorem 1 in [CWM, Ch. X, § 5]. □

Definition A.4.18. Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. A **dense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the F-nerve functor $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. A **dense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a dense functor.

Dually, a **codense functor** is a functor $F : \mathcal{B} \to \mathcal{C}$ such that the opposite functor $F^{\mathrm{op}} : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is dense, and a **codense subcategory** of $\mathcal{C}$ is a subcategory $\mathcal{B}$ such that the inclusion $\mathcal{B} \hookrightarrow \mathcal{C}$ is a codense functor.

Example A.4.19. The Yoneda lemma implies $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is a dense and codense functor.

One may extract an elementary definition for '(co)dense functor' from the following proposition:

Proposition A.4.20. *With notation as in the definition, the following are equivalent:*

(i) *$F : \mathcal{B} \to \mathcal{C}$ is a dense functor.*

(ii) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C, C') \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]\big(\mathrm{N}^F(C), \mathcal{C}(F, C')\big)$$

induced by $\mathrm{N}^F : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted colimit $\mathrm{N}^F(C) \star_{\mathcal{B}} F$ in $\mathcal{C}$.

(iii) *For each object C in $\mathcal{C}$, the tautological cocone to C induced by F is a colimiting cocone.*

(iv) *$\big(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F\big)$ is a pointwise left Kan extension of F along F.*

Dually, the following are equivalent:

(i′) *$F : \mathcal{B} \to \mathcal{C}$ is a codense functor.*

(ii′) *For each object C in $\mathcal{C}$, the maps*

$$\mathcal{C}(C', C) \to [\mathcal{B}, \mathbf{Set}]\big(\mathrm{N}^{F^{\mathrm{op}}}(C), \mathcal{C}(C', F)\big)$$

induced by $\mathrm{N}^{F^{\mathrm{op}}} : \mathcal{C}^{\mathrm{op}} \to [\mathcal{B}, \mathbf{Set}]$ are natural bijections, exhibiting C as a weighted limit $\big\{\mathrm{N}^{F^{\mathrm{op}}}(C), F\big\}^{\mathcal{B}}$ in $\mathcal{C}$.

(iii′) *For each object C in $\mathcal{C}$, the tautological cone from C induced by F is a limiting cone.*

(iv′) $\left(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_F\right)$ *is a pointwise right Kan extension of F along F.*

Proof. (i) ⇔ (ii). The indicated maps are bijections for all C and C' if and only if N^F is fully faithful, by definition.

(ii) ⇔ (iii) ⇔ (iv). This is an application of theorem A.4.11. ■

Definition A.4.21. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. A **densely-defined partial left adjoint** for G is a triple (F, i, η), where $F : \mathcal{B} \to \mathcal{D}$ is a functor, $i : \mathcal{B} \to \mathcal{C}$ is a dense functor, and $\eta : i \Rightarrow GF$ is a natural transformation such that the maps

$$\mathcal{D}(FB, D) \to \mathcal{C}(iB, GD)$$
$$g \mapsto Gg \circ \eta_B$$

are bijections that are natural in B and D.

Dually, given a functor $F : \mathcal{C} \to \mathcal{D}$, a **codensely-defined partial right adjoint** for F is a triple (G, j, ε), where $G : \mathcal{B} \to \mathcal{C}$ is a functor, $j : \mathcal{B} \to \mathcal{C}$ is a codense functor, and $\varepsilon : FG \Rightarrow j$ is a natural transformation such that the maps

$$\mathcal{C}(C, GB) \to \mathcal{D}(FC, jB)$$
$$f \mapsto \varepsilon_B \circ Ff$$

are bijections that are natural in B and C.

Example A.4.22. The Yoneda embedding $\hbar_\bullet : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ has a densely-defined partial left adjoint, namely $\left(\mathrm{id}_{\mathcal{B}}, \hbar_\bullet, \mathrm{id}_{\hbar_\bullet}\right)$.

Remark A.4.23. $\left(F, \mathrm{id}_{\mathcal{C}}, \eta\right)$ is a densely-defined partial left adjoint for G if and only if F is a left adjoint for G in the usual sense, with η being the adjunction unit.

Proposition A.4.24. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ and $\mathcal{D}$ be locally $\mathbf{U}$-small categories. Given functors $G : \mathcal{D} \to \mathcal{C}$, $F : \mathcal{B} \to \mathcal{D}$, and $i : \mathcal{B} \to \mathcal{C}$, the following are equivalent:*

(i) (F, i, η) *is a densely-defined partial left adjoint for G.*

(ii) *The functor $i : \mathcal{B} \to \mathcal{C}$ is dense, and there exists a diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\text{よ}_\bullet} & [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \\ {\scriptstyle G}\downarrow & \swArrow{\scriptstyle\alpha} & \downarrow{\scriptstyle (F^{\mathrm{op}})^*} \\ \mathcal{C} & \xrightarrow[\mathrm{N}^i]{} & [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

where α factors through $\eta^* : \mathrm{N}^{GF} \Rightarrow \mathrm{N}^i$ and is a natural isomorphism.

(iii) *The functor $i : \mathcal{B} \to \mathcal{C}$ is dense, and the diagram*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\text{よ}_\bullet} & [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle (F^{\mathrm{op}})^*} \\ \mathcal{C} & \xrightarrow[\mathrm{N}^i]{} & [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

commutes up to natural isomorphism.

Dually, given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{B} \to \mathcal{C}$, and $j : \mathcal{B} \to \mathcal{D}$, the following are equivalent:

(i′) *(G, j, ε) is a codensely-defined partial right adjoint for F.*

(ii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and there exists a diagram*

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\text{よ}^\bullet} & [\mathcal{C}, \mathbf{Set}] \\ {\scriptstyle F^{\mathrm{op}}}\downarrow & \swArrow{\scriptstyle\beta} & \downarrow{\scriptstyle G^*} \\ \mathcal{D}^{\mathrm{op}} & \xrightarrow[\mathrm{N}^{j^{\mathrm{op}}}]{} & [\mathcal{B}, \mathbf{Set}] \end{array}$$

where β factors through $(\varepsilon^{\mathrm{op}})^* : \mathrm{N}^{F^{\mathrm{op}}G^{\mathrm{op}}} \Rightarrow \mathrm{N}^{j^{\mathrm{op}}}$ and is a natural isomorphism.

(iii′) *The functor $j : \mathcal{B} \to \mathcal{D}$ is codense, and the diagram*

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{\text{よ}^\bullet} & [\mathcal{C}, \mathbf{Set}] \\ {\scriptstyle F^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle G^*} \\ \mathcal{D}^{\mathrm{op}} & \xrightarrow[\mathrm{N}^{j^{\mathrm{op}}}]{} & [\mathcal{B}, \mathbf{Set}] \end{array}$$

commutes up to natural isomorphism.

Proof. (i) ⇒ (ii). This immediately follows from the definition.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). The displayed diagram commutes up to natural isomorphism precisely when there are bijections

$$\alpha_{B,D} : \mathcal{D}(FB, D) \to \mathcal{C}(iB, GD)$$

that are natural in both B and D. Taking $D = FB$, let $\eta_B : iB \to GFB$ be the morphism corresponding to $\mathrm{id}_{FB} : FB \to FB$. Applying the Yoneda lemma, we see that the natural bijection $\alpha_{B,D}$ must be the map $g \mapsto Gg \circ \eta_B$. ■

Corollary A.4.25. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two categories. If a functor $G : \mathcal{D} \to \mathcal{C}$ has a densely-defined partial left adjoint, then G preserves:*

(i) *limits for all diagrams in $\mathcal{D}$,*

(ii) *weighted limits, and*

(iii) *pointwise right Kan extensions.*

Dually, if a functor $F : \mathcal{C} \to \mathcal{D}$ has a codensely-defined partial right adjoint, then F preserves:

(i′) *colimits for all diagrams in $\mathcal{C}$,*

(ii′) *weighted colimts, and*

(iii′) *pointwise left Kan extensions.*

Proof. Choose a universe $\mathbf{U}$ such that the domain of $i : \mathcal{B} \to \mathcal{C}$ is $\mathbf{U}$-small and both $\mathcal{C}$ and $\mathcal{D}$ are locally $\mathbf{U}$-small, and consider the following diagram:

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{h_\bullet} & [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle (F^{\mathrm{op}})^*} \\ \mathcal{C} & \xrightarrow[\mathrm{N}^i]{} & [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}] \end{array}$$

Since i is dense, the i-nerve functor $\mathrm{N}^i : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful. Corollary A.4.13 implies $(F^{\mathrm{op}})^* : [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}] \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is a right adjoint, and the Yoneda embedding $h_\bullet : \mathcal{D} \to [\mathcal{D}^{\mathrm{op}}, \mathbf{Set}]$ preserves all limits and weighted limits (see proposition A.4.8), so we use the fact that N^i reflects limits and weighted limits to conclude that G preserves them. We then apply corollary A.4.14. ■

A.5 Ends and coends

Prerequisites. §§ 0.1, A.4

In this section we use the explicit universe convention.

Definition A.5.1. Let $F, G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be functors. A **dinatural transformation** $\alpha : F \stackrel{\diamond}{\to} G$ is a family $\left(\alpha_C : F(C,C) \to G(C,C) \,\middle|\, C \in \operatorname{ob} \mathcal{C}\right)$ such that the diagram

$$
\begin{array}{ccccc}
 & F(C,C) & \xrightarrow{\alpha_C} & G(C,C) & \\
{\scriptstyle F(\mathrm{id}_C, f)} \nearrow & & & & \searrow {\scriptstyle G(f, \mathrm{id}_C)} \\
F(C,C') & & & & G(C',C) \\
{\scriptstyle F(f, \mathrm{id}_{C'})} \searrow & & & & \nearrow {\scriptstyle G(\mathrm{id}_{C'}, f)} \\
 & F(C',C') & \xrightarrow[\alpha_{C'}]{} & G(C',C') &
\end{array}
$$

commutes for all morphisms $f : C' \to C$ in $\mathcal{C}$.

Example A.5.2. Let $\mathbf{U}$ be a pre-universe, let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets. Consider the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ that sends a pair of objects in $\mathcal{C}$ to their hom-set. For each natural number n, we have an dinatural transformation $\mathrm{Hom}_{\mathcal{C}} \stackrel{\diamond}{\to} \mathrm{Hom}_{\mathcal{C}}$ defined by $e \mapsto e^n$, where e^n denotes the n-fold iterate of the endomorphism e.

Definition A.5.3. A **wedge** from an object D in $\mathcal{D}$ to a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is a dinatural transformation $\Delta D \stackrel{\diamond}{\to} G$, where $\Delta D : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is the constant functor with value D; dually, a **cowedge** from a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ to an object D in $\mathcal{D}$ is a dinatural transformation $F \stackrel{\diamond}{\to} \Delta D$.

Definition A.5.4. An **end** for a functor $G : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a wedge $\lambda : \Delta E \stackrel{\diamond}{\to} G$ with the following universal property:

- For each wedge $\varphi : \Delta D \stackrel{\diamond}{\to} G$, there is a unique morphism $f : D \to E$ in $\mathcal{D}$ such that $\varphi_C = \lambda_C \circ f$ for all objects C in $\mathcal{C}$.

We write the following formula to mean that E is an end for G:

$$E = \int_{C : \mathcal{C}} G(C,C)$$

Dually, a **coend** for a functor $F : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ is an object E and a cowedge $\lambda : F \overset{\diamond}{\to} \Delta E$ with the following universal property:

- For each cowedge $\varphi : F \overset{\diamond}{\to} \Delta D$, there is a unique morphism $f : E \to D$ in $\mathcal{D}$ such that $\varphi_C = f \circ \lambda_C$ for all objects C in $\mathcal{C}$.

We write the following formula to mean that E is a coend for F:

$$E = \int^{C:\mathcal{C}} F(C, C)$$

Remark A.5.5. Let $\mathbf{U}$ be a pre-universe, let $\mathbb{D}$ be a $\mathbf{U}$-small category, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. Then, for all functors $F, G : \mathbb{D} \to \mathcal{C}$, we have a bijection

$$[\mathbb{D}, \mathcal{C}](F, G) \cong \int_{d:\mathbb{D}} \mathcal{C}(Fd, Gd)$$

and this is natural in both F and G. (The size restriction ensures that the LHS is a $\mathbf{U}$-set.) See also lemma A.4.7.

Proposition A.5.6. *Let* $\mathbf{U}$ *be a pre-universe and let* $\mathbb{D}$ *be a* $\mathbf{U}$*-small category. If* $\mathcal{C}$ *is a* $\mathbf{U}$*-complete category, then* $\mathcal{C}$ *has ends for all functors* $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*. Dually, if* $\mathcal{C}$ *is a* $\mathbf{U}$*-cocomplete category, then* $\mathcal{C}$ *has coends for all functors* $A : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*.*

Proof. It is clear from the definition that an end is a special kind of limit, and a coend is a special kind of colimit. To make this precise, one can use Mac Lane's subdivision category $\mathcal{C}^{\S}$: see [CWM, Ch. IX, § 5]. □

Proposition A.5.7. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* $F : \mathcal{C} \to \mathcal{D}$ *be any functor where* $\mathcal{C}$ *and* $\mathcal{D}$ *are locally* $\mathbf{U}$*-small categories.*

(i) *For any functor* $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$*, if the ends* $\int_{\mathcal{J}} A$ *and* $\int_{\mathcal{J}} FA$ *both exist, with* λ *being the universal wedge in* $\mathcal{C}$*, then there is a canonical comparison morphism*

$$F \int_{\mathcal{J}} A \to \int_{\mathcal{J}} FA$$

induced by the wedge $F\lambda$*.*

(ii) *For any object* C *in* $\mathcal{C}$*, the functor* $\mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$ *preserves* all *ends.*

(iii) *The functors* $\mathcal{C}(C, -)$ jointly *reflect ends.*

(iv) *If F has a left adjoint, then F preserves ends.*

Dually:

(i′) *For any functor $A : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, if the coends $\int^{\mathcal{J}} A$ and $\int^{\mathcal{J}} FA$ both exist, with λ being the universal cowedge in $\mathcal{C}$, then there is a canonical comparison morphism*

$$\int^{\mathcal{J}} FA \to F \int^{\mathcal{J}} A$$

induced by the cowedge $F\lambda$.

(ii′) *For any object C in $\mathcal{C}$, the functor $\mathcal{C}(-, C) : \mathcal{C} \to$ **Set** sends any coend in $\mathcal{C}$ to the corresponding end in* **Set**.

(iii′) *The functors $\mathcal{C}(-, C) : \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$* jointly *reflect coends.*

(iv′) *If F has a right adjoint, then F preserves coends.*

Proof. All straightforward. □

Definition A.5.8. Let $\mathbf{U}$ be a pre-universe, let **Set** be the category of $\mathbf{U}$-sets, and let $\mathbb{1}$ be the trivial category with $*$ as its only object. A **tensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted colimit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a tensored $\mathbf{U}$-category, then we write $X \odot C$ for the weighted colimit $W \star_{\mathbb{1}} A$, where $X = W(*)$ and $C = A(*)$.

Dually, a **cotensored U-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ such that, for all weights $W : \mathbb{1} \to \mathbf{Set}$ and all diagrams $A : \mathbb{1} \to \mathbf{Set}$, a W-weighted limit for A exists in $\mathcal{C}$; if $\mathcal{C}$ is a cotensored $\mathbf{U}$-category, then we write $X \pitchfork C$ for the weighted limit $\{W, A\}^{\mathbb{1}}$, where $X = W(*)$ and $C = A(*)$.

Proposition A.5.9 (Tensor–hom–cotensor adjunction)**.** *Let* $\mathbf{U}$ *be a pre-universe, let* **Set** *be the category of* $\mathbf{U}$*-sets, let $\mathcal{C}$ be a locally* $\mathbf{U}$*-small category.*

(i) *If $\mathcal{C}$ is a tensored* $\mathbf{U}$*-category, then the assignment $(X, C) \mapsto X \odot C$ can be extended to a functor* $\mathbf{Set} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object C, we have the following adjunction:*

$$- \odot C \dashv \mathcal{C}(C, -) : \mathcal{C} \to \mathbf{Set}$$

(ii) *If $\mathcal{C}$ is a cotensored* **U**-*category, then the assignment* $(X, C) \mapsto X \pitchfork C$ *can be extended to a functor* $\mathbf{Set}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ *such that, for each object* C*, the functors* $- \pitchfork C : \mathbf{Set}^{\mathrm{op}} \to \mathcal{C}$ *and* $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ *are contravariantly adjoint on the right.*

(iii) *If $\mathcal{C}$ is a tensored and cotensored* **U**-*category, then for each set* X*, we have the following adjunction:*

$$X \odot - \dashv X \pitchfork - : \mathcal{C} \to \mathcal{C}$$

Proof. Claims (i) and (ii) are formally dual and are straightforward applications of the parametrised adjunction theorem.[1] For claim (iii), simply observe that we have bijections

$$\mathcal{C}(X \odot A, B) \cong \mathbf{Set}(X, \mathcal{C}(A, B)) \cong \mathcal{C}(A, X \pitchfork B)$$

and these are natural in A, B, and X. ■

Theorem A.5.10. *Let* **U** *be a pre-universe, let* **Set** *be the category of* **U**-*sets, and let $\mathcal{C}$ be a locally* **U**-*small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a* **U**-*complete category.*

(ii) *$\mathcal{C}$ is a cotensored* **U**-*category and, for all* **U**-*small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, an end for* A *exists in* $\mathcal{C}$.

(iii) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted limit for* A*, provided* $\mathbb{D}$ *is a* **U**-*small category.*

Dually, the following are equivalent:

(i′) *$\mathcal{C}$ is a* **U**-*cocomplete category.*

(ii′) *$\mathcal{C}$ is a tensored* **U**-*category and, for all* **U**-*small categories* $\mathbb{D}$ *and all functors* $B : \mathbb{D}^{\mathrm{op}} \times \mathbb{D} \to \mathcal{C}$*, a coend for* A *exists in* $\mathcal{C}$.

(iii′) *For all weights* $W : \mathbb{D}^{\mathrm{op}} \to \mathbf{Set}$ *and all diagrams* $A : \mathbb{D} \to \mathbf{Set}$*,* $\mathcal{C}$ *has a* W*-weighted colimit for* A*, provided* $\mathbb{D}$ *is a* **U**-*small category.*

[1] See Theorem 3 in [CWM, Ch. IV, § 7].

Proof. (i) ⇒ (ii). It is clear that $X \pitchfork C$ is nothing more than an X-fold product of copies of C, so C is certainly **U**-cotensored if it is **U**-complete, and proposition A.5.6 says $\mathcal{C}$ also has the required ends in that case.

(ii) ⇒ (iii). We have the following natural bijections:

$$\begin{aligned} \mathcal{C}\big(C, \{W, A\}^{\mathbb{D}}\big) &\cong [\mathbb{D}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\ &\cong \int_{d:\mathbb{D}} \mathbf{Set}(Wd, \mathcal{C}(C, Ad)) \\ &\cong \int_{d:\mathbb{D}} \mathcal{C}(C, Wd \pitchfork Ad) \\ &\cong \mathcal{C}\left(C, \int_{d:\mathbb{D}} Wd \pitchfork Ad\right) \end{aligned}$$

Thus, using the Yoneda lemma and *assuming $\mathcal{C}$ is a cotensored* **U**-*category*, the weighted limit $\{W, A\}^{\mathbb{D}}$ exists if and only if the end $\int_{d:\mathbb{D}} Wd \pitchfork Ad$ exists.

(iii) ⇒ (i). Ordinary limits are a special case of weighted limits, as remarked in A.4.6. ∎

Proposition A.5.11. *Let* **U** *be a pre-universe, let* **Set** *be the category of* **U**-*sets, let $\mathcal{C}$ be a locally* **U**-*small category, and let $\mathcal{J}$ be any category. If $\mathcal{C}$ is a tensored* **U**-*category and has weighted limits for all weights $W : \mathcal{J} \to$* **Set** *and diagrams $A : \mathcal{J} \to \mathcal{C}$, then:*

(i) *$(W, A) \mapsto \{W, A\}^{\mathcal{J}}$ extends to a functor $[\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$.*

(ii) *For each diagram $A : \mathcal{J} \to \mathcal{C}$, the functors $\{-, A\}^{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}]^{\mathrm{op}} \to \mathcal{C}$ and $\mathcal{C}(-, A) : \mathcal{C}^{\mathrm{op}} \to [\mathcal{J}, \mathbf{Set}]$ are contravariantly adjoint on the right.*

(iii) *For each weight $W : \mathcal{J} \to$* **Set**, *we have the following adjunction:*

$$W \odot - \dashv \{W, -\}^{\mathcal{J}} : [\mathcal{J}, \mathcal{C}] \to \mathcal{C}$$

Here, $W \odot C : \mathcal{J} \to \mathcal{C}$ is the diagram $j \mapsto Wj \odot C$.

Dually, if $\mathcal{C}$ is a cotensored **U**-*category and has weighted colimits for all weights $W : \mathcal{J}^{\mathrm{op}} \to$* **Set** *and diagrams $A : \mathcal{J} \to \mathcal{C}$, then:*

(i′) *$(W, A) \mapsto W \star_{\mathcal{J}} A$ extends to a functor $[\mathcal{J}^{\mathrm{op}}, \mathbf{Set}] \times \mathcal{C} \to \mathcal{C}$.*

(ii′) *For each diagram $A : \mathcal{J} \to \mathcal{C}$, we have the following adjunction:*

$$- \star_{\mathcal{J}} A \dashv \mathcal{C}(A, -) : \mathcal{C} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]$$

(iii′) *For each weight $W : \mathcal{J}^{\mathrm{op}} \to \mathbf{Set}$, we have the following adjunction:*

$$W \star_{\mathcal{J}} - \dashv W \pitchfork - : \mathcal{C} \to [\mathcal{J}, \mathcal{C}]$$

Here, $W \pitchfork C : \mathcal{J} \to \mathcal{C}$ is the diagram $j \mapsto Wj \pitchfork C$.

Proof. Claim (i) is straightforward, and for claims (ii) and (iii), observe that we have bijections

$$\begin{aligned}
\mathcal{C}\big(C, \{W, A\}^{\mathcal{J}}\big) &\cong [\mathcal{J}, \mathbf{Set}](W, \mathcal{C}(C, A)) \\
&\cong \int_{j:\mathcal{J}} \mathbf{Set}(Wj, \mathcal{C}(C, Aj)) \\
&\cong \int_{j:\mathcal{J}} \mathcal{C}(Wj \odot C, Aj) \\
&\cong [\mathcal{J}, \mathcal{C}](W \odot C, A)
\end{aligned}$$

and these are natural in W, A, and C. ■

Lemma A.5.12. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathbb{I}$ and $\mathbb{J}$ be $\mathbf{U}$-small categories. For all functors $A : \mathbb{I}^{\mathrm{op}} \times \mathbb{J}^{\mathrm{op}} \times \mathbb{I} \times \mathbb{J} \to \mathbf{Set}$:*

(i) *The assignment $(i', i) \mapsto \int_{j:\mathbb{J}} A(i', j, i, j)$ extends to a functor $\mathbb{I}^{\mathrm{op}} \times \mathbb{I} \to \mathbf{Set}$.*

(ii) *There is a unique morphism θ making the diagram below commute for all i and j,*

$$\begin{array}{ccc}
\displaystyle\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i', j', i', j') & \longrightarrow & \displaystyle\int_{j':\mathbb{J}} A(i, j', i, j') \\
{\scriptstyle\theta}\,\big\downarrow & & \big\downarrow \\
\displaystyle\int_{(i',j'):\mathbb{I}\times\mathbb{J}} A(i', j', i', j') & \longrightarrow & A(i, j, i, j)
\end{array}$$

where the unlabelled arrows are the components of the respective universal wedges, and θ is moreover an isomorphism.

(iii) *There is a unique morphism σ making the diagram below commute for all i and j,*

$$\begin{array}{ccccc}
\int_{i':\mathbb{I}} \int_{j':\mathbb{J}} A(i',j',i',j') & \longrightarrow & \int_{j':\mathbb{J}} A(i,j',i,j') & \searrow & \\
{\scriptstyle \sigma}\big\downarrow & & & & A(i,j,i,j) \\
\int_{j':\mathbb{J}} \int_{i':\mathbb{I}} A(i',j',i',j') & \longrightarrow & \int_{i':\mathbb{I}} A(i,j',i,j') & \nearrow &
\end{array}$$

where the unmarked arrows are the components of the respective universal wedges, and σ is moreover an isomorphism.

Proof. See [CWM, Ch. IX, § 8]. □

Theorem A.5.13 (Interchange law for ends and coends)**.** *Let $\mathcal{C}$ be any category and let $A : \mathcal{I}^{\mathrm{op}} \times \mathcal{J}^{\mathrm{op}} \times \mathcal{I} \times \mathcal{J} \to$ **Set** be any functor. If the end $\int_{i:\mathcal{I}} A(i,j',i,j)$ exists in $\mathcal{C}$ for all j' and j in $\mathcal{J}$, and the end $\int_{j:\mathcal{J}} A(i',j,i,j)$ exists in $\mathcal{C}$ for all i' and i in $\mathcal{I}$, then the following are equivalent:*

(i) *The end $\int_{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(ii) *The iterated end $\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(iii) *The iterated end $\int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

In this case, we have a canonical isomorphism in $\mathcal{C}$:

$$\int_{i:\mathcal{I}} \int_{j:\mathcal{J}} A(i,j,i,j) \cong \int_{j:\mathcal{J}} \int_{i:\mathcal{I}} A(i,j,i,j)$$

Dually, if the coend $\int^{i:\mathcal{I}} A(i,j',i,j)$ exists in $\mathcal{C}$ for all j' and j in $\mathcal{J}$, and the coend $\int^{j:\mathcal{J}} A(i',j,i,j)$ exists in $\mathcal{C}$ for all i' and i in $\mathcal{I}$, then the following are equivalent:

(i) *The coend $\int^{(i,j):\mathcal{I}\times\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(ii) *The iterated coend $\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

(iii) *The iterated coend $\int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$ exists in $\mathcal{C}$.*

In this case, we have a canonical isomorphism in $\mathcal{C}$:

$$\int^{i:\mathcal{I}} \int^{j:\mathcal{J}} A(i,j,i,j) \cong \int^{j:\mathcal{J}} \int^{i:\mathcal{I}} A(i,j,i,j)$$

Proof. Choose a pre-universe $\mathbf{U}$ such that $\mathcal{I}$ and $\mathcal{J}$ are $\mathbf{U}$-small categories and $\mathcal{C}$ is a locally $\mathbf{U}$-small category, and use the Yoneda lemma to reduce the claims to the previous lemma. ■

Proposition A.5.14. *Let $\mathbf{U}$ be a pre-universe, let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets, and let $\mathcal{C}$ and $\mathcal{J}$ be locally $\mathbf{U}$-small categories.*

(i) *For all j in $\mathcal{J}$ and all functors $A : \mathcal{J} \to \mathcal{C}$, the Yoneda bijection*

$$\mathcal{C}(C, Aj) \cong [\mathcal{J}, \mathbf{Set}]\big(\hbar^j, \mathcal{C}(C, A)\big)$$

exhibits Aj as the weighted limit $\left\{\hbar^j, A\right\}^{\mathcal{J}}$ in $\mathcal{C}$.

(ii) *If $\mathcal{C}$ is a cotensored $\mathbf{U}$-category, then the end $\int_{j':\mathcal{J}} \mathcal{J}(j, j') \pitchfork Aj'$ exists in $\mathcal{C}$ and can be canonically identified with Aj.*

(iii) *For all functors $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, the weighted limit $\left\{\mathrm{Hom}_{\mathcal{J}}, H\right\}^{\mathcal{J}^{\mathrm{op}} \times \mathcal{J}}$ exists in $\mathcal{C}$ if and only if the end $\int_{j:\mathcal{J}} H(j,j)$ exists in $\mathcal{C}$, and there is a canonical identification of the two.*

Dually:

(i′) *For all j in $\mathcal{J}$ and all functors $A : \mathcal{J} \to \mathcal{C}$, the Yoneda bijection*

$$\mathcal{C}(Aj, C) \cong [\mathcal{J}^{\mathrm{op}}, \mathbf{Set}]\big(\hbar_j, \mathcal{C}(A, C)\big)$$

exhibits Aj as the weighted colimit $\hbar_j \star_{\mathcal{J}} A$ in $\mathcal{C}$.

(ii′) *If $\mathcal{C}$ is a tensored $\mathbf{U}$-category, then the coend $\int^{j':\mathcal{J}} \mathcal{J}(j', j) \odot Aj'$ exists in $\mathcal{C}$ and can be canonically identified with Aj.*

(iii′) *For all functors $H : \mathcal{J}^{\mathrm{op}} \times \mathcal{J} \to \mathcal{C}$, the weighted colimit $\mathrm{Hom}_{\mathcal{J}^{\mathrm{op}}} \star_{\mathcal{J}^{\mathrm{op}} \times \mathcal{J}} H$ exists in $\mathcal{C}$ if and only if the coend $\int^{j:\mathcal{J}} H(j,j)$ exists in $\mathcal{C}$, and there is a canonical identification of the two.*

Proof. (i). This is an immediate consequence of the Yoneda lemma and the definition of weighted limit.

(ii). Use the identification constructed in the proof of theorem A.5.10.

(iii). For all objects C in $\mathcal{C}$, using claim (ii) and the interchange law for ends (theorem A.5.13), there are bijections

$$\begin{aligned}[\mathcal{J}^{\mathrm{op}} \times \mathcal{J}, \mathbf{Set}]\big(\mathrm{Hom}_{\mathcal{J}}, \mathcal{C}(C, H)\big) &\cong \int_{(j',j):\mathcal{J}^{\mathrm{op}}\times\mathcal{J}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(H(j', j))) \\ &\cong \int_{j:\mathcal{J}} \int_{j':\mathcal{J}^{\mathrm{op}}} \mathbf{Set}(\mathcal{J}(j', j), \mathcal{C}(H(j', j))) \\ &\cong \int_{j:\mathcal{J}} \mathcal{C}(C, H(j, j))\end{aligned}$$

and these are natural in C; now apply propositions A.4.8 and A.5.7. ■

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