Notes on homotopical algebra

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Preface

These notes are intended as a kind of annotated index to the various standard references in homotopical algebra: the focus is on definitions and statements of results, *not* proofs.

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Foundations

0.1 Set theory

In category theory it is often convenient to invoke a certain set-theoretic device commonly known as a 'Grothendieck universe', but we shall say simply 'universe', so as to simplify exposition and proofs by eliminating various circumlocutions involving cardinal bounds, proper classes etc.

Definition 0.1.1. A **pre-universe** is a set $\mathbf{U}$ satisfying these axioms:

1. If $x \in y$ and $y \in \mathbf{U}$, then $x \in \mathbf{U}$.
2. If $x \in \mathbf{U}$ and $y \in \mathbf{U}$ (but not necessarily distinct), then $\{x, y\} \in \mathbf{U}$.
3. If $x \in \mathbf{U}$, then $\mathscr{P}(x) \in \mathbf{U}$, where $\mathscr{P}(x)$ denotes the set of all subsets of x.
4. If $x \in \mathbf{U}$ and $f : x \to \mathbf{U}$ is a map, then $\bigcup_{i \in x} f(i) \in \mathbf{U}$.

A **universe** is a pre-universe $\mathbf{U}$ with this additional property:

5. $\omega \in \mathbf{U}$, where ω is the set of all finite (von Neumann) ordinals.

Example 0.1.2. The empty set is a pre-universe, and with very mild assumptions, so is the set **HF** of all hereditarily finite sets.

¶ **0.1.3.** The notion of universe makes sense in any material set theory, but their existence must be postulated. We adopt the following:

- **Grothendieck–Verdier universe axiom.** For each set x, there exists a universe $\mathbf{U}$ with $x \in \mathbf{U}$.

For definiteness, we may take our base theory to be Mac Lane set theory, which is a weak subsystem of Zermelo–Fraenkel set theory with choice (ZFC). Readers interested in the details of Mac Lane set theory are referred to [Mathias, 2001], but in practice as long as one is working at all times *inside some universe*, one may as well be working in ZFC. Indeed:

Proposition 0.1.4. *With the assumptions of Mac Lane set theory, any universe is a transitive model of ZFC.*

Proof. Let $\mathbf{U}$ be a universe. By definition, $\mathbf{U}$ is a transitive set containing pairs, power sets, unions, and ω, so the axioms of extensionality, empty set, pairs, power sets, unions, choice, and infinity are all automatically satisfied. We must show that the axiom schemas of separation and replacement are also satisfied, and in fact it is enough to check that replacement is valid; but this is straightforward using axioms 2 and 4. ■

Definition 0.1.5. Let $\mathbf{U}$ be a pre-universe. A **U-set** is a member of $\mathbf{U}$, a **U-class** is a subset of $\mathbf{U}$, and a **proper U-class** is a $\mathbf{U}$-class that is not a $\mathbf{U}$-set.

Lemma 0.1.6. *A* $\mathbf{U}$*-class* X *is a* $\mathbf{U}$*-set if and only if there exists a* $\mathbf{U}$*-class* Y *such that* $X \in Y$. ■

Proposition 0.1.7. *If* $\mathbf{U}$ *is a universe, then the collection of* $\mathbf{U}$*-classes is a transitive model of Morse–Kelley class–set theory (MK), and so is a transitive model of von Neumann–Bernays–Gödel class–set theory (NBG) in particular.* ■

Definition 0.1.8. A **U-small category** is a category $\mathbb{C}$ such that $\operatorname{ob}\mathbb{C}$ and $\operatorname{mor}\mathbb{C}$ are $\mathbf{U}$-sets. A **locally U-small category** is a category $\mathcal{D}$ satisfying these conditions:

- $\operatorname{ob}\mathcal{D}$ and $\operatorname{mor}\mathcal{D}$ are $\mathbf{U}$-classes, and
- for all objects x and y in $\mathcal{D}$, the hom-set $\mathcal{D}(x, y)$ is a $\mathbf{U}$-set.

An **essentially U-small category** is a category $\mathcal{D}$ for which there exist a $\mathbf{U}$-small category $\mathbb{C}$ and a functor $\mathbb{C} \to \mathcal{D}$ that is fully faithful and essentially surjective on objects.

Proposition 0.1.9. *If* $\mathbb{C}$ *is a* $\mathbf{U}$*-small category and* $\mathcal{D}$ *is a locally* $\mathbf{U}$*-small category, then the functor category* $[\mathbb{C}, \mathcal{D}]$ *is locally* $\mathbf{U}$*-small.*

Proof. Strictly speaking, this depends on the set-theoretic implementation of ordered pairs, categories, functors, etc., but at the very least $[\mathbb{C}, \mathcal{D}]$ should be isomorphic to a locally **U**-small category.

In the context of $[\mathbb{C}, \mathcal{D}]$, we may regard functors $\mathbb{C} \to \mathcal{D}$ as being the pair consisting of the *graph* of the object map $\operatorname{ob} \mathbb{C} \to \operatorname{ob} \mathcal{D}$ and the *graph* of the morphism map $\operatorname{mor} \mathbb{C} \to \operatorname{mor} \mathcal{D}$, and these are **U**-sets by the **U**-replacement axiom. Similarly, if F and G are objects in $[\mathbb{C}, \mathcal{D}]$, then we may regard a natural transformation $\alpha : F \Rightarrow G$ as being the triple (F, G, A), where A is the set of all pairs $\left(c, \alpha_c\right)$. ■

One complication introduced by having multiple universes concerns the existence of (co)limits.

Theorem 0.1.10 (Freyd)**.** *Let $\mathcal{C}$ be a category and let κ be a cardinal such that $|\operatorname{mor} \mathcal{C}| \leq \kappa$. If $\mathcal{C}$ has products for families of size κ, then any two parallel morphisms in $\mathcal{C}$ must be equal.*

Proof. Suppose, for a contradiction, that $f, g : X \to Y$ are distinct morphisms in $\mathcal{C}$. Let Z be the product of κ-many copies of Y in $\mathcal{C}$. The universal property of products implies there are at least 2^κ-many distinct morphisms $X \to Z$; but $\mathcal{C}(X, Z) \subseteq \operatorname{mor} \mathcal{C}$, so this is an absurdity. ■

Definition 0.1.11. Let **U** be a pre-universe. A **U-complete** (resp. **U-cocomplete**) **category** is a category $\mathcal{C}$ with the following property:

- For all **U**-small categories $\mathbb{D}$ and all diagrams $A : \mathbb{D} \to \mathcal{C}$, a limit (resp. colimit) of A exists in $\mathcal{C}$.

We may instead say $\mathcal{C}$ has all **finite limits** (resp. **finite colimits**) in the special case **U** = **HF**.

Proposition 0.1.12. *Let $\mathcal{C}$ be a category and let* **U** *be a non-empty pre-universe. The following are equivalent:*

(i) *$\mathcal{C}$ is* **U***-complete.*

(ii) *$\mathcal{C}$ has all finite limits and products for all families of objects indexed by a* **U***-set.*

(iii) *For each* **U***-small category* $\mathbb{D}$*, there exists an adjunction*

$$\Delta \dashv \varprojlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$$

where ΔX *is the constant functor with value* X*.*

Dually, the following are equivalent:

(i′) $\mathcal{C}$ *is* **U***-cocomplete.*

(ii′) $\mathcal{C}$ *has all finite colimits and coproducts for all families of objects indexed by a* **U***-set.*

(iii′) *For each* **U***-small category* $\mathbb{D}$*, there exists an adjunction*

$$\varinjlim_{\mathbb{D}} \dashv \Delta : \mathcal{C} \to [\mathbb{D}, \mathcal{C}]$$

where ΔX *is the constant functor with value* X*.*

Proof. This is a standard result; but we remark that we do require a sufficiently powerful form of the axiom of choice to pass from (ii) to (iii). □

¶ **0.1.13.** In the **explicit universe convention**, the words ‘set’, ‘class’, etc. have their usual meanings, and in the **implicit universe convention**, these instead abbreviate ‘**U**-set’, ‘**U**-class’, etc. for a fixed (but arbitrary) universe **U**. However, the word ‘category’ always refers to a category that is contained in *some* universe, which may or may not be **U**. In all subsequent chapters, the implicit universe convention should be assumed *unless otherwise stated.*

We now recall some definitions and results about ordinal and cardinal numbers. Readers familiar with axiomatic set theory may wish to skip ahead.

Definition 0.1.14. A **von Neumann ordinal** is a set α with the following properties:

- If $x \in y$ and $y \in \alpha$, then $x \in \alpha$.
- The binary relation $\in$ is strict total ordering of α.
- If S is a subset of α such that
  - $\varnothing \in S$,

- If $\beta \in S$ and $\beta \cup \{\beta\} \in \alpha$, then $\beta \cup \{\beta\} \in S$.
- If $T \subseteq S$, then $\bigcup T \in S$.

then $S = \alpha$.

We identify 0 with the von Neumann ordinal $\varnothing$, and by induction, we identify the natural number $n + 1$ with the von Neumann ordinal $\{0, \dots, n\}$.

Proposition 0.1.15.

(i) *If α is a von Neumann ordinal, then every member of α is an initial segment of α and is in particular a von Neumann ordinal.*

(ii) *If α is a von Neumann ordinal, so is $\alpha \cup \{\alpha\}$. (This is usually denoted by* $\alpha + 1$ *and called the* **successor** *of α.)*

(iii) *The union of a set S of von Neumann ordinals is another von Neumann ordinal. (This is usually denoted by* $\sup S$ *and called the* **supremum** *of S.)*

(iv) *If $\mathbf{U}$ is a pre-universe and $\kappa(\mathbf{U})$ is the set of von Neumann ordinals in $\mathbf{U}$, then $\kappa(\mathbf{U})$ a von Neumann ordinal, but $\kappa(\mathbf{U}) \notin \mathbf{U}$.*

Proof. Claims (i) – (iii) are all easy, and claim (iv) is Burali-Forti's paradox. □

Theorem 0.1.16 (Classification of well-orderings)**.**

(i) *In Zermelo–Fraenkel set theory, every well-ordered set is isomorphic to a unique von Neumann ordinal.*

(ii) *In Mac Lane set theory, if $\mathbf{U}$ is a pre-universe and X is a well-ordered set in $\mathbf{U}$, then X is isomorphic to a unique von Neumann ordinal in $\mathbf{U}$.*

Proof. Claim (i) is a standard result in axiomatic set theory, and claim (ii) is an obvious corollary. □

Definition 0.1.17. Two sets are **equinumerous** if there exists a bijection between them. A **cardinality class** in a pre-universe $\mathbf{U}$ is an equivalence class under the relation of equinumerosity.

Definition 0.1.18. An **ℵ-number** is an infinite von Neumann ordinal κ such that, for any von Neumann ordinal λ such that κ and λ are equinumerous, we have $\kappa \subseteq \lambda$.

Example 0.1.19. The first infinite von Neumann ordinal, i.e. $\omega = \{0, 1, 2, \ldots\}$, is the $\aleph$-number $\aleph_0$.

Theorem 0.1.20 (Classification of cardinalities)**.**

(i) *In Zermelo–Fraenkel set theory, for every well-ordered infinite set X, there exists a unique $\aleph$-number κ such that X and κ are equinumerous.*

(ii) *In Zermelo–Fraenkel set theory with the axiom of choice, the same is true for any infinite set whatsoever.*

(iii) *In Mac Lane set theory, if $\mathbf{U}$ is a universe and X is an infinite set in $\mathbf{U}$, then there exists a unique $\aleph$-number κ in the cardinality class of X.*

(iv) *In Mac Lane set theory with the Grothendieck universe axiom, if $\mathbf{U}$ is a pre-universe and κ is an $\aleph$-number not in $\mathbf{U}$, then the cardinality of $\mathbf{U}$ is at most κ.*

Proof. Claim (i) is a standard fact, whence claims (ii) and (iii), by the well-ordering theorem. Claim (iv) can be proven using axiom 4 for pre-universes. □

¶ **0.1.21.** Henceforth, we identify the cardinality class of a finite set with the unique von Neumann ordinal contained in that class, and similarly we identify the cardinality class of an infinite set with the unique $\aleph$-number in that class. These are the **cardinal numbers**.

Definition 0.1.22. A **cofinal subset** of a partially-ordered set X is a subset $Y \subseteq X$ such that, for all x in X, there exists some y in Y such that $x \le y$. A **regular cardinal number** is an $\aleph$-number κ such that any cofinal subset of κ has cardinality equal to κ. A **singular cardinal number** is an $\aleph$-number that is not regular.

The following helps to motivate the definition of regular cardinal numbers.

Definition 0.1.23. Let $\mathbf{U}$ be a pre-universe. An **arity class** in $\mathbf{U}$ is a $\mathbf{U}$-class K of cardinal numbers satisfying the following conditions:

- $1 \in K$.
- If $\kappa \in K$ and $\lambda : \kappa \to K$ is a function, then the cardinal sum $\sum_{\alpha \in \kappa} \lambda(\alpha)$ is also in K.

- If $\kappa \in K$ and $\lambda : \kappa \to \mathbf{U}$ is a function such that each $\lambda(\alpha)$ is a cardinal number and $\sum_{\alpha \in \kappa} \lambda(\alpha) \in K$, then $\lambda(\alpha) \in K$ as well.

Theorem 0.1.24 (Classification of arity classes)**.** *In Mac Lane set theory, if K is an arity class in a pre-universe* $\mathbf{U}$*, then K must be either*

- $\{1\}$*, or*
- $\{0, 1\}$*, or*
- *of the form $\{\lambda \in \mathbf{U} \mid \lambda$ is a cardinal number and $\lambda < \kappa\}$ for some regular cardinal number κ (possibly not in* $\mathbf{U}$*).*

Proof. The notion of arity class and this result are due to Shulman [2012]. □

Definition 0.1.25. Let κ be a regular cardinal number. A **κ-small category** is a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ has cardinality *less than* κ. A **finite category** is an $\aleph_0$-small category, i.e. a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ is finite. A **finite diagram** (resp. **κ-small diagram**, $\mathbf{U}$-small diagram) in a category $\mathcal{C}$ is a functor $\mathbb{D} \to \mathcal{C}$ where $\mathbb{D}$ is a finite (resp. κ-small, $\mathbf{U}$-small) category.

Theorem 0.1.26. *Let* $\mathbf{U}$ *be a pre-universe, and let* $\mathbf{U}^+$ *be a universe with* $\mathbf{U} \in \mathbf{U}^+$*. Let* **Set** *be the category of* $\mathbf{U}$*-sets, and let* $\mathbf{Set}^+$ *be the category of* $\mathbf{U}^+$*-sets.*

(i) *If $X : \mathbb{D} \to$* **Set** *is a* $\mathbf{U}$*-small diagram, then there exist a limit and a colimit for X in* **Set***.*

(ii) *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *is fully faithful and preserves limits and colimits for all* $\mathbf{U}$*-small diagrams.*

Proof. One can construct products, equalisers, coproducts, coequalisers, and hom-sets in a completely explicit way, making the preservation properties obvious. □

Corollary 0.1.27. *The inclusion* $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ *reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Corollary 0.1.28. *For any* $\mathbf{U}$*-small category* $\mathbb{C}$*:*

(i) *The functor category* $[\mathbb{C}, \mathbf{Set}]$ *is* $\mathbf{U}$*-complete and* $\mathbf{U}$*-cocomplete, with limits and colimits of* $\mathbf{U}$*-small diagrams computed componentwise in* **Set***.*

(ii) *The inclusion* $[\mathbb{C}, \mathbf{Set}] \hookrightarrow [\mathbb{C}, \mathbf{Set}^+]$ *is fully faithful and both preserves and reflects limits and colimits for all* $\mathbf{U}$*-small diagrams.* ■

Definition 0.1.29. An **inaccessible cardinal number** is a regular cardinal number κ such that, for all sets X of cardinality less than κ, the power set $\mathscr{P}(X)$ is also of cardinality less than κ.

Example 0.1.30. $\aleph_0$ is an inaccessible cardinal number and is the only one that can be proven to exist in ZFC. It is more conventional to exclude $\aleph_0$ from the definition of inaccessible cardinal number by demanding that they be uncountable.

Proposition 0.1.31. *In Mac Lane set theory:*

(i) *If* $\mathbf{U}$ *is a non-empty pre-universe, then there exists an inaccessible cardinal number* κ *such that the members of* $\mathbf{U}$ *are all the sets of rank less than* κ*. Moreover, this* κ *is the rank and the cardinality of* $\mathbf{U}$.

(ii) *If* $\mathbf{U}$ *is a universe and* κ *is an inaccessible cardinal number such that* $\kappa \in \mathbf{U}$*, then there exists a* $\mathbf{U}$*-set* $\mathbf{V}_\kappa$ *whose members are all the sets of rank less than* κ*, and* $\mathbf{V}_\kappa$ *is a pre-universe.*

(iii) *If* $\mathbf{U}$ *and* $\mathbf{U}'$ *are pre-universes, then either* $\mathbf{U} \subseteq \mathbf{U}'$ *or* $\mathbf{U}' \subseteq \mathbf{U}$*; and if* $\mathbf{U} \subsetneqq \mathbf{U}'$*, then* $\mathbf{U} \in \mathbf{U}'$.

Proof. (i). Let κ be the set of all von Neumann ordinals in $\mathbf{U}$; this exists by Δ_0-separation applied to $\mathbf{U}$. Since $\mathbf{U}$ is closed under power sets and internally-indexed unions, κ must be an inaccessible cardinal.

We can construct the set all of $\mathbf{U}$-sets of rank less than κ using transfinite recursion on κ as follows: starting with $\mathbf{V}_0 = \varnothing$, for each von Neumann ordinal α less than κ, we set $\mathbf{V}_{\alpha+1} = \mathscr{P}(\mathbf{V}_\alpha)$, and for each ordinal λ that is not a successor, we set $\mathbf{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{V}_\alpha$. The well-foundedness of $\in$ (restricted to $\mathbf{U}$) implies that in fact this must be all of $\mathbf{U}$.

Clearly, every set of rank less than κ is in fact a $\mathbf{U}$-set, and $\mathbf{U}$ is itself a set of rank κ. The cardinality of $\mathbf{U}$ is also κ, since κ is a regular cardinal number and any cardinal number less than κ is a member of $\mathbf{U}$.

(ii). We may construct $\mathbf{V}_\kappa$ using the same method as in (i). By construction $\mathbf{V}_\kappa$ satisfies axiom 1; since κ is infinite, $\mathbf{V}_\kappa$ satisfies axioms 2 and 3; and since κ is inaccessible, $\mathbf{V}_\kappa$ satisfies axiom 4. Thus $\mathbf{V}_\kappa$ is a pre-universe.

(iii). Again, let κ be the rank of $\mathbf{U}$. If $\kappa \in \mathbf{U}'$ then we can show by transfinite induction that $\mathbf{V}_\kappa \in \mathbf{U}'$ and so $\mathbf{U} \subsetneqq \mathbf{U}'$; else we must have $\mathbf{U}' \subseteq \mathbf{V}_\kappa = \mathbf{U}$. ■

0.2 Accessibility and ind-completions

Prerequisites. § 0.1.

A classical technology for controlling size problems in category theory, due to Gabriel and Ulmer [1971], is the notion of accessibility. Though we make use of universes, accessibility remains important and is a crucial tool in verifying the stability of various universal constructions when one passes from one universe to a larger one.

Definition 0.2.1. Let κ be a regular cardinal. A **κ-filtered category** is a category $\mathcal{J}$ satisfying these conditions:

- $\mathcal{J}$ is **inhabited**, i.e. there exists an object in $\mathcal{J}$.
- If λ is a cardinal number strictly less than κ and S is a subset of $\operatorname{ob} \mathcal{J}$ of cardinality λ, then there exist an object j and arrows $f_i : i \to j$ for each object i in S.
- If $f, g : i \to j$ are a pair of parallel arrows in $\mathcal{J}$, then there exist an object k and an arrow $h : j \to k$ such that $h \circ f = h \circ g$.

A **κ-directed preorder** is a preordered set that is κ-filtered when considered as a category; note that the third condition is then vacuous. A **κ-filtered diagram** (resp. **κ-directed diagram**) in a category $\mathcal{C}$ is a functor $\mathbb{D} \to \mathcal{C}$ such that $\mathbb{D}$ is a κ-filtered category (resp. κ-directed preorder). It is conventional to omit mention of κ when $\kappa = \aleph_0$.

Example 0.2.2. The category with one object $*$ and only one non-trivial arrow f is filtered if and only if $f = f \circ f$.

Example 0.2.3. Let X be any set. The set of all finite subsets of X, partially ordered by inclusion, is a directed preorder. More generally, if κ is any regular cardinal, then the set of all subsets of X with cardinality strictly less than κ is a κ-directed preorder.

Theorem 0.2.4. *Let* $\mathbf{U}$ *be a pre-universe, let* $\mathbf{Set}$ *be the category of* $\mathbf{U}$*-sets, and let* κ *be any regular cardinal. Given a* $\mathbf{U}$*-small category* $\mathbb{D}$*, the following are equivalent:*

(i) $\mathbb{D}$ *is a* κ*-filtered category.*

(ii) *The functor* $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathbf{Set}] \to \mathbf{Set}$ *preserves limits of all diagrams that are both* κ*-small and* $\mathbf{U}$*-small.*

Proof. The claim (i) ⇒ (ii) is very well known, and the converse is an exercise in using the Yoneda lemma and manipulating limits and colimits of representable functors. □

Definition 0.2.5. Let κ be a regular cardinal in a universe $\mathbf{U}^+$ containing a pre-universe $\mathbf{U}$. A $(\kappa, \mathbf{U})$**-compact object** in a locally $\mathbf{U}^+$-small category $\mathcal{C}$ is an object A such that the representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}^+$ preserves colimits for all $\mathbf{U}$-small κ-filtered diagrams. A κ**-compact object** is one that is $(\kappa, \mathbf{U})$-compact for all pre-universes $\mathbf{U}$.

Though the above definition is stated using a pre-universe $\mathbf{U}$ contained in a universe $\mathbf{U}^+$, the following lemma shows there is no dependence on $\mathbf{U}^+$.

Lemma 0.2.6. *Let* A *be an object in a locally* $\mathbf{U}^+$*-small category* $\mathcal{C}$*. The following are equivalent:*

(i) A *is a* $(\kappa, \mathbf{U})$*-compact object in* $\mathcal{C}$*.*

(ii) *For all* $\mathbf{U}$*-small* κ*-filtered diagrams* $B : \mathbb{D} \to \mathcal{C}$*, if* $\lambda : B \Rightarrow \Delta C$ *is a colimiting cocone, then for any morphism* $f : A \to C$*, there exist an object* i *in* $\mathbb{D}$ *and a morphism* $f' : A \to Bi$ *in* $\mathcal{C}$ *such that* $f = \lambda_i \circ f'$*; and moreover if* $f = \lambda_j \circ f''$ *for some morphism* $f'' : A \to Bj$ *in* $\mathcal{C}$*, then there exists an object* k *and a pair of arrows* $g : i \to k$*,* $h : i \to k$ *in* $\mathbb{D}$ *such that* $Bg \circ f' = Bh \circ f''$*.*

Proof. Use the explicit description of $\varinjlim_{\mathbb{D}} \mathcal{C}(A, B)$ as a filtered colimit of sets; see Definition 1.1 in [LPAC], or Proposition 5.1.3 in [Borceux, 1994b]. □

Corollary 0.2.7. *Let* $B : \mathbb{D} \to \mathcal{C}$ *be a* $\mathbf{U}$*-small* κ*-filtered diagram, and let* $\lambda : B \Rightarrow \Delta C$ *be a colimiting cocone in* $\mathcal{C}$*. If* C *is a* $(\kappa, \mathbf{U})$*-compact object in* $\mathcal{C}$*, then for some object* i *in* $\mathbb{D}$*,* $\lambda_i : Bi \to C$ *is a* split *epimorphism.* ■

Lemma 0.2.8. *Let A be an object in a category $\mathcal{C}$.*

(i) *If $\mathbf{U}$ is a pre-universe contained in a universe $\mathbf{U}^+$ and κ is a regular cardinal such that A is $(\kappa, \mathbf{U}^+)$-compact, then A is $(\kappa, \mathbf{U})$-compact as well.*

(ii) *If κ is a regular cardinal such that A is $(\kappa, \mathbf{U})$-compact and λ is any regular cardinal such that $\kappa \le \lambda$, then A is also $(\lambda, \mathbf{U})$-compact.*

Proof. Obvious. ■

Lemma 0.2.9. *Let κ and λ be regular cardinals in a universe $\mathbf{U}^+$ containing a pre-universe $\mathbf{U}$. If $\kappa \le \lambda$ and $B : \mathbb{D} \to \mathcal{C}$ is a λ-small diagram of $(\kappa, \mathbf{U})$-compact objects in a locally $\mathbf{U}^+$-small category, then the colimit $\varinjlim_{\mathbb{D}} B$, if it exists, is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{C}$.*

Proof. Use theorem 0.2.4 and the fact that $\mathcal{C}(-, C) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}^+$ maps colimits in $\mathcal{C}$ to limits in $\mathbf{Set}^+$. (Note that any $\mathbf{U}$-small λ-filtered diagram is in particular a $\mathbf{U}$-small κ-filtered diagram.) ■

Corollary 0.2.10. *A retract of a $(\kappa, \mathbf{U})$-compact object is also a $(\kappa, \mathbf{U})$-compact object.*

Proof. Suppose $r : A \to B$ and $s : B \to A$ are morphisms in $\mathcal{C}$ such that $r \circ s = \mathrm{id}_B$. Then $e = s \circ r$ is an idempotent morphism and the diagram below

$$A \underset{e}{\overset{\mathrm{id}_A}{\rightrightarrows}} A \xrightarrow{r} B$$

is a (split) coequaliser diagram in $\mathcal{C}$, so B is $(\kappa, \mathbf{U})$-compact if A is. ■

Proposition 0.2.11. *Let $\mathbf{U}$ be a pre-universe and let $\mathbf{Set}$ be the category of $\mathbf{U}$-sets. For any $\mathbf{U}$-set A, the following are equivalent:*

(i) *A has cardinality less than κ.*

(ii) *The representable functor $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ preserves colimits of $\mathbf{U}$-small κ-filtered diagrams.*

(iii) *The representable functor $\mathbf{Set}(A, -) : \mathbf{Set} \to \mathbf{Set}$ preserves colimits of $\mathbf{U}$-small κ-directed diagrams.*

Proof. The claim (i) $\Rightarrow$ (ii) follows from the theorem, and (ii) $\Rightarrow$ (iii) is obvious. To see (iii) $\Rightarrow$ (i), we may use corollary 0.2.7 and the fact that every set is the directed union of its subsets of cardinality at most κ. ■

Corollary 0.2.12. *A set is κ-compact if and only if its cardinality is $\leq \kappa$.* ■

Definition 0.2.13. Let κ be a regular cardinal in a universe $\mathbf{U}$. A **κ-accessible $\mathbf{U}$-category** is a locally $\mathbf{U}$-small category $\mathcal{C}$ satisfying the following conditions:

- $\mathcal{C}$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams.
- There exists a $\mathbf{U}$-set $\mathcal{G}$ whose members are $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ such that, for every object B in $\mathcal{C}$, there exists a $\mathbf{U}$-small κ-filtered diagram of objects in $\mathcal{G}$ with B as its colimit in $\mathcal{C}$.

We write $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$ for the full subcategory of $\mathcal{C}$ spanned by the $(\kappa, \mathbf{U})$-compact objects.

Example 0.2.14. The category of $\mathbf{U}$-sets is a κ-accessible $\mathbf{U}$-category for any regular cardinal κ in $\mathbf{U}$.

Remark 0.2.15. Lemma 0.2.9 implies that, for each object A in an accessible $\mathbf{U}$-category, there exists a regular cardinal λ in $\mathbf{U}$ such that A is $(\lambda, \mathbf{U})$-compact.

Theorem 0.2.16. *Let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, and let κ be a regular cardinal in $\mathbf{U}$. There exist a locally $\mathbf{U}$-small category $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ and a functor $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ with the following properties:*

(i) *The objects of $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ are $\mathbf{U}$-small κ-filtered diagrams $B : \mathbb{D} \to \mathcal{C}$, and γ sends an object C in $\mathcal{C}$ to the corresponding trivial diagram $\mathbb{1} \to \mathcal{C}$ with value C.*

(ii) *The functor $\gamma : \mathcal{C} \to \mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ is fully faithful, injective on objects, and preserves all limits that exist in $\mathcal{C}$.*

(iii) *$\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$ has colimits for all $\mathbf{U}$-small κ-filtered diagrams.*

(iv) *For every object C in $\mathcal{C}$, the object γC is $(\kappa, \mathbf{U})$-compact in $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$, and for each $\mathbf{U}$-small κ-filtered diagram $B : \mathbb{D} \to \mathcal{C}$, there is a canonical colimiting cocone $\gamma B \Rightarrow \Delta B$ in $\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathcal{C})$.*

(v) *If $\mathcal{D}$ is a category with colimits for all* **U**-*small* κ-*filtered diagrams, then for each functor* $F : \mathcal{C} \to \mathcal{D}$*, there exists a functor* $\bar{F} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *that preserves colimits for all* **U**-*small* κ-*filtered diagrams in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *such that* $\gamma\bar{F} = F$*, and given any functor* $\bar{G} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to \mathcal{D}$ *whatsoever, the induced map* $\mathrm{Nat}(\bar{F}, \bar{G}) \to \mathrm{Nat}(F, \gamma\bar{G})$ *is a bijection.*

The category $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ *is called the* **free** $(\kappa, \mathbf{U})$**-ind-completion** *of* $\mathcal{C}$*, or the* **category of** $(\kappa, \mathbf{U})$**-ind-objects** *in* $\mathcal{C}$*.*

Proof. If $B : \mathbb{D} \to \mathcal{C}$ and $B' : \mathbb{D}' \to \mathcal{C}$ are two **U**-small κ-filtered diagrams, then properties (ii) and (iii) together imply that

$$\mathrm{Hom}(B', B) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and so, taking the RHS as the *definition* of the LHS, we need only find a suitable notion of composition to make $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ into a locally **U**-small category. However, we observe that, if $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, then

$$\mathrm{Hom}\left(\varinjlim_{\mathbb{D}'} \mathrm{N}B', \varinjlim_{\mathbb{D}} \mathrm{N}B\right) \cong \varprojlim_{\mathbb{D}'} \varinjlim_{\mathbb{D}} \mathcal{C}(B', B)$$

and, assuming property (v), the Yoneda embedding $\mathrm{N} : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ must extend along γ to a functor $\bar{\mathrm{N}} : \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C}) \to [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that preserves colimits for **U**-small κ-filtered diagram, so, in consideration of properties (i) and (iv), we may as well *define* the composition in $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ so that $\bar{\mathrm{N}}$ becomes fully faithful. This completes the definition of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ as a category.

It remains to be shown that $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ actually has properties (ii), (iii), (iv), and (v); see Corollary 6.4.14 in [Borceux, 1994a] and Theorem 2.26 in [LPAC]. □

Proposition 0.2.17. *Let* $\mathbb{B}$ *be a* **U**-*small category and let* κ *be a regular cardinal in* **U**.

(i) $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ *is a* κ-*accessible* **U**-*category.*

(ii) *Every* $(\kappa, \mathbf{U})$-*compact object in* $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ *is a retract of an object of the form* γB*, where* $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ *is the canonical embedding.*

(iii) $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}))$ *is an essentially* **U**-*small category.*

Proof. (i). This claim more-or-less follows from the properties of $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ explained in the previous theorem.

(ii). Use corollary 0.2.10.

(iii). Since $\mathbb{B}$ is $\mathbf{U}$-small and $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is locally $\mathbf{U}$-small, claim (ii) implies that $\mathbf{K}^{\mathbf{U}}_{\kappa}\big(\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})\big)$ must be essentially $\mathbf{U}$-small. ■

Definition 0.2.18. Let κ be a regular cardinal in a universe $\mathbf{U}$. A $(\kappa, \mathbf{U})$**-accessible functor** is a functor $F : \mathcal{C} \to \mathcal{D}$ such that

- $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category, and
- F preserves all colimits for $\mathbf{U}$-small κ-filtered diagrams.

We write $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D})$ for the full subcategory of the functor category $[\mathcal{C}, \mathcal{D}]$ spanned by the $(\kappa, \mathbf{U})$-accessible functors. An **accessible functor** is a functor that is $(\kappa, \mathbf{U})$-accessible functor for some regular cardinal κ in some universe $\mathbf{U}$.

Theorem 0.2.19 (Classification of accessible categories)**.** *Let κ be a regular cardinal in a universe $\mathbf{U}$, and let $\mathcal{C}$ be a locally $\mathbf{U}$-small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category.*

(ii) *The inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ extends along the embedding $\gamma : \mathcal{C} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathcal{C})$ to a $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}\big(\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})\big) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

(iii) *There exists a $\mathbf{U}$-small category $\mathbb{B}$ and a functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathcal{C}$ that is fully faithful and essentially surjective on objects.*

Proof. See Theorem 2.26 in [LPAC], or Theorem 5.35 in [Borceux, 1994b]. □

Corollary 0.2.20. *If $\mathcal{C}$ is a κ-accessible $\mathbf{U}$-category and $\mathcal{D}$ is any category, then:*

(i) *The restriction $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D}) \to \big[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}), \mathcal{D}\big]$ is fully faithful and surjective on objects.*

(ii) *In particular, if $\mathcal{D}$ is also locally $\mathbf{U}$-small, then $\mathbf{Acc}^{\mathbf{U}}_{\kappa}(\mathcal{C}, \mathcal{D})$ is equivalent to a locally $\mathbf{U}$-small category.*

(iii) *If $\mathcal{D}$ has colimits for all* **U**-*small* κ-*filtered diagrams, then the inclusion* $\mathbf{Acc}_\kappa^{\mathbf{U}}(\mathcal{C}, \mathcal{D}) \hookrightarrow [\mathcal{C}, \mathcal{D}]$ *has a left adjoint.* ■

Definition 0.2.21. Let κ be a regular cardinal in a universe **U**. A **locally** κ-**presentable U-category** is a κ-accessible **U**-category that is also **U**-cocomplete. A **locally presentable U-category** is one that is a locally κ-presentable **U**-category for some regular cardinal κ in **U**, and we often say **locally finitely presentable** instead of locally $\aleph_0$-presentable.

Example 0.2.22. The category of **U**-sets is a locally κ-presentable **U**-category for any regular cardinal κ in **U**.

Lemma 0.2.23. *Let $\mathcal{C}$ be a locally κ-presentable* **U**-*category.*

(i) *For any regular cardinal λ in* **U**, *if $\kappa \leq \lambda$, then $\mathcal{C}$ is a locally λ-presentable* **U**-*category.*

(ii) *If* $\mathbf{U}^+$ *is any universe with* $\mathbf{U} \in \mathbf{U}^+$, *and $\mathcal{C}$ is a locally κ-presentable* $\mathbf{U}^+$-*category, then $\mathcal{C}$ must be a preorder.*

Proof. (i). See the remark after Theorem 1.20 in [LPAC], or Propositions 5.3.2 and 5.2.3 in [Borceux, 1994b].

(ii). This is a corollary of theorem 0.1.10. ■

Corollary 0.2.24. *A category $\mathcal{C}$ is a locally presentable* **U**-*category for* at most *one universe* **U**, *provided $\mathcal{C}$ is not a preorder.*

Proof. Use proposition 0.1.31 together with the above lemma. ■

Theorem 0.2.25 (Classification of locally presentable categories)**.** *Let κ be a regular cardinal in a universe* **U**, *let* **Set** *be the category of* **U**-*sets, and let $\mathcal{C}$ be a locally* **U**-*small category. The following are equivalent:*

(i) *$\mathcal{C}$ is a locally κ-presentable* **U**-*category.*

(ii) *The restricted Yoneda embedding* $\mathcal{C} \to \left[\mathbf{K}_\kappa^{\mathbf{U}}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ *is fully faithful,* $(\kappa, \mathbf{U})$-*accessible, and has a left adjoint.*

(iii) *There exists a* **U**-*small category* $\mathbb{A}$ *and a fully faithful* $(\kappa, \mathbf{U})$-*accessible functor* $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ *such that* $\mathbb{A}$ *has limits for all* κ-*small diagrams, R has a left adjoint, and R is essentially surjective onto the full subcategory of functors* $\mathbb{A} \to \mathbf{Set}$ *that preserve finite limits.*

(iv) *There exists a* $\mathbf{U}$*-small category* $\mathbb{A}$ *and a fully faithful* $(\kappa, \mathbf{U})$*-accessible functor* $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ *such that* R *has a left adjoint.*

(v) $\mathcal{C}$ *is a* κ*-accessible* $\mathbf{U}$*-category and is* $\mathbf{U}$*-complete.*

Proof. See Proposition 1.27, Corollary 1.28, Theorem 1.46, and Corollary 2.47 in [LPAC], or Theorems 5.2.7 and 5.5.8 in [Borceux, 1994b]. □

Remark 0.2.26. If $\mathcal{C}$ is equivalent to $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ for some $\mathbf{U}$-small category $\mathbb{B}$ that has limits for all κ-small diagrams, then $\mathbb{B}$ must be equivalent to $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})$ by proposition 0.2.17. In other words, every locally κ-presentable $\mathbf{U}$-category is, up to equivalence, the $(\kappa, \mathbf{U})$-ind-completion of an essentially unique $\mathbf{U}$-small κ-complete category.

Example 0.2.27. Obviously, for any $\mathbf{U}$-small category $\mathbb{A}$, the functor category $[\mathbb{A}, \mathbf{Set}]$ is locally finitely presentable. More generally, one may show that for any κ-ary algebraic theory $\mathbb{T}$, possibly many-sorted), the category of $\mathbb{T}$-algebras in $\mathbf{U}$ is a locally κ-presentable $\mathbf{U}$-category. The above theorem can also be used to show that $\mathbf{Cat}$, the category of $\mathbf{U}$-small categories, is a locally finitely presentable $\mathbf{U}$-small category.

Corollary 0.2.28. *Let* $\mathcal{C}$ *be a locally* κ*-presentable* $\mathbf{U}$*-category. For any* $\mathbf{U}$*-small* κ*-filtered diagram* $\mathbb{D}$*,* $\varinjlim_{\mathbb{D}} : [\mathbb{D}, \mathcal{C}] \to \mathcal{C}$ *preserves* κ*-small limits.*

Proof. The claim is certainly true when $\mathcal{C} = [\mathbb{A}, \mathbf{Set}]$, by theorem 0.2.4. In general, choose a $(\kappa, \mathbf{U})$-accessible fully faithful functor $R : \mathcal{C} \to [\mathbb{A}, \mathbf{Set}]$ with a left adjoint, and simply note that R creates limits for all $\mathbf{U}$-small diagrams as well as colimits for all $\mathbf{U}$-small κ-filtered diagrams. ■

Theorem 0.2.29 (Accessible adjoint functor theorem)**.** *Let* κ *and* λ *be regular cardinals in a universe* $\mathbf{U}$*, with* $\kappa \le \lambda$*, let* $\mathcal{C}$ *be a locally* κ*-presentable* $\mathbf{U}$*-category, and let* $\mathcal{D}$ *be a locally* λ*-presentable* $\mathbf{U}$*-category.*

Given a functor $F : \mathcal{C} \to \mathcal{D}$*, the following are equivalent:*

(i) F *has a right adjoint* $G : \mathcal{D} \to \mathcal{C}$*, and* G *is a* $(\lambda, \mathbf{U})$*-accessible functor.*

(ii) F *preserves colimits for all* $\mathbf{U}$*-small diagrams and sends* $(\kappa, \mathbf{U})$*-compact objects in* $\mathcal{C}$ *to* $(\lambda, \mathbf{U})$*-compact objects in* $\mathcal{D}$*.*

(iii) F *has a right adjoint and sends* $(\kappa, \mathbf{U})$*-compact objects in* $\mathcal{C}$ *to* $(\lambda, \mathbf{U})$*-compact objects in* $\mathcal{D}$*.*

On the other hand, given a functor $G : \mathcal{D} \to \mathcal{C}$*, the following are equivalent:*

(iv) *G has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\lambda, \mathbf{U})$-compact objects in $\mathcal{D}$.*

(v) *G is a $(\lambda, \mathbf{U})$-accessible functor and preserves limits for all $\mathbf{U}$-small diagrams.*

(vi) *G is a $(\lambda, \mathbf{U})$-accessible functor and there exist a functor $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ and hom-set bijections*

$$\mathcal{C}(C, GD) \cong \mathcal{D}\big(F_0 C, D\big)$$

natural in D for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, where D varies in $\mathcal{D}$.

Proof. We will need to refer back to the details of the proof of this theorem later, so here is a sketch of the constructions involved.

(i) ⇒ (ii). If F is a left adjoint, then F certainly preserves colimits for all $\mathbf{U}$-small diagrams. Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathbb{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{D}\left(FC, \varinjlim_{\mathbb{J}} B\right) \cong \mathcal{C}\left(C, G \varinjlim_{\mathbb{J}} B\right) &\cong \mathcal{C}\left(C, \varinjlim_{\mathbb{J}} GB\right) \\ &\cong \varinjlim_{\mathbb{J}} \mathcal{C}(C, GB) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(FC, B)\end{aligned}$$

and thus FC is indeed a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$.

(ii) ⇒ (iii). It is enough to show that, for each object D in $\mathcal{D}$, the comma category $(F \downarrow D)$ has a terminal object.[1] Let $\mathcal{G}$ be the full subcategory of $(F \downarrow D)$ spanned by those (A, f) where A is a $(\kappa, \mathbf{U})$-compact object in $\mathcal{C}$; note that proposition 0.2.17 implies $\mathcal{G}$ is an essentially $\mathbf{U}$-small category. Since $\mathcal{C}$ has colimits for all $\mathbf{U}$-small diagrams and F preserves them, $(F \downarrow D)$ is also $\mathbf{U}$-cocomplete.[2] Thus the inclusion $\mathcal{G} \hookrightarrow (F \downarrow D)$ has a colimit, say (C, h). It is not hard to check that (C, h) is a weakly terminal object in $(F \downarrow D)$, so the formal dual of Freyd's initial object lemma[3] gives us a terminal object in $(F \downarrow D)$; explicitly, $\big(GD, \varepsilon_D\big)$ may be constructed as the joint coequaliser of all the endomorphisms of (C, h).

[1] See Theorem 2 in [CWM, Ch. IV, § 1].
[2] See the Lemma in [CWM, Ch. V, § 6].
[3] See Theorem 1 in [CWM, Ch. V, § 6].

(iii) ⇒ (i). Given a $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$ and a $\mathbf{U}$-small λ-filtered diagram $B : \mathbb{J} \to \mathcal{D}$, observe that

$$\begin{aligned}\mathcal{C}\left(C, G \varinjlim_{\mathbb{J}} B\right) &\cong \mathcal{D}\left(FC, \varinjlim_{\mathbb{J}} B\right) \cong \varinjlim_{\mathbb{J}} \mathcal{C}(FC, B) \\ &\cong \varinjlim_{\mathbb{J}} \mathcal{C}(C, GB) \cong \mathcal{C}\left(C, \varinjlim_{\mathbb{J}} GB\right)\end{aligned}$$

because FC is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$; but theorem 0.2.25 says the restricted Yoneda embedding $\mathcal{C} \to \left[\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C})^{\mathrm{op}}, \mathbf{Set}\right]$ is fully faithful, so this is enough to conclude that G preserves colimits of $\mathbf{U}$-small λ-filtered diagrams.

(iv) ⇒ (v). If G is a right adjoint, then G certainly preserves colimits for all $\mathbf{U}$-small diagrams; the rest of the claim is subsumed by (iii) ⇒ (i).

(v) ⇒ (vi). It is enough to show that, for each $(\kappa, \mathbf{U})$-compact object C in $\mathcal{C}$, the comma category $(C \downarrow G)$ has an initial object. Let $\mathcal{F}$ be the full subcategory of $(C \downarrow G)$ spanned by those (B, g) where B is a $(\lambda, \mathbf{U})$-compact object in $\mathcal{D}$. G preserves colimits for all $\mathbf{U}$-small λ-filtered diagrams, so $\mathcal{F}$ must be a weakly initial family in $(C \downarrow G)$; note that proposition 0.2.17 implies $\mathcal{F}$ is an essentially $\mathbf{U}$-small category. Since $\mathcal{D}$ has limits for all $\mathbf{U}$-small diagrams and G preserves them, $(C \downarrow G)$ is also $\mathbf{U}$-complete. Thus the inclusion $\mathcal{F} \hookrightarrow (C \downarrow G)$ has a limit, say $\left(F_0 C, \eta_C\right)$, and it can be shown that this is an initial object in $(C \downarrow G)$.[4] It is clear how to make F_0 into a functor $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$.

(vi) ⇒ (iv). We use theorems 0.2.16 and 0.2.25 to extend $F_0 : \mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \to \mathcal{D}$ along the inclusion $\mathbf{K}^{\mathbf{U}}_{\kappa}(\mathcal{C}) \hookrightarrow \mathcal{C}$ to get $(\kappa, \mathbf{U})$-accessible functor $F : \mathcal{C} \to \mathcal{D}$. We then observe that, for any $\mathbf{U}$-small κ-filtered diagram $A : \mathbb{I} \to \mathcal{C}$ of $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$,

$$\begin{aligned}\mathcal{C}\left(\varinjlim_{\mathbb{I}} A, GD\right) &\cong \varprojlim_{\mathbb{I}} \mathcal{C}(A, GD) \cong \varprojlim_{\mathbb{I}} \mathcal{C}\left(F_0 A, D\right) \\ &\cong \mathcal{C}\left(\varinjlim_{\mathbb{I}} FA, D\right) \cong \mathcal{C}\left(F \varinjlim_{\mathbb{I}} A, D\right)\end{aligned}$$

is a series of bijections natural in D, where D varies in $\mathcal{D}$; but C is a locally κ-presentable $\mathbf{U}$-category, so this is enough to show that F is a left adjoint of G. The remainder of the claim is a corollary of (i) ⇒ (ii). ■

[4] See Theorem 1 in [CWM, Ch. X, § 2].

Corollary 0.2.30. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally presentable* $\mathbf{U}$*-categories. If a functor* $G : \mathcal{D} \to \mathcal{C}$ *has a left adjoint, then there exists a regular cardinal* μ *in* $\mathbf{U}$ *such that* G *is a* $(\mu, \mathbf{U})$*-accessible functor.*

Proof. Suppose $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category, $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category, and $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint for G. Since $\mathbf{K}_{\kappa}^{\mathbf{U}}(\mathcal{C})$ is an essentially $\mathbf{U}$-small category, recalling lemma 0.2.8, there certainly exists a regular cardinal μ in $\mathbf{U}$ such that $\mu \geq \lambda$ and F sends $(\kappa, \mathbf{U})$-compact objects in $\mathcal{C}$ to $(\mu, \mathbf{U})$-compact objects in $\mathcal{D}$. The above theorem, plus lemma 0.2.23, implies G is an $(\mu, \mathbf{U})$-accessible functor. ■

— I —

Simplicial sets

Simplicial sets, like simplicial complexes, are combinatorial models for spaces built up by gluing standard n-simplices together; unlike simplicial complexes, an n-simplex in a simplicial set need not be uniquely determined by its vertices. It is for this reason that simplicial sets were once known by the unwieldy name 'complete semi-simplicial (c.s.s.) complex'.

In the 1960s, it was discovered that one can mimic the definitions and constructions of classical homotopy theory by combinatorial means using simplicial sets, and that the resulting theory is moreover equivalent to the classical theory in a natural, functorial way. More recently, it has been shown that the homotopy theory of simplicial sets is *universal* in a precise sense,[1] so it seems fitting that we begin here.

1.1 Basics

Definition 1.1.1. The **simplex category** is the category $\mathbf{\Delta}$ whose objects are the positive finite ordinals and whose morphisms are the monotone maps. We use the geometer's convention: $[n]$ denotes the ordinal $\{0, 1, \dots, n\}$.

Definition 1.1.2. A **simplicial object** in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, and a **morphism of simplicial objects** in $\mathcal{C}$ is a natural transformation of such functors. The **category of simplicial objects** in $\mathcal{C}$ is the functor category $\left[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{C}\right]$ and is denoted by $\mathbf{s}\mathcal{C}$.

[1] See [Dugger, 2001].

Definition 1.1.3. The **coface maps** in $\mathbf{\Delta}$ are the morphisms $\delta_n^i : [n-1] \to [n]$, where δ_n^i is the unique injective monotone map that misses i; and the **codegeneracy maps** in $\mathbf{\Delta}$ are the morphisms $\sigma_n^i : [n+1] \to [n]$, where σ_n^i is the unique surjective monotone map with $\sigma_n^i(i) = \sigma_n^i(i+1) = i$.

Theorem 1.1.4 (Cosimplicial identities)**.** *The following equations hold in* $\mathbf{\Delta}$*:*

$$\begin{aligned}
\delta_{n+1}^{j+1} \circ \delta_n^i &= \delta_{n+1}^i \circ \delta_n^j && \text{if } 0 \le i \le j \le n \\
\sigma_n^j \circ \sigma_{n+1}^i &= \sigma_n^i \circ \sigma_{n+1}^{j+1} && \text{if } 0 \le i \le j \le n \\
\sigma_{n+1}^{j+1} \circ \delta_{n+1}^i &= \delta_n^i \circ \sigma_n^j && \text{if } 0 \le i \le j \le n \\
\delta_n^{j+1} \circ \sigma_n^i &= \sigma_{n+1}^i \circ \delta_{n+1}^{j+2} && \text{if } 0 \le i < j < n \\
\sigma_n^i \circ \delta_n^i &= \mathrm{id} && \text{if } 0 \le i \le n \\
\sigma_n^{i+1} \circ \delta_n^i &= \mathrm{id} && \text{if } 0 \le i < n
\end{aligned}$$

Equivalently, the following diagrams commute:

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta^i} & [n] \\ {\scriptstyle \delta^j}\big\downarrow & & \big\downarrow{\scriptstyle \delta^{j+1}} \\ [n] & \xrightarrow[\delta^i]{} & [n+1] \end{array} \quad \text{for } 0 \le i \le j \le n$$

$$\begin{array}{ccc} [n+1] & \xrightarrow{\sigma^i} & [n] \\ {\scriptstyle \sigma^{j+1}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma^{j}} \\ [n] & \xrightarrow[\sigma^i]{} & [n-1] \end{array} \quad \text{for } 0 \le i \le j \le n$$

$$\begin{array}{ccc} [n] & \xrightarrow{\delta^i} & [n+1] \\ {\scriptstyle \sigma^{j}}\big\downarrow & & \big\downarrow{\scriptstyle \sigma^{j+1}} \\ [n-1] & \xrightarrow[\delta^i]{} & [n] \end{array} \quad \text{for } 0 \le i \le j \le n$$

$$\begin{array}{ccc} [n] & \xrightarrow{\sigma^i} & [n-1] \\ {\scriptstyle \delta^{j+2}}\big\downarrow & & \big\downarrow{\scriptstyle \delta^{j+1}} \\ [n+1] & \xrightarrow[\sigma^i]{} & [n] \end{array} \quad \text{for } 0 \le i < j < n$$

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta^i} & [n] \\ {\scriptstyle \delta^i}\downarrow & \searrow^{\mathrm{id}} & \downarrow{\scriptstyle \sigma^{i+1}} \\ [n] & \xrightarrow[\sigma^i]{} & [n-1] \end{array} \quad \textit{for } 0 \le i \le n$$

Moreover, every morphism $[n] \to [m]$ *in* $\mathbf{\Delta}$ *is* uniquely *a composite of the form*

$$\delta_m^{j_1} \circ \cdots \circ \delta_k^{j_{m-k}} \circ \sigma_k^{i_{n-k}} \circ \cdots \circ \sigma_n^{i_1}$$

where $k \le \min\{n, m\}$*, and*

$$0 \le i_{n-k} \le \cdots \le i_1 \le n$$
$$0 \le j_{m-k} \le \cdots \le j_1 \le m$$

The category $\mathbf{\Delta}$ *is uniquely characterised by these properties.*

Proof. See [May, 1967, § 2], [GZ, Ch. II, § 2], or [Weibel, 1994, § 8.1]. □

Definition 1.1.5. Let A be a simplicial object in a category $\mathcal{C}$. A **face operator** for A is a morphism of the form $A(\delta_n^i) : A([n]) \to A([n-1])$, and a **degeneracy operator** for A is a morphism of the form $A(\sigma_n^i) : A([n]) \to A([n+1])$. For brevity, we will usually write A_n instead of $A([n])$, d_i^n instead of $A(\delta_n^i)$, and s_i^n instead of $A(\sigma_n^i)$.

Corollary 1.1.6 (Simplicial identities)**.** *The face and degeneracy operators of a simplicial object satisfy the formal duals of the equations in theorem 1.1.4.* ■

Corollary 1.1.7. *A simplicial object* A *is uniquely determined by the sequence of objects* $A_0, A_1, A_2, \ldots$ *together with the face and degeneracy operators. Conversely, any sequence of objects equipped with face and degeneracy operators satisfying the simplicial identities defined a simplicial object.* ■

Definition 1.1.8. A **simplicial set** is a simplicial object in **Set**, and the **category of simplicial sets** is denoted by **sSet**.

Lemma 1.1.9.

(i) *Limits (resp. colimits) in* **sSet** *are constructed degreewise: a cone (resp. cocone) in* **sSet** *over a diagram is limiting (resp. colimiting) if and only if it is so in every degree.*

(ii) *A morphism of* **sSet** *is monic (resp. epic) if and only if it is degreewise injective (resp. surjective).*

Proof. These are standard facts about functor categories. □

Definition 1.1.10. The **standard n-simplex** in **sSet**, denoted by Δ^n, is the representable presheaf $\mathbf{\Delta}(-, [n])$.

Theorem 1.1.11. *Let* $\Delta^\bullet : \mathbf{\Delta} \to \mathbf{sSet}$ *be the functor* $[n] \mapsto \Delta^n$.

(i) *For any simplicial set X, the map* $\mathbf{sSet}(\Delta^n, X) \to X_n$ *defined by* $f \mapsto f_n(\mathrm{id}_{[n]})$ *is a bijection and is moreover natural in* $[n]$ *and* X.

(ii) **sSet** *has limits and colimits for all small diagrams, every epimorphism is effective, and for all morphisms* $f : X \to Y$ *in* **sSet**, *the pullback functor* $f^* : \mathbf{sSet}_{/Y} \to \mathbf{sSet}_{/X}$ *preserves colimits.*

(iii) $\Delta^\bullet$ *is dense, i.e. for any simplicial set X, the tautological cocone from the canonical diagram* $(\Delta^\bullet \downarrow X) \to \mathbf{sSet}$ *to X is colimiting.*

(iv) *Let $\mathcal{E}$ be a locally small category with colimits for all small diagrams. If* $F : \mathbf{sSet} \to \mathcal{E}$ *is a functor that preserves small colimits, then it is left adjoint to the functor* $\mathcal{E} \to \mathbf{sSet}$ *defined by* $E \mapsto \mathcal{E}(F\Delta^\bullet, E)$.

(v) *With $\mathcal{E}$ as above, the functor* $F \mapsto F\Delta^\bullet$ *from the category of colimit-preserving functors* $\mathbf{sSet} \to \mathcal{E}$ *to the category of all functors* $\mathbf{\Delta} \to \mathcal{E}$ *is fully faithful and essentially surjective on objects.*

Proof. Claim (i) is just the Yoneda lemma, claim (ii) follows from the lemma above, and claims (iii)–(v) are a special case of what is discussed in [ML–M, Ch. I, § 5]. □

¶ **1.1.12.** An element of X_n is often called an **n-simplex** in X; in particular, an element of X_0 is a **vertex** of X and an element of X_1 is an **edge** of X. This is justified by statement (i) in the above theorem.

Corollary 1.1.13. *There exists an adjunction*

$$\tau_1 \dashv \mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$$

such that $\tau_1 \Delta^n$ *is just* $[n]$ *regarded as a preorder category, and this adjunction is unique up to unique isomorphism. The functor* N *is moreover fully faithful and exhibits* **Cat** *as a reflective subcategory of* **sSet**.

Proof. In this case it is easier to use the general adjoint functor theorem to construct a left adjoint for the functor N defined by $\mathrm{N}(\mathbb{C})_n = \mathrm{ob}\,[[n], \mathbb{C}]$. ■

Definition 1.1.14. The **fundamental category** of a simplicial set X is the small category $\tau_1 X$, and the **nerve** of a small category $\mathbb{C}$ is the simplicial set $\mathrm{N}(\mathbb{C})$.

Remark 1.1.15. Given a simplicial set X, the fundamental category $\tau_1 X$ admits the following presentation by generators and relations: the objects are the vertices of X, and the morphisms are generated by the edges of X, modulo the relation $d_0(x) \circ d_2(x) = d_1(x)$ for all 2-simplices x in X.

Corollary 1.1.16. *There exists an adjunction*

$$\pi_0 \dashv \mathrm{disc} : \mathbf{Set} \to \mathbf{sSet}$$

such that $\pi_0 \Delta^n = 1$ *for all n, and this adjunction is unique up to unique isomorphism. Explicitly, we may take*

$$(\mathrm{disc}\, Y)_n = Y$$

for all n, with id_Y *for all the face and degeneracy operators. The functor* disc *is moreover fully faithful and exhibits* **Set** *as a reflective subcategory of* **sSet**. ■

Definition 1.1.17. The **set of connected components** of a simplicial set X is the set $\pi_0 X$, and a **discrete simplicial set** is one that is isomorphic to $\mathrm{disc}\, Y$ for some set Y.

¶ **1.1.18.** We will usually not distinguish between Y and $\mathrm{disc}\, Y$ notationally.

Definition 1.1.19. The **standard *n*-simplex** in **Top**, denoted by $|\Delta^n|$, is the topological space

$$|\Delta^n| = \left\{ \left(x_0, \dots, x_n\right) \in [0,1]^{n+1} \,\middle|\, x_0 + \cdots + x_n = 1 \right\}$$

where $[0,1]$ is the closed unit interval with the standard metric. The functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ sends $[n]$ to $|\Delta^n|$ and is defined on morphisms by linearly interpolating the obvious map of vertices.

Corollary 1.1.20. *There exists an adjunction*

$$|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$$

extending the functor $|\Delta^\bullet| : \mathbf{\Delta} \to \mathbf{Top}$ *defined above, and this adjunction is unique up to unique isomorphism.* ■

Definition 1.1.21. The **geometric realisation** of a simplicial set X is the topological space $|X|$, and the **singular set** of a topological space Y is the simplicial set $\mathrm{S}(Y)$.

Definition 1.1.22. Let $\mathcal{C}$ be a category with binary products, and let Y and Z be objects in $\mathcal{C}$. An **exponential object** for Y and Z is an object Z^Y in $\mathcal{C}$ and a morphism $\mathrm{ev}_{Y,Z} : Z^Y \times Y \to Z$ with the following universal property:

- For all morphisms $f : X \times Y \to Z$ in $\mathcal{C}$, there exists a unique morphism $\bar{f} : X \to Z^Y$ such that $\mathrm{ev}_{Y,Z} \circ \left(\bar{f} \times \mathrm{id}_Y\right) = f$.

A **cartesian closed category** is a category with all finite products and exponential objects for all pairs of objects.

Proposition 1.1.23. *Let* **CGHaus** *be the category of compactly-generated Hausdorff spaces*[1] *and continuous maps.*

(i) **sSet** *is a cartesian closed category, with exponential objects given by the formula below:*

$$Z^Y = \mathbf{sSet}(\Delta^\bullet \times Y, Z)$$

(ii) *If Y is a locally compact Hausdorff space, then for all topological spaces Z, the set of all continuous maps $Y \to Z$, equipped with the compact–open topology, is an exponential object Z^Y in* **Top**.

(iii) **CGHaus** *is a cartesian closed category.*

Proof. Claim (i) can be verified by direct calculation, claim (ii) follows from Theorems 46.10 and 46.11 in [Munkres, 2000], and claim (iii) is proved in [GZ, Ch. III, § 2]. □

Theorem 1.1.24.

(i) *The topological standard n-simplex $|\Delta^n|$ is a compact Hausdorff space.*

(ii) *For any simplicial set X, the geometric realisation $|X|$ is a compactly-generated Hausdorff space.*

(iii) *The previously-constructed adjunction $|-| \dashv \mathrm{S} : \mathbf{Top} \to \mathbf{sSet}$ restricts to an adjunction between* **CGHaus** *and* **sSet**, *and moreover the functor $|-| : \mathbf{sSet} \to \mathbf{CGHaus}$ preserves finite limits and reflects isomorphisms.*

[1] — also known as **Kelley spaces**.

Proof. Claim (i) is a standard fact, while claims (ii) and (iii) are proven in [GZ, Ch. III, § 3]. □

1.2 The Quillen model structure

Prerequisites. §§ 1.1, A.1.

In [1967], Quillen constructed an axiomatic framework for doing homotopy theory in abstract categories, which he called 'closed model categories', and showed that **sSet** can be endowed with a model structure such that the resulting homotopy theory is equivalent in a strong sense to the homotopy theory of topological spaces.

Definition 1.2.1. A **horn** is a simplicial subset of the form $\Lambda^n_k \subseteq \Delta^n$, where Λ^n_k is the union of the images of $\delta^0_n, \ldots, \delta^{k-1}_n, \delta^{k+1}_n, \ldots, \delta^n_n : \Delta^{n-1} \to \Delta^n$ in **sSet**. In other words, Λ^n_k is the union of all the faces of Δ^n that include the k-th vertex. The **boundary** of Δ^n is the simplicial subset $\partial\Delta^n \subseteq \Delta^n$ generated by the images of $\delta^0_n, \ldots, \delta^n_n : \Delta^{n-1} \to \Delta^n$.

Definition 1.2.2. A **cofibration** in **sSet** is a monomorphism. A **Kan fibration** is a morphism $f : X \to Y$ in **sSet** that has the right lifting property with respect to the horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$, where $n \geq 1$ and $0 \leq k \leq n$. A **Kan complex** is a simplicial set X such that the unique morphism $X \to 1$ is a Kan fibration.

Proposition 1.2.3. *Let $\mathcal{C}_Q$ and $\mathcal{F}_Q$ be, respectively, the class of cofibrations and the class of Kan fibrations in* **sSet**. *There exists a unique class $\mathcal{W}_Q$ of morphisms in* **sSet** *with the following properties:*

(i) $(\mathcal{C}_Q \cap \mathcal{W}_Q, \mathcal{F}_Q)$ *is a weak factorisation system for* **sSet**.

(ii) $(\mathcal{C}_Q, \mathcal{F}_Q \cap \mathcal{W}_Q)$ *is a weak factorisation system for* **sSet**.

(iii) *A morphism is in $\mathcal{W}_Q$ if and only if it is of the form $f \circ c$, where f is in $\mathcal{F}_Q$ and has the right lifting property with respect to all morphisms in $\mathcal{C}_Q$, and c is in $\mathcal{C}_Q$ and has the left lifting property with respect to all morphisms in $\mathcal{F}_Q$.*

Proof. ??? □

Definition 1.2.4. A **Quillen weak equivalence** in **sSet** is a morphism in the class $\mathcal{W}_Q$ described above. An **anodyne extension**, or **trivial cofibration** in **sSet**, is a cofibration that has the left lifting property with respect to all Kan fibrations. A **trivial Kan fibration** is a Kan fibration that has the right lifting property with respect to all cofibrations.

— II —

Homotopical categories

2.1 Basics

Prerequisites. § A.2.

Definition 2.1.1. A relative category $\mathcal{C}$ is a **category with weak equivalences** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-3 property, and it is a **homotopical category** if $\operatorname{weq} \mathcal{C}$ has the 2-out-of-6 property. A **homotopical functor** is a relative functor between homotopical categories.

Example 2.1.2. Any saturated relative category is automatically a homotopical category, by corollary A.2.14. In particular, any minimal saturated relative category is a homotopical category. On the other hand, any maximal relative category is obviously a homotopical category.

Remark 2.1.3. A relative category $\mathcal{C}$ is a category with weak equivalences or a homotopical category if and only if the opposite relative category $\mathcal{C}^{\mathrm{op}}$ is.

Lemma 2.1.4. *Let A be an object in a homotopical category (resp. category with weak equivalences) $\mathcal{C}$. Then the slice category $\mathcal{C}_{/A}$ is also a homotopical category (resp. category with weak equivalences) if we declare a morphism in $\mathcal{C}_{/A}$ to be a weak equivalence if and only if it is a weak equivalence in $\mathcal{C}$.*

Proof. Use lemma A.2.13 on the projection functor $\mathcal{C}_{/A} \to \mathcal{C}$. ■

Definition 2.1.5. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two (not necessarily relative) functors between relative categories. A **natural weak equivalence** $\alpha : F \Rightarrow G$ is a natural transformation such that $\alpha_C : FC \to GC$ is a weak equivalence in $\mathcal{D}$ for all objects C in $\mathcal{C}$, and we say F and G are **naturally weakly equivalent** if they can be connected by a *zigzag* of natural weak equivalences.

Remark 2.1.6. If F and G are relative functors, then this is precisely the notion of weak equivalence in the relative functor category $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$. Although the definition above applies to all functors, if $H : \mathcal{D} \to \mathcal{E}$ is a functor, then the natural transformation $H\alpha : HF \Rightarrow HG$ is only guaranteed to be a natural weak equivalence if we assume H is a relative functor.

Definition 2.1.7. A **homotopical equivalence** is a relative functor $F : \mathcal{C} \to \mathcal{D}$ for which there exists a relative functor $G : \mathcal{D} \to \mathcal{C}$ such that GF is naturally weakly equivalent to $\mathrm{id}_{\mathcal{C}}$ and FG is naturally weakly equivalent to $\mathrm{id}_{\mathcal{D}}$. Such a G is said to be a **homotopical inverse** of F.

Proposition 2.1.8. *If $F : \mathcal{C} \to \mathcal{D}$ is a homotopical equivalence of relative categories with homotopical inverse $G : \mathcal{D} \to \mathcal{C}$, then* $\mathrm{Ho}\, F : \mathrm{Ho}\, \mathcal{C} \to \mathrm{Ho}\, \mathcal{D}$ *is an equivalence of categories, with quasi-inverse* $\mathrm{Ho}\, G : \mathrm{Ho}\, \mathcal{D} \to \mathrm{Ho}\, \mathcal{C}$. ■

2.2 Homotopical Kan extensions

Prerequisites. § 2.1.

Definition 2.2.1. Let $\mathcal{C}$ be a homotopical category. A **homotopically initial object** in $\mathcal{C}$ is an object A for which there exists a zigzag of natural transformations of the form

$$\Delta A \;\text{———}\; F \xrightarrow{\alpha} G \;\text{———}\; \mathrm{id}_{\mathcal{C}}$$

where $\Delta A : \mathcal{C} \to \mathcal{C}$ is the constant functor with value A, $\alpha_A : FA \to GA$ is a weak equivalence in $\mathcal{C}$, and the unmarked lines denote (possibly trivial) zigzags of natural weak equivalences. Dually, a **homotopically terminal object** in $\mathcal{C}$ is a homotopically initial object in $\mathcal{C}^{\mathrm{op}}$.

Proposition 2.2.2. *Let $\mathcal{C}$ be a homotopical category. If A is a homotopically initial (resp. homotopically terminal) object in $\mathcal{C}$, then:*

(i) *Any object in $\mathcal{C}$ weakly equivalent to A is also a homotopically initial (resp. homotopically terminal) object in $\mathcal{C}$.*

(ii) *A is an initial (resp. terminal) object in* $\mathrm{Ho}\, \mathcal{C}$.

(iii) *If C is a minimal homotopical category, then A is an initial (resp. terminal) object in C as well.*

Conversely, any initial (resp. terminal) object in C is also homotopically initial (resp. homotopically terminal).

Proof. Obvious. (This is Proposition 38.3 in [DHKS].) ■

Definition 2.2.3. A **homotopically contractible category** is a homotopical category C such that the unique (homotopical) functor $C \to \mathbb{1}$ is a homotopical equivalence, where $\mathbb{1}$ is the trivial category with only one object.

Proposition 2.2.4. *Let C be a homotopical category. The following are equivalent:*

(i) *C is homotopically contractible.*

(ii) *C is inhabited, and for every object A in C, the constant functor ΔA is naturally weakly equivalent to id_C.*

(iii) *There exists an object A in C such that ΔA and id_C are naturally weakly equivalent.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ■

Proposition 2.2.5. *Let C be a homotopically contractible category.*

(i) *Every morphism in C is a weak equivalence.*

(ii) *The unique functor $\mathrm{Ho}\, C \to \mathbb{1}$ is an equivalence of categories.*

(iii) *If C is a minimal homotopical category, then $C \to \mathbb{1}$ is also an equivalence of categories.*

(iv) *The opposite homotopical category C^{op} and the homotopical functor category $[\mathcal{D}, C]_{\mathrm{h}}$ (for any homotopical category $\mathcal{D}$) are also homotopically contractible.*

(v) *Every object in C is both homotopically initial and homotopically terminal.*

Proof. Obvious. (This is paragraph 37.6 in [DHKS].) ■

Proposition 2.2.6. *Let $\mathcal{C}$ be a homotopical category. If $\mathcal{D}$ is the full homotopical subcategory of $\mathcal{C}$ spanned by the homotopically initial (or homotopically terminal) objects, then $\mathcal{D}$ is homotopically contractible.*

Proof. See paragraph 38.5 in [DHKS]. □

Definition 2.2.7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two (not necessarily homotopical) functors between homotopical categories. A **homotopical left Kan extension** (resp. **homotopical right Kan extension**) of G along F is a homotopically initial (resp. homotopically terminal) object of the relative category $(G \downarrow F^*)_{\mathrm{h}}$ (resp. $(F^* \downarrow G)_{\mathrm{h}}$) described below:

- The objects are pairs (H, α) where H is a homotopical functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).
- The weak equivalences are the natural weak equivalences.

— III —

Model categories

3.1 Basics

Prerequisites. §§ 2.1, A.1.

In [1967], Quillen introduced the notion of a 'closed model category' (but we shall say simply 'model category') for homotopy theory, so as to formalise the similarities between the homotopy theory of spaces and homological algebra. The idea was that, to do homotopy theory, one really only needed to know which morphisms are cofibrations, which are weak equivalences, and which are fibrations.

Definition 3.1.1. A **model category** is a locally small category $\mathcal{M}$ equipped with three subclasses $\mathcal{C}, \mathcal{W}, \mathcal{F}$ of mor $\mathcal{M}$ satisfying the following axioms:

- **CM1.** $\mathcal{M}$ has finite limits and finite colimits.
- **CM2.** $\mathcal{W}$ has the 2-out-of-3 property.
- **CM3.** $\mathcal{C}$, $\mathcal{W}$, and $\mathcal{F}$ are closed under retracts.
- **CM4.** Given a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

  where i is in $\mathcal{C}$ and p is in $\mathcal{F}$, if at least one of i or p is also in $\mathcal{W}$, then there exists a morphism $B \to X$ making the evident triangles commute.

- **CM5.** Any morphism f in $\mathcal{M}$ may be factored in two ways:
  - $f = p \circ i$ where i is in $\mathcal{C} \cap \mathcal{W}$ and p is in $\mathcal{F}$, and
  - $f = q \circ j$, where j is in $\mathcal{C}$ and q is in $\mathcal{W} \cap \mathcal{F}$.

The triple $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is said to be a **model structure** on $\mathcal{M}$. Given such a model structure on $\mathcal{M}$,

- a **cofibration** is a morphism in $\mathcal{C}$,
- a **weak equivalence** is a morphism in $\mathcal{W}$,
- a **fibration** is a morphism in $\mathcal{F}$,
- a **trivial cofibration** (or **acyclic cofibration**) is a morphism in $\mathcal{C} \cap \mathcal{W}$, and
- a **trivial fibration** (or **acyclic fibration**) is a morphism in $\mathcal{W} \cap \mathcal{F}$;
- a **cofibrant object** in $\mathcal{M}$ is an object X such that the unique morphism $0 \to X$ is a cofibration, and
- a **fibrant object** in $\mathcal{M}$ is an object X such that the unique morphism $X \to 1$ is a fibration.
- a **cofibrant–fibrant object** in $\mathcal{M}$ is an object that is both cofibrant and fibrant.

Remark 3.1.2. The above presentation of the axioms is due to Quillen [1969], and is the one used in [DS] and [GJ]; however, [DHKS], [Hirschhorn, 2003], and [Hovey, 1999] use a variant definition that replaces axioms CM1 and CM5 with stronger ones:

- **CM1′.** $\mathcal{M}$ is complete and cocomplete.
- **CM5′.** The $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$-factorisations can be chosen *functorially* in the sense of definition A.1.9.

Note also that Hovey [1999] considers the functorial factorisations to be a *structure* rather than a property.

Remark 3.1.3. Let $\mathcal{M}$ be a category with finite limits and finite colimits. Then, $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure on $\mathcal{M}$ if and only if $(\mathcal{F}^{\mathrm{op}}, \mathcal{W}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})$ is a model structure on $\mathcal{M}^{\mathrm{op}}$.

Theorem 3.1.4. *Let $\mathcal{M}$ be a locally small category and let $\mathcal{C}, \mathcal{W}, \mathcal{F}$ be subclasses of* $\operatorname{mor} \mathcal{M}$. *Assuming $\mathcal{M}$ has finite limits and finite colimits, the following are equivalent:*

(i) *$(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a model structure for $\mathcal{M}$.*

(ii) *$\mathcal{M}$ is a saturated homotopical category with* $\operatorname{weq} \mathcal{M} = \mathcal{W}$, *and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems for $\mathcal{M}$.*

(iii) *$(\mathcal{M}, \mathcal{W})$ is a category with weak equivalences (as in definition 2.1.1), and both $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are weak factorisation systems for $\mathcal{M}$.*

Proof. (i) ⇒ (ii). The fact that we have two weak factorisation systems follows from Lemma 1.1 in [GJ, Ch. II] or Proposition 7.2.3 in [Hirschhorn, 2003]; and the saturation property follows from Theorems 1.10 and 1.11 in [GJ, Ch. II], or Theorem 8.3.10 in [Hirschhorn, 2003].

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). Use proposition A.1.7. □

Lemma 3.1.5. *Let A be an object in a model category $\mathcal{M}$. Then the slice category $\mathcal{M}_{/A}$ has the* **slice model structure**, *where a morphism in $\mathcal{M}_{/A}$ is a cofibration, weak equivalence, or fibration if it is so in $\mathcal{M}$.*

Proof. Use lemmas 2.1.4 and A.1.6, plus the fact that $\mathcal{M}_{/A}$ has finite limits and finite colimits if $\mathcal{M}$ does. ■

Definition 3.1.6. A **left Quillen functor** is a functor $F : \mathcal{M} \to \mathcal{N}$ between model categories that has a right adjoint and preserves cofibrations and trivial cofibrations; dually, a **right Quillen functor** is a functor $G : \mathcal{N} \to \mathcal{M}$ between model categories that has a left adjoint and preserves fibrations and trivial fibrations. A **Quillen adjunction** is an adjunction

$$F \dashv G : \mathcal{M} \to \mathcal{N}$$

where $\mathcal{M}$ and $\mathcal{N}$ are model categories, such that F is a left Quillen functor and G is a right Quillen functor. A **Quillen equivalence** is a Quillen adjunction as above satisfying this additional condition:

- Given a cofibrant object A in $\mathcal{N}$ and fibrant object X in $\mathcal{M}$, a morphism $FA \to X$ is a weak equivalence in $\mathcal{M}$ if and only if its adjoint transpose $A \to GX$ is a weak equivalence in $\mathcal{N}$.

Proposition 3.1.7. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be an adjunction between model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F is a left Quillen functor.*

(iii) *G is a right Quillen functor.*

(iv) *F preserves cofibrations and G preserves fibrations.*

(v) *F preserves trivial cofibrations and G preserves trivial fibrations.*

Proof. Use proposition A.1.8. ■

Lemma 3.1.8 (Kenneth S. Brown)**.** *Let $\mathcal{M}$ be a model category and let C be a category with weak equivalences. If $F : \mathcal{M} \to C$ sends trivial cofibrations (resp. trivial fibrations) in $\mathcal{M}$ to weak equivalences in C, then F preserves all weak equivalences between cofibrant (resp. fibrant) objects.*

Proof. See Lemma 9.9 in [DS], Lemma 7.7.1 in [Hirschhorn, 2003], or Lemma 14.5 in [DHKS]. □

Corollary 3.1.9. *Let $F \dashv G : \mathcal{M} \to \mathcal{N}$ be a Quillen adjunction.*

(i) *If A and B are cofibrant objects in $\mathcal{N}$ and $f : A \to B$ is a weak equivalence in $\mathcal{N}$, then Ff is a weak equivalence in $\mathcal{M}$.*

(ii) *If X and Y are fibrant objects in $\mathcal{M}$ and $g : X \to Y$ is a weak equivalence in $\mathcal{M}$, then Gg is a weak equivalence in $\mathcal{N}$.* ■

Proposition 3.1.10 (Dugger)**.** *Let $F \dashv G$ be an adjunction between [strong???] model categories. The following are equivalent:*

(i) *$F \dashv G$ is a Quillen adjunction.*

(ii) *F preserves cofibrations between cofibrant objects and all trivial cofibrations.*

(iii) *G preserves fibrations between fibrant objects and all trivial fibrations.*

Proof. This is Proposition 8.5.4 in [Hirschhorn, 2003]. □

Definition 3.1.11. Let X be an object in a model category $\mathcal{M}$.

- A **cofibrant replacement** for X is a pair $(\tilde{X}, p)$ where $\tilde{X}$ is a cofibrant object in $\mathcal{M}$ and p is a weak equivalence $\tilde{X} \to X$.
- A **fibrant replacement** for X is a pair $(\hat{X}, i)$ where $\hat{X}$ is a fibrant object in $\mathcal{M}$ and i is a weak equivalence $X \to \hat{X}$.
- A **fibrant cofibrant replacement** for X is a cofibrant replacement $(\tilde{X}, p)$ where $p : \tilde{X} \to X$ is a trivial fibration.
- A **cofibrant fibrant replacement** for X is a fibrant replacement $(\hat{X}, i)$ where $i : X \to \hat{X}$ is a trivial cofibration.

Remark 3.1.12. Note that a fibrant cofibrant replacement for X is precisely a cofibrant replacement for X that is fibrant as an object in $\mathcal{M}_{/X}$, and a cofibrant fibrant replacement for X is precisely a fibrant replacement for X that is cofibrant as an object in ${}^{X/}\mathcal{M}$.

Moreover, if X is fibrant and $(\tilde{X}, p)$ is a fibrant cofibrant replacement for X, then $\tilde{X}$ is both fibrant and cofibrant in $\mathcal{M}$, and if X is cofibrant and $(\hat{X}, i)$ is a cofibrant fibrant replacement for X, then $\hat{X}$ is both cofibrant and fibrant in $\mathcal{M}$.

Proposition 3.1.13. *Any object in a model category has both a fibrant cofibrant replacement and a cofibrant fibrant replacement.*

Proof. Use axiom CM5. ■

3.2 Left and right homotopy

Prerequisites. § 3.1.

Definition 3.2.1. Let X be an object in a model category $\mathcal{M}$. A **cylinder object** for X is a quadruple $(\mathrm{Cyl}(X), i_0, i_1, p)$, where $\mathrm{Cyl}(X)$ is an object in $\mathcal{M}$, $p : \mathrm{Cyl}(X) \to X$ is a weak equivalence, and $i_0, i_1 : X \to \mathrm{Cyl}(X)$ are sections of p such that the morphism $[i_0, i_1] : X + X \to \mathrm{Cyl}(X)$ is a cofibration. Dually, a **path object** for X is a quadruple $(\mathrm{Path}(X), i, p_0, p_1)$, where $\mathrm{Path}(X)$ is an object in $\mathcal{M}$, $i : X \to \mathrm{Path}(X)$ is a weak equivalence, and $p_0, p_1 : \mathrm{Path}(X) \to X$ are retractions of i such that the morphism $\langle p_0, p_1 \rangle : \mathrm{Path}(X) \to X \times X$ is a fibration.

Proposition 3.2.2. *Let X be an object in a model category $\mathcal{M}$.*

(i) *There exists a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, where the morphism $p : \mathrm{Cyl}(X) \to X$ is a trivial fibration.*

(ii) *There exists a path object $\big(\mathrm{Path}(X), i, p_0, p_1\big)$ for X, where the morphism $i : X \to \mathrm{Path}(X)$ is a trivial cofibration.*

Proof. Use axioms CM1 and CM5. ■

Definition 3.2.3. Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$. A **left homotopy** from f_0 to f_1 with respect to a cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ is a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f_0$ and $H \circ i_1 = f_1$. Dually, a **right homotopy** from f_0 to f_1 with respect to a path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ is a morphism $H : X \to \mathrm{Path}(Y)$ such that $p_0 \circ H = f_0$ and $p_1 \circ H = f_1$. We say f_0 and f_1 are **left homotopic** if there exists a left homotopy from f_0 to f_1 with respect to some cylinder object for X, and we say f_0 and f_1 are **right homotopic** if there exists a right homotopy from f_0 to f_1 with respect to some path object for Y.

Remark 3.2.4. If f_0 and f_1 are either left homotopic or right homotopic, then they must represent the same morphism in $\mathrm{Ho}\,\mathcal{M}$. For definiteness, let us write $\gamma : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ for the universal functor, and suppose $H : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to f_1. Since i_0 and i_1 are both sections of the weak equivalence $p : \mathrm{Cyl}(X) \to X$, we must have $\gamma i_0 = (\gamma p)^{-1} = \gamma i_1$; but $f_0 = H \circ i_0$ and $f_1 = H \circ i_1$, so indeed $\gamma f_0 = \gamma f_1$. This is one of the reasons for calling $\mathrm{Ho}\,\mathcal{M}$ the *homotopy* category of $\mathcal{M}$.

However, it is not quite true that $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are either left homotopic or right homotopic; this only happens in special cases. In general, being left/right homotopic fails to even be an equivalence relation.

Lemma 3.2.5. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *Given any cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, $f_0 \circ p : \mathrm{Cyl}(X) \to Y$ is a left homotopy from f_0 to itself.*

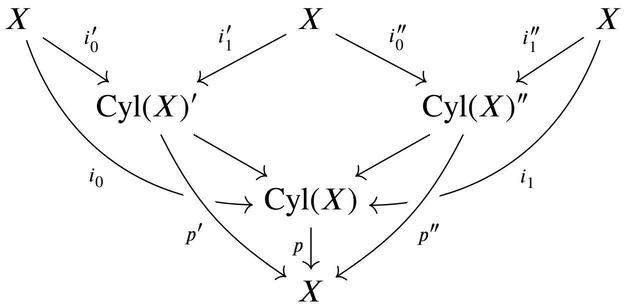
(ii) *Given any path object* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *for* Y*,* $i \circ f_0 : X \to \mathrm{Path}(Y)$ *is a right homotopy from* f_0 *to itself.*

(iii) *If* $H : \mathrm{Cyl}(X) \to Y$ *is a left homotopy from* f_0 *to* f_1 *with respect to a cylinder object* $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ *for* X*, then the same* H *is a left homotopy from* f_1 *to* f_0 *for the cylinder object* $\big(\mathrm{Cyl}(X), i_1, i_0, p\big)$.

(iv) *If* $H : X \to \mathrm{Path}(Y)$ *is a right homotopy from* f_0 *to* f_1 *with respect to a path object* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *for* Y*, then the same* H *is a right homotopy from* f_1 *to* f_0 *for the path object* $\big(\mathrm{Path}(Y), i, p_1, p_0\big)$.

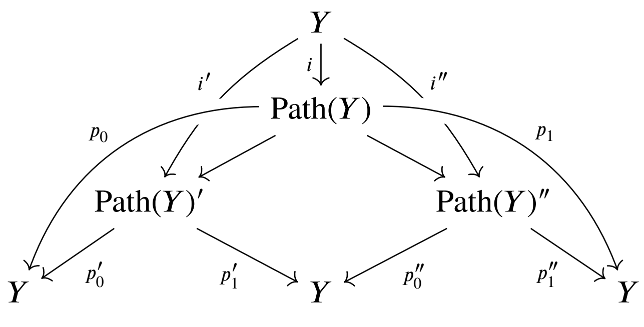
Proof. Obvious. ■

Lemma 3.2.6. *Let* X *be a cofibrant object in a model category* $\mathcal{M}$*. Given two cylinder objects for* X*, say* $\big(\mathrm{Cyl}(X)', i'_0, i'_1, p'\big)$ *and* $\big(\mathrm{Cyl}(X)'', i''_0, i''_1, p''\big)$*, there exists a third cylinder object* $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ *such that the diagram below commutes,*



and the diamond is a pushout diagram.

Dually, if Y *is a fibrant object in* $\mathcal{M}$*, and we have two path objects for* Y*, say* $\big(\mathrm{Path}(Y)', i', p'_0, p'_1\big)$ *and* $\big(\mathrm{Path}(Y)'', i'', p''_0, p''_1\big)$*, then there exists a third path object* $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ *such that the diagram below commutes,*



and the diamond is a pullback diagram.

Proof. See Lemma 1.5 in [GJ, Ch. II], or Lemma 7.4.2 in [Hirschhorn, 2003]. □

Corollary 3.2.7. *Let $f_0, f_1, f_2 : X \to Y$ be three parallel morphisms in a model category $\mathcal{M}$.*

(i) *If f_0 and f_1 are left homotopic, and f_1 and f_2 are left homotopic, then f_0 and f_2 are also left homotopic.*

(ii) *If f_0 and f_1 are right homotopic, and f_1 and f_2 are right homotopic, then f_0 and f_2 are also right homotopic.* ■

Lemma 3.2.8. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *If X is cofibrant, and f_0 and f_1 are left homotopic, given any path object $\big(\mathrm{Path}(Y), i, p_0, p_1\big)$ for Y, there is a right homotopy $H : X \to \mathrm{Path}(Y)$ from f_0 to f_1.*

(ii) *If Y is fibrant, and f_0 and f_1 are right homotopic, given any cylinder object $\big(\mathrm{Cyl}(X), i_0, i_1, p\big)$ for X, there is a left homotopy $H : \mathrm{Cyl}(X) \to Y$ from f_0 to f_1.*

Proof. See Proposition 1.8 in [GJ, Ch. II], or Proposition 7.4.7 in [Hirschhorn, 2003]. □

Proposition 3.2.9. *Let X and Y be objects in a model category $\mathcal{M}$.*

(i) *If X is cofibrant, then being left homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(ii) *If Y is fibrant, then being right homotopic is an equivalence relation on the hom-set $\mathcal{M}(X, Y)$.*

(iii) *If X is cofibrant and Y is fibrant, then these two equivalence relations on $\mathcal{M}(X, Y)$ coincide.*

Proof. Use the preceding lemmas. ■

Lemma 3.2.10. *Let $f_0, f_1 : X \to Y$ be a parallel pair of morphisms in a model category $\mathcal{M}$.*

(i) *If* f_0 *and* f_1 *are right homotopic and* $g : W \to X$ *is any morphism in* $\mathcal{M}$, *then* $f_0 \circ g$ *and* $f_1 \circ g$ *are also right homotopic.*

(ii) *If* f_0 *and* f_1 *are left homotopic and* $g : Y \to Z$ *is any morphism in* $\mathcal{M}$, *then* $g \circ f_0$ *and* $g \circ f_1$ *are also left homotopic.*

Proof. Obvious. ■

Corollary 3.2.11. *Let* $\mathcal{M}$ *be a model category, and let* $\mathcal{M}_{\mathrm{cf}}$ *be the full subcategory spanned by the cofibrant–fibrant objects. Then the equivalence relation induced by homotopy is a congruence on* $\mathcal{M}_{\mathrm{cf}}$*; in particular, there exist a locally small category* $\mathcal{M}'$ *and a full functor* $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ *with these properties:*

- *The objects of* $\mathcal{M}'$ *are those of* $\mathcal{M}_{\mathrm{cf}}$.
- *The hom-set* $\mathcal{M}'(X, Y)$ *is* $\mathcal{M}(X, Y)$ *modulo homotopy.*
- *The functor* $\mathcal{M}_{\mathrm{cf}} \to \mathcal{M}'$ *sends each morphism in* $\mathcal{M}'$ *to its homotopy class.* ■

The next result is a version of Whitehead's theorem; however, this is a purely formal consequence of the model category axioms and has no real content, unlike the original theorem.

Proposition 3.2.12. *Let* X *and* Y *be cofibrant–fibrant objects in a model category* $\mathcal{M}$. *If* $f : X \to Y$ *is a weak equivalence, then* f *has a* **homotopy inverse** *in* $\mathcal{M}$, *i.e. a morphism* $g : Y \to X$ *such that* $g \circ f$ *and* id_X *are homotopic, and* $f \circ g$ *and* id_Y *are homotopic.*

Proof. See Theorem 1.10 in [GJ, Ch. II], or Theorem 7.5.10 in [Hirschhorn, 2003]. □

Corollary 3.2.13. *Let* W, X, Y, Z *be cofibrant–fibrant objects in a model category* $\mathcal{M}$, *and let* $f_0, f_1 : X \to Y$ *be a parallel pair of morphisms.*

(i) *If* $g : W \to X$ *is a weak equivalence such that* $f_0 \circ g$ *and* $f_1 \circ g$ *are homotopic, then* f_0 *and* f_1 *are homotopic.*

(ii) *If* $g : Y \to Z$ *is a weak equivalence such that* $g \circ f_0$ *and* $g \circ f_1$ *are homotopic, then* f_0 *and* f_1 *are homotopic.*

Proof. Use a homotopy inverse to cancel g. ■

3.3 The homotopy category

Prerequisites. §§ 3.1, 3.2, A.2.

Definition 3.3.1. The **Quillen homotopy category** (or, more simply, **homotopy category**) of a model category $\mathcal{M}$ is the category $\operatorname{Ho}\mathcal{M}$ obtained by freely inverting the weak equivalences in $\mathcal{M}$, as in definition A.2.9.

Theorem 3.3.2. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho}\mathcal{M}$ be the universal functor.*

(i) *$\operatorname{Ho}\mathcal{M}$ is equivalent to the locally small category $\mathcal{M}'$ defined in corollary 3.2.11, and $\mathcal{M}$ is a saturated homotopical category.*

(ii) *If X and Y are cofibrant–fibrant objects in $\mathcal{M}$, then the hom-class map $\mathcal{M}(X,Y) \to \operatorname{Ho}\mathcal{M}(X,Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

(iii) *For any two objects X and Y in $\mathcal{M}$, every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{M}$ can be represented as a zigzag of the form*

$$X \xleftarrow{\;p\;} \tilde{X} \longrightarrow \hat{Y} \xleftarrow{\;i\;} Y$$

where $(\tilde{X}, p)$ is any cofibrant replacement for X and $(\hat{Y}, i)$ is any fibrant replacement for Y.

Proof. (i). This is Theorem 1.11 in [GJ, Ch. II], or Proposition 5.8 in [DS].

(ii). Implied by claim (i).

(iii). Using claim (ii), every morphism $X \to Y$ in $\operatorname{Ho}\mathcal{M}$ can be represented as a zigzag of the form

$$\begin{array}{ccccccc} X & \xleftarrow{\;p\;} & \tilde{X} & & \hat{Y} & \xleftarrow{\;i\;} & Y \\ & & {\scriptstyle i'}\downarrow & & \uparrow{\scriptstyle p'} & & \\ & & R\tilde{X} & \xrightarrow{\;f'\;} & Q\hat{Y} & & \end{array}$$

where $(R\tilde{X}, i')$ is a cofibrant fibrant replacement for $\tilde{X}$ and $(Q\hat{Y}, p')$ is a fibrant cofibrant replacement for $\hat{Y}$; but such a zigzag is manifestly equivalent to the zigzag

$$X \xleftarrow{p} \tilde{X} \xrightarrow{f} \hat{Y} \xleftarrow{i} Y$$

where $f = p' \circ f' \circ i'$. □

Corollary 3.3.3. *Let $\mathcal{M}$ be a model category and let $\gamma : \mathcal{M} \to \operatorname{Ho} \mathcal{M}$ be the universal functor. If X is a cofibrant object in $\mathcal{M}$ and Y is a fibrant object in $\mathcal{M}$, then the hom-class map $\mathcal{M}(X, Y) \to \operatorname{Ho} \mathcal{M}(X, Y)$ induced by γ is surjective; and moreover for any parallel pair $f_0, f_1 : X \to Y$ in $\mathcal{M}$, we have $\gamma f_0 = \gamma f_1$ if and only if f_0 and f_1 are homotopic.*

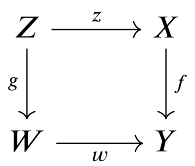
Proof. As noted in remark 3.2.4, if $f_0, f_1 : X \to Y$ are homotopic, then we must have $\gamma f_0 = \gamma f_1$. Conversely, suppose $\gamma f_0 = \gamma f_1$ with X cofibrant and Y fibrant. Let (RX, i') be a cofibrant fibrant replacement for X and (QY, p') be a fibrant cofibrant replacement for Y. Then, there exists morphisms $f_0', f_1' : RX \to QY$ such that $f_0 = p' \circ f_0' \circ i'$ and $f_1 = p' \circ f_1' \circ i'$. Since $i' : X \to RX$ and $p' : QY \to Y$ are weak equivalences, we must have $\gamma f_0' = \gamma f_1'$ in $\operatorname{Ho} \mathcal{M}$. The theorem then implies f_0' and f_1' are homotopic; thus f_0 and f_1 are also homotopic, by lemmas 3.2.8 and 3.2.10. ■

— A —

Generalities

A.1 Factorisation systems

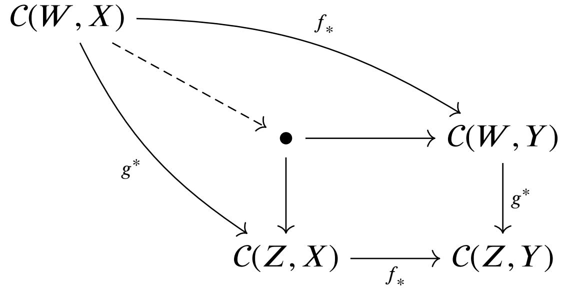
Definition A.1.1. Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a category $\mathcal{C}$. Given a commutative square in $\mathcal{C}$,



a **lift** is a morphism $h : W \to X$ such that $f \circ h = w$ and $h \circ g = z$.

We say g has the **left lifting property** with respect to f and f has the **right lifting property** with respect to g if every commutative square in $\mathcal{C}$ of the form above has a lift; and we say f is **left orthogonal** to g and g is **right orthogonal** to f if lifts exist *and* are unique.

Lemma A.1.2. *Let $f : X \to Y$ and $g : Z \to W$ be morphisms in a locally small category $\mathcal{C}$. Consider the commutative diagram in* **Set** *shown below,*



where the inner square is a pullback diagram.

(i) *The dashed arrow is a surjection if and only if g has the left lifting property with respect to f.*

(ii) *The dashed arrow is a bijection if and only if g is left orthogonal to f.*

Proof. This is just a restatement of the definition. ■

Lemma A.1.3. *Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. The following are equivalent:*

(i) *f is an isomorphism.*

(ii) *f is right orthogonal to any morphism in $\mathcal{C}$.*

(iii) *f has the right lifting property with respect to any morphism in $\mathcal{C}$.*

(iv) *f has the right lifting property with respect to itself.*

Dually, the following are equivalent:

(i′) *f is an isomorphism.*

(ii′) *f is left orthogonal to any morphism in $\mathcal{C}$.*

(iii′) *f has the left lifting property with respect to any morphism in $\mathcal{C}$.*

(iv′) *f has the left lifting property with respect to itself.*

Proof. (i) ⇒ (ii). Suppose $r : Y \to X$ is a morphism such that $r \circ f = \mathrm{id}_X$. Then, for any commutative square as below,

$$\begin{array}{ccc} Z & \xrightarrow{z} & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

we have $(r \circ w) \circ g = r \circ f \circ z = z$; but if $f \circ r = \mathrm{id}_Y$ as well, then $f \circ (r \circ w) = w$; thus $r \circ w : W \to X$ is the required lift. It is clearly unique, as f is monic.

(ii) ⇒ (iii), (iii) ⇒ (iv). Obvious.

(iv) ⇒ (i). Consider the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}} & X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

Since f has the right lifting property with respect to itself, there exists a morphism $h : Y \to X$ such that $h \circ f = \mathrm{id}_X$ and $f \circ h = \mathrm{id}_Y$. ■

Definition A.1.4. A **weak factorisation system** for a category $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of subclasses of $\operatorname{mor} \mathcal{C}$ satisfying these conditions:

- For each morphism f in $\mathcal{C}$ there exists a pair (g, h) with $g \in \mathcal{L}$ and $h \in \mathcal{R}$ such that $f = h \circ g$. Such a pair is a $(\mathcal{L}, \mathcal{R})$**-factorisation** of f.
- A morphism is in $\mathcal{L}$ if and only if it has the left lifting property with respect to every morphism in $\mathcal{R}$.
- A morphism is in $\mathcal{R}$ if and only if it has the right lifting property with respect to every morphism in $\mathcal{L}$.

An **orthogonal factorisation system** is the same thing as a weak factorisation system, except for replacing '… has the left/right lifting property with respect to …' with '… is left/right orthogonal to …'.

Remark A.1.5. Obviously, $(\mathcal{L}, \mathcal{R})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}$ if and only if $(\mathcal{R}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}})$ is a weak (resp. orthogonal) factorisation system for $\mathcal{C}^{\mathrm{op}}$.

Lemma A.1.6. *Let A be an object in a category $\mathcal{C}$ with a weak (resp. orthgonal) factorisation system $(\mathcal{L}, \mathcal{R})$. Then the slice category $\mathcal{C}_{/A}$ has a weak (resp. orthogonal) factorisation system where a morphism is in the left or right class if and only if it is so in $\mathcal{C}$.*

Proof. The projection $\mathcal{C}_{/A} \to \mathcal{C}$ induces a bijection between solutions for lifting problems in $\mathcal{C}_{/A}$ and solutions for the corresponding lifting problems in $\mathcal{C}$. ■

Proposition A.1.7. *Let $(\mathcal{L}, \mathcal{R})$ be a weak or orthogonal factorisation system for a category $\mathcal{C}$.*

(i) *Given a pullback diagram in $\mathcal{C}$ as below,*

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[q]{} & Y \end{array}$$

if the morphism f is in $\mathcal{R}$, then f' is also in $\mathcal{R}$.

(ii) *Let I be a set. If $f_i : X_i \to Y_i$ is a morphism in $\mathcal{R}$ for all i in I and the product $\prod_i f_i : \prod_i X_i \to \prod_i Y_i$ exists in $\mathcal{C}$, then $\prod_i f_i$ is also in $\mathcal{R}$.*

(iii) *Given a commutative diagram of the form*

$$\begin{array}{ccccc} X' & \xrightarrow{i_X} & X & \xrightarrow{r_X} & X' \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y' & \xrightarrow[i_Y]{} & Y & \xrightarrow[r_Y]{} & Y' \end{array}$$

where $r_X \circ i_X = \mathrm{id}_{X'}$ and $r_Y \circ i_Y = \mathrm{id}_{Y'}$, if f is in $\mathcal{R}$, then so is f'; in other words, $\mathcal{R}$ is closed under retracts.

(iv) *$\mathcal{L}$ is closed under composition.*

(v) *Let γ be an ordinal and let $Z : \gamma \to \mathcal{C}$ be a functor that preserves sequential colimits. We write Z_α for $Z(\alpha)$, where $\alpha < \gamma$, and $g_{\alpha,\beta} : Z_\alpha \to Z_\beta$ for the morphism $Z(\alpha \to \beta)$, where $\alpha < \beta < \gamma$. If λ is a colimiting cocone from Z to W and each $g_{\alpha,\beta}$ is in $\mathcal{L}$, then each component $\lambda_\alpha : Z_\alpha \to W$ is also in $\mathcal{L}$.*

Proof. (i). Suppose g is in $\mathcal{L}$ and consider the following commutative diagram:

$$\begin{array}{ccccc} Z & \xrightarrow{z} & X' & \xrightarrow{p} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y' & \xrightarrow[q]{} & Y \end{array}$$

There exists $h : Z \to X$ such that $h \circ g = p \circ z$ and $f \circ h = q \circ w$. In particular, there exists a unique morphism $h' : Z \to X'$ such that $f' \circ h' = w$ and $p \circ h' = h$, by the universal property of pullbacks. Thus $p \circ h' \circ g = h \circ g = p \circ z$ and

$f' \circ h' \circ g = w \circ g = f' \circ z$, but p and f' are jointly monic, so $h' \circ g = z$. Thus we have the required lift, and h' is unique if h is.

(ii). We may construct the required lift componentwise.

(iii). Suppose g is in $\mathcal{L}$ and consider the following commutative diagram:

$$\begin{array}{ccccccc} Z & \xrightarrow{z} & X' & \xrightarrow{i_X} & X & \xrightarrow{r_X} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ W & \xrightarrow[w]{} & Y' & \xrightarrow[i_Y]{} & Y & \xrightarrow[r_Y]{} & Y' \end{array}$$

There exists $h : Z \to X$ such that $h \circ g = i_X \circ z$ and $f \circ h = i_Y \circ w$, and so for $h' = r_X \circ h$:

$$h' \circ g = r_X \circ i_X \circ z = z$$
$$f' \circ h' = f' \circ r_X \circ h = r_Y \circ f \circ h = r_Y \circ i_Y \circ w = w$$

Thus $h' : Z \to X'$ is the required lift, and h' is unique if h is (because i_X is split monic).

(iv). Suppose $g' : Z' \to Z$ and $g : Z \to W$ are in $\mathcal{L}$ and $f : X \to Y$ is in $\mathcal{R}$. Consider the following commutative diagram:

$$\begin{array}{ccc} Z' & \xrightarrow{z'} & X \\ {\scriptstyle g'}\downarrow & & \\ Z & & \downarrow{\scriptstyle f} \\ {\scriptstyle g}\downarrow & & \\ W & \xrightarrow[w]{} & Y \end{array}$$

There must exist a morphism $z : Z \to X$ such that $z \circ g' = z'$ and $f \circ z' = w \circ g$, and hence a morphism $h : W \to X$ such that $h \circ g = z$ and $f \circ h = w$. Obviously, $h \circ (g' \circ g) = z'$, so h is the required lift and is moreover unique if $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system.

(v). We may assume without loss of generality that $\alpha = 0$, since any non-empty terminal segment of γ is cofinal in γ. Suppose $f : X \to Y$ is in $\mathcal{R}$ and consider the following commutative diagram:

$$\begin{array}{ccc} Z_0 & \xrightarrow{z_0} & X \\ {\scriptstyle \lambda_0}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow[w]{} & Y \end{array}$$

For each $\alpha < \gamma$, given z_α making the following diagram commute,

$$\begin{array}{ccccc} Z_\alpha & & \xrightarrow{z_\alpha} & & X \\ {\scriptstyle g_{\alpha,\alpha+1}}\downarrow & & & & \downarrow{\scriptstyle f} \\ Z_{\alpha+1} & \xrightarrow[\lambda_{\alpha+1}]{} & W & \xrightarrow[w]{} & Y \end{array}$$

choose a lift $z_{\alpha+1} : Z_{\alpha+1} \to X$; for each limit ordinal $\beta < \gamma$, let $z_\beta : Z_\beta \to X$ be the unique morphism such that $z_\beta \circ g_{\alpha,\beta} = z_\alpha$ for all $\alpha < \beta$. (Such z_β exist and are unique because $Z_\beta = \varinjlim_{\alpha<\beta} Z_\alpha$.) Note that the universal property of W then guarantees that $w \circ \lambda_\beta = f \circ z_\beta$.

Having constructed morphisms $z_\alpha : Z_\alpha \to X$ for all $\alpha < \gamma$ as above, we may now obtain $h : W \to X$ as the unique morphism such that $h \circ \lambda_\alpha = z_\alpha$ for all $\alpha < \gamma$, and again we automatically have $f \circ h = w$. It is also clear that h is unique if $(\mathcal{L}, \mathcal{R})$ is an orthogonal factorisation system. ■

Proposition A.1.8. *Let $(\mathcal{L}, \mathcal{R})$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}$, and let $(\mathcal{L}', \mathcal{R}')$ be a weak (resp. orthogonal) factorisation system for a category $\mathcal{C}'$. Given an adjunction*

$$F \dashv U : \mathcal{C}' \to \mathcal{C}$$

the following are equivalent:

(i) *F sends morphisms in $\mathcal{L}$ to morphisms in $\mathcal{L}'$.*

(ii) *U sends morphisms in $\mathcal{R}'$ to morphisms in $\mathcal{R}$.*

Proof. The adjunction induces a bijection between solutions to the two lifting problems shown below:

$$\begin{array}{ccc} FZ & \longrightarrow & X \\ {\scriptstyle Fg}\downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle f} \\ FW & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} Z & \longrightarrow & UX \\ {\scriptstyle g}\downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle Uf} \\ W & \longrightarrow & UY \end{array}$$

Thus, Fg has the left lifting property (resp. is left orthogonal) with respect to f if and only if Uf has the right lifting property (resp. is right orthogonal) with respect to g. ■

Definition A.1.9. A **functorial factorisation system** on a category $\mathcal{C}$ is a pair of functors $L, R : [\mathbb{2}, \mathcal{C}] \to [\mathbb{2}, \mathcal{C}]$ satisfying the following equations:

$$\mathrm{dom} \circ L = \mathrm{dom} \qquad \mathrm{codom} \circ L = \mathrm{dom} \circ R \qquad \mathrm{codom} \circ R = \mathrm{codom}$$

Here, dom and codom are considered as functors $[\mathbb{2}, \mathcal{C}] \to \mathcal{C}$. A **functorial weak** (resp. **orthogonal**) **factorisation system** on $\mathcal{C}$ is a weak (resp. orthogonal) factorisation system $(\mathcal{L}, \mathcal{R})$ together with a functorial factorisation system (L, R) such that $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$ for all morphisms f in $\mathcal{C}$.

Proposition A.1.10. *Any orthogonal factorisation system can be extended to a functorial one.*

Proof. For each morphism f in a category $\mathcal{C}$ with an orthogonal factorisation system $(\mathcal{L}, \mathcal{R})$, choose a factorisation $f = Rf \circ Lf$ with $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$. Given a commutative square in $\mathcal{C}$, say

$$\begin{array}{ccc} Z & \xrightarrow{g} & W \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[f]{} & Y \end{array}$$

the lifting property ensures that the dashed arrow in the diagram below exists,

$$\begin{array}{ccccc} Z & \xrightarrow{Lg} & \bullet & \xrightarrow{Rg} & W \\ {\scriptstyle z}\downarrow & & \downarrow & & \downarrow{\scriptstyle w} \\ X & \xrightarrow[Lf]{} & \bullet & \xrightarrow[Rf]{} & Y \end{array}$$

and orthogonality ensures uniqueness and hence functoriality. ■

Proposition A.1.11. *Let A be an object in a category $\mathcal{C}$ and let $\Sigma_A : \mathcal{C}_{/A} \to \mathcal{C}$ be the projection from the slice category.*

(i) *For each functorial factorisation system (L, R) on $\mathcal{C}$, there exists a unique functorial factorisation system (L_A, R_A) on $\mathcal{C}_{/A}$ such that*

$$[\mathbb{2}, \Sigma_A] \circ L_A = L \circ [\mathbb{2}, \Sigma_A] \qquad [\mathbb{2}, \Sigma_A] \circ R_A = R \circ [\mathbb{2}, \Sigma_A]$$

where $[\mathbb{2}, \Sigma_A] : [\mathbb{2}, \mathcal{C}_{/A}] \to [\mathbb{2}, \mathcal{C}]$ is the evident induced functor.

(ii) *If* (L, R) *is part of a functorial weak or orthogonal factorisation system on* $\mathcal{C}$*, then* (L_A, R_A) *is compatible with the induced weak or orthogonal factorisation system on* $\mathcal{C}_{/A}$ *as well.*

Proof. Obvious. ■

A.2 Relative categories

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.2.1. A **relative category** $\mathcal{C}$ consists of a category $\operatorname{und} \mathcal{C}$ and a subcategory $\operatorname{weq} \mathcal{C}$ such that $\operatorname{ob} \operatorname{und} \mathcal{C} = \operatorname{ob} \operatorname{weq} \mathcal{C}$. We say $\operatorname{und} \mathcal{C}$ is the **underlying category** of $\mathcal{C}$, and that the morphisms in $\operatorname{weq} \mathcal{C}$ are the **weak equivalences** in $\mathcal{C}$.

Remark A.2.2. The subcategory $\operatorname{weq} \mathcal{C}$ is entirely determined by $\operatorname{mor} \operatorname{weq} \mathcal{C}$, so a relative category may equivalently be defined as a category equipped with a distinguished subset of morphisms closed under composition and containing all the identity morphisms.

For brevity, we will write $\operatorname{ob} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, $\operatorname{mor} \mathcal{C}$ for $\operatorname{ob} \operatorname{und} \mathcal{C}$, and we may occasionally abuse notation and write $\operatorname{weq} \mathcal{C}$ instead of $\operatorname{mor} \operatorname{weq} \mathcal{C}$.

Remark A.2.3. Every category $\mathcal{C}$ can be endowed with the structure of a relative category in two ways: we can make it into a **minimal relative category** $\min \mathcal{C}$ by taking $\operatorname{weq} \min \mathcal{C}$ to be the set of identity morphisms in $\mathcal{C}$; or we could make it into a **maximal relative category** $\max \mathcal{C}$ by taking $\operatorname{weq} \max \mathcal{C} = \operatorname{mor} \mathcal{C}$. We may also define the **minimal saturated relative category** $\min^+ \mathcal{C}$ by taking $\operatorname{weq} \min^+ \mathcal{C}$ to be the set of all isomorphisms in $\mathcal{C}$.

Definition A.2.4. Given a relative category $\mathcal{C}$, the **opposite relative category** $\mathcal{C}^{\mathrm{op}}$ is defined by $\operatorname{und} \mathcal{C}^{\mathrm{op}} = (\operatorname{und} \mathcal{C})^{\mathrm{op}}$ and $\operatorname{weq} \mathcal{C}^{\mathrm{op}} = (\operatorname{weq} \mathcal{C})^{\mathrm{op}}$.

Definition A.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be relative categories. A **relative functor** $\mathcal{C} \to \mathcal{D}$ is a functor $\operatorname{und} \mathcal{C} \to \operatorname{und} \mathcal{D}$ that sends weak equivalences in $\mathcal{C}$ to weak equivalences in $\mathcal{D}$. The **relative functor category** $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ is the full subcategory of $[\operatorname{und} \mathcal{C}, \operatorname{und} \mathcal{D}]$ spanned by the relative functors, and the weak equivalences in $[\mathcal{C}, \mathcal{D}]_{\mathrm{h}}$ are defined to be the natural transformations that are componentwise weak equivalences in $\mathcal{D}$.

Definition A.2.6. Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$. A **localisation of $\mathcal{C}$ away from $\mathcal{W}$** is a category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\gamma : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ with the following universal property:

- Given a functor $F : \mathcal{C} \to \mathcal{D}$ such that Ff is an isomorphism for all f in $\mathcal{W}$, there exists a unique functor $\overline{F} : \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{D}$ such that $\overline{F}\gamma = F$.

Remark A.2.7. The universal property in the above definition is strict; as such, $\mathcal{C}[\mathcal{W}^{-1}]$ is unique up to unique isomorphism. Nonetheless, $\mathcal{C}[\mathcal{W}^{-1}]$ automatically has a 2-universal property: if $F, G : \mathcal{C} \to \mathcal{D}$ both factor through $\mathcal{C}[\mathcal{W}^{-1}]$, then so do all natural transformations $F \Rightarrow G$.

Proposition A.2.8. *If $\mathcal{C}$ is a* **U***-small category, then there exists a* **U***-small category with the universal property of $\mathcal{C}[\mathcal{W}^{-1}]$.*

Proof. Use the general adjoint functor theorem. ■

Definition A.2.9. The **homotopy category** of a relative category $\mathcal{C}$ is a localisation of $\operatorname{und} \mathcal{C}$ away from $\operatorname{weq} \mathcal{C}$ and is denoted $\operatorname{Ho} \mathcal{C}$. A **semi-saturated relative category** is a relative category in which every isomorphism is a weak equivalence. A **saturated relative category** is a relative category $\mathcal{C}$ such that the weak equivalences in $\mathcal{C}$ are precisely the ones that become isomorphisms in $\operatorname{Ho} \mathcal{C}$.

Remark A.2.10. Obviously, there is no loss of generality in considering semi-saturated relative categories and their homotopy categories instead of localisations $\mathcal{C}[\mathcal{W}^{-1}]$ for arbitrary subsets $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$.

Remark A.2.11. Clearly, every saturated relative category is semi-saturated, and a minimal saturated relative category is indeed saturated in the sense above.

Definition A.2.12. Let $\mathcal{C}$ be a category and let $\mathcal{W}$ be a subset of $\operatorname{mor} \mathcal{C}$. The **2-out-of-3 property** for $\mathcal{W}$ says:

- Given any two morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, if any two of f, g, or $g \circ f$ are in $\mathcal{W}$, then all of them are.

The **2-out-of-6 property** for $\mathcal{W}$ says:

- Given any three morphisms $f : X \to Y$, $g : Y \to Z$, $h : Y \to Z$ in $\mathcal{C}$, if both $h \circ g$ and $g \circ f$ are in $\mathcal{W}$, then so too are f, g, h, and $h \circ g \circ f$.

Lemma A.2.13. *Let $\mathcal{C}$ be a category and let $\mathcal{W} \subseteq \operatorname{mor} \mathcal{C}$.*

(i) *If $\mathcal{W}$ has the 2-out-of-6 property, then it also has the 2-out-of-3 property.*

(ii) *The set of all isomorphisms in $\mathcal{C}$ has the 2-out-of-6 property.*

(iii) *If $F : \mathcal{C}' \to \mathcal{C}$ is a functor and $\mathcal{W}$ has either the 2-out-of-3 property or the 2-out-of-6 property, then $F^{-1}\mathcal{W}$ has the same property.*

Proof. (i). Consider the three cases $f = \mathrm{id}$, $g = \mathrm{id}$, $h = \mathrm{id}$ in turn.

(ii). If $h \circ g$ and $g \circ f$ are isomorphisms, then g must be split epic and split monic; thus g itself is an isomorphism, hence so too are f and h.

(iii). Obvious. ■

Corollary A.2.14. *If $\mathcal{C}$ is a saturated relative category, then* $\operatorname{weq} \mathcal{C}$ *has the 2-out-of-6 property.* ■

Lemma A.2.15. *Let* **RelCat** *be the category of* **U**-*small relative categories and relative functors, let* **SsRelCat** *be the full subcategory of semi-saturated relative categories, and let* **Cat** *be the category of* **U**-*small categories and functors.*

(i) **RelCat** *has finite products and is cartesian closed, where the product of $\mathcal{C}$ and $\mathcal{D}$ is the cartesian product $\mathcal{C} \times \mathcal{D}$ with weak equivalences taken componentwise, and the exponential of $\mathcal{E}$ by $\mathcal{D}$ is the relative functor category $[\mathcal{D}, \mathcal{E}]_{\mathrm{h}}$; moreover,* **SsRelCat** *is closed under finite products in* **RelCat**, *and if $\mathcal{E}$ is semi-saturated, then so is $[\mathcal{D}, \mathcal{E}]_{\mathrm{h}}$.*

(ii) *If $\mathcal{D}$ is an ordinary category and $\mathcal{E}$ is a relative category, then the inclusion*

$$\operatorname{und} [\min \mathcal{D}, \mathcal{E}]_{\mathrm{h}} \hookrightarrow [\mathcal{D}, \operatorname{und} \mathcal{E}]$$

is an isomorphism of ordinary categories, natural in $\mathcal{D}$ and $\mathcal{E}$; and if $\mathcal{E}$ is moreover semi-saturated, then the inclusion

$$\operatorname{und} \left[\min^{+} \mathcal{D}, \mathcal{E}\right]_{\mathrm{h}} \hookrightarrow [\mathcal{D}, \operatorname{und} \mathcal{E}]$$

is also an isomorphism of ordinary categories, natural in $\mathcal{D}$ and $\mathcal{E}$.

(iii) *If $\mathcal{D}$ is a relative category and $\mathcal{E}$ is an ordinary category, then the canonical relative functor*

$$\min^{+} [\operatorname{Ho} \mathcal{D}, \mathcal{E}] \to \left[\mathcal{D}, \min^{+} \mathcal{E}\right]_{\mathrm{h}}$$

is an isomorphism of relative categories, natural in $\mathcal{D}$ and $\mathcal{E}$; in particular, the **U***-small minimal saturated relative categories form an exponential ideal in* **RelCat***.*

(iv) *The functors* min : **Cat** → **RelCat** *and* $\min^+$: **Cat** → **SsRelCat** *are cartesian closed functors, i.e. they preserves finite products, and the canonical relative functors*

$$\min\,[\mathcal{D},\mathcal{E}] \to [\min \mathcal{D}, \min \mathcal{E}]_{\mathrm{h}}$$
$$\min^+\,[\mathcal{D},\mathcal{E}] \to \left[\min^+ \mathcal{D}, \min^+ \mathcal{E}\right]_{\mathrm{h}}$$

are isomorphisms of relative categories, natural in $\mathcal{D}$ and $\mathcal{E}$.

(v) *If $\mathcal{C}$ and $\mathcal{D}$ are relative categories, then the canonical functor*

$$\mathrm{Ho}(\mathcal{C} \times \mathcal{D}) \to \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{D})$$

is an isomorphism of ordinary categories, natural in $\mathcal{C}$ and $\mathcal{D}$.

Proof. Claims (i) – (iv) are straightforward from the definitions. For claim (v), we simply observe that we have a chain of isomorphisms

$$\begin{aligned}
\mathrm{Fun}(\mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{D}), \mathcal{E}) &\cong \mathrm{Fun}(\mathrm{Ho}\,\mathcal{C}, [\mathrm{Ho}\,\mathcal{D}, \mathcal{E}]) \\
&\cong \mathrm{RelFun}\big(\mathcal{C}, \min^+ [\mathrm{Ho}\,\mathcal{D}, \mathcal{E}]\big) \\
&\cong \mathrm{RelFun}\big(\mathcal{C}, \left[\mathcal{D}, \min^+ \mathcal{E}\right]_{\mathrm{h}}\big) \\
&\cong \mathrm{RelFun}\big(\mathcal{C} \times \mathcal{D}, \min^+ \mathcal{E}\big) \\
&\cong \mathrm{Fun}(\mathrm{Ho}(\mathcal{C} \times \mathcal{D}), \mathcal{E})
\end{aligned}$$

natural in $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, so by taking $\mathcal{E} = \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{D})$ and $\mathcal{E} = \mathrm{Ho}(\mathcal{C} \times \mathcal{D})$ in turn, we see that Ho preserves binary products. ■

Proposition A.2.16. *Let* **SsRelCat** *be the category of* **U***-small semi-saturated relative categories and relative functors, and let* **Cat** *be the category of* **U***-small categories and functors. There is then a string of adjoint functors*

$$\mathrm{Ho} \dashv \min^+ \dashv \mathrm{und} \dashv \max \dashv \mathrm{weq} : \mathbf{SsRelCat} \to \mathbf{Cat}$$

where Ho *sends a relative category to its homotopy category,* $\min^+$ *makes an ordinary category into a minimal saturated relative category,* und *sends a relative category to its underlying category,* max *makes an ordinary category into a* *maximal relative category, and* weq *sends a relative category to its subcategory of weak equivalences. Moreover, both* $\min^+$ *and* max *are fully faithful, and* Ho *preserves finite products.*

Proof. Obvious. ■

Definition A.2.17. A **zigzag type** is a relative category T where und T is the free category on an inhabited finite planar graph of the form

$$\bullet \;\text{———}\; \bullet \;\text{———}\; \cdots \;\text{———}\; \bullet \;\text{———}\; \bullet$$

where the edges are arrows that point either left or right, and weq T consists of all identities and all composites of left-pointing arrows. A **morphism of zigzag types** is a relative functor that maps the leftmost object to the leftmost object and the rightmost object to the rightmost object. We write $\mathbf{T}$ for the category of zigzag types.[1]

A **zigzag** of type T in a relative category C is a relative functor $T \to C$. Given objects X and Y in C, we denote by $C^T(X,Y)$ the category whose objects are the zigzags starting at X and ending at Y and whose morphisms are commutative diagrams in C of the form

$$\begin{array}{ccccccccc}
X & \text{———} & \bullet & \text{———} & \cdots & \text{———} & \bullet & \text{———} & Y \\
{\scriptstyle \mathrm{id}}\downarrow & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
X & \text{———} & \bullet & \text{———} & \cdots & \text{———} & \bullet & \text{———} & Y
\end{array}$$

where the rows are zigzags of type T and the unmarked columns are weak equivalences.

Remark A.2.18. It is clear that $C^T(X,Y)$ is a subcategory of the relative functor category $[T,C]_{\mathrm{h}}$. Thus, if C is a $\mathbf{U}$-small relative category, precomposition makes the assignment $T \mapsto C^T(X,Y)$ into a functor $\mathbf{T}^{\mathrm{op}} \to \mathbf{Cat}$, which we denote by $C^*(X,Y)$. The Grothendieck construction applied to this functor yields the following $\mathbf{U}$-small category $C^{(\mathbf{T})}(X,Y)$:

- Its objects are pairs (T,f), where T is a zigzag type and f is a zigzag of type T in C.

[1] Warning: This is the *opposite* of the category $\mathbf{T}$ defined in [DHKS, § 34].

- A morphism $(T', f') \to (T, f)$ is a pair (α, β) where $\alpha : T' \to T$ is a morphism in $\mathbf{T}$ and $\beta : f' \to \alpha^* f$ is a morphism in $\mathcal{C}^{T'}(X, Y)$.
- The composite of a pair of morphisms $(\alpha', \beta') : (T'', f'') \to (T', f')$ and $(\alpha, \beta) : (T', f') \to (T, f)$ is given by $(\alpha \circ \alpha', \alpha'^* \beta \circ \beta')$.

There is an evident projection functor $\mathcal{C}^{(\mathbf{T})}(X, Y) \to \mathbf{T}$, and by construction it is a Grothendieck fibration with a canonical splitting.

Example A.2.19. If $f : X \to Y$ is a weak equivalence in a relative category $\mathcal{C}$, then we have commutative diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{id}} & X & \xleftarrow{\mathrm{id}} & X \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\ X & \xrightarrow[f]{} & Y & \xleftarrow[f]{} & X \end{array} \qquad \begin{array}{ccccc} Y & \xleftarrow{f} & X & \xrightarrow{f} & Y \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\ Y & \xleftarrow[\mathrm{id}]{} & Y & \xrightarrow[\mathrm{id}]{} & Y \end{array}$$

and these correspond to morphisms of zigzags in $\mathcal{C}$.

Theorem A.2.20. *Let X and Y be objects in a relative category $\mathcal{C}$.*

(i) *For each zigzag type T, the map that sends an object in $\mathcal{C}^T(X, Y)$ to the corresponding composite in* $\operatorname{Ho}\mathcal{C}(X, Y)$ *is a functor when the latter is regarded as a discrete category.*

(ii) *The functors described above constitute a jointly surjective cocone from the diagram $\mathcal{C}^*(X, Y)$ to* $\operatorname{Ho}\mathcal{C}(X, Y)$.

(iii) *The induced functor $\mathcal{C}^{(\mathbf{T})}(X, Y) \to$* $\operatorname{Ho}\mathcal{C}(X, Y)$ *is surjective, and moreover two objects in $\mathcal{C}^{(\mathbf{T})}(X, Y)$ become equal in* $\operatorname{Ho}\mathcal{C}$ *if and only if they are in the same connected component.*

Proof. All obvious except for the last part of claim (iii), for which we refer to paragraphs 33.8 and 33.10 in [DHKS]. ■

Definition A.2.21. Two objects in a relative category are **weakly equivalent** if they can be connected by a *zigzag* of weak equivalences.

Remark A.2.22. If X and Y are weakly equivalent in a relative category $\mathcal{C}$, then they are isomorphic in $\operatorname{Ho}\mathcal{C}$.

A.3 Kan extensions

Prerequisites. § 0.1.

In this section we use the explicit universe convention.

Definition A.3.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{E}$ be two functors. A **left Kan extension** (resp. **right Kan extension**) of G along F is an initial (resp. terminal) object of the category $(G \downarrow F^*)$ (resp. $(F^* \downarrow G)$) described below:

- The objects are pairs (H, α) where H is a functor $\mathcal{D} \to \mathcal{E}$ and α is a natural transformation of type $G \Rightarrow HF$ (resp. $HF \Rightarrow G$).
- The morphisms $(H', \alpha') \to (H, \alpha)$ are those natural transformations $\beta : H' \Rightarrow H$ such that $\beta F \bullet \alpha' = \alpha$ (resp. $\alpha \bullet \beta F = \alpha'$).

¶ A.3.2. Clearly, Kan extensions are unique up to unique isomorphism if they exist. We write $(\operatorname{Lan}_G F, \eta)$ for the left Kan extension of F along G, and $(\operatorname{Ran}_G F, \varepsilon)$ for the right Kan extension of F along G.

Definition A.3.3. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{C} \to \mathcal{E}$, and $L : \mathcal{E} \to \mathcal{F}$ be three functors. We say L **preserves** a left (resp. right) Kan extension (H, α) of F along G if $(KH, K\alpha)$ is a left (resp. right) Kan extension of KF along G.

Let **Set** be the category of **U**-small sets, and suppose $\mathcal{E}$ is locally **U**-small. We say a left Kan extension $(\operatorname{Lan}_G F, \eta)$ is **pointwise** if it is preserved by all functors of the form $\mathcal{E}(-, E) : \mathcal{E} \to \mathbf{Set}^{\mathrm{op}}$; dually, we say a right Kan extension $(\operatorname{Ran}_G F, \varepsilon)$ is **pointwise** if it is preserved by all functors of the form $\mathcal{E}(E, -) : \mathcal{E} \to \mathbf{Set}$. If a Kan extension is preserved by *all* functors, then it is said to be **absolute**.

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