Algebraic Theories

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PREFACE

These notes are intended as an introduction to the category-theoretic approach to algebraic theories, with a view toward the generalised algebraic geometry developed by Durov [2007].

I shall assume the reader is familiar with basic category theory. The usual reference is Mac Lane's *Categories for the working mathematician* [CWM]. At minimum, the reader should know the following topics: categories, functors, and natural transformations [Ch. I]; basic constructions on categories [Ch. II]; universal properties, limits, and the Yoneda lemma [Ch. III]; adjunctions [Ch. IV].

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MONADS

Monads[1] are a powerful generalisation of the monoids of abstract algebra and are ubiquitous in structural mathematics: indeed, every adjoint pair of functors induces a monad, as discovered by Huber [1961]. Although monads were first introduced by Godement [1958, App., № 3] for the purpose of generating cosimplicial resolutions for cohomology, it is now understood that there is a deep connection between monads on **Set** (and similar categories) and varieties (in the sense of universal algebra).

1 Introduction

First of all, what *is* a monad?

Definition 1.1.1. Let $\mathcal{C}$ be any category. A **monad** $\mathbb{T}$ on $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ together with natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ satisfying the equations

$$\mu \bullet T\eta = \mathrm{id}_T \qquad \mu \bullet \eta T = \mathrm{id}_T \qquad \mu \bullet T\mu = \mu \bullet \mu T$$

where $\bullet$ denotes vertical composition of natural transformations. Equivalently, the three diagrams

$$\begin{tikzcd} TM \arrow[r, "T\eta_M"] \arrow[dr, "\mathrm{id}"'] & T^2M \arrow[d, "\mu_M"] \\ & TM \end{tikzcd} \qquad \begin{tikzcd} TM \arrow[r, "\eta_{TM}"] \arrow[dr, "\mathrm{id}"'] & T^2M \arrow[d, "\mu_M"] \\ & TM \end{tikzcd} \qquad \begin{tikzcd} T^3M \arrow[r, "T\mu_M"] \arrow[d, "\mu_{TM}"'] & T^2M \arrow[d, "\mu_M"] \\ T^2M \arrow[r, "\mu_M"'] & TM \end{tikzcd}$$

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[1] Or 'triples', according to some authors, e.g. Barr and Wells [TTT].

commute for every object M in C. We may sometimes abuse notation and simply say that T is a monad, leaving η and μ unspecified.

The three axioms are sometimes called the 'right unit,' 'left unit,' and 'associativity' (respectively), by analogy with the axioms for ordinary monoids. Also, just like ordinary monoids, there is a notion of an object equipped with a (left) action of a monad:

Definition 1.1.2. Let C be a category, and let $\mathbb{T} = (T, \eta, \mu)$ be a monad on C. A $\mathbb{T}$-**module**[1] is a pair (M, α) consisting of an object M of C and a morphism $\alpha : TM \to M$ satisfying the following equations:

$$\alpha \circ \eta_M = \mathrm{id}_M \qquad \alpha \circ T\alpha = \alpha \circ \mu_M$$

$$\begin{tikzcd}
M \arrow[r, "\eta_M"] \arrow[dr, "\mathrm{id}"'] & TM \arrow[d, "\alpha"] \\
& TM
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2M \arrow[r, "T\alpha"] \arrow[d, "\mu_M"'] & TM \arrow[d, "\alpha"] \\
TM \arrow[r, "\alpha"'] & TM
\end{tikzcd}$$

Given two $\mathbb{T}$-modules (M, α) and (M', α'), a **homomorphism of $\mathbb{T}$-modules** $f : (M, \alpha) \to (M', \alpha')$ is a morphism $f : M \to M'$ in C such that the equation below holds:

$$f \circ \alpha = \alpha' \circ Tf$$

$$\begin{tikzcd}
TM \arrow[r, "Tf"] \arrow[d, "\alpha"'] & TM' \arrow[d, "\alpha'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

The **Eilenberg–Moore category for** $\mathbb{T}$ is the category of all $\mathbb{T}$-modules and is denoted by $C^{\mathbb{T}}$.

Extending the monoid analogy, the two axioms for T-modules are sometimes called '(left) unitality' and 'compatibility' (respectively), and the single axiom for T-module homomorphisms corresponds to the preservation of the action.

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[1] Strictly speaking, this is the definition of a *left* $\mathbb{T}$-module, but we will not encounter right $\mathbb{T}$-modules here.

Example 1.1.3. Let $C = \mathbf{Set}$, and let A be a monoid. We define an endofunctor $T : C \to C$ by

$$\begin{aligned} TM &= A \times M \\ Tf &= \mathrm{id}_A \times f \end{aligned}$$

We also define natural transformations $\eta : \mathrm{id} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ by

$$\begin{aligned} \eta_M(m) &= (e, m) \\ \mu_M(a, b, m) &= (a \cdot b, m) \end{aligned}$$

where e is the unit element of A and $\cdot$ is the monoid operation. It is easy to verify that $\mathbb{T} = (T, \eta, \mu)$ is a monad: the unit and associativity axioms for $\mathbb{T}$ correspond precisely to the unit and associativity axioms for A. Similarly, the Eilenberg–Moore category $C^{\mathbb{T}}$ is easily seen to be isomorphic to the category of left A-sets (i.e. sets equipped with a left action of A).

We may also formally dualise these notions by reversing arrows.

Definition 1.1.4. A **comonad** $\mathbb{G}$ on a category C is consists of an endofunctor $G : C \to C$ together with natural transformations $\varepsilon : G \Rightarrow \mathrm{id}_C$ and $\delta : G \Rightarrow G^2$ such that $\mathbb{G}^{\mathrm{op}} = (G^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \delta^{\mathrm{op}})$ is a monad on C^{op}. An $\mathbb{G}$-**comodule** is a pair (C, α) consisting of an object C of C and a morphism $\alpha : C \to WC$ such that (C, α) is a $\mathbb{G}^{\mathrm{op}}$-module in C^{op}.

Example 1.1.5. Let $\mathcal{B}$ be a monoidal category, and let B be the unit of the tensor product $\otimes$. For convenience, we will assume $\mathcal{B}$ is a strict monoidal category. Let E be a comonoid for $\otimes$: that is, we have a counit $p : E \to B$ and a comultiplication $\Delta : E \to E \otimes E$, satisfying the counit axioms

$$(\mathrm{id}_E \otimes p) \circ \Delta = \mathrm{id}_E \qquad\qquad (p \otimes \mathrm{id}_E) \circ \Delta = \mathrm{id}_E$$

and the coassociativity axiom

$$(\mathrm{id}_E \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_E) \circ \Delta$$

This gives rise to a comonad on $\mathcal{B}$ as follows. Let $W : \mathcal{B} \to \mathcal{B}$ be the functor defined by

$$WC = E \otimes C \qquad\qquad Wf = \mathrm{id}_E \otimes f$$

We define two natural transformations, $\varepsilon : W \Rightarrow \mathrm{id}_{\mathcal{B}}$ and $\delta : W \Rightarrow W^2$, by

$$\varepsilon_C = p \otimes \mathrm{id}_C \qquad\qquad \delta_C = \Delta \otimes \mathrm{id}_C$$

The triple (W, ε, δ) is a comonad precisely because (E, p, Δ) is a comonoid.

Let us now recall the definition of an adjunction of categories.

Definition 1.1.6. Let $\mathcal{D}$ and C be categories. An **adjunction of categories** consists of two functors, $F : C \to \mathcal{D}$ and $U : \mathcal{D} \to C$ and two natural transformations, $\eta : \mathrm{id}_C \Rightarrow UF$ and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$, satisfying the **triangle identities**:

$$F\varepsilon \bullet \eta F = \mathrm{id}_F \qquad\qquad \varepsilon U \bullet \varepsilon U = \mathrm{id}_U$$

The functor F is said to be **left adjoint** to U, and U is said to be **right adjoint** to F. The **unit** of the adjunction is the natural transformation η, and the **counit** is ε. We often abbreviate this by writing

$$F \dashv U : \mathcal{D} \to C$$

and omit mention of η and ε.

In the introduction, it was noted that every adjunction of categories induces a monad. To be precise:

Theorem 1.1.7 (Huber). *Let $F \dashv U : \mathcal{D} \to C$ be an adjunction of categories, with unit $\eta : \mathrm{id}_{\mathcal{B}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{E}}$. Define $T = UF$ and $\mu = U\varepsilon F$. Then, the triplet (T, η, μ) is a monad on C. Dually, if $W = FU$ and $\delta = F\eta U$, then the triplet (W, ε, δ) is a comonad on $\mathcal{D}$.*

Proof. The proof is straightforward algebra of natural transformations. For the left unit axiom,

$$\begin{aligned} \mu \bullet \eta T &= U\varepsilon F \bullet \eta UF && \text{by definition} \\ &= (U\varepsilon \bullet \eta U)F && \text{by regrouping} \\ &= \mathrm{id}_U F = \mathrm{id}_T && \text{by the right triangle identity} \end{aligned}$$

and similarly, for the right unit axiom,

$$\begin{aligned} \mu \bullet T\eta &= U\varepsilon F \bullet UF\eta && \text{by definition} \\ &= U(\varepsilon F \bullet F\eta) && \text{by regrouping} \\ &= U\mathrm{id}_F = \mathrm{id}_T && \text{by the left triangle identity} \end{aligned}$$

Finally, for the associativity axiom, we appeal to the naturality of ε:

$$\begin{aligned}
\mu \bullet T\mu &= U\varepsilon F \bullet UFU\varepsilon F && \text{by definition} \\
&= U(\varepsilon \bullet FU\varepsilon)F && \text{by regrouping} \\
&= U(\varepsilon \bullet \varepsilon FU)F && \text{by naturality} \\
&= U\varepsilon F \bullet U\varepsilon FUF && \text{by regrouping} \\
&= \mu \bullet \mu T && \text{by definition} \qquad \blacksquare
\end{aligned}$$

The claim for the comonad follows by duality.

In fact, the converse is also true: every monad arises from an adjunction, and in two universal ways. First, we look at the construction given by Eilenberg and Moore [1965]:

Theorem 1.1.8 (Eilenberg–Moore). *Let* $\mathbb{T} = (T, \eta, \mu)$ *be a monad on* $\mathcal{C}$*, and let* $\mathcal{C}^{\mathbb{T}}$ *be the Eilenberg–Moore category of* $\mathbb{T}$*-modules. Let* $F^{\mathbb{T}} : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$ *be the free* $\mathbb{T}$*-module functor defined by*

$$F^{\mathbb{T}} X = (TX, \mu_X) \qquad\qquad F^{\mathbb{T}} f = Tf$$

and let $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ *be the forgetful functor defined by*

$$U^{\mathbb{T}}(M, \alpha) = M \qquad\qquad U^{\mathbb{T}} f = f$$

There is an adjunction

$$F^{\mathbb{T}} \dashv U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$$

and (T, η, μ) *is precisely the monad induced by* $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$*. Moreover, it is universal in the following sense: for each adjunction* $F \dashv U : \mathcal{D} \to \mathcal{C}$ *inducing the monad* $\mathbb{T}$*, there is a* unique *functor* $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ *such that*

$$KF = F^{\mathbb{T}} \qquad\qquad U = U^{\mathbb{T}} K$$

Proof. First, we must check that we have really defined an adjunction. Let X be an object in $\mathcal{C}$; then the left unit and associativity axioms for $\mathbb{T}$ imply the unit and compatibility axiom for (TX, μ_X), so $F^{\mathbb{T}} X$ is indeed a $\mathbb{T}$-module. Naturality of μ implies that a morphism $f : X \to Y$ in $\mathcal{C}$ gives a homomorphism $Tf : (TX, \mu_X) \to (TY, \mu_Y)$, and functoriality of T makes $F^{\mathbb{T}}$ into a functor. On the other hand, it is clear that $U^{\mathbb{T}}$ is a functor.

The compatibility axiom for $\mathbb{T}$-modules gives us a natural transformation $\varepsilon^{\mathbb{T}} : F^{\mathbb{T}} U^{\mathbb{T}} \Rightarrow \mathrm{id}_{\mathcal{C}^{\mathbb{T}}}$, given in components by the structural morphism of the $\mathbb{T}$-module:

$$\varepsilon^{\mathbb{T}}_{(A,\alpha)} = \alpha$$

We will now show that $F^{\mathbb{T}}$ is left adjoint to $U^{\mathbb{T}}$ with unit η and counit $\varepsilon^{\mathbb{T}}$. The right unit axiom for $\mathbb{T}$ implies the left triangle identity:

$$\begin{aligned}\left(\varepsilon F^{\mathbb{T}} \bullet F^{\mathbb{T}} \eta\right)_X &= \varepsilon_{\left(TX,\mu_X\right)} \circ T\eta_X \\ &= \mu_X \circ T\eta_X = \mathrm{id}_X\end{aligned}$$

On the other hand, the unit axiom for $\mathbb{T}$-modules gives the right triangle identity:

$$\begin{aligned}\left(U^{\mathbb{T}} \varepsilon \bullet \eta U^{\mathbb{T}}\right)_{(A,\alpha)} &= \varepsilon_{(A,\alpha)} \circ \eta_A \\ &= \alpha \circ \eta_A = \mathrm{id}_A\end{aligned}$$

Thus, we have the required adjunction.

Finally, suppose $F \dashv U : \mathcal{D} \to \mathcal{C}$ is any adjunction with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$, such that $UF = T$ and $U\varepsilon F = \mu$. For $A \in \mathrm{ob}\,\mathcal{D}$, observe that UA together with the morphism $U\varepsilon_A : UFUA \to UA$ satisfy the conditions to be a $\mathbb{T}$-module, so we may set $KA = (UA, U\varepsilon_A)$.

$$\begin{tikzcd} UA \arrow[r, "\eta_{UA}"] \arrow[dr, "\mathrm{id}"'] & TUA \arrow[d, "U\varepsilon_A"] \\ & UA \end{tikzcd} \qquad \begin{tikzcd} T^2UA \arrow[r, "TU\varepsilon_A"] \arrow[d, "\mu_{UA}"'] & TUA \arrow[d, "U\varepsilon_A"] \\ TUA \arrow[r, "U\varepsilon_A"'] & UA \end{tikzcd}$$

Similarly, for $f \in \mathcal{D}(A, B)$, the diagram

$$\begin{tikzcd} TUA \arrow[r, "TUf"] \arrow[d, "U\varepsilon_A"'] & TUB \arrow[d, "U\varepsilon_B"] \\ UA \arrow[r, "Uf"'] & UB \end{tikzcd}$$

commutes by naturality of ε, so if we set $Kf = Uf$, we obtain the desired functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$. If $K' : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is any other functor such that $K'F = F^{\mathbb{T}}$ and $U = U^{\mathbb{T}} K'$, then we must have $K'f = U^{\mathbb{T}} K' f = Uf$, and since $TU\varepsilon_A$ is a split epimorphism, $K'F = F^{\mathbb{T}}$ and naturality of ε implies $K'A = (UA, U\varepsilon_A)$; thus, $K' = K$, as required. ■

The construction of Kleisli [1965] has the dual universal property:

Theorem 1.1.9 (Kleisli). *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on $\mathcal{C}$. There exists a category $\mathcal{C}_\mathbb{T}$ and an adjunction*

$$F_\mathbb{T} \dashv U_\mathbb{T} : \mathcal{C}_\mathbb{T} \to \mathcal{C}$$

inducing the monad $\mathbb{T}$, such that for any other adjunction $F \dashv U : \mathcal{D} \to \mathcal{C}$ inducing the same monad, there is a unique *functor $K : \mathcal{C}_\mathbb{T} \to \mathcal{D}$ such that*

$$F = K F_\mathbb{T} \qquad\qquad UK = U_\mathbb{T}$$

and in particular, for $\mathcal{D} = \mathcal{C}^\mathbb{T}$, $F = F^\mathbb{T}$ and $U = U^\mathbb{T}$ as in the Eilenberg–Moore theorem, the functor K is fully faithful and injective on objects. The category $\mathcal{C}_\mathbb{T}$ is called the **Kleisli category** *for $\mathbb{T}$ and is unique up to unique isomorphism.*

Proof. The Kleisli category is notable as the primary example of a category with a “funny” composition law. To be precise, $\mathcal{C}_\mathbb{T}$ is the category whose objects are those of $\mathcal{C}$, but we define $\mathcal{C}_\mathbb{T}(X, Y) = \mathcal{C}(X, TY)$, where we compose $f : X \to TY$ and $g : Y \to TZ$ according to the rule below:

$$g \circ_\mathbb{T} f = \mu_Z \circ Tg \circ f$$

The monad unit $\eta_X : X \to TX$ acts as the identity under this composition law:

$$\begin{aligned} f \circ_\mathbb{T} \eta_X &= \mu_Y \circ Tf \circ \eta_X && \text{by definition} \\ &= \mu_Y \circ \eta_{TX} \circ f && \text{by naturality} \\ &= f && \text{by the left unit law} \end{aligned}$$

$$\begin{aligned} \eta_Y \circ_\mathbb{T} f &= \mu_Y \circ T\eta_Y \circ f && \text{by definition} \\ &= f && \text{by the right unit law} \end{aligned}$$

The partial binary operation $\circ_\mathbb{T}$ is associative because μ is associative and natural: given $h : Z \to TW$, we have

$$\begin{aligned} (h \circ_\mathbb{T} g) \circ_\mathbb{T} f &= \mu_W \circ T(\mu_W \circ Th \circ g) \circ f && \text{by definition} \\ &= (\mu_W \circ T\mu_W) \circ \left(T^2 h \circ Tg \circ f\right) && \text{by regrouping} \\ &= \mu_W \circ \left(\mu_{TW} \circ T^2 h\right) \circ Tg \circ f && \text{by associativity} \\ &= \mu_W \circ Th \circ (\mu_Z \circ Tg \circ f) && \text{by naturality} \\ &= h \circ_\mathbb{T} (g \circ_\mathbb{T} f) && \text{by definition} \end{aligned}$$

as required; so $\mathcal{C}_{\mathbb{T}}$ is indeed a well-defined category.

We define functors $F_{\mathbb{T}} : \mathcal{C} \to \mathcal{C}_{\mathbb{T}}$ and $U_{\mathbb{T}} : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ as follows. For an object X in $\mathcal{C}$, $F_{\mathbb{T}} X = X$, and for an object X in $\mathcal{C}_{\mathbb{T}}$, $U_{\mathbb{T}} X = TX$. For a morphism $f : X \to Y$ in $\mathcal{C}$, we set $Ff = Tf \circ \eta_X$; the unit law for $\mathbb{T}$ implies $F_{\mathbb{T}}$ is a well-defined functor. For $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, we define $U_{\mathbb{T}} = \mu_Y \circ Tf$; the monad axioms for $\mathbb{T}$ then make $U_{\mathbb{T}}$ into a well-defined functor. Clearly, $T = U_{\mathbb{T}} F_{\mathbb{T}}$. We define a natural transformation $\varepsilon_{\mathbb{T}} : F_{\mathbb{T}} U_{\mathbb{T}} \Rightarrow \mathrm{id}_{\mathcal{C}_{\mathbb{T}}}$ by setting $(\varepsilon_{\mathbb{T}})_X = \mathrm{id}_{TX}$; it is then straightforward to check that the triangle identities hold, and so we have $F_{\mathbb{T}} \dashv U_{\mathbb{T}}$ inducing the monad $\mathbb{T}$, as required.

Now, let $F \dashv U : \mathcal{D} \to \mathcal{C}$ be any adjunction inducing the monad $\mathbb{T}$. We define the functor $K : \mathcal{C}_{\mathbb{T}} \to \mathcal{D}$ by setting $KX = FX$ for each object X in $\mathcal{C}_{\mathbb{T}}$, and for each f in $\mathcal{C}_{\mathbb{T}}(X, Y)$, we define $Kf : FX \to FY$ by the formula

$$Kf = \varepsilon_{FY} \circ Ff$$

and observe that this is a well-defined functor: given $g \in \mathcal{C}_{\mathbb{T}}(Y, Z)$, we have

$$\begin{aligned}
K(g \circ_{\mathbb{T}} f) &= K(\mu_Z \circ Tg \circ f) && \text{by definition} \\
&= \varepsilon_{FZ} \circ F(\mu_Z \circ Tg \circ f) && \text{by definition} \\
&= (\varepsilon_{FZ} \circ F\mu_Z) \circ (FTg \circ Ff) && \text{by regrouping} \\
&= (\varepsilon_{FZ} \circ FU\varepsilon_{FZ}) \circ (FUFg \circ Ff) && \text{by hypothesis} \\
&= \varepsilon_{FZ} \circ (\varepsilon_{FUFZ} \circ FUFg) \circ Ff && \text{by naturality} \\
&= (\varepsilon_{FZ} \circ Fg) \circ (\varepsilon_{FY} \circ Ff) && \text{by naturality} \\
&= Kg \circ Kf && \text{by definition}
\end{aligned}$$

and $K\eta_X = \varepsilon_{FX} \circ F\eta_X = \mathrm{id}_{FX}$ by the left triangle identity, as required. Suppose $K' : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}$ is any other functor such that $F = K' F_{\mathbb{T}}$ and $UK' = U_{\mathbb{T}}$. Since $F_{\mathbb{T}}$ is surjective on objects, we must have $K'X = FX$. On the other hand, for $f \in \mathcal{C}_{\mathbb{T}}(X, Y)$, we have

$$\begin{aligned}
K'f &= K'f \circ \varepsilon_{FX} \circ F\eta_X && \text{by the left triangle identity} \\
&= \varepsilon_{FY} \circ FUK'f \circ F\eta_X && \text{by naturality} \\
&= \varepsilon_{FY} \circ FU_{\mathbb{T}} f \circ F\eta_X && \text{by hypothesis} \\
&= \varepsilon_{FY} \circ F(\mu_Y \circ Tf) \circ F\eta_X && \text{by definition} \\
&= (\varepsilon_{FY} \circ FU\varepsilon_{FY}) \circ (FUFf \circ F\eta_X) && \text{by hypothesis} \\
&= \varepsilon_{FY} \circ (\varepsilon_{FUFY} \circ F\eta_{UFY}) \circ Ff && \text{by naturality} \\
&= \varepsilon_{FY} \circ Ff && \text{by the left triangle identity}
\end{aligned}$$

so $K' = K$, as required.

Finally, suppose $\mathcal{D} = \mathcal{C}^{\mathbb{T}}$, with $F = F^{\mathbb{T}}$ and $U = U^{\mathbb{T}}$ as in the Eilenberg–Moore theorem. Then $KX = (TX, \mu_X)$, so K is injective on objects. By the adjunction we know that $\mathbb{T}$-module homomorphisms $(TX, \mu_X) \to (TY, \mu_Y)$ correspond to morphisms $X \to TY$ in $\mathcal{C}$ (by composing with $\eta_X : X \to TX$), and for $f : X \to TY$ we have

$$\mu_Y \circ Tf \circ \eta_X = \mu_Y \circ \eta_{TY} \circ f = f$$

so K is indeed fully faithful. ■

Remark 1.1.10. The last part of the above theorem shows that we could equivalently define the Kleisli category $\mathcal{C}_{\mathbb{T}}$ as the *full* subcategory of $\mathcal{C}^{\mathbb{T}}$ spanned by the image of $F^{\mathbb{T}} : \mathcal{C} \to \mathcal{C}^{\mathbb{T}}$. Thus, we may think of $\mathcal{C}_{\mathbb{T}}$ as a category of *free* $\mathbb{T}$-modules.

2 Monadicity

In this section, we consider the properties of module categories. We will see that these categories inherit some of the good properties of the base category through the free–forgetful adjunction, and this leads to naturally to the problem of finding necessary and sufficient conditions for a category to be equivalent to a category of modules for a monad. More precisely, we seek to characterise these functors:

Definition 1.2.1. A **monadic functor** (resp. **strictly monadic functor**) is a functor $U : \mathcal{D} \to \mathcal{C}$ with the following properties:

- U has a left adjoint $F : \mathcal{C} \to \mathcal{D}$, and
- the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$, where $\mathbb{T}$ is the monad induced by $F \dashv U : \mathcal{D} \to \mathcal{C}$, is fully faithful and essentially surjective on objects (resp. an isomorphism of categories).

Example 1.2.2. Let **Grp** be the category of groups. The forgetful functor U : **Grp** → **Set** is strictly monadic. Indeed, it has a left adjoint F : **Set** → **Grp**, namely the free group functor, and it is not hard to see that a group structure on a set X is the same thing as a $\mathbb{T}$-module structure on X for the monad $\mathbb{T}$

induced by $F \dashv U$: after all, an element of FX is just a formal word and a group structure on X tells us how to evaluate formal words.

Notice that strict monadicity is *not* invariant under equivalence of categories: if U is equivalent to a strictly monadic functor, then U is in general only a monadic functor. For most purposes, a monadic functor is as good as a strictly monadic functor, so we will at first focus on the conditions necessary for monadicity.

Definition 1.2.3. A **conservative functor** is a functor that reflects isomorphisms.

Proposition 1.2.4. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. The forgetful functor $U : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is conservative.*

Proof. Let $f : (A, \alpha) \to (B, \beta)$ be a $\mathbb{T}$-module homomorphism and suppose $f : A \to B$ is an isomorphism in $\mathcal{C}$ with inverse $g : B \to A$. Then,

$$g \circ \beta = g \circ \beta \circ Tf \circ Tg = g \circ f \circ \alpha \circ Tg = \alpha \circ Tg$$

so $g : (B, \beta) \to (A, \alpha)$ is also a $\mathbb{T}$-module homomorphism; thus f is an isomorphism in $\mathcal{C}^{\mathbb{T}}$ with inverse g. ■

Recall that a functor $U : \mathcal{D} \to \mathcal{C}$ that has a left adjoint must preserve all limits that exist in $\mathcal{D}$; however, if U is monadic, then it even *creates* all limits that exist in $\mathcal{C}$:

Proposition 1.2.5. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$. Given a diagram $(A_\bullet, \alpha_\bullet) : \mathcal{J} \to \mathcal{C}^{\mathbb{T}}$, if $\lambda : \Delta B \Rightarrow A_\bullet$ is a limiting cone in $\mathcal{C}$, then there exists a unique $\mathbb{T}$-module structure $\beta : TB \to B$ such that $\lambda : \Delta(B, \beta) \Rightarrow (A_\bullet, \alpha_\bullet)$ is a limiting cone in $\mathcal{C}^{\mathbb{T}}$.*

Proof. First, note that by composing $T\lambda$ with α, we get a cone from TB to $A_\bullet$:

$$\begin{tikzcd}
TB \arrow[r, "T\lambda_i"] \arrow[dr, "T\lambda_j"'] & TA_i \arrow[r, "\alpha_i"] \arrow[d] & A_i \arrow[d] \\
 & TA_j \arrow[r, "\alpha_j"'] & A_j
\end{tikzcd}$$

Thus, there is a unique $\beta : TB \to B$ in $\mathcal{C}$ such that $\alpha_i \circ T\lambda_i = \lambda_i \circ \beta$; in other words, there is at most one $\mathbb{T}$-module structure on B making each $\lambda_i : B \to A_i$

into a $\mathbb{T}$-module homomorphism. We need to check that β *is* a $\mathbb{T}$-module structure:

$$\begin{aligned}
(\lambda_i \circ \beta) \circ \eta_B &= \alpha_i \circ (T\lambda_i \circ \eta_B) && \text{by definition} \\
&= \alpha_i \circ \eta_{A_i} \circ \lambda_i && \text{by naturality} \\
&= \lambda_i && \text{by unitality}
\end{aligned}$$

$$\begin{aligned}
(\lambda_i \circ \beta) \circ \mu_B &= \alpha_i \circ (T\lambda_i \circ \mu_B) && \text{by definition} \\
&= (\alpha_i \circ \mu_{A_i}) \circ T^2\lambda_i && \text{by naturality} \\
&= \alpha_i \circ \left(T\alpha_i \circ T^2\lambda_i\right) && \text{by transitivity} \\
&= (\alpha_i \circ T\lambda_i) \circ T\beta && \text{by definition} \\
&= \lambda_i \circ (\beta \circ T\beta) && \text{by definition}
\end{aligned}$$

Since $\lambda : \Delta B \Rightarrow A_\bullet$ is a limiting cone in C, it follows that $\beta \circ \eta_B = \mathrm{id}_B$ and $\beta \circ \mu_B = \beta \circ T\beta$, as required for a $\mathbb{T}$-module structure.

It remains to be shown that this makes λ into a limiting cone in $C^{\mathbb{T}}$, so let (C, γ) be any $\mathbb{T}$-module, and suppose we have a cone $\nu : \Delta(C, \gamma) \Rightarrow (A_\bullet, \alpha_\bullet)$. Since λ is a limiting cone in C, ν must factor through λ in C uniquely as $\nu_i = \lambda_i \circ f$ for some $f : C \to B$. Now,

$$\lambda_i \circ f \circ \gamma = \nu_i \circ \gamma = \alpha_i \circ T\nu_i = \alpha_i \circ T\lambda_i \circ Tf = \lambda_i \circ \beta \circ Tf$$

for all vertices i in $\mathcal{J}$, so by uniqueness we must have $f \circ \gamma = \beta \circ Tf$; so ν factors through λ uniquely in $C^{\mathbb{T}}$ as well, and therefore λ is a limiting cone in $C^{\mathbb{T}}$. ■

The situation with colimits is only slightly more complicated. Roughly speaking, a monadic functor creates all colimits that the monad preserves:

Proposition 1.2.6. *Let* $\mathbb{T} = (T, \eta, \mu)$ *be a monad on a category* C*. Given a diagram* $(A_\bullet, \alpha_\bullet) : \mathcal{J} \to C^{\mathbb{T}}$*, if both* $\lambda : A_\bullet \Rightarrow \Delta B$ *and* $T\lambda : TA_\bullet \Rightarrow \Delta TB$ *are colimiting cocones in* C *and* $T^2\lambda : T^2A_\bullet \Rightarrow \Delta T^2B$ *is jointly epimorphic, then there exists a unique* $\mathbb{T}$*-module structure* $\beta : TB \to B$ *such that* $\lambda : (A_\bullet, \alpha_\bullet) \Rightarrow \Delta(B, \beta)$ *is a colimiting cocone in* $C^{\mathbb{T}}$*.*

Proof. By composing α with λ, we get a cocone from $TA_\bullet$ to B:

$$\begin{tikzcd}
TA_i \arrow[r, "\alpha_i"] \arrow[d] & A_i \arrow[d] \arrow[dr, "\lambda_i"] & \\
TA_j \arrow[r, "\alpha_j"'] & A_j \arrow[r, "\lambda_j"'] & B
\end{tikzcd}$$

Thus, there is a unique $\beta : TB \to B$ in C such that $\beta \circ T\lambda_i = \lambda_i \circ \alpha_i$; in other words, there is at most one $\mathbb{T}$-module structure on B making each $\lambda_i : A_i \to B$ into a $\mathbb{T}$-module homomorphism. We need to check that β *is* a $\mathbb{T}$-module structure:

$$\begin{aligned}
\beta \circ (\eta_B \circ \lambda_i) &= (\beta \circ T\lambda_i) \circ \eta_{A_i} && \text{by naturality} \\
&= \lambda_i \circ (\alpha_i \circ \eta_{A_i}) && \text{by definition} \\
&= \lambda_i && \text{by unitality}
\end{aligned}$$

$$\begin{aligned}
\beta \circ \left(\mu_B \circ T^2\lambda_i\right) &= (\beta \circ T\lambda_i) \circ \mu_{A_i} && \text{by naturality} \\
&= \lambda_i \circ (\alpha_i \circ \mu_{A_i}) && \text{by definition} \\
&= (\lambda_i \circ \alpha_i) \circ T\alpha_i && \text{by transitivity} \\
&= \beta \circ (T\lambda_i \circ T\alpha_i) && \text{by definition} \\
&= (\beta \circ T\beta) \circ T^2\lambda_i && \text{by definition}
\end{aligned}$$

Since both $\lambda : A_\bullet \Rightarrow \Delta B$ and $T^2\lambda : T^2 A_\bullet \Rightarrow \Delta T^2 B$ are jointly epimorphic in C, it follows that $\beta \circ \eta_B = \mathrm{id}_B$ and $\beta \circ \mu_B = \beta \circ T\beta$, as required for a $\mathbb{T}$-module structure.

It remains to be shown that this makes λ into a colimiting cocone in $C^{\mathbb{T}}$, so let (C, γ) be any $\mathbb{T}$-module, and suppose we have a cocone $\nu : (A_\bullet, \alpha_\bullet) \Rightarrow \Delta(C, \gamma)$. Since λ is a colimiting cocone in C, ν must factor through λ in C uniquely as $\nu_i = f \circ \lambda_i$ for some $f : B \to C$. Now,

$$\gamma \circ Tf \circ T\lambda_i = \gamma \circ T\nu_i = \nu_i \circ \alpha_i = f \circ \lambda_i \circ \alpha_i = f \circ \beta \circ T\lambda_i$$

for all vertices i in $\mathcal{J}$, so by uniqueness we must have $\gamma \circ Tf = f \circ \beta$; so ν factors through λ uniquely in $C^{\mathbb{T}}$ as well, and therefore λ is a colimiting cocone in $C^{\mathbb{T}}$. ∎

In the context of modules for a monad, one particular class of colimits is especially important. In abstract algebra, any group G has a free presentation: that means there exists a coequaliser diagram of the form

$$\begin{tikzcd} F_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & F_0 \arrow[r, "\varepsilon"] & G \end{tikzcd}$$

where F_1 and F_0 are free groups. We think of $\varepsilon : F_0 \to G$ as enumerating the generators of G and $\langle d_0, d_1 \rangle : F_1 \to F_0 \times F_0$ as enumerating the relations in G. We will now see that something similar is true for modules over any monad.

Definition 1.2.7. A **split coequaliser diagram** is a diagram of the form

$$\begin{tikzcd} X_2 \arrow[r, shift left, "d^2_0"] \arrow[r, shift right, "d^2_1"'] & X_1 \arrow[r, "d^1_0"] \arrow[l, bend left, "s^1_1"] & X_0 \arrow[l, bend left, "s^0_0"] \end{tikzcd}$$

satisfying the following identities:

$$d^1_0 \circ s^0_0 = \mathrm{id}_{X_0} \qquad s^0_0 \circ d^1_0 = d^2_0 \circ s^1_1 \qquad d^2_1 \circ s^1_1 = \mathrm{id}_{X_1} \qquad d^1_0 \circ d^2_0 = d^1_0 \circ d^2_1$$

Remark. Readers familiar with simplicial objects should recognise these equations as special cases of the simplicial identities.

Lemma 1.2.8. *In a split coequaliser diagram as above, d^1_0 is the coequaliser of d^2_0 and d^2_1.*

Proof. Let $f : X_1 \to Y$ be any morphism such that $f \circ d^2_0 = f \circ d^2_1$; we shall show that f factors as $f = g \circ d^1_0$ for a unique $g : X_0 \to Y$. Indeed, if there were such a morphism g, we must have $g = f \circ s^0_0$ by the first identity; but if $g = f \circ s^0_0$, then

$$\begin{aligned} g \circ d^1_0 &= f \circ s^0_0 \circ d^1_0 && \text{by definition} \\ &= f \circ d^2_0 \circ s^1_1 && \text{by the second identity} \\ &= f \circ d^2_1 \circ s^1_1 && \text{by hypothesis} \\ &= f && \text{by the third identity} \end{aligned}$$

as required. ■

Definition 1.2.9. A **reflexive pair** is a pair of morphisms $g, h : C \to B$ for which a common right inverse exists, i.e. there is a morphism $f : B \to C$ such that $g \circ f = \mathrm{id}_B = h \circ f$.

Proposition 1.2.10. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category C, and let ε be the counit of the free-forgetful adjunction $F \dashv U : C^{\mathbb{T}} \to C$. For any $\mathbb{T}$-module (A, α), the diagram*

$$\begin{tikzcd}
T^2A \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_A"'] & TA \arrow[l, bend left=50, "\eta_{TA}"] \arrow[r, "\alpha"] & A \arrow[l, bend left=50, "\eta_A"]
\end{tikzcd}$$

is a split coequaliser diagram in C, and the diagram

$$\begin{tikzcd}
FUFU(A,\alpha) \arrow[r, shift left, "FU\varepsilon_{(A,\alpha)}"] \arrow[r, shift right, "\varepsilon_{FU(A,\alpha)}"'] & FU(A,\alpha) \arrow[r, "\varepsilon_{(A,\alpha)}"] & (A,\alpha)
\end{tikzcd}$$

is a coequaliser diagram in $C^{\mathbb{T}}$, where the pair $(FU\varepsilon_{A,\alpha}, \varepsilon_{FU(A,\alpha)})$ is reflexive.

Proof. Checking that the first diagram is a split coequaliser diagram in C amounts to checking that the identities

$$\alpha \circ \eta_A = \mathrm{id}_A \qquad \eta_A \circ \alpha = T\alpha \circ \eta_{TA} \qquad \mu_A \circ \eta_{TA} = \mathrm{id}_{TA} \qquad \alpha \circ T\alpha = \alpha \circ \mu_A$$

hold and these follow straightforwardly from the axioms. A split coequaliser diagram is preserved by any functor whatsoever, so in particular

$$\begin{tikzcd}
T^3A \arrow[r, shift left, "T^2\alpha"] \arrow[r, shift right, "T\mu_A"'] & T^2A \arrow[r, "T\alpha"] & TA
\end{tikzcd}$$

is also a coequaliser diagram in C; but by proposition 1.2.6, this implies

$$\begin{tikzcd}
(T^2A, \mu_{TA}) \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu_A"'] & (TA, \mu_A) \arrow[r, "\alpha"] & (A, \alpha)
\end{tikzcd}$$

is a coequaliser diagram in $C^{\mathbb{T}}$, and this is simply a relabelling of the second diagram in the statement of the proposition. Finally, observe that

$$FU\varepsilon_{(A,\alpha)} \circ F\eta_{U(A,\alpha)} = \mathrm{id}_{FU(A,\alpha)} = \varepsilon_{FU(A,\alpha)} \circ F\eta_{U(A,\alpha)}$$

by the triangle identities, and

$$F\eta_{U(A,\alpha)} \circ \varepsilon_{FU(A,\alpha)} = \varepsilon_{FUFU(A,\alpha)} \circ FUF\eta_{U(A,\alpha)}$$

so $F\eta_{U(A,\alpha)}$ is a $\mathbb{T}$-module homomorphism and $(FU\varepsilon_{(A,\alpha)}, \varepsilon_{FU(A,\alpha)})$ is a reflexive pair in $C^{\mathbb{T}}$. ■

The proposition plays a key role in determining whether or not a functor is monadic. Indeed, a celebrated theorem of Beck [1967] states the following:

Theorem 1.2.11 (Beck's monadicity theorem). *Let $U : \mathcal{D} \to \mathcal{C}$ be any functor. A U-***split pair** *is a pair of morphisms $g, h : C \to B$ in $\mathcal{D}$ such that there exists a split coequaliser diagram of the form*

$$\begin{tikzcd} UC \arrow[r, shift left, "Ug"] \arrow[r, shift right, "Uh"'] & UB \arrow[l, bend left] \arrow[r] & X \arrow[l, bend left] \end{tikzcd}$$

in $\mathcal{C}$. The functor $U : \mathcal{D} \to \mathcal{C}$ is monadic if and only if

(a) *U has a left adjoint $F : \mathcal{C} \to \mathcal{D}$,*

(b) *U is conservative,*

(c) *$\mathcal{D}$ has coequalisers for all reflexive U-split pairs and U preserves them.*

Since it needs only a little more effort, we prove a more precise theorem:

Theorem 1.2.12 (Beck). *Let $U : \mathcal{D} \to \mathcal{C}$ be a functor with left adjoint $F : \mathcal{C} \to \mathcal{D}$, and let $\mathbb{T} = (T, \eta, \mu)$ be the induced monad. Suppose $\mathcal{D}$ has coequalisers for all reflexive U-split pairs.*

(i) *The comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ itself has a left adjoint $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$.*

(ii) *If U* preserves *coequalisers of reflexive U-split pairs, then the* unit *of the adjunction $L \dashv K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is a natural isomorphism.*

(iii) *If U* reflects *coequalisers of reflexive U-split pairs, then the* counit *of the adjunction $L \dashv K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is a natural isomorphism.*

Proof. A monadic functor must have a left adjoint by definition, is conservative by proposition 1.2.4, and satisfies condition (c) as a special case of proposition 1.2.6. Now, let us show that both hypotheses of (iii) hold when U satisfies conditions (b) and (c). Let $g, h : C \to B$ constitute a reflexive U-split pair, and suppose we have $f' : B \to A'$ in $\mathcal{D}$ such that $f' \circ g = f' \circ h$ and $Uf' : UB \to UA'$ is a coequaliser of Ug and Uh in $\mathcal{C}$. We wish to show that $f' : B \to A'$ is a coequaliser of g and h in $\mathcal{D}$. $\mathcal{D}$ has coequalisers of reflexive U-split pairs, so let $f : B \to A$ be a coequaliser of g and h in $\mathcal{D}$; there then exists a unique morphism $k : A \to A'$ such that $f' = k \circ f$. U preserves coequalisers of reflexive U-split pairs, $Uk : UA \to UA'$ must be an isomorphism in $\mathcal{C}$; and U reflects isomorphisms, so $k : A \to A'$ is an isomorphism in $\mathcal{D}$. Thus, $f' : B \to A'$ is also a coequaliser of g and h in $\mathcal{D}$, as required for (iii). In particular, if U satisfies conditions (a), (b), and (c), then the conclusions of (i), (ii), and (iii) all hold, and so $L \dashv K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an adjoint equivalence of categories.

Henceforth, we will assume $F : \mathcal{C} \to \mathcal{D}$ is the left adjoint of $U : \mathcal{D} \to \mathcal{C}$. Let $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$ be the counit of this adjunction, and suppose $\mathcal{D}$ has coequalisers for all reflexive U-split pairs. We define a functor $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$ as follows: for a $\mathbb{T}$-module (X, ξ), we choose an object $L(X, \xi)$ in $\mathcal{D}$ so that

$$\begin{tikzcd}
FUFX \arrow[r, shift left, "F\xi"] \arrow[r, shift right, "\varepsilon_{FX}"'] & FX \arrow[r, dashed, "\varphi_{(X,\xi)}"] & L(X,\xi)
\end{tikzcd}$$

is a coequaliser diagram in $\mathcal{D}$;[1] for a homomorphism $f : (X, \xi) \to (Y, \theta)$, we choose a morphism $Lf : L(X, \xi) \to L(Y, \theta)$ in $\mathcal{D}$ so that the right square in the diagram below commutes:

$$\begin{tikzcd}
FUFX \arrow[r, shift left, "F\xi"] \arrow[r, shift right, "\varepsilon_{FX}"'] \arrow[d, "FUFf"'] & FX \arrow[r, "\varphi_{(X,\xi)}"] \arrow[d, "Ff"'] & L(X,\xi) \arrow[d, dashed, "Lf"] \\
FUFY \arrow[r, shift left, "F\theta"] \arrow[r, shift right, "\varepsilon_{FY}"'] & FY \arrow[r, "\varphi_{(Y,\theta)}"'] & L(Y,\theta)
\end{tikzcd}$$

The rows of the diagram above are coequalisers, so Lf exists and is unique; a standard argument then shows that L is a functor. Since the diagram

$$\begin{tikzcd}
UFUFX \arrow[r, shift left, "UF\xi"] \arrow[r, shift right, "U\varepsilon_{FX}"'] & UFX \arrow[r, "\xi"] & X
\end{tikzcd}$$

is a coequaliser in $\mathcal{C}$, there is a unique morphism $\eta^{KL}_{(X,\xi)} : X \to UL(X, \xi)$ such that $U\varphi_{(X,\xi)} = \eta^{KL}_{(X,\xi)} \circ \xi$. Thus,

$$\begin{aligned}
\left(U\varepsilon_{L(X,\xi)} \circ UF\eta^{KL}_{(X,\xi)}\right) \circ UF\xi &= U\varepsilon_{L(X,\xi)} \circ UF\left(\eta^{KL}_{(X,\xi)} \circ \xi\right) \\
&= U\varepsilon_{L(X,\xi)} \circ UFU\varphi_{(X,\xi)} \\
&= U\varphi_{(X,\xi)} \circ U\varepsilon_{FX} \\
&= U\varphi_{(X,\xi)} \circ UF\xi \\
&= \left(\eta^{KL}_{(X,\xi)} \circ \xi\right) \circ UF\xi
\end{aligned}$$

---

[1] This makes sense because $F\xi$ and ε_{FX} constitute a reflexive U-split pair.

but $UF\xi$ is a split epimorphism, so we have $\eta^{KL}_{(X,\xi)} \circ \xi = U\varepsilon_{L(X,\xi)} \circ UF\eta^{KL}_{(X,\xi)}$, and therefore we have a $\mathbb{T}$-module homomorphism $\eta^{KL}_{(X,\xi)} : (X,\xi) \to KL(X,\xi)$. Note that this defines a natural transformation $\eta^{KL} : \mathrm{id}_{\mathcal{C}^{\mathbb{T}}} \Rightarrow KL$. Now, take $(X,\xi) = KA = (UA, U\varepsilon_A)$. There is a unique morphism $\varepsilon^{LK}_A : LKA \to A$ in $\mathcal{D}$ making the diagram below commute:

$$\begin{tikzcd}
FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] \arrow[d, "\mathrm{id}"'] & FUA \arrow[r, "\varphi_{KA}"] \arrow[d, "\mathrm{id}"'] & LKA \arrow[d, dashed, "\varepsilon^{LK}_A"] \\
FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"'] & A
\end{tikzcd}$$

A standard argument then shows we have a natural transformation $\varepsilon^{LK} : LK \Rightarrow \mathrm{id}_{\mathcal{D}}$. We must now verify the triangle identities for η^{KL} and ε^{LK}. First, take $A = L(X,\xi)$ and consider the diagram below:

$$\begin{tikzcd}
FUFX \arrow[r, shift left, "F\xi"] \arrow[r, shift right, "\varepsilon_{FX}"'] \arrow[d, "FUF\eta^{KL}_{(X,\xi)}"'] & FX \arrow[r, "\varphi_{(X,\xi)}"] \arrow[d, "F\eta^{KL}_{(X,\xi)}"'] & L(X,\xi) \arrow[d, dashed, "L\eta^{KL}_{(X,\xi)}"] \\
FUFUL(X,\xi) \arrow[r, shift left, "FU\varepsilon_{L(X,\xi)}"] \arrow[r, shift right, "\varepsilon_{FUL(X,\xi)}"'] \arrow[d, "\mathrm{id}"'] & FUL(X,\xi) \arrow[r, "\varphi_{KL(X,\xi)}"] \arrow[d, "\mathrm{id}"'] & LKL(X,\xi) \arrow[d, dashed, "\varepsilon^{LK}_{L(X,\xi)}"] \\
FUFUL(X,\xi) \arrow[r, shift left, "FU\varepsilon_{L(X,\xi)}"] \arrow[r, shift right, "\varepsilon_{FUL(X,\xi)}"'] & FUL(X,\xi) \arrow[r, "\varepsilon_{L(X,\xi)}"'] & L(X,\xi)
\end{tikzcd}$$

Since the top two rows are coequalisers, $\varepsilon^{LK}_{L(X,\xi)} \circ L\eta^{KL}_{(X,\xi)}$ is the unique morphism in $\mathcal{D}$ such that

$$\varepsilon^{LK}_{L(X,\xi)} \circ L\eta^{KL}_{(X,\xi)} \circ \varphi_{(X,\xi)} = \varepsilon_{L(X,\xi)} \circ F\eta^{KL}_{(X,\xi)}$$

but $F\xi$ is a split epimorphism and we have

$$\begin{aligned}
\varepsilon_{L(X,\xi)} \circ F\eta^{KL}_{(X,\xi)} \circ F\xi &= \varepsilon_{L(X,\xi)} \circ F\left(\eta^{KL}_{(X,\xi)} \circ \xi\right) \\
&= \varepsilon_{L(X,\xi)} \circ FU\varphi_{(X,\xi)} \\
&= \varphi_{(X,\xi)} \circ \varepsilon_{FX} \\
&= \varphi_{(X,\xi)} \circ F\xi
\end{aligned}$$

thus $\varepsilon^{LK}_{L(X,\xi)} \circ L\eta^{KL}_{(X,\xi)} = \mathrm{id}_{L(X,\xi)}$. Also,

$$\begin{aligned} U\varepsilon^{LK}_A \circ \eta^{KL}_{KA} &= U\varepsilon^{LK}_A \circ \eta^{KL}_{KA} \circ \mathrm{id}_{UA} \\ &= U\varepsilon^{LK}_A \circ \left(\eta^{KL}_{KA} \circ U\varepsilon_A\right) \circ \eta_{UA} \\ &= \left(U\varepsilon^{LK}_A \circ U\varphi_{KA}\right) \circ \eta_{UA} \\ &= U\varepsilon_A \circ \eta_{UA} = \mathrm{id}_{UA} \end{aligned}$$

so η^{KL} and ε^{LK} indeed satisfy the triangle identities. This completes the proof of (i).

If U preserves coequalisers of reflexive U-split pairs, then

$$\begin{tikzcd} UFUFX \arrow[r, shift left, "UF\xi"] \arrow[r, shift right, "U\varepsilon_{FX}"'] & UFX \arrow[r, "U\varphi_{(X,\xi)}"] & UL(X,\xi) \end{tikzcd}$$

is a coequaliser diagram in $\mathcal{C}$; this implies $\eta^{KL}_{(X,\xi)} : X \to UL(X,\xi)$ must be an isomorphism in $\mathcal{C}$, so η^{KL} is a natural isomorphism, as claimed in (ii). If U instead reflects reflexive coequalisers of U-split pairs, then

$$\begin{tikzcd} FUFUA \arrow[r, shift left, "FU\varepsilon_A"] \arrow[r, shift right, "\varepsilon_{FUA}"'] & FUA \arrow[r, "\varepsilon_A"] & A \end{tikzcd}$$

is a coequaliser diagram in $\mathcal{D}$; this implies $\varepsilon^{LK}_A : LKA \to A$ is an isomorphism in $\mathcal{D}$, as claimed in (iii). ■

Corollary 1.2.13. *Let $F \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction and let $\mathbb{T}$ be the induced monad. If $\mathcal{D}$ has coequalisers for all U-split pairs and U reflects them, then the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is fully faithful, and $\mathcal{D}$ is a reflective subcategory of $\mathcal{C}^{\mathbb{T}}$.*

Proof. The hypotheses of the corollary are the same as those for (iii) in the above theorem, so under these conditions, K has a left adjoint $L : \mathcal{C}^{\mathbb{T}} \to \mathcal{D}$ and the counit $\varepsilon^{LK} : LK \Rightarrow \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism. The right triangle identity

$$K\varepsilon^{LK}_A \circ \eta^{KL}_{KA} = \mathrm{id}_{KA}$$

then implies we must have $K\left(\varepsilon^{LK}_A\right)^{-1} = \left(K\varepsilon^{LK}_A\right)^{-1} = \eta_{KA}$. Let f be any $\mathbb{T}$-module homomorphism $KA \to KB$. We have

$$\begin{aligned} f &= K\varepsilon^{LK}_B \circ \eta^{KL}_{KB} \circ f \\ &= K\varepsilon^{LK}_B \circ KLf \circ \eta_{KA} \\ &= K\varepsilon^{LK}_B \circ KLf \circ K\left(\varepsilon^{LK}_A\right)^{-1} \\ &= K\left(\varepsilon^{LK}_B \circ Lf \circ \left(\varepsilon^{LK}_A\right)^{-1}\right) \end{aligned}$$

and thus K is a full functor. On the other hand, for any morphism $g : A \to B$ in $\mathcal{D}$, we have

$$g = g \circ \varepsilon_A^{LK} \circ \left(\varepsilon_A^{LK}\right)^{-1} = \varepsilon_B^{LK} \circ LKg \circ \left(\varepsilon_A^{LK}\right)^{-1}$$

so g is uniquely determined by Kg, i.e. K is a faithful functor. ■

Unfortunately, the composite of two monadic functors need not be monadic. One remedy for this problem is to find some sufficient conditions for monadicity that define a class of functors closed under composition. Observe that the composition of two right adjoints is another right adjoint, so the property of having a left adjoint is good in this sense; similarly, the composite of two conservative functors is a conservative functor. The real problem is in the existence and preservation of coequalisers of reflexive U-split pairs, and this motivates the 'imprecise' monadicity theorems:

Corollary 1.2.14 (Imprecise monadicity theorems). *A conservative functor $U : \mathcal{D} \to \mathcal{C}$ with a left adjoint is monadic if any of the following* additional *conditions are satisfied:*

- (CTT). *$\mathcal{D}$ has coequalisers of all reflexive pairs whose images under U admit a coequaliser, and U preserves them.*
- (VTT). *$\mathcal{D}$ has* split *coequalisers of all reflexive U-split pairs.*
- (WTT). *$\mathcal{D}$ has coequalisers of all parallel pairs and U preserves them.*

Moreover, the composite of any two functors satisfying the CTT / VTT / WTT condition also satisfies the CTT / VTT / WTT condition, respectively.

Proof. This is a straightforward exercise. ◊

Proposition 1.2.15. *If $U : \mathcal{D} \to \mathcal{C}$ satisfies the CTT condition and $V : \mathcal{C} \to \mathcal{E}$ is monadic, then the composite $VU : \mathcal{D} \to \mathcal{E}$ is also monadic.*

Proof. We shall apply Beck's monadicity theorem. It is easy to check that VU is a conservative functor with a left adjoint, and it remains to be shown that $\mathcal{D}$ coequalisers for all reflexive VU-split pairs and VU preserves them. But if $g, h : C \to B$ constitute a reflexive VU-split pair, then $Ug, Uh : UC \to UB$ constitute a reflexive V-split pair, so by Beck's monadicity theorem, Ug, Uh must have a coequaliser in $\mathcal{C}$ that is preserved by V. The CTT condition then says g, h has a coequaliser in $\mathcal{D}$ that is preserved by U, and hence, by VU. ■

Finally, let us return to the question of strict monadicity.

Definition 1.2.16. An **amnestic functor** is a functor $U : \mathcal{D} \to \mathcal{C}$ such that, for any isomorphism f in $\mathcal{D}$, $U f = \mathrm{id}$ implies $f = \mathrm{id}$. An **isofibration** is a functor $U : \mathcal{D} \to \mathcal{C}$ with isomorphism lifting property: for every object D in $\mathcal{D}$ and every isomorphism $f : C \to UD$ in $\mathcal{C}$, there exists an isomorphism $\tilde{f} : \tilde{C} \to D$ in $\mathcal{D}$ such that $U\tilde{f} = f$.

Lemma 1.2.17. *Let $K : \mathcal{D} \to \mathcal{C}'$ and $U' : \mathcal{C}' \to \mathcal{C}$ be functors, and let $U = U'K$ be their composite.*

(i) *If U' is an amnestic isofibration, then for any object A in $\mathcal{C}'$ and any isomorphism $f : C \to UA$ in $\mathcal{C}$, there is a* unique *isomorphism $\tilde{f} : \tilde{C} \to A$ such that $U\tilde{f} = f$.*

(ii) *If U is amnestic, then so is K.*

(iii) *If U is an isofibration and U' is an amnestic isofibration, then K is also an isofibration.*

(iv) *If K is amnestic and fully faithful, then K is injective on objects.*

(v) *If K is an isofibration and essentially surjective on objects, then K is strictly surjective on objects.*

Proof. (i). If U' is an isofibration then there is at least one such $\tilde{f}$, so suppose $\tilde{f}' : \tilde{C}' \to A$ were another. Then, $U'\left(\tilde{f}^{-1} \circ \tilde{f}'\right) = \mathrm{id}_C$, so if U' is also amnestic, we must have $\tilde{f} = \tilde{f}'$.

(ii). Let f be an isomorphism in $\mathcal{D}$ and suppose $Kf = \mathrm{id}$. Then $Uf = U'Kf = \mathrm{id}$ as well, so $f = \mathrm{id}$, as required.

(iii). Let D be an object in $\mathcal{D}$ and let $f : A \to KD$ be an isomorphism in $\mathcal{C}'$. Then $U'f : U'A \to UD$ is also an isomorphism, so there exists an isomorphism $\tilde{f} : \tilde{A} \to D$ in $\mathcal{D}$ such that $U\tilde{f} = U'K\tilde{f} = U'f$. But U' is an amnestic isofibration, so we must have $K\tilde{f} = f$.

(iv). Suppose $KD' = KD$. Since K is fully faithful, there is an *isomorphism* $f : D' \to D$ such that $Kf = \mathrm{id}_{KD}$; but if K is amnestic, then $f = \mathrm{id}$, so $D' = D$.

(v). Let A be any object in $\mathcal{C}'$. Since K is essentially surjective on objects, there is some isomorphism $f : A \to KD$ in $\mathcal{C}'$; so if K is an isofibration, then there is some isomorphism $\tilde{f} : \tilde{A} \to D$ in $\mathcal{D}$ such that $K\tilde{f} = f$ and, in particular, $K\tilde{A} = A$. ■

Proposition 1.2.18. *A monadic functor is an amnestic isofibration if and only if it is strictly monadic.*

Proof. Let $U : \mathcal{D} \to \mathcal{C}$ be a monadic functor inducing the monad $\mathbb{T}$. The forgetful functor $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is clearly an amnestic isofibration, so if the comparison functor $K : \mathcal{D} \to \mathcal{C}^{\mathbb{T}}$ is an isomorphism of categories, $U : \mathcal{D} \to \mathcal{C}$ must also be an amnestic isofibration. The converse follows from lemma above and the fact that $U^{\mathbb{T}}$ is an amnestic isofibration. ■

3 Colimits in the Eilenberg–Moore category

In this section we study conditions under which the category of modules for a monad has certain colimits. Since we have already mentioned them, let us start by analysing the case where the category of modules has coequalisers for all reflexive pairs.

Lemma 1.3.1. *Let $\mathcal{C}$ be a category with coequalisers for all reflexive pairs. If $\mathcal{C}$ has binary coproducts, then it also has coequalisers for arbitrary parallel pairs.*

Proof. Let $g, h : A \to B$ be a parallel pair in $\mathcal{C}$, and let $A + B$ be the coproduct of A and B in $\mathcal{C}$. Let $[g, \mathrm{id}_B] : A + B \to B$ be the morphism determined by g when restricted to A and id_B when restricted to B, and define the morphism $[h, \mathrm{id}_B] : A + B \to B$ similarly. Clearly, $[g, \mathrm{id}_B]$ and $[h, \mathrm{id}_B]$ constitute a reflexive pair in $\mathcal{C}$, so we may form the coequaliser diagram below:

$$\begin{tikzcd} A + B \arrow[r, shift left, "{[g, \mathrm{id}_B]}"] \arrow[r, shift right, "{[h, \mathrm{id}_B]}"'] & B \arrow[r, dashed, "f"] & C \end{tikzcd}$$

Now, for any morphism $f' : B \to C'$, we have $f' \circ g = f' \circ h$ if and only if $f' \circ [g, \mathrm{id}_B] = f' \circ [h, \mathrm{id}_B]$, so f is also the coequaliser of g and h. ■

The following result of Linton [1969b] tells us when the category of modules for a monad on a cocomplete category is itself cocomplete.

Theorem 1.3.2 (Linton)**.** *Let $\mathbb{T}$ be a monad on a finitely cocomplete (resp. cocomplete) category $\mathcal{S}$. The following are equivalent:*

(i) *$\mathcal{S}^{\mathbb{T}}$ has coequalisers for all reflexive pairs.*

(ii) *$\mathcal{S}^{\mathbb{T}}$ has all finite (resp. small) colimits.*

Proof. (i) ⇒ (ii). Let $\mathbb{T} = (T, \eta, \mu)$ and let $F \dashv U : \mathcal{S}^{\mathbb{T}} \to \mathcal{S}$ be the free–forgetful adjunction. Let $((A_i, \alpha_i) \mid i \in I)$ be a finite (resp. small) family of $\mathbb{T}$-modules, and let $\coprod_i A_i$ be the coproduct in $\mathcal{S}$. Since F is a left adjoint, $F\left(\coprod_i A_i\right)$ must be (isomorphic to) the coproduct $\sum_i FA_i$ in $\mathcal{S}^{\mathbb{T}}$; similarly, $F\left(\coprod_i UFA_i\right)$ must be (isomorphic to) $\sum_i FUFA_i$. Since each $\left(F\alpha_i, \varepsilon_{FA_i}\right)$ is a reflexive pair in $\mathcal{S}^{\mathbb{T}}$, we may form the coequaliser diagram below in $\mathcal{S}^{\mathbb{T}}$:

$$\begin{tikzcd}
\sum_i FUFA_i \arrow[r, shift left, "\sum_i F\alpha_i"] \arrow[r, shift right, "\sum_i \varepsilon_{FA_i}"'] & \sum_i A_i \arrow[r, dashed] & (A, \alpha)
\end{tikzcd}$$

We claim (A, α) is the coproduct $\sum_i (A_i, \alpha_i)$ in $\mathcal{S}^{\mathbb{T}}$. Indeed, for each (A_i, α_i), we have a unique homomorphism $j_i : (A_i, \alpha_i) \to (A, \alpha)$ making the diagram

$$\begin{tikzcd}
FUFA_i \arrow[r, shift left, "F\alpha_i"] \arrow[r, shift right, "\varepsilon_{FA_i}"'] \arrow[d] & FA_i \arrow[r, "\alpha_i"] \arrow[d] & (A_i, \alpha_i) \arrow[d, dashed, "j_i"] \\
\sum_i FUFA_i \arrow[r, shift left, "\sum_i F\alpha_i"] \arrow[r, shift right, "\sum_i \varepsilon_{FA_i}"'] & \sum_i A_i \arrow[r] & (A, \alpha)
\end{tikzcd}$$

commute, where the two left vertical arrows are the coproduct insertions. Given any family of homomorphisms $\left(f_i : (A_i, \alpha_i) \to (B, \beta) \,\middle|\, i \in I\right)$, there exists a unique homomorphism $f : (A, \alpha) \to (B, \beta)$ making the diagram

$$\begin{tikzcd}
\sum_i FUFA_i \arrow[r, shift left, "\sum_i F\alpha_i"] \arrow[r, shift right, "\sum_i \varepsilon_{FA_i}"'] \arrow[d, "{[FUFf_i]}"'] & \sum_i FA_i \arrow[r] \arrow[d, "{[Ff_i]}"'] & (A, \alpha) \arrow[d, dashed, "f"] \\
FUFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\varepsilon_{FB}"'] & FB \arrow[r, "\beta"'] & (B, \beta)
\end{tikzcd}$$

commute, where $[Ff_i]$ is the unique homomorphism $\sum_i FA_i \to FB$ that restricts to $Ff_i : FA_i \to FB$, and similarly for $[FUFf_i]$. On the other hand, any homomorphism $f : (A, \alpha) \to (B, \beta)$ determines a family $\left(f_i = f \circ j_i \,\middle|\, i \in I\right)$ for which the diagram below commutes,

$$\begin{tikzcd}
FUFA_i \arrow[r, shift left, "F\alpha_i"] \arrow[r, shift right, "\varepsilon_{FA_i}"'] \arrow[d, "FUFf_i"'] & FA_i \arrow[r] \arrow[d, "Ff_i"'] & (A_i, \alpha_i) \arrow[d, "f_i"] \\
FUFB \arrow[r, shift left, "F\beta"] \arrow[r, shift right, "\varepsilon_{FB}"'] & FB \arrow[r, "\beta"'] & (B, \beta)
\end{tikzcd}$$

so f can be recovered from its restrictions, and therefore (A, α) is the coproduct $\sum_i (A_i, \alpha_i)$ in $\mathcal{S}$.

Thus, $\mathcal{S}^{\mathbb{T}}$ has all finite (resp. small) coproducts and coequalisers for all reflexive pairs; so, by the preceding lemma, $\mathcal{S}^{\mathbb{T}}$ has coequalisers of arbitrary parallel pairs and therefore has all finite (resp. small) colimits.

(ii) ⇒ (i). Obvious. ■

But when does $\mathcal{S}^{\mathbb{T}}$ has coequalisers in the first place? For this, we must place additional hypotheses on $\mathcal{S}$. Although what follows is applicable to (say) any complete and cocomplete well-powered regular category $\mathcal{S}$ where all regular epimorphisms split, in order to expedite the exposition, we shall assume $\mathcal{S} = \mathbf{Set}$.

Definition 1.3.3. Let $\mathcal{C}$ be any category with pullbacks, and let A be an object in $\mathcal{C}$. A **congruence** on A is a parallel pair $k_0, k_1 : K \to A$ with the following properties:

- (Relational). The morphisms k_0 and k_1 are jointly monic.
- (Reflexivity). There is a (unique) morphism $r : A \to K$ such that

$$k_0 \circ r = \mathrm{id}_A = k_1 \circ r$$

- (Symmetry). There is a (unique) morphism $s : K \to K$ such that

$$k_0 \circ s = k_1 \qquad\qquad k_1 \circ s = k_0$$

- (Transitivity). Given a pullback square as below,

$$\begin{tikzcd} K \times_A K \arrow[r, "q_1"] \arrow[d, "q_0"'] & K \arrow[d, "k_0"] \\ K \arrow[r, "k_1"'] & A \end{tikzcd}$$

  there is a (unique) morphism $t : K \times_A K \to K$ such that

$$k_0 \circ q_0 = k_0 \circ t \qquad\qquad k_1 \circ q_1 = k_1 \circ t$$

The **quotient** of a congruence (k_0, k_1) is the coequaliser of k_0 and k_1, if it exists.

Remark 1.3.4. If C has binary products, a pair of morphisms $k_0, k_1 : K \to A$ in C is jointly monic if and only if $\langle k_0, k_1 \rangle : K \to A \times A$ is a monomorphism. Thus, a congruence on A is a special kind of subobject of $A \times A$. Of course, a congruence on a *set* A is essentially the same thing as an equivalence relation on A.

Definition 1.3.5. Let $f : A \to B$ be a morphism in a category C. The **kernel pair** of f is a pair of morphisms $k_0, k_1 : K \to A$ in C with the following universal property: for any pair $g_0, g_1 : X \to A$ such that $f \circ g_0 = f \circ g_1$, there is a unique morphism $h : X \to K$ such that $g_0 = k_0 \circ h$ and $g_0 = k_1 \circ h$.

Lemma 1.3.6. *Let C be any category with pullbacks, and let $f : A \to B$ be any morphism in C. If there exists $c : B \to A$ such that $f \circ c = \mathrm{id}_B$, then there is a split coequaliser diagram of the form*

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \arrow[l, bend left=50] \arrow[r, "f"] & B \arrow[l, bend left=50, "c"] \end{tikzcd}$$

in C, where $k_0, k_1 : K \to A$ is the kernel pair of f.

Proof. Since $f = f \circ c \circ f$, there exists a unique morphism $c' : A \to K$ such that $k_0 \circ c' = \mathrm{id}_A$ and $k_1 \circ c' = c \circ f$. Thus, we have

$$f \circ c = \mathrm{id}_B \qquad c \circ f = k_1 \circ c' \qquad k_0 \circ c' = \mathrm{id}_A \qquad f \circ k_0 = f \circ k_1$$

as required for a split coequaliser. ■

Corollary 1.3.7. *Let A be a set, and let $k_0, k_1 : K \to A$ be an equivalence relation on A. Assuming the axiom of choice, there is a split coequaliser diagram*

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \arrow[l, bend left=50] \arrow[r, "f"] & B \arrow[l, bend left=50] \end{tikzcd}$$

where B is the quotient of A by K and $f : A \to B$ is the quotient map.

Proof. Note that $f : A \to B$ is the coequaliser of k_0 and k_1 even without the axiom of choice; in particular, f is a surjection. However, under the axiom of choice, there exists a map $c : B \to A$ such that $f \circ c = \mathrm{id}_A$, so the claim amounts to checking that the kernel pair of f is indeed $k_0, k_1 : K \to A$, which is easy. ■

Proposition 1.3.8. *Let $\mathbb{T}$ be a monad on* **Set**. *If $k_0, k_1 : (K, \kappa) \to (A, \alpha)$ is a congruence in* $\mathbf{Set}^{\mathbb{T}}$*, then the coequaliser*

$$\begin{tikzcd}(K,\kappa) \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & (A,\alpha) \arrow[r, "f"] & (B,\beta)\end{tikzcd}$$

exists in $\mathbf{Set}^{\mathbb{T}}$ *and is created by the forgetful functor* $U : \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$.

Proof. By proposition 1.2.5, U preserves limits, so $Uk_0, Uk_1 : K \to A$ is an equivalence relation in **Set** and has a split coequaliser $f : A \to B$; but then proposition 1.2.6 implies there is a unique $\mathbb{T}$-module structure on B making $f : A \to B$ into a coequaliser in $\mathbf{Set}^{\mathbb{T}}$. ■

Definition 1.3.9. A **regular epimorphism** in a category $\mathcal{C}$ is any morphism that is the coequaliser of some parallel pair in $\mathcal{C}$.

Proposition 1.3.10. *In a category with kernel pairs, a morphism is a regular epimorphism if and only if it is the coequaliser of its kernel pair.*

Proof. Let $\mathcal{C}$ be a category with kernel pairs, and suppose $f : A \to B$ is the coequaliser of $g_0, g_1 : C \to A$ in $\mathcal{C}$. Let $k_0, k_1 : K \to A$ be the kernel pair of f; since $f \circ g_0 = f \circ g_1$, there is a unique morphism $g : C \to K$ in $\mathcal{C}$ such that $g_0 = k_0 \circ g$ and $g_1 = k_1 \circ g$. Now, let $h : A \to X$ be any morphism in $\mathcal{C}$ such that $h \circ k_0 = h \circ k_1$; then $h \circ g_0 = h \circ g_1$, so h factors uniquely as $h = \bar{h} \circ f$ for some $\bar{h} : B \to X$. Thus, f is also the coequaliser of k_0 and k_1. ■

Proposition 1.3.11. *Let $\mathcal{C}$ be any category with pullbacks, and let $f : A \to B$ be any morphism in $\mathcal{C}$. If $k_0, k_1 : K \to A$ is the kernel pair of f, then the pair (k_0, k_1) is a congruence on A.*

Proof. Let $A \times_B A \times_B A$ be the threefold fibre product of A over B, with projections $p_0, p_1, p_2 : A \times_B A \times_B A \to A$. Let $q_0 = \langle p_0, p_1 \rangle$ and let $q_1 = \langle p_1, p_2 \rangle$. It is easy to verify that the following diagram is a pullback square:

$$\begin{tikzcd}A \times_B A \times_B A \arrow[r, "q_0"] \arrow[d, "q_1"'] & K \arrow[d, "k_1"] \\ K \arrow[r, "k_0"'] & A\end{tikzcd}$$

Thus, $K \times_A K \cong A \times_B A \times_B A$. The morphisms r, s, t required by the axioms of equivalence relations exist and are unique: indeed, the axioms together with the universal property of pullbacks imply that

$$r = \langle \mathrm{id}_A, \mathrm{id}_A \rangle \qquad s = \langle k_1, k_0 \rangle \qquad t = \langle p_0, p_2 \rangle$$

but it is easily checked that these do indeed work. ■

Corollary 1.3.12. *Let $\mathbb{T}$ be a monad on* **Set**. *A $\mathbb{T}$-module homomorphism $f : (A, \alpha) \to (B, \beta)$ is a regular epimorphism in* $\mathbf{Set}^{\mathbb{T}}$ *if and only if $f : A \to B$ is an epimorphism in* **Set**.

Proof. It is well-known that $f : A \to B$ is an epimorphism in **Set** if and only if it is a regular epimorphism in **Set**, so propositions 1.2.5, 1.3.8, 1.3.10 and 1.3.11 together imply the claim. ■

Lemma 1.3.13. *Let C be a complete category and let A be an object in C. The intersection of a small family of congruences on A (considered as subobjects of $A \times A$) is also a congruence on A.*

Proof. Let $\{K_i \rightarrowtail A \times A \mid i \in I\}$ be a small family of congruences on A, and let $K \rightarrowtail A \times A$ be the intersection. It is clear that the diagonal $\Delta : A \to A \times A$ factors through $K \rightarrowtail A \times A$. Suppose $\langle x_0, x_1 \rangle : X \to A \times A$ factors through $K \rightarrowtail A \times A$; then $\langle x_0, x_1 \rangle$ factors through $K_i \rightarrowtail A \times A$ for all i in I, and so by the symmetry axiom, $\langle x_1, x_0 \rangle$ factors through each $K_i \rightarrowtail A \times A$, and thus, through $K \rightarrowtail A \times A$. A similar argument shows that $K \rightarrowtail A \times A$ satisfies the transitivity axiom. ■

Theorem 1.3.14. *If $\mathbb{T} = (T, \eta, \mu)$ is a monad on* **Set**, *then* $\mathbf{Set}^{\mathbb{T}}$ *has all small colimits.*

Proof. By Linton's theorem (1.3.2), it is enough to show that $\mathbf{Set}^{\mathbb{T}}$ has coequalisers for all parallel pairs. Let $g, h : (A, \alpha) \to (B, \beta)$ be a pair of $\mathbb{T}$-module homomorphisms. Note that if $k : K \rightarrowtail B \times B$ is a monomorphism, then there is at most one $\kappa : TK \to K$ such that

$$\begin{tikzcd}
TK \arrow[r, "Tk"] \arrow[d, dashed, "\kappa"'] & T(B \times B) \arrow[d] \\
K \arrow[r, "k"'] & B \times B
\end{tikzcd}$$

commutes, so (up to isomorphism) there is only a set of congruences on (B,β) in $\mathbf{Set}^{\mathbb{T}}$. In particular, we may form the intersection of all congruences on (B,β) through which $\langle g,h\rangle : (C,\gamma) \to (B,\beta)$ factor, and this is a congruence (K,κ) on (B,β) by the preceding lemma. Let $f : (B,\beta) \to (C,\gamma)$ be the quotient of this congruence; this exists by proposition 1.3.8. We claim f is the coequaliser of g and h. Indeed, for any $f' : (B,\beta) \to (C',\gamma')$ such that $f' \circ g = f' \circ h$, observe that the congruence (K,κ) factors through the kernel pair of f'; thus, f' must factor uniquely through f by the universal property of f as the quotient of (K,κ), and so f is the coequaliser of g and h. ■

4 Morphisms of monads

So far, we have been studying monads one at a time—so much so that we have not even discussed the possibility of morphisms between two monads! We should therefore immediately rectify the situation by considering the following definition:

Definition 1.4.1. Let $\mathbb{S} = (S,\eta^{\mathbb{S}},\mu^{\mathbb{S}})$ be a monad on $\mathcal{C}$ and let $\mathbb{T} = (T,\eta^{\mathbb{T}},\mu^{\mathbb{T}})$ be a monad on $\mathcal{D}$. A **morphism of monads** $\mathbb{S} \to \mathbb{T}$ consists of

- a functor $F : \mathcal{C} \to \mathcal{D}$ and
- a natural transformation $\varphi : TF \Rightarrow FS$

satisfying the following equations:

$$\varphi \bullet \eta^{\mathbb{T}} F = F\eta^{\mathbb{S}} \qquad\qquad \varphi \bullet \mu^{\mathbb{T}} F = F\mu^{\mathbb{S}} \bullet \varphi S \bullet T\varphi$$

$$\begin{tikzcd}
FA \arrow[r, "\eta^{\mathbb{T}}_{FA}"] \arrow[dr, "F\eta^{\mathbb{S}}_A"'] & TFA \arrow[d, "\varphi_A"] \\
& FSA
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2FA \arrow[r, "T\varphi_A"] \arrow[d, "\mu^{\mathbb{T}}_{FA}"'] & TFSA \arrow[r, "\varphi_{SA}"] & FS^2A \arrow[d, "F\mu^{\mathbb{S}}_A"] \\
TFA \arrow[rr, "\varphi_A"'] & & FSA
\end{tikzcd}$$

Remark 1.4.2. In the special case $\mathcal{C} = \mathcal{D}$ and $F = \mathrm{id}_{\mathcal{C}}$, the monad morphism is entirely determined by the natural transformation part $T \Rightarrow S$—note the direction! For this reason, some authors define monad morphisms using the opposite convention.

Example 1.4.3. Let A and B be two monoids in **Set**, and let $\mathbb{S}$ and $\mathbb{T}$ be the corresponding monoids, as in example 1.1.3. If $f : B \to A$ is a homomorphism of monoids, then we get a morphism of monoids $(\mathrm{id}, \varphi) : \mathbb{S} \to \mathbb{T}$ given by the formula below:

$$\varphi_M(b,m) = (f(b), m)$$

Indeed, $\varphi \bullet \eta^{\mathbb{T}} = \eta^{\mathbb{S}}$ since f preserves the unit element, and $\varphi \bullet \mu^{\mathbb{T}} = \mu^{\mathbb{S}} \bullet \varphi S \bullet T\varphi$ because f respects the monoid operation.

Conversely, every monad morphism $(\mathrm{id}, \varphi) : \mathbb{S} \to \mathbb{T}$ is of this form for some monoid homomorphism f: the component $\varphi_1 : S1 \to T1$ is a monoid homomorphism $f : B \to A$, and each element m of M determines a map $\bar{m} : 1 \to M$; but naturality means $\varphi_M \circ T\bar{m} = S\bar{m} \circ \varphi_1$, so by varying m we conclude $\varphi_M(b,m) = (f(b), m)$.

Proposition 1.4.4. *If $(F, \varphi) : \mathbb{S} \to \mathbb{T}$ and $(G, \psi) : \mathbb{T} \to \mathbb{V}$ are morphisms of monads, then*

$$(G, \psi) \circ (F, \varphi) = (GF, G\varphi \bullet \psi F)$$

defines a monad morphism $\mathbb{S} \to \mathbb{V}$, and this is an associative composition with identity morphisms.

Proof. We verify that $(GF, \psi F \bullet G\varphi)$ satisfies the axioms:

$$G\varphi \bullet \psi F \bullet \eta^{\mathbb{V}} GF = G\varphi \bullet \left(\psi \bullet \eta^{\mathbb{V}} G\right) F = G\varphi \bullet G\eta^{\mathbb{T}} F = G\left(\varphi \bullet \eta^{\mathbb{T}} F\right) = GF\eta^{\mathbb{S}}$$

$$\begin{aligned}
G\varphi \bullet \psi F \bullet \mu^{\mathbb{V}} GF &= G\varphi \bullet \left(\psi \bullet \mu^{\mathbb{V}} G\right) F \\
&= G\varphi \bullet \left(G\mu^{\mathbb{T}} \bullet \psi T \bullet V\psi\right) F \\
&= G\left(\varphi \bullet \mu^{\mathbb{T}} F\right) \bullet \psi TF \bullet V\psi F \\
&= G\left(F\mu^{\mathbb{S}} \bullet \varphi S \bullet T\varphi\right) \bullet \psi TF \bullet V\psi F \\
&= GF\mu^{\mathbb{S}} \bullet G\varphi S \bullet (GT\varphi \bullet \psi TF) \bullet V\psi F \\
&= GF\mu^{\mathbb{S}} \bullet (G\varphi S \bullet \psi FS) \bullet (VG\varphi \bullet V\psi F) \\
&= GF\mu^{\mathbb{S}} \bullet (G\varphi \bullet \psi F)S \bullet V(G\varphi \bullet \psi F)
\end{aligned}$$

It is clear that the composition is associative because composition of functors and natural transformations is associative, and the identity morphism is given by (id, id). ■

We refrain from saying that this defines a "category of monads", for much the same reason that we do not say there is a "category of categories". Informally, however, it is useful to pretend there is such a thing as a "category of monads"; then one might summarise the following result by saying that the Eilenberg–Moore construction is a "functor" from the "category of monads" to the "category of categories".

Proposition 1.4.5. *A morphism of monads* $(F, \varphi) : \mathbb{S} \to \mathbb{T}$ *induces a functor* $F^{\varphi} : \mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$*, given by*

$$F^{\varphi}(A, \alpha) = \left(FA, F\alpha \circ \varphi_A\right)$$

for an $\mathbb{S}$*-module* (A, α) *and*

$$F^{\varphi} f = Ff$$

for an $\mathbb{S}$*-module homomorphism* $f : (A, \alpha) \to (B, \beta)$*. If* $(G, \psi) : \mathbb{T} \to \mathbb{V}$ *is another morphism of monads, then we have*

$$(GF)^{G\varphi \bullet \psi F} = G^{\psi} F^{\varphi}$$

and the functor induced by $(\mathrm{id}, \mathrm{id})$ *is the identity functor.*

Proof. First, we check that $\left(FA, F\alpha \circ \varphi_A\right)$ is indeed a $\mathbb{T}$-module. We have

$$F\alpha \circ \varphi_A \circ \eta^{\mathbb{T}}_{FA} = F\alpha \circ F\eta^{\mathbb{S}}_{A} = \mathrm{id}_{FA}$$

by the first axiom, and the second axiom together with naturality gives

$$\begin{aligned} F\alpha \circ \left(\varphi_A \circ \mu^{\mathbb{T}}_{A}\right) &= \left(F\alpha \circ F\mu^{\mathbb{S}}_{A}\right) \circ \varphi_{SA} \circ T\varphi_A \\ &= F\alpha \circ \left(FS\alpha \circ \varphi_{SA}\right) \circ T\varphi_A \\ &= \left(F\alpha \circ \varphi_A\right) \circ \left(TF\alpha \circ T\varphi_A\right) \end{aligned}$$

as required for $F^{\varphi}(A, \alpha) = \left(FA, F\alpha \circ \varphi_A\right)$ to be a $\mathbb{T}$-module. Now, we must check that Ff is a $\mathbb{T}$-module homomorphism $F^{\varphi}(A, \alpha) \to F^{\varphi}\left(B, \beta\right)$:

$$\left(Ff \circ F\alpha\right) \circ F\varphi_A = F\beta \circ \left(FSf \circ \varphi_A\right) = \left(F\beta \circ \varphi_B\right) \circ TFf$$

by naturality of φ. Since $F : \mathcal{C} \to \mathcal{D}$ is a functor, we may now conclude that F^{φ} is a functor $\mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$.

We now check that $(GF)^{G\varphi \bullet \psi F}$ and $G^{\psi} F^{\varphi}$ agree on objects:

$$GF\alpha \circ G\varphi_A \circ \psi_{FA} = G\left(F\alpha \circ \varphi_A\right) \circ \psi_{FA}$$

Since the action of $(GF)^{G\varphi \bullet \psi F}$ and $G^{\psi} F^{\varphi}$ on $\mathbb{S}$-module homomorphisms are both defined by GF, we conclude that $(GF)^{G\varphi \bullet \psi F} = G^{\psi} F^{\varphi}$. It is also clear that (id, id) induces the identity functor, so we are done. ■

In the converse direction, we have the following result:

Proposition 1.4.6. *Let $\mathbb{S} = \left(S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}}\right)$ be a monad on $\mathcal{C}$, let $\mathbb{T} = \left(T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}}\right)$ be a monad on $\mathcal{D}$, and let $U^{\mathbb{S}} : \mathcal{C}^{\mathbb{S}} \to \mathcal{C}$ and $U^{\mathbb{T}} : \mathcal{D}^{\mathbb{T}} \to \mathcal{D}$ be the respective forgetful functors. If $\Phi : \mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$ is a functor such that there exists a functor $F : \mathcal{C} \to \mathcal{D}$ for which $U^{\mathbb{T}}\Phi = FU^{\mathbb{S}}$, then $\Phi = F^{\varphi}$ for a unique monad morphism $\left(F, \varphi\right) : \mathbb{S} \to \mathbb{T}$.*

Proof. For each $\mathbb{S}$-module (A, α), we must have $\Phi(A, \alpha) = \left(A, \beta_{(A,\alpha)}\right)$ for some $\beta_{(A,\alpha)} : TA \to A$. Since Φ takes $\mathbb{S}$-module homomorphisms to $\mathbb{T}$-module homomorphisms, β must be a natural transformation $TU^{\mathbb{T}}\Phi \Rightarrow U^{\mathbb{T}}\Phi$.

We seek a monad morphism $\left(F, \varphi\right) : \mathbb{S} \to \mathbb{T}$ such that $\beta_{(A,\alpha)} = F\alpha \circ \varphi_A$ for all $\mathbb{S}$-modules (A, α), but if there is such a monad morphism, then it must satisfy

$$\varphi_A \circ \eta^{\mathbb{T}}_A = F\eta^{\mathbb{S}}_A \qquad \varphi_A \circ \mu^{\mathbb{T}}_A = F\mu^{\mathbb{S}}_A \circ \varphi_{SA} \circ T\varphi_A \qquad F\mu^{\mathbb{S}}_A \circ \varphi_{SA} = \beta_{\left(SA, \mu^{\mathbb{S}}_A\right)}$$

and therefore we must have

$$\varphi_A = \left(\varphi_A \circ \mu^{\mathbb{T}}_A\right) \circ T\eta^{\mathbb{T}}_A = \left(F\mu^{\mathbb{S}}_A \circ \varphi_{SA}\right) \circ \left(T\varphi_A \circ T\eta^{\mathbb{T}}_A\right) = \beta_{\left(SA, \mu^{\mathbb{S}}_A\right)} \circ TF\eta^{\mathbb{S}}_A$$

hence $\left(F, \varphi\right) : \mathbb{S} \to \mathbb{T}$ is unique if it exists.

On the other hand, if we *define* $\varphi_A = \beta_{\left(SA, \mu^{\mathbb{S}}_A\right)} \circ TF\eta^{\mathbb{S}}_A$, then we have

$$\begin{aligned} \varphi_A \circ \eta^{\mathbb{T}}_A &= \beta_{\left(SA, \mu^{\mathbb{S}}_A\right)} \circ \left(TF\eta^{\mathbb{S}}_A \circ \eta^{\mathbb{T}}_A\right) \\ &= \left(\beta_{\left(SA, \mu^{\mathbb{S}}_A\right)} \circ \eta^{\mathbb{T}}_{SA}\right) \circ F\eta^{\mathbb{S}}_A \\ &= \mathrm{id}_{FSA} \circ F\eta^{\mathbb{S}}_A = \eta^{\mathbb{S}}_A \end{aligned}$$

and $F\mu^{\mathbb{S}}_A$ is a $\mathbb{T}$-module homomorphism $\Phi\left(S^2A, \mu^{\mathbb{S}}_{SA}\right) \to \Phi\left(SA, \mu^{\mathbb{S}}_A\right)$, i.e.

$$\beta_{\left(SA, \mu^{\mathbb{S}}_A\right)} \circ TF\mu^{\mathbb{S}}_A = F\mu^{\mathbb{S}}_A \circ \beta_{\left(S^2A, \mu^{\mathbb{S}}_{SA}\right)}$$

so therefore

$$\begin{aligned}\varphi_A \circ \mu^{\mathbb{T}}_A &= \beta_{(SA,\mu^{\mathbb{S}}_A)} \circ \left(TF\eta^{\mathbb{S}}_A \circ \mu^{\mathbb{T}}_A\right)\\ &= \left(\beta_{(SA,\mu^{\mathbb{S}}_A)} \circ \mu^{\mathbb{T}}_{FSA}\right) \circ T^2F\eta^{\mathbb{S}}_A\\ &= \beta_{(SA,\mu^{\mathbb{S}}_A)} \circ \left(T\beta_{(SA,\mu^{\mathbb{S}}_A)} \circ T^2F\eta^{\mathbb{S}}_A\right)\\ &= \beta_{(SA,\mu^{\mathbb{S}}_A)} \circ T\varphi_A\\ &= \beta_{(SA,\mu^{\mathbb{S}}_A)} \circ \mathrm{id}_{TFSA} \circ T\varphi_A\\ &= \left(\beta_{(SA,\mu^{\mathbb{S}}_A)} \circ TF\mu^{\mathbb{S}}_A\right) \circ TF\eta^{\mathbb{S}}_{SA} \circ T\varphi_A\\ &= F\mu^{\mathbb{S}}_A \circ \left(\beta_{(S^2A,\mu^{\mathbb{S}}_{SA})} \circ TF\eta^{\mathbb{S}}_{SA}\right) \circ T\varphi_A\\ &= F\mu^{\mathbb{S}}_A \circ \varphi_{SA} \circ T\varphi_A\end{aligned}$$

and φ defines a natural transformation $T \Rightarrow S$ since it is the composite of two natural transformations; moreover, since $F\alpha$ is a $\mathbb{T}$-module homomorphism $\Phi(SA, \mu^{\mathbb{S}}_A) \to \Phi(A, \alpha)$, we have

$$F\alpha \circ \varphi_A = \left(F\alpha \circ \beta_{(SA,\mu^{\mathbb{S}}_A)}\right) \circ T\eta^{\mathbb{S}}_A = \beta_{(A,\alpha)} \circ \left(TF\alpha \circ TF\eta^{\mathbb{S}}_A\right) = \beta_{(A,\alpha)}$$

thus we have the required monad morphism. ■

Corollary 1.4.7. *Let $\mathbb{I}$ be the identity monad on the terminal category. For any monad $\mathbb{T}$ on any category $\mathcal{C}$, to give a monad morphism $\mathbb{I} \to \mathbb{T}$ is the same thing as to give a $\mathbb{T}$-module.* ■

Remark 1.4.8. Not every functor $\Phi : \mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$ admits a functor $F : \mathcal{C} \to \mathcal{D}$ such that $U^{\mathbb{T}}\Phi = FU^{\mathbb{S}}$ (not even up to isomorphism). For example, let $\mathcal{C} = \mathcal{D} = \mathbf{Set}$, and let $\mathbb{S} = \mathbb{T}$ be the free group monad, so that $\mathcal{C}^{\mathbb{S}} = \mathcal{D}^{\mathbb{T}} \cong \mathbf{Grp}$, and consider the functor $\Phi : \mathbf{Grp} \to \mathbf{Grp}$ taking a group to its torsion subgroup: it maps the group $\mathbb{Z}$ to the trivial group 1, while the group $G = (\mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}$ is mapped to $\mathbb{Z}/2\mathbb{Z}$; noting that 1 and $\mathbb{Z}/2\mathbb{Z}$ have different cardinalities even though $\mathbb{Z}$ and G have the same cardinality, we conclude that Φ is *not* induced by any morphism of monads.

Of course, there is also a 2-categorical structure here:

Definition 1.4.9. Let $(F, \varphi), (F', \varphi') : \mathbb{S} \to \mathbb{T}$ be two morphisms of monads. A **transformation of monad morphisms** $\psi : (F, \varphi) \Rightarrow (F, \varphi')$ is a natural transformation of functors $\theta : F \Rightarrow F'$ satisfying the following equation:

$$\varphi' \bullet T\theta = \theta S \bullet \varphi$$

$$\begin{tikzcd}
TFA \arrow[r, "T\theta_A"] \arrow[d, "\varphi_A"'] & TF'A \arrow[d, "\varphi'_A"] \\
FSA \arrow[r, "\theta_{SA}"'] & F'SA
\end{tikzcd}$$

Proposition 1.4.10.

(i) *Given two transformations of monad morphisms* $\theta : (F,\varphi) \Rightarrow (F',\varphi')$, $\theta' : (F',\varphi') \Rightarrow (F'',\varphi'')$, *the vertical composite* $\theta' \bullet \theta$ *is a transformation of monad morphisms* $(F,\varphi) \Rightarrow (F,\varphi'')$.

(ii) *Given monad morphisms* $(F,\varphi),(F',\varphi') : \mathbb{S} \to \mathbb{T}$, $(G,\psi),(G',\psi') : \mathbb{T} \to \mathbb{V}$ *and transformations* $\theta : (F,\varphi) \Rightarrow (F',\varphi')$, $\sigma : (G,\psi) \Rightarrow (G',\psi')$, *the horizontal composite*

$$\sigma \circ \theta = \sigma F' \bullet G\theta = G'\theta \bullet \sigma F$$

is a transformation $(G,\psi) \circ (F,\varphi) \Rightarrow (G',\psi') \circ (F',\varphi')$.

Proof. (i). This is straightforward algebra of natural transformations:

$$\begin{aligned}
\varphi'' \bullet T(\theta' \bullet \theta) &= (\varphi'' \bullet T\theta') \bullet T\theta \\
&= \theta' S \bullet (\varphi' \bullet T\theta) \\
&= (\theta' S \bullet \theta S) \bullet \varphi \\
&= (\theta' \bullet \theta) S \bullet \varphi
\end{aligned}$$

(ii). For clarity, we take $\sigma \circ \theta = \sigma F' \bullet G\theta$. Then,

$$\begin{aligned}
(G'\varphi' \bullet \psi' F') \bullet V(\sigma F' \bullet G\theta) &= G'\varphi' \bullet (\psi' \bullet V\sigma) F' \bullet VG\theta \\
&= G'\varphi' \bullet (\sigma T \bullet \psi) F' \bullet VG\theta \\
&= (G'\varphi' \bullet \sigma T F') \bullet (\psi F' \bullet VG\theta) \\
&= (\sigma F' S \bullet G\varphi') \bullet (GT\theta \bullet \psi F) \\
&= \sigma F' S \bullet G(\varphi' \bullet T\theta) \bullet \psi F \\
&= \sigma F' S \bullet G(\theta S \bullet \varphi) \bullet \psi F \\
&= (\sigma F' \bullet G\theta) S \bullet (G\varphi \bullet \psi F)
\end{aligned}$$

as required. ■

Informally, we may speak of the “2-category of monads”. Unsurprisingly, the Eilenberg–Moore construction extends to a (strict!) “2-functor” from the “2-category of monads” to the “2-category of categories”.

Proposition 1.4.11. *A transformation of monad morphisms* $\theta : (F, \varphi) \Rightarrow (G, \psi)$ *induces a natural transformation of functors* $\theta_* : F^{\varphi} \Rightarrow G^{\psi}$*, where*

$$(\theta_*)_{(A,\alpha)} = \theta_A$$

for each $\mathbb{S}$*-module* (A, α)*, and this respects horizontal and vertical composition of transformations.*

Proof. Since θ is already a natural transformation $F \Rightarrow G$, it is enough to show that θ_A is a $\mathbb{T}$-module homomorphism $F^{\varphi}(A, \alpha) \to G^{\psi}(A, \alpha)$ for any $\mathbb{S}$-module structure $\alpha : SA \to A$, and we have

$$(\theta_A \circ F)\alpha \circ \varphi_A = G\alpha \circ (\theta_{SA} \circ \varphi_A) = (G\alpha \circ \psi_A) \circ T\theta_A$$

as required. Since the horizontal/vertical composition of transformations of monad morphisms is just the horizontal/vertical composition of the underlying natural transformations of functors, it is clear that this construction respects horizontal/vertical composition. ■

Again, we have a converse:[1]

Proposition 1.4.12. *Let* $(F, \varphi), (G, \psi) : \mathbb{S} \to \mathbb{T}$ *be a morphism of monads. If* $\bar{\theta} : F^{\varphi} \Rightarrow G^{\psi}$ *is a natural transformation such that there exists a natural transformation* $\theta : F \Rightarrow G$ *for which*

$$\bar{\theta}_{(A,\alpha)} = \theta_A$$

for all $\mathbb{S}$*-modules* (A, α)*, then* θ *is a transformation* $(F, \varphi) \Rightarrow (G, \psi)$*.*

Proof. From the calculations in proposition 1.4.6, we know that

$$\varphi_A = F\mu^{\mathbb{S}}_A \circ \varphi_{SA} \circ TF\eta^{\mathbb{S}}_A$$
$$\psi_A = G\mu^{\mathbb{S}}_A \circ \psi_{SA} \circ TG\eta^{\mathbb{S}}_A$$

---

[1] This proof was suggested by Todd Trimble [2012].

and we have a homomorphism $\theta_{SA} : \left(FSA, F\mu^{\mathbb{S}}_A \circ \varphi_{SA}\right) \to \left(GSA, G\mu^{\mathbb{S}}_A \circ \psi_{SA}\right)$ for each A, therefore

$$\theta_{SA} \circ F\mu^{\mathbb{S}}_A \circ \varphi_{SA} = G\mu^{\mathbb{S}}_A \circ \psi_{SA} \circ T\theta_{SA}$$

but $\theta : F \Rightarrow G$ is a natural transformation, so

$$\theta_{SA} \circ F\eta^{\mathbb{S}}_A = G\eta^{\mathbb{S}}_A \circ \theta_A$$

and so we have

$$\begin{aligned} \psi_A \circ T\theta_A &= G\mu^{\mathbb{S}}_A \circ \psi_{SA} \circ \left(TG\eta^{\mathbb{S}}_A \circ T\theta_A\right) \\ &= \left(G\mu^{\mathbb{S}}_A \circ \psi_{SA} \circ T\theta_{SA}\right) \circ TF\eta^{\mathbb{S}}_A \\ &= \theta_{SA} \circ \left(F\mu^{\mathbb{S}}_A \circ \varphi_{SA} \circ TF\eta^{\mathbb{S}}_A\right) \\ &= \theta_{SA} \circ \varphi_A \end{aligned}$$

as required for a transformation $\theta : (F, \varphi) \Rightarrow (G, \psi)$. ■

Corollary 1.4.13. *Let $\mathbb{I}$ be the identity monad on the terminal category. For any monad $\mathbb{T}$ on any category $\mathcal{C}$, the category of monad morphisms $\mathbb{I} \to \mathbb{T}$ together with their transformations is isomorphic to the category $\mathcal{C}^{\mathbb{T}}$.*

Remark 1.4.14. Not every natural transformation $F^{\varphi} \Rightarrow G^{\psi}$ is of the form θ_* for some transformation $\theta : (F, \varphi) \Rightarrow (G, \psi)$. For example, let $\mathcal{C} = \mathcal{D} = \mathbf{Set}$, and let $\mathbb{S} = \mathbb{T}$ be the free group monad, so that $\mathcal{C}^{\mathbb{S}} = \mathcal{D}^{\mathbb{T}} \cong \mathbf{Grp}$, and consider the natural transformation $\mathrm{id}_{\mathbf{Grp}} \Rightarrow \mathrm{id}_{\mathbf{Grp}}$ that maps every group to its identity element. This clearly depends on the group structure, so cannot be of the form θ_* for some $\theta : (\mathrm{id}, \mathrm{id}) \Rightarrow (\mathrm{id}, \mathrm{id})$.

Definition 1.4.15. Let U, R, U', R' be functors as in the diagram below,

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "R'"] \arrow[d, "U"'] \arrow[dr, phantom, "\sigma\Nearrow"] & \mathcal{D}' \arrow[d, "U'"] \\ \mathcal{C} \arrow[r, "R"'] & \mathcal{D} \end{tikzcd}$$

where the natural transformation $\sigma : RU \Rightarrow U'R'$ is an isomorphism, and let $F \dashv U$ and $F' \dashv U'$ be adjunctions. Define $\hat{\sigma} : F'R \Rightarrow R'F$ to be the natural transformation

$$\hat{\sigma} = \varepsilon^{F'U'} R'F \bullet F'\sigma F \bullet F'R\eta^{UF}$$

where $\varepsilon^{F'U'} : F'U' \Rightarrow \mathrm{id}_{\mathcal{D}'}$ is the counit of $F' \dashv U'$ and $\eta^{UF} : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ is the unit of $F \dashv U$. We say the **Beck–Chevalley condition** holds for the diagram above when $\hat{\sigma}$ is a natural isomorphism, and the **strict Beck–Chevalley condition** holds when both σ and $\hat{\sigma}$ are identities.

Proposition 1.4.16. *Let $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ be a monad on $\mathcal{C}$ and let $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ be a monad on $\mathcal{D}$. Let $R^{\varphi} : \mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$ be the functor induced by a morphism of monads $(R, \varphi) : \mathbb{S} \to \mathbb{T}$. The Beck–Chevalley condition holds for the diagram*

$$\begin{tikzcd}
\mathcal{C}^{\mathbb{S}} \arrow[r, "R^{\varphi}"] \arrow[d, "U^{\mathbb{S}}"'] & \mathcal{D}^{\mathbb{T}} \arrow[d, "U^{\mathbb{T}}"] \\
\mathcal{C} \arrow[r, "R"'] & \mathcal{D}
\end{tikzcd}$$

if and only if $\varphi : TR \Rightarrow RS$ is a natural isomorphism, and the strict Beck–Chevalley conditions holds if and only if $TR = RS$ and $\varphi = \mathrm{id}$.

Proof. In this case, $\sigma = \mathrm{id}$ because the square in question commutes strictly, and

$$\hat{\sigma} = \varepsilon^{\mathbb{T}} R^{\varphi} F^{\mathbb{S}} \bullet F^{\mathbb{T}} R \eta^{\mathbb{S}}$$

but $U^{\mathbb{T}} : \mathcal{D}^{\mathbb{T}} \to \mathcal{D}$ is conservative and amnestic , so $\hat{\sigma} : F^{\mathbb{T}} R \Rightarrow R^{\varphi} F^{\mathbb{S}}$ is an isomorphism or an identity if and only if $U^{\mathbb{T}} \hat{\sigma} : U^{\mathbb{T}} F^{\mathbb{T}} R \Rightarrow U^{\mathbb{T}} R^{\varphi} F^{\mathbb{S}}$ is. Let X be any object in $\mathcal{C}$; then

$$R^{\varphi} F^{\mathbb{S}} X = R^{\varphi} \left(SX, \mu^{\mathbb{S}}_X \right) = \left(RSX, R\mu^{\mathbb{S}}_X \circ \varphi_{SX} \right)$$

hence $U^{\mathbb{T}} \varepsilon^{\mathbb{T}}_{R^{\varphi} F^{\mathbb{S}} X} = R\mu^{\mathbb{S}}_X \circ \varphi_{SX}$, and so

$$U^{\mathbb{T}} \hat{\sigma} = U^{\mathbb{T}} \varepsilon^{\mathbb{T}} R^{\varphi} F^{\mathbb{S}} \bullet U^{\mathbb{T}} F^{\mathbb{T}} R \eta^{\mathbb{S}} = R\mu^{\mathbb{S}} \bullet \left(\varphi S \bullet TR\eta^{\mathbb{S}} \right) = \left(R\mu^{\mathbb{S}} \bullet RS\eta^{\mathbb{S}} \right) \bullet \varphi = \varphi$$

thence follow the claims. ■

Proposition 1.4.17. *Let $(R, \varphi) : \mathbb{S} \to \mathbb{T}$ be a morphism of monads. If R is fully faithful and $\varphi : TR \Rightarrow RS$ is a natural isomorphism, then:*

(i) *A $\mathbb{T}$-module (X, ξ) is in the essential image of R^{φ} if and only if X is in the essential image of R.*

(ii) *A $\mathbb{T}$-module (X, ξ) is in the strict image of R^{φ} if and only if X is in the strict image of R.*

Proof. Suppose we have an isomorphism $f : X \to RA$. Then, there is a unique morphism $\alpha : SA \to A$ in $\mathcal{C}$ such that the diagram below commutes:

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "\xi"'] & TRA \arrow[r, "\varphi_A"] & RSA \arrow[d, "R\alpha"] \\
X \arrow[rr, "f"'] & & RA
\end{tikzcd}$$

If $\alpha : SA \to A$ is an $\mathbb{S}$-module structure on A, then it immediately follows that f is a $\mathbb{T}$-module isomorphism $(X, \xi) \to R^{\varphi}(A, \alpha)$, so it is enough to just check the $\mathbb{S}$-module axioms. Since R is fully faithful, we need only show that the equations

$$R\left(\alpha \circ \eta^{\mathbb{S}}_A\right) = \mathrm{id}_{RA} \qquad\qquad R\left(\alpha \circ \mu^{\mathbb{S}}_A\right) = R(\alpha \circ S\alpha)$$

hold in $\mathcal{D}$. But we have

$$\begin{aligned}
R(\alpha \circ \eta_A) &= R\alpha \circ R\eta^{\mathbb{S}}_A \\
&= (R\alpha \circ \varphi_A) \circ \eta^{\mathbb{T}}_{RA} \\
&= f \circ \xi \circ \left(Tf^{-1} \circ \eta^{\mathbb{T}}_{RA}\right) \\
&= f \circ \left(\xi \circ \eta^{\mathbb{T}}_X\right) \circ f^{-1} \\
&= f \circ f^{-1} = \mathrm{id}_{RA}
\end{aligned}$$

and also

$$\begin{aligned}
R\left(\alpha \circ \mu^{\mathbb{S}}_A\right) &= R\alpha \circ R\mu^{\mathbb{S}}_A \\
&= R\alpha \circ \left(R\mu^{\mathbb{S}}_A \circ \varphi_{SA} \circ T\varphi_A\right) \circ T\varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= (R\alpha \circ \varphi_A) \circ \mu^{\mathbb{T}}_{RA} \circ \varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= f \circ \xi \circ \left(Tf^{-1} \circ \mu^{\mathbb{T}}_{RA}\right) \circ \varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= f \circ \left(\xi \circ \mu^{\mathbb{T}}_X\right) \circ T^2 f^{-1} \circ T\varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= (f \circ \xi) \circ \left(T\xi \circ T^2 f^{-1}\right) \circ T\varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= \left(f \circ \xi \circ Tf^{-1}\right) \circ \left(Tf \circ T\xi \circ T^2 f^{-1}\right) \circ T\varphi_A^{-1} \circ \varphi_{SA}^{-1} \\
&= R\alpha \circ (\varphi_A \circ TR\alpha) \circ \left(T\varphi_A \circ T\varphi_A^{-1}\right) \circ \varphi_{SA}^{-1} \\
&= R\alpha \circ RS\alpha \circ \left(\varphi_{SA} \circ \varphi_{SA}^{-1}\right) \\
&= R(\alpha \circ S\alpha)
\end{aligned}$$

as required for claim (i). For claim (ii), we simply take $X = RA$ and $f = \mathrm{id}_X$ in the above argument. ■

5 Descent and base change

Let A and B be rings, not necessarily commutative, and suppose we have a ring homomorphism $f : B \to A$. This induces an adjunction between the respective categories of left modules:

$$A \otimes_B (-) \dashv \mathrm{Hom}_A(A, -) : \mathbf{Mod}(A) \to \mathbf{Mod}(B)$$

The right adjoint is in fact (isomorphic to a functor) induced by the morphism of monads induced by the ring homomorphism, and we will see later in this section that the functor induced by a morphism of monads has a left adjoint under mild assumptions.

Now, consider a general adjunction of categories:

$$F \dashv U : \mathcal{D} \to \mathcal{C}$$

We think of U as being "restriction of scalars" and F as being "extension of scalars". The **descent problem**[1] for F asks for a characterisation of those objects A in $\mathcal{D}$ that arise as FX for some object X in $\mathcal{C}$, and a way to reconstruct X given FX. The formal dual of theorem 1.1.8 immediately gives a necessary condition: A must have the structure of a comodule for the induced comonad $\mathbb{G} = (FU, \varepsilon, F\eta U)$. Indeed, if $A = FX$, then the adjunction unit $\eta_X : X \to UFX$ gives us a morphism $\xi = F\eta_X : A \to FUA$ such that

$$\varepsilon_A \circ \xi = \mathrm{id}_A \qquad\qquad F\eta_{UA} \circ \xi = FU\xi \circ \xi$$

exactly as required for (A, ξ) to be a $\mathbb{G}$-comodule. We think of this as a 'descent datum' for F. Notice that we have a coreflexive fork in $\mathcal{C}$ for each object X:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] & UFX \arrow[r, shift left=2, "UF\eta_X"] \arrow[r, shift right=2, "\eta_{UFX}"'] & UFUFX \arrow[l]
\end{tikzcd}$$

$$UF\eta_X \circ \eta_X = \eta_{UFX} \circ \eta_X \qquad UF\eta_X \circ U\varepsilon_{FX} = \mathrm{id}_{UFX} \qquad \eta_{UFX} \circ U\varepsilon_{FX} = \mathrm{id}_{UFX}$$

Moreover, its image under F is a reflexive split equaliser in $\mathcal{D}$:

$$\begin{tikzcd}
FX \arrow[r, "F\eta_X"] & FUFX \arrow[l, bend left, "\varepsilon_{FX}"] \arrow[r, shift left=2, "FUF\eta_X"] \arrow[r, shift right=2, "F\eta_{UFX}"'] & FUFUFX \arrow[l] \arrow[l, bend left, "\varepsilon_{FUFX}"]
\end{tikzcd}$$

---

[1] Or, perhaps, 'codescent problem': our setup is formally dual to that of Bénabou and Roubaud [1970] and of Janelidze and Tholen [1994].

The formal dual of proposition 1.2.6 then implies

$$\begin{tikzcd}(FX, F\eta_X) \arrow[r, "F\eta_X"] & (FUFX, F\eta_{UFX}) \arrow[r, shift left, "FUF\eta_X"] \arrow[r, shift right, "F\eta_{UFX}"'] & (FUFUFX, F\eta_{UFUFX})\end{tikzcd}$$

is an equaliser diagram in the category $\mathcal{D}_{\mathbb{G}}$ of comodules for $\mathbb{G}$. Thus, if $\mathcal{D}_{\mathbb{G}}$ is to be a 'good' model of $\mathcal{C}$, the adjunction unit $\eta_X : X \to UFX$ should be the equaliser of $UF\eta_X, \eta_{UFX} : UFX \to UFUFX$. This motivates the following definition:

Definition 1.5.1. A **functor of comonadic descent type** is a functor $F : \mathcal{C} \to \mathcal{D}$ with the following properties:

- F has a right adjoint $U : \mathcal{D} \to \mathcal{C}$, and
- for each object X in $\mathcal{C}$, the adjunction unit $\eta_X : X \to UFX$ is the equaliser of $UF\eta_X, \eta_{UFX} : UFX \to UFUFX$.

Let $\mathbb{G}$ be the comonad induced by the adjunction $F \dashv U$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor of comonadic descent type *and* every $\mathbb{G}$-comodule is isomorphic to one of the form $(FX, F\eta_X)$, then we say comonadic descent is **effective** for F.

Proposition 1.5.2. *Let $F \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction of categories with unit $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow UF$ and counit $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{D}}$. Let $\mathbb{G}$ be the induced comonad on $\mathcal{D}$, and let $K : \mathcal{C} \to \mathcal{D}_{\mathbb{G}}$ be the comparison functor induced by the formal dual of the Eilenberg–Moore theorem (1.1.8). The following are equivalent:*

(i) *For each object X in $\mathcal{C}$, the adjunction unit $\eta_X : X \to UFX$ is a regular monomorphism.*[2]

(ii) *$F : \mathcal{C} \to \mathcal{D}$ is a functor of comonadic descent type.*

(iii) *$K : \mathcal{C} \to \mathcal{D}_{\mathbb{G}}$ is fully faithful.*

In particular, comonadic descent is effective for F if and only if F is a comonadic functor.

---

[2] A **regular monomorphism** is a morphism that is an equaliser of some parallel pair.

Proof. (i) ⇒ (ii). Suppose $\eta_X : X \to UFX$ is the equaliser of $g, h : UFX \to Y$ in $\mathcal{C}$. Given $f : Z \to UFX$ such that $UF\eta_X \circ f = \eta_{UFX} \circ f$, the triangle identities and our hypotheses imply

$$\begin{aligned} UFg \circ UFf &= UFg \circ U\varepsilon_{FUFX} \circ (UF\eta_{UFX} \circ UFf) \\ &= UFg \circ (U\varepsilon_{FUFX} \circ UFUF\eta_X) \circ UFf \\ &= (UFg \circ UF\eta_X) \circ U\varepsilon_{FX} \circ UFf \\ &= UFh \circ (UF\eta_X \circ U\varepsilon_{FX}) \circ UFf \\ &= UFh \circ U\varepsilon_{FUFX} \circ (UFUF\eta_X \circ UFf) \\ &= UFh \circ (U\varepsilon_{FUFX} \circ UF\eta_{UFX}) \circ UFf \\ &= UFh \circ UFf \end{aligned}$$

and therefore, by naturality,

$$\eta_Y \circ (g \circ f) = (UFg \circ UFf) \circ \eta_X = (UFh \circ UFf) \circ \eta_X = \eta_Y \circ (h \circ f)$$

but $\eta_Y : Y \to UFY$ is a (regular) monomorphism, so we must have $g \circ f = h \circ f$; hence, f factors as $f = \eta_X \circ \tilde{f}$ for a unique morphism $\tilde{f} : Z \to X$ in $\mathcal{C}$, as required.

(ii) ⇒ (iii). Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If the adjunction unit $\eta_Y : Y \to UFY$ is the equaliser of $UF\eta_Y, \eta_{UFY} : UFY \to UFUFY$, then $f : X \to Y$ is the unique morphism making the diagram below commute:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, dashed, "f"'] & UFX \arrow[r, shift left, "UF\eta_X"] \arrow[r, shift right, "\eta_{UFX}"'] \arrow[d, "UFf"] & UFUFX \arrow[d, "UFUFf"] \\ Y \arrow[r, "\eta_X"] & UFY \arrow[r, shift left, "UF\eta_Y"] \arrow[r, shift right, "\eta_{UFY}"'] & UFUFY \end{tikzcd}$$

In particular, f is uniquely determined by UFf, so $K : \mathcal{C} \to \mathcal{D}_{\mathbb{G}}$ must be faithful. Given any $\mathbb{G}$-comodule homomorphism $h : (FX, F\eta_X) \to (FY, F\eta_Y)$, i.e. a morphism $h : FX \to FY$ in $\mathcal{D}$ such that

$$F\eta_Y \circ h = FUh \circ F\eta_X$$

there must be a unique morphism $f : X \to Y$ in $\mathcal{C}$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, dashed, "f"'] & UFX \arrow[r, shift left, "UF\eta_X"] \arrow[r, shift right, "\eta_{UFX}"'] \arrow[d, "Uh"] & UFUFX \arrow[d, "UFUh"] \\ Y \arrow[r, "\eta_X"] & UFY \arrow[r, shift left, "UF\eta_Y"] \arrow[r, shift right, "\eta_{UFY}"'] & UFUFY \end{tikzcd}$$

and so $Uh = UFf$, but $h : FX \to FY$ is the unique morphism in $\mathcal{D}$ making the diagram

$$\begin{tikzcd}
FX \arrow[r, "F\eta_X"] \arrow[d, dashed, "h"'] & FUFX \arrow[r, shift left, "FUF\eta_X"] \arrow[r, shift right, "F\eta_{UFX}"'] \arrow[d, "FUh"] & FUFUFX \arrow[d, "FUFUh"] \\
FY \arrow[r, "F\eta_Y"] & FUFY \arrow[r, shift left, "FUF\eta_Y"] \arrow[r, shift right, "F\eta_{UFY}"'] & FUFUFY
\end{tikzcd}$$

commute, so $h = Ff$; thus $K : C \to \mathcal{D}_{\mathbb{G}}$ is full.

(iii) ⇒ (i). If $K : C \to \mathcal{D}_{\mathbb{G}}$ is fully faithful, then it reflects all limits that exist in $\mathcal{D}_{\mathbb{G}}$; in particular, $\eta_X : X \to UFX$ must be the equaliser of $UF\eta_X, \eta_{UFX} : UFX \to UFUFX$, since its image under K is the equaliser of $FUF\eta_X, F\eta_{UFX} : (FUFX, F\eta_{UFX}) \to (FUFUFX, F\eta_{UFUFX})$. ■

Since a functor $F : C \to \mathcal{D}$ is of effective comonadic descent type if and only if it is comonadic, the formal dual of Beck's theorem gives us necessary and sufficient conditions for comonadic descent to be effective for F:

Theorem 1.5.3. *Let $F : C \to \mathcal{D}$ be a functor with a right adjoint $U : \mathcal{D} \to C$. Here, by F-***split pair** *we mean a parallel pair in C whose image under F fits into a split* equaliser *diagram in $\mathcal{D}$.*

(i) *F is a comonadic functor if and only if F is conservative, C has equalisers for all reflexive F-split pairs and F preserves them.*

(ii) *F is a functor of comonadic descent type if C has equalisers for all reflexive F-split pairs and the left adjoint F* reflects *them.*

Proof. First, note that passing to the opposite category reverses left and right adjunctions; that is, the adjunction $F \dashv U$ induces the adjunction below:

$$U^{\mathrm{op}} \dashv F^{\mathrm{op}} : C^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$$

As such, (i) is a straightforward translation of theorem 1.2.11, and (ii) is the formal dual of corollary 1.2.13. ■

Example 1.5.4. Let B be a commutative ring. We say that a B-algebra A is faithfully flat just if the functor $A \otimes_B (-) : \mathbf{Mod}(B) \to \mathbf{Mod}(A)$ preserves and reflects all exact sequences. Since $\mathbf{Mod}(A)$ and $\mathbf{Mod}(B)$ are abelian categories, $A \otimes_B (-)$ preserves and reflects all exact sequences if and only if it preserves and reflects all finite limits and all finite colimits. Thus, the extension-of-scalars functor $A \otimes_B (-)$ is comonadic when A is a faithfully flat B-algebra.

We now give the promised construction of a left adjoint for the functor induced by a morphism of monads $(R, \varphi) : \mathbb{S} \to \mathbb{T}$. We may think of this as being a 'base change' functor, for the following reason.[3] Suppose $L_\varphi : \mathcal{D}^\mathbb{T} \to \mathcal{C}^\mathbb{S}$ is a left adjoint to $R^\varphi : \mathcal{C}^\mathbb{S} \to \mathcal{D}^\mathbb{T}$. By the uniqueness of left adjoints up to isomorphism, we must have $L_\varphi F^\mathbb{T} \cong F^\mathbb{S} L$, therefore $L_\varphi(TA, \mu^\mathbb{T}_A)$ must be isomorphic to $(SLA, \mu^\mathbb{S}_{LA})$. Moreover, for any $\mathbb{T}$-module (A, α), we have a reflexive coequaliser diagram

$$\begin{tikzcd}
(T^2A, \mu^\mathbb{T}_{TA}) \arrow[r, shift left=2, "T\alpha"] \arrow[r, shift right=2, "\mu^\mathbb{T}_A"'] & (TA, \mu^\mathbb{T}_A) \arrow[l] \arrow[r, "\alpha"] & (A, \alpha)
\end{tikzcd}$$

in $\mathcal{D}^\mathbb{T}$, so its image under L_φ must be a reflexive coequaliser diagram in $\mathcal{C}^\mathbb{S}$:

$$\begin{tikzcd}
(SLTA, \mu^\mathbb{S}_{LTA}) \arrow[r, shift left=2, "SL\alpha"] \arrow[r, shift right=2] & (SLA, \mu^\mathbb{S}_{LA}) \arrow[l] \arrow[r] & L_\varphi(A, \alpha)
\end{tikzcd}$$

This is the essential idea behind the construction of L_φ: roughly speaking, $L_\varphi(A, \alpha)$ is built using the same plan as (A, α), but with the free $\mathbb{T}$-modules replaced by the corresponding free $\mathbb{S}$-modules.

Theorem 1.5.5. *Let $\mathbb{S} = (S, \eta^\mathbb{S}, \mu^\mathbb{S})$ be a monad on $\mathcal{C}$, let $\mathbb{T} = (T, \eta^\mathbb{T}, \mu^\mathbb{T})$ be a monad on $\mathcal{D}$, and let $U^\mathbb{S} : \mathcal{C}^\mathbb{S} \to \mathcal{C}$ and $U^\mathbb{T} : \mathcal{D}^\mathbb{T} \to \mathcal{D}$ be the respective forgetful functors.*

Let $(R, \varphi) : \mathbb{S} \to \mathbb{T}$ be a morphism of monads, and suppose $\mathcal{C}^\mathbb{S}$ has coequalisers for all reflexive pairs.

(i) *If $R : \mathcal{C} \to \mathcal{D}$ has a left adjoint $L : \mathcal{D} \to \mathcal{C}$, then $R^\varphi : \mathcal{C}^\mathbb{S} \to \mathcal{D}^\mathbb{T}$ has a left adjoint $L_\varphi : \mathcal{D}^\mathbb{T} \to \mathcal{C}^\mathbb{S}$.*

(ii) *If $\mathcal{C} = \mathcal{D}$ and $R = \mathrm{id}_\mathcal{C}$, then $R^\varphi : \mathcal{C}^\mathbb{S} \to \mathcal{C}^\mathbb{T}$ is strictly monadic.*

Proof. Let $\eta^{RL} : \mathrm{id}_\mathcal{D} \Rightarrow RL$ and $\varepsilon^{LR} : LR \Rightarrow \mathrm{id}_\mathcal{C}$ be the unit and counit of the adjunction $L \dashv R : \mathcal{C} \to \mathcal{D}$. Given a natural transformation $\varphi : TR \Rightarrow RS$, the mate construction yields a natural transformation $\psi : LT \Rightarrow SL$:

$$\psi = \varepsilon^{LR} SL \bullet L\varphi L \bullet LT\eta^{RL}$$

Let (A, α) be an $\mathbb{S}$-module. We use ψ to construct a reflexive pair in $\mathcal{C}^\mathbb{S}$:

$$\begin{tikzcd}
(SLTA, \mu^\mathbb{S}_{LTA}) \arrow[r, shift left=1, "SL\alpha"] \arrow[r, shift right=1, "\mu^\mathbb{S}_{LA} \circ S\psi_A"'] & (SLA, \mu^\mathbb{S}_{LA})
\end{tikzcd}$$

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[3] However, note that 'base change' in the sense of pullbacks is a *right* adjoint!

These are $\mathbb{S}$-module homomorphisms because the $\mathbb{S}$-modules in question are free. We claim $SL\eta_A^{\mathbb{T}} : (SLA, \mu_{LA}^{\mathbb{S}}) \to (SLTA, \mu_{LTA}^{\mathbb{S}})$ is the required common right inverse: indeed, $SL\alpha \circ SL\eta^{\eta_{\mathbb{A}}^{\mathbb{T}}} = \mathrm{id}_{SLA}$ by the $\mathbb{T}$-module axioms, and

$$\begin{tikzcd}
SLA \arrow[r, "SL\eta_A^{RL}"] \arrow[d, "SL\eta_A^{\mathbb{T}}"'] & SLRLA \arrow[r, "\mathrm{id}"] \arrow[d, "SL\eta_{RLA}^{\mathbb{T}}"'] & SLRLA \arrow[r, "S\varepsilon_{LA}^{LR}"] \arrow[d, "SLR\eta_{LA}^{\mathbb{S}}"'] & SLA \arrow[d, "S\eta_{LA}^{\mathbb{S}}"'] \arrow[dr, "\mathrm{id}"] & \\
SLTA \arrow[r, "SLT\eta_A^{RL}"'] & SLTRLA \arrow[r, "SL\varphi_{LA}"'] & SLRSLA \arrow[r, "S\varepsilon_{SLA}^{LR}"'] & S^2LA \arrow[r, "\mu_{LA}^{\mathbb{S}}"'] & SLA
\end{tikzcd}$$

$$\begin{aligned}
\mu_{LA}^{\mathbb{S}} \circ S\psi_A \circ SL\eta_A^{\mathbb{T}} &= \mu_{LA}^{\mathbb{S}} \circ S\varepsilon_{SLA}^{LR} \circ SL\varphi_{LA} \circ \left(SLT\eta_A^{RL} \circ SL\eta_A^{\mathbb{T}}\right) \\
&= \mu_{LA}^{\mathbb{S}} \circ S\varepsilon_{SLA}^{LR} \circ \left(SL\varphi_{LA} \circ SL\eta_{RLA}^{\mathbb{T}}\right) \circ SL\eta_A^{RL} \\
&= \mu_{LA}^{\mathbb{S}} \circ \left(S\varepsilon_{SLA}^{LR} \circ SLR\eta_{LA}^{\mathbb{S}}\right) \circ SL\eta_A^{RL} \\
&= \left(\mu_{LA}^{\mathbb{S}} \circ S\eta_A^{\mathbb{S}}\right) \circ \left(S\varepsilon_{LA}^{LR} \circ SL\eta_A^{RL}\right) \\
&= \mathrm{id}_{SLA}
\end{aligned}$$

as required; therefore we may form the coequaliser in $\mathcal{C}^{\mathbb{T}}$:

$$\begin{tikzcd}
(SLTA, \mu_{LTA}^{\mathbb{S}}) \arrow[r, shift left=2, "SL\alpha"] \arrow[r, shift right=2, "\mu_{LA}^{\mathbb{S}} \circ S\psi_A"'] & (SLA, \mu_{LA}^{\mathbb{S}}) \arrow[l] \arrow[r, dashed, "\theta_{(A,\alpha)}"] & L_\varphi(A, \alpha)
\end{tikzcd}$$

Given a $\mathbb{T}$-module homomorphism $f : (A, \alpha) \to (B, \beta)$, there is a unique $\mathbb{S}$-module homomorphism $L_\varphi f : L_\varphi(A, \alpha) \to L_\varphi(B, \beta)$ making the diagram below commute:

$$\begin{tikzcd}
(SLTA, \mu_{LTA}^{\mathbb{S}}) \arrow[r, shift left, "SL\alpha"] \arrow[r, shift right, "\mu_{LA}^{\mathbb{S}} \circ S\psi_A"'] \arrow[d, "SLTf"'] & (SLA, \mu_{LA}^{\mathbb{S}}) \arrow[r, "\theta_{(A,\alpha)}"] \arrow[d, "SLf"'] & L_\varphi(A, \alpha) \arrow[d, dashed, "L_\varphi f"] \\
(SLTB, \mu_{LTB}^{\mathbb{S}}) \arrow[r, shift left, "SL\beta"] \arrow[r, shift right, "\mu_{LB}^{\mathbb{S}} \circ S\psi_B"'] & (SLB, \mu_{LB}^{\mathbb{S}}) \arrow[r, "\theta_{(B,\beta)}"'] & L_\varphi(B, \beta)
\end{tikzcd}$$

By standard arguments, this defines a functor $L_\varphi : \mathcal{D}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{S}}$.

Let (A, α) be any $\mathbb{T}$-module. By naturality, the diagram

$$\begin{tikzcd}
T^2A \arrow[r, "T\alpha"] \arrow[d, "T\eta_{TA}^{RL}"'] & TA \arrow[d, "T\eta_A^{RL}"] \\
TRLTA \arrow[r, "TRL\alpha"] \arrow[d, "\varphi_{LTA}"'] & TRLA \arrow[d, "\varphi_{LA}"] \\
RSLTA \arrow[r, "RSL\alpha"'] & RSLA
\end{tikzcd}$$

commutes, hence we have

$$\left(\varphi_{LA} \circ T\eta^{RL}_{A}\right) \circ T\alpha = RSL\alpha \circ \left(\varphi_{LTA} \circ T\eta^{RL}_{TA}\right)$$

and naturality also yields

$$\begin{aligned}\left(\varphi_{LA} \circ T\eta^{RL}_{A}\right) \circ \mu^{\mathbb{T}}_{A} &= \left(\varphi_{LA} \circ \mu^{\mathbb{T}}_{A}\right) \circ T^2\eta^{RL}_{A} \\ &= \left(R\mu^{\mathbb{S}}_{LA} \circ \varphi_{SLA}\right) \circ \left(T\varphi_{LA} \circ T^2\eta^{RL}_{A}\right) \\ &= \left(R\mu^{\mathbb{S}}_{LA} \circ \varphi_{SLA}\right) \circ T\left(\varphi_{LA} \circ T\eta^{RL}_{A}\right)\end{aligned}$$

so $\varphi_{LA} \circ T\eta^{RL}_{A}$ is a $\mathbb{T}$-module homomorphism $\left(TA, \mu^{\mathbb{T}}_{A}\right) \to R^{\varphi}\left(SLA, \mu^{\mathbb{S}}_{LA}\right)$, and similarly $\varphi_{LTA} \circ T\eta^{RL}_{TA}$ is a homomorphism $\left(T^2, \mu^{\mathbb{T}}_{TA}\right) \to R^{\varphi}\left(SLTA, \mu^{\mathbb{S}}_{LTA}\right)$. The diagram below also commutes:

$$\begin{tikzcd}
T^2A \arrow[r, "T^2\eta^{RL}_{A}"] \arrow[d, "T\eta^{RL}_{TA}"'] & T^2RLA \arrow[r, "T\varphi_{LA}"] \arrow[d, "T\eta^{RL}_{TRLA}"'] & TRSLA \arrow[d, "T\eta^{RL}_{RSLA}"'] & \\
TRLTA \arrow[r, "TRLT\eta^{RL}_{A}"] \arrow[d, "\varphi_{LTA}"'] & TRLTRLA \arrow[r, "TRL\varphi_{LA}"] \arrow[d, "\varphi_{LTRLA}"'] & TRLRSLA \arrow[r, "TR\varepsilon^{LR}_{SLA}"] \arrow[d, "\varphi_{LRSLA}"'] & TRSLA \arrow[d, "\varphi_{SLA}"'] \\
RSLTA \arrow[r, "RSLT\eta^{RL}_{A}"'] & RSLTRLA \arrow[r, "RSL\varphi_{LA}"'] & RSLRSLA \arrow[r, "RS\varepsilon^{LR}_{SLA}"'] & RS^2LA
\end{tikzcd}$$

Thus, we have

$$\begin{aligned}\left(\varphi_{LA} \circ T\eta^{RL}_{A}\right) \circ \mu^{\mathbb{T}}_{A} &= \left(R\mu^{\mathbb{S}}_{LA} \circ \varphi_{SLA}\right) \circ \left(T\varphi_{LA} \circ T^2\eta^{RL}_{A}\right) \\ &= R\mu^{\mathbb{S}}_{LA} \circ \varphi_{SLA} \circ TR\varepsilon^{LR}_{SLA} \circ T\eta^{RL}_{RSLA} \circ T\varphi_{LA} \circ T^2\eta^{RL}_{A} \\ &= R\mu^{\mathbb{S}}_{LA} \circ RS\left(\varepsilon^{LR}_{SLA} \circ L\varphi_{LA} \circ LT\eta^{RL}_{A}\right) \circ \left(\varphi_{LTA} \circ T\eta^{RL}_{TA}\right) \\ &= R\left(\mu^{\mathbb{S}}_{LA} \circ S\psi_{RX}\right) \circ \left(\varphi_{LTA} \circ T\eta^{RL}_{TA}\right)\end{aligned}$$

so there is a unique $\mathbb{T}$-module homomorphism $\eta^{\varphi}_{(A,\alpha)} : (A,\alpha) \to R^{\varphi}L_{\varphi}(A,\alpha)$ making the diagram below commute:

$$\begin{tikzcd}
\left(T^2A, \mu^{\mathbb{T}}_{TA}\right) \arrow[r, shift left, "T\alpha"] \arrow[r, shift right, "\mu^{\mathbb{T}}_{A}"'] \arrow[d, "\varphi_{LTA} \circ T\eta^{RL}_{TA}"'] & \left(TA, \mu^{\mathbb{T}}_{A}\right) \arrow[r, "\alpha"] \arrow[d, "\varphi_{LA} \circ T\eta^{RL}_{A}"'] & (A,\alpha) \arrow[d, dashed, "\eta^{\varphi}_{(A,\alpha)}"] \\
R^{\varphi}\left(SLTA, \mu^{\mathbb{S}}_{LTA}\right) \arrow[r, shift left, "RSL\alpha"] \arrow[r, shift right, "R\mu^{\mathbb{S}}_{LA} \circ RS\psi_{A}"'] & R^{\varphi}\left(SLA, \mu^{\mathbb{S}}_{LA}\right) \arrow[r, "R\theta_{(A,\alpha)}"'] & R^{\varphi}L_{\varphi}(A,\alpha)
\end{tikzcd}$$

Now, let $(A,\alpha) = R^{\varphi}(X,\xi) = \big(RX, R\xi \circ \varphi_X\big)$ for some $\mathbb{S}$-module (X,ξ). By naturality, the diagrams below commute:

$$\begin{tikzcd}
SLRSX \arrow[r, "SLR\xi"] \arrow[d, "S\varepsilon^{LR}_{SX}"'] & SLRX \arrow[d, "S\varepsilon^{LR}_{X}"] \\
S^2X \arrow[r, "S\xi"'] & SX
\end{tikzcd}$$

$$\begin{tikzcd}
SLTRLRX \arrow[r, "SL\varphi_{LRX}"] \arrow[d, "SLTR\varepsilon^{LR}_{X}"'] & SLRSLRX \arrow[r, "S\varepsilon^{LR}_{SLRX}"] \arrow[d, "SLRS\varepsilon^{LR}_{X}"'] & S^2LRX \arrow[r, "\mu^{\mathbb{S}}_{LRX}"] \arrow[d, "S^2\varepsilon^{LR}_{X}"'] & SLRX \arrow[d, "S\varepsilon^{LR}_{X}"] \\
SLTRX \arrow[r, "SL\varphi_X"'] & SLRSX \arrow[r, "S\varepsilon^{LR}_{SX}"'] & S^2X \arrow[r, "\mu^{\mathbb{S}}_{X}"'] & SX
\end{tikzcd}$$

Hence,

$$\begin{aligned}
S\varepsilon^{LR}_{X} \circ \big(SLR\xi \circ SL\varphi_X\big) &= S\xi \circ \Big(S\varepsilon^{LR}_{SX} \circ SL\varphi_X\Big) \\
S\varepsilon^{LR}_{X} \circ \Big(\mu^{\mathbb{S}}_{LRX} \circ S\psi_{RX}\Big) &= \mu^{\mathbb{S}}_{X} \circ \Big(S\varepsilon^{LR}_{SX} \circ SL\varphi_X\Big)
\end{aligned}$$

so we have a unique $\mathbb{S}$-module homomorphism $\varepsilon^{\varphi}_{(X,\xi)} : L_{\varphi}R^{\varphi}(X,\xi) \to (X,\xi)$ making the diagram below commute:

$$\begin{tikzcd}
\big(SLTRX, \mu^{\mathbb{S}}_{LTRX}\big) \arrow[r, shift left, "SLR\xi \circ SL\varphi_X"] \arrow[r, shift right, "\mu^{\mathbb{S}}_{LRX} \circ S\psi_{RX}"'] \arrow[d, "S\varepsilon^{LR}_{SX} \circ SL\varphi_X"'] & \big(SLRX, \mu^{\mathbb{S}}_{LRX}\big) \arrow[r, "\theta_{R^{\varphi}(X,\xi)}"] \arrow[d, "S\varepsilon^{LR}_{X}"'] & L_{\varphi}R^{\varphi}(X,\xi) \arrow[d, dashed, "\varepsilon^{\varphi}_{(X,\xi)}"] \\
\Big(S^2X, \mu^{\mathbb{S}}_{SX}\Big) \arrow[r, shift left, "S\xi"] \arrow[r, shift right, "\mu^{\mathbb{S}}_{X}"'] & \big(SX, \mu^{\mathbb{S}}_{X}\big) \arrow[r, "\xi"'] & (X,\xi)
\end{tikzcd}$$

We claim these are the unit and counit of an adjunction $L_{\varphi} \dashv R^{\varphi} : \mathcal{C}^{\mathbb{S}} \to \mathcal{D}^{\mathbb{T}}$. A standard argument shows that $\eta^{\varphi}_{(A,\alpha)}$ is natural in (A,α) and that $\varepsilon^{\varphi}_{(X,\xi)}$ is natural in (X,ξ). Consider the diagram below:

$$\begin{tikzcd}
\big(SLTA, \mu^{\mathbb{S}}_{LTA}\big) \arrow[r, shift left, "SL\alpha"] \arrow[r, shift right, "\mu^{\mathbb{S}}_{LA} \circ S\psi_A"'] \arrow[d, "SLT\eta^{\varphi}_{(A,\alpha)}"'] & \big(SLA, \mu^{\mathbb{S}}_{LA}\big) \arrow[r, "\theta_{(A,\alpha)}"] \arrow[d, "SL\eta^{\varphi}_{(A,\alpha)}"'] & L_{\varphi}(A,\alpha) \arrow[d, dashed, "L_{\varphi}\eta^{\varphi}_{(A,\alpha)}"] \\
F^{\mathbb{S}}LTRU^{\mathbb{S}}L_{\varphi}(A,\alpha) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "S\varepsilon^{LR}_{SL_{\varphi}(A,\alpha)} \circ SL\varphi_{L_{\varphi}(A,\alpha)}"'] & F^{\mathbb{S}}LRU^{\mathbb{S}}L_{\varphi}(A,\alpha) \arrow[r] \arrow[d, "S\varepsilon^{LR}_{L_{\varphi}(A,\alpha)}"'] & L_{\varphi}R^{\varphi}L_{\varphi}(A,\alpha) \arrow[d, dashed, "\varepsilon^{\varphi}_{L_{\varphi}(A,\alpha)}"] \\
F^{\mathbb{S}}U^{\mathbb{S}}F^{\mathbb{S}}U^{\mathbb{S}}L_{\varphi}(A,\alpha) \arrow[r, shift left, "S\varepsilon^{\mathbb{S}}_{L_{\varphi}(A,\alpha)}"] \arrow[r, shift right, "\mu^{\mathbb{S}}_{L_{\varphi}(A,\alpha)}"'] & F^{\mathbb{S}}U^{\mathbb{S}}L_{\varphi}(A,\alpha) \arrow[r, "\varepsilon^{\mathbb{S}}_{L_{\varphi}(A,\alpha)}"'] & L_{\varphi}(A,\alpha)
\end{tikzcd}$$

By the universal property of coequalisers, $\varepsilon^{\varphi}_{L_\varphi(A,\alpha)} \circ L_\varphi \eta^{\varphi}_{(A,\alpha)}$ is the unique $\mathbb{S}$-module homomorphism such that

$$\left(\varepsilon^{\varphi}_{L_\varphi(A,\alpha)} \circ L_\varphi \eta^{\varphi}_{(A,\alpha)}\right) \circ \theta_{(A,\alpha)} = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ \left(S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ SL\eta^{\varphi}_{(A,\alpha)}\right)$$

but we have

$$\begin{aligned}
&\varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ \left(S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ SL\eta^{\varphi}_{(A,\alpha)}\right) \\
&\qquad = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ SL\eta^{\varphi}_{(A,\alpha)} \circ \mathrm{id}_{SLA} \\
&\qquad = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ \left(SL\eta^{\varphi}_{(A,\alpha)} \circ SL\alpha\right) \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ SL\left(R\theta_{(A,\alpha)} \circ \varphi_{LA} \circ T\eta^{RL}_{A}\right) \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ \left(S\varepsilon^{LR}_{L_\varphi(A,\alpha)} \circ SLR\theta_{(A,\alpha)}\right) \circ SL\varphi_{LA} \circ SLT\eta^{RL}_{A} \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ S\theta_{(A,\alpha)} \circ \left(S\varepsilon^{LR}_{SLA} \circ SL\varphi_{LA} \circ SLT\eta^{RL}_{A}\right) \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \left(\varepsilon^{\mathbb{S}}_{L_\varphi(A,\alpha)} \circ S\theta_{(A,\alpha)}\right) \circ S\psi_{A} \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \left(\theta_{(A,\alpha)} \circ \mu^{\mathbb{S}}_{LA} \circ S\psi_{A}\right) \circ SL\eta^{\mathbb{T}}_{A} \\
&\qquad = \theta_{(A,\alpha)} \circ \left(SL\alpha \circ SL\eta^{\mathbb{T}}_{A}\right) \\
&\qquad = \theta_{(A,\alpha)}
\end{aligned}$$

and therefore the left triangle identity holds:

$$\varepsilon^{\varphi}_{L_\varphi(A,\alpha)} \circ L_\varphi \eta^{\varphi}_{(A,\alpha)} = \mathrm{id}_{L_\varphi(A,\alpha)}$$

On the other hand, by considering the diagram below,

$$\begin{tikzcd}
\left(T^2RX, \mu^{\mathbb{T}}_{TRX}\right) \arrow[r, shift left, "TR\xi \circ T\varphi_X"] \arrow[r, shift right, "\mu^{\mathbb{T}}_{TRX}"'] \arrow[d, "\varphi_{LTRX} \circ T\eta^{RL}_{TRX}"'] & \left(TRX, \mu^{\mathbb{T}}_{RX}\right) \arrow[r, "R\xi \circ \varphi_X"] \arrow[d, "\varphi_{LRX} \circ T\eta^{RL}_{RX}"'] & R^{\varphi}(X,\xi) \arrow[d, dashed, "\eta^{\varphi}_{R^{\varphi}(X,\xi)}"] \\
R^{\varphi}\left(SLTRX, \mu^{\mathbb{S}}_{LTRX}\right) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "RS\varepsilon^{LR}_{SX} \circ RSL\varphi_X"'] & R^{\varphi}\left(SLRX, \mu^{\mathbb{S}}_{LRX}\right) \arrow[r] \arrow[d, "RS\varepsilon^{LR}_{X}"'] & R^{\varphi} L_\varphi R^{\varphi}(X,\xi) \arrow[d, dashed, "R^{\varphi}\varepsilon^{\varphi}_{(X,\xi)}"] \\
R^{\varphi}\left(S^2X, \mu^{\mathbb{S}}_{SX}\right) \arrow[r, shift left, "RS\xi"] \arrow[r, shift right, "R\mu^{\mathbb{S}}_{X}"'] & R^{\varphi}\left(SX, \mu^{\mathbb{S}}_{X}\right) \arrow[r, "R\xi"'] & R^{\varphi} L_\varphi R^{\varphi}(X,\xi)
\end{tikzcd}$$

we see that $R^{\varphi}\varepsilon^{\varphi}_{(X,\xi)} \circ \eta^{\varphi}_{R^{\varphi}(X,\xi)}$ is the unique $\mathbb{T}$-module homomorphism such that

$$\left(R^{\varphi}\varepsilon^{\varphi}_{(X,\xi)} \circ \eta^{\varphi}_{R^{\varphi}(X,\xi)}\right) \circ \left(R\xi \circ \varphi_X\right) = R\xi \circ \left(RS\varepsilon^{LR}_{X} \circ \varphi_{LRX} \circ T\eta^{RL}_{RX}\right)$$

but we have

$$R\xi \circ \left(RS\varepsilon^{LR}_X \circ \varphi_{LRX}\right) \circ T\eta^{RL}_{RX} = R\xi \circ \varphi_X \circ \left(TR\varepsilon^{LR}_X \circ T\eta^{RL}_{RX}\right) = R\xi \circ \varphi_X$$

therefore the right triangle identity also holds:

$$R^\varphi \varepsilon^\varphi_{(X,\xi)} \circ \eta^\varphi_{R^\varphi(X,\xi)} = \mathrm{id}_{R^\varphi(X,\xi)}$$

This completes the proof of (i).

Now, assume $\mathcal{C} = \mathcal{D}$ and $R = \mathrm{id}_\mathcal{C}$, as in (ii). We just showed that the functor $R^\varphi : \mathcal{C}^\mathbb{S} \to \mathcal{C}^\mathbb{T}$ has a left adjoint, and it is clear that R^φ reflects isomorphisms under these assumptions. It is also clear that reflexive R^φ-split pairs in $\mathcal{C}^\mathbb{S}$ are reflexive $U^\mathbb{S}$-split pairs in particular, so R^φ must preserve coequalisers of reflexive R^φ-split pairs, since $U^\mathbb{S} = U^\mathbb{T} R^\varphi$ preserves and $U^\mathbb{T}$ reflects them. The monadicity theorem (1.2.12) then implies R^φ is monadic. It is clear that R^φ is also an amnestic isofibration, so by proposition 1.2.18, R^φ is strictly monadic. ■

Remark 1.5.6. Statement (ii) of the above theorem can also be regarded as a form of 'base change': it says that, given monads $\mathbb{S}$ and $\mathbb{T}$ on the same category $\mathcal{S}$ such that $\mathcal{S}^\mathbb{S}$ has reflexive coequalisers, if there is even one monad morphism $\mathbb{S} \to \mathbb{T}$, then $\mathcal{S}^\mathbb{S}$ is equivalent to the category of modules for a monad on $\mathcal{S}^\mathbb{T}$. It is tempting to write $\mathbb{S}/\mathbb{T}$ for this monad, but we should refrain from doing so as it is not analogous to the quotient construction for ordinary monoids.

Corollary 1.5.7. *Let $\mathbb{T}$ be a monad on a category $\mathcal{C}$, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}^\mathbb{T}$. If $\mathcal{C}'$ has coequalisers for all reflexive pairs and the restriction of the forgetful functor $U^\mathbb{T} : \mathcal{C}^\mathbb{T} \to \mathcal{C}$ to $\mathcal{C}'$ is monadic, then $\mathcal{C}'$ is a reflective subcategory of $\mathcal{C}^\mathbb{T}$.*

Proof. By proposition 1.4.6, there is a morphism of monads inducing the inclusion $\mathcal{C}' \hookrightarrow \mathcal{C}^\mathbb{T}$, and by (i) in the theorem above, such a functor has a left adjoint. ■

Example 1.5.8. **Ab**, the category of abelian groups, is monadic over **Set** and has all coequalisers, so it is a reflective subcategory of **Grp**, the category of groups.

6 Constructing monads

In this section we will look at ways of constructing new monads from various data. Since monads are supposed to be generalised algebraic theories, we look to universal algebra for inspiration.

One of the simplest ways of extending an algebraic theory is to add new constant symbols to the language. It is easiest to describe what we want to achieve in terms of the category of modules: if $\mathbb{S}$ is a given monad on a category $\mathcal{C}$ and X is an object in $\mathcal{C}$, the category of $\mathbb{S}$-modules equipped with an X-indexed family of "constants" is the comma category $(X \downarrow U^{\mathbb{S}})$; assuming the evident forgetful functor $(X \downarrow U^{\mathbb{S}}) \to \mathcal{C}$ is monadic, we wish to find an explicit description of the monad it induces.

Proposition 1.6.1. *Let $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ be a monad on a category $\mathcal{C}$ with binary coproducts. Let X be an object in $\mathcal{C}$. If we define a functor $T : \mathcal{C} \to \mathcal{C}$ by*

$$TA = S(A \amalg X) \qquad\qquad Tf = S(f \amalg \mathrm{id}_X)$$

and natural transformations $\eta^{\mathbb{T}} : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu^{\mathbb{T}} : T^2 \Rightarrow T$ by

$$\eta^{\mathbb{T}}_A = \eta^{\mathbb{S}}_{A \amalg X} \circ i_A \qquad\qquad \mu^{\mathbb{T}}_A = \mu^{\mathbb{S}}_{A \amalg X} \circ S\big(\big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A \amalg X} \circ j_A\big]\big)$$

where $i_A : A \to A \amalg X$ and $j_A : X \to A \amalg X$ are the coproduct insertions, then $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ is a monad on $\mathcal{C}$, and $(\mathrm{id}, Si) : \mathbb{T} \to \mathbb{S}$ is a morphism of monads.

Moreover, the forgetful functor $(X \downarrow \mathcal{C}^{\mathbb{S}}) \to \mathcal{C}$ is strictly monadic and induces the monad $\mathbb{T}$ constructed above.

Proof. First, note that the coproduct insertions define natural transformations $i : \mathrm{id}_{\mathcal{C}} \Rightarrow (-) \amalg X$ and $j : \Delta X \Rightarrow (-) \amalg X$, so $\eta^{\mathbb{T}}$ and $\mu^{\mathbb{T}}$ are indeed natural transformations. First, we verify the unit axioms:

$$\begin{aligned}
\mu^{\mathbb{T}}_A \circ T\eta^{\mathbb{T}}_A &= \mu^{\mathbb{S}}_{A \amalg X} \circ S\big(\big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A \amalg X} \circ j_A\big]\big) \circ S\big(\big(\eta^{\mathbb{S}}_{A \amalg X} \circ i_A\big) \amalg \mathrm{id}_X\big) \\
&= \mu^{\mathbb{S}}_{A \amalg X} \circ S\big(\big[\eta^{\mathbb{S}}_{A \amalg X} \circ i_A, \eta^{\mathbb{S}}_{A \amalg X} \circ j_A\big]\big) \\
&= \mu^{\mathbb{S}}_{A \amalg X} \circ S\big(\eta^{\mathbb{S}}_{A \amalg X} \circ [i_A, j_A]\big) \\
&= \mu^{\mathbb{S}}_{A \amalg X} \circ S\eta^{\mathbb{S}}_{A \amalg X} \\
&= \mathrm{id}_{S(A \amalg X)} = \mathrm{id}_{TA}
\end{aligned}$$

$$
\begin{aligned}
\mu^{\mathbb{T}}_A \circ \eta^{\mathbb{T}}_{TA} &= \mu^{\mathbb{S}}_{A\amalg X} \circ S\Big(\big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big]\Big) \circ \eta^{\mathbb{S}}_{TA\amalg X} \circ i_{TA} \\
&= \mu^{\mathbb{S}}_{A\amalg X} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ \big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big] \circ i_{TA} \\
&= \mu^{\mathbb{S}}_{A\amalg X} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ \mathrm{id}_{TA} \\
&= \mathrm{id}_{S(A\amalg X)} \circ \mathrm{id}_{TA} = \mathrm{id}_{TA}
\end{aligned}
$$

Now, let us write $k_A = [\mathrm{id}_{TA}, \eta^{\mathbb{S}} \circ j_A]$; this defines a natural transformation $k : T(-) \amalg X \Rightarrow T$, and $\mu^{\mathbb{T}} = \mu^{\mathbb{S}} \bullet Sk$. The coproduct axioms and naturality imply

$$
\begin{aligned}
Sk_A \circ \eta^{\mathbb{S}}_{TA\amalg X} \circ j_{TA} &= \eta^{\mathbb{S}}_{TA} \circ k_A \circ j_{TA} \\
&= \eta^{\mathbb{S}}_{TA} \circ \big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big] \circ j_{TA} \\
&= \eta^{\mathbb{S}}_{TA} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A
\end{aligned}
$$

and similarly, using the coproduct and monad axioms, we have

$$
\begin{aligned}
k_A \circ \big(\mu^{\mathbb{S}}_{A\amalg X} \amalg \mathrm{id}_X\big) &= \big[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big] \circ \big(\mu^{\mathbb{S}}_{A\amalg X} \amalg \mathrm{id}_X\big) \\
&= \big[\mu^{\mathbb{S}}_{A\amalg X}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big] \\
&= \big[\mu^{\mathbb{S}}_{A\amalg X}, \mu^{\mathbb{S}}_{A\amalg X} \circ \eta^{\mathbb{S}}_{TA} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big] \\
&= \mu^{\mathbb{S}}_{A\amalg X} \circ \big[\mathrm{id}_{STA}, \eta^{\mathbb{S}}_{TA} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\big]
\end{aligned}
$$

thus,

$$
\mu^{\mathbb{S}}_{A\amalg X} \circ \big[\mathrm{id}, Sk_A \circ \eta^{\mathbb{S}} \circ j_{TA}\big] = k_A \circ \big(\mu^{\mathbb{S}}_{A\amalg X} \amalg \mathrm{id}\big)
$$

and by the coproduct axioms,

$$
\begin{aligned}
Sk_A \circ k_{TA} &= Sk_A \circ \big[\mathrm{id}_{T^2A}, \eta^{\mathbb{S}}_{TA\amalg X} \circ j_{TA}\big] \\
&= \big[Sk_A, Sk_A \circ \eta^{\mathbb{S}}_{TA\amalg X} \circ j_{TA}\big] \\
&= \big[\mathrm{id}_{STA}, Sk_A \circ \eta^{\mathbb{S}}_{TA\amalg X} \circ j_{TA}\big] \circ (Sk_A \amalg \mathrm{id}_X)
\end{aligned}
$$

therefore the monad axioms make the diagram below commute:

$$\begin{tikzcd}
T^3A \arrow[r, "TSk_A"] \arrow[d, swap, "Sk_{TA}"] & TSTA \arrow[r, "T\mu^{\mathbb{S}}_{A\amalg X}"] \arrow[d] & T^2A \arrow[d, "Sk_A"] \\
ST^2A \arrow[r, "S^2k_A"] \arrow[d, swap, "\mu^{\mathbb{S}}_{TA\amalg X}"] & S^2TA \arrow[r, "S\mu^{\mathbb{S}}_{A\amalg X}"] \arrow[d, swap, "\mu^{\mathbb{S}}_{TA}"] & STA \arrow[d, "\mu^{\mathbb{S}}_{A\amalg X}"] \\
T^2A \arrow[r, swap, "Sk_A"] & STA \arrow[r, swap, "\mu^{\mathbb{S}}_{A\amalg X}"] & TA
\end{tikzcd}$$

The claim that $(\mathrm{id}, Si) : \mathbb{T} \to \mathbb{S}$ is a morphism of monads amounts to checking the commutativity of the diagrams below,

$$\begin{tikzcd}
A \arrow[r, "\eta^{\mathbb{S}}_A"] \arrow[d, "i_A"'] & SA \arrow[d, "Si_A"] \\
A \amalg X \arrow[r, "\eta^{\mathbb{S}}_{A\amalg X}"'] & TA
\end{tikzcd}$$

$$\begin{tikzcd}
S^2A \arrow[r, "S^2 i_A"] \arrow[dd, "\mu^{\mathbb{S}}_A"'] & STA \arrow[r, "Si_{TA}"] \arrow[dr, "\mathrm{id}"'] & T^2A \arrow[d, "Sk_A"] \\
 & & STA \arrow[d, "\mu^{\mathbb{S}}_{A\amalg X}"] \\
SA \arrow[rr, "Si_A"'] & & TA
\end{tikzcd}$$

and this is straightforward.

Now, let (A, α) be an $\mathbb{S}$-module and $e : X \to A$ be any morphism. Define $\bar{e} : A \amalg X \to A$ by $\bar{e} = [\mathrm{id}_A, e]$ and $\bar{\alpha} : TA \to A$ by $\bar{\alpha} = \alpha \circ S\bar{e}$. Then,

$$\begin{aligned}
S\bar{e} \circ k_A &= S\bar{e} \circ \left[\mathrm{id}_{TA}, \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\right] \\
&= \left[S\bar{e}, S\bar{e} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\right] \\
&= \left[\mathrm{id}_{SA}, S\bar{e} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A\right] \circ (S\bar{e} \amalg \mathrm{id}_X)
\end{aligned}$$

by the coproduct axioms, and also

$$\begin{aligned}
\bar{e} \circ (\alpha \amalg \mathrm{id}_X) &= [\mathrm{id}_A, e] \circ (\alpha \amalg \mathrm{id}_X) \\
&= [\alpha, e] \\
&= \left[\alpha, \alpha \circ \eta^{\mathbb{S}}_A \circ e\right] \\
&= \alpha \circ \left[\mathrm{id}_{SA}, \eta^{\mathbb{S}}_A \circ e\right]
\end{aligned}$$

but we have

$$S\bar{e} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A = \eta^{\mathbb{S}}_A \circ \bar{e} \circ j_A = \eta^{\mathbb{S}}_A \circ e$$

so the $\mathbb{S}$-module axioms imply diagram below commutes:

$$\begin{tikzcd}
T^2A \arrow[r, "TS\bar{e}"] \arrow[d, "Sk_A"'] & TSA \arrow[r, "T\alpha"] \arrow[d] & TA \arrow[d, "S\bar{e}"] \\
STA \arrow[r, "S^2\bar{e}"] \arrow[d, "\mu^{\mathbb{S}}_{A\amalg X}"'] & S^2A \arrow[r, "S\alpha"] \arrow[d, "\mu^{\mathbb{S}}_{A}"'] & SA \arrow[d, "\alpha"] \\
TA \arrow[r, "S\bar{e}"'] & SA \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

Thus $\bar{\alpha} = \alpha \circ S\bar{e}$ is indeed a $\mathbb{T}$-module structure on A. Now, let $\bar{\alpha}$ be given, and set

$$\alpha = \bar{\alpha} \circ Si_A \qquad\qquad e = \bar{\alpha} \circ \eta^{\mathbb{S}}_{A\amalg X} \circ j_A$$

then, writing $\bar{e}$ for $[\mathrm{id}_A, e]$, the coproduct axioms imply

$$\alpha \circ S\bar{e} = \bar{\alpha} \circ Si_A \circ S\bar{e} = \bar{\alpha} \circ \mathrm{id}_{TA}$$

and on the other hand, if $\bar{\alpha} = \alpha \circ S\bar{e}$, then

$$\bar{\alpha} \circ Si_A = \alpha \circ S\bar{e} \circ Si_A = \alpha \circ \mathrm{id}_{SA}$$

by the coproduct axioms, since $\bar{e} = [\mathrm{id}_A, e]$, and

$$\bar{\alpha} \circ \eta^{\mathbb{S}}_{A\amalg X} = \alpha \circ S\bar{e} \circ \eta^{\mathbb{S}}_{A\amalg X} = \alpha \circ \eta^{\mathbb{S}}_{A} \circ \bar{e} = \mathrm{id}_A \circ \bar{e}$$

by naturality and the $\mathbb{S}$-module axioms, so every $\mathbb{T}$-module $(A, \bar{\alpha})$ is of the form $\bar{\alpha} = \alpha \circ S\bar{e}$ for a unique $\mathbb{S}$-module (A, α) and morphism $e : X \to A$.

Now, let (A, α) and (B, β) be $\mathbb{S}$-modules, and let $e : X \to A$, $f : X \to B$ be morphisms. Suppose $h : (A, \alpha) \to (B, \beta)$ is a $\mathbb{S}$-module homomorphism such that $h \circ e = f$. Then,

$$h \circ \bar{\alpha} = h \circ \alpha \circ S\bar{e} = \beta \circ Sh \circ S\bar{e} = \beta \circ S\bar{f}$$

so h is also a $\mathbb{T}$-module homomorphism $h : (A, \bar{\alpha}) \to (B, \bar{\beta})$. Conversely, if h is a $\mathbb{T}$-module homomorphism, then

$$h \circ \alpha = h \circ \bar{\alpha} \circ Si_A = \bar{\beta} \circ Th \circ Si_A = \bar{\beta} \circ Si_B \circ Sh = \beta \circ Sh$$

so h is also a $\mathbb{S}$-module homomorphism, and by naturality,

$$\begin{aligned} h \circ e &= h \circ \bar{\alpha} \circ \eta^{\mathbb{S}}_{A \amalg X} \circ j_A \\ &= \bar{\beta} \circ Th \circ \eta^{\mathbb{S}}_{A \amalg X} \circ j_A \\ &= \bar{\beta} \circ Th \circ Sj_A \circ \eta^{\mathbb{S}}_X \\ &= \bar{\beta} \circ Sj_B \circ \eta^{\mathbb{S}}_X \\ &= \bar{\beta} \circ \eta^{\mathbb{S}}_{B \amalg X} \circ j_B = f \end{aligned}$$

as claimed. Thus $\mathcal{C}^{\mathbb{T}}$ is isomorphic to $(X \downarrow \mathcal{C}^{\mathbb{S}})$ as a category over $\mathcal{C}$. ■

Another natural way of extending an algebraic theory is to add a new n-ary operation that commutes with all the existing operations. For example, for the case $n = 1$, if A is a commutative ring, the polynomial algebra $A[x]$ can be characterised as the commutative ring such that giving an $A[x]$-module is the same thing as giving an A-module equipped with a distinguished A-linear endomorphism.

More generally, starting from some category $\mathcal{C}$, we form the category $\mathcal{D}$ whose objects are pairs (A, α_1), where A is an object of $\mathcal{C}$ and $\alpha_1 : A^n \to A$ is a morphism in $\mathcal{C}$; a morphism $f : (A, \alpha_1) \to (B, \beta_1)$ is then defined to be a morphism $f : A \to B$ in $\mathcal{C}$ such that $f \circ \alpha_1 = \beta_1 \circ (f \times \cdots \times f)$. This motivates the next definition:

Definition 1.6.2. Let J be an endofunctor on a category $\mathcal{C}$. A J-**module** consists of an object A in $\mathcal{C}$ and a morphism $\alpha_1 : JA \to A$, and a **homomorphism of J-modules** $f : (A, \alpha_1) \to (B, \beta_1)$ is any morphism $f : A \to B$ in $\mathcal{C}$ such that $f \circ \alpha_1 = \beta_1 \circ Jf$. The **category of J-modules** is denoted by $\mathcal{C}^J$.

Example 1.6.3. If $\mathcal{C} = \mathbf{Set}$ and J is the endofunctor $X \mapsto S \times X$ for a fixed set S, then a J-module is the same thing as a set equipped with an S-indexed family of endomorphisms, or equivalently, an action of the free monoid on S-many generators.

Since there are no axioms imposed on the structure of a J-module beyond the mere choice of a morphism $JA \to A$, one expects the algebraic theory it generates to be "free" in some sense. The following definition is one way of making this precise:

Definition 1.6.4. Let $J : \mathcal{C} \to \mathcal{C}$ be a functor. A **free monad on** J is any monad $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ on $\mathcal{C}$ equipped with a natural transformation $\iota^J : J \Rightarrow T$ with the following universal property: for any monad $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{T}})$ on $\mathcal{C}$ and any natural transformation $\varphi : J \Rightarrow S$, there is a unique morphism of monads[1] $(\mathrm{id}, \tilde{\varphi}) : \mathbb{S} \to \mathbb{T}$ such that $\varphi = \tilde{\varphi} \bullet \iota^J$.

Under certain circumstances, it is possible to construct the free monad on J by mimicking the usual construction of the free monoid on a set of generators. For example, if $\mathcal{C}$ has countable coproducts and J preserves them, then we could define

$$T = \mathrm{id} \amalg J \amalg J^2 \amalg \cdots$$

and equip T with the structure of a monad $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ in the same way as one equips the set of words with the structure of a monoid. In general, a more subtle approach is required. Consider the category of $\mathbb{T}$-modules for $\mathbb{T}$ as constructed above: by expanding the coproducts, we see that an $\mathbb{T}$-module is an object A in $\mathcal{C}$ together with a sequence $(\alpha_n : J^n A \to A \mid n \in \mathbb{N})$ such that

$$\alpha_0 \circ \mathrm{id}_A = \mathrm{id}_A$$
$$\alpha_m \circ J^m \alpha_n = \alpha_{n+m}$$

for all natural numbers n and m; hence, $\alpha_0 = \mathrm{id}_A$ and in general

$$\alpha_{n+1} = \alpha_1 \circ J\alpha_1 \circ \cdots \circ J^n \alpha_1$$

so to specify an $\mathbb{T}$-module it is enough to give an object A in $\mathcal{C}$ together with any morphism $\alpha_1 : JA \to A$. Thus, the category of J-modules is isomorphic to the category of $\mathbb{T}$-modules when J and $\mathbb{T}$ are as above, and this motivates the main construction in the following theorem of Barr [1970]:

Theorem 1.6.5 (Barr). *Let J be an endofunctor on a category $\mathcal{C}$.*

(i) *The forgetful functor $U^J : \mathcal{C}^J \to \mathcal{C}$ is strictly monadic if and only if it has a left adjoint.*

(ii) *If the forgetful functor $U^J : \mathcal{C}^J \to \mathcal{C}$ is (strictly) monadic, then the monad it induces is the free monad on J.*

---

[1] Note the direction!

Proof. U^J is clearly conservative and creates coequalisers for all reflexive U^J-split pairs, so Beck's monadicity theorem (1.2.11) implies U^J is monadic if and only if it has a left adjoint $F^J : \mathcal{C} \to \mathcal{C}^J$. It is also clear that U^J is an amnestic isofibration, so proposition 1.2.18 implies it is monadic if and only if it is strictly monadic. This proves claim (i).

Now, let $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ be the induced monad on $\mathcal{C}$, and let $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ be any monad on $\mathcal{C}$. Given a natural transformation $\varphi : J \Rightarrow S$, we get a functor $\mathrm{id}^{\varphi} : \mathcal{C}^{\mathbb{S}} \to \mathcal{C}^J$ sending a $\mathbb{S}$-module (A, α) to the J-module $(A, \alpha \circ \varphi_J)$; thus, by proposition 1.4.6, there is a unique morphism of monads $(\mathrm{id}, \tilde{\varphi}) : \mathbb{S} \to \mathbb{T}$ inducing the same functor.

Consider the left adjoint $F^J : \mathcal{C} \to \mathcal{C}^J$: for each object X in $\mathcal{C}$, there is a morphism $\gamma_X : JTX \to TX$ such that $F^J X = (TX, \gamma_X)$. Since F^J is a functor, this defines a natural transformation $\gamma : JT \Rightarrow T$, so we may define a natural transformation $\iota^J : J \Rightarrow T$ by setting

$$\iota^J = \gamma \bullet J\eta^{\mathbb{T}}$$

and we claim this is the required universal natural transformation. Indeed, the adjunction counit $\varepsilon^J : F^J U^J \Rightarrow \mathrm{id}_{\mathcal{C}^J}$ gives a J-module homomorphism $\varepsilon^J_{(A,\alpha_1)} : (TA, \gamma_A) \to (A, \alpha)$ for each J-module (A, α_1), i.e. $\varepsilon^J_{(A,\alpha_1)} \circ \gamma_A = \alpha_1 \circ J\varepsilon^J_{(A,\alpha_1)}$ and for each $\mathbb{S}$-module (A, α), we have the factorisation $\varepsilon^J_{(A,\alpha\circ\varphi_A)} = \alpha \circ \tilde{\varphi}_A$ because $(\mathrm{id}, \tilde{\varphi}) : \mathbb{S} \to \mathbb{T}$ induces the functor $\mathrm{id}^{\varphi} : \mathcal{C}^{\mathbb{S}} \to \mathcal{C}^J$; in particular, the diagram below commutes:

$$\begin{tikzcd}
JTSA \arrow[r, "J\tilde{\varphi}_{SA}"] \arrow[dd, "\gamma_{SA}"'] & JS^2A \arrow[r, "J\mu^{\mathbb{S}}_A"] & JSA \arrow[d, "\varphi_{SA}"] \\
 & & S^2A \arrow[d, "\mu^{\mathbb{S}}_A"] \\
TSA \arrow[r, "\tilde{\varphi}_{SA}"'] & S^2A \arrow[r, "\mu^{\mathbb{S}}_A"'] & SA
\end{tikzcd}$$

Thus,

$$\begin{aligned}
\tilde{\varphi} \bullet \left(\gamma \bullet J\eta^{\mathbb{T}}\right) &= \mu^{\mathbb{S}} \bullet \left(S\eta^{\mathbb{S}} \bullet \tilde{\varphi}\right) \bullet \gamma \bullet J\eta^{\mathbb{T}} && \text{by the monad axioms} \\
&= \mu^{\mathbb{S}} \bullet \tilde{\varphi}S \bullet \left(T\eta^{\mathbb{S}} \bullet \gamma\right) \bullet J\eta^{\mathbb{T}} && \text{by naturality} \\
&= \left(\mu^{\mathbb{S}} \bullet \tilde{\varphi}S \bullet \gamma S\right) \bullet JT\eta^{\mathbb{S}} \bullet J\eta^{\mathbb{T}} && \text{by naturality}
\end{aligned}$$

$$\begin{aligned}
&= \mu^{\mathbb{S}} \bullet \varphi S \bullet J\mu^{\mathbb{S}} \bullet \left(J\tilde{\varphi}S \bullet JT\eta^{\mathbb{S}}\right) \bullet J\eta^{\mathbb{T}} && \text{by above diagram} \\
&= \mu^{\mathbb{S}} \bullet \varphi S \bullet \left(J\mu^{\mathbb{S}} \bullet JS\eta^{\mathbb{S}}\right) \bullet J\tilde{\varphi} \bullet J\eta^{\mathbb{T}} && \text{by naturality} \\
&= \mu^{\mathbb{S}} \bullet \varphi S \bullet J\tilde{\varphi} \bullet J\eta^{\mathbb{T}} && \text{by the monad axioms}
\end{aligned}$$

But $(\mathrm{id}, \tilde{\varphi}) : \mathbb{S} \to \mathbb{T}$ is a morphism of monads, so

$$\begin{aligned}
\tilde{\varphi} \bullet \left(\gamma \bullet J\eta^{\mathbb{T}}\right) &= \mu^{\mathbb{S}} \bullet \varphi S \bullet \left(J\tilde{\varphi} \bullet J\eta^{\mathbb{T}}\right) \\
&= \mu^{\mathbb{S}} \bullet \left(\varphi S \bullet J\eta^{\mathbb{S}}\right) \\
&= \left(\mu^{\mathbb{S}} \bullet S\eta^{\mathbb{S}}\right) \bullet \varphi \\
&= \varphi
\end{aligned}$$

i.e. $\varphi = \tilde{\varphi} \bullet \iota^J$, as required. In particular, $\iota^J : J \Rightarrow T$ induces the identity morphism $(\mathrm{id}, \mathrm{id}) : \mathbb{T} \to \mathbb{T}$, by uniqueness. This proves claim (ii). ■

Example 1.6.6. Let $J : \mathbf{Set} \to \mathbf{Set}$ be the functor $X \mapsto 1 \amalg X$. A J-module is the same thing as a set A together with an element $x_0 \in X$ and a map $\alpha_1 : A \to A$. We may identify the set of natural numbers with the initial object $(\mathbb{N}, 0, \mathrm{suc})$ in $\mathbf{Set}^J$. The forgetful functor $U^J : \mathbf{Set}^J \to \mathbf{Set}$ has a left adjoint, namely the functor $X \mapsto (\mathbb{N} \amalg X, 0, \mathrm{suc} \amalg \mathrm{id}_X)$; thus, the functor $U^J : \mathbf{Set}^J \to \mathbf{Set}$ is monadic.

Example 1.6.7. If $\mathcal{C} = \mathbf{Set}$ and J is the endofunctor $X \mapsto X \times X$, then a J-module is the same thing as a magma, so the free monad on J must be the free magma monad.

Example 1.6.8. The category of $A[x]$-modules is equivalent to the category of modules for the identity endofunctor $\mathrm{id} : \mathbf{Mod}(A) \to \mathbf{Mod}(A)$. But $\mathbf{Mod}(A[x])$ is certainly monadic over $\mathbf{Mod}(A)$ (by e.g. theorem 1.5.5), so the free monad generated by the identity endofunctor exists.

We also have a partial converse to Barr's theorem. This proof is due to Kelly [1980] and requires some technical devices which we describe now.

Definition 1.6.9. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, and let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{A} \to \mathcal{C}$ be functors. A **right Kan extension** of G along F is a functor $\mathrm{Ran}_F G : \mathcal{B} \to \mathcal{C}$ equipped with a natural transformation $\varepsilon : (\mathrm{Ran}_F G)F \Rightarrow G$ with the following universal property: for any functor $H : \mathcal{B} \to \mathcal{C}$ and any natural transformation $\varphi : HF \Rightarrow G$, there exists a unique natural transformation $\tilde{\varphi} : H \Rightarrow \mathrm{Ran}_F G$ such that $\varphi = \varepsilon \bullet \tilde{\varphi} F$.

Theorem 1.6.10 (Kan). *If $\mathcal{A}$ is small and $\mathcal{C}$ is complete, then the right Kan extension of any functor $G : \mathcal{A} \to \mathcal{C}$ along any functor $F : \mathcal{A} \to \mathcal{B}$ exists.*

Proof. See [CWM, Ch. X, § 3, Cor. 2]. □

Proposition 1.6.11. *Let $G : \mathcal{A} \to \mathcal{C}$ and suppose the right Kan extension of G along G exists. Let $T = \operatorname{Ran}_G G$ and let $\varepsilon : TG \Rightarrow G$ be the universal natural transformation. If $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are the unique natural transformations satisfying the equations below,*

$$\varepsilon \bullet \eta G = \mathrm{id}_G \qquad\qquad \varepsilon \bullet \mu G = \varepsilon \bullet T\varepsilon$$

then $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$, called the **codensity monad** *for $G : \mathcal{A} \to \mathcal{C}$.*

Proof. Clearly, $\mathrm{id}_G : G \Rightarrow G$ and $\varepsilon \bullet T\varepsilon : T^2 G \Rightarrow G$ are natural transformations, so the natural transformations $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ exist by the universal property of T as a right Kan extension. Note that

$$\begin{aligned}
\varepsilon \bullet (\mu \bullet T\eta) G &= (\varepsilon \bullet \mu F) \bullet T\eta G \\
&= \varepsilon \bullet (T\varepsilon \bullet T\eta G) && \text{by definition of } \mu \\
&= \varepsilon \bullet \mathrm{id}_{TG} && \text{by definition of } \eta
\end{aligned}$$

hence $\mu \bullet T\eta = \mathrm{id}_T$, by uniqueness. Similarly,

$$\begin{aligned}
\varepsilon \bullet (\mu \bullet \eta T) G &= (\varepsilon \bullet \mu F) \bullet \eta TG \\
&= \varepsilon \bullet (T\varepsilon \bullet \eta TG) && \text{by definition of } \mu \\
&= (\varepsilon \bullet \eta G) \bullet \varepsilon && \text{by naturality} \\
&= \mathrm{id}_{TG} \bullet \varepsilon && \text{by definition of } \eta
\end{aligned}$$

so $\mu \bullet \eta T = \mathrm{id}_T$, and

$$\begin{aligned}
\varepsilon \bullet (\mu \bullet T\mu) G &= (\varepsilon \bullet \mu G) \bullet T\mu G \\
&= \varepsilon \bullet (T\varepsilon \bullet T\mu G) && \text{by definition of } \mu \\
&= (\varepsilon \bullet T\varepsilon) \bullet T^2\varepsilon && \text{by definition of } \mu \\
&= \varepsilon \bullet \left(\mu G \bullet T^2\varepsilon\right) && \text{by definition of } \mu \\
&= (\varepsilon \bullet T\varepsilon) \bullet \mu TG && \text{by naturality} \\
&= \varepsilon \bullet (\mu \bullet \mu T) G && \text{by definition of } \mu
\end{aligned}$$

so $\mu \bullet T\mu = \mu \bullet \mu T$, as required for a monad. ■

It is worth noting that $\operatorname{Ran}_F G$ can exist even when the domain of F is not essentially small. Here is one situation of particular interest to us:

Proposition 1.6.12. *Let $U : \mathcal{A} \to \mathcal{B}$ be a functor. If $F : \mathcal{B} \to \mathcal{A}$ is a* left *adjoint of F and $G : \mathcal{A} \to \mathcal{C}$ is any functor, then $GF : \mathcal{B} \to \mathcal{C}$ is a* right *Kan extension of G along F.*

Proof. Let $\eta : \mathrm{id}_{\mathcal{A}} \Rightarrow UF$ and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{B}}$ be the unit and counit of the adjunction $F \dashv U$. The universal natural transformation making GF into a right Kan extension is $G\varepsilon : GFU \Rightarrow G$. Indeed, suppose $H : \mathcal{A} \to \mathcal{C}$ is any functor and $\varphi : HU \Rightarrow G$ is any natural transformation. Then, by the right triangle identity,

$$\varphi = \varphi \bullet H(U\varepsilon \bullet \eta U) = G\varepsilon \bullet (\varphi F \bullet H\eta)U$$

so $\varphi F \bullet H\eta$ is a natural transformation $\tilde{\varphi} : H \Rightarrow GF$ such that $\varphi = G\varepsilon \bullet \tilde{\varphi}U$; but if $\tilde{\varphi}$ is such a natural transformation, the left triangle identity gives us

$$\varphi F \bullet H\eta = (G\varepsilon \bullet \tilde{\varphi}U)F \bullet H\eta = G\varepsilon F \bullet GF\eta \bullet \tilde{\varphi} = \tilde{\varphi}$$

and so $\tilde{\varphi}$ is uniquely determined by φ and the equation $\varphi = G\varepsilon \bullet \tilde{\varphi}U$. ■

Corollary 1.6.13. *If $U : \mathcal{A} \to \mathcal{C}$ has a left adjoint $F : \mathcal{C} \to \mathcal{A}$, then the codensity monad for U exists and is the monad induced by the adjunction $F \dashv U$.*

Proof. Let $\eta : \mathrm{id}_{\mathcal{A}} \Rightarrow UF$ and $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathcal{B}}$ be the unit and counit of the adjunction $F \dashv U$, and let $\mathbb{T} = (T, \eta, \mu)$ be the induced monad. It is not hard to check that

$$U\varepsilon \bullet \eta U = \mathrm{id}_U \qquad\qquad U\varepsilon \bullet \mu U = U\varepsilon \bullet UFU\varepsilon$$

and so the calculations above and proposition 1.6.11 imply that $\mathbb{T}$ is indeed the codensity monad for $U : \mathcal{A} \to \mathcal{C}$. ■

Thus, the codensity monad for general functor $G : \mathcal{A} \to \mathcal{C}$, if it exists, can be thought of as the monad $\mathbb{T}$ such that $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ is the "best approximation" of G by a functor with a left adjoint. This idea is discussed in detail in [Linton, 1969a], but we briefly indicate the sense in which it is the *best* approximation:

Proposition 1.6.14. *Let $\mathcal{C}$ be a complete category and let $G : \mathcal{A} \to \mathcal{C}$ be a functor. If the codensity monad $\mathbb{E} = (E, \eta^{\mathbb{E}}, \mu^{\mathbb{E}})$ for G exists, then the universal natural transformation $\varepsilon^G : EG \Rightarrow G$ induces the following bijections:*

(i) *For any endofunctor* J *and any functor* $\tilde{G} : \mathcal{A} \to \mathcal{C}^J$ *such that* $G = U^J \tilde{G}$, *there is a unique natural transformation* $\tilde{\varphi} : J \Rightarrow E$ *such that* $\tilde{G}A = \left(GA, \varepsilon^G_A \circ \tilde{\varphi}_{GA}\right)$, *and vice versa.*

(ii) *For any monad* $\mathbb{T} = (T, \eta, \mu)$ *and any functor* $\tilde{G} : \mathcal{A} \to \mathcal{C}^{\mathbb{T}}$ *such that* $G = U^{\mathbb{T}} \tilde{G}$, *there is a unique monad morphism* $(\mathrm{id}, \tilde{\alpha}) : \mathbb{E} \to \mathbb{T}$ *such that* $\tilde{G}A = \left(GA, \varepsilon^G_A \circ \tilde{\alpha}_{GA}\right)$, *and vice versa.*

Proof. (i). Giving a functor $\tilde{G} : \mathcal{A} \to \mathcal{C}^J$ such that $G = U^J \tilde{G}$ is the same thing as giving a natural transformation $\varphi : JG \Rightarrow G$ such that $\tilde{G}A = (GA, \varphi_A)$, and the universal property of E as a right Kan extension of G along G gives us a unique natural transformation $\tilde{\varphi} : J \Rightarrow E$ such that $\varepsilon^G \bullet \tilde{\varphi} G = \varphi$.

(ii). Similarly, giving a functor $\tilde{G} : \mathcal{A} \to \mathcal{C}^{\mathbb{T}}$ such that $G = U^{\mathbb{T}} \tilde{G}$ is the same thing as giving a natural transformation $\alpha : TG \Rightarrow G$ such that $\tilde{G}A = (GA, \alpha_A)$, so there is a unique natural transformation $\tilde{\alpha} : T \Rightarrow E$ such that $\alpha = \varepsilon^G \bullet \tilde{\alpha} G$. Moreover, because α_A is a natural $\mathbb{T}$-module structure on A, we have

$$\begin{aligned}
\varepsilon^G \bullet \left(\tilde{\alpha} \bullet \mu^{\mathbb{T}}\right) G &= \alpha \bullet \mu^{\mathbb{T}} G \\
&= \alpha \bullet T\alpha \\
&= \left(\varepsilon^G \bullet \tilde{\alpha} G\right) \bullet \left(T\varepsilon^G \bullet T\tilde{\alpha} G\right) \\
&= \varepsilon^G \bullet E\varepsilon^G \bullet \tilde{\alpha} EG \bullet T\tilde{\alpha} G \\
&= \varepsilon^G \bullet \left(\mu^{\mathbb{E}} \bullet \tilde{\alpha} E \bullet T\tilde{\alpha}\right) G
\end{aligned}$$

so we must have $\tilde{\alpha} \bullet \mu^{\mathbb{T}} = \mu^{\mathbb{E}} \bullet \tilde{\alpha} E \bullet T\tilde{\alpha}$ by universality of ε^G, and similarly

$$\varepsilon^G \bullet \left(\tilde{\alpha} \bullet \eta^{\mathbb{T}}\right) G = \alpha \bullet \eta^{\mathbb{T}} G = \mathrm{id}_G = \varepsilon^G \bullet \eta^{\mathbb{E}} G$$

so $\tilde{\alpha} \bullet \eta^{\mathbb{T}} = \eta^{\mathbb{E}}$. Thus, $(\mathrm{id}, \tilde{\alpha}) : \mathbb{E} \to \mathbb{T}$ is indeed a monad morphism. The converse is obtained by reversing the above argument. ■

The above result is reminiscent of the representation-theoretic definition of actions: after all, if M is a monoid and X is a set, then to give a left M-action on X is the same thing as to give a monoid homomorphism $M \to \mathrm{End}(X)$. Accordingly, some authors refer to the codensity monad for G as the **endomorphism monad** of G.

Under mild hypotheses on the category $\mathcal{C}$, we can give a completely explicit description of the codensity monad for G when G is the inclusion of an object A into $\mathcal{C}$:

Proposition 1.6.15. *Let C be a locally small category. Suppose C has X-fold products for all sets X, i.e. assume that for every object A in C and every set X, there is an object A^X in C and a bijection*

$$\mathbf{Set}(X, C(B,A)) \cong C\left(B, A^X\right)$$

that is natural in A, B, and X. Then, the codensity monad for $\{A\} \hookrightarrow C$ exists for all objects A in C and is given by $\mathbb{E}\mathrm{nd}(A) = \left(E^A, \eta^A, \mu^A\right)$, where $E^A A = A^{C(B,A)}$, $\eta^A_B : B \longrightarrow A^{C(B,A)}$ is given in component $f : B \longrightarrow A$ by $f : B \longrightarrow A$ itself, and $\mu^A_B : A^{C\left(E^A B, A\right)} \longrightarrow A^{C(B,A)}$ is given in component $f : B \longrightarrow A$ by projecting to component $\pi_f : A^{C(B,A)} \longrightarrow A$ of $A^{C\left(E^A B, A\right)}$.

Proof. Let $\pi_{\mathrm{id}_A} : A^{C(A,A)} \longrightarrow A$ be the projection to the $\mathrm{id}_A : A \longrightarrow A$ component. We claim this is the universal morphism making E^A the right Kan extension of $\{A\} \hookrightarrow C$ along itself. Let $H : C \longrightarrow C$ be any functor and let $h : HA \longrightarrow A$ be any morphism. If $\tilde{\varphi} : H \Rightarrow E^A$ is any natural transformation such that $h = \pi_{\mathrm{id}_A} \circ \tilde{\varphi}_A$, then the diagram below

$$\begin{tikzcd}
HB \arrow[r, "\tilde{\varphi}_B"] \arrow[d, "Hf"'] & A^{C(B,A)} \arrow[r, "\pi_f"] \arrow[d, "A^{C(f,A)}"'] & A \arrow[d, "\mathrm{id}"] \\
HA \arrow[r, "\tilde{\varphi}_A"'] & A^{C(A,A)} \arrow[r, "\pi_{\mathrm{id}_A}"'] & A
\end{tikzcd}$$

commutes for all morphisms $f : B \longrightarrow A$ in C, so $\tilde{\varphi}_B : HB \longrightarrow A^{C(B,A)}$ must be given in component $f : B \longrightarrow A$ by $h \circ Hf$. Conversely, it is not hard to check that $\tilde{\varphi}_B$ so defined constitute a natural transformation $\tilde{\varphi} : H \Rightarrow E^A$ such that $h = \pi_{\mathrm{id}_A} \circ \tilde{\varphi}_A$.

Now, to show that $\left(E^A, \eta^A, \mu^A\right)$ is the codensity monad for $\{A\} \hookrightarrow C$, by proposition 1.6.11 it is enough to check that these equations hold:

$$\pi_{\mathrm{id}_A} \circ \eta^A_A = \mathrm{id}_A \qquad\qquad \pi_{\mathrm{id}_A} \circ \mu^A_A = \pi_{\mathrm{id}_A} \circ A^{C\left(\pi_{\mathrm{id}_A}, A\right)}$$

The first equation holds by construction of η^A, and the LHS of the second equation is $\pi_{\pi_{\mathrm{id}_A}}$, but $\pi_f = \pi_{\mathrm{id}_A} \circ A^{C(f,A)}$ for all morphisms $f : B \longrightarrow A$ in C, so the second equation also holds. Note that the argument in the above paragraph implies

$$\pi_f \circ \eta^A_B = f$$
$$\pi_f \circ \mu^A_B = \pi_{\pi_{\mathrm{id}_A}} \circ A^{C\left(A^{C(f,A)}, A\right)} = \pi_{\pi_f}$$

and so η^A and μ^A must be defined as in the statement of the proposition. ■

Corollary 1.6.16. *If $\mathcal{C}$ is a complete category and $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ is the free monad generated by an endofunctor J, then the functor $\mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{J}$ induced by the universal natural transformation $\iota^J : J \Rightarrow T$ is an isomorphism of categories. In particular, $\mathbb{T}$ must be constructed as in Barr's theorem (1.6.5).*

Proof. Let $\mathcal{A}$ be any small subcategory of $\mathcal{C}^J$, and let $G : \mathcal{A} \to \mathcal{C}$ be the composite of the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}^J$ and the forgetful functor $U^J : \mathcal{C}^J \to \mathcal{C}$. Let $\mathbb{E} = (E, \eta^{\mathbb{E}}, \mu^{\mathbb{E}})$ be the codensity monad for G; this exists by theorem 1.6.10, since $\mathcal{A}$ is small. By proposition 1.6.14, the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}^J$ corresponds to unique natural transformation $\tilde{\varphi} : J \Rightarrow E$, and the universal property of $\mathbb{T}$ gives a unique monad morphism $(\mathrm{id}, \tilde{\alpha}) : \mathbb{E} \to \mathbb{T}$ such that $\tilde{\alpha} \bullet \iota^J = \tilde{\varphi}$, and this monad morphism in turn corresponds to a unique functor $\tilde{G} : \mathcal{A} \to \mathcal{C}^{\mathbb{T}}$.

Inspecting the construction of $\tilde{G}$, we see that for each (X, ξ_1) in $\mathcal{A}$ there is a (X, ξ) in $\mathcal{C}^{\mathbb{T}}$ such that $\xi_1 = \xi \circ \iota^J$, and $\tilde{G}(X, \xi_1) = (X, \xi)$, while $\tilde{G}$ acts as the identity on morphisms. Thus, by varying $\mathcal{A}$, we obtain a two-sided inverse for the functor $\mathcal{C}^{\mathbb{T}} \to \mathcal{C}^J$. ■

7 Arities for monads

Classical algebraic theories are usually described in terms of a set of operations of finite arity: for example, the theory of groups has a constant (the unit), a unary operation (inversion), and a binary operation (the group operation). However, if we think of monads on **Set** as generalised algebraic theories, then we also have to generalise our notions of 'operation' and 'arity' in order to accommodate the algebraic theories embodied by some monads. We do not even have to go very far to find an example of a monad that cannot be described in terms of finitary operations:

Example 1.7.1. Let $T : \mathbf{Set} \to \mathbf{Set}$ be the covariant powerset functor. Let $\eta : \mathrm{id} \Rightarrow T$ be the singleton map $x \mapsto \{x\}$, and let $\mu : T^2 \Rightarrow T$ be the union operation $Y \mapsto \bigcup_{Z \in Y} Z$. It is straightforward to verify that we have a monad $\mathbb{T} = (T, \eta, \mu)$, and with a little more work, we see that a $\mathbb{T}$-module is the same thing as a cocomplete join semilattice. We will later[1] show precisely how $\mathbb{T}$ fails to be a finitary algebraic theory, but for now we appeal to intuition: if

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[1] See example 2.4.23.

$\mathbb{T}$ were finitary, then we would have criteria for checking whether a homomorphism of semilattices is cocontinuous purely in terms of finitary joins, which is absurd.

The road to generalised arities for monads begins with the following observations.[2] Let $\mathcal{C}$ be a category of algebraic structures, which for the purposes of this discussion means a locally small category with finite products and equipped with a free–forgetful adjunction $F \dashv U : \mathcal{C} \to \mathbf{Set}$. What do we mean by an 'n-ary operation' ω in $\mathcal{C}$? Of course, it must be a family of maps $\omega_A : (UA)^n \to UA$ for each object A in $\mathcal{C}$, and it must be preserved by homomorphisms in the sense that

$$Uf \circ \omega_A = \omega_B \circ (Uf \times \cdots \times Uf)$$

for all homomorphisms $f : A \to B$ in $\mathcal{C}$. Thus, ω is a natural transformation $(U-)^n \Rightarrow U$. But the functor $(U-)^n$ is represented by the free object Fn, so the Yoneda lemma implies that there is a natural bijection between n-ary operations and elements of UFn, or equivalently, between n-ary operations and morphisms $F1 \to Fn$ in $\mathcal{C}$.

If $\mathcal{C}$ is a category of *finitary* algebraic structures, then we expect these n-ary operations to be exhaustive, in the sense that any map $UA \to UB$ commuting with *all* n-ary operations (for all natural numbers n) should come from a unique morphism $A \to B$ in $\mathcal{C}$. More formally, if $\mathcal{A}$ is the full subcategory of $\mathcal{C}$ spanned by the objects Fn, then we are asserting that $\mathcal{C}$ embeds as a *full* subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ by sending an object A to (the restriction of) the representable functor $\mathcal{C}(-, A)$. This will be the prototype for our notion of 'arities' for a monad.

One difficulty that arises when passing from finitary algebraic theories to the generalised algebraic theories of monads is in the size of the category $\mathcal{A}$. Notice that if $\mathcal{A}$ is not small, then the "category" $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ in general fails to be locally small, and cannot even be formally defined within von Neumann–Bernays–Gödel class–set theory. As such, we must use some clever circumlocution to capture the notion of a "fully faithful embedding" in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ without mentioning the "category" $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ directly.

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[2] This exposition is based on a seminar talk given by Richard Garner at Cambridge in March 2012.

Definition 1.7.2. Let $F : \mathcal{B} \to C$ be a functor, let X be an object in C, and let $U^X : (F \downarrow X) \to C$ be the forgetful functor sending an object $(A, f : FA \to X)$ in the comma category $(F \downarrow X)$ to FA in C. The **canonical diagram** of shape $(F \downarrow X)$ is U^X, and the **canonical cocone** to X is the cocone $\lambda^X : U^X \Rightarrow \Delta X$ given by $\lambda^X_{(A,f)} = f$. A **dense functor** is a functor $F : \mathcal{B} \to C$ such that the canonical cocone $\lambda^X : U^X \Rightarrow \Delta X$ is colimiting for every object X in C.

Proposition 1.7.3. *Let $\mathcal{B}$ be a small category and C be a locally small category. A functor $F : \mathcal{B} \to C$ is dense if and only if the functor $H_\bullet : C \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ defined by $H_X = C(F(-), X)$ is fully faithful.*

Proof. Let X and Y be objects in C, and consider a natural transformation $\alpha : H_X \Rightarrow H_Y$. Each element of $H_X(A)$ is a morphism $f : FA \to X$ in C, and is mapped to a morphism $\alpha_A(f) : FA \to Y$ by $\alpha_A : H_X(A) \to H_Y(A)$. Let $U^X : (F \downarrow X) \to C$ be the canonical diagram of shape $(F \downarrow X)$. Naturality of α implies, for each commutative triangle of the form

$$\begin{tikzcd}
FA \arrow[dd, "Fa"'] \arrow[dr, "f"] & \\
& X \\
FA' \arrow[ur, "f'"'] &
\end{tikzcd}$$

in C, the corresponding triangle

$$\begin{tikzcd}
FA \arrow[dd, "Fa"'] \arrow[dr, "\alpha_A(f)"] & \\
& X \\
FA' \arrow[ur, "\alpha_{A'}(f')"'] &
\end{tikzcd}$$

also commutes; in other words, α defines a cocone $U^X \Rightarrow \Delta Y$. Conversely, any cocone $U^X \Rightarrow \Delta Y$ defines a natural transformation $H_X \Rightarrow H_Y$. On the other hand, if $\alpha = H_h$ for some morphism $h : X \to Y$, then by definition that means $\alpha_A(f) = h \circ f$. So $H_\bullet : C \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful if and only if the canonical cocone from U^X to X is colimiting for every object X in C. ■

Example 1.7.4. If C is a small category, then the above proposition together with the Yoneda lemma implies the identity functor $\mathrm{id} : C \to C$ is dense. This is true even when C is a large category: indeed, in this case, the comma category $(\mathrm{id} \downarrow X)$ is just the slice category $(C \downarrow X)$, but $\mathrm{id} : X \to X$ is the terminal object in $(C \downarrow X)$, so the canonical cocone is certainly colimiting.

The definition below generalises the notion studied by Leinster [2004]; it was originally suggested by Steve Lack and first published by Weber [2007].

Definition 1.7.5. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a locally small category C. A **category of arities** for $\mathbb{T}$ is a subcategory $\mathcal{A}$ of C with the following properties:

- The inclusion $\mathcal{A} \hookrightarrow C$ is dense.
- If $\lambda^X : U^X \Rightarrow \Delta X$ is the canonical cocone on the diagram of shape $(\mathcal{A} \downarrow X)$ and A is an object in $\mathcal{A}$, then $C(A, T\lambda^X) : C(A, TU^X) \Rightarrow \Delta C(A, TX)$ is a colimiting cocone in **Set**.

A **monad with arities** is a monad equipped with a *small* category of arities.

Example 1.7.6. Let M be a monoid and let $\mathbb{T} = (M \times (-), \eta, \mu)$ be the induced monad on **Set**, as in example 1.1.3. By proposition 1.7.3, the full subcategory $\{1\}$ of **Set** is a dense subcategory, and since $M \times (-)$ preserves all coproducts, $\{1\}$ is a category of arities for $\mathbb{T}$. This should come as no surprise to us since the theory of M-sets can be described as an equational theory with M-many unary operations and no constants.

Example 1.7.7. A monad may admit many categories of arities. For example, if $\mathbb{T}$ is any monad on a category C, then C itself is trivially a category of arities for $\mathbb{T}$. The notion is most useful when the category of arities is small, which is why we have a size restriction in the definition of 'monad with arities.'

Lemma 1.7.8. *Let C be a locally small category and let $F : \mathcal{B} \to C$ be a dense functor. Let $D : \mathcal{J} \to C$ be a diagram and let $\lambda : D \Rightarrow \Delta X$ be a cocone in C. If $C(FB, \lambda) : C(FB, D) \Rightarrow \Delta C(FB, X)$ is a colimiting cocone in* **Set** *for each object B in $\mathcal{B}$, then:*

(i) *The cocone $\lambda : D \Rightarrow \Delta X$ is colimiting in C.*

(ii) *For any morphism $g : FB \to X$ in C, there is a j in $\mathcal{J}$ and a morphism $\tilde{g} : FB \to Dj$ in C such that $g = \lambda_j \circ \tilde{g}$.*

Proof. (i). In the case when $\mathcal{B}$ is small, this is an easy corollary of the pointwise construction of colimits in $[\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$, the fact that $\mathcal{C}(F, -) : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful (proposition 1.7.3), and the fact that a fully faithful functor reflects all colimits. Spelling all this out in elementary terms proves the general case where $\mathcal{B}$ is not necessarily small.

(ii). Colimiting cocones are jointly epimorphic sinks, so if $g \in \mathcal{C}(FB, X)$, for some j in $\mathcal{J}$, there exists $\tilde{g}$ in $\mathcal{C}(FB, Dj)$ such that $\lambda_j \circ \tilde{g}$, as required. ■

Corollary 1.7.9. *If $\mathbb{T} = (T, \eta, \mu)$ is a monad on a locally small category $\mathcal{C}$, $\mathcal{A}$ is a category of arities for $\mathbb{T}$, and $\lambda^X : U^X \Rightarrow \Delta X$ is the canonical cocone on the diagram of shape $(\mathcal{A} \downarrow X)$, then:*

(i) *The cocone $T\lambda^X : TU^X \Rightarrow \Delta TX$ is colimiting in $\mathcal{C}$.*

(ii) *If B is any object in $\mathcal{A}$ and $g : B \to TX$ is a morphism, then there exists a factorisation $g = Tf \circ \hat{g}$ with $f \in \operatorname{ob}(\mathcal{A} \downarrow X)$.* ■

Proposition 1.7.10. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a locally small category $\mathcal{C}$. Let $\mathcal{A}$ be any subcategory of $\mathcal{C}$, and let $\mathbf{\Theta}_{\mathcal{A}}$ be the full subcategory of $\mathcal{C}^{\mathbb{T}}$ spanned by those $\mathbb{T}$-modules (strictly) of the form $F^{\mathbb{T}}A = (TA, \mu_A)$ for some A in $\mathcal{A}$. If $\mathcal{A}$ is a category of arities for $\mathbb{T}$, then the inclusion $\mathbf{\Theta}_{\mathcal{A}} \hookrightarrow \mathcal{C}^{\mathbb{T}}$ is dense.*

Proof. By the free–forgetful adjunction, there is a natural bijection

$$\mathcal{C}^{\mathbb{T}}\big((TA, \mu_A), (X, \xi)\big) \cong \mathcal{C}(A, X)$$

so each object of $(\mathbf{\Theta}_{\mathcal{A}} \downarrow (X, \xi))$ is of the form $\bar{f} = \xi \circ Tf : (TA, \mu_A) \to (X, \xi)$ for a unique $f = \bar{f} \circ \eta_A : A \to X$ in $(\mathcal{A} \downarrow X)$. We shall make frequent use of this bar notation throughout this proof.

Let $\varphi : U^{(X,\xi)} \Rightarrow \Delta(Y, \theta)$ be a cocone on the canonical diagram of shape $(\mathbf{\Theta}_{\mathcal{A}} \downarrow (X, \xi))$. That means whenever the inner triangle in the diagram below commutes,

$$\begin{tikzcd}
(TA, \mu_A) \arrow[dd] \arrow[dr, "\bar{f}"] \arrow[drr, bend left, "\varphi_{(A,f)}"] & & \\
& (X, \xi) & (Y, \theta) \\
(TA', \mu_{A'}) \arrow[ur, "\bar{f}'"'] \arrow[urr, bend right, "\varphi_{(A',f')}"'] & &
\end{tikzcd}$$

the outer triangle also commutes. By composing with the natural transformation $\eta : \mathrm{id}_C \Rightarrow T$, we obtain the outer triangle in the diagram below,

$$\begin{tikzcd}
A \arrow[dd] \arrow[dr, "f"] \arrow[drr, bend left, "\varphi_{(A,f)} \circ \eta_A"] & & \\
& X \arrow[r, dashed, "h"] & Y \\
A' \arrow[ur, "f'"'] \arrow[urr, bend right, "\varphi_{(A',f')} \circ \eta_{A'}"'] & &
\end{tikzcd}$$

and so there is a unique morphism $h : X \to Y$ in C such that

$$h \circ f = \varphi_{(A,f)} \circ \eta_A$$

for all objects $f : A \to X$ in $(\mathcal{A} \downarrow X)$ because $\mathcal{A}$ is dense in C. This is clearly the only possible candidate for a $\mathbb{T}$-module homomorphism $h : (X, \xi) \to (Y, \theta)$ such that

$$h \circ \bar{f} = \varphi_{(A,f)}$$

for all objects $\bar{f} : (TA, \mu_A) \to (X, \xi)$ in $(\mathbf{\Theta}_{\mathcal{A}} \downarrow (X, \xi))$; indeed, if h were a $\mathbb{T}$-module homomorphism, then

$$h \circ \bar{f} = h \circ \xi \circ Tf = \theta \circ Th \circ Tf = \theta \circ \varphi_{(A,f)} \circ T\eta_A = \varphi_{(A,f)} \circ \mu_A \circ T\eta_A = \varphi_{(A,f)}$$

as required for (X, ξ) to be the colimit of $U^{(X,\xi)}$, so it is enough to verify that h is a $\mathbb{T}$-module homomorphism.

Now, recall proposition 1.2.10: for each $\mathbb{T}$-module (X, ξ), the diagram

$$\begin{tikzcd}
(T^2 X, \mu_{TX}) \arrow[r, shift left, "T\xi"] \arrow[r, shift right, "\mu_X"'] & (TX, \mu_X) \arrow[r, "\xi"] & (X, \xi)
\end{tikzcd}$$

is a coequaliser in $C^{\mathbb{T}}$, so to show that a morphism $h : X \to Y$ in C lifts to a $\mathbb{T}$-module homomorphism $h : (X, \xi) \to (Y, \theta)$, it is enough to check that

$$T\xi \circ \bar{h} = \mu_X \circ \bar{h}$$

where $\bar{h} : (TX, \mu_X) \to (Y, \theta)$ is given by $\bar{h} = \theta \circ Th$. So let $g : B \to TX$ be an object of $(\mathcal{A} \downarrow TX)$, and use the preceding corollary to obtain a factorisation $g = Tf \circ \hat{g}$ where $f : A \to X$ is an object of $(\mathcal{A} \downarrow X)$. Then,

$$\mu_X \circ Tg = \mu_X \circ T^2 f \circ T\hat{g} = Tf \circ \mu_A \circ T\hat{g}$$

and since $h \circ f = h \circ \bar{f} \circ \eta_A = \varphi_{(A,f)} \circ \eta_A$,

$$\begin{aligned}\bar{h} \circ \mu_X \circ Tg &= \theta \circ (Th \circ Tf) \circ \mu_A \circ T\hat{g} \\ &= \theta \circ T\varphi_{(A,f)} \circ T\eta_A \circ \mu_A \circ T\hat{g} \\ &= \varphi_{(A,f)} \circ \mu_A \circ T\eta_A \circ \mu_A \circ T\hat{g} \\ &= \varphi_{(A,f)} \circ \mu_A \circ T\hat{g} \\ &= \varphi_{(B,\bar{f}\circ\hat{g})}\end{aligned}$$

where we have used the commutativity of the diagram below:

$$\begin{tikzcd}
(TB,\mu_B) \arrow[rd] \arrow[rrd, bend left, "\varphi_{(B,\bar{f}\circ\hat{g})}"] \arrow[dd, "\mu_A \circ T\hat{g}"'] & & \\
& (X,\xi) & (Y,\theta) \\
(TA,\mu_A) \arrow[ru, "\bar{f}"'] \arrow[rru, bend right, "\varphi_{(A,f)}"'] & &
\end{tikzcd}$$

Also, factorise $g' = T\xi \circ \eta_{TX} \circ g$ as $Tf' \circ \hat{g}'$ for some $f' : A' \to X$ in $(\mathcal{A} \downarrow X)$ and $\hat{g}' : B \to A'$ in C, so that

$$T\xi \circ Tg \circ \eta_B = T\xi \circ \eta_{TX} \circ g = Tf' \circ \hat{g}'$$

and thus, by adjoint transposition,

$$T\xi \circ Tg = \mu_X \circ T^2 f' \circ T\hat{g}' = Tf' \circ \mu_{A'} \circ T\hat{g}'$$

whence

$$\begin{aligned}\bar{h} \circ T\xi \circ Tg &= \theta \circ (Th \circ Tf') \circ \mu_{A'} \circ T\hat{g}' \\ &= \theta \circ T\varphi_{(A',f')} \circ T\eta_{A'} \circ \mu_{A'} \circ T\hat{g}' \\ &= \varphi_{(A',f')} \circ \mu_{A'} \circ T\eta_{A'} \circ \mu_{A'} \circ T\hat{g}' \\ &= \varphi_{(B,\bar{f}'\circ\hat{g}')}\end{aligned}$$

but by the $\mathbb{T}$-module axioms, we have

$$\begin{aligned}\bar{f}' \circ \hat{g}' &= \xi \circ (Tf' \circ \hat{g}') \\ &= (\xi \circ T\xi) \circ \eta_{TX} \circ Tg \\ &= \xi \circ (\mu_X \circ \eta_{TX}) \circ Tg \\ &= \xi \circ Tg \\ &= (\xi \circ Tf) \circ \hat{g} \\ &= \bar{f} \circ \hat{g}\end{aligned}$$

and therefore $\varphi_{(B,\bar{f}\circ\hat{g})} = \varphi_{(B,\bar{f}'\circ\hat{g}')}$. Since $g : B \to TX$ was an arbitrary object in $(\mathcal{A} \downarrow TX)$, the fact that the canonical cocone on the diagram of shape $(\mathcal{A} \downarrow TX)$ is colimiting implies

$$\bar{h} \circ T\xi = \bar{h} \circ \mu_X$$

as required for $h : (X, \xi) \to (Y, \theta)$ to be a $\mathbb{T}$-module homomorphism. ■

Corollary 1.7.11. *If $\mathbb{T}$ is a monad on a locally small category $\mathcal{C}$, then the comparison functor $K : \mathcal{C}_{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ from the Kleisli category to the Eilenberg–Moore category is dense.*

Proof. We already remarked that $\mathcal{C}$ itself is always a category of arities for $\mathbb{T}$, and theorem 1.1.9 tells us K is an isomorphism from $\mathcal{C}_{\mathbb{T}}$ to $\mathbf{\Theta}_{\mathcal{C}}$. ■

Remark 1.7.12. The corollary is also true in the case where $\mathcal{C}$ is not locally small: this can be established by a simple proof using proposition 1.2.10 directly.

Definition 1.7.13. Let $\mathbb{T}$ be a monad with arities $\mathcal{A}$ on a locally small category $\mathcal{C}$, and let $\mathbf{\Theta}_{\mathcal{A}}$ be as in proposition 1.7.10. The **$\mathcal{A}$-nerve** of a $\mathbb{T}$-module (X, ξ) is the functor $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}X : (\mathbf{\Theta}_{\mathcal{A}})^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}X = \mathcal{C}^{\mathbb{T}}(-, (X, \xi))$$

and the **$\mathcal{A}$-nerve functor** is the evident functor $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} : \mathcal{C}^{\mathbb{T}} \to [(\mathbf{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}]$.

Theorem 1.7.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be small categories, and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Let $\mathcal{S}$ be any category and let $F^* : [\mathcal{B}, \mathcal{S}] \to [\mathcal{A}, \mathcal{S}]$ be defined by $H \mapsto HF$.*

(i) *If $\mathcal{S}$ is a cocomplete category, then the left adjoint of F^* exists and is the global left Kan extension functor* $\mathrm{Lan}_F : [\mathcal{A}, \mathcal{S}] \to [\mathcal{B}, \mathcal{S}]$.

(ii) *If $\mathcal{S}$ has coequalisers for all reflexive pairs and $F : \mathcal{A} \to \mathcal{B}$ is essentially surjective on objects, then the functor $F^* : [\mathcal{B},\mathcal{S}] \to [\mathcal{A},\mathcal{S}]$ satisfies the CTT condition*[3] *if and only if it has a left adjoint.*

(iii) *If $F : \mathcal{A} \to \mathcal{B}$ is* bijective *on objects, then $F^* : [\mathcal{B},\mathcal{S}] \to [\mathcal{A},\mathcal{S}]$ is an amnestic isofibration.*

Proof. (i). A left Kan extension of a functor $G : \mathcal{A} \to \mathcal{S}$ along $F : \mathcal{A} \to \mathcal{B}$ is, by definition, a right Kan extension of of $G^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}$ along $F^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \to \mathcal{S}^{\mathrm{op}}$, so the existence of $\mathrm{Lan}_F G$ when $\mathcal{A}$ is small and $\mathcal{S}$ is cocomplete follows from the formal dual of theorem 1.6.10. Functoriality and adjointness of $\mathrm{Lan}_F : [\mathcal{A},\mathcal{S}] \to [\mathcal{B},\mathcal{S}]$ follows from the universal property of left Kan extensions.

(ii). A natural transformation $\alpha : H' \Rightarrow H$ is a natural isomorphism if and only if each component $\alpha_B : H'B \to HB$ is an isomorphism, so if $F : \mathcal{A} \to \mathcal{B}$ is essentially surjective on objects, $\alpha : H' \Rightarrow H$ is a natural isomorphism if and only if $\alpha F : H'F \Rightarrow HF$ is a natural isomorphism: in other words, F^* is conservative. Since $\mathcal{S}$ has coequalisers of reflexive pairs, the coequaliser of any reflexive pair in $[\mathcal{B},\mathcal{S}]$ exists and can be computed pointwise; the same is true of $[\mathcal{A},\mathcal{S}]$, so F^* preserves coequalisers of all reflexive pairs in $[\mathcal{B},\mathcal{S}]$. Thus F^* satisfies the CTT condition if and only if it has a left adjoint.

(iii). A natural transformation $\alpha : H' \Rightarrow H$ is the identity if and only if each component is an identity morphism in $\mathcal{S}$, so if $F : \mathcal{A} \to \mathcal{B}$ is strictly surjective on objects, $\alpha : H' \Rightarrow H$ is the identity if and only if $\alpha F : H'F \Rightarrow HF$ is the identity. Now, suppose we have a natural isomorphism $\bar{\alpha} : \bar{H} \Rightarrow HF$ in $[\mathcal{A},\mathcal{S}]$. If $F : \mathcal{A} \to \mathcal{B}$ is bijective on objects, then there is a unique functor $H' : \mathcal{B} \to \mathcal{S}$ such that $H'F = \bar{H}$: clearly, for each object B in $\mathcal{B}$, we must have $H'B = \bar{H}A$ for the unique object A in $\mathcal{A}$ such that $FA = B$, and for each morphism $f : B \to B'$, we must have $H'f = \bar{\alpha}_{A'}^{-1} \circ Hf \circ \bar{\alpha}_A$. Thus, $\bar{\alpha} : \bar{H} \Rightarrow HF$ lifts to a natural isomorphism $\alpha : H' \Rightarrow H$ as required. ■

Corollary 1.7.15. *If $\mathcal{B}$ is a small category and $\mathcal{S}$ is cocomplete, then the forgetful functor $[\mathcal{B},\mathcal{S}] \to [\mathrm{ob}\,\mathcal{B},\mathcal{S}]$ is strictly monadic and satisfies the CTT condition.*

Proof. Use corollary 1.2.14 and proposition 1.2.18. ■

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[3] See corollary 1.2.14.

Theorem 1.7.16 (Weber's nerve theorem). *Let* $\mathbb{T} = \big(T, \eta, \mu\big)$ *be a monad with arities* $\mathcal{A}$ *on a locally small category* C. *Let* $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} : C^{\mathbb{T}} \to \big[(\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\big]$ *be the* $\mathcal{A}$*-nerve functor, and define a functor* $\mathrm{N}_{\mathcal{A}} : C \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *by* $\mathrm{N}_{\mathcal{A}} X = C(-, X)$. *Let* $J : \mathcal{A}^{\mathrm{op}} \to (\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}}$ *be the functor defined by* $A \mapsto F^{\mathbb{T}} A$ *and consider the diagram below:*

$$\begin{tikzcd}
C^{\mathbb{T}} \arrow[r, "\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}"] \arrow[d, "U^{\mathbb{T}}"'] & \big[(\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\big] \arrow[d, "J^*"] \\
C \arrow[r, "\mathrm{N}_{\mathcal{A}}"'] & [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]
\end{tikzcd}$$

(i) *The functor* $J^* : \big[(\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\big] \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *is strictly monadic.*

(ii) *The diagram commutes up to natural isomorphism and satisfies the Beck–Chevalley condition.*

(iii) *The two functors* $\mathrm{N}_{\mathcal{A}} : C \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ *and* $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} : C^{\mathbb{T}} \to \big[(\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\big]$ *are both fully faithful and preserve all limits.*

(iv) *A functor* $Q : (\boldsymbol{\Theta}_{\mathcal{A}})^{\mathrm{op}} \to \mathbf{Set}$ *is in the essential image of* $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}$ *if and only if the composite* $QJ : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ *is in the essential image of* $\mathrm{N}_{\mathcal{A}}$.

Proof. (i). J is bijective on objects, so this follows from the theorem above.

(ii). Let (X, ξ) be a $\mathbb{T}$-algebra, and let A be any object in $\mathcal{A}$. On one hand,

$$\Big(J^* \mathrm{N}^{\mathbb{T}}_{\mathcal{A}}(X, \xi)\Big)(A) = \Big(\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}(X, \xi)\Big)\Big(F^{\mathbb{T}} A\Big) = C^{\mathbb{T}}\Big(F^{\mathbb{T}} A, (X, \xi)\Big)$$

while on the other,

$$\Big(\mathrm{N}_{\mathcal{A}} U^{\mathbb{T}}(X, \xi)\Big)(A) = \Big(\mathrm{N}_{\mathcal{A}} X\Big)(A) = C(A, X)$$

and these two sets are in natural bijection because we have the free–forgetful adjunction $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$. Thus, we have a natural isomorphism $\sigma : \mathrm{N}_{\mathcal{A}} U^{\mathbb{T}} \overset{\sim}{\Rightarrow} J^* \mathrm{N}^{\mathbb{T}}_{\mathcal{A}}$.

Let $J_! = \mathrm{Lan}_J$ be the left adjoint of J^*, and let $\varepsilon^J : J_! J^* \Rightarrow \mathrm{id}$ be the counit of this adjunction. The Beck–Chevalley condition asks that the natural transformation

$$\varphi = \varepsilon^J \mathrm{N}^{\mathbb{T}}_{\mathcal{A}} F^{\mathbb{T}} \bullet J_! \sigma F^{\mathbb{T}} \bullet J_! \mathrm{N}_{\mathcal{A}} \eta^{\mathbb{T}}$$

be a natural isomorphism, and since J^* is conservative, it is enough to check that

$$\sigma^{-1}_{F^{\mathbb{T}} X} \circ J^* \varphi_X = \sigma^{-1}_{F^{\mathbb{T}} X} \circ J^* \varepsilon^J_{\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} F^{\mathbb{T}} X} \circ J^* J_! \sigma_{F^{\mathbb{T}} X} \circ J^* J_! \mathrm{N}_{\mathcal{A}} \eta^{\mathbb{T}}_X$$

is an isomorphism $J^* J_! \mathcal{C}(-, X) \to \mathcal{C}(-, TX)$ for each object X in $\mathcal{C}$.

Observe that $J^* J_! \mathcal{C}(-, TX)$ is defined by a colimit over a diagram of shape $(\mathcal{A} \downarrow TX)$: in fact, it is

$$J^* J_! \mathcal{C}(-, TX) \cong \varinjlim_{(\mathcal{A} \downarrow TX)} \mathcal{C}\left(-, TU^{TX}\right) \cong \mathcal{C}\left(-, T^2 X\right)$$

because the functors $\mathcal{C}(A, T(-))$ preserve the colimit of the canonical diagram $U^{TX} : (\mathcal{A} \downarrow TX) \to \mathcal{C}$, when $\mathcal{A}$ is a category of arities for $\mathbb{T}$ and A is an object in $\mathcal{A}$. It is then clear that the natural transformation

$$J^* \varepsilon^J_{\mathcal{C}^{\mathbb{T}}\left(-, F^{\mathbb{T}} X\right)} : J^* J_! J^* \mathcal{C}^{\mathbb{T}}\left(-, F^{\mathbb{T}} X\right) \to J^* \mathcal{C}^{\mathbb{T}}\left(-, F^{\mathbb{T}} X\right)$$

is identified with the natural transformation

$$\mathcal{C}(-, \mu_X) : \mathcal{C}\left(-, T^2 X\right) \to \mathcal{C}(-, TX)$$

via these natural isomorphisms, and so the Beck–Chevalley condition is indeed satisfied.

(iii). By proposition 1.7.3, $\mathrm{N}_{\mathcal{A}} : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful because the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$ is dense; and by proposition 1.7.10, the inclusion $\mathbf{\Theta}_{\mathcal{A}} \hookrightarrow \mathcal{C}^{\mathbb{T}}$ is also dense, so $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} : \mathcal{C}^{\mathbb{T}} \to \left[(\mathbf{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\right]$ is also fully faithful. Preservation of limits is automatic because limits are computed pointwise in functor categories and representable functors preserve all limits.

(iv). Since $J^* : \left[(\mathbf{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\right] \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is an isofibration, we may replace $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}}$ with an isomorphic functor making the diagram commute strictly. The claim then follows from propositions 1.4.16 and 1.4.17. ■

The next example is a somewhat frivolous application of the above result but serves to show how the theorem generalises previously-known results.

Example 1.7.17. Let M be a monoid and let $\mathbb{T} = (M \times (-), \eta, \mu)$ be the induced monad on **Set**. Recall example 1.7.6: $\mathcal{A} = \{1\}$ is a category of arities for $\mathbb{T}$. It is easy to see that $\mathbf{\Theta}_{\mathcal{A}}$ is isomorphic to the opposite monoid M^{op} considered as a one-object category, and the embedding $\mathrm{N}_{\mathcal{A}} : \mathbf{Set} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is an isomorphism of categories in this case, so the above theorem implies the category of sets equipped with a left M-action is isomorphic to the functor category $\left[(\mathbf{\Theta}_{\mathcal{A}})^{\mathrm{op}}, \mathbf{Set}\right]$. This can also be checked directly.

Example 1.7.18. Let C be a small category and let $\mathbb{T}$ be the identity monad. Then C itself is a small category of arities for $\mathbb{T}$, so the theorem implies C has a limit-preserving embedding in the functor category $[C^{\mathrm{op}}, \mathbf{Set}]$: in other words, the Yoneda embedding is recovered as a special case of the above theorem.

— II —

Finitary theories

In this chapter we focus on ‘finitary’ algebraic theories, which can be regarded as a class of particularly well-behaved monads on **Set** and related categories.

1 Notions of finiteness

First, we should make precise the sense in which some functor is ‘finitary’, or the sense in which an object of a general category is ‘finite’. The definitive solution to this problem was given by Gabriel and Ulmer [1971] and is, in a way, a generalisation of the old order-theoretic notion of an ‘algebraic lattice’. We shall motivate the definition by studying the base case of **Set**.

Let **FinSet** be the category of all hereditarily finite sets,[1] and consider the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$. We claim this is dense, and moreover the categories $(\mathbf{FinSet} \downarrow X)$ have a special property:

Definition 2.1.1. A **(finitely) filtered category** is a category $\mathcal{J}$ satisfying the following conditions:

(i) $\mathcal{J}$ is inhabited, i.e. there exists an object in $\mathcal{J}$.

(ii) For any two objects i and j in $\mathcal{J}$, there is an object k such that there exist arrows $i \to k$ and $j \to k$ in $\mathcal{J}$.

(iii) For any parallel pair $f, g : i \to j$ in $\mathcal{J}$, there exists an arrow $h : j \to k$ such that $h \circ f = h \circ g$.

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[1] A set is **hereditarily finite** if it is either the empty set, or a finite set of other hereditarily finite sets. Note that the class of hereditarily finite sets is a *small* set (of rank ω).

Proposition 2.1.2. *For each set X, the category* $(\mathbf{FinSet} \downarrow X)$ *is a filtered category, and the canonical cocone on the diagram of shape* $(\mathbf{FinSet} \downarrow X)$ *is colimiting. In particular,* $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ *is dense.*

Proof. The empty set $\emptyset$ is initial in **Set**, so there is a unique map $\emptyset \to X$; thus $\mathcal{J} = (\mathbf{FinSet} \downarrow X)$ is inhabited. If $A \to X$ and $B \to X$ are two maps, we can form the union C of their images and obtain a commutative diagram

$$\begin{tikzcd}
 & B \arrow[d] \arrow[ddr, bend left] & \\
A \arrow[r] \arrow[drr, bend right] & C \arrow[dr] & \\
 & & X
\end{tikzcd}$$

in **Set**; if A and B are moreover finite, then C is also finite, so $\mathcal{J}$ satisfies axiom (ii). Similarly, given a parallel pair $f, g : A \to B$, we can form their coequaliser $h : B \to C$ and obtain $h \circ f = h \circ g$; and if B is finite, then C is also finite, so $\mathcal{J}$ satisfies axiom (iii). Thus $\mathcal{J}$ is a filtered category.

Let $\mathcal{K}$ be the subcategory consisting of all *injective* maps from a finite set to X. It is not hard to see that the canonical cocone on the diagram of shape $\mathcal{K}$ is colimiting: after all, every set is the directed union of its finite subsets. But for every object $f : A \to X$ in $\mathcal{J}$, there exists an object $\bar{f} : \bar{A} \rightarrowtail X$ in $\mathcal{K}$ and a surjective map $g : A \twoheadrightarrow \bar{A}$ such that $\bar{f} \circ g = f$, so any cocone on the canonical diagram of shape $\mathcal{J}$ is uniquely determined by the sub-cocone on the subdiagram of shape $\mathcal{K}$. Thus the canonical cocone on $\mathcal{J}$ is also colimiting. ■

We also record here a generalisation of the trick used in the proof above:

Definition 2.1.3. A **cofinal functor**[2] is a functor $F : \mathcal{K} \to \mathcal{J}$ with the following properties:

- For any object j in $\mathcal{J}$, there exists an object k in $\mathcal{K}$ and a morphism $f : j \to Fk$ in $\mathcal{J}$.
- For any two arrows $f : j \to Fk$ and $f' : j \to Fk'$ in $\mathcal{J}$, there exist arrows $g : k \to k''$ and $g' : k' \to k''$ in $\mathcal{K}$ such that $Fg \circ f = Fg' \circ f'$.

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[2] Or 'final functor', according to Mac Lane [CWM, Ch. IX, § 3].

Dually, a **coinitial functor** is a functor $F : \mathcal{K} \to \mathcal{J}$ such that $F^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \to \mathcal{J}^{\mathrm{op}}$ is cofinal.

Proposition 2.1.4. *Let C be a category and let $A : \mathcal{J} \to C$ be a diagram. If we have a cofinal functor $F : \mathcal{K} \to \mathcal{J}$, then F induces a natural bijection between cocones on A and cocones on AF:*

$$\begin{aligned} \mathrm{Nat}(A, \Delta X) &\to \mathrm{Nat}(AF, \Delta X) \\ \varphi &\mapsto \varphi F \end{aligned}$$

In particular, $\varinjlim_{\mathcal{J}} A$ exists in C if and only if $\varinjlim_{\mathcal{K}} AF$ exists in C, and the two colimits agree.

Proof. Let $\psi : AF \Rightarrow \Delta X$ be a cocone. Note that for any $h : j \to j'$ in $\mathcal{J}$, given $f : j \to Fk$ and $f' : j' \to Fk'$, there must be $g : k \to k''$ and $g' : k' \to k''$ in C such that $Fg \circ f = Fg' \circ f' \circ h$. Taking $h = \mathrm{id}_j$, we see that it makes sense to define $\varphi_j : Aj \to X$ by setting $\varphi_j = \psi_k \circ Af$, and for general $h : j \to j'$, we get

$$\varphi_{j'} \circ Ah = \psi_{k''} \circ AFg' \circ Af' \circ Ah = \psi_{k''} \circ AFg \circ Af = \varphi_j$$

so $\varphi : A \Rightarrow \Delta X$ is a cocone. Taking $f = \mathrm{id}_{Fk}$ shows $\varphi F = \psi$. If $\psi = \theta F$ for some cocone $\theta : A \Rightarrow \Delta X$, then

$$\varphi_j = \psi_k \circ Af = \theta_{Fk} \circ Af = \theta_j$$

so $\varphi = \theta$. Thus $\varphi \mapsto \varphi F$ is indeed a bijection, and naturality is obvious. ■

Colimits of filtered diagrams in **Set** are straightforward to compute: if $A : \mathcal{J} \to \mathbf{Set}$ is a diagram where $\mathcal{J}$ is a small filtered category, then $\varinjlim_{\mathcal{J}} A$ is, roughly speaking, the set $\coprod_j A_j$ modulo the equivalence relation

$$a \sim a' \text{ if and only if } Af(a) = Af'(a') \text{ for some } f : j \to k, f' : j' \to k \text{ in } \mathcal{J}$$

where $a \in Aj$ and $a' \in Aj'$; this *is* an equivalence relation because of axiom (ii) in the definition of a filtered category. This concrete characterisation is useful for proving the following:

Theorem 2.1.5. *Let $\mathcal{J}$ be a small category, and let $\varinjlim_{\mathcal{J}} : [\mathcal{J}, \mathbf{Set}] \to \mathbf{Set}$ be the functor sending a diagram $A : \mathcal{J} \to \mathbf{Set}$ to its colimit $\varinjlim_{\mathcal{J}} A$ in* **Set**. *The following are equivalent:*

(i) *$\mathcal{J}$ is a filtered category.*

(ii) *For any finite diagram $A : \mathcal{L} \to [\mathcal{J}, \mathbf{Set}]$, if $\lambda : \Delta X \Rightarrow A$ is a limiting cone in $[\mathcal{J}, \mathbf{Set}]$, then $\varinjlim_{\mathcal{J}} \lambda : \Delta\left(\varinjlim_{\mathcal{J}} X\right) \Rightarrow \varinjlim_{\mathcal{J}} A$ is a limiting cone in* **Set**; *i.e. $\varinjlim_{\mathcal{J}}$ preserves all finite limits.*

Proof. (i) ⇒ (ii). This is classical and well-known: see e.g. [CWM, Ch. IX, § 2, Thm 1] or [Borceux, 1994a, Thm 2.13.4].

(ii) ⇒ (i). First, consider the empty diagram $\varnothing \to [\mathcal{J}, \mathbf{Set}]$: its limit must be the constant functor $\Delta 1$, and $\varinjlim_{\mathcal{J}} \Delta 1 \cong 1$ if and only if $\mathcal{J}$ is connected and inhabited. (If $\mathcal{J}$ is empty, then $\varinjlim_{\mathcal{J}} \Delta 1 = \varnothing$, and if $\mathcal{J}$ has X-many connected components, then $\varinjlim_{\mathcal{J}} \Delta 1 \cong X$.) Henceforth, we shall assume $\mathcal{J}$ is inhabited.

Observe that for any object j in $\mathcal{J}$, the Yoneda lemma implies there is a (natural) bijection between cocones $\mathcal{J}(j,-) \Rightarrow \Delta X$ and elements of X itself; thus, $\varinjlim_{\mathcal{J}} \mathcal{J}(j,-)$ must be a singleton set. So let i and j be any two objects in $\mathcal{J}$, and consider the product functor $\mathcal{J}(i,-) \times \mathcal{J}(j,-) : \mathcal{J} \to \mathbf{Set}$. Since $\varinjlim_{\mathcal{J}} \mathcal{J}(i,-)$ and $\varinjlim_{\mathcal{J}} \mathcal{J}(j,-)$ are both inhabited, it follows that $\varinjlim_{\mathcal{J}} \left(\mathcal{J}(i,-) \times \mathcal{J}(j,-)\right)$ is also inhabited; hence, there must be some object k in $\mathcal{J}$ for which both $\mathcal{J}(i,k)$ and $\mathcal{J}(j,k)$ are inhabited.

Finally, let $f, g : i \to j$ be a parallel pair in $\mathcal{J}$, and consider the equaliser diagram below:

$$\begin{tikzcd}
E(k) \arrow[r] & \mathcal{J}(j,k) \arrow[r, shift left, "\mathcal{J}(f,k)"] \arrow[r, shift right, "\mathcal{J}(g,k)"'] & \mathcal{J}(i,k)
\end{tikzcd}$$

Note that $E(k)$ is inhabited if and only if there is some morphism $h : j \to k$ in $\mathcal{J}$ such that $h \circ f = h \circ g$. But its colimit $\varinjlim_{\mathcal{J}} E$ is certainly inhabited since

$$\begin{tikzcd}
1 \arrow[r] & \varinjlim_{\mathcal{J}} \mathcal{J}(j,-) \arrow[r, shift left] \arrow[r, shift right] & \varinjlim_{\mathcal{J}} \mathcal{J}(i,-)
\end{tikzcd}$$

is an equaliser diagram, thus, there is such a morphism $h : j \to k$. □

Corollary 2.1.6. *The representable functor* $\mathbf{Set}(A,-) : \mathbf{Set} \to \mathbf{Set}$ *preserves all small filtered colimits if and only if A is a finite set.*

Proof. If A is an n-element set, then $\mathbf{Set}(A,X)$ is just the n-fold product X^n. The formation of finite products commutes with small filtered colimits by the above theorem, so $\mathbf{Set}(A,-)$ indeed preserves small filtered colimits.

Conversely, suppose $\mathbf{Set}(A,-)$ preserves small filtered colimits. Then, for the canonical diagram $U^A : (\mathbf{FinSet} \downarrow A) \to \mathbf{Set}$, we must have

$$\mathbf{Set}(A,A) \cong \varinjlim_{(\mathbf{FinSet}\downarrow A)} \mathbf{Set}\big(A, U^A\big)$$

since $(\mathbf{FinSet} \downarrow A)$ is filtered and $\varinjlim_{(\mathbf{FinSet}\downarrow A)} U^A \cong A$ by proposition 2.1.2. In particular, this means we have $\mathrm{id}_A = f \circ g$ for some map $f : B \to A$ where B is finite; but $f : B \to A$ is a (split) surjection, so A is also finite. ■

Let us also recall the definition of 'compact subset' in general topology: a subset Y of a topological space X is compact just if, for every family $\mathfrak{U}$ of open subsets of X such that $Y \subseteq \bigcup_{U\in\mathfrak{U}} U$, there exists a *finite* subfamily $\mathfrak{V} \subseteq \mathfrak{U}$ for which $Y \subseteq \bigcup_{V\in\mathfrak{V}} V$; in short, Y is a compact subset when every open cover of Y has a finite subcover.

Now let $\mathrm{Ouv}(X)$ be the lattice of open subsets of X, and observe the following: a subset Y is compact if and only if for every filtered diagram $U_\bullet : \mathcal{J} \to \mathrm{Ouv}(X)$ with $Y \subseteq \varinjlim_{\mathcal{J}} U_\bullet$, there exists some j such that $Y \subseteq U_j$, because a filtered diagram on a poset corresponds to a non-empty subset that is closed under finite unions. This together with the previous results concerning the inclusion $\mathbf{FinSet} \hookrightarrow \mathbf{Set}$ motivates the following definition:

Definition 2.1.7. A **compact object** in a category $\mathcal{C}$ is an object A with the following properties:

- For any small filtered diagram $X : \mathcal{J} \to \mathcal{C}$, if $\bar{X} = \varinjlim_{\mathcal{J}} X$ exists in $\mathcal{C}$ and $f : A \to \bar{X}$ is any morphism, then there exists a factorisation $f = \lambda_j \circ \tilde{f}$ where $\tilde{f}$ is a morphism $A \to Xj$ for some object j in $\mathcal{J}$ and $\lambda : X \Rightarrow \Delta\bar{X}$ is the colimiting cocone.

- If we have two such factorisations $f = \lambda_j \circ \tilde{f} = \lambda_{j'} \circ \tilde{f}'$, then there exist arrows $h : j \to k$ and $h' : j' \to k$ in $\mathcal{J}$ for which $Xh \circ \tilde{f} = Xh' \circ \tilde{f}'$.

We relate this to the previous results by using the next proposition:

Proposition 2.1.8. *Let $\mathcal{C}$ be a locally small category. An object A is compact in $\mathcal{C}$ if and only if the representable functor $\mathcal{C}(A,-) : \mathcal{C} \to$ **Set** preserves all small filtered colimits.*

Proof. Let $X : \mathcal{J} \to C$ be a small filtered diagram, and suppose $\bar{X} = \varinjlim_{\mathcal{J}} X$ exists in C. Let $\lambda : X \Rightarrow \Delta \bar{X}$ be the colimiting cocone, and consider a morphism $f : A \to \bar{X}$ in C. By the concrete description of filtered colimits in **Set**, we see that $\varinjlim_{\mathcal{J}} C(A, X)$ is a set whose elements are morphisms $g : A \to Xj$, modulo the equivalence relation

$$g \sim g' \text{ if and only if } Xh \circ g = Xh' \circ g' \text{ for some } h : j \to k, h' : j \to k \text{ in } \mathcal{J}$$

where $g \in C(A, Xj)$ and $g' \in C(A, Xj')$. Thus, $C(A, -)$ preserves the colimit of $X : \mathcal{J} \to C$ if and only if we can factorise every morphism $f : A \to \bar{X}$ as in the definition above. ■

Corollary 2.1.9. *A set is a compact object in* **Set** *if and only if it is finite.* ■

Now, let us examine the notion of compactness in the setting of algebra.

Example 2.1.10. Let R be a ring, not necessarily commutative or noetherian, and let M be a left R-module. It is easy to see that the partially ordered set of finitely-generated submodules of M is a filtered category $\mathrm{Sub}^{\mathrm{fg}}(M)$, and with a little more effort, we can show that the canonical cocone on the diagram of shape $\mathrm{Sub}^{\mathrm{fg}}(M)$ is colimiting. So if M is a compact object in $\mathbf{Mod}(R)$, then by factorising id_M we see that M is finitely generated.

But more is true: using proposition 1.7.3 we can show that the category $\mathbf{\Theta}^R$ of finitely-generated *free* left R-modules is dense in $\mathbf{Mod}(R)$, and it easy to see that the category $(\mathbf{\Theta}^R \downarrow M)$ is filtered, so factorising id_M gives a split epimorphism $p : R^n \to M$, where n is some natural number. The splitting lemma of commutative algebra[3] then yields a split epimorphism $q : R^n \to K$, where K is the kernel of $p : R^n \to M$; thus M is finitely *presented*.

On the other hand, it is clear that every object is compact in a category that has no filtered colimits, so we should restrict our attention to categories with enough filtered colimits to make the notion of compactness profitable.

Definition 2.1.11. A **finitely-accessible category** is a locally small category C with the following properties:

- Every small filtered diagram in C has a colimit.

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[3] See [Borceux, 1994b, Prop. 1.8.7].

- There exists a small set $\mathcal{B}$ of compact objects in $\mathcal{C}$ such that every object in $\mathcal{C}$ is a colimit of a small filtered diagram involving objects in $\mathcal{B}$.

A **locally finitely-presentable category**, or **l.f.p. category** for short, is a finitely-accessible category that also has finite coproducts and coequalisers for parallel pairs.

Note that we do *not* require objects in finitely-accessible category $\mathcal{C}$ to be *canonical* colimits of objects in $\mathcal{B}$, so (the full subcategory spanned by) $\mathcal{B}$ need not be dense in $\mathcal{C}$. Nonetheless, something very close is true, and we need a little lemma to prove what we want.

Lemma 2.1.12. *Let $\mathcal{C}$ be any locally small category.*

(i) *Let $A : \mathcal{K} \to \mathcal{C}$ be a finite diagram of compact objects in $\mathcal{C}$. If $\varinjlim_{\mathcal{K}} A$ exists in $\mathcal{C}$, then it is compact in $\mathcal{C}$.*

(ii) *If $p : B \twoheadrightarrow A$ is a split epimorphism in $\mathcal{C}$ and B is a compact object, then A is also compact in $\mathcal{C}$.*

Proof. (i). Let $X : \mathcal{J} \to \mathcal{C}$ be a small filtered diagram and suppose $\varinjlim_{\mathcal{J}} X$ exists in $\mathcal{C}$. By theorem 2.1.5, small filtered colimits preserve finite limits, so

$$\begin{aligned}
\varinjlim_{\mathcal{J}} \mathcal{C}\Big(\varinjlim_{\mathcal{K}} A, X\Big) &\cong \varinjlim_{\mathcal{J}} \varprojlim_{\mathcal{K}} \mathcal{C}(A, X) \\
&\cong \varprojlim_{\mathcal{K}} \varinjlim_{\mathcal{J}} \mathcal{C}(A, X) \\
&\cong \varprojlim_{\mathcal{K}} \mathcal{C}\Big(A, \varinjlim_{\mathcal{J}} X\Big) \\
&\cong \mathcal{C}\Big(\varinjlim_{\mathcal{K}} A, \varinjlim_{\mathcal{J}} X\Big)
\end{aligned}$$

canonically, and thus $\varinjlim_{\mathcal{K}} A$ is a compact object.

(ii). We shall exhibit A as a colimit of a finite diagram involving B. By definition, there must be a monomorphism $i : A \rightarrowtail B$ such that $p \circ i = \mathrm{id}_A$, so $i \circ p : B \to B$ is a split idempotent. Thus,

$$\begin{tikzcd}
B \arrow[r, shift left, "i \circ p"] \arrow[r, shift right, "\mathrm{id}"'] & B \arrow[l, bend left, "\mathrm{id}"] \arrow[r, "p"] & A \arrow[l, bend left, "i"]
\end{tikzcd}$$

is a split coequaliser diagram in $\mathcal{C}$, so by (i), B is compact in $\mathcal{C}$ if A is. ■

Proposition 2.1.13. *Let C be a finitely-accessible category, and let $\mathbf{K}(C)$ be the full subcategory of C spanned by the compact objects in C.*

(i) *If $B : \mathcal{J} \to C$ is a (small) filtered diagram of compact objects in C and $\lambda : B \Rightarrow \Delta X$ is a colimiting cocone, then $j \mapsto \lambda_j$ defines a cofinal functor $\mathcal{J} \to (\mathbf{K}(C) \downarrow X)$.*

(ii) *The inclusion $\mathbf{K}(C) \hookrightarrow C$ is dense.*

(iii) *$\mathbf{K}(C)$ is an essentially small category.*[4]

Proof. (i). For any morphism $f : A \to X$ in C, if A is compact, then there must be a factorisation $f = \lambda_j \circ \tilde{f}$ for some $\tilde{f} : A \to Bj$, and if $f = \lambda_{j'} \circ \tilde{f}'$ for some $\tilde{f}' : A \to Bj'$, then there must be some $h : j \to k$ and $h' : j' \to k$ in $\mathcal{J}$ for which $Bh \circ \tilde{f} = Bh' \circ \tilde{f}'$; thus we see that $j \mapsto \lambda_j$ indeed defines a cofinal functor $\mathcal{J} \to (\mathbf{K}(C) \downarrow X)$.

(ii). Any object X *is* a filtered colimit of compact objects in C, so by (i) and proposition 2.1.4, the canonical cocone on the diagram of shape $(\mathbf{K}(C) \downarrow X)$ is colimiting.

(iii). By definition, there is a small set $\mathcal{B}$ of objects that generates C under filtered colimits, so for any compact object A, there exists a factorisation $\mathrm{id}_A = p \circ i$ with $p : B \twoheadrightarrow A$ having domain in $\mathcal{B}$. We have seen that A can be recovered by taking the coequaliser of id_B and $i \circ p$, so the compact objects in C are classified up to isomorphism by the split idempotents in $\mathcal{B}$. Since C is locally small, there is only a small set of split idempotents in $\mathcal{B}$, and therefore $\mathbf{K}(C)$ must be essentially small. ■

Proposition 2.1.14. *Let C be a category with colimits for small filtered diagrams. If C has finite coproducts, then C also has coproducts of all small families of objects.*

Proof. Let $A : I \to C$ be a small family of objects, and let $\mathcal{J}$ be the lattice of all finite subsets of I. Clearly, $\mathcal{J}$ is a filtered category. Consider the diagram $B : \mathcal{J} \to C$ defined by $BJ = \coprod_{j \in J} Aj$; this makes sense because each J is a finite subset of I, and $C = \varinjlim_{\mathcal{J}} B$ exists because C has small filtered colimits. We claim $C \cong \coprod_{i \in I} Ai$. Indeed, for any object J in $\mathcal{J}$, the family $\left(Aj \to BJ \,\middle|\, j \in J\right)$ is a jointly epimorphic sink, so any cocone on B is uniquely determined by

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[4] An **essentially small category** is a category that is equivalent to a small one.

its restriction to the subdiagram A, and therefore their colimits must be isomorphic. ■

Corollary 2.1.15. *Any locally finitely-presentable category has colimits for all small diagrams.*

Proof. If C is an l.f.p. category, then it has finite coproducts and colimits for small filtered diagrams, so it must also have all small coproducts. By definition C also has coequalisers of parallel pairs, and it is well-known that all small colimits can be constructed from small coproducts and coequalisers of parallel pairs. ■

We presently return to the problem of defining finiteness for functors. What should we mean when we say that a functor $F : \mathbf{Set} \to C$ is finitary? If F is known to preserve all colimits, then it is true that F is determined by its restriction to **FinSet**: after all, by proposition 2.1.2, every set is the colimit of a canonical filtered diagram of finite sets. In fact, it is enough to know the action of F on a singleton set 1, since every set is canonically a coproduct of copies of 1. The (formal dual of the) special adjoint functor theorem also implies F has a right adjoint, so all in all, cocontinuity is far too much to ask for.

A slightly better answer to this question is given by left Kan extension: let us provisionally declare F to be finitary if it is a left Kan extension of the restriction $F : \mathbf{FinSet} \to C$ along the inclusion $\mathbf{FinSet} \to C$. Let us also assume that C is cocomplete. Then, the explicit description of left Kan extensions tells us that

$$FX \cong \varinjlim_{(\mathbf{FinSet} \downarrow X)} FU^X$$

where $U^X : (\mathbf{FinSet} \downarrow X) \to \mathbf{Set}$ is the canonical diagram, so we can compute FX by taking the filtered colimit of FA as A varies over the lattice of finite subsets of X. However, this description of left Kan extension only applies when C has all small colimits, so this definition is perhaps too weak for practical purposes.

We would also like to have a notion of 'finitary functor' that makes sense when the domain is any finitely-accessible category. Thus we are led to the following:

Definition 2.1.16. A **finitary functor** is a functor between finitely-accessible categories that preserves colimits for all small filtered diagrams.

Note, however, that a finitary functor need not map compact objects to compact objects.

Example 2.1.17. Let $\mathbb{T}$ be the free monoid monad on **Set** and let T be the underlying endofunctor. Since

$$TX = \coprod_{n \in \mathbb{N}} X^n = 1 \amalg X \amalg X^2 \amalg \cdots$$

it follows immediately from theorem 2.1.5 that T is a finitary functor in the above sense. Since finite sets are compact objects in **Set**, we also see that **FinSet** is a small category of arities for $\mathbb{T}$.

Example 2.1.18. Let R be any ring. Since filtered colimits of left R-modules exist and are computed as in **Set**,[5] the forgetful functor $U : \mathbf{Mod}(R) \to \mathbf{Set}$ preserves them. The argument of example 2.1.10 also implies that $\mathbf{Mod}(R)$ is a finitely-accessible category, so U is therefore a finitary functor.

Theorem 2.1.19 (Finitary adjoint functor theorem)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be finitely-accessible categories, and suppose $\mathcal{D}$ has all small limits. For a* finitary *functor $U : \mathcal{D} \to \mathcal{C}$, the following are equivalent:*

(i) *U preserves limits for all diagrams.*

(ii) *U preserves limits for all small diagrams.*

(iii) *U has a left adjoint.*

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). We will show that the solution set condition of the general adjoint functor theorem[6] is satisfied. First, let X be a compact object in $\mathcal{C}$ and consider the comma category $(X \downarrow U)$. Since every object in $\mathcal{D}$ can be written as a filtered colimit of a small set $\mathcal{B}$ of compact objects in $\mathcal{D}$, compactness of X implies every morphism $f : X \to UA$ must factor through UB for some object B in $\mathcal{B}$. Since $\mathcal{C}$ is locally small, there is therefore a small weakly initial

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[5] See e.g. [Eisenbud, 1995, App. 6, Prop. A6.3] for a proof.

[6] See [CWM, Ch. V, § 6] for details.

family in $(X \downarrow U)$, and the initial object lemma implies $(X \downarrow U)$ has an initial object, say (FX, η_X). If $g : X \to X'$ is any morphism in $\mathcal{C}$ where X' is compact, then there is a unique morphism $Fg : FX \to FX'$ in $\mathcal{D}$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[d, "g"'] & UFX \arrow[d, "UFg"] \\
X' \arrow[r, "\eta_{X'}"'] & UFX'
\end{tikzcd}$$

commute in $\mathcal{C}$, and the usual argument can be used to show that this defines a functor $F : \mathbf{K}(\mathcal{C}) \to \mathcal{D}$. (Note the domain!)

Now, let Y be a general object in $\mathcal{C}$. Since $\mathcal{C}$ is finitely accessible, there is a small filtered diagram $X : \mathcal{J} \to \mathcal{C}$ such that $\varinjlim_{\mathcal{J}} X \cong Y$, and thus we may define an object $FY = \varinjlim_{\mathcal{J}} FX$ in $\mathcal{D}$. (If Y is itself a compact object, set $FY = FX$ instead.) This is independent of the choice of X by statement (i) of proposition 2.1.13 and proposition 2.1.4. Since $U : \mathcal{D} \to \mathcal{C}$ preserves small filtered colimits, there is a unique morphism $\eta_Y : Y \to UFY$ in $\mathcal{C}$ making the diagram

$$\begin{tikzcd}
Xj \arrow[r, "\eta_{Xj}"] \arrow[d, "\lambda_j"'] & UFXj \arrow[d, "UF\lambda_j"] \\
Y \arrow[r, dashed, "\eta_Y"'] & UFY
\end{tikzcd}$$

commute for each object j in $\mathcal{J}$, where $\lambda : X \Rightarrow \Delta Y$ and $F\lambda : FX \Rightarrow \Delta FY$ are defined[7] to be the respective colimiting cocones. We claim (FY, η_Y) is an initial object in $(Y \downarrow U)$. Let $f : Y \to UA$ be any morphism in $\mathcal{C}$. For each j, there exists a unique $\varphi_j : FXj \to A$ in $\mathcal{D}$ such that $f \circ \lambda_j = U\varphi_j \circ \eta_{Xj}$, and uniqueness implies that the outer triangle in the diagram

$$\begin{tikzcd}
FXj \arrow[dd, "FXh"'] \arrow[dr, "F\lambda_j"] \arrow[drr, bend left, "\varphi_j"] & & \\
& FY \arrow[r, dashed, "\bar{f}"] & A \\
FXj' \arrow[ur, "F\lambda_{j'}"'] \arrow[urr, bend right, "\varphi_{j'}"'] & &
\end{tikzcd}$$

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[7] Caution: F is not defined on all of $\mathcal{C}$ yet!

commute for every morphism $h : j \to j'$ in $\mathcal{J}$. The universal property of FY then yields a unique morphism $\tilde{f} : FY \to A$ such that $\varphi_j = \tilde{f} \circ F\lambda_j$ for each j, so

$$U\tilde{f} \circ \eta_Y \circ \lambda_j = U\tilde{f} \circ UF\lambda_j \circ \eta_{Xj} = U\varphi_j \circ \eta_{Xj} = f \circ \lambda_j$$

for all j; therefore $U\tilde{f} \circ \eta_Y = f$, and $\tilde{f} : FY \to A$ is the unique such morphism in $\mathcal{D}$. This establishes the solution set condition for $U : \mathcal{D} \to \mathcal{C}$, and it follows that we have a left adjoint $F : \mathcal{C} \to \mathcal{D}$.

(iii) ⇒ (i). It is well-known that right adjoints preserve all limits. ■

Remark 2.1.20. The core of the proof above is a straightforward generalisation of the usual cardinality argument for showing that the forgetful functor from some category of algebraic structures satisfies the solution set condition.

It is not quite true that every functor between finitely-accessible categories that has a left adjoint is necessarily finitary. Instead, we have the following result:

Proposition 2.1.21. *Let $F \dashv U : \mathcal{D} \to \mathcal{C}$ be an adjunction between locally small categories.*

(i) *If $\mathcal{C}$ is finitely accessible and F sends compact objects in $\mathcal{C}$ to compact objects in $\mathcal{D}$, then U preserves any small filtered colimits that exist in $\mathcal{D}$.*

(ii) *If U preserves all small filtered colimits that exist in $\mathcal{D}$, then F sends compact objects in $\mathcal{C}$ to compact objects in $\mathcal{D}$.*

Proof. Let X be a compact object in $\mathcal{C}$, and let $A : \mathcal{J} \to \mathcal{D}$ be a small filtered diagram. Suppose $\varinjlim_{\mathcal{J}} A$ exists in $\mathcal{D}$.

(i). If FX is a compact object in $\mathcal{D}$, then we have natural bijections as below:

$$\begin{aligned}
\mathcal{C}\Big(X, U\varinjlim_{\mathcal{J}} A\Big) &\cong \mathcal{D}\Big(FX, \varinjlim_{\mathcal{J}} A\Big) && \text{by adjunction} \\
&\cong \varinjlim_{\mathcal{J}} \mathcal{D}(FX, A) && \text{by compactness} \\
&\cong \varinjlim_{\mathcal{J}} \mathcal{C}(X, UA) && \text{by adjunction} \\
&\cong \mathcal{C}\Big(X, \varinjlim_{\mathcal{J}} UA\Big) && \text{by compactness}
\end{aligned}$$

Since X is arbitrary and $\mathcal{C}$ is finitely accessible, propositions 2.1.13 and 1.7.3 together imply $U \varinjlim_{\mathcal{J}} A \cong \varinjlim_{\mathcal{J}} UA$.

(ii). On the other hand, if U preserves small filtered colimits, then,

$$\begin{aligned}
\mathcal{D}\left(FX, \varinjlim_{\mathcal{J}} A\right) &\cong \mathcal{C}\left(X, U \varinjlim_{\mathcal{J}} A\right) && \text{by adjunction} \\
&\cong \mathcal{C}\left(X, \varinjlim_{\mathcal{J}} UA\right) && \text{because } U \text{ is finitary} \\
&\cong \varinjlim_{\mathcal{J}} \mathcal{C}(X, UA) && \text{by compactness} \\
&\cong \varinjlim_{\mathcal{J}} \mathcal{D}(FX, A) && \text{by adjunction}
\end{aligned}$$

and $A : \mathcal{J} \to \mathcal{D}$ is arbitrary, so FX is compact. ■

Corollary 2.1.22. *Let $\mathcal{C}$ be a finitely-accessible category, and let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$. If the inclusion $R : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint $L : \mathcal{C} \to \mathcal{D}$ and R preserves all small filtered colimits, then $\mathcal{D}$ is a finitely-accessible category.*

Proof. Because R is fully faithful, $LR \cong \mathrm{id}_{\mathcal{D}}$. Thus, given a small filtered diagram $A : \mathcal{J} \to \mathcal{D}$, we have the isomorphisms

$$L \varinjlim_{\mathcal{J}} RA \cong \varinjlim_{\mathcal{J}} LRA \cong \varinjlim_{\mathcal{J}} A$$

and so $\mathcal{D}$ has colimits for all small filtered diagrams. The proposition implies L maps compact objects in $\mathcal{C}$ to compact objects in $\mathcal{D}$, and every object of $\mathcal{C}$ is a colimit of some small filtered diagram of compact objects in $\mathcal{C}$, so the same must be true in $\mathcal{D}$. ■

2 Presheaves and torsors

Perhaps the most important l.f.p. categories are the categories $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ for small categories $\mathcal{A}$. It will be convenient to give these a name:

Definition 2.2.1. A **presheaf** on a category $\mathcal{A}$ is any functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$. If X is a presheaf on $\mathcal{A}$, for $x \in X(A)$ and $f : A' \to A$ in $\mathcal{C}$, we will usually write $x \restriction f$ instead of $X(f)(x)$. A **presheaf topos** is any category of the form $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ for some *small* category $\mathcal{A}$.

We will prove that any presheaf topos is indeed a l.f.p. category, but for this we require two technical lemmas:

Lemma 2.2.2. *Let C be a locally small category with colimits for all finite diagrams. If $\mathcal{A}$ is a small full subcategory of C, then there exists a category $\mathcal{B}$ with all of the following properties:*

- *$\mathcal{A} \subseteq \mathcal{B}$, and $\mathcal{B}$ is a small full subcategory of C.*
- *Colimits for finite diagrams exist in $\mathcal{B}$ and are preserved by the inclusion $\mathcal{B} \hookrightarrow C$.*
- *There is a $\mathbb{N}$-indexed sequence of full subcategories*

  $$\mathcal{A} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}$$

  such that each object in $\mathcal{B}_{n+1}$ can be obtained as a colimit in $\mathcal{B}$ of a finite diagram in $\mathcal{B}_n$, and $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$.

Proof. This is a standard induction argument. To begin, set $\mathcal{B}_0 = \mathcal{A}$. For each natural number n, let $\mathcal{B}_{n+1}$ be the full subcategory of C obtained by adding a colimit for each finite diagram in $\mathcal{B}_n$. Clearly, $\mathcal{B}_{n+1}$ is a small category if $\mathcal{B}_n$ is. Let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$. This is also a small category, and by construction $\mathcal{A} \subseteq \mathcal{B} \subseteq C$. We claim $\mathcal{B}$ is closed under finite colimits in C: indeed, if $X : \mathcal{J} \to \mathcal{B}$ is a finite diagram, then X must factor through an inclusion $\mathcal{B}_n \hookrightarrow \mathcal{B}$ for some natural number n, in which case $\varinjlim_{\mathcal{J}} X$ exists in $\mathcal{B}_{n+1}$. ■

Lemma 2.2.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of a category C. If $\mathcal{A} \subseteq \mathcal{B} \subseteq C$ and $\mathcal{A} \hookrightarrow C$ is dense, then $\mathcal{B} \hookrightarrow C$ is also dense.*

Proof. Let $f : B \to X$ be an object in $(\mathcal{B} \downarrow X)$. Since $\mathcal{A} \hookrightarrow C$ is dense, the canonical cocone on the diagram of shape $(\mathcal{A} \downarrow B)$ is colimiting. In particular, since $\mathcal{B}$ is a full subcategory of C, there exists a colimiting cocone on f in $(\mathcal{B} \downarrow X)$ from objects in $(\mathcal{A} \downarrow X)$. Thus, any cocone on the diagram of shape $(\mathcal{B} \downarrow X)$ is uniquely and freely determined by its components on the subdiagram of shape $(\mathcal{A} \downarrow X)$. ■

Proposition 2.2.4. *Let C be a locally small category. The following are equivalent:*

(i) *C is locally finitely presentable.*

(ii) *C has colimits for all small diagrams, and there exists a small full subcategory $\mathcal{A}$ such that the inclusion $\mathcal{A} \hookrightarrow C$ is dense and every object in $\mathcal{A}$ is a compact object in C.*

Proof. (i) ⇒ (ii). By proposition 2.1.13, we know $\mathbf{K}(\mathcal{C})$ is equivalent to a small full subcategory $\mathcal{A}$ and $\mathbf{K}(\mathcal{C})$ is dense, so $\mathcal{A} \hookrightarrow \mathcal{C}$ is also dense. Corollary 2.1.15 says $\mathcal{C}$ has all small colimits, so we are done here.

(ii) ⇒ (i). Let $\mathcal{B}$ be the finite colimit closure of $\mathcal{A}$ as constructed in the first lemma. By lemma 2.1.12, every object in $\mathcal{B}$ is a compact object in $\mathcal{C}$. The slice category $(\mathcal{B} \downarrow X)$ must be a filtered category because $\mathcal{B}$ has finite colimits, and since $\widehat{\mathcal{A}}$ is locally small and $\mathcal{B}$ is small, $(\mathcal{B} \downarrow X)$ is also small. $\mathcal{A} \hookrightarrow \mathcal{C}$ is dense by hypothesis, so the second lemma implies $\mathcal{B}$ is dense. Thus, X is a colimit of a (canonical) small filtered diagram of compact objects drawn from a small set, and therefore $\mathcal{C}$ is an l.f.p. category. ■

Corollary 2.2.5. *A presheaf topos is a locally finitely-presentable category.*

Proof. Let $\mathcal{A}$ be a small category and let $\widehat{\mathcal{A}} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Small colimits in presheaf toposes are computed componentwise since $\mathbf{Set}$ is cocomplete, so $\widehat{\mathcal{A}}$ certainly has all colimits. Let $h_\bullet : \mathcal{A} \to \widehat{\mathcal{A}}$ be the Yoneda embedding. The Yoneda lemma implies $\widehat{\mathcal{A}}(h_a, -) : \widehat{\mathcal{A}} \to \mathbf{Set}$ preserves all small colimits, so h_a is certainly a compact object in $\mathcal{A}$, and proposition 1.7.3 implies $h_\bullet : \mathcal{A} \to \widehat{\mathcal{A}}$ is a dense functor, so the claim follows from the proposition. ■

Presheaf toposes enjoy the following universal property:

Definition 2.2.6. Let $\mathcal{A}$ be a locally small category. A **free cocompletion** of $\mathcal{A}$ is a locally small category $\widehat{\mathcal{A}}$ equipped with a universal functor $H : \mathcal{A} \to \widehat{\mathcal{A}}$ such that, for any locally small and cocomplete category $\mathcal{C}$ and any functor $F : \mathcal{A} \to \mathcal{C}$, there exists a cocontinuous functor $\tilde{F} : \widehat{\mathcal{A}} \to \mathcal{C}$ such that $\tilde{F} H \cong F$, and $\tilde{F}$ is unique up to natural isomorphism.

Theorem 2.2.7 (Kan). *Let $\mathcal{A}$ be a small category and let $\widehat{\mathcal{A}}$ be the presheaf topos* $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.

(i) *Let $\mathcal{C}$ be a locally small and cocomplete category and let $F : \mathcal{A} \to \mathcal{C}$ be any functor. If $F^\sharp : \mathcal{C} \to \widehat{\mathcal{A}}$ is the functor defined by $F^\sharp C = \mathcal{C}(F(-), C)$, then $F^\sharp$ has a left adjoint $F_\sharp : \widehat{\mathcal{A}} \to \mathcal{C}$ such that $\tilde{F} h_\bullet \cong F$, where $h_\bullet : \mathcal{A} \to \widehat{\mathcal{A}}$ is the Yoneda embedding.*

(ii) *$\widehat{\mathcal{A}}$ is a free cocompletion of $\mathcal{A}$, where the universal functor is $h_\bullet : \mathcal{A} \to \widehat{\mathcal{A}}$.*

Proof. By proposition 1.7.3, the Yoneda embedding $h_\bullet : \mathcal{A} \to \widehat{\mathcal{A}}$ is dense, so there is *at most* one extension $F_\sharp : \widehat{\mathcal{A}} \to \mathcal{C}$ up to natural isomorphism. Noting that a left adjoint automatically preserves all colimits, we see that to prove claim (ii) it is enough to prove claim (i).

For convenience, we will usually identify $\mathcal{A}$ with its image under $h_\bullet$. Since $\mathcal{C}$ is cocomplete, it makes sense to define $F_\sharp X = \varinjlim_{(\mathcal{A}\downarrow X)} FU^X$, where U^X is the canonical diagram of shape $(\mathcal{A}\downarrow X)$. Using the Yoneda lemma, we obtain bijections

$$\mathcal{C}\big(F_\sharp X, C\big) \cong \varinjlim_{(\mathcal{A}\downarrow X)} \mathcal{C}\big(FU^X, C\big) \cong \varinjlim_{(\mathcal{A}\downarrow X)} \widehat{\mathcal{A}}\big(U^X, F^\sharp C\big) \cong \widehat{\mathcal{A}}\big(X, F^\sharp C\big)$$

that are natural in C. In particular there is a morphism $\eta_X : X \to F^\sharp F_\sharp X$ corresponding to $\mathrm{id} : F_\sharp X \to F_\sharp X$ in $\mathcal{C}$. We claim $\big(F_\sharp X, \eta_X\big)$ is an initial object in the comma category $\big(X \downarrow F^\sharp\big)$. Indeed, for any presheaf morphism $f : X \to F^\sharp C$, there is a corresponding morphism $\tilde{f} : F_\sharp X \to C$ in $\mathcal{C}$, and naturality implies $\tilde{f}$ is the unique morphism $F_\sharp X \to C$ such that $F^\sharp \tilde{f} \circ \eta_X = f$. Thus, the general adjoint functor theorem makes $F_\sharp$ into a functor $\widehat{\mathcal{A}} \to \mathcal{C}$ that is left adjoint to $F^\sharp$, as required. ■

Corollary 2.2.8. *Let $\mathcal{A}$ be a small category and let $L : \mathcal{A} \to \mathcal{B}$ be a functor. Let $L^\sharp : \mathcal{B} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ be the functor defined by $L^\sharp B = \mathcal{B}(L(-), B)$. If L is fully faithful and essentially surjective on objects, then $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is a free cocompletion of $\mathcal{B}$ with universal functor $L^\sharp$.*

Proof. Assuming a sufficiently powerful axiom of choice, $L : \mathcal{A} \to \mathcal{B}$ has a right adjoint $R : \mathcal{B} \to \mathcal{A}$ such that the unit and counit of the adjunction are natural isomorphisms. By adjunction, we have isomorphisms $L^\sharp B \cong h_{RB}$ natural in B, where $h_\bullet : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding.

Let $\mathcal{C}$ be any locally small and cocomplete category. Because $LR \cong \mathrm{id}_{\mathcal{B}}$, every functor $F : \mathcal{B} \to \mathcal{C}$ is naturally isomorphic to one that factors through R, namely FLR, and $FL : \mathcal{A} \to \mathcal{C}$ factors as $(FL)_\sharp h_\bullet$ where $(FL)_\sharp : [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{C}$ is a cocontinuous functor that is unique up to isomorphism, thus:

$$(FL)_\sharp L^\sharp \cong (FL)_\sharp h_{R(-)} \cong FLR \cong F$$

On the other hand, given any cocontinuous functor $\tilde{F} : [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathcal{C}$ such that $\tilde{F} L^\sharp \cong F$, we must have

$$FL \cong \tilde{F} L^\sharp L \cong \tilde{F} h_{RL(-)} \cong \tilde{F} h_\bullet$$

and therefore $\tilde{F} \cong (FL)_\sharp$, as required. ■

The importance of presheaf toposes *qua* finitely-accessible categories is highlighted by the following theorem:

Theorem 2.2.9 (Representation of finitely-accessible categories). *Let $\mathcal{C}$ be any finitely-accessible category, and let $\mathbf{K}(\mathcal{C})$ be the full subcategory spanned by compact objects in $\mathcal{C}$.*

(i) *$\mathbf{K}(\mathcal{C})$ is an essentially small category of arities for the identity monad on $\mathcal{C}$.*

(ii) *There exists a presheaf topos $\widehat{\mathbf{K}(\mathcal{C})} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and a fully faithful* finitary *functor $\mathrm{N}_{\mathbf{K}(\mathcal{C})} : \mathcal{C} \to \widehat{\mathbf{K}(\mathcal{C})}$.*

(iii) *A presheaf is in the essential image of $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ if and only if it is a colimit of some small filtered diagram of representable presheaves.*

Proof. (i). $\mathbf{K}(\mathcal{C}) \hookrightarrow \mathcal{C}$ is dense by statement (ii) of proposition 2.1.13. For each object B in $\mathcal{C}$, there is a cofinal small filtered category of $(\mathbf{K}(\mathcal{C}) \downarrow B)$, and for any object A in $\mathbf{K}(\mathcal{C}) \hookrightarrow \mathcal{C}$, $\mathcal{C}(A, -) : \mathcal{C} \to \mathbf{Set}$ preserves small filtered colimits by proposition 2.1.8, so proposition 2.1.4 implies $\mathcal{C}(A, -)$ maps the canonical cocone on the canonical diagram of shape $(\mathbf{K}(\mathcal{C}) \downarrow B)$ to a colimiting cocone in $\mathbf{Set}$.

(ii). By statement (iii) of proposition 2.1.13, $\mathbf{K}(\mathcal{C})$ is equivalent to some small full subcategory $\mathcal{A}$, so we can define $\mathrm{N}_{\mathbf{K}(\mathcal{C})} = L^{\sharp}$, where $L : \mathcal{A} \hookrightarrow \mathcal{B}$ is the inclusion. $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ so defined is fully faithful by proposition 1.7.3, and it is finitary because small colimits in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ are computed componentwise and every object in $\mathcal{A}$ is a compact object in $\mathcal{C}$.

(iii). Every object in $\mathcal{C}$ is a colimit (in $\mathcal{C}$) of a small filtered diagram of objects in $\mathcal{A}$, so the claim follows immediately from (ii). ■

In other words, every finitely-accessible category is equivalent to a full subcategory of a presheaf topos spanned by presheaves with a certain property. We may give an intrinsic description of these presheaves as follows:

Definition 2.2.10. Let $\mathcal{A}$ be a category. A **right $\mathcal{A}$-torsor** is a presheaf X on $\mathcal{A}$ with the following properties:

- X is non-empty, in the sense that there is some object A such that the set $X(A)$ is inhabited.

- Given $a \in X(A)$ and $b \in X(B)$, there are arrows $f : A \to C$ and $g : B \to C$ in $\mathcal{A}$ and an element c of $X(C)$ such that $c \upharpoonright f = a$ and $c \upharpoonright g = b$.

- Given $b \in X(B)$, if $f, g : A \to B$ are arrows such that $b \upharpoonright f = b \upharpoonright g$, then there exist an arrow $h : B \to C$ in $\mathcal{A}$ and an element c of $X(C)$ such that $h \circ f = h \circ g$ and $c \upharpoonright h = b$.

A **left $\mathcal{A}$-torsor** is a right $\mathcal{A}^{\mathrm{op}}$-torsor. When $\mathcal{A}$ is small, we write $\mathbf{Tors}(\mathcal{A}, \mathbf{Set})$ for the full subcategory of $[\mathcal{A}, \mathbf{Set}]$ spanned by *left* $\mathcal{A}$-torsors.

Example 2.2.11. We should give an example justifying the name. Let G be a group, and let $\mathcal{A}$ be G considered as a category with one object—say, $*$. Clearly, a presheaf on $\mathcal{A}$ is the same thing as a set equipped with a right G-action. We claim a right $\mathcal{A}$-torsor is the same thing as an inhabited set equipped with a freely transitive right G-action.

Indeed, first suppose X is a right $\mathcal{A}$-torsor. Then, $X(*)$ must be an inhabited set because $*$ is the only object in $\mathcal{A}$. For any two elements a and b in $X(*)$, there exist elements f and g in the group G and an element c in $X(*)$ such that $c \upharpoonright f = a$ and $c \upharpoonright g = b$, but since f is invertible, $a \upharpoonright f^{-1} \circ g = c \upharpoonright g = b$, so G acts transitively on $X(*)$. Also, if $b \upharpoonright f = b \upharpoonright g$, then there exist an element h in the group G and an element c in $X(*)$ such that $h \circ f = h \circ g$, but h is invertible, so $f = g$, so G acts freely on $X(*)$.

Conversely, suppose $X(*)$ is inhabited and G acts freely and transitively on $X(*)$. Then, for any two elements a and b in $X(*)$, there exist an element g in the group G such that $a \upharpoonright g = b$, and obviously $a \upharpoonright e = a$ when e is the unit of G. Similarly, if $b \upharpoonright f = b \upharpoonright g$, then $f = g$, so we have $e \circ f = e \circ g$ and $b \upharpoonright e = b$. Thus X is a right $\mathcal{A}$-torsor.

Proposition 2.2.12. *Let $\mathcal{A}$ be a small category.*

(i) *For each object X, the representable presheaf h_X is a right $\mathcal{A}$-torsor.*

(ii) *If $X : \mathcal{J} \to \mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is a small filtered diagram, then $\bar{X} = \varinjlim_{\mathcal{J}} h_{X(-)}$ exists in $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ and is computed as in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$.*

Proof. (i). Since $\mathrm{id}_X \in h_X(X) = \mathcal{A}(X, X)$, h_X is non-empty. Let $a \in h_X(A)$ and $b \in h_X(B)$. Then, $a = \mathrm{id}_X \circ a$ and $b = \mathrm{id}_X \circ b$, so $c = \mathrm{id}_X \in h_X(X)$ is the required element such that $c \upharpoonright a = a$ and $c \upharpoonright b = b$. Similarly, if $b \in h_X(B)$ and $b \circ f = b \circ g$, then $c = \mathrm{id}_X$ is the required element such that $c \upharpoonright b = b$. Thus $\mathcal{h}_X$ is a right $\mathcal{A}$-torsor.

(ii). Let $\lambda : X \Rightarrow \Delta\bar{X}$ be the colimiting cocone. Since $\mathcal{J}$ has at least one object j and Xj is non-empty, $\bar{X}$ is non-empty as well.

Let $a \in \bar{X}(A)$ and $b \in \bar{X}(B)$. Since $\mathcal{J}$ is filtered, we can choose an object j in $\mathcal{J}$ such that $a = \lambda_j(\tilde{a})$ and $b = \lambda_j\left(\tilde{b}\right)$ for some $\tilde{a}$ in $Xj(A)$ and $\tilde{b}$ in $Xj(B)$. Since Xj is a right $\mathcal{A}$-torsor, there are then arrows $f : A \to C$ and $g : B \to C$ and an element $\tilde{c}$ of $Xj(C)$ such that $\tilde{c} \upharpoonright f = \tilde{a}$ and $\tilde{c} \upharpoonright g = \tilde{b}$; and $\lambda_j : Xj \to \bar{X}$ is a presheaf morphism, so we obtain the required element $c = \lambda_j(\tilde{c})$ such that $c \upharpoonright f = a$ and $c \upharpoonright g = b$.

Now let $b \in \bar{X}(B)$ and suppose $f, g : A \to B$ are arrows in $\mathcal{A}$ such that $b \upharpoonright f = b \upharpoonright g$. As before, there exists an object j in $\mathcal{J}$ such that $b = \lambda_j\left(\tilde{b}\right)$ for some arrow $\tilde{b} : B \to Xj$ in $\mathcal{A}$, but by the concrete description of filtered colimits in **Set**, there must be arrows $k, l : j \to j'$ in $\mathcal{J}$ such that $Xk\left(\tilde{b} \upharpoonright f\right) = Xl\left(\tilde{b} \upharpoonright g\right)$. Since $\mathcal{J}$ is filtered, there is an arrow $m : j' \to j''$ in $\mathcal{J}$ such that $m \circ k = m \circ l$, so we obtain an element $\tilde{b}'' = X(m \circ k)\left(\tilde{b}\right) = X(m \circ l)\left(\tilde{b}\right)$ of $Xj''(B)$ such that $\tilde{b}'' \upharpoonright f = \tilde{b}'' \upharpoonright g$. Since Xj'' is a right $\mathcal{A}$-torsor, we obtain an arrow $h : B \to C$ and an element $\tilde{c}$ of $Xj''(C)$ such that $\tilde{c} \upharpoonright h = \tilde{b}''$, and so $c = \lambda_{j''}(\tilde{c})$ is the required element such that $c \upharpoonright h = b$. ■

We also have a converse:

Lemma 2.2.13. *Let $\mathcal{A}$ be a small category, and let $\mathcal{h}_\bullet : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ be the Yoneda embedding. A presheaf X is a right $\mathcal{A}$-torsor if and only if the comma category $(\mathcal{h}_\bullet \downarrow X)$ is filtered.*

Proof. This is an easy exercise in using the Yoneda lemma. ◊

These two results allow us to strengthen our earlier representation theorem to obtain a complete classification of finitely accessible categories:

Theorem 2.2.14 (Classification of finitely-accessible categories). *Let C be a category. The following are equivalent:*

(i) *C is finitely accessible.*

(ii) *C is equivalent to* $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$*, where $\mathcal{A}$ is a small category equivalent to* $\mathbf{K}(C)$*.*

(iii) *C is equivalent to* $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ *for some small category $\mathcal{A}$.*

Proof. (i) ⇒ (ii). This follows from the proposition above and the representation theorem for finitely-accessible categories.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (i). The proposition shows that $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is closed under small filtered colimits in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, and the lemma shows that every right $\mathcal{A}$-torsor the colimit of a small filtered diagram of representable presheaves. Since $\mathcal{A}$ is small and representable presheaves are compact objects in $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$, it follows that $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ is finitely-accessible category. ■

What can we say about l.f.p. categories? Recall lemma 2.1.12: the colimit of a finite diagram in $\mathbf{K}(\mathcal{C})$ can be computed in $\mathcal{C}$ whenever it exists in $\mathcal{C}$. Thus, when $\mathcal{C}$ is an l.f.p. category, $\mathbf{K}(\mathcal{C})$ has all finite colimits. We will shortly see that right $\mathcal{A}$-torsors on a finitely cocomplete category $\mathcal{A}$ have a particularly simple description.

Definition 2.2.15. A **left-exact functor** is a functor $F : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{A}$ has limits for all finite diagrams and F preserves them. Dually, a **right-exact functor** is a functor $F : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{A}$ has colimits for all finite diagrams and F preserves them. An **exact functor** is a functor that is both left exact and right exact.

For variety, we will state the proposition for left torsors; the case for right torsors is obtained by dualisation as usual.

Proposition 2.2.16. *Let $\mathcal{A}$ be a small category with limits for all finite diagrams. The following are equivalent for a functor $F : \mathcal{A} \to \mathbf{Set}$:*

(i) *F is a left-exact functor.*

(ii) *F is a left $\mathcal{A}$-torsor.*

Proof. (i) ⇒ (ii). Since $\mathcal{A}$ is finitely complete, it has a terminal object 1, and $F(1)$ must be a terminal object in $\mathbf{Set}$; in particular, $F(1)$ is inhabited. Also, given $a \in F(A)$ and $b \in F(B)$, if $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ are the respective projections in $\mathcal{A}$, preservation of binary products implies there is an element c of $F(A \times B)$ such that $a = F(\pi_1)(c)$ and $b = F(\pi_2)(c)$. Similarly, given $b \in F(B)$ and a parallel pair $f, g : B \to A$ in $\mathcal{A}$ such that $F(f)(b) = F(g)(b)$, we may form the equaliser $h : C \to B$ of f and g in $\mathcal{A}$, and preservation of equalisers implies there is a c in $F(C)$ such that $b = F(h)(c)$. So F is a left $\mathcal{A}$-torsor.

(ii) ⇒ (i). By lemma 2.2.13, F is the colimit in $[\mathcal{A}, \mathbf{Set}]$ of a (canonical) small filtered diagram of representable functors. It is well-known that representable functors preserve all limits, and theorem 2.1.5 says small filtered colimits in **Set** preserve finite limits, so F must also preserve finite limits. ■

Theorem 2.2.17 (Representation of locally finitely-presentable categories). *Let $\mathcal{C}$ be any locally finitely-presentable category, and let $\mathbf{K}(\mathcal{C})$ be the full subcategory spanned by compact objects in $\mathcal{C}$.*

(i) *There exists a presheaf topos $\widehat{\mathbf{K}(\mathcal{C})} = [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and a fully faithful finitary functor $\mathrm{N}_{\mathbf{K}(\mathcal{C})} : \mathcal{C} \to \widehat{\mathbf{K}(\mathcal{C})}$, where $\mathcal{A}$ has colimits for all finite diagrams and $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ has a right adjoint.*

(ii) *A presheaf is in the essential image of $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ if and only if it is left exact as a functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$.*

Proof. (i). Let F be the inclusion $\mathcal{A} \hookrightarrow \mathcal{C}$. In the notation of theorem 2.2.7, $\mathrm{N}_{\mathbf{K}(\mathcal{C})} = F^\sharp$, and $\mathcal{C}$ is cocomplete by corollary 2.1.15, so $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ has a left adjoint.

(ii). Lemma 2.1.12 says $\mathcal{A}$ has finite colimits, so the category $(\mathcal{A} \downarrow C)$ is filtered for every object C in $\mathcal{C}$; the conclusion follows by lemma 2.2.13, proposition 2.2.16, and the fact that $\mathrm{N}_{\mathcal{A}}$ preserves small filtered colimits. ■

Theorem 2.2.18 (Classification of locally finitely-presentable categories). *Let $\mathcal{C}$ be a category. The following are equivalent:*

(i) *$\mathcal{C}$ is a locally finitely-presentable category.*

(ii) *$\mathcal{C}$ is equivalent to $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$, where $\mathcal{A}$ is a small category with colimits for all finite diagrams.*

(iii) *There exists a category $\mathcal{A}$ and a fully faithful functor $R : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ such that R has a left adjoint and preserves colimits for small filtered diagrams.*

Proof. (i) ⇒ (ii). This is just the representation theorem for l.f.p. categories.

(ii) ⇒ (iii). We know that the inclusion $R : \mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set}) \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is fully faithful and finitary. Since limits preserve limits, the limit (in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$) of any small diagram of left exact functors $\mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is again a left exact functor, so by proposition 2.2.16, $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ has all small limits and R preserves them. The finitary adjoint functor theorem (2.1.19) then gives a left adjoint for R.

(iii) ⇒ (i). Let $L : [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] \to C$ be the left adjoint of R. C is automatically cocomplete: the colimit of a small diagram $A : \mathcal{J} \to C$ may be computed as $L\left(\varinjlim_{\mathcal{J}} RA\right)$. By corollary 2.1.22, C is also finitely accessible, so C is an l.f.p. category. ■

The representation and classification of l.f.p. categories implies a somewhat surprising result:

Corollary 2.2.19. *Let C be a finitely-accessible category. The following are equivalent:*

(i) *C is locally finitely presentable.*

(ii) *C has limits for all small diagrams.*

Proof. (i) ⇒ (ii). Let C be an l.f.p. category. By the representation theorem for l.f.p. categories, there is a finitely cocomplete category $\mathcal{A}$ such that C is equivalent to $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$; but as argued above, $\mathbf{Tors}(\mathcal{A}^{\mathrm{op}}, \mathbf{Set})$ has all small limits when $\mathcal{A}$ has all finite colimits.

(ii) ⇒ (i). Let C be a finitely-accessible category with all small limits. By the representation theorem for finitely-accessible categories, there is a fully faithful functor $R : C \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ where $\mathcal{A}$ is small and R is finitary and preserves all small limits. The finitary adjoint functor theorem (2.1.19) gives a left adjoint for R, so statement (iii) of the classification theorem for l.f.p. categories implies C must be an l.f.p. category. ■

We also get a less surprising corollary:

Corollary 2.2.20. *Let $\mathcal{J}$ be a small category. If C is locally finitely presentable, then so is the functor category $[\mathcal{J}, C]$.*

Proof. By the representation theorem for l.f.p. categories, there is an adjunction $L \dashv R : C \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ where $\mathcal{A}$ is a small category and R is fully faithful and finitary. Since $[\mathcal{J}, -]$ is a 2-functor, there is then an induced adjunction $L_* \dashv R_* : [\mathcal{J}, C] \to [\mathcal{J}, [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]]$, where $L_*(X)$ is defined to be the composite LX and $R_*(A)$ is the composite RA. C has all small colimits, so small colimits exist in $[\mathcal{J}, C]$ and are computed componentwise, and it follows from the fact that R is finitary that R_* preserves all small filtered colimits. The category $[\mathcal{J}, [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]]$ is isomorphic to the presheaf topos $[\mathcal{J} \times \mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ by exponential transposition, and R_* is clearly fully faithful, so the classification theorem for l.f.p. categories implies $[\mathcal{J}, \mathcal{C}]$ is locally finitely presentable. ■

Here is another fact about l.f.p. categories obtained using the representation theorem:

Proposition 2.2.21. *If $\mathcal{C}$ is a locally finitely-presentable category, then small filtered colimits preserve limits of all finite diagrams in $\mathcal{C}$.*

Proof. We have seen in the course of proving corollary 2.2.19 that there is a small category $\mathcal{A}$ and an adjunction $L \dashv R : \mathcal{C} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ where R creates all small limits and all small filtered colimits. Since limits and colimits are computed componentwise in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, theorem 2.1.5 implies small filtered colimits preserve finite limits in $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. The claim for $\mathcal{C}$ follows. ■

3 Quotients and images

We pause here to collect some technical results needed for the following sections. First, recall the following definition:

Definition 2.3.1. A **subobject** of an object X in a category $\mathcal{C}$ is any monomorphism $Y \rightarrowtail X$, considered as an object of the slice category $(\mathcal{S} \downarrow X)$. A **well-powered category** is a category $\mathcal{C}$ such that every object has only a small family of subobjects up to isomorphism.

The relevance of well-poweredness can be seen in, for example, the special adjoint functor theorem[1] and the proof of theorem 1.3.14. It turns out that a slightly weakened form of the dual notion is sufficient for our purposes.

Definition 2.3.2. A **regular quotient** of an object X in a category $\mathcal{C}$ is any regular epimorphism $X \twoheadrightarrow Y$, considered as an object of the coslice category $(X \downarrow \mathcal{S})$. A **regularly well-copowered category** is a category $\mathcal{C}$ such that every object has only a small family of regular quotients up to isomorphism.

Example 2.3.3. Any set X has a powerset $\mathscr{P}(X)$, and the monomorphisms in **Set** are precisely the injective maps, so **Set** is well-powered. On the other

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[1] See [CWM, Ch. V, § 8] for details.

hand, the powerset of any (regular) quotient of a set X is isomorphic to some sublattice of $\mathscr{P}(X)$, so **Set** is regularly well-copowered.

Proposition 2.3.4.

(i) *A well-powered category with kernel pairs and binary products is regularly well-copowered.*

(ii) *A finitely-accessible category is well-powered.*

(iii) *A locally finitely-presentable category is both well-powered and regularly well-copowered.*

Proof. (i). Let $\mathcal{C}$ be a well-powered category with kernel pairs and binary products. Proposition 1.3.10 implies that regular quotients of an object X in $\mathcal{C}$ are classified by their kernel pairs $K \rightrightarrows X$; but $\mathcal{C}$ has products and is well-powered by (i), so by remark 1.3.4 and proposition 1.3.11, $\mathcal{C}$ must be regularly well-copowered.

(ii). Let $\mathcal{C}$ be a finitely-accessible category. By theorem 2.2.17, there is a fully faithful continuous functor $\mathrm{N}_{\mathbf{K}(\mathcal{C})} : \mathcal{C} \to \widehat{\mathbf{K}(\mathcal{C})}$, where $\widehat{\mathbf{K}(\mathcal{C})}$ is a presheaf topos. Since limits are computed objectwise in a presheaf topos, $\widehat{\mathbf{K}(\mathcal{C})}$ inherits well-poweredness from **Set**. $\mathrm{N}_{\mathbf{K}(\mathcal{C})}$ is conservative and preserves monomorphisms, so $\mathcal{C}$ inherits well-poweredness from $\widehat{\mathbf{K}(\mathcal{C})}$.

(iii). Let $\mathcal{C}$ be an l.f.p. category. Corollary 2.2.19 implies $\mathcal{C}$ has all finite limits, so the conclusion follows from (i) and (ii). ■

Remark 2.3.5. Every finitely-accessible category is in fact well-copowered in the strong sense of having only a small family of quotients for each object up to isomorphism—notice the omission of the word 'regular'! A proof of this can be found in [LPAC, Thm 2.49].

Definition 2.3.6. Let $\mathcal{C}$ be a category with kernel pairs and coequalisers for kernel pairs. The **regular coimage** of a morphism $f : X \to Y$ in $\mathcal{C}$ is the coequaliser of the kernel pair of f. The **image** of f is a monomorphism $m : Z \to Y$ in $\mathcal{C}$ satisfying the following conditions:

- There exists a factorisation $f = m \circ e$ in $\mathcal{C}$.
- If $m' : Z' \to Y$ is any monomorphism such that $f = m' \circ e'$ for some $e' : X \to Z'$, then there exists a (unique) morphism $h : Z \to Z'$ such that $m' = m \circ h$.

Lemma 2.3.7. *Let $f : X \to Y$ be a morphism in a category $\mathcal{C}$ with kernel pairs and coequalisers for kernel pairs.*

(i) *There exists a unique morphism $m : Z \to Y$ in $\mathcal{C}$ such that $f = m \circ e$, where $e : X \to Z$ is the regular coimage of f.*

(ii) *For any factorisation $f = m' \circ e'$ in $\mathcal{C}$ where $m' : Z' \to X$ is a monomorphism, there exists a unique morphism $h : Z \to Z'$ such that $e' = h \circ e$ and $m = m' \circ h$.*

(iii) *For any factorisation $f = m' \circ e'$ in $\mathcal{C}$ where $e' : X \to Z'$ is a regular epimorphism, there exists a unique morphism $h' : Z' \to Z$ such that $e = h' \circ e'$ and $m' = m \circ h'$.*

In particular, if $m : Z \to X$ is a monomorphism, then it is the image of f.

Proof. (i). Let $k_0, k_1 : K \to X$ be the kernel pair of f. Since $f \circ k_0 = f \circ k_1$, the universal property of Z as a coequaliser gives a unique morphism $m : Z \to X$ such that $f = m \circ e$.

(ii). First, suppose $m' : Z' \to X$ is a monomorphism and $f = m' \circ e'$. Since

$$m' \circ (e' \circ k_0) = f \circ k_0 = f \circ k_1 = m' \circ (e' \circ k_1)$$

we must have $e' \circ k_0 = e' \circ k_1$, and so there is a unique morphism $h : Z \to Z'$ for which $e' = h \circ e$, but then

$$(m' \circ h) \circ e = m' \circ e' = f = m \circ e$$

so we get $m' \circ h = m$ because $e : X \to Z$ is a (regular) epimorphism.

(iii). Now suppose instead $e' : X \to Z'$ is a regular epimorphism and $f = m' \circ e'$. Then, by proposition 1.3.10, e' is the coequaliser of its kernel pair $k'_0, k'_1 : K' \to X$. But

$$f \circ k'_0 = m' \circ \left(e' \circ k'_0\right) = m' \circ \left(e' \circ k'_1\right) = f \circ k'_1$$

so the universal property of K gives a unique morphism $l : K' \to K$ such that $k'_0 = k_0 \circ l$ and $k'_1 = k_1 \circ l$, and therefore

$$e \circ k'_0 = (e \circ k_0) \circ l = (e \circ k_1) \circ l = e \circ k'_1$$

so the universal property of Z' as a coequaliser gives a unique morphism $h' : Z' \to Z$ such that $e = h' \circ e'$. Since

$$(m \circ h') \circ e' = m \circ e = m' \circ e'$$

it follows from the fact that e' is a (regular) epimorphism that $m \circ h' = m'$. ■

This is about as much as we can say about regular coimages without extra assumptions. In particular, we cannot even say that the morphism $m : Z \to Y$ appearing in the factorisation $f = m \circ e$ is a monomorphism, so we cannot say that m is the image of f in general!

Definition 2.3.8. A **regular category** is a category C with the following properties:

- C has limits for all finite diagrams.
- C has coequalisers for all kernel pairs.
- Given a pullback square in C as below,

$$\begin{tikzcd} X' \arrow[r, "e'"] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "e"'] & Z \end{tikzcd}$$

  if $e : X \to Z$ is a regular epimorphism, then so is $e' : X' \to Z'$.

Example 2.3.9. Since a map in **Set** is a (regular) epimorphism if and only if it is a surjection, one can show directly that **Set** is a regular category. Alternatively, one may prove by abstract nonsense that any locally cartesian closed category with coequalisers for kernel pairs is regular.

Because products are intimately related to pullbacks, it is tempting to say that regular epimorphisms are "obviously" preserved by products in regular categories. However, one must be more careful:

Lemma 2.3.10. *Let C be a regular category.*

(i) *Regular epimorphisms in C are closed under composition.*

(ii) *If* $f : X \to Y$ *is a regular epimorphism in* C*, then so is the morphism* $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$.

(iii) *If* $f : X \to Y$ *and* $g : Z \to W$ *are regular epimorphisms in* C*, then their product* $f \times g : X \times Z \to Y \times W$ *is also a regular epimorphism in* C.

Proof. (i). Let $f : X \to Y$ and $g : Y \to Z$ be regular epimorphisms in C. Let $k_0, k_1 : X \times_Z X \to X$ be the kernel pair of the composite $g \circ f$, and let $p_0, p_1 : X \times_Y X \to X$ and $q_0, q_1 : Y \times_Z Y \to Y$ be the kernel pairs of f and g respectively. Since $(g \circ f) \circ p_0 = (g \circ f) \circ p_1$, there is a unique morphism $p : X \times_Y X \to X \times_Z X$ such that $p_0 = k_0 \circ p$ and $p_1 = k_1 \circ p$. Similarly, $g \circ (f \circ p_0) = g \circ (f \circ p_1)$, so there is a unique morphism $h : X \times_Z X \to Y \times_Z Y$ such that $f \circ p_0 = q_0 \circ h$ and $f \circ p_1 = q_1 \circ h$.

Consider the diagram below, where each square is a pullback in C:

$$\begin{tikzcd}
X \times_Z X \arrow[r] \arrow[d] \arrow[rd, dashed, "h"] & Y \times_Z X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
X \times_Z Y \arrow[r] \arrow[d] & Y \times_Z Y \arrow[r, "q_1"] \arrow[d, "q_0"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

By the pullback pasting lemma, the outer square is also a pullback, so we can identify the composite $X \times_Z X \to X \times_Z Y \to X$ with k_0 and $X \times_Z X \to Y \times_Z X \to X$ with k_1. It follows that $h : X \times_Z X \to Y \times_Z Y$ is the diagonal arrow indicated in the diagram. Each of the horizontal and vertical arrows appearing in the diagram is a regular epimorphism because C is regular, so $h : X \times_Z X \to Y \times_Z Y$ must be an epimorphism, since the class of epimorphisms in any category is closed under composition.

Now suppose $x : X \to W$ is a morphism such that $x \circ k_0 = x \circ k_1$. Then,

$$x \circ p_0 = (x \circ k_0) \circ p = (x \circ k_1) \circ p = x \circ p_1$$

so, recalling proposition 1.3.10, the universal property of Y gives a unique morphism $y : Y \to W$ such that $y \circ f = x$. Similarly,

$$(y \circ q_0) \circ h = (y \circ f) \circ p_0 = x \circ p_0 = x \circ p_1 = (y \circ f) \circ p_1 = (y \circ q_1) \circ h$$

and $h : X \times_Z X \to Y \times_Z Y$ is an epimorphism, so $y \circ q_0 = y \circ q_1$, and there is a unique morphism $z : Z \to W$ such that $z \circ g = y$. Since $g \circ f$ is automatically an epimorphism, z is the unique morphism such that $z \circ (g \circ f) = x$, so $g \circ f$ is the coequaliser of k_0 and k_1, as required.

(ii). This is an immediate consequence of the fact that the diagram below is a pullback square in C:

$$\begin{tikzcd}
X \times Z \arrow[r, "f \times \mathrm{id}_Z"] \arrow[d, "\pi_1"'] & Y \times Z \arrow[d, "\pi_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Here, π_1 denotes the first projection of a binary product.

(iii). By (ii), $f \times \mathrm{id}_Z : X \times Z \to Y \times Z$ and $\mathrm{id}_Y \times g : Y \times Z \to Y \times W$ are both regular epimorphisms, so their composite $(\mathrm{id}_Y \times g) \circ (f \times \mathrm{id}_Z) = f \times g$ must also be a regular epimorphism by (i). ■

For us, the main point of interest in regular categories is the following factorisation theorem:

Proposition 2.3.11. *Let $f : X \to Y$ be a morphism in a regular category C.*

(i) *If $e : X \to Z$ is the regular coimage of f and $f = m \circ e$, then $m : Z \to Y$ is a monomorphism and is the image of f.*

(ii) *If $f = m' \circ e'$ for a regular epimorphism $e' : X \to Z'$ and a monomorphism $m' : Z' \to Y$, then there is a unique isomorphism $h : Z \to Z'$ such that $e' = h \circ e$ and $m = m' \circ h$.*

In particular, every morphism in a regular category factors as a regular epimorphism followed by a monomorphism, and this factorisation is unique up to unique isomorphism.

Proof. (i). Let $k_0, k_1 : K \to X$ be the kernel pair of f. Suppose $a_0, a_1 : A \to Z$ are morphisms such that $m \circ a_0 = m \circ a_1$. Consider the following pullback square in C:

$$\begin{tikzcd}
B \arrow[r, "c"] \arrow[d, "\langle b_0, b_1 \rangle"'] & A \arrow[d, "\langle a_0, a_1 \rangle"] \\
X \times X \arrow[r, "e \times e"'] & Z \times Z
\end{tikzcd}$$

Since $e : X \to Z$ is a regular epimorphism, the lemma says $e \times e : X \times X \to Z \times Z$ is also a regular epimorphism, so $c : B \to A$ must be a (regular) epimorphism. Now,

$$f \circ b_0 = m \circ (e \circ b_0) = (m \circ a_0) \circ c = (m \circ a_1) \circ c = m \circ (e \circ b_1) = f \circ b_1$$

so the universal property of K gives a unique morphism $l : B \to K$ such that $b_0 = k_0 \circ l$ and $b_1 = k_1 \circ l$. Hence,

$$a_0 \circ c = e \circ b_0 = (e \circ k_0) \circ l = (e \circ k_1) \circ l = e \circ b_1 = a_1 \circ c$$

and thus we must have $a_0 = a_1$, as required for m to be monic.

(ii). By the lemma, there are unique morphisms $h : Z \to Z'$, $h' : Z' \to Z$ such that

$$\begin{aligned} e' &= h \circ e & \qquad m &= m' \circ h \\ e &= h' \circ e' & \qquad m' &= m \circ h' \end{aligned}$$

and therefore

$$e' = h \circ e = (h \circ h') \circ e' \qquad\qquad m = m' \circ h = m \circ (h' \circ h)$$

but e' is epic and m is monic, so $h \circ h' = \mathrm{id}_{Z'}$ and $h' \circ h = \mathrm{id}_Z$. ■

Remark 2.3.12. The upshot of the proposition is that regular epimorphisms and (ordinary) monomorphisms form an orthogonal factorisation system in any regular category.

We have an ample supply of regular categories:

Proposition 2.3.13.

(i) *For any small category $\mathcal{A}$, if $\mathcal{C}$ is a regular category, then the functor category $[\mathcal{A}, \mathcal{C}]$ is also a regular category.*

(ii) *In particular, any presheaf topos is a regular category.*

Proof. Since kernel pairs and their coequalisers are computed componentwise in $[\mathcal{A}, \mathcal{C}]$, proposition 1.3.10 implies we can check whether a morphism in $[\mathcal{A}, \mathcal{C}]$ is a regular epimorphism by checking it componentwise; but pullbacks are also computed componentwise, so $[\mathcal{A}, \mathcal{C}]$ is a regular category when $\mathcal{C}$ is. ■

Proposition 2.3.14. *Let $\mathcal{C}$ be a regular category.*

(i) *If $U : \mathcal{D} \to \mathcal{C}$ is a functor that creates limits for all finite diagrams and both preserves and reflects regular epimorphisms, then $\mathcal{D}$ is a regular category if and only if $\mathcal{D}$ has coequalisers for all kernel pairs.*

(ii) *If J is an endofunctor on $\mathcal{C}$ that preserves coequalisers for kernel pairs, then $\mathcal{C}^J$ is a regular category.*

(iii) *If $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{C}$ such that the endofunctor T preserves coequalisers for kernel pairs, then $\mathcal{C}^{\mathbb{T}}$ is a regular category.*

(iv) *If all regular epimorphisms split in $\mathcal{C}$, then $\mathcal{C}^J$ is regular for any endofunctor J, and $\mathcal{C}^{\mathbb{T}}$ is regular for any monad $\mathbb{T}$.*

Proof. (i). If $U : \mathcal{D} \to \mathcal{C}$ creates finite limits, then $\mathcal{D}$ has finite limits when $\mathcal{C}$ does, and U preserves them. Consider a pullback diagram in $\mathcal{D}$:

$$\begin{tikzcd}
A' \arrow[r, "e'"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "e"'] & B
\end{tikzcd}$$

Suppose $e : A \to B$ is a regular epimorphism in $\mathcal{D}$. Then, $Ue : UA \to UB$ is a regular epimorphism in $\mathcal{C}$ by our hypothesis, and we have a pullback square in $\mathcal{C}$ as below:

$$\begin{tikzcd}
UA' \arrow[r, "Ue'"] \arrow[d] & UB' \arrow[d] \\
UA \arrow[r, "Ue"'] & UB
\end{tikzcd}$$

Since $\mathcal{C}$ is a regular category, $Ue' : UA' \to UB'$ must also be a regular epimorphism in $\mathcal{C}$; and U reflects regular epimorphisms, so $e' : A' \to B'$ is a regular epimorphism in $\mathcal{D}$. Thus $\mathcal{D}$ is a regular category if and only if it has coequalisers for all kernel pairs.

(ii). A modification of proposition 1.2.5 shows that the forgetful functor $U^J : \mathcal{C}^J \to \mathcal{C}$ creates all limits that exist in $\mathcal{C}$. If J maps regular epimorphisms to epimorphisms, then U^J reflects regular epimorphisms. Indeed, suppose $f : (A, \alpha_1) \to (B, \beta_1)$ is a J-module homomorphism such that $f : A \to B$ is a regular epimorphism in $\mathcal{C}$. Form the kernel pair $k_0, k_1 : (K, \kappa_1) \to (A, \alpha_1)$ in $\mathcal{C}^J$, and note that the diagram

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

is a coequaliser diagram in $\mathcal{C}$ by proposition 1.3.10. Given any J-module homomorphism $h : (A, \alpha_1) \to (C, \gamma_1)$ such that $h \circ k_0 = h \circ k_1$, the universal property of B gives a unique morphism $g : B \to C$ in $\mathcal{C}$ such that $h = g \circ f$, and

$$(\gamma_1 \circ J g) \circ J f = \gamma_1 \circ J h = h \circ \alpha_1 = g \circ (f \circ \alpha_1) = (g \circ \beta_1) \circ J f$$

but $J f : J A \to J B$ is an epimorphism in $\mathcal{C}$ by our hypothesis, so $\gamma_1 \circ J g = g \circ \beta_1$ as required for g to be a J-module homomorphism $(B, \beta_1) \to (C, \gamma_1)$. Thus $f : (A, \alpha_1) \to (B, \beta_1)$ is the coequaliser of $k_0, k_1 : (K, \kappa_1) \to (A, \alpha_1)$ in $\mathcal{C}^J$.

On the other hand, *given* a kernel pair $k_0, k_1 : (K, \kappa_1) \to (A, \alpha_1)$ of some unspecified morphism in $\mathcal{C}^J$, $k_0, k_1 : K \to A$ is a kernel pair in $\mathcal{C}$, so we may form its coequaliser $f : A \to B$ in $\mathcal{C}$. If $J f : J A \to J B$ is the coequaliser of $J k_0, J k_1 : J K \to J A$, then there *is* a unique $\beta_1 : J B \to B$ making f into a J-module homomorphism. The argument above then shows $f : (A, \alpha_1) \to (B, \beta_1)$ is the coequaliser of $k_0, k_1 : (K, \kappa_1) \to (A, \alpha_1)$, so $U^J : \mathcal{C}^J \to \mathcal{C}$ creates coequalisers for all kernel pairs. Proposition 1.3.10 then implies U^J preserves regular epimorphisms. We conclude $\mathcal{C}^J$ is a regular category by using (i).

(iii). The first part of the proof above goes through without modification, so $U^{\mathbb{T}} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}$ creates all finite limits and reflects regular epimorphisms. Under our hypotheses, T preserves regular epimorphisms, so proposition 1.2.6 implies $U^{\mathbb{T}}$ also creates coequalisers for all kernel pairs. Thus, $\mathcal{C}^{\mathbb{T}}$ is a regular category by (i).

(iv). If $f : A \to B$ is a split epimorphism in $\mathcal{C}$, then lemma 1.3.6 says there is a *split* coequaliser diagram in $\mathcal{C}$ of the form below,

$$\begin{tikzcd} K \arrow[r, shift left, "k_0"] \arrow[r, shift right, "k_1"'] & A \arrow[l, bend left] \arrow[r, "f"] & B \arrow[l, bend left] \end{tikzcd}$$

where $k_0, k_1 : K \to A$ is the kernel pair of f. Thus, J (resp. T and T^2) must preserve this coequaliser, so $\mathcal{C}^J$ (resp. $\mathcal{C}^{\mathbb{T}}$) is a regular category by (ii) (resp. (iii)). ■

Theorem 2.3.15. *Let $\mathcal{C}$ be regular category with limits for all small diagrams, and let $\mathfrak{D}$ be a locally small full subcategory that is closed under small limits* *in $\mathcal{C}$. If $\mathcal{C}$ is regularly well-copowered and $\mathcal{D}$ is closed under subobjects in $\mathcal{C}$, then the inclusion $U : \mathcal{D} \to \mathcal{C}$ has a left adjoint.*

Proof. Let X be any object of $\mathcal{C}$, and consider the comma category $(X \downarrow U)$. Any morphism $f : X \to UA$ in $\mathcal{C}$ must factor as $f = m \circ e$ in $\mathcal{C}$ for an essentially unique regular epimorphism $e : X \to Z$ and a monomorphism $m : Z \to UA$. Since $\mathcal{D}$ is closed under subobjects, $Z = UB$ for some object B in $\mathcal{D}$; but $\mathcal{D}$ is well-copowered, so the class of all regular epimorphisms $X \twoheadrightarrow UB$ is only a small family up to isomorphism. Since $U : \mathcal{D} \to \mathcal{C}$ preserves all small limits by hypothesis, the general adjoint functor theorem gives the required left adjoint for U. ■

Example 2.3.16. Recall the following definitions: a **magma** is a set equipped with a binary operation, and a **semigroup** is a magma whose binary operation is associative. Clearly, any submagma of a semigroup is also a semigroup. The category of semigroups is closed under small limits in the category of magmas, and the category of magmas is complete, so we infer from the theorem that the category of semigroups is a reflective subcategory of the category of magmas.

4 Equational theories

We now recall the classical notion of an algebraic theory presented by operations and equations.

Definition 2.4.1. An **algebraic signature** Σ consists of a set I_Σ of **sorts** and, for each sort T, a set $\Omega_{\Sigma,T}$ of T-**valued operations**, together with a function assigning a cardinal $\nu_S(\omega)$, called the S-**arity** of ω, to each operation ω and each sort S. Σ is a **finitary algebraic signature** if I_Σ is a finite set and $\nu_S(\omega)$ is finite for each operation ω and each sort S.

Example 2.4.2. The signature of the theory of groups has:

- one sort, G,
- three operations, e, i, and m,
- with arities $\nu_G(e) = 0$, $\nu_G(i) = 1$, and $\nu_G(m) = 2$.

In the intended interpretation, e is the identity element of G, i is the inversion operation, and m is the group operation.

Example 2.4.3. The signature of theory of directed multigraphs has:

- two sorts, V and E,
- two V-valued operations, s and t,
- with arities $\nu_V(s) = 0$, $\nu_V(t) = 0$, $\nu_E(s) = 1$, and $\nu_E(t) = 1$.

In the intended interpretation, V is the vertex set, E is the edge set, $s(e)$ is the source of the edge e, and $t(e)$ is the target of e.

Example 2.4.4. The signature of the theory of left modules for a ring has:

- two sorts, R and M,
- five R-valued operations, 0_R, 1_R, n_R, a_R, m_R,
- four M-valued operations, 0_M, n_M, a_M, m_M,
- with arities as below:

$$\begin{array}{rrrr}
\nu_R(0_R) = 0 & \nu_M(0_R) = 0 & \nu_R(0_M) = 0 & \nu_M(0_M) = 0 \\
\nu_R(1_R) = 0 & \nu_M(1_R) = 0 & & \\
\nu_R(n_R) = 1 & \nu_M(n_R) = 0 & \nu_R(n_M) = 0 & \nu_M(n_M) = 1 \\
\nu_R(a_R) = 2 & \nu_M(a_R) = 0 & \nu_R(a_M) = 0 & \nu_M(a_M) = 2 \\
\nu_R(m_R) = 2 & \nu_M(m_R) = 0 & \nu_R(m_M) = 1 & \nu_M(m_M) = 1
\end{array}$$

In the intended interpretation, R is a ring, M is a left R-module, 0_R, 1_R, 0_M are the various constants, n_R and n_M are the respective negation operations, a_R and a_M are the respective addition operations, and m_R and m_M are the respective multiplication operations.

Definition 2.4.5. Let Σ be an algebraic signature (resp. finitary algebraic signature) and let $\mathcal{S}$ be a category with small products (resp. finite products). A Σ-**structure** A in $\mathcal{S}$ consists of the following data:

- for each sort S in I_Σ, an object $A(S)$ in $\mathcal{S}$, and

- for each T-valued operation ω in $\Omega_{\Sigma,T}$, a morphism in $\mathcal{S}$ of the type below:
$$\omega_A : \prod_{S \in I_\Sigma} A(S)^{\nu_S(\omega)} \to A(T)$$

A **homomorphism of Σ-structures** $f : A \to A'$ is a morphism $f_S : A(S) \to A'(S)$ in $\mathcal{S}$ for each sort S, such that the diagram

$$\begin{tikzcd}
\prod_{S \in I_\Sigma} A(S)^{\nu_S(\omega)} \arrow[r, "\omega_A"] \arrow[d, "\prod_{S \in I_\Sigma} f_S^{\nu_S(\omega)}"'] & A(T) \arrow[d, "f_T"] \\
\prod_{S \in I_\Sigma} A'(S)^{\nu_S(\omega)} \arrow[r, "\omega_{A'}"'] & A(T)
\end{tikzcd}$$

commutes for every T-valued operation ω. The **category of Σ-structures in** $\mathcal{S}$ is denoted by $\mathbf{Str}(\Sigma, \mathcal{S})$. When $\mathcal{S} = \mathbf{Set}$, we may omit mention of $\mathcal{S}$ and simply write $\mathbf{Str}(\Sigma)$.

Proposition 2.4.6. *Let Σ be an algebraic signature (resp. finitary algebraic signature) and let $\mathcal{S}$ be a category with small products (resp. finite products). Let $[I_\Sigma, \mathcal{S}]$ be the category of I_Σ-indexed objects of $\mathcal{S}$.*

(i) *If $\mathcal{S}$ has small coproducts, then* $\mathbf{Str}(\Sigma, \mathcal{S})$ *is isomorphic to the category of modules for an endofunctor J on $[I_\Sigma, \mathcal{S}]$.*

(ii) *Suppose $\mathcal{S}$ also has colimits for all small filtered diagrams. If Σ is finitary and small filtered colimits preserve finite products in $\mathcal{S}$, then the endofunctor J in (i) preserves all small filtered colimits.*

Proof. (i). Let $J : [I_\Sigma, \mathcal{S}] \to [I_\Sigma, \mathcal{S}]$ be the functor that sends an object $A : I_\Sigma \to \mathcal{S}$ to the object
$$T \mapsto \coprod_{\omega \in \Omega_{\Sigma,T}} \prod_{S \in I_\Sigma} A(S)^{\nu_S(\omega)}$$
and define J on morphisms in the obvious way. The universal property of coproducts implies a morphism $\alpha_1 : JA \to A$ is defined by a $\coprod_T \Omega_{\Sigma,T}$-indexed family of morphisms of the exactly types necessary to specify a Σ-structure on A, and it is then evident that a homomorphism of J-modules is the same thing as a homomorphism of Σ-structures.

(ii). If Σ is a finitary algebraic signature, then only finite products appear in the definition of J, so the claim is an immediate consequence of the hypotheses and the fact that colimits in $[I_\Sigma, \mathcal{S}]$ are computed componentwise. ∎

Proposition 2.4.7. *Let $\mathcal{S}$ be a category with colimits for all countable diagrams. If J is an endofunctor on $\mathcal{S}$ that preserves countable filtered colimits, then,*

(i) *the forgetful functor $U^J : \mathcal{S}^J \to \mathcal{S}$ is strictly monadic, and*

(ii) *the category $\mathcal{S}^J$ has coequalisers for all parallel pairs.*

Proof. (i). Let $T_0 = \mathrm{id}_\mathcal{S}$ and $T_1 = \mathrm{id}_\mathcal{S} \amalg J$, and define $i_0 : T_0 \Rightarrow T_1$, $j_0 : J T_0 \Rightarrow T_1$ to be the respective coproduct insertions. For each natural number n, define a functor $T_{n+1} : \mathcal{S} \to \mathcal{S}$ so that the diagram

$$\begin{tikzcd}
J T_n X \arrow[r, "J i_{n,X}"] \arrow[d, "j_{n,X}"'] & J T_{n+1} X \arrow[d, dashed, "j_{n+1,X}"] \\
T_{n+1} X \arrow[r, dashed, "i_{n+1,X}"'] & T_{n+2} X
\end{tikzcd}$$

is a natural pushout square, in the sense that $i_{n+1,X}$ and $j_{n+1,X}$ respectively define natural transformations $T_{n+1} \Rightarrow T_{n+2}$ and $J T_{n+1} \Rightarrow T_{n+2}$. Then define $T : \mathcal{S} \to \mathcal{S}$ by a filtered colimit as in the diagram below:

$$\begin{tikzcd}
T_0 X \arrow[r, "i_{0,X}"] \arrow[drrrr, bend right, "\lambda_{0,X}"'] & T_1 X \arrow[r, "i_{1,X}"] \arrow[drrr, bend right, "\lambda_{1,X}"'] & T_2 X \arrow[r, "i_{2,X}"] \arrow[drr, "\lambda_{2,X}"'] & \cdots \arrow[r] \arrow[dr] & \cdots \\
 & & & & TX
\end{tikzcd}$$

Since J preserves countable filtered colimits, there is a unique morphism $j_X : JTX \to TX$ such that the diagram

$$\begin{tikzcd}
J T_n X \arrow[r, "J\lambda_{n,X}"] \arrow[d, "j_{n,X}"'] & JTX \arrow[d, dashed, "j_X"] \\
T_{n+1} X \arrow[r, "\lambda_{n+1,X}"'] & TX
\end{tikzcd}$$

commutes for every natural number n. Clearly, j is a natural transformation $JT \Rightarrow T$, and $X \mapsto (TX, j_X)$ defines a functor $F^J : \mathcal{S} \to \mathcal{S}^J$. We claim F^J is left adjoint to $U^J : \mathcal{S}^J \to \mathcal{S}$, with unit $\eta : \mathrm{id}_{\mathcal{S}} \Rightarrow U^J F^J$ given by $\eta_X = \lambda_{0,X}$. Indeed, given a J-module (A, α_1) and a morphism $f : X \to A$ in $\mathcal{S}$, we may construct a J-module homomorphism $\tilde{f} : (TX, j_X) \to (A, \alpha_1)$ as follows: set $f_0 = f$, $f_1 = [f_0, (\alpha_1 \circ J f_0)]$, and for each natural number n, let $f_{n+2} : T_{n+2}X \to A$ be the unique morphism in $\mathcal{S}$ making the diagram

$$\begin{tikzcd}
J T_n X \arrow[r, "J i_{n,X}"] \arrow[d, "j_{n,X}"'] & J T_{n+1} X \arrow[d, "j_{n+1,X}"] \arrow[ddr, bend left, "\alpha_1 \circ J f_{n+1}"] & \\
T_{n+1} X \arrow[r, "i_{n+1,X}"'] \arrow[drr, bend right, "f_{n+1}"'] & T_{n+2} X \arrow[dr, dashed, "f_{n+2}"] & \\
 & & A
\end{tikzcd}$$

commute, and define $\tilde{f} : TX \to A$ to be the unique morphism in $\mathcal{S}$ such that $\tilde{f} \circ \lambda_{n,X} = f_n$ for each natural number n. This makes sense because

$$\alpha_1 \circ J f_1 \circ J i_{0,X} = \alpha_1 \circ J f_0 = f_1 \circ j_0$$

by construction, so by induction we have

$$\alpha_1 \circ J f_{n+1} \circ J i_{n,X} = \alpha_1 \circ J f_n = f_{n+1} \circ j_{n,X}$$

as required to define f_{n+2}, and we also have

$$f_{n+1} \circ i_{n,X} = f_n$$

as required to define $\tilde{f}$. Since $\left(J\lambda_{n,X} \,\middle|\, n \in \mathbb{N}\right)$ is a jointly epimorphic sink on JTX, the fact that

$$\alpha_1 \circ J \tilde{f} \circ J \lambda_{n,X} = \alpha_1 \circ J f_{n,X} = f_{n+1} \circ j_{n+1,X} = \tilde{f} \circ \lambda_{n+1} \circ j_{n,X} = \tilde{f} \circ j_X \circ J \lambda_{n,X}$$

for all natural numbers n implies $\alpha_1 \circ J\tilde{f} = \tilde{f} \circ j_X$, so $\tilde{f}$ is indeed a J-module homomorphism $(TX, j_X) \to (A, \alpha_1)$, and clearly $\tilde{f} \circ \eta_X = f_0 = f$. On the other hand, if $\tilde{f} : (TX, j_X) \to (A, \alpha_1)$ is any J-module homomorphism such that $\tilde{f} \circ \eta_X = f$, then commutativity of the diagram below

$$\begin{tikzcd}
J T_n X \arrow[r, "J i_{n,X}"] \arrow[d, "j_{n,X}"'] & J T_{n+1} X \arrow[r, "J \lambda_{n+1,X}"] \arrow[d, "j_{n+1,X}"'] & J T X \arrow[r, "J \tilde{f}"] \arrow[d, "j_X"'] & J A \arrow[d, "\alpha_1"] \\
T_{n+1} X \arrow[r, "i_{n+1,X}"'] & T_{n+2} X \arrow[r, "\lambda_{n+2,X}"'] & T X \arrow[r, "\tilde{f}"'] & A
\end{tikzcd}$$

forces $\tilde{f}$ to be defined as in the above construction, so $\tilde{f}$ is *unique* J-module homomorphism such that $\tilde{f} \circ \eta_X = f$. Thus $((TX, j_X), \eta_X)$ is an initial object in the comma category $(X \downarrow U^J)$, and thus F^J is indeed left adjoint to U^J with unit η. Hence, by Barr's theorem (1.6.5), U^J must be strictly monadic.

(ii). Let $f, g : (A, \alpha_1) \to (B, \beta_1)$ be a parallel pair in $\mathcal{S}^J$. Let $Q_0 = B$, and form the coequaliser $p_0 : Q_0 \to Q_1$ of $f, g : A \to B$ in $\mathcal{S}$. Let $q_0 : JQ_0 \to Q_1$ be the composite $p_0 \circ \beta_1$. For each natural number n, form the pushout diagram

$$\begin{tikzcd}
JQ_n \arrow[r, "Jp_n"] \arrow[d, "q_n"'] & JQ_{n+1} \arrow[d, dashed, "q_{n+1}"] \\
Q_{n+1} \arrow[r, dashed, "p_{n+1}"'] & Q_{n+2}
\end{tikzcd}$$

in $\mathcal{S}$, and then define Q by the filtered colimit shown below:

$$\begin{tikzcd}
Q_0 \arrow[r, "p_0"] \arrow[drrrrr, bend right, "\lambda_0"'] & Q_1 \arrow[r, "p_1"] \arrow[drrrr, bend right, "\lambda_1"'] & Q_2 \arrow[r, "p_2"] \arrow[drrr, "\lambda_2"] & \cdots \arrow[drr] & \arrow[r] & \cdots \\
 & & & & & Q
\end{tikzcd}$$

Since J preserves countable filtered colimits, there is a unique morphism $q : JQ \to Q$ such that $q \circ J\lambda_n = \lambda_{n+1} \circ q_n$ for every natural number n. In particular,

$$q \circ J\lambda_0 = \lambda_1 \circ q_0 = \lambda_1 \circ p_0 \circ \beta_1 = \lambda_0 \circ \beta_1$$

so λ_0 is a J-module homomorphism $(B, \beta_1) \to (Q, q)$. We claim λ_0 is the coequaliser of $f, g : (A, \alpha_1) \to (B, \beta_1)$ in $\mathcal{S}^J$.

Indeed, suppose $h : (B, \beta_1) \to (C, \gamma_1)$ satisfies the equation $h \circ f = h \circ g$. Let $h_0 = h$. Since p_0 is the coequaliser of f and g in $\mathcal{S}$, there is a unique morphism $h_1 : Q_1 \to C$ in $\mathcal{S}$ such that $h_1 \circ p_0 = h_0$. Noting that $h_0 \circ \beta_1 = \gamma_1 \circ J h_0$, by induction, there is a unique morphism $h_{n+2} : Q_{n+2} \to C$ in $\mathcal{S}$ such that

$$h_{n+2} \circ p_{n+1} = h_{n+1} \qquad\qquad h_{n+2} \circ q_{n+1} = \gamma_1 \circ J h_{n+1}$$

because for each natural number n we have

$$h_{n+1} \circ q_n = \gamma_1 \circ J h_n = (\gamma_1 \circ J h_{n+1}) \circ J p_{n+1}$$

as required to construct such a morphism by the universal property of Q_{n+2}. Then, by the universal property of Q, there is a unique morphism $\tilde{h} : Q \to C$ in $\mathcal{S}$ such that $\tilde{h} \circ \lambda_n = h_n$ for each natural number n. Moreover, because

$$\tilde{h} \circ q \circ J\lambda_n = \tilde{h} \circ \lambda_{n+1} \circ q_n = h_{n+1} \circ q_n = \gamma_1 \circ J h_n = \gamma_1 \circ J\tilde{h} \circ J\lambda_n$$

for all natural numbers n, we have $\tilde{h} \circ q = \gamma_1 \circ J\tilde{h}$, and thus $\tilde{h}$ is a J-module homomorphism $(Q, q) \to (C, \gamma_1)$ such that $\tilde{h} \circ \lambda_0 = h$. On the other hand, given any J-module homomorphism $\tilde{h} : (Q, q) \to (C, \gamma_1)$ such that $\tilde{h} \circ \lambda_0 = h$, commutativity of the diagram below

$$\begin{tikzcd}
JQ_n \arrow[r, "Jp_n"] \arrow[d, "q_n"'] & JQ_{n+1} \arrow[r, "J\lambda_{n+1}"] \arrow[d, "q_{n+1}"'] & JQ \arrow[r, "J\tilde{h}"] \arrow[d, "q"'] & JC \arrow[d, "\gamma_1"] \\
Q_{n+1} \arrow[r, "p_{n+1}"'] & Q_{n+2} \arrow[r, "\lambda_{n+2}"'] & Q \arrow[r, "\tilde{h}"'] & C
\end{tikzcd}$$

forces $\tilde{h}$ to be defined as in the above construction, so $\tilde{h}$ is the *unique* J-module homomorphism such that $\tilde{h} \circ \lambda_0 = h$. Thus $\lambda_0 : (B, \beta_1) \to (Q, q)$ is the coequaliser of $f, g : (A, \alpha_1) \to (B, \beta_1)$, as claimed. ■

Theorem 2.4.8. *Let $\mathcal{S}$ be a locally finitely-presentable category. If J is a finitary endofunctor on $\mathcal{S}$, then:*

(i) *The forgetful functor $U^J : \mathcal{S}^J \to \mathcal{S}$ is strictly monadic and creates colimits for all small filtered diagrams.*

(ii) *The category $\mathbf{K}(\mathcal{S})$ of compact objects in $\mathcal{S}$ is an essentially small category of arities for the induced monad.*

(iii) *$\mathcal{S}^J$ is a locally finitely-presentable category.*

Proof. (i). An l.f.p. category has all small colimits and a finitary endofunctor preserves small filtered colimits, so the proposition implies U^J is strictly monadic. The creation of small filtered colimits can be proved the same way as in proposition 1.2.6.

(ii). Let $\mathbb{T} = (T, \eta, \mu)$ be the monad induced by U^J. Since U^J preserves small filtered colimits, $T = U^J F^J$ is a finitary endofunctor on $\mathcal{S}$. The objects in $\mathbf{K}(\mathcal{C})$ are compact objects in $\mathcal{C}$, so propositions 2.1.8 and 2.1.13 together imply $\mathbf{K}(\mathcal{C})$ is an essentially small category of arities for $\mathbb{T}$.

(iii). Statement (ii) of proposition 2.4.7 says $\mathcal{S}^J$ has coequalisers for all parallel pairs, and corollary 2.1.15 says $\mathcal{S}$ has all small colimits, so $\mathcal{S}^J$ is cocomplete by theorem 1.3.2. Propositions 1.7.10 and 2.2.4 then imply $\mathcal{S}^J$ is an l.f.p. category. ■

Corollary 2.4.9. *Let Σ be a finitary algebraic signature and let $\mathcal{S}$ be a locally finitely-presentable category.*

(i) *The forgetful functor $U^\Sigma : \mathbf{Str}(\Sigma, \mathcal{S}) \to [I_\Sigma, \mathcal{S}]$ is strictly monadic and creates colimits for all small filtered diagrams.*

(ii) *The category $\mathbf{K}([I_\Sigma, \mathcal{S}])$ of compact objects in $[I_\Sigma, \mathcal{S}]$ is an essentially small category of arities for the induced monad.*

(iii) *$\mathbf{Str}(\Sigma, \mathcal{S})$ is a locally finitely-presentable category.*

Proof. Corollary 2.2.20 tell us that $[I_\Sigma, \mathcal{S}]$ is an l.f.p. category, while propositions 2.2.21 and 2.4.6 together imply U^Σ is isomorphic to the forgetful functor $U^J : [I_\Sigma, \mathcal{S}]^J \to [I_\Sigma, \mathcal{S}]$ for a finitary endofunctor J on $\mathcal{S}$, so we just have to apply the theorem above. ■

We should also say something about $\mathbf{Str}(\Sigma, \mathcal{S})$ when $\mathcal{S}$ is a regular category; after all, we have just dedicated a whole section to the matter!

Proposition 2.4.10. *Let Σ be a finitary algebraic signature and let $\mathcal{S}$ be a category with limits for all finite diagrams.*

(i) *The forgetful functor $U^\Sigma : \mathbf{Str}(\Sigma, \mathcal{S}) \to [I_\Sigma, \mathcal{S}]$ creates all finite limits.*

(ii) *If $\mathcal{S}$ is well-powered, then so is $\mathbf{Str}(\Sigma, \mathcal{S})$.*

(iii) *If $\mathcal{S}$ is a regular category, then U^Σ also creates coequalisers for kernel pairs, and $\mathbf{Str}(\Sigma, \mathcal{S})$ is a regular category.*

Proof. (i). Because finite limits in $[I_\Sigma, \mathcal{S}]$ exist and can be computed componentwise, a straightforward if tedious modification of proposition 1.2.5 shows that the same is true for $\mathbf{Str}(\Sigma, \mathcal{S})$.

(ii). Since a morphism in $[I_\Sigma, \mathcal{S}]$ is a monomorphism if and only if its components are monomorphisms, $[I_\Sigma, \mathcal{S}]$ is well-powered if $\mathcal{S}$ is. The forgetful functor $U^\Sigma : \mathbf{Str}(\Sigma, \mathcal{S}) \to [I_\Sigma, \mathcal{S}]$ is conservative and preserves monomorphisms by (i), so $\mathbf{Str}(\Sigma, \mathcal{S})$ is well-powered if $[I_\Sigma, \mathcal{S}]$ is.

(iii). First, observe that the kernel pair of a product is the product of the kernel pairs. (This is just the fact that limits preserve limits.) Lemma 2.3.10 implies the product of a finite family of regular epimorphisms in $\mathcal{S}$ is again a regular epimorphism, so we can use proposition 1.3.10 to conclude that the product of the coequalisers of a finite family of kernel pairs is the coequaliser of the product of the kernel pairs. This shows that U^Σ creates coequalisers for kernel pairs. In particular, U^Σ must both preserve and reflect regular epimorphisms, so we apply proposition 2.3.14 to obtain the desired conclusion. ■

Now, we turn to the titular subject of this section. Because we have constructed the free Σ-structure functor for finitary algebraic signatures Σ, we can define an equational theory over the language of Σ in very concisely:

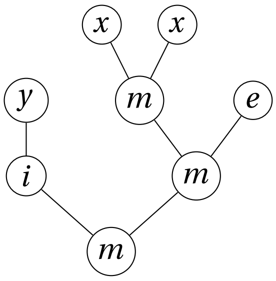
Definition 2.4.11. Let Σ be a finitary algebraic signature. If T is a sort in I_Σ, then a **term of type** T in the language of Σ with variables in an I_Σ-indexed finite set X is an element of $T^\Sigma X(T)$, where $(T^\Sigma, \eta^\Sigma, \mu^\Sigma)$ is the free Σ-structure monad on $[I_\Sigma, \mathbf{Set}]$. An **equation** in the language of Σ is a pair of terms of the same type with variables in the same I_Σ-indexed finite set. An **equational theory** $\mathbb{T}$ over Σ is a (possibly infinite) set of equations in the language of Σ, called the **axioms** of $\mathbb{T}$.

We should explain what this means more concretely. To simplify the notation, we assume Σ is a one-sorted finitary algebraic signature. Let X be any set, and assume X is disjoint from the set Ω_Σ of operations in Σ. By unwinding the construction of F^Σ in proposition 2.4.7, we see that an element of $F^\Sigma X$ the same thing as an ordered tree satisfying the following axioms:

- Each node is labelled by either an operation in Ω or an element of X.
- If a node is labelled by an element of X, then it has no children.
- If a node is labelled by an operation ω, then it has exactly $\nu(\omega)$ children.
- Every branch is finite, i.e. there is no infinite simple path in the tree.

Note that König's lemma implies such a tree has only finitely many nodes when Σ is a finitary algebraic signature.

Example 2.4.12. If Σ is the signature of the theory of groups, then an element of $F^{\Sigma}(\{x,y\})$ might look like this:



In conventional notation, one represents this tree as the word below:

$$m(i(y), m(m(x,x), e))$$

The **theory of groups** is the equational theory with the following axioms:

$$\begin{aligned} m(e,x) &= m(x) \\ m(x,e) &= m(x) \\ m(m(x,y),z) &= m(x, m(y,z)) \\ m(i(x),x) &= e \\ m(x, i(x)) &= e \end{aligned}$$

It is not immediately clear from our definition of equational theory how we can define a model of an equational theory in a general category with finite products. Nonetheless, we press on by initially thinking about the special case of **Set**. Before we can even talk about models of an equational theory, we need to be able to interpret terms in the language of its signature Σ in arbitrary Σ-structures in **Set**:

Definition 2.4.13. Let Σ be a finitary algebraic signature, and let t be a term of type T in the language of Σ with variables in the I_{Σ}-indexed finite set X. Write $F^{\Sigma} \dashv U^{\Sigma} : \mathbf{Str}(\Sigma, \mathbf{Set}) \to [I_{\Sigma}, \mathbf{Set}]$ for the free–forgetful adjunction. Let A be a Σ-structure in **Set**. An **assignment** of the variables of t in A is any I_{Σ}-indexed map $x : X \to U^{\Sigma}A$. The **interpretation** of a term t with respect to an assignment $x : X \to U^{\Sigma}A$ is the element of $U^{\Sigma}A(T)$ given by the formula

$$t_A(x) = \left(U^{\Sigma}\left(\varepsilon^{\Sigma}_{A} \circ F^{\Sigma}x\right)\right)_T(t)$$

where $\varepsilon^{\Sigma} : F^{\Sigma}U^{\Sigma} \Rightarrow \mathrm{id}$ is the counit of the adjunction.

Naturality of the counit ε immediately implies this lemma:

Lemma 2.4.14. *For each term t of type T in a finitary algebraic signature Σ with variables in X, there is a natural transformation*

$$t_\bullet : [I_\Sigma, \mathbf{Set}]\big(X, U^\Sigma(-)\big) \Rightarrow U^\Sigma(-)(T)$$

of functors $\mathbf{Str}(\Sigma, \mathbf{Set}) \to \mathbf{Set}$*, defined by the map that takes an assignment $x : X \to U^\Sigma A$ to the interpretation $t_A(x)$.* ■

In other words, homomorphisms of Σ-structures preserve the interpretation of all terms, as one expects. (The reader should think about why this *not* obvious from our definition of the interpretation of a term!)

We provisionally define a **model of** $\mathbb{T}$ in **Set** to be an object A in $\mathbf{Str}(\Sigma, \mathbf{Set})$ satisfying the following condition:

- For each pair of terms t_1, t_2 of type T with variables in an I_Σ-indexed finite set X, if $t_1 = t_2$ is an axiom of $\mathbb{T}$, then their interpretations with respect to *all* assignments $x : X \to U^\Sigma A$ are equal.

The trouble with this definition lies in the notion of assignment: X is always an I_Σ-indexed *set*, but in general $U^\Sigma A$ is an object in the category $[I_\Sigma, \mathcal{S}]$ and not necessarily an I_Σ-indexed set itself! In order to transport this definition to a general category with products, we will use the following obvious but important observation:

Proposition 2.4.15. *Let Σ be an algebraic signature (resp. finitary algebraic signature), and let $\mathcal{S}$ and $\mathcal{S}'$ be categories with all small (resp. finite) products. If $R : \mathcal{S} \to \mathcal{S}'$ is any functor that preserves small (resp. finite) products, then there is a unique functor $R^\Sigma : \mathbf{Str}(\Sigma, \mathcal{S}) \to \mathbf{Str}(\Sigma, \mathcal{S}')$ such that the diagram of functors*

$$\begin{tikzcd}
\mathbf{Str}(\Sigma, \mathcal{S}) \arrow[r, "R^\Sigma"] \arrow[d, "U^\Sigma"'] & \mathbf{Str}(\Sigma, \mathcal{S}') \arrow[d, "U'^\Sigma"] \\
{[I_\Sigma, \mathcal{S}]} \arrow[r, "{[I_\Sigma, R]}"'] & {[I_\Sigma, \mathcal{S}']}
\end{tikzcd}$$

commutes strictly, where the vertical arrows are the respective forgetful functors. Moreover, any natural transformation $\varphi : R \Rightarrow R'$ induces a unique natural transformation $\varphi^\Sigma : R^\Sigma \Rightarrow R'^\Sigma$ such that $U'^\Sigma \varphi^\Sigma = [I_\Sigma, \varphi] U^\Sigma$. ■

It is well-known that representable functors preserve products, so for any finitary algebraic signature Σ and any object E in a locally small category $\mathcal{S}$ with finite products, the representable functor $\mathcal{S}(E,-) : \mathcal{S} \to \mathbf{Set}$ induces a functor $\Gamma(E,-) : \mathbf{Str}(\Sigma,\mathcal{S}) \to \mathbf{Str}(\Sigma,\mathbf{Set})$. Thus we may provisionally define a **model of** $\mathbb{T}$ in $\mathcal{S}$ to be a Σ-structure A for which $\Gamma(E,A)$ is a model of $\mathbb{T}$ in **Set** for all objects E in $\mathcal{S}$.

This seems to be an extremely unwieldy definition, so we should contemplate it for longer. Note that if X is a I_Σ-indexed finite set, then for each object $A : I_\Sigma \to \mathcal{S}$ there is an object A^X in $\mathcal{S}$ such that we have a bijection

$$[I_\Sigma, \mathbf{Set}](X, \mathcal{S}(E, A(-))) \cong \mathcal{S}\big(E, A^X\big)$$

that is natural in E, A, and X: indeed, this is just a sophisticated way of saying

$$A^X = \prod_{S \in I_\Sigma} A(S)^{X(S)}$$

but it is important to emphasise the functor that it represents here. Thus, when A is a Σ-structure in $\mathcal{S}$, each I_Σ-indexed map $x : X \to U^\Sigma \Gamma(E,A)$ can be naturally identified with a morphism $\vec{x} : E \to A^X$ in $\mathcal{S}$ and vice versa.

In particular, there is a I_Σ-indexed map $a : X \to U^\Sigma \Gamma\big((U^\Sigma A)^X, A\big)$ that corresponds to the morphism $\mathrm{id} : (U^\Sigma A)^X \to (U^\Sigma A)^X$, so for each term t of type T, we can naturally identify its interpretation $t_{\Gamma((U^\Sigma A)^X, A)}(a)$ with a morphism $t_A : (U^\Sigma A)^X \to U^\Sigma A(T)$ in $\mathcal{S}$. The yoga of the Yoneda lemma then implies it is enough to check $(t_1)_A = (t_2)_A$ for every axiom $t_1 = t_2$ in $\mathbb{T}$.

$$\begin{tikzcd}
{[I_\Sigma, \mathbf{Set}]\big(X, U^\Sigma \Gamma\big((U^\Sigma A)^X, A\big)\big)} \arrow[r] \arrow[d, "\cong"'] & {[I_\Sigma, \mathbf{Set}]\big(X, U^\Sigma \Gamma(E,A)\big)} \arrow[d, "\cong"] \\
{\mathcal{S}\big((U^\Sigma A)^X, (U^\Sigma A)^X\big)} \arrow[r, "\vec{x}^*"] \arrow[d, "(t_A)_*"'] & {\mathcal{S}\big(E, (U^\Sigma A)^X\big)} \arrow[d, "(t_A)_*"] \\
{\mathcal{S}\big((U^\Sigma A)^X, U^\Sigma A(T)\big)} \arrow[r, "\vec{x}^*"'] \arrow[d, "="'] & {\mathcal{S}\big(E, U^\Sigma A(T)\big)} \arrow[d, "="] \\
{U^\Sigma \Gamma\big((U^\Sigma A)^X, A\big)(T)} \arrow[r] & {U^\Sigma \Gamma(E,A)(T)}
\end{tikzcd}$$

Indeed, given an I_Σ-indexed map $x : X \to U^\Sigma \Gamma(E,A)$, if $\vec{x} : E \to \big(U^\Sigma A\big)^X$ is the morphism in $\mathcal{S}$ corresponding to x, then by chasing the element a around the commutative diagram above, from the assumption $(t_1)_A = (t_2)_A$ we may deduce the following equality in $U^\Sigma \Gamma(E,A)(T)$:

$$(t_1)_{\Gamma(E,A)}(x) = (t_2)_{\Gamma(E,A)}(x)$$

Conversely, if this equality holds for all x and E, then in particular we must have $(t_1)_A = (t_2)_A$. Thus we are led to the final form of our definition:

Definition 2.4.16. Let $\mathbb{T}$ be an equational theory over a finitary algebraic signature Σ. A **model of** $\mathbb{T}$ in a locally small category $\mathcal{S}$ with finite products is a Σ-structure A in $\mathcal{S}$ with the following property:

- For each pair of terms t_1, t_2 of type T with variables in an I_Σ-indexed finite set X, if $t_1 = t_2$ is an axiom of $\mathbb{T}$, then the canonical interpretations of t_1, t_2 as morphisms $(t_1)_A, (t_2)_A : (U^\Sigma A)^X \to U^\Sigma A(T)$ are equal in $\mathcal{S}$.

The **category of** $\mathbb{T}$**-models in** $\mathcal{S}$ is the full subcategory $\mathbf{Mod}(\mathbb{T}, \mathcal{S})$ of $\mathbf{Str}(\Sigma, \mathcal{S})$ spanned by the models of $\mathbb{T}$ in $\mathcal{S}$.

Remark 2.4.17. It is possible to spell out exactly what this means in elementary terms by thinking of a term t as an ordered tree and defining the interpretation t_A by structural recursion. This also allows us to define $\mathbb{T}$-models in categories that are not locally small. The details are left to the interested reader.

A Yoneda-type argument of the same kind as in the preceding discussion proves the obvious extensions of lemma 2.4.14 and proposition 2.4.15:

Lemma 2.4.18. *Let $\mathcal{S}$ be a (locally small) category with finite products. For each term t of type T in a finitary algebraic signature Σ with variables in X, there is a natural transformation*

$$t_\bullet : \left(U^\Sigma(-)\right)^X \Rightarrow U^\Sigma(-)(T)$$

of functors $\mathbf{Str}(\Sigma, \mathcal{S}) \to \mathcal{S}$*, defined for each Σ-structure A by the canonical interpretation of t as a morphism $t_A : (U^\Sigma A)^X \to U^\Sigma A(T)$ in $\mathcal{S}$.* ■

Proposition 2.4.19. *Let $\mathbb{T}$ be an equational theory over a finitary algebraic signature Σ, and let $\mathcal{S}$ and $\mathcal{S}'$ be categories with all finite products. If $R : \mathcal{S} \to \mathcal{S}'$*

is any functor that preserves finite products, then there is a unique functor $R^{\mathbb{T}} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to \mathbf{Mod}(\mathbb{T},\mathcal{S}')$ *such that the diagram of functors*

$$\begin{tikzcd}
\mathbf{Mod}(\mathbb{T},\mathcal{S}) \arrow[r, "R^{\mathbb{T}}"] \arrow[d, "U^{\mathbb{T}}"'] & \mathbf{Mod}(\mathbb{T},\mathcal{S}') \arrow[d, "U'^{\mathbb{T}}"] \\
{[I_\Sigma,\mathcal{S}]} \arrow[r, "{[I_\Sigma, R]}"'] & {[I_\Sigma,\mathcal{S}']}
\end{tikzcd}$$

commutes strictly, where the vertical arrows are the respective forgetful functors. Moreover, any natural transformation $\varphi : R \Rightarrow R'$ *induces a unique natural transformation* $\varphi^{\mathbb{T}} : R^{\mathbb{T}} \Rightarrow R'T$ *such that* $U'T\varphi^{\mathbb{T}} = [I_\Sigma, \varphi]\, U^{\mathbb{T}}$. ■

Finally, we come to a much anticipated theorem: we shall show that the category of models in **Set** for an equational theory $\mathbb{T}$ over a finitary algebraic signature Σ is strictly monadic over $[I_\Sigma, \mathbf{Set}]$; in particular, if Σ is one-sorted, then the forgetful functor $U^{\mathbb{T}} : \mathbf{Mod}(\mathbb{T}, \mathbf{Set}) \to \mathbf{Set}$ is strictly monadic. We will do this using the machinery developed earlier.

Proposition 2.4.20. *Let* $\mathbb{T}$ *be an equational theory over a finitary algebraic signature* Σ*. For any (locally small) category* $\mathcal{S}$ *with finite products, let*

$$U^{\mathbb{T}}_{\Sigma} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to \mathbf{Str}(\Sigma,\mathcal{S}) \quad U^{\mathbb{T}} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to [I_\Sigma,\mathcal{S}] \quad U^{\Sigma} : \mathbf{Str}(\Sigma,\mathcal{S}) \to [I_\Sigma,\mathcal{S}]$$

be the various forgetful functors.

(i) *The inclusion* $U^{\mathbb{T}}_{\Sigma}$ *reflects all limits and colimits.*

(ii) *If* U^{Σ} *preserves all monomorphisms, then* $\mathbf{Mod}(\mathbb{T},\mathcal{S})$ *is closed under subobjects in* $\mathbf{Str}(\Sigma,\mathcal{S})$*.*

(iii) *If* $\mathcal{S}$ *has limits for all diagrams of shape* $\mathcal{J}$*, then the forgetful functors* $U^{\mathbb{T}}$ *and* U^{Σ} *create limits for all diagrams of shape* $\mathcal{J}$*.*

(iv) *Given a diagram* $A : \mathcal{J} \to \mathbf{Mod}(\mathbb{T},\mathcal{S})$*, if* $\lambda : U^{\mathbb{T}}_{\Sigma} A \Rightarrow \Delta\bar{A}$ *is a colimiting cocone in* $\mathbf{Str}(\Sigma,\mathcal{S})$ *and* $(U^{\Sigma}\lambda)^X : (U^{\mathbb{T}}A)^X \Rightarrow \Delta\big(U^{\Sigma}\bar{A}\big)^X$ *is a jointly epimorphic sink in* $\mathcal{S}$*, then* $\bar{A}$ *is a model of* $\mathbb{T}$ *in* $\mathcal{S}$*.*

(v) *If* U^{Σ} *maps regular epimorphisms in* $\mathbf{Str}(\Sigma,\mathcal{S})$ *to epimorphisms in* $[I_\Sigma,\mathcal{S}]$*, then* $U^{\mathbb{T}}_{\Sigma}$ *creates coequalisers for all parallel pairs whose coequaliser exists in* $\mathbf{Str}(\Sigma,\mathcal{S})$*.*

Proof. (i). $U^{\mathbb{T}}_{\Sigma}$ is a fully faithful functor by definition, and it is well-known that fully faithful functors reflect all limits and colimits.

(ii). Let B be a model of $\mathbb{T}$ in $\mathcal{S}$, and let $f : A \rightarrowtail U^{\mathbb{T}}_{\Sigma} B$ be a monomorphism in $\mathbf{Str}(\Sigma, \mathcal{S})$. Let t be a term of type T with variables in X, and consider the diagram below:

$$\begin{tikzcd}
(U^{\Sigma}A)^X \arrow[r, "(U^{\Sigma}f)^X"] \arrow[d, "t_A"'] & (U^{\mathbb{T}}B)^X \arrow[d, "t_B"] \\
U^{\Sigma}A(T) \arrow[r, "(U^{\Sigma}f)_T"'] & U^{\mathbb{T}}B(T)
\end{tikzcd}$$

Since U^{Σ} preserves all monomorphisms, $(U^{\Sigma}f)_T : U^{\Sigma}A(T) \to U^{\mathbb{T}}B(T)$ must be a monomorphism. Thus, t_B is the unique morphism $(U^{\Sigma}A)^X \to U^{\Sigma}A(T)$ in $\mathcal{S}$ making the diagram commute; so if $(t_1)_B = (t_2)_B$, we also have $(t_1)_A = (t_2)_A$. Hence, A must be a model of $\mathbb{T}$ if B is.

(iii). The case of U^{Σ} is straightforward and amounts to the fact that limits preserve products in any category. Since $U^{\mathbb{T}}_{\Sigma}$ reflects limits, to prove the claim for $U^{\mathbb{T}}$ it is enough to show that the limit in $\mathbf{Str}(\Sigma, \mathcal{S})$ of a diagram of $\mathbb{T}$-models is also a $\mathbb{T}$-model.

Let $A : \mathcal{J} \to \mathbf{Mod}(\mathbb{T}, \mathcal{S})$ be a diagram, and suppose the limit $\hat{A} = \varprojlim_{\mathcal{J}} U^{\mathbb{T}}A$ exists in $\mathbf{Str}(\Sigma, \mathcal{S})$ and is preserved by U^{Σ}. Let $\lambda : \Delta\hat{A} \Rightarrow A$ be the limiting cone, let t be a term of type T with variables in X, and consider the diagram below:

$$\begin{tikzcd}
\left(U^{\Sigma}\hat{A}\right)^X \arrow[r, "\left(U^{\Sigma}\lambda_j\right)^X"] \arrow[d, "t_{\hat{A}}"'] & (U^{\Sigma}Aj)^X \arrow[d, "t_{Aj}"] \\
U^{\Sigma}\hat{A}(T) \arrow[r, "\left(U^{\Sigma}\lambda_j\right)_T"'] & U^{\Sigma}Aj(T)
\end{tikzcd}$$

By naturality of $t_{\bullet} : (U^{\Sigma}(-))^X \to U^{\Sigma}(-)(T)$, the diagram commutes. Since $\left(U^{\Sigma}\lambda_{\bullet}\right)_T : \Delta U^{\Sigma}\hat{A}(T) \Rightarrow U^{\Sigma}A(-)(T)$ is jointly monomorphic, if $(t_1)_{Aj} = (t_2)_{Aj}$ for all j in $\mathcal{J}$, then $(t_1)_{\hat{A}} = (t_2)_{\hat{A}}$ as well. Hence $\hat{A}$ is a model of $\mathbb{T}$.

(iv). This is essentially dual to (ii). Since $U^{\mathbb{T}}_{\Sigma}$ reflects colimits, to prove the claim for $U^{\mathbb{T}}$ it is enough to show that the colimit in $\mathbf{Str}(\Sigma, \mathcal{S})$ of a diagram of $\mathbb{T}$-models is also a $\mathbb{T}$-model.

Let $A : \mathcal{J} \to \mathbf{Mod}(\mathbb{T}, \mathcal{S})$ be a diagram, and suppose the colimit $\bar{A} = \varinjlim_{\mathcal{J}} U^{\mathbb{T}}A$ exists in $\mathbf{Str}(\Sigma, \mathcal{S})$. Let $\lambda : A \Rightarrow \Delta\bar{A}$ be the colimiting cocone. Let t be a term of type T with variables in X, and consider the diagram below:

$$\begin{tikzcd}
\left(U^{\Sigma} A j\right)^{X} \arrow[r, "\left(U^{\Sigma}\lambda_j\right)^{X}"] \arrow[d, "t_{Aj}"'] & \left(U^{\Sigma}\bar{A}\right)^{X} \arrow[d, "t_{\bar{A}}"] \\
U^{\Sigma} A j(T) \arrow[r, "\left(U^{\Sigma}\lambda_j\right)_{T}"'] & U^{\Sigma}\bar{A}(T)
\end{tikzcd}$$

By naturality of $t_\bullet : (U^{\Sigma}(-))^X \to U^{\Sigma}(-)(T)$, the diagram commutes. Since $(U^{\Sigma}\lambda)^X : (U^{\mathbb{T}}A)^X \Rightarrow \Delta\left(U^{\Sigma}\bar{A}\right)^X$ is jointly epimorphic sink by assumption, if $(t_1)_{Aj} = (t_2)_{Aj}$ for all j in $\mathcal{J}$, then $(t_1)_{\bar{A}} = (t_2)_{\bar{A}}$ as well. Hence $\bar{A}$ is a model of $\mathbb{T}$.

(v). This is a special case of (iv). ∎

Theorem 2.4.21. *Let $\mathbb{T}$ be an equational theory over a finitary algebraic signature Σ and let $\mathcal{S}$ be a locally small regular category with limits for all small diagrams.*

(i) $\mathbf{Mod}(\mathbb{T},\mathcal{S})$ *is a regular category.*

(ii) *If $\mathcal{S}$ is well-powered, then the inclusion $U^{\mathbb{T}}_{\Sigma} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to \mathbf{Str}(\Sigma,\mathcal{S})$ satisfies the CTT condition and is strictly monadic.*

If $\mathcal{S}$ is in addition a locally finitely-presentable category, then:

(iii) *The forgetful functor $U^{\mathbb{T}} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to [I_{\Sigma},\mathcal{S}]$ is strictly monadic and creates colimits for small filtered diagrams.*

(iv) $\mathbf{K}([I_{\Sigma},\mathcal{S}])$ *is an essentially small category of arities for the monad induced by $U^{\mathbb{T}}$.*

(v) $\mathbf{Mod}(\mathbb{T},\mathcal{S})$ *is a locally finitely-presentable category.*

Proof. (i). Proposition 2.4.10 implies that $U^{\Sigma} : \mathbf{Str}(\Sigma,\mathcal{S}) \to [I_{\Sigma},\mathcal{S}]$ preserves regular epimorphisms, so $U^{\mathbb{T}}_{\Sigma} : \mathbf{Mod}(\mathbb{T},\mathcal{S}) \to \mathbf{Str}(\Sigma,\mathcal{S})$ creates all finite limits and coequalisers for all kernel pairs by the above proposition. Since $\mathbf{Str}(\Sigma,\mathcal{S})$ is already known to be regular, we use proposition 2.3.14 to conclude that $\mathbf{Mod}(\mathbb{T},\mathcal{S})$ is also regular.

(ii). Propositions 2.4.10 and 2.3.4 together tell us $\mathbf{Str}(\Sigma,\mathcal{S})$ is a regularly well-copowered category. The above proposition implies the hypotheses of theorem 2.3.15 are satisfied, so $U^{\mathbb{T}}_{\Sigma}$ has a left adjoint. Thus, $U^{\mathbb{T}}_{\Sigma}$ is the inclusion of a reflective subcategory, and statement (v) of the proposition implies $U^{\mathbb{T}}_{\Sigma}$ satisfies the CTT condition. It is clear that $U^{\mathbb{T}}_{\Sigma}$ is an amnestic isofibration, so corollary 1.2.14 and proposition 1.2.18 imply $U^{\mathbb{T}}_{\Sigma}$ is strictly monadic.

(iii). By proposition 2.3.4, $\mathcal{S}$ is well-powered, so (ii), corollary 2.4.9 and proposition 1.2.15 together imply $U^{\mathbb{T}}$ is monadic. Again, it is clear that $U^{\mathbb{T}}$ is an amnestic isofibration, so $U^{\mathbb{T}}$ is strictly monadic by proposition 1.2.18. To see that $U^{\mathbb{T}}$ creates small filtered colimits, we use the fact that U^{Σ} is finitary, proposition 2.2.21, and statement (iv) of the proposition above.

(iv). Since $U^{\mathbb{T}}$ creates small filtered colimits, $U^{\mathbb{T}} F^{\mathbb{T}}$ preserves them. We then apply propositions 2.1.8 and 2.1.13 to see that $\mathbf{K}([I_{\Sigma}, \mathcal{S}])$ is an essentially small category of arities.

(v). We know $\mathbf{Str}(\Sigma, \mathcal{S})$ is cocomplete and $\mathbf{Mod}(\mathbb{T}, \mathcal{S})$ is a reflective subcategory of $\mathbf{Str}(\Sigma, \mathcal{S})$, so $\mathbf{Mod}(\mathbb{T}, \mathcal{S})$ must also be cocomplete. We may then apply proposition 2.2.4 to show it is an l.f.p. category. ■

Corollary 2.4.22. *For any algebraic theory $\mathbb{T}$ over a finitary algebraic signature Σ:*

(i) $\mathbf{Mod}(\mathbb{T}, \mathbf{Set})$ *is a regular locally finitely-presentable category.*

(ii) *The forgetful functor* $U^{\mathbb{T}} : \mathbf{Mod}(\mathbb{T}, \mathbf{Set}) \to [I_{\Sigma}, \mathbf{Set}]$ *is strictly monadic and creates colimits for small filtered diagrams.*

(iii) $[I_{\Sigma}, \mathbf{FinSet}]$ *is a small category of arities for the induced monad.* ■

Example 2.4.23. With this result at hand, we can now conclusively show that the theory of complete semilattices is *not* finitary. Indeed, if it were, then the covariant powerset functor would be finitary; but $\mathscr{P}(\mathbb{N})$ is uncountable while $\mathscr{P}(n)$ is finite for each n, so $\mathscr{P}(-)$ cannot preserve filtered colimits.

5 Lawvere theories

One defect in the syntactic approach of equational theories is the difficulty in defining morphisms of theories. For example, given an algebraic signature Σ, we can define a new signature Σ' by adding a unary operation to Σ; clearly, there will be a forgetful functor $\mathbf{Str}(\Sigma', \mathcal{S}) \to \mathbf{Str}(\Sigma, \mathcal{S})$ that is natural in $\mathcal{S}$ in the obvious sense. Similarly, if an algebraic theory $\mathbb{T}'$ is obtained by adding axioms to another algebraic theory $\mathbb{T}$, then there is a natural inclusion $\mathbf{Mod}(\mathbb{T}',\mathcal{S}) \to \mathbf{Mod}(\mathbb{T},\mathcal{S})$. We would like to say this is because $\mathbf{Mod}(-,\mathcal{S})$ is a functor, but on what category?

The seminal category-theoretic account of (one-sorted) finitary algebraic theories and their semantics was given by Lawvere [1963] in his doctoral thesis (republished as [Lawvere, 2004]). We will now study their many-sorted generalisation.

Throughout this section, I is a fixed finite set that enumerates the sorts in the theories we are interested in. Let **FinCard** be the full subcategory of **FinSet** spanned by the finite cardinals $0, 1, 2, \ldots$. Note that it is a small category with a *canonical* choice of finite coproducts; note also that every isomorphism in **FinCard** is an automorphism. Clearly, for each finite set I, the category $[I, \mathbf{FinCard}]$ has the same properties. Though it is possible in principle to define Lawvere theories using $[I, \mathbf{FinSet}]$ as a base, it is traditional and conceptually clearer to use $[I, \mathbf{FinCard}]$.

Definition 2.5.1. An **I-sorted finitary Lawvere theory** is a small category $\mathcal{T}$ equipped with a functor $F : [I, \mathbf{FinCard}] \to \mathcal{T}$ that is bijective on objects and preserves finite coproducts. Abusing notation, we will usually suppress the functor F and speak of the category $\mathcal{T}$ itself as being a finitary Lawvere theory.

A **morphism of Lawvere theories** is any functor $G : \mathcal{T}' \to \mathcal{T}$ such that $GF' = F$, where F and F' are the structural functors of $\mathcal{T}$ and $\mathcal{T}'$, respectively. A **transformation of morphisms of Lawvere theories** is any natural transformation between the underlying functors of two such morphisms. We write $\mathbf{Law}_I$ for the **category of I-sorted finitary Lawvere theories**.

Definition 2.5.2. A **cartesian monoidal functor** is any functor $\mathcal{A} \to \mathcal{B}$ that preserves finite products, where $\mathcal{A}$ and $\mathcal{B}$ are assumed to be categories with finite products.

Definition 2.5.3. Let $\mathcal{T}$ be a Lawvere theory, and let $\mathcal{S}$ be any category with finite products. A **model of $\mathcal{T}$ in $\mathcal{S}$** is any cartesian monoidal functor $\mathcal{T}^{\mathrm{op}} \to \mathcal{S}$. The **underlying object** of a $\mathcal{T}$-model $A : \mathcal{T} \to \mathcal{S}$ is the object of $[I,\mathcal{S}]$ defined by $S \mapsto A(1_S)$, where 1_S is the object of $[I, \mathbf{FinCard}]$ that is 1 in the component S and 0 in all other components.

A **homomorphism of models of $\mathcal{T}$** is any natural transformation between such functors. We write $\mathbf{Mod}(\mathcal{T},\mathcal{S})$ for the **category of $\mathcal{T}$-models in $\mathcal{S}$**. The

forgetful functor is the functor $U^{\mathcal{T}} : \mathbf{Mod}(\mathcal{T},\mathcal{S}) \to [I,\mathcal{S}]$ that sends a model of $\mathcal{T}$ to its underlying object.

Proposition 2.5.4.

(i) *Given a morphism of Lawvere theories* $G : \mathcal{T}' \to \mathcal{T}$ *and a cartesian monoidal functor* $R : \mathcal{S} \to \mathcal{S}'$ *between categories with finite products, there is a functor* $\mathbf{Mod}(G,R) : \mathbf{Mod}(\mathcal{T},\mathcal{S}) \to \mathbf{Mod}(\mathcal{T}',\mathcal{S}')$*, taking a* $\mathcal{T}$*-model* $A : \mathcal{T}^{\mathrm{op}} \to \mathcal{S}$ *to the* $\mathcal{T}'$*-model* $RAG^{\mathrm{op}} : \mathcal{T}'^{\mathrm{op}} \to \mathcal{S}'$*.*

(ii) *Given a transformation* $\varphi : G' \Rightarrow G$ *of morphisms of Lawvere theories and a natural transformation* $\psi : R \Rightarrow R'$ *of functors, there is an induced natural transformation* $\mathbf{Mod}(\varphi,\psi) : \mathbf{Mod}(G,R) \Rightarrow \mathbf{Mod}(G',R')$*, defined by* $\psi A \varphi^{\mathrm{op}} : RAG^{\mathrm{op}} \Rightarrow R'AG'^{\mathrm{op}}$ *on each* $\mathcal{T}$*-model* A*.*

(iii) *Let* $F : [I,\mathbf{FinCard}] \to \mathcal{T}$ *and* $F' : [I,\mathbf{FinCard}] \to \mathcal{T}'$ *be the structural functors of the Lawvere theories* $\mathcal{T}$ *and* $\mathcal{T}'$*, respectively. Given a functor* $H : \mathbf{Mod}(\mathcal{T},\mathbf{Set}) \to \mathbf{Mod}(\mathcal{T}',\mathbf{Set})$ *such that* $H(A)F'^{\mathrm{op}} = AF^{\mathrm{op}}$ *for all* $\mathcal{T}$*-models* A *in* $\mathcal{S}$*, there exists a unique morphism* $G : \mathcal{T}' \to \mathcal{T}$ *such that* $H = \mathbf{Mod}(G,\mathrm{id})$*.*

(iv) *For any two morphisms* $G, G' : \mathcal{T}' \to \mathcal{T}$*, any natural transformation* $\theta : \mathbf{Mod}(G,\mathrm{id}) \Rightarrow \mathbf{Mod}(G',\mathrm{id})$ *of functors* $\mathbf{Mod}(\mathcal{T},\mathbf{Set}) \to \mathbf{Mod}(\mathcal{T}',\mathbf{Set})$ *is of the form* $\mathbf{Mod}(\varphi,\mathrm{id})$ *for a unique transformation* $\varphi : G' \Rightarrow G$*.*

Proof. (i). The definition of 'morphism of Lawvere theories' forces $G^{\mathrm{op}} : \mathcal{T}'^{\mathrm{op}} \to \mathcal{T}^{\mathrm{op}}$ to be a cartesian monoidal functor, and the rest is clear from the definition of 'model of a Lawvere theory.'

(ii). This is obvious from the definition of 'homomorphism of models of a Lawvere theory.'

(iii). It is well-known that representable functors preserve products, so the Yoneda embedding factors through $\mathbf{Mod}(\mathcal{T},\mathbf{Set}) \hookrightarrow [\mathcal{T}^{\mathrm{op}},\mathbf{Set}]$ as a functor $\hbar_{\bullet} : \mathcal{T} \to \mathbf{Mod}(\mathcal{T},\mathbf{Set})$. Consider the $\mathcal{T}'$-models $H(\hbar_{FY})$ for each I-indexed finite cardinal Y. By hypothesis, $H(\hbar_{FY})F'^{\mathrm{op}} = \hbar_{FY}F^{\mathrm{op}}$, so in particular

$$H(\hbar_{FY})(F'X) = \mathcal{T}(FX,FY)$$

for each object Y in $[I,\mathbf{FinCard}]$, and we may define an arrow $Gf : FX \to FY$ in $\mathcal{T}$ for each arrow $f : F'X \to F'Y$ in $\mathcal{T}'$ by setting $Gf = H(\hbar_{FY})(f)(\mathrm{id}_{FY})$. The commutativity of the diagram

$$\begin{tikzcd}
 & & H\big(\hslash_{FZ}\big)(F'Z) \arrow[d, "H(\hslash_{FZ})(g)"] \\
 & H\big(\hslash_{FY}\big)(F'Y) \arrow[r, "\big(H\big(\hslash_{Gg}\big)\big)_{F'Y}"] \arrow[d, "H(\hslash_{FY})(f)"'] & H\big(\hslash_{FZ}\big)(F'Y) \arrow[d, "H(\hslash_{FZ})(f)"] \\
H\big(\hslash_{FX}\big)(F'Z) \arrow[r, "\big(H\big(\hslash_{Gf}\big)\big)_{F'Z}"'] & H\big(\hslash_{FY}\big)(F'Z) \arrow[r, "\big(H\big(\hslash_{Gg}\big)\big)_{F'Z}"'] & H\big(\hslash_{FZ}\big)(F'Z)
\end{tikzcd}$$

ensures that $G(g \circ f) = Gg \circ Gf$ for all $f : F'X \to F'Y$ and $g : F'Y \to F'Z$ in $\mathcal{T}'$, and thus we obtain a functor $G : \mathcal{T}' \to \mathcal{T}$. Now let A be any object in $\mathbf{Mod}(\mathcal{T}, \mathbf{Set})$, and let $f : F'X \to F'Y$ be an arrow in $\mathcal{T}'$. The Yoneda lemma says there is a unique natural transformation $\alpha : \hslash_{FY} \Rightarrow A$ such that $\alpha_{FY}(\mathrm{id}_{FY}) = a$ for each element a of $A(FY) = H(A)(F'Y)$, so the commutativity of the diagram of sets below

$$\begin{tikzcd}
H\big(\hslash_{FY}\big)(F'Y) \arrow[r, "(H(\alpha))_{F'Y}"] \arrow[d, "H(\hslash_{FY})(f)"'] & H(A)(F'Y) \arrow[d, "H(A)(f)"] \\
H\big(\hslash_{FY}\big)(F'X) \arrow[r, "(H(\alpha))_{F'X}"'] & H(A)(F'X)
\end{tikzcd}$$

for all such natural transformations α implies $H(A)(f) = A(Gf)$. Thus, H must agree with $\mathbf{Mod}(G, \mathrm{id})$. Conversely, if $H = \mathbf{Mod}(G, \mathrm{id})$ for some morphism $G : \mathcal{T}' \to \mathcal{T}$, then

$$Gf = \hslash_{FY}(Gf)(\mathrm{id}_{FY}) = H\big(\hslash_{FY}\big)(f)(\mathrm{id}_{FY})$$

so G must be the functor constructed above.

(iv). Let $\theta : \mathbf{Mod}(G, \mathrm{id}) \Rightarrow \mathbf{Mod}(G', \mathrm{id})$ be a natural transformation. For each object X in $[I, \mathbf{FinCard}]$, set $\varphi_{F'X} = \big(\theta_{\hslash_{FX}}\big)_{F'X}(\mathrm{id}_{FX})$. This defines a natural transformation $\varphi : G' \Rightarrow G$ because both diagrams below commute for every arrow $f : F'X \to F'Y$ in $\mathcal{T}'$:

$$\begin{tikzcd}
\hslash_{FX}(FX) \arrow[r, "\big(\theta_{\hslash_{FX}}\big)_{F'X}"] \arrow[d, "\big(\hslash_{Gf}\big)_{FX}"'] & \hslash_{FX}(FX) \arrow[d, "\big(\hslash_{Gf}\big)_{FX}"] & & \hslash_{FY}(FY) \arrow[r, "\big(\theta_{\hslash_{FY}}\big)_{F'Y}"] \arrow[d, "\hslash_{FY}(Gf)"'] & \hslash_{FY}(FY) \arrow[d, "\hslash_{FY}(G'f)"] \\
\hslash_{FY}(FX) \arrow[r, "\big(\theta_{\hslash_{FY}}\big)_{F'X}"'] & \hslash_{FY}(FX) & & \hslash_{FY}(FX) \arrow[r, "\big(\theta_{\hslash_{FY}}\big)_{F'X}"'] & \hslash_{FY}(FX)
\end{tikzcd}$$

We claim $\theta_A = A\varphi^{\mathrm{op}}$ for each $\mathcal{T}$-model A. Indeed, for each element a of $A(FX)$, the Yoneda lemma gives a unique natural transformation $\alpha : \hslash_{FX} \Rightarrow A$ such that $\alpha_{FX}(\mathrm{id}_{FX}) = a$, and the diagram below

$$\begin{tikzcd}
\hslash_{FX}(FX) \arrow[r, "(\theta_{\hslash_{FX}})_{F'X}"] \arrow[d, "\alpha_{FX}"'] & \hslash_{FX}(FX) \arrow[d, "\alpha_{FX}"] \\
A(FX) \arrow[r, "(\theta_A)_{F'X}"'] & A(FX)
\end{tikzcd}$$

commutes for each such α, so $(\theta_A)_{F'X}(a) = A(\varphi_{FX})(a)$ for all a, as required. Conversely, if $\theta_A = A\varphi^{\mathrm{op}}$ for some natural transformation $\varphi : G' \Rightarrow G$, then

$$\varphi_{F'X} = \hslash_{FX}(\varphi_{F'X})(\mathrm{id}_{FX}) = (\theta_{\hslash_{FX}})_{F'X}(\mathrm{id}_{FX})$$

so φ must be the natural transformation constructed above. ■

Remark 2.5.5. Claim (iii) cannot be generalised, albeit for trivial reasons. For example, if $R : \mathbf{Set} \to \mathbf{Set}$ is the functor mapping every set X to the terminal set 1, then $\mathbf{Mod}(G, R) = \mathbf{Mod}(G', R)$ for *any* two morphisms $G, G' : \mathcal{T}' \to \mathcal{T}$. Alternatively, we may take $\mathcal{S}'$ to be the terminal category; then in that case $\mathbf{Mod}(\mathcal{T}', \mathcal{S}')$ is also the terminal category, for any Lawvere theory $\mathcal{T}'$.

So, given an equational theory $\mathbb{T}$ over a finitary algebraic signature, how do we "complete" $\mathbb{T}$ to obtain a Lawvere theory? The most straightforward way of doing so goes via the free–forgetful adjunction, so we may as well state the construction purely in terms of monads.

Definition 2.5.6. Let $\mathbb{T}$ be a monad on $[I, \mathbf{Set}]$. The **finitary Lawvere theory** of $\mathbb{T}$ is the full subcategory $\boldsymbol{\Theta}^{\mathbb{T}}$ of the Eilenberg–Moore category $[I, \mathbf{Set}]^{\mathbb{T}}$ spanned by the image of $[I, \mathbf{FinCard}]$ under the free $\mathbb{T}$-module functor $F^{\mathbb{T}}$, equipped with the evident structural functor $[I, \mathbf{FinCard}] \to \boldsymbol{\Theta}^{\mathbb{T}}$.

This makes sense because $F^{\mathbb{T}}$ is a left adjoint and so preserves all coproducts that exist in $[I, \mathbf{FinCard}]$. The notation recalls our earlier work with monads with arities, and for good reason:

Theorem 2.5.7. *Let $\mathbb{T}$ be a monad on $[I, \mathbf{Set}]$. If $\mathcal{A} = [I, \mathbf{FinCard}]$ is a category of arities for $\mathbb{T}$, then the nerve functor $\mathrm{N}^{\mathbb{T}}_{\mathcal{A}} : [I, \mathbf{Set}]^{\mathbb{T}} \to \left[(\boldsymbol{\Theta}^{\mathbb{T}})^{\mathrm{op}}, \mathbf{Set}\right]$ factors through the inclusion $\mathbf{Mod}(\boldsymbol{\Theta}^{\mathbb{T}}, \mathbf{Set}) \hookrightarrow \left[(\boldsymbol{\Theta}^{\mathbb{T}})^{\mathrm{op}}, \mathbf{Set}\right]$ as a functor that is fully faithful and essentially surjective on objects.*

Proof. By Weber's nerve theorem (1.7.16), $\mathrm{N}_{\mathcal{A}}^{\mathbb{T}} : [I, \mathbf{Set}]^{\mathbb{T}} \to \left[(\mathbf{\Theta}^{\mathbb{T}})^{\mathrm{op}}, \mathbf{Set} \right]$ is a fully faithful functor, and a presheaf $X : (\mathbf{\Theta}^{\mathbb{T}})^{\mathrm{op}} \to \mathbf{Set}$ is in the essential image of $\mathrm{N}_{\mathcal{A}}^{\mathbb{T}}$ if and only if $XF : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is in the essential image of the functor $\mathrm{N}_{\mathcal{A}} : [I, \mathbf{Set}] \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$; however, it is clear that a presheaf is in the essential image of $\mathrm{N}_{\mathcal{A}}$ if and only if it is cartesian monoidal as a functor. ■

In other words, if $\mathbb{T}$ is a monad with arities $[I, \mathbf{FinCard}]$ then the category of $\mathbb{T}$-modules canonically equivalent to the category of models for the finitary Lawvere theory $\mathbf{\Theta}^{\mathbb{T}}$ it induces. In particular:

Corollary 2.5.8. *If $\mathbb{T}$ is an equational theory over a finitary algebraic signature Σ, then $\mathbf{Mod}(\mathbb{T}, \mathbf{Set})$ is canonically equivalent to $\mathbf{Mod}(\mathbf{\Theta}^{\mathbb{T}}, \mathbf{Set})$, where $\mathbf{\Theta}^{\mathbb{T}}$ is the finitary I_{Σ}-sorted Lawvere theory induced by the free $\mathbb{T}$-model monad on $[I_{\Sigma}, \mathbf{Set}]$.*

Proof. Use corollary 2.4.22. ■

We are now led to wonder, is the converse true? That is, is every finitary Lawvere theory induced by an equational theory over a finitary algebraic signature? Unsurprisingly, the answer is yes.

Definition 2.5.9. Let $\mathcal{T}$ be a finitary I-sorted Lawvere theory with structural functor $F : [I, \mathbf{FinCard}] \to \mathcal{T}$. The **algebraic signature of** $\mathcal{T}$ has I sorts and one T-valued operation ω_f for each arrow $f : F1_T \to FX$ in $\mathcal{T}$, whose arities are given by the formula $\nu_S(\omega_f) = X(S)$. The **equational theory** of $\mathcal{T}$ has one equation of the form

$$\omega_{F\iota_j}(x_\bullet) = x_j$$

for each coproduct insertion $\iota_j : 1_T \to X$ in $[I, \mathbf{FinCard}]$, and one equation of the form

$$\omega_f(\omega_k(z_\bullet), \ldots) = \omega_g(\omega_h(z_\bullet), \ldots)$$

for each commutative diagram of the form below in $\mathcal{T}$:

$$\begin{tikzcd}
F1_T \arrow[r, "g"] \arrow[d, "f"'] & FY \arrow[d, "{[h,\ldots]}"] \\
FX \arrow[r, "{[k,\ldots]}"'] & FZ
\end{tikzcd}$$

Theorem 2.5.10. *Let $\mathcal{T}$ be a Lawvere theory and let $\mathbb{T}$ be its equational theory. There is a canonical functor* $\mathbf{Mod}(\mathbb{T}, \mathbf{Set}) \to \mathbf{Mod}(\mathcal{T}, \mathbf{Set})$ *that is fully faithful and essentially surjective on objects.*

Proof. This is clear from the fact that every object in $\mathcal{T}$ is canonically a coproduct of objects of the form $F1_S$ for various sorts S in I; the details are left to the reader. ◊

Corollary 2.5.11. *If $\mathbb{T}$ is a monad on* $[I, \mathbf{Set}]$ *such that* $[I, \mathbf{FinCard}]$ *is a category of arities for $\mathbb{T}$, then $\mathbb{T}$ is isomorphic to the free model monad for an I-sorted equational theory over a finitary algebraic signature.* ■

Note, however, that we are not saying the equational theory $\mathbb{T}'$ of the finitary Lawvere theory induced by an equational theory $\mathbb{T}$ is necessarily "the same" as $\mathbb{T}$ in the sense of having the same operations and axioms. This is quite clearly not the case: $\mathbb{T}'$ *always* has infinitely many axioms and infinitely many operations.

Instead, we should think of equational theories as *presentations* of Lawvere theories; from this point of view, the fact that two different equational theories sometimes correspond to the same Lawvere theory is no more surprising than the fact that a group can have two different presentations in terms of generators and relations. A concrete example of this phenomenon is the well-known equivalence between boolean algebras and boolean rings.

The value of Lawvere theories, then, is in proposition 2.5.4: two equational theories over a finitary algebraic signature have categories of models in **Set** that are equivalent as categories over $[I, \mathbf{Set}]$ if and only if the Lawvere theories they present are isomorphic. Yet, this is not the only possible form of equivalence: for example, if k is a field and R is the ring of $n \times n$ matrices with entries in k, then the category of k-vector spaces and the category of left R-modules are equivalent—but not as categories over **Set** if $n > 1$.[1] Indeed, while in the theory of k-vector spaces, all unary operations commute with each other, there are non-commutative unary operations in the theory of left R-modules, so their corresponding Lawvere theories cannot possibly be isomorphic. We will address this point again later.

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[1] In other words, k and R are Morita-equivalent as rings. To see this, one simply verifies that every left R-module is of the form $k^{\oplus n} \otimes_k V$ for a unique k-vector space V up to isomorphism. See [Weibel, 1994, Prop. 9.5.2].

6 Finitary and algebraic monads

Let us now return to the study of monads. We have just seen that equational theories over finitary algebraic signatures and finitary Lawvere theories are both equivalent to a certain class of monads, so it seems reasonable to focus on them, or at least a class of monads containing them.

Definition 2.6.1. A **finitary monad** is a monad on an l.f.p. category whose underlying endofunctor is finitary. An **algebraic endofunctor** is a finitary endofunctor on a regular l.f.p. category that preserves coequalisers for kernel pairs. An **algebraic monad** is a monad on a regular l.f.p. category whose underlying endofunctor is algebraic.

Remark 2.6.2. This terminology is non-standard: most authors consider 'finitary monad' and 'algebraic monad' to be synonyms. Fortunately, when the base category is any regular l.f.p. category in which all regular epimorphisms split—such as **Set**—then any finitary monad is automatically algebraic by lemma 1.3.6.

Recall also the results of the previous sections: the monad on a regular l.f.p. category induced by any equational theory over a finitary algebraic signature is always an algebraic monad, so we see that the definition is not overly restrictive.

Proposition 2.6.3. *Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a regular l.f.p. category $\mathcal{S}$. The following are equivalent:*

(i) *The forgetful functor $U^{\mathbb{T}} : \mathcal{S}^{\mathbb{T}} \to \mathcal{S}$ creates coequalisers for kernel pairs and colimits for all small filtered diagrams.*

(ii) *$\mathbb{T}$ is an algebraic monad on $\mathcal{S}$.*

(iii) *$\mathcal{S}^{\mathbb{T}}$ is a regular l.f.p. category and $U^{\mathbb{T}} : \mathcal{S}^{\mathbb{T}} \to \mathcal{S}$ preserves regular epimorphisms and colimits for all small filtered diagrams.*

Proof. (i) ⇒ (ii). The free $\mathbb{T}$-module functor $F^{\mathbb{T}} : \mathcal{S} \to \mathcal{S}^{\mathbb{T}}$ preserves all colimits because it is a left adjoint, and $T = U^{\mathbb{T}} F^{\mathbb{T}}$, so T must preserve coequalisers for kernel pairs and colimits for small filtered diagrams if $U^{\mathbb{T}}$ creates them.

(ii) ⇒ (iii). If T preserves coequalisers for kernel pairs, then it also preserves regular epimorphisms, so proposition 1.2.6 implies $U^{\mathbb{T}}$ creates coequalisers for kernel pairs, and hence, preserves regular epimorphisms. A direct application of the same proposition shows $U^{\mathbb{T}}$ creates colimits for all small filtered diagrams.

Now, consider the category of modules for the *endofunctor* T. By theorem 2.4.8 and proposition 2.3.14, $\mathcal{S}^T$ is a regular l.f.p. category. It is clear that $\mathcal{S}^{\mathbb{T}}$ can be embedded as a full subcategory of $\mathcal{S}^T$, and an argument similar to proposition 2.4.20 shows $\mathcal{S}^{\mathbb{T}}$ is closed under subobjects in $\mathcal{S}^T$, because $U^T : \mathcal{S}^T \to \mathcal{S}$ preserves monomorphisms. $U^T : \mathcal{S}^T \to \mathcal{S}$ is also conservative, so $\mathcal{S}^T$ is well-powered and hence regularly well-copowered by proposition 2.3.4. We then apply theorem 2.3.15 to obtain $\mathcal{S}^{\mathbb{T}}$ as a reflective subcategory of $\mathcal{S}^T$. The inclusion $\mathcal{S}^{\mathbb{T}} \hookrightarrow \mathcal{S}^T$ preserves small filtered colimits, so corollary 2.1.22 and proposition 2.3.14 together imply $\mathcal{S}^{\mathbb{T}}$ is a regular l.f.p. category.

(iii) ⇒ (i). $U^{\mathbb{T}}$ is conservative by proposition 1.2.4, so $U^{\mathbb{T}}$ reflects all colimits it preserves. Since $\mathcal{S}^{\mathbb{T}}$ is assumed to have coequalisers for kernel pairs and colimits for small filtered diagrams, $U^{\mathbb{T}}$ creates them. ■

We would like to prove something similar for finitary monads on a general l.f.p. category, but we have not yet shown that the category of modules for a finitary monad on an l.f.p. category is again an l.f.p. category. The following proof is based on [Borceux, 1994b, Prop. 4.3.6]; it is roughly the same as the proof of proposition 2.4.7, but the induction step is different.

Proposition 2.6.4. *Let $\mathcal{S}$ be a category with colimits for countable diagrams. If $\mathbb{T} = (T, \eta, \mu)$ is a monad on $\mathcal{S}$ such that the endofunctor T preserves countable filtered diagrams, then $\mathcal{S}^{\mathbb{T}}$ has coequalisers for all parallel pairs.*

Proof. Let $f, g : (A, \alpha) \to (B, \beta)$ be a parallel pair in $\mathcal{S}^{\mathbb{T}}$. Set $Q_0 = B$, and form the coequaliser $p_0 : Q_0 \to Q_1$ of $f, g : A \to B$ in $\mathcal{S}$. Let $q_0 : TQ_0 \to Q_1$ be the composite $p_0 \circ \beta$. For each natural number n, form the coequaliser diagram

$$\begin{tikzcd}
T^2 Q_n \arrow[r, shift left, "Tq_n"] \arrow[r, shift right, "Tp_n \circ \mu_{Q_n}"'] & TQ_{n+1} \arrow[r, "q_{n+1}"] & Q_{n+2}
\end{tikzcd}$$

in $\mathcal{S}$, set $p_{n+1} = q_{n+1} \circ \eta_{Q_{n+1}}$, and define Q by the filtered colimit shown below:

$$\begin{tikzcd}
Q_0 \arrow[r, "p_0"] \arrow[rrrrd, bend right, "\lambda_0"'] & Q_1 \arrow[r, "p_1"] \arrow[rrrd, bend right, "\lambda_1"'] & Q_2 \arrow[r, "p_2"] \arrow[rrd, "\lambda_2"'] & \cdots \arrow[r] \arrow[rd] & \cdots \\
 & & & & Q
\end{tikzcd}$$

Observe that, for each natural number n, we have

$$\begin{aligned} p_{n+1} \circ q_n &= q_{n+1} \circ \eta_{Q_{n+1}} \circ q_n \\ &= q_{n+1} \circ Tq_n \circ \eta_{TQ_n} \\ &= q_{n+1} \circ Tp_n \circ \mu_{Q_n} \circ \eta_{TQ_n} \\ &= q_{n+1} \circ Tp_n \end{aligned}$$

and T preserves countable filtered colimits, so there is a unique morphism $q : TQ \to Q$ such that $q \circ T\lambda_n = \lambda_{n+1} \circ q_n$ for every natural number n. In particular,

$$q \circ T\lambda_0 = \lambda_1 \circ q_0 = \lambda_1 \circ p_0 \circ \beta = \lambda_0 \circ \beta$$

so λ_0 would be a $\mathbb{T}$-module homomorphism if $q : TQ \to Q$ were a $\mathbb{T}$-module structure. We claim q *is* a $\mathbb{T}$-module structure and λ_0 is the coequaliser of $f, g : (A, \alpha) \to (B, \beta)$ in $\mathcal{S}^{\mathbb{T}}$.

Indeed, first note that, for each natural number n,

$$\begin{aligned} q \circ (\eta_Q \circ \lambda_n) &= (q \circ T\lambda_n) \circ \eta_{Q_n} \\ &= \lambda_{n+1} \circ (q_n \circ \eta_{Q_n}) \\ &= \lambda_{n+1} \circ p_n \\ &= \lambda_n \end{aligned}$$

and $(\lambda_n : Q_n \to Q \mid n \in \mathbb{N})$ is jointly epimorphic, so $q \circ \eta_Q = \mathrm{id}_Q$. Similarly,

$$\begin{aligned} q \circ \left(\mu_Q \circ T^2\lambda_n\right) &= (q \circ T\lambda_n) \circ \mu_{Q_n} \\ &= \lambda_{n+1} \circ q_n \circ \mu_{Q_n} \\ &= \lambda_{n+2} \circ (p_{n+1} \circ q_n) \circ \mu_{Q_n} \\ &= \lambda_{n+2} \circ (q_{n+1} \circ Tp_n \circ \mu_{Q_n}) \\ &= (\lambda_{n+2} \circ q_{n+1}) \circ Tq_n \\ &= q \circ (T\lambda_{n+1} \circ Tq_n) \\ &= (q \circ Tq) \circ T^2\lambda_n \end{aligned}$$

and $(T^2\lambda_n : T^2Q_n \to T^2Q \,\big|\, n \in \mathbb{N})$ is also jointly surjective, so $q \circ \mu_Q = q \circ Tq$. Thus $q : TQ \to Q$ is indeed a $\mathbb{T}$-module structure, and λ_0 is a $\mathbb{T}$-module homomorphism $(B, \beta) \to (Q, q)$.

Suppose $h : (B, \beta) \to (C, \gamma)$ satisfies the equation $h \circ f = h \circ g$. Let $h_0 = h$. Since p_0 is the coequaliser of f and g in $\mathcal{S}$, there exists a unique morphism

$h_1 : Q_1 \to C$ in $\mathcal{S}$ such that $h_1 \circ p_0 = h_0$. Note that $h_0 \circ \beta = \gamma \circ Th_0$. Now, given a morphism $h_{n+1} : Q_{n+1} \to C$ such that

$$\begin{aligned} h_{n+1} \circ p_n &= h_n \\ h_{n+1} \circ q_n &= \gamma \circ Th_n \end{aligned}$$

there is a unique $h_{n+2} : Q_{n+2} \to C$ in $\mathcal{S}$ such that $h_{n+2} \circ q_{n+1} = \gamma \circ Th_{n+1}$ because we have

$$\begin{aligned} (\gamma \circ T_{n+1}) \circ Tq_n &= \gamma \circ T(h_{n+1} \circ q_n) \\ &= (\gamma \circ T\gamma) \circ T^2 h_n \\ &= \gamma \circ \left(\mu_C \circ T^2 h_n\right) \\ &= \gamma \circ Th_n \circ \mu_{Q_n} \\ &= (\gamma \circ Th_{n+1}) \circ (Tp_n \circ \mu_{Q_n}) \end{aligned}$$

as required to construct such a morphism by the universal property of Q_{n+2}, and

$$h_{n+2} \circ p_{n+1} = h_{n+2} \circ q_{n+1} \circ \eta_{Q_{n+1}} = \gamma \circ Th_{n+1} \circ \eta_{Q_{n+1}} = \gamma \circ \eta_C \circ h_{n+1} = h_{n+1}$$

so by induction we obtain a family $(h_n : Q_n \to C \mid n \in \mathbb{N})$ such that

$$h_{n+1} \circ p_n = h_n$$

for all natural numbers n. So, by the universal property of Q, there exists a unique morphism $\tilde{h} : Q \to C$ in $\mathcal{S}$ such that $\tilde{h} \circ \lambda_n = h_n$ for each natural number n. Moreover, because

$$\tilde{h} \circ q \circ T\lambda_n = \tilde{h} \circ \lambda_{n+1} \circ q_n = h_{n+1} \circ q_n = \gamma \circ Th_n = \gamma \circ T\tilde{h} \circ T\lambda_n$$

for all natural numbers n, we have $\tilde{h} \circ q = \gamma \circ T\tilde{h}$, and thus $\tilde{h}$ is a $\mathbb{T}$-module homomorphism $(Q, q) \to (C, \gamma)$ such that $\tilde{h} \circ \lambda_0 = h$. On the other hand, given any $\mathbb{T}$-module homomorphism $\tilde{h} : (Q, q) \to (C, \gamma_1)$ such that $\tilde{h} \circ \lambda_0 = h$, commutativity of the diagram below

$$\begin{tikzcd}
T^2 Q_n \arrow[r, shift left, "Tq_n"] \arrow[r, shift right, "Tp_n \circ \mu_{Q_n}"'] \arrow[d, "T^2\lambda_n"'] & TQ_{n+1} \arrow[r, "q_{n+1}"] \arrow[d, "T\lambda_{n+1}"] & Q_{n+2} \arrow[d, "\lambda_{n+2}"] \\
T^2 Q \arrow[r, shift left, "Tq"] \arrow[r, shift right, "\mu_Q"'] \arrow[d, "T^2\tilde{h}"'] & TQ \arrow[r, "q"] \arrow[d, "T\tilde{h}"] & Q \arrow[d, "\tilde{h}"] \\
T^2 C \arrow[r, shift left, "T\gamma"] \arrow[r, shift right, "\mu_C"'] & TC \arrow[r, "\gamma"] & C
\end{tikzcd}$$

forces $\tilde{h}$ to be defined as in the above construction, so $\tilde{h}$ is the *unique* $\mathbb{T}$-module homomorphism such that $\tilde{h} \circ \lambda_0 = h$. Thus $\lambda_0 : (B, \beta) \to (Q, q)$ is the coequaliser of $f, g : (A, \alpha) \to (B, \beta)$, as claimed. ■

Theorem 2.6.5. *Let $\mathcal{S}$ be a locally finitely-presentable category. If $\mathbb{T}$ is a finitary monad on $\mathcal{S}$, then:*

(i) *The forgetful functor $U^{\mathbb{T}} : \mathcal{S}^{\mathbb{T}} \to \mathcal{S}$ creates colimits for all small filtered diagrams.*

(ii) *The category $\mathbf{K}(\mathcal{S})$ of compact objects in $\mathcal{S}$ is an essentially small category of arities for $\mathbb{T}$.*

(iii) *$\mathcal{S}^{\mathbb{T}}$ is a locally finitely-presentable category.*

Proof. (i). An l.f.p. category has all small colimits and a finitary endofunctor preserves small filtered colimits, so proposition 1.2.6 implies $U^{\mathbb{T}}$ creates colimits for all small filtered diagrams.

(ii). $\mathbf{K}(\mathcal{S})$ consists of compact objects in $\mathcal{S}$, so propositions 2.1.8 and 2.1.13 together imply $\mathbf{K}(\mathcal{S})$ is an essentially small category of arities for $\mathbb{T}$.

(iii). The previous proposition says $\mathcal{S}^{\mathbb{T}}$ has all coequalisers, and corollary 2.1.15 says $\mathcal{S}$ has all small colimits, so $\mathcal{S}^{\mathbb{T}}$ is cocomplete by theorem 1.3.2. Propositions 1.7.10 and 2.2.4 then imply $\mathcal{S}^{\mathbb{T}}$ is an l.f.p. category. ■

The most important point to make about finitary monads is that they are determined by an essentially small amount of data. After all, if $\mathcal{S}$ is a finitely-accessible category, then $\mathbf{K}(\mathcal{S}) \hookrightarrow \mathcal{S}$ is dense, so any finitary endofunctor on $\mathcal{S}$ is determined up to isomorphism by its restriction to $\mathbf{K}(\mathcal{S})$. Moreover:

Lemma 2.6.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be any two finitely-accessible categories. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors, and let $\alpha, \beta : F \Rightarrow G$ be natural transformations. If F is finitary and $\alpha_X = \beta_X$ for all compact objects X in $\mathcal{C}$, then $\alpha = \beta$.*

Proof. Let Y be an arbitrary object in $\mathcal{C}$, and consider a morphism $f : X \to Y$ in $\mathcal{C}$ where X is compact. By naturality, the diagrams below commute in $\mathcal{D}$:

$$\begin{tikzcd}
FX \arrow[r, "\alpha_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\alpha_Y"'] & GY
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\beta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\
FY \arrow[r, "\beta_Y"'] & GY
\end{tikzcd}$$

Since F is finitary, $\left(F f : F X \to F Y \,\middle|\, X \in \operatorname{ob} \mathbf{K}(\mathcal{C})\right)$ is a jointly epimorphic sink in $\mathcal{D}$, so if $\alpha_X = \beta_X$ for all compact objects X, then $\alpha_Y = \beta_Y$ as well. ■

So, modulo some set-theoretic circumlocution, it should be clear that the category of finitary functors between two finitely-accessible categories is essentially small. Actually, we can say something a little better:

Proposition 2.6.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be l.f.p. categories.*

(i) *The category of finitary functors $\mathcal{C} \to \mathcal{D}$ is equivalent to the category of (all) functors $\mathbf{K}(\mathcal{C}) \to \mathcal{D}$.*

(ii) *The category of finitary functors $\mathcal{C} \to \mathcal{D}$ is locally finitely presentable.*

(iii) *The full subcategory of finitary right adjoints is essentially small.*

Proof. (i). It is clear that any finitary functor $\mathcal{C} \to \mathcal{D}$ extending a functor $\mathbf{K}(\mathcal{C}) \to \mathcal{D}$ is unique up to isomorphism if it exists, and the lemma says that natural transformations of finitary functors $\mathcal{C} \to \mathcal{D}$ are entirely determined by their components on $\mathbf{K}(\mathcal{C})$, so to prove the claim it is enough to check that any functor $F : \mathbf{K}(\mathcal{C}) \to \mathcal{D}$ can be extended to a finitary functor. By the universal property of presheaf toposes (theorem 2.2.7), there is a unique cocontinuous functor $\tilde{F} : \widehat{\mathbf{K}(\mathcal{C})} \to \mathcal{D}$ extending F. Since the universal functor $\mathbf{K}(\mathcal{C}) \to \widehat{\mathbf{K}(\mathcal{C})}$ extends to a finitary functor $\mathcal{C} \to \widehat{\mathbf{K}(\mathcal{C})}$ by theorem 2.2.17, composing with $\tilde{F}$ gives the required finitary functor $\mathcal{C} \to \mathcal{D}$.

(ii). This an immediate consequence of (i) and corollary 2.2.20.

(iii). Right adjoints are determined up to isomorphism by their left adjoints, but (i) and proposition 2.1.21 imply the category of left adjoints whose right adjoints are finitary is essentially small. ■

Definition 2.6.8. The **category of finitary monads** on an l.f.p. category $\mathcal{S}$ is the category $\mathbf{FinMon}(\mathcal{S})$ whose objects are finitary monads on $\mathcal{S}$ and whose morphisms are morphisms of monads of the form (id, φ) for a natural transformation φ. The **category of algebraic monads** on a regular l.f.p. category $\mathcal{S}$, denoted by $\mathbf{AlgMon}(\mathcal{S})$, is defined analogously.

The reader is reminded that a morphism of monads is defined so that it *covariantly* induces a functor on module categories; proposition 1.4.6 then implies $\mathbf{FinMon}(\mathcal{S})$ is equivalent to a full subcategory of the category of l.f.p. categories equipped with a strictly monadic functor to $\mathcal{S}$.

Definition 2.6.9. Let $\mathcal{C}$ and $\mathcal{D}$ be finitely accessible categories. A **finitary coreflection** of a functor $F : \mathcal{C} \to \mathcal{D}$ is a finitary functor $\bar{F} : \mathcal{C} \to \mathcal{D}$ equipped with a universal natural transformation $\varepsilon : \bar{F} \Rightarrow F$ such that any natural transformation $F' \Rightarrow F$ with F' finitary factors through ε in a unique way.

Proposition 2.6.10. *If $\mathcal{C}$ and $\mathcal{D}$ are l.f.p. categories, then any functor $F : \mathcal{C} \to \mathcal{D}$ has a finitary coreflection $\bar{F} : \mathcal{C} \to \mathcal{D}$, and $\bar{F}$ is a left Kan extension of $F|_{\mathbf{K}(\mathcal{C})}$ along the inclusion $\mathbf{K}(\mathcal{C}) \hookrightarrow \mathcal{C}$.*

Proof. Proposition 2.6.7 implies there exists a finitary functor $\bar{F} : \mathcal{C} \to \mathcal{D}$ such that $\bar{F}\big|_{\mathbf{K}(\mathcal{C})} = F|_{\mathbf{K}(\mathcal{C})}$, and $\bar{F}$ is determined uniquely up to unique isomorphism by this property. The universal natural transformation $\varepsilon : \bar{F} \Rightarrow F$ is defined as follows: for each object X in $\mathcal{C}$, if $\lambda^X : U^X \Rightarrow \Delta X$ is the canonical cocone, then $\varepsilon_X : \bar{F}X \to FX$ is the unique morphism in $\mathcal{D}$ such that $\varepsilon_X \circ \bar{F}\lambda^X_{(A,f)} = F\lambda^X_{(A,f)}$ for all objects (A, f) in $(\mathbf{K}(\mathcal{C}) \downarrow X)$. This makes sense because $\bar{F}$ and F agree on $\mathbf{K}(\mathcal{C})$ and $\bar{F}$ is finitary.

Having constructed a finitary coreflection $\bar{F}$ of F, we appeal to proposition 2.6.7 again to conclude that $\bar{F}$ equipped with ε has the universal property of a left Kan extension of $F|_{\mathbf{K}(\mathcal{C})}$ along $\mathbf{K}(\mathcal{C}) \hookrightarrow \mathcal{C}$. ■

Proposition 2.6.11. *Let $\mathbb{T} = (T, \eta, \mu)$ be any monad on a finitely accessible category $\mathcal{C}$. If $\bar{T}$ is a finitary coreflection of T, then there are unique natural transformations $\bar{\eta} : \mathrm{id}_{\mathcal{C}} \Rightarrow \bar{T}$ and $\bar{\mu} : \bar{T}^2 \Rightarrow \bar{T}$ such that $\bar{\mathbb{T}} = \left(\bar{T}, \bar{\eta}, \bar{\mu}\right)$ is a finitary monad on $\mathcal{C}$, and moreover the universal natural transformation $\varepsilon : \bar{T} \Rightarrow T$ induces a monad morphism $(\mathrm{id}, \varepsilon) : \mathbb{T} \to \bar{\mathbb{T}}$.*

Proof. Finitary endofunctors are closed under composition, so the universal property of $\bar{T}$ ensures there exist unique $\bar{\eta}$ and $\bar{\mu}$ such that the equations below hold:

$$\varepsilon \bullet \bar{\eta} = \eta \qquad\qquad \varepsilon \bullet \bar{\mu} = \mu \bullet \varepsilon T \bullet \bar{T}\varepsilon$$

Direct calculation then shows $\bar{\mathbb{T}} = \left(\bar{T}, \bar{\eta}, \bar{\mu}\right)$ is a finitary monad on $\mathcal{C}$. Indeed,

$$\begin{aligned} \varepsilon \bullet \bar{\mu} \bullet \bar{T}\bar{\eta} &= \mu \bullet \varepsilon T \bullet \left(\bar{T}\varepsilon \bullet \bar{T}\bar{\eta}\right) \\ &= \mu \bullet \varepsilon T \bullet \bar{T}\eta \\ &= \mu \bullet T\eta \bullet \varepsilon = \varepsilon \end{aligned}$$

$$\begin{aligned}
\varepsilon \bullet \bar{\mu} \bullet \bar{\eta}\bar{T} &= \mu \bullet \varepsilon T \bullet \left(\bar{T}\varepsilon \bullet \bar{\eta}\bar{T}\right) \\
&= \mu \bullet \left(T\varepsilon \bullet T\bar{\eta}\right) \bullet \varepsilon \\
&= \mu \bullet T\eta \bullet \varepsilon = \varepsilon
\end{aligned}$$

so $\bar{\mu} \bullet \bar{T}\bar{\eta} = \mathrm{id}_{\bar{T}}$ and $\bar{\mu} \bullet \bar{\eta}\bar{T} = \mathrm{id}_{\bar{T}}$ by uniqueness, and

$$\begin{aligned}
\varepsilon \bullet \bar{\mu} \bullet \bar{T}\bar{\mu} &= \mu \bullet \varepsilon T \bullet \left(\bar{T}\varepsilon \bullet \bar{T}\bar{\mu}\right) \\
&= \mu \bullet \varepsilon T \bullet \bar{T}\mu \bullet \bar{T}T\varepsilon \bullet \bar{T}\varepsilon\bar{T} \\
&= \left(\mu \bullet T\mu\right) \bullet \left(\varepsilon T^2 \bullet \bar{T}T\varepsilon\bar{T} \bullet \bar{T}\varepsilon\bar{T}\right) \\
&= \left(\mu \bullet T\mu\right) \bullet (\varepsilon \circ \varepsilon \circ \varepsilon) \\
&= \left(\mu \bullet \mu T\right) \bullet (\varepsilon \circ \varepsilon \circ \varepsilon) \\
&= \mu \bullet \mu T \bullet T^2\varepsilon \bullet \varepsilon T\bar{T} \bullet \bar{T}\varepsilon\bar{T} \\
&= \mu \bullet T\varepsilon \bullet \left(\mu \bullet \varepsilon T \bullet \bar{T}\varepsilon\right)\bar{T} \\
&= \mu \bullet \left(T\varepsilon \bullet \varepsilon\bar{T}\right) \bullet \bar{\mu}\bar{T} \\
&= \left(\mu \bullet \varepsilon T \bullet \bar{T}\varepsilon\right) \bullet \bar{\mu}\bar{T} \\
&= \varepsilon \bullet \bar{\mu} \bullet \bar{\mu}\bar{T}
\end{aligned}$$

so $\bar{\mu} \bullet \bar{T}\bar{\mu} = \bar{\mu} \bullet \bar{\mu}\bar{T}$ as well. Thus $\bar{\mathbb{T}}$ is a monad, and $(\mathrm{id}, \varepsilon) : \mathbb{T} \to \bar{\mathbb{T}}$ is a morphism of monads. ■

Example 2.6.12. Let $\mathbb{T}$ be the free complete semilattice monad on **Set**. Its underlying endofunctor is the covariant powerset functor $\mathscr{P}(-)$, and the finitary coreflection of $\mathscr{P}(-)$ is the subfunctor that maps a set X to the set of *finite* subsets of X. The associated finitary monad $\bar{\mathbb{T}}$ is isomorphic to the free semilattice monad.

Note also that the finitary Lawvere theories of $\mathbb{T}$ and $\bar{\mathbb{T}}$ are isomorphic by construction. This is one sense in which the theory of semilattices is the "best finitary approximation" of the theory of complete semilattices.

Lemma 2.6.13. *Let $F : \mathcal{B} \to \mathcal{A}$ and $G : \mathcal{C} \to \mathcal{A}$ be functors.*

$$\begin{tikzcd}
\mathcal{D} \arrow[r, "Q"] \arrow[d, "P"'] \arrow[rd, "R"] & \mathcal{C} \arrow[d, "G"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{A}
\end{tikzcd}$$

If the diagram above is a (strict) pullback square, then the three functors P, Q, and R jointly *reflect limits and colimits.*

Proof. Let $D : \mathcal{J} \to \mathcal{D}$ be a diagram, and let $\lambda : \Delta\hat{D} \Rightarrow D$ be a cone in $\mathcal{D}$. Suppose $P\lambda$, $Q\lambda$, and $R\lambda$ are limiting cones in $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{A}$, respectively. Then, given a cone $\varphi : \Delta X \Rightarrow D$ in $\mathcal{D}$, there are unique morphisms $f : PX \to P\hat{D}$, $g : QX \to Q\hat{D}$, and $h : RX \to R\hat{D}$ such that

$$P\varphi_j = P\lambda_j \circ f \qquad Q\varphi_j = Q\lambda_j \circ g \qquad R\varphi_j = R\lambda_j \circ h$$

in $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{A}$, respectively, for each object j in $\mathcal{J}$. This implies $Ff = Gg = h$, so there is a unique morphism $k : X \to \hat{D}$ in $\mathcal{D}$ such that $Pk = f$ and $Qk = g$. Moreover, $\lambda_j \circ k = \varphi_j$ for all j, thus λ is indeed a limiting cone in $\mathcal{D}$.

The claim for colimits follows by replacing the categories in the proof above with their opposites. ■

Proposition 2.6.14. *Let $\mathcal{S}$ be a regular l.f.p. category.*

(i) *The identity monad on $\mathcal{S}$ is the terminal object in* **AlgMon**($\mathcal{S}$).

(ii) *Given monad morphisms $\mathbb{B} \to \mathbb{A}$ and $\mathbb{C} \to \mathbb{A}$ in* **AlgMon**($\mathcal{S}$)*, their pullback $\mathbb{D}$ exists in* **AlgMon**($\mathcal{S}$)*, and $\mathcal{S}^{\mathbb{D}}$ is isomorphic to the fibre product of $\mathcal{S}^{\mathbb{B}}$ and $\mathcal{S}^{\mathbb{C}}$ over $\mathcal{S}^{\mathbb{A}}$.*

(iii) *Given a small family of monads $(\mathbb{T}_i \mid i \in I)$, their product $\prod_{i \in I} \mathbb{T}_i$ exists in* **AlgMon**($\mathcal{S}$)*, and the category of modules for $\prod_{i \in I} \mathbb{T}_i$ is isomorphic to the* fibre *product of the module categories $\left(\mathcal{S}^{\mathbb{T}_i} \,\middle|\, i \in I\right)$ over $\mathcal{S}$.*

(iv) **AlgMon**($\mathcal{S}$) *has limits for all small diagrams, and the category of modules for the limit of a diagram of algebraic monads is the limit of the corresponding diagram of categories of modules (considered as categories over $\mathcal{S}$).*

Proof. (i). This is an immediate consequence of proposition 1.4.6.

(ii). Let $\mathcal{D}$ be the indicated fibre product of categories:

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] & \mathcal{S}^{\mathbb{C}} \arrow[d] \\
\mathcal{S}^{\mathbb{B}} \arrow[r] & \mathcal{S}^{\mathbb{A}}
\end{tikzcd}$$

If we can show that $U : \mathcal{D} \to \mathcal{S}$ is strictly monadic, then the claim follows by proposition 1.4.6. Note that U creates limits for all small diagrams, coequalisers for kernel pairs, and colimits for all small filtered diagrams because these are all computed as in $\mathcal{S}$, so if U is strictly monadic, then the monad it induces is an algebraic monad on $\mathcal{S}$ by the lemma above.

Let $T^{\mathbb{A}}$, $T^{\mathbb{B}}$, and $T^{\mathbb{C}}$ be the underlying endofunctors of $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ respectively. Define an endofunctor J on $\mathcal{S}$ so that we have a componentwise pushout square:

$$\begin{tikzcd}
T^{\mathbb{A}} \arrow[r] \arrow[d] & T^{\mathbb{B}} \arrow[d] \\
T^{\mathbb{C}} \arrow[r] & J
\end{tikzcd}$$

$T^{\mathbb{A}}$, $T^{\mathbb{B}}$, and $T^{\mathbb{C}}$ all preserve coequalisers for kernel pairs by assumption, and pushouts preserve coequalisers, so J also preserves coequalisers for kernel pairs. An argument similar to that of proposition 2.6.3 then exhibits $\mathcal{D}$ as a reflective subcategory of $\mathcal{S}^J$. Pushouts also preserve colimits of small filtered diagrams, so J is finitary, and thus the forgetful functor $\mathcal{S}^J \to \mathcal{S}$ is (strictly) monadic by theorem 2.4.8. The inclusion $\mathcal{D} \to \mathcal{S}^J$ satisfies the CTT condition, so proposition 1.2.15 implies $U : \mathcal{D} \to \mathcal{S}^J$ is monadic; but U is automatically an amnestic isofibration because $\mathcal{S}^{\mathbb{B}} \to \mathcal{S}^{\mathbb{A}}$, $\mathcal{S}^{\mathbb{C}} \to \mathcal{S}^{\mathbb{A}}$, and $\mathcal{S}^{\mathbb{A}} \to \mathcal{S}$ are, so U is also strictly monadic by proposition 1.2.18.

(iii). The argument used to prove (ii) can be adapted for this case.

(iv). It is well-known that a category with a terminal object and pullbacks also has equalisers for parallel pairs. Since we also have products for small families, we can construct limits for all small diagrams. ■

We should stop to consider what it means to take the limit of a diagram of algebraic monads. According to the proposition, to give a module structure on an object X for the monad $\mathbb{B} \times_{\mathbb{A}} \mathbb{C}$ is the same thing as to give a $\mathbb{B}$-module and a $\mathbb{C}$-module structure on X, such that the two induced $\mathbb{A}$-module structures agree. In terms of equational theories, we are essentially taking the union of $\mathbb{B}$ and $\mathbb{C}$ amalgamated along $\mathbb{A}$. For example:

Example 2.6.15. Let $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ respectively be the free monoid monad, the free group monad, and the free commutative monoid monad on **Set**. Then $\mathbb{B} \times_{\mathbb{A}} \mathbb{C}$ is (isomorphic to) the free abelian group monad.

The construction of colimits requires a different approach, but it is just as straightforward: it turns out that (the opposite of) the category of monads is *almost* monadic over the category of endofunctors—the only problematic point is the existence of a left adjoint to the forgetful functor, and this can be resolved when we restrict our attention to finitary and algebraic monads.

Proposition 2.6.16. *Let $\mathcal{S}$ be a regular l.f.p. category, and let* $\mathbf{AlgEnd}(\mathcal{S})$ *be the full subcategory of* $[\mathcal{S},\mathcal{S}]$ *spanned by the algebraic endofunctors on* $\mathcal{S}$*.*

(i) *Every algebraic endofunctor on $\mathcal{S}$ generates an algebraic free monad as in Barr's theorem (1.6.5).*

(ii) *The forgetful functor $T^{\bullet} : \mathbf{AlgMon}(\mathcal{S})^{\mathrm{op}} \to \mathbf{AlgEnd}(\mathcal{S})$ is strictly monadic.*

Proof. (i). Use propositions 2.4.7 and 2.6.3.

(ii). The definition of 'free monad' means $T^{\bullet}$ has a left adjoint precisely when every algebraic endofunctor generates an algebraic free monad. $T^{\bullet}$ is clearly a conservative amnestic isofibration, and so to apply Beck's monadicity theorem (theorem 1.2.11) we need only check that $T^{\bullet}$ creates coequalisers for reflexive $T^{\bullet}$-split pairs.

Let $(\mathrm{id}, \varphi), (\mathrm{id}, \psi) : \mathbb{B} \to \mathbb{C}$ be a reflexive pair in $\mathbf{AlgMon}(\mathcal{S})$, and suppose the diagram below is a split reflexive coequaliser diagram in $\mathbf{AlgEnd}(\mathcal{S})$:

$$\begin{tikzcd}
T^{\mathbb{C}} \arrow[r, shift left=2, "\varphi"] \arrow[r, leftarrow] \arrow[r, shift right=2, "\psi"'] \arrow[r, leftarrow, bend right=50, "\theta"'] & T^{\mathbb{B}} \arrow[r, "\chi"] \arrow[r, leftarrow, bend right=40, "\sigma"'] & T^{\mathbb{A}}
\end{tikzcd}$$

We shall exhibit a monad structure on $T^{\mathbb{A}}$, which is *a priori* only an algebraic endofunctor. Let $\rho : T^{\mathbb{B}} \Rightarrow T^{\mathbb{C}}$ be the common splitting of φ and ψ, and define two natural transformations $\eta^{\mathbb{A}} : \mathrm{id}_{\mathcal{S}} \Rightarrow T^{\mathbb{A}}$ and $\mu^{\mathbb{A}} : T^{\mathbb{A}} T^{\mathbb{A}} \Rightarrow T^{\mathbb{A}}$:

$$\eta^{\mathbb{A}} = \chi \bullet \eta^{\mathbb{B}} \qquad\qquad \mu^{\mathbb{A}} = \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma)$$

Here, $\circ$ denotes the horizontal composition of natural transformations. One may verify that

$$\begin{aligned}
\chi \bullet \mu^{\mathbb{B}} \bullet T^{\mathbb{B}}(\sigma \bullet \chi) &= \chi \bullet \mu^{\mathbb{B}} \bullet T^{\mathbb{B}}(\varphi \bullet \theta) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet ((\varphi \bullet \rho) \circ (\varphi \bullet \theta)) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\varphi \circ \varphi) \bullet (\rho \circ \theta) \\
&= \chi \bullet \varphi \bullet \mu^{\mathbb{C}} \bullet (\rho \circ \theta) \\
&= \chi \bullet \psi \bullet \mu^{\mathbb{C}} \bullet (\rho \circ \theta)
\end{aligned}$$

$$\begin{aligned}
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\psi \circ \psi) \bullet (\rho \circ \theta) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet ((\psi \bullet \rho) \circ (\psi \bullet \theta)) \\
&= \chi \bullet \mu^{\mathbb{B}}
\end{aligned}$$

and a similar argument shows $\chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \bullet \chi) T^{\mathbb{B}} = \chi \bullet \mu^{\mathbb{B}}$; thus, we see that $(T^{\mathbb{A}}, \eta^{\mathbb{A}}, \mu^{\mathbb{A}})$ satisfy the monad axioms:

$$\begin{aligned}
\mu^{\mathbb{A}} \bullet T^{\mathbb{A}} \eta^{\mathbb{A}} &= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma) \bullet T^{\mathbb{A}} \chi \bullet T^{\mathbb{A}} \eta^{\mathbb{B}} \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet \left(\sigma \circ \left(\sigma \bullet \chi \bullet \eta^{\mathbb{B}}\right)\right) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet T^{\mathbb{B}} (\sigma \bullet \chi) \bullet \left(\sigma \circ \eta^{\mathbb{B}}\right) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet \left(\sigma \circ \eta^{\mathbb{B}}\right) \\
&= \chi \bullet \sigma = \mathrm{id}
\end{aligned}$$

$$\begin{aligned}
\mu^{\mathbb{A}} \bullet \eta^{\mathbb{A}} T^{\mathbb{A}} &= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma) \bullet \chi T^{\mathbb{A}} \bullet \eta^{\mathbb{B}} T^{\mathbb{A}} \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet \left(\left(\sigma \bullet \chi \bullet \eta^{\mathbb{B}}\right) \circ \sigma\right) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \bullet \chi) T^{\mathbb{B}} \bullet \left(\eta^{\mathbb{B}} \circ \sigma\right) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet \left(\eta^{\mathbb{B}} \circ \sigma\right) \\
&= \chi \bullet \sigma = \mathrm{id}
\end{aligned}$$

$$\begin{aligned}
\mu^{\mathbb{A}} \bullet T^{\mathbb{A}} \mu^{\mathbb{A}} &= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma) \bullet T^{\mathbb{A}} \left(\chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma)\right) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet T^{\mathbb{B}} (\sigma \circ \chi) \bullet T^{\mathbb{B}} \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma \circ \sigma) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet T^{\mathbb{B}} \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma \circ \sigma) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet \mu^{\mathbb{B}} T^{\mathbb{B}} \bullet (\sigma \circ \sigma \circ \sigma) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \chi) T^{\mathbb{B}} \bullet \mu^{\mathbb{B}} T^{\mathbb{B}} \bullet (\sigma \circ \sigma \circ \sigma) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma) \bullet \left(\chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma)\right) T^{\mathbb{A}} \\
&= \mu^{\mathbb{A}} \bullet \mu^{\mathbb{A}} T^{\mathbb{A}}
\end{aligned}$$

Note also that $\chi \bullet \eta^{\mathbb{B}} = \eta^{\mathbb{A}}$ by construction, and

$$\begin{aligned}
\mu^{\mathbb{A}} \bullet (\chi \circ \chi) &= \chi \bullet \mu^{\mathbb{B}} \bullet (\sigma \circ \sigma) \bullet (\chi \circ \chi) \\
&= \chi \bullet \mu^{\mathbb{B}} \bullet (\varphi \circ \varphi) \bullet (\theta \circ \theta) \\
&= \chi \bullet \varphi \bullet \mu^{\mathbb{C}} \bullet (\theta \circ \theta) \\
&= \chi \bullet \psi \bullet \mu^{\mathbb{C}} \bullet (\theta \circ \theta) \\
&= \chi \bullet \mu^{\mathbb{B}} \circ (\psi \circ \psi) \bullet (\theta \circ \theta) \\
&= \chi \bullet \mu^{\mathbb{B}}
\end{aligned}$$

as required for $\chi : T^{\mathbb{B}} \to T^{\mathbb{A}}$ to be a monad morphism $(\mathrm{id}, \chi) : \mathbb{A} \to \mathbb{B}$. We claim this is the equaliser of (id, φ) and (id, ψ) in $\mathbf{AlgMon}(\mathcal{S})$. Since $T^{\mathbb{A}}$ is defined as a coequaliser, to give an $\mathbb{A}$-module structure $\alpha : T^{\mathbb{A}} X \to X$ is the same as giving a morphism $\beta : T^{\mathbb{B}} X \to X$ in $\mathcal{S}$ such that $\alpha \circ \chi_X = \beta$ and $\beta \circ \varphi_X = \beta \circ \psi_X$. Since χ is a morphism of monads, $\beta : T^{\mathbb{B}} X \to X$ is automatically a $\mathbb{B}$-module structure. Conversely, if $\beta : T^{\mathbb{B}} X \to X$ is a $\mathbb{B}$-module structure such that $\beta \circ \varphi_X = \beta \circ \psi_X$, then there is a unique morphism $\alpha : T^{\mathbb{A}} X \to X$ such that $\alpha \circ \chi_X = \beta$, and α is an $\mathbb{A}$-module structure:

$$\alpha \circ \eta^{\mathbb{A}}_X = (\alpha \circ \chi_X) \circ \eta^{\mathbb{B}}_X = \beta \circ \eta^{\mathbb{B}}_X = \mathrm{id}_X$$

$$\begin{aligned}
\alpha \circ \mu^{\mathbb{A}}_X &= (\alpha \circ \chi_X) \circ \mu^{\mathbb{B}}_X \circ T^{\mathbb{B}} \sigma_X \circ \sigma_{T^{\mathbb{A}} X} \\
&= \left(\beta \circ \mu^{\mathbb{B}}_X\right) \circ T^{\mathbb{B}} \sigma_X \circ \sigma_{T^{\mathbb{A}} X} \\
&= \beta \circ T^{\mathbb{B}} \beta \circ T^{\mathbb{B}} \sigma_X \circ \sigma_{T^{\mathbb{A}} X} \\
&= \beta \circ T^{\mathbb{B}} \alpha \circ \left(T^{\mathbb{B}} \chi_X \circ T^{\mathbb{B}} \sigma_X\right) \circ \sigma_{T^{\mathbb{A}} X} \\
&= \beta \circ T^{\mathbb{B}} \alpha \circ \sigma_{T^{\mathbb{A}} X} \\
&= \alpha \circ \left(\chi_X \circ T^{\mathbb{B}} \alpha\right) \circ \sigma_{T^{\mathbb{A}} X} \\
&= \alpha \circ T^{\mathbb{A}} \alpha \circ \left(\chi_{T^{\mathbb{A}} X} \circ \sigma_{T^{\mathbb{A}} X}\right) \\
&= \alpha \circ T^{\mathbb{A}} \alpha
\end{aligned}$$

Hence, we have an equaliser diagram of categories over $\mathcal{S}$,

$$\begin{tikzcd}
\mathcal{S}^{\mathbb{A}} \arrow[r] & \mathcal{S}^{\mathbb{B}} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{S}^{\mathbb{C}}
\end{tikzcd}$$

and therefore the corresponding diagram of monads is also an equaliser diagram by proposition 1.4.6. ■

Proposition 2.6.17. *Let $\mathcal{S}$ be an l.f.p. category, and let* $\mathbf{FinEnd}(\mathcal{S})$ *be the full subcategory of* $[\mathcal{S}, \mathcal{S}]$ *spanned by the finitary endofunctors on* $\mathcal{S}$*.*

(i) *Every finitary endofunctor on $\mathcal{S}$ generates a finitary free monad as in Barr's theorem (1.6.5).*

(ii) *The forgetful functor* $T^{\bullet} : \mathbf{FinMon}(\mathcal{S})^{\mathrm{op}} \to \mathbf{FinEnd}(\mathcal{S})$ *is strictly monadic.*

Proof. The previous proof also works here. ■

Corollary 2.6.18. *Let $\mathcal{S}$ be an l.f.p. category.*

(i) **FinMon**($\mathcal{S}$) *has colimits for all small diagrams, and the forgetful functor* $T^\bullet : \mathbf{FinMon}(\mathcal{S})^{\mathrm{op}} \to \mathbf{FinEnd}(\mathcal{S})$ *maps small colimits in* **FinMon**($\mathcal{S}$) *to the corresponding limits in* **FinEnd**($\mathcal{S}$).

(ii) *Colimits for finite diagrams in* **FinMon**($\mathcal{S}$) *can be computed as the corresponding (componentwise) limits in* $[\mathcal{S}, \mathcal{S}]$.

Proof. (i). By proposition 1.2.5, it is enough to show that **FinEnd**($\mathcal{S}$) has all small limits. Since $\mathcal{S}$ is complete, small limits exist in $[\mathbf{K}(\mathcal{S}), \mathcal{S}]$ and can be computed componentwise. Proposition 2.6.7 then implies **FinEnd**($\mathcal{S}$) is also complete.

(ii). Proposition 2.2.21 says small filtered colimits preserve finite limits in $\mathcal{S}$, so the left Kan extension functor $\mathrm{Lan}_{\mathbf{K}(\mathcal{S}) \hookrightarrow \mathcal{S}} : [\mathbf{K}(\mathcal{S}), \mathcal{S}] \to [\mathcal{S}, \mathcal{S}]$ preserves finite limits; thus finite limits in **FinEnd**($\mathcal{S}$) can be computed componentwise. ■

We cannot say as much about colimits in **AlgMon**($\mathcal{S}$) because algebraic endofunctors are less well-behaved with respect to finite limits. On the other hand, while it *seems* as if we cannot say much about limits in **FinMon**($\mathcal{S}$), we can at least prove they exist. We shall demonstrate that **FinMon**($\mathcal{S}$) is both complete and cocomplete as a corollary of a result of Kelly and Power [1993, § 4], which basically says that a finitary monad on $\mathcal{S}$ is the same thing as a module for a certain finitary monad on **FinEnd**($\mathcal{S}$).

Lemma 2.6.19. *If $\mathcal{J}$ is a filtered category, then the diagonal functor $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ is cofinal.*

Proof. Let i and j be objects in $\mathcal{J}$. Since $\mathcal{J}$ is filtered, there are arrows $f : i \to k$ and $g : j \to k$ in $\mathcal{J}$, hence, an arrow $(f, g) : (i, j) \to \Delta k$ in $\mathcal{J} \times \mathcal{J}$. Moreover, given any other arrow $(f', g') : (i, j) \to \Delta k'$ in $\mathcal{J} \times \mathcal{J}$, filteredness of $\mathcal{J}$ allows us to construct two commutative squares

$$\begin{tikzcd}
i \arrow[r, "f'"] \arrow[d, "f"'] & k' \arrow[d] \\
k \arrow[r] & k''
\end{tikzcd}
\qquad
\begin{tikzcd}
i \arrow[r, "f'"] \arrow[d, "f"'] & k' \arrow[d] \\
k \arrow[r] & k''
\end{tikzcd}$$

as required to make $\Delta : \mathcal{J} \to \mathcal{J} \times \mathcal{J}$ a cofinal functor. ■

Theorem 2.6.20 (Kelly–Power)**.** *Let $\mathcal{S}$ be an l.f.p. category.*

(i) *If $\mathcal{J}$ is a small filtered category and $\mathbb{T}_\bullet = (T_\bullet, \eta_\bullet, \mu_\bullet) : \mathcal{J}^{\mathrm{op}} \to$ **FinMon**$(\mathcal{S})$ is a diagram, then $\bar{\mathbb{T}} = \left(\bar{T}, \bar{\eta}, \bar{\mu}\right) = \varprojlim_{\mathcal{J}^{\mathrm{op}}} \mathbb{T}_\bullet$ exists in **FinMon**$(\mathcal{S})$ and $\mathcal{S}^{\bar{\mathbb{T}}}$ is isomorphic to the limit of the diagram $\mathcal{S}^{\mathbb{T}_\bullet}$. In particular, the forgetful functor $T^\bullet :$ **FinMon**$(\mathcal{S})^{\mathrm{op}} \to$ **FinEnd**$(\mathcal{S})$ creates colimits for small filtered diagrams.*

(ii) *The free finitary monad monad on* **FinEnd**$(\mathcal{S})$ *is a finitary monad.*

(iii) **FinMon**$(\mathcal{S})^{\mathrm{op}}$ *is an l.f.p. category.*

Proof. **FinEnd**$(\mathcal{S})$ is an l.f.p. category by proposition 2.6.7, and by proposition 2.6.17, $T^\bullet$: **FinMon**$(\mathcal{S})^{\mathrm{op}} \to$ **FinEnd**$(\mathcal{S})$ is strictly monadic, so to prove claims (ii) and (iii), we just check claim (i) and apply theorem 2.6.5.

Let $\bar{T} = \varinjlim_{\mathcal{J}} T_\bullet$, and let $\lambda_\bullet : T_\bullet \Rightarrow \bar{T}$ be the colimiting cocone. Since each T_j is a finitary endofunctor, the lemma implies

$$\bar{T}^2 = \bar{T}\bar{T} = \varinjlim_{\mathcal{J}} T_\bullet \left(\varinjlim_{\mathcal{J}} T_\bullet \right) \cong \varinjlim_{\mathcal{J}\times\mathcal{J}} T_\bullet T_\bullet \cong \varinjlim_{\mathcal{J}} T_\bullet^2$$

and colimits preserve colimits, so $\bar{T}$ is a finitary endofunctor. Thus, we may define $\bar{\mu} : \bar{T}^2 \Rightarrow \bar{T}$ to be the unique natural transformation making the diagram below commute for each j in $\mathcal{J}$:

$$\begin{tikzcd}
T_j^2 \arrow[r, "\lambda_j \circ \lambda_j"] \arrow[d, "\mu_j"'] & \bar{T}^2 \arrow[d, dashed, "\bar{\mu}"] \\
T_j \arrow[r, "\lambda_j"'] & \bar{T}
\end{tikzcd}$$

Here, $\circ$ denotes the horizontal composition of natural transformations. This makes sense since the diagram

$$\begin{tikzcd}
T_i^2 \arrow[r, "T_f \circ T_f"] \arrow[d, "\mu_i"'] & T_j^2 \arrow[d, "\mu_j"] \\
T_i \arrow[r, "T_f"'] & T_j
\end{tikzcd}$$

commutes for any arrow $f : i \to j$ in $\mathcal{J}$. We define $\bar{\eta} : \mathrm{id}_{\mathcal{S}} \Rightarrow \bar{T}$ by setting

$$\bar{\eta} = \lambda_j \bullet \eta_j$$

for an arbitrary object j in $\mathcal{J}$; this makes sense because $\mathcal{J}$ is inhabited and

$$\begin{tikzcd}
\mathrm{id}_{\mathcal{S}} \arrow[r, "\eta_i"] \arrow[dr, "\eta_j"'] & T_i \arrow[d, "T_f"] \arrow[dr, "\lambda_i"] & \\
& T_j \arrow[r, "\lambda_j"'] & \bar{T}
\end{tikzcd}$$

commutes for any arrow $f : i \to j$ in $\mathcal{J}$. We claim $\bar{\mathbb{T}} = \left(\bar{T}, \bar{\mu}, \bar{\eta}\right)$ is a monad. Indeed:

$$\begin{aligned}
\left(\bar{\mu} \bullet \bar{T}\bar{\eta}\right) \bullet \lambda_j &= \bar{\mu} \bullet \lambda_j \bar{T} \bullet T_j \bar{\eta} \\
&= \bar{\mu} \bullet \left(\lambda_j \bar{T} \bullet T_j \lambda_j\right) \bullet T_j \eta_j \\
&= \bar{\mu} \bullet \left(\lambda_j \circ \lambda_j\right) \bullet T_j \eta_j \\
&= \lambda_j \bullet \left(\mu_j \circ T_j \eta_j\right) \\
&= \lambda_j
\end{aligned}$$

$$\begin{aligned}
\left(\bar{\mu} \bullet \bar{\eta}\bar{T}\right) \bullet \lambda_j &= \bar{\mu} \bullet \bar{T}\lambda_j \bullet \bar{\eta} T_j \\
&= \bar{\mu} \bullet \left(\bar{T}\lambda_j \bullet \lambda_j T_j\right) \bullet \eta_j T_j \\
&= \bar{\mu} \bullet \left(\lambda_j \circ \lambda_j\right) \bullet \eta_j T_j \\
&= \lambda_j \bullet \left(\mu_j \bullet \eta_j T_j\right) \\
&= \lambda_j
\end{aligned}$$

$$\begin{aligned}
\left(\bar{\mu} \bullet \bar{T}\bar{\mu}\right) \bullet \left(\lambda_j \circ \lambda_j \circ \lambda_j\right) &= \bar{\mu} \bullet \lambda_j \bar{T} \bullet T_j \bar{\mu} \bullet T_j \left(\lambda_j \circ \lambda_j\right) \\
&= \bar{\mu} \bullet \left(\lambda_j \bar{T} \bullet T_j \lambda_j\right) \bullet T_j \mu_j \\
&= \bar{\mu} \bullet \left(\lambda_j \circ \lambda_j\right) \bullet T_j \mu_j \\
&= \lambda_j \bullet \left(\mu_j \bullet T_j \mu_j\right) \\
&= \lambda_j \bullet \left(\mu_j \bullet \mu_j T_j\right) \\
&= \bar{\mu} \bullet \left(\lambda_j \circ \lambda_j\right) \bullet \mu_j T_j \\
&= \bar{\mu} \bullet \left(\bar{T}\lambda_j \bullet \lambda_j T_j\right) \bullet \mu_j T_j \\
&= \bar{\mu} \bullet \bar{T}\lambda_j \bullet \bar{\mu} T_j \bullet \left(\lambda_j \circ \lambda_j\right) T_j \\
&= \left(\bar{\mu} \bullet \bar{\mu}\bar{T}\right) \bullet \left(\lambda_j \circ \lambda_j \circ \lambda_j\right)
\end{aligned}$$

Hence, $\bar{\mu} \bullet \bar{T}\bar{\eta} = \mathrm{id}$, $\bar{\mu} \bullet \bar{\eta}\bar{T} = \mathrm{id}$, and $\bar{\mu} \bullet \bar{T}\bar{\mu} = \bar{\mu} \bullet \bar{\mu}\bar{T}$, and so $\bar{\mathbb{T}}$ is a finitary monad. Note that each λ_j defines a monad morphism $\left(\mathrm{id}, \lambda_j\right) : \bar{\mathbb{T}} \to \mathbb{T}_j$ by construction of $\bar{\eta}$ and $\bar{\mu}$.

Since $\bar{T}$ is defined by a colimit, giving a morphism $\bar{\alpha} : \bar{T}X \to X$ is the same thing as giving a family $\left(\alpha_j : T_j X \to X \,\middle|\, j \in \mathrm{ob}\,\mathcal{J}\right)$ such that $\alpha_j = \bar{\alpha} \circ \lambda_j$ for each j; since each $\left(\mathrm{id}, \lambda_j\right)$ is a monad morphism, if $\bar{\alpha} : \bar{T}X \to X$ is a $\bar{\mathbb{T}}$-module structure, then each $\alpha_j : T_j X \to X$ is a $\mathbb{T}_j$-module structure. Conversely, given a $\mathbb{T}_j$-module structure α_j for each j, if $\alpha_j \circ \left(T_f\right)_X = \alpha_i$ for each arrow $f : i \to j$ in $\mathcal{J}$, then there is a unique morphism $\bar{\alpha} : \bar{T}X \to X$ such that $\alpha_j = \bar{\alpha} \circ \lambda_j$ for each j, and we have

$$\bar{\alpha} \circ \bar{\eta}_X = \bar{\alpha} \circ \lambda_j \circ \left(\eta_j\right)_X = \alpha_j \circ \left(\eta_j\right)_X = \mathrm{id}_X$$

$$\begin{aligned} \bar{\alpha} \circ \bar{\mu}_X \circ \left(\lambda_j \circ \lambda_j\right)_X &= \bar{\alpha} \circ \left(\lambda_j\right)_X \circ \left(\mu_j\right)_X \\ &= \alpha_j \circ \left(\mu_j\right)_X \\ &= \alpha_j \circ T_j \alpha_j \\ &= \bar{\alpha} \circ \left(\lambda_j\right)_X \circ T_j \bar{\alpha} \circ \left(T_j \lambda_j\right)_X \\ &= \bar{\alpha} \circ \bar{T}\bar{\alpha} \circ \left(\lambda_j \circ \lambda_j\right)_X \end{aligned}$$

but $\left(\left(\lambda_j \circ \lambda_j\right)_X : T_j^2 X \to \bar{T}^2 X \,\middle|\, j \in \mathrm{ob}\,\mathcal{J}\right)$ is jointly epimorphic, so we also have $\bar{\alpha} \circ \bar{\mu}_X = \bar{\alpha} \circ \bar{T}\bar{\alpha}$, as required for $\bar{\alpha}$ to be a $\bar{\mathbb{T}}$-module structure. Hence, $\mathcal{S}^{\bar{\mathbb{T}}}$ is isomorphic to the limit of the diagram $\mathcal{S}^{\mathbb{T}_\bullet}$, and $\bar{\mathbb{T}}$ is the limit of the diagram $\mathbb{T}_\bullet$ by proposition 1.4.6. ■

Corollary 2.6.21. *For any l.f.p. category $\mathcal{S}$,* **FinMon**($\mathcal{S}$) *has limits and colimits for all small diagrams.*

Proof. Use corollaries 2.1.15 and 2.2.19. ■

Remark 2.6.22. In fact, small limits in **FinMon**($\mathcal{S}$) can be computed the same way as in **AlgMon**($\mathcal{S}$). The proof is the essentially the same as that of corollary 1.6.16, though where we used the codensity monad we must replace it with its finitary coreflection.

7 Distributive laws

The theory of rings can be succinctly described as the union of the theory of abelian groups and the theory of monoids, plus a "distributive law" that controls the interaction between the two theories.

Can we see this in terms of monads? We can—and quite straightforwardly too. Let $\mathbb{T}^{\Sigma} = (\Sigma, \eta^{\Sigma}, \mu^{\Sigma})$ be the free abelian group monad on **Set**, and $\mathbb{T}^{\Pi} = (\Pi, \eta^{\Pi}, \mu^{\Pi})$ be the free monoid monad on **Set**. The distributivity of products over sums is a natural transformation $\nu : \Pi\Sigma \Rightarrow \Sigma\Pi$, and ν moreover has properties which endow the composite $\Sigma\Pi$ with a canonical monad structure making it the free ring monad. This is readily generalised.

Definition 2.7.1. Let $\mathbb{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{C}$, and let J be an endofunctor on $\mathcal{C}$. A **distributive law** for $\mathbb{T}$ over J is a natural transformation $\nu : TJ \Rightarrow JT$ such that (J, ν) constitute a monad morphism $\mathbb{T} \to \mathbb{T}$.

Remark 2.7.2. Proposition 1.4.6 implies that there is a bijection between distributive laws for $\mathbb{T}$ over J and functors $\tilde{J} : \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$ such that $U^{\mathbb{T}} \tilde{J} = J U^{\mathbb{T}}$.

Example 2.7.3. If $\mathcal{C}$ has binary products, then the endofunctor J defined by $JX = X \times X$ certainly can be lifted to an endofunctor $\tilde{J}$ on $\mathcal{C}^{\mathbb{T}}$ for any monad $\mathbb{T}$ on $\mathcal{C}$, so $\mathbb{T}$ distributes over J. A $\tilde{J}$-module is then a $\mathbb{T}$-module equipped with a homomorphic binary operation: for example, if $\mathcal{C} = \mathbf{Set}$ and $\mathbb{T}$ is the free group monad, then a $\tilde{J}$-module is a group $(G, e, \cdot)$ equipped with an extra binary operation $*$ such that these two equations hold:

$$e * e = e$$
$$(g \cdot g') * (h \cdot h') = (g * h) \cdot (g' * h')$$

The second equation is also known as the **middle-four interchange law**. We will see it again in the next chapter: see remark 3.4.6.

The following definition is the main focus of this section, and is originally due to Beck [1969]:

Definition 2.7.4. Let $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ and $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ be two monads on $\mathcal{C}$. A **distributive law** for $\mathbb{T}$ over $\mathbb{S}$ is a natural transformation $\nu : TS \Rightarrow ST$ such that the four equations below hold:

$$\nu \bullet \eta^{\mathbb{T}} S = S\eta^{\mathbb{T}} \qquad \nu \bullet \mu^{\mathbb{T}} S = S\mu^{\mathbb{T}} \bullet \nu T \bullet T\nu$$

$$\nu \bullet T\eta^{\mathbb{S}} = \eta^{\mathbb{S}} T \qquad \nu \bullet T\mu^{\mathbb{S}} = \mu^{\mathbb{S}} T \bullet S\nu \bullet \nu S$$

Remark 2.7.5. The first pair of equations say that (S, ν) is a monad morphism $\mathbb{T} \to \mathbb{T}$, so a distributive law for $\mathbb{T}$ over the monad $\mathbb{S}$ is in particular a distributive law for $\mathbb{T}$ over the endofunctor S. The second pair of equations tell us that $\eta^{\mathbb{S}} : (\mathrm{id}, \mathrm{id}_T) \Rightarrow (S, \nu)$ and $\mu^{\mathbb{S}} : (S^2, S\nu \bullet \nu S) \Rightarrow (S, \nu)$ are transformations of monad morphisms.

Theorem 2.7.6. *Let* $\mathbb{T} = (T, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ *and* $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ *be monads on a category* C*, and let* $\nu : TS \Rightarrow ST$ *be a distributive law for* $\mathbb{T}$ *over* S*.*

(i) $\mathbb{ST} = (ST, \eta^{\mathbb{ST}}, \mu^{\mathbb{ST}})$ *defines a monad on* C*, where the unit and multiplication are as given below:*

$$\eta^{\mathbb{ST}} = \eta^{\mathbb{S}} \circ \eta^{\mathbb{T}} \qquad \mu^{\mathbb{ST}} = \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet S\nu T$$

Here, $\circ$ *denotes the horizontal composition of natural transformations.*

(ii) *The units of* $\mathbb{T}$ *and* $\mathbb{S}$ *induce monad morphisms* $(\mathrm{id}, S\eta^{\mathbb{T}}) : \mathbb{ST} \to \mathbb{S}$ *and* $(\mathrm{id}, \eta^{\mathbb{S}} T) : \mathbb{ST} \to \mathbb{T}$*, giving a commutative diagram of forgetful functors:*

$$\begin{tikzcd}
C^{\mathbb{ST}} \arrow[r, "U^{\mathbb{ST}}_{\mathbb{S}}"] \arrow[d, "U^{\mathbb{ST}}_{\mathbb{T}}"'] & C^{\mathbb{S}} \arrow[d, "U^{\mathbb{S}}"] \\
C^{\mathbb{T}} \arrow[r, "U^{\mathbb{T}}"'] & C
\end{tikzcd}$$

(iii) *The functor* $U^{\mathbb{ST}}_{\mathbb{T}} : C^{\mathbb{ST}} \to C^{\mathbb{T}}$ *is strictly monadic, and the diagram above satisfies the strict Beck–Chevalley condition.*

Proof. (i). The proof of this claim is straightforward calculation. For right unitality, we have

$$\begin{aligned}
\mu^{\mathbb{ST}} \bullet ST\eta^{\mathbb{ST}} &= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet S\nu T \bullet ST\left(\eta^{\mathbb{S}} \circ \eta^{\mathbb{T}}\right) \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \left(\mathrm{id}_S \circ \eta^{\mathbb{S}} \circ \mathrm{id}_T \circ \eta^{\mathbb{T}}\right) \\
&= \mathrm{id}_S \circ \mathrm{id}_T = \mathrm{id}_{ST}
\end{aligned}$$

and a similar calculation gives us left unitality:

$$\mu^{\mathbb{S}\mathbb{T}} \bullet \eta^{\mathbb{S}\mathbb{T}} ST = \mathrm{id}_{ST}$$

For associativity, first observe that

$$\begin{aligned}
S\nu T \bullet ST\left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) &= S\left(\nu \bullet T\mu^{\mathbb{S}}\right) T \bullet STS^2\mu^{\mathbb{T}} \\
&= S\left(\mu^{\mathbb{S}} T \bullet S\nu \bullet \nu S\right) T \bullet STS^2\mu^{\mathbb{T}} \\
&= S\mu^{\mathbb{S}} T^2 \bullet S^2\nu T \bullet \left(S\nu ST \bullet STS^2\mu^{\mathbb{T}}\right) \\
&= S\mu^{\mathbb{S}} T^2 \bullet \left(S^2\nu T \bullet S^2 TS\mu^{\mathbb{T}}\right) \bullet S\nu ST^2 \\
&= \left(S\mu^{\mathbb{S}} T^2 \bullet S^3 T\mu^{\mathbb{T}}\right) \bullet \left(S^2\nu \bullet S\nu S\right) T^2 \\
&= \left(\mathrm{id}_S \circ \mu^{\mathbb{S}} \circ \mathrm{id}_T \circ \mu^{\mathbb{T}}\right) \bullet \left(S^2\nu \bullet S\nu S\right) T^2
\end{aligned}$$

and similarly

$$S\nu T \bullet \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right)ST = \left(\mu^{\mathbb{S}} \circ \mathrm{id}_S \circ \mu^{\mathbb{T}} \circ \mathrm{id}_T\right) \bullet S^2\left(\nu T^2 \bullet T\nu T\right)$$

but naturality of ν implies

$$S^2\nu T^2 \bullet S\nu ST^2 \bullet STS\nu T = S^2\nu T^2 \bullet S(\nu \circ \nu)T = S^2\nu T^2 \bullet S^2 T\nu T \bullet S\nu TST$$

and so:

$$\begin{aligned}
\mu^{\mathbb{S}\mathbb{T}} \bullet ST\mu^{\mathbb{S}\mathbb{T}} &= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet S\nu T \bullet ST\left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet STS\nu T \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \left(\mathrm{id}_S \circ \mu^{\mathbb{S}} \circ \mathrm{id}_T \circ \mu^{\mathbb{T}}\right) \bullet \left(S^2\nu T^2 \bullet S\nu ST^2 \bullet STS\nu T\right) \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \left(\mu^{\mathbb{S}} \circ \mathrm{id}_S \circ \mu^{\mathbb{T}} \circ \mathrm{id}_T\right) \bullet \left(S^2\nu T^2 \bullet S^2 T\nu T \bullet S\nu TST\right) \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet S\nu T \bullet \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right)ST \bullet S\nu TST \\
&= \mu^{\mathbb{S}\mathbb{T}} \bullet \mu^{\mathbb{S}\mathbb{T}} ST
\end{aligned}$$

(ii). This is again a straightforward calculation. First, note that

$$S\eta^{\mathbb{T}} \bullet \eta^{\mathbb{S}} = \eta^{\mathbb{S}\mathbb{T}} = \eta^{\mathbb{S}} T \bullet \eta^{\mathbb{T}}$$

by definition, so we only need to check that $S\eta^{\mathbb{T}}$ and $\eta^{\mathbb{S}} T$ are compatible with the monad operations:

$$\begin{aligned}
\mu^{\mathbb{S}\mathbb{T}} \bullet S\eta^{\mathbb{T}} ST \bullet S^2\eta^{\mathbb{T}} &= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \left(S\nu T \bullet S\eta^{\mathbb{T}} ST\right) \bullet S^2\eta^{\mathbb{T}} \\
&= \mu^{\mathbb{S}} T \bullet S^2\left(\mu^{\mathbb{T}} \bullet \eta^{\mathbb{T}} T \bullet \eta^{\mathbb{T}}\right) \\
&= \mu^{\mathbb{S}} T \bullet S^2\eta^{\mathbb{T}} \\
&= S\eta^{\mathbb{T}} \bullet \mu^{\mathbb{S}}
\end{aligned}$$

$$\begin{aligned}
\mu^{\mathbb{S}\mathbb{T}} \bullet \eta^{\mathbb{S}} TST \bullet T\eta^{\mathbb{S}} T &= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \left(S\nu T \bullet \eta^{\mathbb{S}} TST\right) \bullet T\eta^{\mathbb{S}} T \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \eta^{\mathbb{S}} ST^2 \bullet \left(\nu \bullet T\eta^{\mathbb{S}}\right) T \\
&= \left(\mu^{\mathbb{S}} \circ \mu^{\mathbb{T}}\right) \bullet \eta^{\mathbb{S}} ST^2 \bullet \eta^{\mathbb{S}} T^2 \\
&= S\mu^{\mathbb{T}} \bullet \left(\mu^{\mathbb{S}} \bullet \eta^{\mathbb{S}} S \bullet \eta^{\mathbb{S}}\right) T^2 \\
&= S\mu^{\mathbb{T}} \bullet \eta^{\mathbb{S}} T^2 \\
&= \eta^{\mathbb{S}} T \bullet \mu^{\mathbb{T}}
\end{aligned}$$

Thus $(\mathrm{id}, S\eta^{\mathbb{T}}) : \mathbb{S}\mathbb{T} \to \mathbb{S}$ and $(\mathrm{id}, \eta^{\mathbb{S}} T) : \mathbb{S}\mathbb{T} \to \mathbb{T}$ are indeed morphisms of monads.

(iii). As earlier remarked, $(S, \nu) : \mathbb{T} \to \mathbb{T}$ is a morphism of monads, so proposition 1.4.5 gives us a functor $S^{\nu} : C^{\mathbb{T}} \to C^{\mathbb{T}}$ such that $U^{\mathbb{T}} S^{\nu} = SU^{\mathbb{T}}$, while proposition 1.4.11 gives us a pair of natural transformations $\eta^{\mathbb{S}\mathbb{T}}_{\mathbb{T}} : \mathrm{id}_{C^{\mathbb{T}}} \Rightarrow S^{\nu}$ and $\mu^{\mathbb{S}\mathbb{T}}_{\mathbb{T}} : S^{\nu} S^{\nu} \Rightarrow S^{\nu}$ such that $U^{\mathbb{T}} \eta^{\mathbb{S}\mathbb{T}}_{\mathbb{T}} = \eta U^{\mathbb{T}}$ and $U^{\mathbb{T}} \mu^{\mathbb{S}\mathbb{T}}_{\mathbb{T}} = \mu^{\mathbb{S}}$, and thus we have a monad $\mathbb{S}_{\mathbb{T}} = (S^{\nu}, \eta^{\mathbb{S}\mathbb{T}}_{\mathbb{T}}, \mu^{\mathbb{S}\mathbb{T}}_{\mathbb{T}})$.

Notice that a $\mathbb{S}_{\mathbb{T}}$-module is a triplet $(A, \alpha^{\mathbb{T}}, \alpha^{\mathbb{S}})$ such that these three diagrams commute:

$$\begin{tikzcd}
T^2A \arrow[r, "T\alpha^{\mathbb{T}}"] \arrow[d, "\mu^{\mathbb{T}}_A"'] & TA \arrow[d, "\alpha^{\mathbb{T}}"] \\
TA \arrow[r, "\alpha^{\mathbb{T}}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
TSA \arrow[r, "T\alpha^{\mathbb{S}}"] \arrow[d, "S\alpha^{\mathbb{T}} \circ \nu_A"'] & TA \arrow[d, "\alpha^{\mathbb{T}}"] \\
SA \arrow[r, "\alpha^{\mathbb{S}}"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
S^2A \arrow[r, "S\alpha^{\mathbb{S}}"] \arrow[d, "\mu^{\mathbb{S}}_A"'] & SA \arrow[d, "\alpha^{\mathbb{S}}"] \\
SA \arrow[r, "\alpha^{\mathbb{S}}"'] & A
\end{tikzcd}$$

We claim this data is equivalent to the data of a $\mathbb{S}\mathbb{T}$-module: if we define

$$\alpha^{\mathbb{S}\mathbb{T}} = \alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}}$$

then we have

$$\alpha^{\mathbb{S}\mathbb{T}} \circ \eta^{\mathbb{S}\mathbb{T}} = \alpha^{\mathbb{S}} \circ \left(S\alpha^{\mathbb{T}} \circ S\eta^{\mathbb{T}}_A\right) \circ \eta^{\mathbb{S}}_A = \alpha^{\mathbb{S}} \circ \eta^{\mathbb{S}}_A = \mathrm{id}_A$$

so the unit axiom is satisfied, and

$$\begin{aligned}
\alpha^{\mathbb{S}\mathbb{T}} \circ \mu^{\mathbb{S}\mathbb{T}} &= \alpha^{\mathbb{S}} \circ S\left(\alpha^{\mathbb{T}} \circ \mu^{\mathbb{T}}_A\right) \circ \mu^{\mathbb{S}}_{T^2A} \circ S\nu_{TA} \\
&= \left(\alpha^{\mathbb{S}} \circ \mu^{\mathbb{S}}_A\right) \circ S^2\left(\alpha^{\mathbb{T}} \circ \mu^{\mathbb{T}}_A\right) \circ S\nu_{TA} \\
&= \left(\alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{S}}\right) \circ S^2\left(\alpha^{\mathbb{T}} \circ T\alpha^{\mathbb{T}}\right) \circ S\nu_{TA} \\
&= \alpha^{\mathbb{S}} \circ S\left(\alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}} \circ \nu_A \circ TS\alpha^{\mathbb{T}}\right)
\end{aligned}$$

$$\begin{aligned}
&= \alpha^{\mathbb{S}} \circ S\left(\alpha^{\mathbb{T}} \circ T\alpha^{\mathbb{S}} \circ TS\alpha^{\mathbb{T}}\right) \\
&= \left(\alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}}\right) \circ ST\left(\alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}}\right) \\
&= \alpha^{\mathbb{ST}} \circ ST\alpha^{\mathbb{ST}}
\end{aligned}$$

thus $\alpha^{\mathbb{ST}}$ is a $\mathbb{ST}$-module structure. Given $\alpha^{\mathbb{ST}}$, we can recover $\alpha^{\mathbb{S}}$ and $\alpha^{\mathbb{T}}$:

$$\begin{aligned}
\alpha^{\mathbb{ST}} \circ S\eta^{\mathbb{T}}_A &= \alpha^{\mathbb{S}} \circ \left(S\alpha^{\mathbb{T}} \circ S\eta^{\mathbb{T}}_A\right) \\
&= \alpha^{\mathbb{S}}
\end{aligned}$$

$$\begin{aligned}
\alpha^{\mathbb{ST}} \circ \eta^{\mathbb{S}}_{TA} &= \alpha^{\mathbb{S}} \circ \left(S\alpha^{\mathbb{T}} \circ \eta^{\mathbb{S}}_{TA}\right) \\
&= \left(\alpha^{\mathbb{S}} \circ \eta^{\mathbb{S}}_A\right) \circ \alpha^{\mathbb{T}} \\
&= \alpha^{\mathbb{T}}
\end{aligned}$$

On the other hand, since $(\mathrm{id}, S\eta^{\mathbb{T}})$ and $(\mathrm{id}, \eta^{\mathbb{S}}T)$ are monad morphisms, any $\mathbb{ST}$-module structure $\alpha^{\mathbb{ST}}$ induces an $\mathbb{S}$-module structure

$$\alpha^{\mathbb{S}} = \alpha^{\mathbb{ST}} \circ S\eta^{\mathbb{T}}_A$$

and a $\mathbb{T}$-module structure

$$\alpha^{\mathbb{T}} = \alpha^{\mathbb{ST}} \circ \eta^{\mathbb{S}}_{TA}$$

and these suffice to recover $\alpha^{\mathbb{ST}}$:

$$\begin{aligned}
\alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}} &= \alpha^{\mathbb{ST}} \circ S\left(\eta^{\mathbb{T}}_A \circ \alpha^{\mathbb{ST}}\right) \circ S\eta^{\mathbb{S}}_{TA} \\
&= \alpha^{\mathbb{ST}} \circ S\left(T\alpha^{\mathbb{ST}} \circ \eta^{\mathbb{T}}_{STA} \circ \eta^{\mathbb{S}}_{TA}\right) \\
&= \alpha^{\mathbb{ST}} \circ \mu^{\mathbb{ST}}_A \circ S\eta^{\mathbb{T}}_{STA} \circ S\eta^{\mathbb{S}}_{TA} \\
&= \alpha^{\mathbb{ST}} \circ \left(\mu^{\mathbb{S}}_{TA} \circ S^2\mu^{\mathbb{T}}_A \circ S\nu_{TA}\right) \circ S\eta^{\mathbb{T}}_{STA} \circ S\eta^{\mathbb{S}}_{TA} \\
&= \alpha^{\mathbb{ST}} \circ \mu^{\mathbb{S}}_{TA} \circ S^2\left(\mu^{\mathbb{T}}_A \circ \eta^{\mathbb{T}}_{TA}\right) \circ S\eta^{\mathbb{S}}_{TA} \\
&= \alpha^{\mathbb{ST}} \circ \mu^{\mathbb{S}}_{TA} \circ S\eta^{\mathbb{S}}_{TA} \\
&= \alpha^{\mathbb{ST}}
\end{aligned}$$

The compatibility condition between $\alpha^{\mathbb{S}}$, $\alpha^{\mathbb{T}}$, and ν_A is also satisfied when

$\alpha^{\mathbb{ST}}$ is known to be a $\mathbb{ST}$-module structure:

$$\begin{aligned}
\alpha^{\mathbb{T}} \circ T\alpha^{\mathbb{S}} &= \alpha^{\mathbb{ST}} \circ \eta^{\mathbb{S}}_{TA} \circ T\alpha^{\mathbb{ST}} \circ TS\eta^{\mathbb{T}}_{A} \\
&= \alpha^{\mathbb{ST}} \circ ST\alpha^{\mathbb{ST}} \circ \eta^{\mathbb{S}}_{TSTA} \circ TS\eta^{\mathbb{T}}_{A} \\
&= \alpha^{\mathbb{ST}} \circ \mu^{\mathbb{ST}}_{A} \circ \eta^{\mathbb{S}}_{TSTA} \circ TS\eta^{\mathbb{T}}_{A} \\
&= \alpha^{\mathbb{ST}} \circ \left(\mu^{\mathbb{S}}_{TA} \circ S^2\mu^{\mathbb{T}}_{A} \circ S\nu_{TA}\right) \circ \eta^{\mathbb{S}}_{TSTA} \circ TS\eta^{\mathbb{T}}_{A} \\
&= \alpha^{\mathbb{ST}} \circ S\mu^{\mathbb{T}}_{A} \circ \left(\mu^{\mathbb{S}}_{T^2A} \circ \eta^{\mathbb{S}}_{ST^2A}\right) \circ \left(\nu_{TA} \circ TS\eta^{\mathbb{T}}_{A}\right) \\
&= \alpha^{\mathbb{ST}} \circ \left(S\mu^{\mathbb{T}}_{A} \circ ST\eta^{\mathbb{T}}_{A}\right) \circ \nu_{A} \\
&= \alpha^{\mathbb{ST}} \circ \nu_{A} \\
&= \alpha^{\mathbb{S}} \circ S\alpha^{\mathbb{T}} \circ \nu_{A}
\end{aligned}$$

Now, let $(A, \alpha^{\mathbb{ST}})$ and $(B, \beta^{\mathbb{ST}})$ be $\mathbb{ST}$-modules, and consider a morphism $f : A \to B$ in $\mathcal{C}$. If f is both a $\mathbb{S}$-module homomorphism and a $\mathbb{T}$-module homomorphism, then,

$$\begin{aligned}
f \circ \alpha^{\mathbb{ST}} &= \left(f \circ \alpha^{\mathbb{S}}\right) \circ S\alpha^{\mathbb{T}} \\
&= \beta^{\mathbb{S}} \circ S\left(f \circ \alpha^{\mathbb{T}}\right) \\
&= \beta^{\mathbb{S}} \circ S\beta^{\mathbb{T}} \circ STf \\
&= \beta^{\mathbb{ST}} \circ STf
\end{aligned}$$

so f is also a $\mathbb{ST}$-module homomorphism; conversely, if f is a $\mathbb{ST}$-module homomorphism, then f is automatically a $\mathbb{S}$-module homomorphism and a $\mathbb{T}$-module homomorphism, because $(\mathrm{id}, S\eta^{\mathbb{T}})$ and $(\mathrm{id}, \eta^{\mathbb{S}}T)$ are morphisms of monads. This establishes an isomorphism between the category of $\mathbb{ST}$-modules and the category of $\mathbb{S}_{\mathbb{T}}$-modules, and therefore the functor $U^{\mathbb{ST}}_{\mathbb{T}} : \mathcal{C}^{\mathbb{ST}} \to \mathcal{C}^{\mathbb{T}}$ induced by $(\mathrm{id}, \eta^{\mathbb{S}}T)$ must be strictly monadic. Finally, note that the functor $U^{\mathbb{ST}}_{\mathbb{S}} : \mathcal{C}^{\mathbb{ST}} \to \mathcal{C}^{\mathbb{S}}$ induced by the monad morphism $(\mathrm{id}, \eta^{\mathbb{S}}T)$ is identified under this isomorphism with the functor induced by the monad morphism $(U^{\mathbb{T}}, \mathrm{id}_{SU^{\mathbb{T}}})$, so proposition 1.4.16 implies indicated diagram of forgetful functors satisfies the strict Beck–Chevalley condition. ■

There are several converses to the above theorem, and we will now work through each one in turn. We start with the easiest one:

Proposition 2.7.7. *Let $\mathbb{T} = (T, \eta^{\mathbb{T}}, \eta^{\mathbb{T}})$ and $\mathbb{S} = (S, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ be monads on a category $\mathcal{C}$, and let $\tilde{\mathbb{S}} = \left(\tilde{S}, \tilde{\eta}, \tilde{\mu}\right)$ be a monad on $\mathcal{C}^{\mathbb{T}}$. If the equations below hold,*

$$U^{\mathbb{T}}\tilde{S} = SU^{\mathbb{T}} \qquad U^{\mathbb{T}}\tilde{\eta} = \eta U^{\mathbb{T}} \qquad U^{\mathbb{T}}\tilde{\mu} = \mu U^{\mathbb{T}}$$

then there exists a unique natural transformation $v : TS \Rightarrow TS$ *such that:*

(i) v *is a distributive law for* $\mathbb{T}$ *over* $\mathbb{S}$*,*

(ii) *the functor* $\tilde{S}$ *is the one induced by the monad morphism* $(S, v) : \mathbb{T} \to \mathbb{T}$*, and*

(iii) *the natural transformations* $\tilde{\eta}$ *and* $\tilde{\mu}$ *are the ones induced by the transformations* $\eta : (\mathrm{id}, \mathrm{id}_T) \Rightarrow (S, v)$ *and* $\mu : (S^2, Sv \bullet vS) \Rightarrow (S, v)$*.*

Proof. Since $U^{\mathbb{T}} V = SU^{\mathbb{T}}$, we can apply proposition 1.4.6 to immediately establish claim (ii) and the uniqueness of v. Proposition 1.4.12 then implies claim (iii), and claim (i) follows by remark 2.7.5. ■

Example 2.7.8. Let C be a category with binary coproducts and fix an object X in C. It is not hard to verify that the coslice category $(X \downarrow C)$ is strictly monadic over C. Let $\mathbb{T}$ be the monad it induces. We claim that any monad $\mathbb{S} = (S, \eta, \mu)$ on C can be lifted to $(X \downarrow C)$ in the sense of the above proposition in a unique way: indeed, one is forced to define $\tilde{S}$ of an object $a : X \to A$ of $(X \downarrow C)$ to be the object $\eta_A \circ a : X \to SA$. Thus, there is a unique distributive law for $\mathbb{T}$ over $\mathbb{S}$, and the composite monad $\mathbb{S}\mathbb{T}$ is in fact the monad constructed in proposition 1.6.1.

— III —

COMMUTATIVE THEORIES

Consider a commutative ring A. What special properties does A have, *vis-à-vis* a non-commutative ring R? Since we are working category-theoretically, perhaps the first thing to point out is the fact that the hom-set $\mathrm{Hom}_A(X, Y)$ is naturally an A-module when X and Y are A-modules, but $\mathrm{Hom}_R(X, Y)$ is in general only an abelian group. This is actually a direct consequence of commutativity: for, given a left R-module homomorphism $f : X \to Y$ and two elements r and s of R, we only have

$$r \cdot f(sx) = rs \cdot f(x)$$

and therefore the map $x \mapsto r \cdot f(x)$ need not be a left R-module homomorphism. In particular, if $rs \neq sr$, then $x \mapsto r \cdot x$ fails to be a left R-module homomorphism $R \to R$.

Another problem concerns the tensor product. Given A-modules X, Y, and Z, the tensor product $X \otimes_A Y$ is defined to an A-module such that

$$\mathrm{Hom}_A(X \otimes_A Y, Z) \cong \left\{ f : X \times Y \to Z \,\middle|\, f \text{ is } A\text{-bilinear} \right\}$$

naturally in X, Y, and Z. This makes **Mod**(A) into a symmetric monoidal category. This *mostly* works when we replace A with a non-commutative ring R: we still get a symmetric functorial product $\tilde{\otimes}_R$,[1] but R fails to be the monoidal unit of $\tilde{\otimes}_R$. Indeed, for a left R-module M, we have

$$(rs \cdot m) \otimes 1 = m \otimes (rs) = (r \cdot m) \otimes s = (sr \cdot m) \otimes 1$$

in $M \tilde{\otimes}_R R$, so the left R-module homomorphism $M \to M \tilde{\otimes}_R R$ defined by $m \mapsto m \otimes 1$ fails to be an isomorphism for $M = R$ itself!

---

[1] This is *not* the usual tensor product of R-modules!

Thus, it seems as if the commutativity of A is closely connected to the fact that **Mod**(A) is a symmetric monoidal closed category with A as its monoidal unit. This is the sense in which commutative theories generalise commutative rings.

1 Monoidal categories

To fix notation, we will quickly review the main definitions in the theory of monoidal categories.

Definition 3.1.1. A **strict monoidal category** is a category C together with an object I and a functor $(-)\otimes(-): C\times C\to C$ satisfying the following axioms:

- (Left unit). $I\otimes(-)=\mathrm{id}_C$.
- (Right unit). $(-)\otimes I=\mathrm{id}_C$.
- (Associativity). For all objects X, Y, and Z in C,

  $$(X\otimes Y)\otimes Z = X\otimes(Y\otimes Z)$$

  and similarly for morphisms in C.

I is called the **monoidal unit**, and $\otimes$ is called the **monoidal product**.

In other words, a strict monoidal functor is a model of the theory of monoids in the category of all categories.

Example 3.1.2. For any (small)[1] category C, the endofunctor category $[C,C]$ is a strict monoidal category with id_C as the monoidal unit and endofunctor composition as the monoidal product.

Despite the above example, strict monoidal categories turn out to be less useful than one might hope: not even **Set** equipped with the usual cartesian product is a strict monoidal category. The problem is in the *equations* we have imposed in the axioms above: in naturally-occurring examples, we do not get *identities* but only natural isomorphisms. This observation led Bénabou [1963] to propose the following notion instead:

---

[1] There are set-theoretic difficulties in formally defining $[C,C]$ when C is not small.

Definition 3.1.3. A **monoidal category** is a category C together with an object I, a functor $(-)\otimes(-) : C \times C \to C$, and three natural isomorphisms λ, ρ, and α,[2] of type

$$\lambda_X : I \otimes X \xrightarrow{\sim} X$$
$$\rho_X : X \otimes I \xrightarrow{\sim} X$$
$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$$

such that the following diagrams commute for all choices of objects in C:

$$\begin{tikzcd}
& (W \otimes (X \otimes Y)) \otimes Z \arrow[dr, "\alpha_{W,X\otimes Y,Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[ur, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "\alpha_{W\otimes X,Y,Z}"'] & & (W \otimes ((X \otimes Y) \otimes Z)) \arrow[d, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"] \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[rr, "\alpha_{W,X,Y\otimes Z}"'] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes I) \otimes Y \arrow[rr, "\alpha_{X,I,Y}"] \arrow[dr, "\rho_X \otimes \mathrm{id}_Y"'] & & X \otimes (I \otimes Y) \arrow[dl, "\lambda_X \otimes \mathrm{id}_Y"] \\
& X \otimes Y &
\end{tikzcd}$$

The natural isomorphisms λ, ρ, and α are called, respectively, the **left unitor**, **right unitor**, and **associator** of the monoidal category C.

Remark 3.1.4. Since λ, ρ, and α are natural *isomorphisms*, a monoidal structure on C induces a monoidal structure on C^{op}. Less obviously, we can define a monoidal category C^{rev} whose underlying category is the same as C, but $X \otimes^{\mathrm{rev}} Y = Y \otimes X$, $\lambda^{\mathrm{rev}} = \rho$, $\rho^{\mathrm{rev}} = \lambda$, and $\alpha^{\mathrm{rev}} = \alpha^{-1}$.

A fairly non-trivial theorem of Mac Lane [1963] and Kelly [1964] essentially states that these two axioms are enough to prove that "all diagrams involving only λ, ρ, and α commute". For example, using the pentagon axiom

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[2] Beware: Mac Lane [CWM, Ch. VII] uses the opposite convention for α.

and the triangle axiom, we may derive

$$\begin{tikzcd}
(I \otimes X) \otimes Y \arrow[rr, "\alpha_{I,X,Y}"] \arrow[dr, "\lambda_X \otimes \mathrm{id}_Y"'] & & I \otimes (X \otimes Y) \arrow[dl, "\lambda_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}$$

from which the equation (!) below can be obtained:

$$\lambda_I = \rho_I$$

We will make the claim precise by using a notion of functor that respects monoidal structures in a suitable way.

Definition 3.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A **lax monoidal functor** $\mathcal{C} \to \mathcal{D}$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ of the underlying categories, together with a morphism $\eta : I \to FI$ in $\mathcal{D}$ and a natural transformation $\mu : F(-) \otimes F(-) \to F(- \otimes -)$ making these diagrams commute:

$$\begin{tikzcd}
I_{\mathcal{D}} \otimes_{\mathcal{D}} FX \arrow[r, "\eta \otimes_{\mathcal{D}} \mathrm{id}_{FX}"] \arrow[d, "\lambda_{FX}"'] & FI_{\mathcal{C}} \otimes_{\mathcal{D}} FX \arrow[d, "\mu_{I_{\mathcal{C}},X}"] \\
FX & F(I_{\mathcal{C}} \otimes_{\mathcal{C}} X) \arrow[l, "F\lambda_X"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \eta"] \arrow[d, "\rho_{FX}"'] & FX \otimes_{\mathcal{D}} FI_{\mathcal{C}} \arrow[d, "\mu_{X,I_{\mathcal{C}}}"] \\
FX & F(X \otimes_{\mathcal{C}} I_{\mathcal{C}}) \arrow[l, "F\rho_X"]
\end{tikzcd}$$

$$\begin{tikzcd}
(FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} FZ \arrow[r, "\alpha_{FX,FY,FZ}"] \arrow[d, "\mu_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{FZ}"'] & FX \otimes_{\mathcal{D}} (FY \otimes_{\mathcal{D}} FZ) \arrow[d, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \mu_{Y,Z}"] \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} FZ \arrow[d, "\mu_{X \otimes_{\mathcal{C}} Y, Z}"'] & FX \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \arrow[d, "\mu_{X, Y \otimes_{\mathcal{C}} Z}"] \\
F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z) \arrow[r, "F\alpha_{X,Y,Z}"'] & F(X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z))
\end{tikzcd}$$

An **oplax monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax monoidal functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. A **strong monoidal functor** is a lax monoidal functor such that η and μ are isomorphisms. A **strict monoidal functor** is a lax monoidal functor such that η and μ are *identities*.

Note that if $\mathcal{C}$ and $\mathcal{D}$ are both strict monoidal categories, then the diagrams above simplify to more familiar ones:

$$\begin{tikzcd}
FX \arrow[r, "\eta \otimes_{\mathcal{D}} \mathrm{id}_{FX}"] \arrow[dr, "\mathrm{id}_{FX}"'] & FI_{\mathcal{C}} \otimes_{\mathcal{D}} FX \arrow[d, "\mu_{I_{\mathcal{C}},X}"] \\
 & FX
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \eta"] \arrow[dr, "\mathrm{id}_{FX}"'] & FX \otimes_{\mathcal{D}} FI_{\mathcal{C}} \arrow[d, "\mu_{X,I_{\mathcal{C}}}"] \\
 & FX
\end{tikzcd}$$

$$\begin{tikzcd}
 & FX \otimes_{\mathcal{D}} FY \otimes_{\mathcal{D}} FZ \arrow[dl, "\mu_{X,Y} \otimes_{\mathcal{D}} \mathrm{id}_{FZ}"'] \arrow[dr, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \mu_{Y,Z}"] & \\
F(X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{D}} FZ \arrow[dr, "\mu_{X \otimes_{\mathcal{C}} Y,Z}"'] & & FX \otimes_{\mathcal{D}} F(Y \otimes_{\mathcal{C}} Z) \arrow[dl, "\mu_{X,Y \otimes_{\mathcal{C}} Z}"] \\
 & F(X \otimes_{\mathcal{C}} Y \otimes_{\mathcal{C}} Z) &
\end{tikzcd}$$

Thus, we see one reason for defining lax monoidal functors as we have done: if $\mathcal{C}$ is the terminal category, then a lax monoidal functor $\mathcal{C} \to \mathcal{D}$ is the same thing as a monoid[3] in $\mathcal{D}$.

Theorem 3.1.6 (Coherence for monoidal categories)**.** *For any monoidal category $\mathcal{C}$, there exist a strict monoidal category $\mathcal{C}_s$ and a pair of strong monoidal functors $F : \mathcal{C} \to \mathcal{C}_s$, $G : \mathcal{C}_s \to \mathcal{C}$ constituting an equivalence of categories.*

Proof. See [CWM, Ch. VII, § 2; Ch. XI, § 3, Thm 1]. □

When we say "all diagrams commute", what we mean is the following. Given any lax monoidal functor $F : \mathcal{C} \to \mathcal{D}$, by recursion one may construct from η and μ a canonical natural transformation for every possible *formal* monoidal product, e.g.

$$(FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} I_{\mathcal{D}} \to F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} I_{\mathcal{C}})$$

while naturality and the lax monoidal functor axioms applied inductively make diagrams such as

$$\begin{tikzcd}
(FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r] \arrow[d, "\rho_{FX \otimes_{\mathcal{D}} FY}"'] & F((X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} I_{\mathcal{C}}) \arrow[d, "F\rho_{X \otimes_{\mathcal{C}} Y}"] \\
FX \otimes_{\mathcal{D}} FY \arrow[r] & F(X \otimes_{\mathcal{C}} Y)
\end{tikzcd}$$

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[3] — in the monoidal category sense, of course; see definition 3.3.7.

$$\begin{tikzcd}
(FX \otimes_{\mathcal{D}} FY) \otimes_{\mathcal{D}} I_{\mathcal{D}} \arrow[r] \arrow[d, "\alpha_{FX,FY,I_{\mathcal{D}}}"'] & F((X \otimes_C Y) \otimes_C I_C) \arrow[d, "F\alpha_{X,Y,I_C}"] \\
FX \otimes_{\mathcal{D}} (FY \otimes_{\mathcal{D}} I_{\mathcal{D}}) \arrow[r] \arrow[d, "\mathrm{id}_{FX} \otimes_{\mathcal{D}} \rho_{FY}"'] & F(X \otimes_C (Y \otimes_C I_C)) \arrow[d, "F(\mathrm{id}_X \otimes_C \rho_Y)"] \\
FX \otimes_{\mathcal{D}} FY \arrow[r] & F(X \otimes_C Y)
\end{tikzcd}$$

commute for all choices of objects X and Y in C. Hence, if F is a faithful strong monoidal functor and $\mathcal{D}$ is strict, we would be able to conclude that the diagram below commutes in C,

$$\begin{tikzcd}
(X \otimes Y) \otimes I \arrow[rr, "\alpha_{X,Y,I}"] \arrow[dr, "\rho_{X \otimes Y}"'] & & X \otimes (Y \otimes I) \arrow[dl, "\mathrm{id}_X \otimes \rho_{X \otimes Y}"] \\
& X \otimes Y &
\end{tikzcd}$$

but the existence of such F and $\mathcal{D}$ is guaranteed by the coherence theorem. Thus, for most intents and purposes, we can replace a given monoidal category with a monoidally equivalent strict monoidal category.

The coherence theorem allows us to construct a "left regular representation" analogous to that for monoids:

Proposition 3.1.7 (Bénabou)**.** *For any (small) monoidal category C, there is a faithful strong monoidal functor $F : C \to [C, C]$ defined by the following data:*

$$FX = X \otimes (-)$$
$$\eta = \lambda^{-1}$$
$$\left(\mu_{X,Y}\right)_Z = \alpha^{-1}_{X,Y,Z}$$

Proof. F is clearly a faithful functor. In this case, the strong monoidal functor axioms become the following diagrams:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\lambda^{-1}_{X \otimes Y}"] \arrow[d, "\mathrm{id}_{X \otimes Y}"'] & I \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}_{I,X,Y}"] \\
X \otimes Y & (I \otimes X) \otimes Y \arrow[l, "\lambda_X \otimes \mathrm{id}_Y"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id}_X \otimes \lambda^{-1}_Y"] \arrow[d, "\mathrm{id}_{X \otimes Y}"'] & X \otimes (I \otimes Y) \arrow[d, "\alpha^{-1}_{X,I,Y}"] \\
X \otimes Y & (X \otimes I) \otimes Y \arrow[l, "\rho_X \otimes \mathrm{id}_Y"]
\end{tikzcd}$$

$$\begin{tikzcd}
& W\otimes(X\otimes(Y\otimes Z)) \arrow[dl, swap, "\alpha^{-1}_{W,X,Y\otimes Z}"] \arrow[dr, "\mathrm{id}_W\otimes\alpha^{-1}_{X,Y,Z}"] & \\
(W\otimes X)\otimes(Y\otimes Z) \arrow[d, swap, "\alpha^{-1}_{W\otimes X,Y,Z}"] & & W\otimes((X\otimes Y)\otimes Z) \arrow[d, "\alpha^{-1}_{W,X\otimes Y,Z}"] \\
((W\otimes X)\otimes Y)\otimes Z \arrow[rr, swap, "\alpha_{W,X,Y}\otimes\mathrm{id}_Z"] & & (W\otimes(X\otimes Y))\otimes Z
\end{tikzcd}$$

The left square commutes by the coherence theorem, while the right square and the pentagon are seen to be immediate consequences of the triangle and pentagon axioms, respectively. ■

Remark 3.1.8. One can check the commutativity of the diagrams in the above proof directly without appealing to the coherence theorem; by our earlier remarks, this is already enough to prove that "all diagrams commute".

Many natural examples of monoidal categories have a "commutative" monoidal product. For example, the cartesian product in any category satisfies $X\times Y\cong Y\times X$. As usual, to do anything useful, we must demand not only the existence of such isomorphisms but also that they be natural and coherent in the following sense:

Definition 3.1.9. A **braided monoidal category** is a monoidal category C equipped with a natural isomorphism γ of type

$$\gamma_{X,Y} : X\otimes Y \xrightarrow{\sim} Y\otimes X$$

such that the following diagrams commute for all choices of objects in C:

$$\begin{tikzcd}
& X\otimes(Y\otimes Z) \arrow[dr, "\gamma_{X,Y\otimes Z}"] & \\
(X\otimes Y)\otimes Z \arrow[ur, "\alpha_{X,Y,Z}"] \arrow[d, swap, "\gamma_{X,Y}\otimes\mathrm{id}_Z"] & & (Y\otimes Z)\otimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y\otimes X)\otimes Z \arrow[dr, swap, "\alpha_{Y,X,Z}"] & & Y\otimes(Z\otimes X) \\
& Y\otimes(X\otimes Z) \arrow[ur, swap, "\mathrm{id}_Y\otimes\gamma_{X,Z}"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (X \otimes Y) \otimes Z \arrow[dr, "\gamma_{X \otimes Y, Z}"] & \\
X \otimes (Y \otimes Z) \arrow[ur, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "\mathrm{id}_X \otimes \gamma_{Y,Z}"'] & & Z \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}_{Y,Z,X}"] \\
X \otimes (Z \otimes Y) \arrow[dr, "\alpha^{-1}_{Y,X,Z}"'] & & (Z \otimes X) \otimes Y \\
 & (X \otimes Z) \otimes Y \arrow[ur, "\gamma_{X,Z} \otimes \mathrm{id}_Y"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
I \otimes X \arrow[rr, "\gamma_{I,X}"] \arrow[dr, "\lambda_X"'] & & X \otimes I \arrow[dl, "\rho_X"] \\
 & X &
\end{tikzcd}$$

The natural isomorphism γ is called the **braiding** of $\mathcal{C}$. A **symmetric monoidal category** is a braided monoidal category $\mathcal{C}$ satisfying the following additional axiom:

$$\gamma \bullet \gamma = \mathrm{id}_{\mathcal{C}}$$

A **braided / symmetric strict monoidal category** is a braided / symmetric monoidal category that is strict as a monoidal category.

There is a coherence theorem for braided and symmetric monoidal categories as well, but in the braided case it is somewhat subtle compared to the coherence theorem for monoidal categories—we cannot be so cavalier as to say that "all diagrams commute" in a braided monoidal category. Instead, just as before, every braided / symmetric monoidal category is equivalent to a strict one via functors respecting the various structural isomorphisms.

Definition 3.1.10. Let $\mathcal{C}$ and $\mathcal{D}$ be braided monoidal categories. A **lax / oplax / strong / strict braided monoidal functor** $\mathcal{C} \to \mathcal{D}$ is a lax / oplax / strong / strict monoidal functor $F : \mathcal{C} \to \mathcal{D}$ making the diagram below commute:

$$\begin{tikzcd}
FX \otimes_{\mathcal{D}} FY \arrow[r, "\mu_{X,Y}"] \arrow[d, "\gamma_{FX,FY}"'] & F(X \otimes_{\mathcal{C}} Y) \arrow[d, "F\gamma_{X,Y}"] \\
FY \otimes_{\mathcal{D}} FX \arrow[r, "\mu_{Y,X}"'] & F(Y \otimes_{\mathcal{C}} X)
\end{tikzcd}$$

Theorem 3.1.11 (Coherence for braided / symmetric monoidal categories)**.** *For any braided / symmetric monoidal category C, there exist a braided / symmetric strict monoidal category C_s and a pair of strong braided monoidal functors $F : C \to C_s$, $G : C_s \to C$ constituting an equivalence of categories.*

Proof. This is a consequence of the classical coherence theorems of Mac Lane [1963] and Kelly [1964]. □

Here is an example of an equation that does *not* necessarily hold in a braided monoidal category, even though they have the same domain and codomain:

$$\gamma_{X,Y} \overset{?}{=} \gamma_{Y,X}^{-1}$$

Indeed, if it were true, then every braided monoidal category would be a symmetric monoidal category! On the other hand, in a symmetric strict monoidal category, it is true that any two composites of braiding operations with the same domain and codomain are equal—provided each object is identified with a different letter, so that we do not get absurdities like this:

$$\gamma_{X,X} \overset{?}{=} \mathrm{id}_{X \otimes X}$$

A similar restriction applies to our claim that "all diagrams commute" in a monoidal category, so it is not unreasonable to say the same is true in a symmetric monoidal category.

We pause briefly to indicate an important special case of a symmetric monoidal category:

Proposition 3.1.12.

(i) *A category with all finite products is automatically a symmetric monoidal category, with the terminal object 1 as its monoidal unit and the cartesian product $\times$ as the monoidal product.*

(ii) *If C and $\mathcal{D}$ are two categories with finite products regarded as symmetric monoidal categories, then every functor $C \to \mathcal{D}$ can be equipped with a canonical oplax braided monoidal functor structure.*

(iii) *A cartesian monoidal functor is canonically equipped with the structure of a strong braided monoidal functor.*

Proof. (i). The verification of the symmetric monoidal category axioms is straightforward and left to the reader as an exercise.

(ii). Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The universal property of the terminal object gives a unique morphism $\varepsilon : F1 \to 1$ in $\mathcal{D}$, and the universal property of binary products gives a canonical morphism $\delta_{X,Y} : F(X \times Y) \to FX \times FY$. It can be shown that the diagrams below commute,

$$\begin{tikzcd}
F(1_{\mathcal{C}} \times_{\mathcal{C}} X) \arrow[r, "\delta_{1_{\mathcal{C}},X}"] \arrow[d, "F\lambda_X"'] & F1_{\mathcal{C}} \times_{\mathcal{D}} FX \arrow[d, "\varepsilon \times_{\mathcal{D}} \mathrm{id}_{FX}"] \\
FX & 1_{\mathcal{D}} \times_{\mathcal{D}} FX \arrow[l, "\lambda_{FX}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X \times_{\mathcal{C}} 1_{\mathcal{C}}) \arrow[r, "\delta_{X,1_{\mathcal{C}}}"] \arrow[d, "F\rho_X"'] & FX \times_{\mathcal{D}} F1_{\mathcal{C}} \arrow[d, "\mathrm{id}_{FX} \times_{\mathcal{D}} \varepsilon"] \\
FX & FX \times_{\mathcal{D}} 1_{\mathcal{D}} \arrow[l, "\rho_{FX}"]
\end{tikzcd}$$

$$\begin{tikzcd}
F((X \times_{\mathcal{C}} Y) \times_{\mathcal{C}} Z) \arrow[r, "F\alpha_{X,Y,Z}"] \arrow[d, "\delta_{X \times_{\mathcal{C}} Y,Z}"'] & F(X \times_{\mathcal{C}} (Y \times_{\mathcal{C}} Z)) \arrow[d, "\delta_{X,Y \times_{\mathcal{C}} Z}"] \\
F(X \times_{\mathcal{C}} Y) \times_{\mathcal{D}} FZ \arrow[d, "\delta_{X,Y} \times_{\mathcal{D}} \mathrm{id}_{FZ}"'] & FX \times_{\mathcal{D}} F(Y \times_{\mathcal{C}} Z) \arrow[d, "\mathrm{id}_{FX} \times_{\mathcal{D}} \delta_{Y,Z}"] \\
(FX \times_{\mathcal{D}} FY) \times_{\mathcal{D}} FZ \arrow[r, "\alpha_{FX,FY,FZ}"'] & FX \times_{\mathcal{D}} (FY \times_{\mathcal{D}} FZ)
\end{tikzcd}$$

$$\begin{tikzcd}
F(X \times_{\mathcal{C}} Y) \arrow[r, "\delta_{X,Y}"] \arrow[d, "F\gamma_{X,Y}"'] & FX \times_{\mathcal{D}} FY \arrow[d, "\gamma_{FX,FY}"] \\
F(Y \otimes_{\mathcal{C}} X) \arrow[r, "\delta_{Y,X}"'] & FY \otimes_{\mathcal{D}} FX
\end{tikzcd}$$

making F into an oplax braided monoidal functor $\mathcal{C} \to \mathcal{D}$.

(iii). A functor is cartesian monoidal precisely if ε and δ as defined above are isomorphisms. ◊

Henceforth, we refer to categories with all finite products as **cartesian monoidal categories**.

Here is the last definition of this section:

Definition 3.1.13. Let $\mathcal{C}$ be a monoidal category, and let Y and Z be objects in $\mathcal{C}$. A **right internal hom**, denoted variously by $\mathcal{H}om(Y,Z)$, $[Y \to Z]$, or Z^Y, is an object of $\mathcal{C}$ together with a morphism $\mathrm{ev}_{Y,Z} : \mathcal{H}om(Y,Z) \otimes Y \to Z$ having the following universal property: for all morphisms $f : X \otimes Y \to Z$ in $\mathcal{C}$, there is a unique morphism $\tilde{f} : X \to \mathcal{H}om(Y,Z)$ in C such that $\mathrm{ev}_{Y,Z} \circ \left(\tilde{f} \otimes \mathrm{id}_Y\right) = f$; equivalently, we have bijections

$$C(X \otimes Y, Z) \cong C\big(X, \mathcal{H}om(Y,Z)\big)$$

that are natural for each object X in C. A **left internal hom**, provisionally denoted by $Y \pitchfork Z$, is a right internal hom in the reverse monoidal structure on C; equivalently, we have bijections

$$C(Y \otimes X, Z) \cong C(X, Y \pitchfork Z)$$

that are natural for each object X in C.

A **closed monoidal category** is a monoidal category that has right internal homs for all pairs of objects. A **biclosed monoidal category** is a monoidal category that has both left and right internal homs. Note that in a symmetric monoidal category, $Y \pitchfork Z$ and $\mathcal{H}om(Y,Z)$ are isomorphic if they exist.

Proposition 3.1.14. *Let C be a closed monoidal category.*

(i) *The assignment $(Y,Z) \mapsto \mathcal{H}om(Y,Z)$ extends to a functor $C^{\mathrm{op}} \times C \to C$ making the bijection*

$$C(X \otimes Y, Z) \cong C\big(X, \mathcal{H}om(Y,Z)\big)$$

natural in X, Y, and Z.

(ii) *For each object Y, we have an adjunction*

$$(-) \otimes Y \dashv \mathcal{H}om(Y,-) : C \to C$$

whose counit is $\mathrm{ev}_{Y,-} : \mathcal{H}om(Y,-) \otimes Y \Rightarrow \mathrm{id}_C$.

(iii) *If I is the monoidal unit of C, then there is a bijection*

$$C(Y,Z) \cong C\big(I, \mathcal{H}om(Y,Z)\big)$$

that is natural in Y and Z.

Proof. (i). This is a straightforward example of an adjunction with a parameter: see [CWM, Ch. IV, § 7].

(ii). This is clear from the definition of $\mathcal{H}om(Y,Z)$ and $\mathrm{ev}_{Y,-}$.

(iii). The left unitor $\lambda_Y : Y \xrightarrow{\sim} I \otimes Y$ induces the required bijection. ■

Example 3.1.15. **Set** is a closed symmetric monoidal category when regarded as a cartesian monoidal category. Such a category is said to be a **cartesian closed category**. Internal hom objects in a cartesian closed category are usually called **exponential objects**.

2 Enriched categories

Having mentioned monoidal closed categories, it is inevitable that we discuss enriched categories.

Definition 3.2.1. Let $\mathcal{V}$ be a monoidal category. A **$\mathcal{V}$-enriched category**, or simply **$\mathcal{V}$-category**, consists of the following data:

- A class of objects, $\mathrm{ob}\, C$.
- For each pair of objects X, Y, a hom-object $C(X, Y)$ in $\mathcal{V}$.
- For each object X, a morphism $\ulcorner \mathrm{id}_X \urcorner : I \to C(X, X)$ in $\mathcal{V}$.
- For each triplet of objects X, Y, Z, a morphism in $\mathcal{V}$ of type

$$m_{X,Y,Z} : C(Y,Z) \otimes C(X,Y) \to C(X,Z)$$

We additionally require the following diagrams commute in $\mathcal{V}$:

$$\begin{tikzcd}
 & C(Y,Y) \otimes C(X,Y) \arrow[d, "m_{X,Y,Y}"] \\
I \otimes C(X,Y) \arrow[ur, "\ulcorner \mathrm{id}_Y \urcorner \otimes \mathrm{id}"] \arrow[r, "\lambda"'] & C(X,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C(X,Y) \otimes C(X,X) \arrow[d, "m_{X,X,Y}"] \\
C(X,Y) \otimes I \arrow[ur, "\mathrm{id} \otimes \ulcorner \mathrm{id}_X \urcorner"] \arrow[r, "\rho"'] & C(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
(C(Y,Z) \otimes C(X,Y)) \otimes C(W,X) \arrow[rr, "\alpha"] \arrow[d, "m_{X,Y,Z} \otimes \mathrm{id}"'] & & C(Y,Z) \otimes (C(X,Y) \otimes C(W,X)) \arrow[d, "\mathrm{id} \otimes m_{W,X,Y}"] \\
C(X,Z) \otimes C(W,X) \arrow[dr, "m_{W,X,Z}"'] & & C(Y,Z) \otimes C(W,Y) \arrow[dl, "m_{W,Y,Z}"] \\
 & C(W,Z) &
\end{tikzcd}$$

Remark 3.2.2. Of course, a locally small category is the same thing as a **Set**-enriched category, where **Set** is regarded as a cartesian monoidal category.

Proposition 3.2.3. *Every $\mathcal{V}$-category $\mathcal{C}$ has an* **underlying ordinary category** *$\mathcal{C}_0$ consisting of the following data:*

- $\operatorname{ob}\mathcal{C}_0 = \operatorname{ob}\mathcal{C}$.
- $\mathcal{C}_0(X,Y) = \mathcal{V}(I, \mathcal{C}(X,Y))$.
- *Given $\ulcorner f \urcorner \in \mathcal{C}_0(X,Y)$ and $\ulcorner g \urcorner \in \mathcal{C}_0(Y,Z)$, we define $\ulcorner g \circ f \urcorner$ in $\mathcal{C}_0(X,Z)$ by*

  $$\ulcorner g \circ f \urcorner = m_{X,Y,Z} \circ \left(\ulcorner g \urcorner \otimes \ulcorner f \urcorner\right) \circ \Delta$$

  where $\Delta = \lambda_I = \rho_I : I \xrightarrow{\sim} I \otimes I$.

Where no confusion can arise, we may simply write f instead of $\ulcorner f \urcorner$. ◊

Remark 3.2.4. Passing to the underlying ordinary category can be a lossy process—after all, $\mathcal{V}(I, -) : \mathcal{V} \to$ **Set** need not be a faithful functor.

Definition 3.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-categories. A **$\mathcal{V}$-enriched functor**, or simply **$\mathcal{V}$-functor**, consists of the following data:

- For each object X in $\mathcal{C}$, an object FX in $\mathcal{D}$.
- For each pair of objects X, Y in $\mathcal{C}$, a morphism in $\mathcal{V}$ of type

  $$f_{X,Y} : \mathcal{C}(X,Y) \to \mathcal{D}(FX, FY)$$

These are required to make the following diagrams commute in $\mathcal{V}$:

$$\begin{tikzcd}
I \arrow[r, "\ulcorner \mathrm{id}_X \urcorner"] \arrow[dr, "\ulcorner \mathrm{id}_{FX} \urcorner"'] & \mathcal{C}(X,X) \arrow[d, "f_{X,X}"] \\
 & \mathcal{D}(FX,FX)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) \arrow[r, "m_{X,Y,Z}"] \arrow[d, "f_{Y,Z} \otimes f_{X,Y}"'] & \mathcal{C}(X,Z) \arrow[d, "f_{X,Z}"] \\
\mathcal{D}(FY,FZ) \otimes \mathcal{C}(FX,FY) \arrow[r, "m_{FX,FY,FZ}"'] & \mathcal{D}(FX,FZ)
\end{tikzcd}$$

Proposition 3.2.6. *A $\mathcal{V}$-functor between two $\mathcal{V}$-categories induces a functor between the underlying ordinary categories, and this respects composition of functors.* ◊

Proposition 3.2.7. *If $\mathcal{V}$ is a monoidal closed category, then there is a canonical $\mathcal{V}$-category whose underlying ordinary category is $\mathcal{V}$.*

Proof. Let $\operatorname{ob} C = \operatorname{ob} \mathcal{V}$, and let $C(X,Y) = \mathcal{H}om(X,Y)$. It is clear how to define the identity arrows in C. Composition $m_{X,Y,Z} : C(Y,Z) \otimes C(X,Y) \to C(X,Z)$ is defined to be the adjoint transpose of

$$\mathrm{ev}_{Y,Z} \circ (\mathrm{id} \otimes \mathrm{ev}_{X,Y}) \circ \alpha : (\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y)) \otimes X \to Z$$

and the verification of the $\mathcal{V}$-category axioms is left as an exercise. ◊

Remark 3.2.8. The natural bijection of the tensor–hom adjunction

$$\mathcal{V}(X \otimes Y, Z) \cong \mathcal{V}(X, \mathcal{H}om(Y,Z))$$

lifts to a natural isomorphism of internal hom objects:

$$\mathcal{H}om(X \otimes Y, Z) \cong \mathcal{H}om(X, \mathcal{H}om(Y,Z))$$

Indeed, using the associator, we have a series of natural bijections

$$\begin{aligned} \mathcal{V}(W, \mathcal{H}om(X \otimes Y, Z)) &\cong \mathcal{V}(W \otimes (X \otimes Y), Z) \\ &\cong \mathcal{V}((W \otimes X) \otimes Y, Z) \\ &\cong \mathcal{V}(W \otimes X, \mathcal{H}om(Y,Z)) \\ &\cong \mathcal{V}(W, \mathcal{H}om(X, \mathcal{H}om(Y,Z))) \end{aligned}$$

and the Yoneda lemma gives us the required natural isomorphism.

We have so far defined enriched categories and enriched functors, but what about enriched natural transformations?

Definition 3.2.9. Let $F, G : C \to \mathcal{D}$ be two $\mathcal{V}$-functors. A **$\mathcal{V}$-enriched natural transformation** $\varphi : F \Rightarrow G$ is a family $\left(\ulcorner \varphi_X \urcorner : I \to \mathcal{D}(FX, GX) \,\middle|\, X \in \operatorname{ob} C\right)$ of morphisms in $\mathcal{V}$ such that the diagram below commutes for all objects:

$$\begin{tikzcd}
I \otimes C(X,Y) \arrow[r, "\ulcorner\varphi_Y\urcorner \otimes f_{X,Y}"] & \mathcal{D}(FY,GY) \otimes \mathcal{D}(FX,FY) \arrow[dr, "m_{FX,FY,GY}"] & \\
C(X,Y) \arrow[u, "\lambda^{-1}_{C(X,Y)}"] \arrow[d, "\rho^{-1}_{C(X,Y)}"'] & & \mathcal{D}(FX,GY) \\
C(X,Y) \otimes I \arrow[r, "g_{X,Y} \otimes \ulcorner\varphi_X\urcorner"'] & \mathcal{D}(GX,GY) \otimes \mathcal{D}(FX,GX) \arrow[ur, "m_{FX,GX,GY}"'] &
\end{tikzcd}$$

Since a morphism $I \to \mathcal{D}(FX, GX)$ in $\mathcal{V}$ is the same thing as an arrow $FX \to GX$ in the underlying ordinary category of $\mathcal{D}$, a $\mathcal{V}$-natural transformation is just an ordinary natural transformation satisfying an additional compatibility condition.

So far, we have not needed to assume that $\mathcal{V}$ is a *symmetric* monoidal category. This turns out to be necessary if we want to have an enriched duality principle: notice that without a braiding operation, we cannot even construct the opposite of a $\mathcal{V}$-category; of course, the braiding operation has to be involutary if we want duality. If $\mathcal{V}$ does not have a braiding operation, then the opposite of a $\mathcal{V}$-category is at best a $\mathcal{V}^{\mathrm{rev}}$-category.

Less obviously, we need $\mathcal{V}$ to be symmetric monoidal so that we can define the monoidal product of two $\mathcal{V}$-categories. This is needed to make sense of e.g. $\mathcal{V}$-functors of two variables.

Definition 3.2.10. Let $\mathcal{V}$ be a symmetric monoidal category. The **monoidal product** of two $\mathcal{V}$-categories $\mathcal{C}$ and $\mathcal{D}$ is the $\mathcal{V}$-category $\mathcal{C} \otimes \mathcal{D}$ defined by the following data: $\mathrm{ob}\, \mathcal{C} \otimes \mathcal{D} = (\mathrm{ob}\, \mathcal{C}) \times (\mathrm{ob}\, \mathcal{D})$, $\mathcal{C} \otimes \mathcal{D}((A, B), (A', B')) = \mathcal{C}(A, A') \otimes \mathcal{D}(B, B')$, and composition in $\mathcal{C} \otimes \mathcal{D}$ is defined by the diagram below:

$$\begin{tikzcd}
(\mathcal{C}(A', A'') \otimes \mathcal{D}(B', B'')) \otimes (\mathcal{C}(A, A') \otimes \mathcal{D}(B, B')) \arrow[d] \\
(\mathcal{C}(A', A'') \otimes \mathcal{C}(A, A')) \otimes (\mathcal{D}(B', B'') \otimes \mathcal{D}(B, B')) \arrow[d, "m_{A,A',A''} \otimes m_{B,B',B''}"] \\
\mathcal{C}(A, A'') \otimes \mathcal{D}(B, B'')
\end{tikzcd}$$

In the diagram above, the first arrow is the middle-four interchange defined using the evident composite of associators and braiding operations.

Proposition 3.2.11. *$\mathcal{C} \otimes \mathcal{D}$ so defined is indeed a $\mathcal{V}$-category.* ◊

Proposition 3.2.12. *Let $\mathcal{C}$ be a $\mathcal{V}$-category. If $\mathcal{V}$ is symmetric monoidal, then the assignment $(X, Y) \mapsto \mathcal{C}(X, Y)$ extends to a $\mathcal{V}$-functor $\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \to \mathcal{V}$.*

Proof. For each quadruplet X, X', Y, Y' of objects in $\mathcal{C}$, define a morphisms in $\mathcal{V}$ of type

$$h_{(X,Y),(X',Y')} : \mathcal{C}(X', X) \otimes \mathcal{C}(Y, Y') \to \mathcal{H}om(\mathcal{C}(X, Y), \mathcal{C}(X', Y'))$$

to be adjoint transpose of the evident composition operation of type

$$\big(\mathcal{C}(X',X)\otimes\mathcal{C}(Y,Y')\big)\otimes\mathcal{C}(X,Y)\to\mathcal{C}(X',Y')$$

namely, $m_{X',X,Y'}\circ\big(m_{X,Y,Y'}\otimes\mathrm{id}\big)\circ(\mathrm{id}\otimes\gamma)\circ\alpha\circ(\gamma\otimes\mathrm{id})$. It can then be shown that $h_{(X,Y),(X',Y')}$ makes $\mathcal{C}(-,-)$ into a $\mathcal{V}$-functor: see [Kelly, 2005, § 1.6]. □

While limits and colimits are usually defined in elementary terms for ordinary categories using functors and natural transformations, for various reasons this does not generalise to the enriched setting. As such, we postpone the development of enriched limits until later. However, we indicate here a special degenerate case:

Definition 3.2.13. Let $\mathcal{V}$ be a symmetric monoidal closed category, and let $\mathcal{C}$ be a $\mathcal{V}$-category. The **cotensor product** of an object X in $\mathcal{V}$ and an object A in $\mathcal{C}$ is an object $X \pitchfork A$ in $\mathcal{C}$ for which we have isomorphisms

$$\mathcal{C}(B, X \pitchfork A) \cong \mathcal{H}om(X, \mathcal{C}(B,A))$$

that are $\mathcal{V}$-natural in B. A **$\mathcal{V}$-cotensored category** is a $\mathcal{V}$-category $\mathcal{C}$ for which $X \pitchfork A$ exists for all X in $\mathcal{V}$ and all A in $\mathcal{C}$.

Dually, the **tensor product** of X and A in $\mathcal{C}$ is an object $X \odot A$ in $\mathcal{C}$ for which we have isomorphisms

$$\mathcal{C}(X \odot A, B) \cong \mathcal{H}om(X, \mathcal{C}(A,B))$$

that are $\mathcal{V}$-natural in B. A **$\mathcal{V}$-tensored category** is a $\mathcal{V}$-category $\mathcal{C}$ for which $X \odot A$ exists for all X in $\mathcal{V}$ and all A in $\mathcal{C}$.

Remark 3.2.14. A locally small category that has all small products / coproducts is cotensored / tensored over **Set**. The converses of both statements are false: simply consider the discrete category with two objects and no non-identity morphisms.

Proposition 3.2.15. *A symmetric monoidal closed category $\mathcal{V}$ has all tensor and cotensor products.*

Proof. The natural isomorphism of the tensor–hom adjunction

$$\mathcal{H}om(X\otimes Y, Z) \cong \mathcal{H}om\big(X, \mathcal{H}om(Y,Z)\big)$$

is in fact $\mathcal{V}$-natural in X, Y, and Z: see [Kelly, 2005, § 1.8]. This implies we can take $X \odot Y = X \otimes Y$. Similarly, we have $\mathcal{V}$-natural isomorphisms

$$\mathcal{H}om(X,\mathcal{H}om(Y,Z)) \cong \mathcal{H}om(X \otimes Y,Z) \cong \mathcal{H}om(Y \otimes X,Z) \cong \mathcal{H}om(Y,\mathcal{H}om(X,Z))$$

and so we may set $X \pitchfork Z = \mathcal{H}om(X,Z)$. ☐

The following notion plays an important role in the next section.

Definition 3.2.16. Let C be a monoidal category. A **monoidal strength** on a functor $T : C \to C$ is a natural transformation $\sigma : (-) \otimes T(-) \to T(- \otimes -)$ making the diagrams below commute:

$$\begin{tikzcd}
I \otimes TX \arrow[r, "\sigma_{I,X}"] \arrow[dr, "\lambda_{TX}"'] & T(I \otimes X) \arrow[d, "T\lambda_X"] \\
& TX
\end{tikzcd}$$

$$\begin{tikzcd}
& X \otimes (Y \otimes TZ) \arrow[dr, "\mathrm{id}_X \otimes \sigma_{Y,Z}"] & \\
(X \otimes Y) \otimes TZ \arrow[ur, "\alpha_{X,Y,TZ}"] \arrow[d, "\sigma_{X \otimes Y,Z}"'] & & X \otimes T(Y \otimes Z) \arrow[d, "\sigma_{X,Y \otimes Z}"] \\
T((X \otimes Y) \otimes Z) \arrow[rr, "T\alpha_{X,Y,Z}"'] & & T(X \otimes (Y \otimes Z))
\end{tikzcd}$$

A result of Kock [1972] explains the significance of monoidal strengths:

Proposition 3.2.17 (Kock). *Let $\mathcal{V}$ be a monoidal closed category and let T be an endofunctor on $\mathcal{V}$. There is a bijection between natural transformations of type*

$$\sigma_{X,Y} : X \otimes TY \to T(X \otimes Y)$$

and natural transformations of type

$$t_{X,Y} : \mathcal{H}om(X, Y) \to \mathcal{H}om(TX, TY)$$

such that the natural transformation σ is a monoidal strength for T if and only if the natural transformation t makes T into a $\mathcal{V}$-functor.

Proof. Let $\lambda_{X,Y} : X \to \mathcal{H}om(Y, X \otimes Y)$ be the unit of the tensor–hom adjunction. Since this is an adjunction with a parameter, we get extranatural transformations for each $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{V}$:[1]

$$\begin{aligned} \mathrm{ev}_{X',Y} \circ (\mathrm{id} \otimes f) &= \mathrm{ev}_{X,Y} \circ (\mathcal{H}om(f,Y) \otimes \mathrm{id}) \\ \mathcal{H}om(g, X \otimes Y') \circ \lambda_{X,Y'} &= \mathcal{H}om(Y, \mathrm{id} \otimes g) \circ \lambda_{X,Y} \end{aligned}$$

Given the natural transformation t, define $\sigma_{X,Y} : X \otimes TY \to T(X \otimes Y)$ to be the following composite:

$$\sigma_{X,Y} = \mathrm{ev}_{TY,T(X \otimes Y)} \circ \left((t_{Y,X\otimes Y} \circ \lambda_{X,Y}) \otimes \mathrm{id}_{TY} \right)$$

For each morphism $g : Y \to Y'$ in $\mathcal{V}$, we have

$$\begin{aligned} & T(\mathrm{id}_X \otimes g) \circ \sigma_{X,Y} \\ &\qquad = T(\mathrm{id}_X \otimes g) \circ \mathrm{ev}_{TY,T(X\otimes Y)} \circ \left((t_{Y,X\otimes Y} \circ \lambda_{X,Y}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY,T(X\otimes Y')} \circ \left((\mathcal{H}om(TY, T(\mathrm{id}_X \otimes g)) \circ t_{Y,X\otimes Y} \circ \lambda_{X,Y}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY,T(X\otimes Y')} \circ \left((t_{Y,X\otimes Y'} \circ \mathcal{H}om(Y, \mathrm{id}_X \otimes g) \circ \lambda_{X,Y}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY,T(X\otimes Y')} \circ \left((t_{Y,X\otimes Y'} \circ \mathcal{H}om(g, X \otimes Y') \circ \lambda_{X,Y'}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY,T(X\otimes Y')} \circ \left((\mathcal{H}om(Tg, T(X \otimes Y')) \circ t_{Y',X\otimes Y'} \circ \lambda_{X,Y'}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY',T(X\otimes Y')} \circ (\mathrm{id}_{\mathcal{H}om(TY',T(X\otimes Y'))} \otimes Tg) \circ \left((t_{Y',X\otimes Y'} \circ \lambda_{X,Y'}) \otimes \mathrm{id}_{TY} \right) \\ &\qquad = \mathrm{ev}_{TY',T(X\otimes Y')} \circ \left((t_{Y',X\otimes Y'} \circ \lambda_{X,Y'}) \otimes \mathrm{id}_{TY'} \right) \circ (\mathrm{id}_X \otimes Tg) \\ &\qquad = \sigma_{X,Y'} \circ (\mathrm{id}_X \otimes Tg) \end{aligned}$$

as required for naturality in Y; naturality in X is straightforward.

Conversely, given the natural transformation σ, define the morphism $t_{X,Y} : \mathcal{H}om(X,Y) \to \mathcal{H}om(TX,TY)$ to be the following composite:

$$t_{X,Y} = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X,Y}) \circ \sigma_{\mathcal{H}om(X,Y),X}\right) \circ \lambda_{\mathcal{H}om(X,Y),TX}$$

For each morphism $f : X \to X'$ in $\mathcal{V}$, we have

$$\begin{aligned} & t_{X,Y} \circ \mathcal{H}om(f,Y) \\ &\qquad = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X,Y}) \circ \sigma_{\mathcal{H}om(X,Y),X}\right) \circ \lambda_{\mathcal{H}om(X,Y),TX} \circ \mathcal{H}om(f,Y) \\ &\qquad = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X,Y}) \circ \sigma_{\mathcal{H}om(X,Y),X} \circ (\mathcal{H}om(f,Y) \otimes \mathrm{id})\right) \circ \lambda_{\mathcal{H}om(X',Y),TX} \\ &\qquad = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X,Y} \circ (\mathcal{H}om(f,Y) \otimes \mathrm{id})) \circ \sigma_{\mathcal{H}om(X',Y),X}\right) \circ \lambda_{\mathcal{H}om(X',Y),TX} \\ &\qquad = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X',Y} \circ (\mathrm{id} \otimes f)) \circ \sigma_{\mathcal{H}om(X',Y),X}\right) \circ \lambda_{\mathcal{H}om(X',Y),TX} \\ &\qquad = \mathcal{H}om\left(TX, T(\mathrm{ev}_{X',Y}) \circ \sigma_{\mathcal{H}om(X',Y),X'} \circ (\mathrm{id} \otimes Tf)\right) \circ \lambda_{\mathcal{H}om(X',Y),TX} \end{aligned}$$

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[1] See [CWM, Ch. IX, § 4].

but by extranaturality,

$$\mathcal{H}om(TX, \mathrm{id} \otimes Tf) \circ \lambda_{\mathcal{H}om(X',Y),TX} = \mathcal{H}om(Tf, \mathcal{H}om(X',Y) \otimes TX') \circ \lambda_{\mathcal{H}om(X',Y),TX'}$$

thus,

$$\begin{aligned}
&\mathcal{H}om(Tf, TY) \circ t_{X',Y} \\
&\qquad = \mathcal{H}om(Tf, TY) \circ \mathcal{H}om(TX', T(\mathrm{ev}_{X',Y}) \circ \sigma_{\mathcal{H}om(X',Y),X'}) \circ \lambda_{\mathcal{H}om(X',Y),TX'} \\
&\qquad = \mathcal{H}om(TX, T(\mathrm{ev}_{X',Y}) \circ \sigma_{\mathcal{H}om(X',Y),X'}) \circ \mathcal{H}om(Tf, \mathcal{H}om(X',Y) \otimes TX') \\
&\qquad\qquad \circ \lambda_{\mathcal{H}om(X',Y),TX'} \\
&\qquad = \mathcal{H}om(TX, T(\mathrm{ev}_{X',Y}) \circ \sigma_{\mathcal{H}om(X',Y),X'} \circ (\mathrm{id} \otimes Tf)) \circ \lambda_{\mathcal{H}om(X',Y),TX} \\
&\qquad = t_{X,Y} \circ \mathcal{H}om(f, Y)
\end{aligned}$$

as required for naturality in X; naturality in Y is straightforward.

These constructions are mutually inverse. Indeed, the triangle identities for the tensor–hom adjunction tell us

$$\begin{aligned}
\mathrm{ev}_{Y,X\otimes Y} \circ (\lambda_{X,Y} \otimes \mathrm{id}_Y) &= \mathrm{id}_{X\otimes Y} \\
\mathcal{H}om(Y, \mathrm{ev}_{Y,Z}) \circ \lambda_{\mathcal{H}om(Y,Z),Y} &= \mathrm{id}_{\mathcal{H}om(Y,Z)}
\end{aligned}$$

so, given a natural transformation t, for σ as constructed above,

$$\begin{aligned}
&T(\mathrm{ev}_{X,Y}) \circ \sigma_{\mathcal{H}om(X,Y),X} \\
&\qquad = T(\mathrm{ev}_{X,Y}) \circ \mathrm{ev}_{TX,T\mathcal{H}om(X,Y)} \circ ((t_{X,\mathcal{H}om(X,Y)\otimes X} \circ \lambda_{\mathcal{H}om(X,Y),X}) \otimes \mathrm{id}_{TX}) \\
&\qquad = \mathrm{ev}_{TX,TY} \circ ((\mathcal{H}om(TX, T(\mathrm{ev}_{X,Y})) \circ t_{X,\mathcal{H}om(X,Y)\otimes X} \circ \lambda_{\mathcal{H}om(X,Y),X}) \otimes \mathrm{id}_{TX}) \\
&\qquad = \mathrm{ev}_{TX,TY} \circ ((t_{X,Y} \circ \mathcal{H}om(X, \mathrm{ev}_{X,Y}) \circ \lambda_{\mathcal{H}om(X,Y),X}) \otimes \mathrm{id}_{TX}) \\
&\qquad = \mathrm{ev}_{TX,TY} \circ (t_{X,Y} \otimes \mathrm{id}_{TX})
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathcal{H}om(TX, T(\mathrm{ev}_{X,Y}) \circ \sigma_{\mathcal{H}om(X,Y),X}) \circ \lambda_{\mathcal{H}om(X,Y),TX} \\
&\qquad\qquad = \mathcal{H}om(TX, \mathrm{ev}_{TX,TY} \circ (t_{X,Y} \otimes \mathrm{id}_{TX})) \circ \lambda_{\mathcal{H}om(X,Y),TX} \\
&\qquad\qquad = \mathcal{H}om(TX, \mathrm{ev}_{TX,TY}) \circ \lambda_{\mathcal{H}om(TX,TY),TX} \circ t_{X,Y} = t_{X,Y}
\end{aligned}$$

while given a natural transformation σ, for t as constructed above,

$$\begin{aligned}
t_{Y,X\otimes Y} \circ \lambda_{X,Y} &= \mathcal{H}om(TY, T(\mathrm{ev}_{Y,X\otimes Y}) \circ \sigma_{\mathcal{H}om(Y,X\otimes Y),Y}) \circ \lambda_{\mathcal{H}om(Y,X\otimes Y),TY} \circ \lambda_{X,Y} \\
&= \mathcal{H}om(TY, T(\mathrm{ev}_{Y,X\otimes Y}) \circ \sigma_{\mathcal{H}om(Y,X\otimes Y),Y} \circ (\lambda_{X,Y} \otimes \mathrm{id}_{TY})) \circ \lambda_{X,TY} \\
&= \mathcal{H}om(TY, T(\mathrm{ev}_{Y,X\otimes Y} \circ (\lambda_{X,Y} \otimes \mathrm{id}_Y)) \circ \sigma_{X,Y}) \circ \lambda_{X,TY} \\
&= \mathcal{H}om(TY, \sigma_{X,Y}) \circ \lambda_{X,TY}
\end{aligned}$$

and so:

$$\begin{aligned}
&\mathrm{ev}_{TY,T(X\otimes Y)}\circ\big((t_{Y,X\otimes Y}\circ\lambda_{X,Y})\otimes\mathrm{id}_{TY}\big)\\
&\qquad=\mathrm{ev}_{TY,T(X\otimes Y)}\circ\big((\mathcal{H}om(TY,\sigma_{X,Y})\circ\lambda_{X,TY})\otimes\mathrm{id}_{TY}\big)\\
&\qquad=\sigma_{X,Y}\circ\mathrm{ev}_{TY,X\otimes TY}\circ(\lambda_{X,TY}\otimes\mathrm{id}_{TY})=\sigma_{X,Y}
\end{aligned}$$

Thus, we have the required bijection.

It remains to be shown that σ satisfies the equations in the proposition if and only if t makes T into a $\mathcal{V}$-functor. First, assume T is a $\mathcal{V}$-functor. Then,

$$\begin{aligned}
T\lambda_X\circ\sigma_{I,X}&=T\lambda_X\circ\mathrm{ev}_{TX,T(I\otimes X)}\circ\big((t_{X,I\otimes X}\circ\lambda_{I,X})\otimes\mathrm{id}_{TX}\big)\\
&=\mathrm{ev}_{TX,TX}\circ\big((\mathcal{H}om(TX,T\lambda_X)\circ t_{X,I\otimes X}\circ\lambda_{I,X})\otimes\mathrm{id}_{TX}\big)\\
&=\mathrm{ev}_{TX,TX}\circ\big((t_{X,X}\circ\mathcal{H}om(X,\lambda_X)\circ\lambda_{I,X})\otimes\mathrm{id}_{TX}\big)\\
&=\mathrm{ev}_{TX,TX}\circ\big((t_{X,X}\circ\ulcorner\mathrm{id}_X\urcorner)\otimes\mathrm{id}_{TX}\big)\\
&=\mathrm{ev}_{TX,TX}\circ(\ulcorner\mathrm{id}_{TX}\urcorner\otimes\mathrm{id}_{TX})\\
&=\lambda_{TX}
\end{aligned}$$

and one may check that

$$\begin{aligned}
&\sigma_{X,Y\otimes Z}\circ(\mathrm{id}_X\otimes\sigma_{Y,Z})\circ\alpha_{X,Y,TZ}=\mathrm{ev}_{TZ,T(X\otimes(Y\otimes Z))}\\
&\quad\circ\big((m_{TZ,T(Y\otimes Z),T(X\otimes(Y\otimes Z))}\circ(t_{Y\otimes Z,X\otimes(Y\otimes Z)}\otimes t_{Z,Y\otimes Z})\circ(\lambda_{X,Y\otimes Z}\otimes\lambda_{Y,Z}))\otimes\mathrm{id}_{TZ}\big)
\end{aligned}$$

whence follows

$$\begin{aligned}
&\sigma_{X,Y\otimes Z}\circ(\mathrm{id}_X\otimes\sigma_{Y,Z})\circ\alpha_{X,Y,TZ}=\mathrm{ev}_{TZ,T(X\otimes(Y\otimes Z))}\\
&\qquad\qquad\circ\big((t_{Z,X\otimes(Y\otimes Z)}\circ m_{Z,Y\otimes Z,X\otimes(Y\otimes Z)}\circ(\lambda_{X,Y\otimes Z}\otimes\lambda_{Y,Z}))\otimes\mathrm{id}_{TZ}\big)
\end{aligned}$$

but it is clear that

$$m_{Z,Y\otimes Z,X\otimes(Y\otimes Z)}\circ(\lambda_{X,Y\otimes Z}\otimes\lambda_{Y,Z})=\mathcal{H}om(Z,\alpha_{X,Y,Z})\circ\lambda_{X\otimes Y,Z}$$

and so

$$\begin{aligned}
&\sigma_{X,Y\otimes Z}\circ(\mathrm{id}_X\otimes\sigma_{Y,Z})\circ\alpha_{X,Y,TZ}\\
&\quad=\mathrm{ev}_{TZ,T(X\otimes(Y\otimes Z))}\circ\big((t_{Z,X\otimes(Y\otimes Z)}\circ\mathcal{H}om(Z,\alpha_{X,Y,Z})\circ\lambda_{X\otimes Y,Z})\otimes\mathrm{id}_{TZ}\big)\\
&\quad=\mathrm{ev}_{TZ,T(X\otimes(Y\otimes Z))}\circ\big((\mathcal{H}om(TZ,T\alpha_{X,Y,Z})\circ t_{Z,(X\otimes Y)\otimes Z}\circ\lambda_{X\otimes Y,Z})\otimes\mathrm{id}_{TZ}\big)\\
&\quad=T\alpha_{X,Y,Z}\circ\mathrm{ev}_{TZ,T((X\otimes Y)\otimes Z)}\circ\big((t_{Z,(X\otimes Y)\otimes Z}\circ\lambda_{X\otimes Y,Z})\otimes\mathrm{id}_{TZ}\big)\\
&\quad=T\alpha_{X,Y,Z}\circ\sigma_{X\otimes Y,Z}
\end{aligned}$$

as required. Now, assume σ satisfies the equations in the proposition. Then,

$$\begin{aligned} t_{X,X} \circ \ulcorner \mathrm{id}_X \urcorner &= \mathcal{H}om\big(TX, T\big(\mathrm{ev}_{X,X}\big) \circ \sigma_{\mathcal{H}om(X,X),X}\big) \circ \lambda_{\mathcal{H}om(X,X),TX} \circ \ulcorner \mathrm{id}_X \urcorner \\ &= \mathcal{H}om\big(TX, T\big(\mathrm{ev}_{X,X}\big) \circ \sigma_{\mathcal{H}om(X,X),X} \circ (\ulcorner \mathrm{id}_X \urcorner \otimes \mathrm{id}_{TX})\big) \circ \lambda_{I,TX} \\ &= \mathcal{H}om\big(TX, T\big(\mathrm{ev}_{X,X} \circ (\ulcorner \mathrm{id}_X \urcorner \otimes \mathrm{id}_X)\big) \circ \sigma_{I,X}\big) \circ \lambda_{I,TX} \\ &= \mathcal{H}om\big(TX, T\lambda_X \circ \sigma_{I,X}\big) \circ \lambda_{I,TX} \\ &= \mathcal{H}om(TX, \lambda_{TX}) \circ \lambda_{I,TX} = \ulcorner \mathrm{id}_{TX} \urcorner \end{aligned}$$

and, by adjoint transposition,

$$\mathrm{ev}_{TX,TY} \circ \big(t_{X,Y} \otimes \mathrm{id}_{TX}\big) = T\big(\mathrm{ev}_{X,Y}\big) \circ \sigma_{\mathcal{H}om(X,Y),X}$$

so we have

$$\begin{aligned} &\mathrm{ev}_{TY,TZ} \circ \big(t_{Y,Z} \otimes \big(\mathrm{ev}_{TX,TY} \circ \big(t_{X,Y} \otimes \mathrm{id}_{TX}\big)\big)\big) \\ &= T\big(\mathrm{ev}_{Y,Z}\big) \circ \sigma_{\mathcal{H}om(Y,Z),Y} \circ \big(\mathrm{id}_{\mathcal{H}om(Y,Z)} \otimes \big(T\big(\mathrm{ev}_{X,Y}\big) \circ \sigma_{\mathcal{H}om(X,Y),X}\big)\big) \\ &= T\big(\mathrm{ev}_{Y,Z} \circ \big(\mathrm{id}_{\mathcal{H}om(Y,Z)} \otimes \mathrm{ev}_{X,Y}\big)\big) \circ \sigma_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y) \otimes X} \circ \big(\mathrm{id}_{\mathcal{H}om(Y,Z)} \otimes \sigma_{\mathcal{H}om(X,Y),X}\big) \end{aligned}$$

but by assumption

$$\begin{aligned} \sigma_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y) \otimes X} \circ \big(\mathrm{id}_{\mathcal{H}om(Y,Z)} \otimes \sigma_{\mathcal{H}om(X,Y),X}\big) \circ \alpha_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y),TX} \\ = T\alpha_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y),X} \circ \sigma_{\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y),X} \end{aligned}$$

thus,

$$\begin{aligned} &\mathrm{ev}_{TY,TZ} \circ \big(t_{Y,Z} \otimes \big(\mathrm{ev}_{TX,TY} \circ \big(t_{X,Y} \otimes \mathrm{id}_{TX}\big)\big)\big) \circ \alpha_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y),TX} \\ &= T\big(\mathrm{ev}_{Y,Z} \circ \big(\mathrm{id}_{\mathcal{H}om(Y,Z)} \otimes \mathrm{ev}_{X,Y}\big) \circ \alpha_{\mathcal{H}om(Y,Z),\mathcal{H}om(X,Y),X}\big) \circ \sigma_{\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y),X} \\ &= T\big(\mathrm{ev}_{X,Z} \circ \big(m_{X,Y,Z} \otimes \mathrm{id}_Z\big)\big) \circ \sigma_{\mathcal{H}om(Y,Z) \otimes \mathcal{H}om(X,Y),X} \\ &= T\big(\mathrm{ev}_{X,Z}\big) \circ \sigma_{\mathcal{H}om(X,Z),X} \circ \big(m_{X,Y,Z} \otimes \mathrm{id}_{TZ}\big) \\ &= \mathrm{ev}_{TX,TZ} \circ \big(\big(t_{X,Z} \circ m_{X,Y,Z}\big) \otimes \mathrm{id}_{TX}\big) \end{aligned}$$

as required for T to be a $\mathcal{V}$-functor. ■

Remark 3.2.18. Actually, Kock [1972] proved a more general result. Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{V}$-tensored categories, and let $F : \mathcal{C} \to \mathcal{D}$ be an ordinary functor. A **tensorial strength** for F is defined to be an ordinary natural transformation

$\sigma : (-) \odot T(-) \to T(- \odot -)$ making the evident diagrams commute:

$$\begin{tikzcd}
I \odot TA \arrow[r, "\sigma_{I,A}"] \arrow[dr] & T(I \odot X) \arrow[d] \\
& TA
\end{tikzcd}$$

$$\begin{tikzcd}
& X \odot (Y \odot TA) \arrow[dr, "\mathrm{id}_X \odot \sigma_{Y,A}"] & \\
(X \otimes Y) \odot TA \arrow[ur] \arrow[d, "\sigma_{X \otimes Y,A}"'] & & X \odot T(Y \odot A) \arrow[d, "\sigma_{X,Y \odot A}"] \\
T((X \otimes Y) \odot A) \arrow[rr] & & T(X \odot (Y \odot A))
\end{tikzcd}$$

Essentially the same proof shows that $F : \mathcal{C} \to \mathcal{D}$ admits a tensorial strength if and only if it is $\mathcal{V}$-enriched.

Proposition 3.2.19. *Let $\mathcal{V}$ be a monoidal closed category, and let $F, G : \mathcal{V} \to \mathcal{V}$ be $\mathcal{V}$-functors. An ordinary natural transformation $\varphi : F \Rightarrow G$ is $\mathcal{V}$-enriched if and only if the diagram*

$$\begin{tikzcd}
X \otimes FY \arrow[r, "\sigma^F_{X,Y}"] \arrow[d, "\mathrm{id}_X \otimes \varphi_Y"'] & F(X \otimes Y) \arrow[d, "\varphi_{X \otimes Y}"] \\
X \otimes GY \arrow[r, "\sigma^G_{X,Y}"'] & G(X \otimes Y)
\end{tikzcd}$$

commutes for all objects X and Y in $\mathcal{V}$, where σ^F and σ^G are the monoidal strengths of F and G, respectively.

Proof. First, assume $\varphi : F \Rightarrow G$ is a $\mathcal{V}$-natural transformation. By adjoint transposition, the square in the proposition commutes if and only if the following diagram commutes:

$$\begin{tikzcd}
 & & \mathcal{H}om(FY,F(X\otimes Y)) \arrow[dr, "\mathcal{H}om(FY,\varphi_{X\otimes Y})"] & \\
X \arrow[r, "\lambda"] & \mathcal{H}om(Y,X\otimes Y) \arrow[ur, "f_{Y,X\otimes Y}"] \arrow[dr, "g_{Y,X\otimes Y}"'] & & \mathcal{H}om(FY,G(X\otimes Y)) \\
 & & \mathcal{H}om(GY,G(X\otimes Y)) \arrow[ur, "\mathcal{H}om(\varphi_Y,G(X\otimes Y))"'] &
\end{tikzcd}$$

Observing that

$$\mathcal{H}om\big(FY,\varphi_{X\otimes Y}\big) = m_{FY,F(X\otimes Y),G(X\otimes Y)} \circ \big(\ulcorner\varphi_{X\otimes Y}\urcorner \otimes \mathrm{id}_{\mathcal{H}om(FY,F(X\otimes Y))}\big) \circ \lambda^{-1}_{\mathcal{H}om(FY,F(X\otimes Y))}$$
$$\mathcal{H}om\big(\varphi_Y,G(X\otimes Y)\big) = m_{FY,GY,G(X\otimes Y)} \circ \big(\mathrm{id}_{\mathcal{H}om(GY,G(X\otimes Y))} \otimes \ulcorner\varphi_Y\urcorner\big) \circ \rho^{-1}_{\mathcal{H}om(GY,G(X\otimes Y))}$$

we see that the diagram above commutes when φ is $\mathcal{V}$-enriched.

Conversely, suppose the square commutes for all X and Y in $\mathcal{V}$. The observation implies φ is a $\mathcal{V}$-natural transformation if and only if the diagram

$$\begin{tikzcd}
 & \mathcal{H}om(FX,FY) \arrow[dr, "\mathcal{H}om(FX,\varphi_Y)"] & \\
\mathcal{H}om(X,Y) \arrow[ur, "f_{X,Y}"] \arrow[dr, "g_{X,Y}"'] & & \mathcal{H}om(FX,GY) \\
 & \mathcal{H}om(GX,GY) \arrow[ur, "\mathcal{H}om(\varphi_X,GY)"'] &
\end{tikzcd}$$

commutes for all X and Y. From the proof of proposition 3.2.17 we know that

$$f_{X,Y} = \mathcal{H}om\Big(FX, F\big(\mathrm{ev}_{X,Y}\big) \circ \sigma^F_{\mathcal{H}om(X,Y),X}\Big) \circ \lambda_{\mathcal{H}om(X,Y),FX}$$
$$g_{X,Y} = \mathcal{H}om\Big(GX, G\big(\mathrm{ev}_{X,Y}\big) \circ \sigma^G_{\mathcal{H}om(X,Y),X}\Big) \circ \lambda_{\mathcal{H}om(X,Y),GX}$$

and so,

$$\begin{aligned}
&\mathcal{H}om(FX,\varphi_Y) \circ f_{X,Y} \\
&\qquad = \mathcal{H}om\Big(FX, \varphi_Y \circ F\big(\mathrm{ev}_{X,Y}\big) \circ \sigma^F_{\mathcal{H}om(X,Y),X}\Big) \circ \lambda_{\mathcal{H}om(X,Y),FX} \\
&\qquad = \mathcal{H}om\Big(FX, G\big(\mathrm{ev}_{X,Y}\big) \circ \varphi_{\mathcal{H}om(X,Y)\otimes X} \circ \sigma^F_{\mathcal{H}om(X,Y),X}\Big) \circ \lambda_{\mathcal{H}om(X,Y),FX} \\
&\qquad = \mathcal{H}om\Big(FX, G\big(\mathrm{ev}_{X,Y}\big) \circ \sigma^G_{\mathcal{H}om(X,Y),X} \circ \big(\mathrm{id}_{\mathcal{H}om(X,Y)} \otimes \varphi_X\big)\Big) \circ \lambda_{\mathcal{H}om(X,Y),FX}
\end{aligned}$$

but by extranaturality,

$$\mathcal{H}om\big(FX, \mathrm{id}_{\mathcal{H}om(X,Y)} \otimes \varphi_X\big) \circ \lambda_{\mathcal{H}om(X,Y),FX} \\ = \mathcal{H}om\big(\varphi_X, \mathcal{H}om(X,Y) \otimes GX\big) \circ \lambda_{\mathcal{H}om(X,Y),GX}$$

thus we have

$$\begin{aligned} \mathcal{H}om\big(FX, \varphi_Y\big) \circ f_{X,Y} &= \mathcal{H}om\Big(\varphi_X, G\big(\mathrm{ev}_{X,Y}\big) \circ \sigma^G_{\mathcal{H}om(X,Y),X}\Big) \circ \lambda_{\mathcal{H}om(X,Y),GX} \\ &= \mathcal{H}om\big(\varphi_X, GY\big) \circ g_{X,Y} \end{aligned}$$

as required for φ to be a $\mathcal{V}$-natural transformation. ■

3 Strong monads

Although it is possible to define commutative theories as Lawvere theories satisfying a certain condition, such a definition is inelegant and does not generalise easily to monads. It turns out the right setting for studying "commutative" monads is that of symmetric monoidal closed categories; however, in order to define what we mean by 'commutative', we will need an extra structure on our monads.

Definition 3.3.1. Let C be a monoidal category. A **strong monad** on C is a quadruplet $\mathbb{T} = (T, \eta, \mu, \sigma)$ consisting of a monad (T, η, μ) on C and a monoidal strength σ for T making the diagrams below commute for all choices of objects in C:

$$\begin{tikzcd}
X \otimes Y \arrow[r, "\mathrm{id}_X \otimes \eta_Y"] \arrow[dr, "\eta_{X\otimes Y}"'] & X \otimes TY \arrow[d, "\sigma_{X,Y}"] \\
 & T(X \otimes Y)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes T^2 Y \arrow[r, "\sigma_{X,TY}"] \arrow[d, "\mathrm{id}_X \otimes \mu_Y"'] & T(X \otimes TY) \arrow[r, "T\sigma_{X,Y}"] & T^2(X \otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
X \otimes TY \arrow[rr, "\sigma_{X,Y}"'] & & T(X \otimes Y)
\end{tikzcd}$$

The natural transformation σ is called the **(monoidal) strength** of $\mathbb{T}$.

Proposition 3.3.2. *Let $\mathcal{V}$ be a monoidal closed category. An ordinary monad (T, η, μ) on $\mathcal{V}$ admits a strength if and only if $T : \mathcal{V} \to \mathcal{V}$ underlies a $\mathcal{V}$-functor and both $\eta : \mathrm{id}_{\mathcal{V}} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ are $\mathcal{V}$-natural transformations, and there is a bijection between such $\mathcal{V}$-enrichments of $\mathbb{T}$ and strengths for $\mathbb{T}$.*

Proof. Use propositions 3.2.17 and 3.2.19. ■

Corollary 3.3.3. *Every monad on* **Set** *has a unique strength.* ■

Example 3.3.4. It is important to appreciate that not every monad admits a strength. For example, consider the tensor algebra monad on **Ab**:

$$TM = \mathbb{Z} \oplus M \oplus (M \otimes_{\mathbb{Z}} M) \oplus (M \otimes_{\mathbb{Z}} M \otimes_{\mathbb{Z}} M) \oplus \cdots$$

Ab is a symmetric monoidal closed category, but T is evidently not an additive functor, so it cannot admit a monoidal strength.

Example 3.3.5. Even algebraic monads need not be strong. Consider the two-sorted equational theory $\mathbb{T}$ of modules over rings. It is clear that the category $[2, \mathbf{Set}]$ is cartesian closed with the obvious internal hom

$$\mathcal{H}om((X, Z), (Y, W)) = \left(Z^X, W^Y\right)$$

but the free $\mathbb{T}$-model monad on $[2, \mathbf{Set}]$ is not strong. Indeed, if $f : X \to Y$ is a map of the generators of the ring part, then $T(f, Z) : T(X, Z) \to T(Y, Z)$ acts on both the ring part and the module part; but if T were enriched, then $T(f, Z)$ would have to act as the identity on the module part.

The following lemma essentially says that tensor products (in the sense of definition 3.2.13) of free modules over a strong monad exist, but as we have not yet shown that $\mathcal{V}^{\mathbb{T}}$ is a $\mathcal{V}$-category, we will state only the part of the claim that concerns ordinary category theory.

Lemma 3.3.6. *Let $\mathbb{T} = (T, \eta, \mu, \sigma)$ be a strong monad on a monoidal category $\mathcal{C}$. For each morphism $f : Z \to W$ in $\mathcal{C}$ and each $\mathbb{T}$-module homomorphism $h : (TX, \mu_X) \to (TY, \mu_Y)$, there exists a unique $\mathbb{T}$-module homomorphism $f \odot h : (T(Z \otimes X), \mu_{Z \otimes X}) \to (T(W \otimes Y), \mu_{W \otimes Y})$ in $\mathcal{C}^{\mathbb{T}}$ making the diagram below commute in $\mathcal{C}$:*

$$\begin{tikzcd}
Z \otimes TX \arrow[r, "f \otimes h"] \arrow[d, "\sigma_{Z,X}"'] & W \otimes TY \arrow[d, "\sigma_{W,Y}"] \\
T(Z \otimes X) \arrow[r, dashed, "f \odot h"'] & T(W \otimes Y)
\end{tikzcd}$$

Proof. We prove uniqueness first. Since σ is a strength for $\mathbb{T}$, we have

$$(f \odot h) \circ \eta_{Z \otimes X} = (f \odot h) \circ \sigma_{Z,X} \circ (\mathrm{id}_Z \otimes \eta_X) = \sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X)$$

but if $f \odot h$ is a $\mathbb{T}$-module homomorphism, then,

$$f \odot h = (f \odot h) \circ \mu_{Z \otimes X} \circ T\eta_{Z \otimes X} = \mu_{W \otimes Y} \circ T((f \odot h) \circ \eta_{Z \otimes X})$$

hence:

$$f \odot h = \mu_{W \otimes Y} \circ T(\sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X))$$

Conversely, suppose $f \odot h$ is defined by the above equation. The diagram commutes, because

$$\begin{aligned}(f \odot h) \circ \sigma_{Z,X} &= \mu_{W \otimes Y} \circ T(\sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X)) \circ \sigma_{Z,X} \\ &= \mu_{W \otimes Y} \circ T\sigma_{W,Y} \circ \sigma_{W,TY} \circ (f \otimes T(h \circ \eta_X)) \\ &= \sigma_{W,Y} \circ (\mathrm{id}_W \otimes \mu_Y) \circ (f \otimes T(h \circ \eta_X)) \\ &= \sigma_{W,Y} \circ (f \otimes h)\end{aligned}$$

where in the last line we used the fact that h is a homomorphism, and

$$\begin{aligned}(f \odot h) \circ \mu_{Z \otimes X} &= \mu_{W \otimes Y} \circ T(\sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X)) \circ \mu_{Z \otimes X} \\ &= \mu_{W \otimes Y} \circ \mu_{T(W \otimes Y)} \circ T^2(\sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X)) \\ &= \mu_{W \otimes Y} \circ T(\mu_{W \otimes Y} \circ T(\sigma_{W,Y} \circ (f \otimes h) \circ (\mathrm{id}_Z \otimes \eta_X)) \circ \mu_{Z \otimes X}) \\ &= \mu_{W \otimes Y} \circ T(f \odot h)\end{aligned}$$

so $f \odot h$ is indeed a $\mathbb{T}$-module homomorphism. ■

It was earlier stated that monads generalise monoids, and we have seen that every monoid in **Set** induces a (strong) monad in a functorial way. This can be generalised:

Definition 3.3.7. Let C be a monoidal category. A **monoid** or **algebra** in C is a triplet (M, e, m) consisting of an object M and morphisms $e : I \to M$, $m : M \otimes M \to M$ such that the following diagrams commute:

$$\begin{tikzcd} I \otimes M \arrow[r, "e \otimes \mathrm{id}_M"] \arrow[dr, "\lambda_M"'] & M \otimes M \arrow[d, "m"] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M \otimes I \arrow[r, "\mathrm{id}_M \otimes e"] \arrow[dr, "\rho_M"'] & M \otimes M \arrow[d, "m"] \\ & M \end{tikzcd}$$

$$\begin{tikzcd}
(M \otimes M) \otimes M \arrow[rr, "\alpha_{M,M,M}"] \arrow[d, "m \otimes \mathrm{id}_M"'] & & M \otimes (M \otimes M) \arrow[d, "\mathrm{id}_M \otimes m"] \\
M \otimes M \arrow[dr, "m"'] & & M \otimes M \arrow[dl, "m"] \\
& M &
\end{tikzcd}$$

If (M, e, m) and (M', e', m') are algebras in C, a **homomorphism of algebras** $(M, e, m) \to (M', e', m')$ is a morphism $f : M \to M'$ in C making the diagrams below commute:

$$\begin{tikzcd}
I \arrow[d, "e"'] \arrow[dr, "e'"] & \\
M \arrow[r, "f"'] & M'
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes M \arrow[r, "f \otimes f"] \arrow[d, "m"] & M' \otimes M' \arrow[d, "m'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

We will often abuse notation by saying M is an algebra, without mention of e and m.

Example 3.3.8. An algebra in **Ab** (considered as a monoidal category via the tensor product) is the same thing as a ring. More generally, if A is a commutative ring, then an algebra in **Mod**(A) is a (not necessarily commutative) associative unital A-algebra.

Proposition 3.3.9. *If (M, e, m) is an algebra in a monoidal category C, then the data*

$$\begin{aligned}
TX &= X \otimes M \\
\eta_X &= (\mathrm{id}_X \otimes e) \circ \rho_X^{-1} \\
\mu_X &= (\mathrm{id}_X \otimes m) \circ \alpha_{X,M,M} \\
\sigma_{X,Y} &= \alpha_{X,Y,M}^{-1}
\end{aligned}$$

defines a strong monad $\mathbb{T} = (T, \eta, \mu, \sigma)$ on C. A **right M-module** *is defined to be a module for this monad.*

Proof. It is clear that η, μ, σ so defined are indeed natural transformations. The monad axioms for (T, η, μ) are satisfied because (M, e, m) is an algebra, and it is easy to show that σ is a strength for $\mathbb{T}$ using the coherence theorem (3.1.6). ■

Example 3.3.10. Of course, if A is a commutative ring and (M, e, m) is an A-algebra, then a right M-module in the sense of the above proposition is the same thing as a right M-module in the traditional sense.

Conversely, each strong monad induces an algebra—though in general this process loses much information.

Proposition 3.3.11. *Let* $\mathbb{T} = (T, \eta, \mu, \sigma)$ *be a strong monad on a monoidal category* C*. If* $M = TI$, $e = \eta_I$, $m = \mu_I \circ T\rho_{TI} \circ \sigma_{TI,I}$*, then* (M, e, m) *is an algebra in* C*.*

Proof. Direct calculation shows the unit axioms are satisfied:

$$\begin{aligned} m \circ (e \otimes \mathrm{id}_M) &= \mu_I \circ T\rho_{TI} \circ \sigma_{TI,I} \circ (\eta_I \otimes \mathrm{id}_{TI}) \\ &= \mu_I \circ T(\rho_{TI} \circ (\eta_I \otimes \mathrm{id}_I)) \circ \sigma_{I,I} \\ &= (\mu_I \circ T\eta_I) \circ (T\rho_I \circ \sigma_{I,I}) \\ &= T\rho_I \circ \sigma_{I,I} \\ &= T\lambda_I \circ \sigma_{I,I} \\ &= \lambda_{TI} \end{aligned}$$

$$\begin{aligned} m \circ (\mathrm{id}_M \otimes e) &= \mu_I \circ T\rho_{TI} \circ \sigma_{TI,I} \circ (\mathrm{id}_{TI} \otimes \eta_I) \\ &= \mu_I \circ T\rho_{TI} \circ \eta_{TI \otimes I} \\ &= (\mu_I \circ \eta_{TI}) \circ \rho_{TI} \\ &= \rho_{TI} \end{aligned}$$

The proof of associativity is more lengthy. First, observe that

$$\begin{aligned} & m \circ (\mathrm{id}_M \otimes m) \circ \alpha_{M,M,M} \\ &= \mu_I \circ T\rho_{TI} \circ \sigma_{TI,I} \circ (\mathrm{id}_{TI} \otimes (\mu_I \circ T\rho_{TI} \circ \sigma_{TI,I})) \circ \alpha_{TI,TI,TI} \\ &= \mu_I \circ T\rho_{TI} \circ \mu_{TI \otimes I} \circ T\sigma_{TI,I} \circ \sigma_{TI,TI} \circ (\mathrm{id}_{TI} \otimes (T\rho_{TI} \circ \sigma_{TI,I})) \circ \alpha_{TI,TI,TI} \end{aligned}$$

by the strong monad axioms, and we have

$$\begin{aligned} \sigma_{TI,TI} \circ (\mathrm{id}_{TI} \otimes (T\rho_{TI} \circ \sigma_{TI,I})) \circ \alpha_{TI,TI,TI} \qquad & \\ = T(\mathrm{id}_{TI} \otimes \rho_{TI}) \circ \sigma_{TI,TI \otimes I} \circ (\mathrm{id}_{TI} \otimes \sigma_{TI,TI}) \circ \alpha_{TI,TI,TI} & \\ = T((\mathrm{id}_{TI} \otimes \rho_{TI}) \circ \alpha_{TI,TI,I}) \circ \sigma_{TI \otimes TI,I} & \\ = T\rho_{TI \otimes TI} \circ \sigma_{TI \otimes TI,I} & \end{aligned}$$

by the axioms for σ, α, and ρ, so:

$$m \circ (\mathrm{id}_M \otimes m) \circ \alpha_{M,M,M} = \mu_I \circ T\rho_{TI} \circ \mu_{TI\otimes I} \circ T\big(\sigma_{TI,I} \circ \rho_{TI\otimes TI}\big) \circ \sigma_{TI\otimes TI,I}$$

On the other hand,

$$\begin{aligned} m \circ (m \otimes \mathrm{id}_M) &= \mu_I \circ T\rho_{TI} \circ \sigma_{TI,I} \circ \big((\mu_I \circ T\rho_{TI} \circ \sigma_{TI,I}) \otimes \mathrm{id}_{TI}\big) \\ &= \mu_I \circ T\big(\rho_{TI} \circ ((\mu_I \circ T\rho_{TI} \circ \sigma_{TI,I}) \otimes \mathrm{id}_I)\big) \circ \sigma_{TI\otimes TI,I} \\ &= \mu_I \circ T\big(\mu_I \circ T\rho_{TI} \circ \sigma_{TI,I} \circ \rho_{TI\otimes TI}\big) \circ \sigma_{TI\otimes TI,I} \\ &= \mu_I \circ \mu_{TI} \circ T^2\rho_{TI} \circ T\big(\sigma_{TI,I} \circ \rho_{TI\otimes TI}\big) \circ \sigma_{TI\otimes TI,I} \\ &= \mu_I \circ T\rho_{TI} \circ \mu_{TI\otimes I} \circ T\big(\sigma_{TI,I} \circ \rho_{TI\otimes TI}\big) \circ \sigma_{TI\otimes TI,I} \end{aligned}$$

so we indeed have $m \circ (m \otimes \mathrm{id}_M) = m \circ (\mathrm{id}_M \otimes m) \circ \alpha_{M,M,M}$. ■

Remark 3.3.12. Let $\mathbb{T} = (T, \eta, \mu, \sigma)$ be a strong monad on a monoidal category C. It is not hard to check that $\mathbb{T}$ is isomorphic to the free right M-module monad for some algebra (M, e, m) in C if and only if σ is a natural isomorphism, in which case the above construction recovers (M, e, m) up to isomorphism.

Remark 3.3.13. If $\mathbb{T}$ is a (strong) monad on **Set**, then $\mathbf{Set}^{\mathbb{T}}\big((T1, \mu_1), (T1, \mu_1)\big)$ has a natural monoid structure, and with a little work one can show that it is isomorphic to the monoid (M, e, m) constructed above. This should not be surprising: if $\mathbb{T}$ is the free *right* M-set monad, and $\overline{x} : M \to M$ is the unique M-equivariant map sending e to x, then $\overline{x} \circ \overline{y} = \overline{x \cdot y}$, and so $\mathbf{Set}^{\mathbb{T}}\big((T1, \mu_1), (T1, \mu_1)\big)$ is isomorphic to M in this case.

We shall now prove that the Eilenberg–Moore category of a strong monad on a symmetric monoidal category has a symmetric functorial product, provided it has reflexive coequalisers.

Lemma 3.3.14. *Let $\mathbb{T} = (T, \eta, \mu, \sigma)$ be a strong monad on a symmetric monoidal category C. Let σ' be the natural transformation whose components $\sigma'_{X,Y} : TX \otimes Y \to T(X \otimes Y)$ are given by*

$$\sigma'_{X,Y} = T\gamma_{Y,X} \circ \sigma_{Y,X} \circ \gamma_{TX,Y}$$

for each pair X, Y in C. The following diagrams commute:

$$\begin{tikzcd}
TX\otimes I \arrow[r, "\sigma'_{X,I}"] \arrow[dr, "\rho_{TX}"'] & T(X\otimes I) \arrow[d, "T\rho_X"] \\
& TX
\end{tikzcd}
\qquad
\begin{tikzcd}
X\otimes Y \arrow[r, "\eta_X\otimes \mathrm{id}_Y"] \arrow[dr, "\eta_{X\otimes Y}"'] & TX\otimes Y \arrow[d, "\sigma'_{X,Y}"] \\
& T(X\otimes Y)
\end{tikzcd}$$

$$\begin{tikzcd}
& T(X\otimes Y)\otimes Z \arrow[dr, "\sigma'_{X\otimes Y,Z}"] & \\
(TX\otimes Y)\otimes Z \arrow[ur, "\sigma'_{X,Y}\otimes \mathrm{id}_Z"] \arrow[d, "\alpha_{TX,Y,Z}"'] & & T((X\otimes Y)\otimes Z) \arrow[d, "T\alpha_{X,Y,Z}"] \\
TX\otimes(Y\otimes Z) \arrow[rr, "\sigma'_{X,Y\otimes Z}"'] & & T(X\otimes(Y\otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
T^2X\otimes Y \arrow[r, "\sigma'_{TX,Y}"] \arrow[d, "\mu_C\otimes \mathrm{id}_Y"'] & T(TX\otimes Y) \arrow[r, "T\sigma'_{X,Y}"] & T^2(X\otimes Y) \arrow[d, "\mu_{X\otimes Y}"] \\
TX\otimes Y \arrow[rr, "\sigma'_{X,Y}"'] & & T(X\otimes Y)
\end{tikzcd}$$

Proof. These follow straightforwardly from the axioms for a strong monad on a symmetric monoidal category. ■

Definition 3.3.15. Let $\mathbb{T} = (T,\eta,\mu,\sigma)$ be a strong monad on a symmetric monoidal category C and let σ' be the natural transformation defined in the lemma. Let (A,α), (B,β), and (C,γ) be $\mathbb{T}$-modules. A **bihomomorphism of $\mathbb{T}$-modules** $f : (A,\alpha)\otimes(B,\beta) \to (C,\gamma)$ is a morphism $f : A\otimes B \to C$ in C making both diagrams below commute:

$$\begin{tikzcd}
A\otimes TB \arrow[r, "\sigma_{A,B}"] \arrow[d, "\mathrm{id}_A\otimes\beta"'] & T(A\otimes B) \arrow[r, "Tf"] & TC \arrow[d, "\gamma"] \\
A\otimes B \arrow[rr, "f"'] & & C
\end{tikzcd}$$

$$\begin{tikzcd}
TA \otimes B \arrow[r, "\sigma'_{A,B}"] \arrow[d, "\alpha \otimes \mathrm{id}_B"'] & T(A \otimes B) \arrow[r, "Tf"] & TC \arrow[d, "\gamma"] \\
A \otimes B \arrow[rr, "f"'] & & C
\end{tikzcd}$$

For each positive integer n, we may define **n-fold homomorphisms of $\mathbb{T}$-modules** $(A_1,\alpha_1)\otimes\cdots\otimes(A_n,\alpha_n)\to(C,\gamma)$ analogously using the associator and braiding operation of C. We refer to these generically as multihomomorphisms, and we write $\mathrm{Multi}_{\mathbb{T}}((A_1,\alpha_1),\ldots,(A_n,\alpha_n);(C,\gamma))$ for the collection of all n-fold homomorphisms $(A_1,\alpha_1)\otimes\cdots\otimes(A_n,\alpha_n)\to(C,\gamma)$.[1]

Lemma 3.3.16. *Let $\mathbb{T}=\left(T,\eta,\mu,\sigma\right)$ be a strong monad on a symmetric monoidal category C. Let (A,α), (B,β), and (C,γ) be three $\mathbb{T}$-modules. Given a morphism $f:A\otimes B\to C$ in C, f is a bihomomorphism $(A,\alpha)\otimes(B,\beta)\to(C,\gamma)$ if and only if the following equation holds:*

$$(*)\qquad \gamma\circ Tf\circ\sigma_{A,B}\circ(\alpha\otimes\mathrm{id}_{TB})=\gamma\circ Tf\circ\sigma'_{A,B}\circ\left(\mathrm{id}_{TA}\otimes\beta\right)$$

Proof. First, note that

$$\gamma\circ Tf\circ\sigma_{A,B}\circ(\alpha\otimes\mathrm{id}_{TB})\circ\left(\eta_A\otimes\mathrm{id}_{TB}\right)=\gamma\circ Tf\circ\sigma_{A,B}$$

by the monad axioms and functoriality of $\otimes$, but

$$\begin{aligned}
\gamma\circ Tf\circ\sigma'_{A,B}\circ\left(\mathrm{id}_{TA}\otimes\beta\right)\circ\left(\eta_A\otimes\mathrm{id}_{TB}\right) &= \gamma\circ Tf\circ\sigma'_{A,B}\circ\left(\eta_A\otimes\mathrm{id}_B\right)\circ\left(\mathrm{id}_A\otimes\beta\right)\\
&= \gamma\circ Tf\circ\eta_{A\otimes B}\circ\left(\mathrm{id}_A\otimes\beta\right)\\
&= \gamma\circ\eta_C\circ f\circ\left(\mathrm{id}_A\otimes\beta\right)\\
&= f\circ\left(\mathrm{id}_A\otimes\beta\right)
\end{aligned}$$

by the axioms for γ and σ', so

$$f\circ\left(\mathrm{id}_A\otimes\beta\right)=\gamma\circ Tf\circ\sigma_{A,B}$$

if the equation $(*)$ holds, and by symmetry

$$f\circ(\alpha\otimes\mathrm{id}_B)=\gamma\circ Tf\circ\sigma'_{A,B}$$

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[1] We should choose an explicit bracketing of the monoidal products appearing in this paragraph, but the coherence theorem guarantees that any two choices will yield canonically isomorphic definitions.

also holds when $(*)$ holds. On the other hand, if f is a bihomomorphism as claimed, then we have

$$\begin{aligned}\gamma \circ Tf \circ \sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB}) &= f \circ \left(\mathrm{id}_A \otimes \beta\right) \circ (\alpha \otimes \mathrm{id}_{TB}) \\ &= f \circ (\alpha \otimes \mathrm{id}_B) \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \\ &= \gamma \circ Tf \circ \sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right)\end{aligned}$$

which is precisely the equation $(*)$. ■

Theorem 3.3.17. *Let* $\mathbb{T} = (T, \eta, \mu, \sigma)$ *be a strong monad on a symmetric monoidal category* C. *If* $C^{\mathbb{T}}$ *has coequalisers for reflexive pairs, then, given two* $\mathbb{T}$*-modules* $(A, \alpha), (B, \beta)$*, there exist a* $\mathbb{T}$*-module* $(A, \alpha) \otimes_{\mathbb{T}} (B, \beta)$*, called the* **tensor product** *of* (A, α) *and* (B, β)*, and bijections*

$$C^{\mathbb{T}}((A, \alpha) \otimes_{\mathbb{T}} (B, \beta), (C, \gamma)) \cong \mathrm{Multi}_{\mathbb{T}}((A, \alpha), (B, \beta); (C, \gamma))$$

that are natural in (A, α)*,* (B, β)*, and* (C, γ)*.*

Proof. Define two morphisms $d_0, d_1 : T(TA \otimes TB) \to T(A \otimes B)$ in C as below:

$$\begin{aligned}d_0 &= \mu_{A\otimes B} \circ T\left(\sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB})\right) \\ d_1 &= \mu_{A\otimes B} \circ T\left(\sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right)\right)\end{aligned}$$

These are homomorphisms $(T(TA \otimes TB), \mu_{TA\otimes TB}) \to (T(A \otimes B), \mu_{A\otimes B})$. They have a common splitting in $C^{\mathbb{T}}$:

$$\begin{aligned}d_0 \circ T\left(\eta_A \otimes \eta_B\right) &= \mu_{A\otimes B} \circ T\left(\sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB}) \circ \left(\eta_A \otimes \eta_B\right)\right) \\ &= \mu_{A\otimes B} \circ T\left(\sigma_{A,B} \circ \left(\mathrm{id}_A \otimes \eta_B\right)\right) \\ &= \mu_{A\otimes B} \circ T\eta_{A\otimes B} \\ &= \mathrm{id}_{T(A\otimes B)}\end{aligned}$$

$$\begin{aligned}d_1 \circ T\left(\eta_A \otimes \eta_B\right) &= \mu_{A\otimes B} \circ T\left(\sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \circ \left(\eta_A \otimes \eta_B\right)\right) \\ &= \mu_{A\otimes B} \circ T\left(\sigma'_{A,B} \circ \left(\eta_A \otimes \mathrm{id}_B\right)\right) \\ &= \mu_{A\otimes B} \circ T\eta_{A\otimes B} \\ &= \mathrm{id}_{T(A\otimes B)}\end{aligned}$$

Thus, we may construct the following coequaliser in $C^{\mathbb{T}}$:

$$\begin{tikzcd}(T(TA \otimes TB), \mu_{TA\otimes TB}) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & (T(A \otimes B), \mu_{A\otimes B}) \arrow[r, "p"] & (A, \alpha) \otimes_{\mathbb{T}} (B, \beta)\end{tikzcd}$$

Note that the monad axioms and naturality of η imply

$$p \circ \sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB}) = p \circ \sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right)$$

and the axioms for σ and σ' imply the following diagram commutes,

$$\begin{tikzcd}
 & A \otimes TB \arrow[dr, "\sigma_{A,B}"] & \\
A \otimes B \arrow[ur, "\mathrm{id}_A \otimes \eta_B"] \arrow[rr, "\eta_{A \otimes B}"] \arrow[dr, "\eta_A \otimes \mathrm{id}_B"'] & & T(A \otimes B) \\
 & TA \otimes B \arrow[ur, "\sigma'_{A,B}"'] &
\end{tikzcd}$$

therefore the equations below hold:

$$\begin{aligned}
p \circ \eta_{A \otimes B} \circ \left(\alpha \otimes \beta\right) &= p \circ \sigma_{A,B} \circ \left(\mathrm{id}_A \otimes \eta_B\right) \circ \left(\alpha \otimes \beta\right) \\
&= p \circ \sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB}) \circ \left(\mathrm{id}_{TA} \otimes \eta_B\right) \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \\
&= p \circ \sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \circ \left(\mathrm{id}_{TA} \otimes \eta_B\right) \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \\
&= p \circ \sigma'_{A,B} \circ \left(\mathrm{id}_{TA} \otimes \beta\right) \\
&= p \circ \sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB})
\end{aligned}$$

If $f : (A, \alpha) \otimes (B, \beta) \to (C, \gamma)$ is a bihomomorphism, then the diagram below commutes:

$$\begin{tikzcd}
 & A \otimes TB \arrow[r, "\sigma_{A,B}"] \arrow[dr, "\mathrm{id}_A \otimes \beta"] & T(A \otimes B) \arrow[r, "Tf"] & TC \arrow[d, "\gamma"] \\
TA \otimes TB \arrow[ur, "\alpha \otimes \mathrm{id}_{TB}"] \arrow[dr, "\mathrm{id}_{TA} \otimes \beta"'] & & A \otimes B \arrow[r, "f"] & C \\
 & TA \otimes B \arrow[r, "\sigma'_{A,B}"'] \arrow[ur, "\alpha \otimes \mathrm{id}_B"] & T(A \otimes B) \arrow[r, "Tf"'] & TC \arrow[u, "\gamma"']
\end{tikzcd}$$

It is easy to check that $\gamma \circ Tf$ is a homomorphism $(T(A \otimes B), \mu_{A \otimes B}) \to (C, \gamma)$, and the commutativity of the diagram above implies $\gamma \circ Tf \circ d_0 = \gamma \circ Tf \circ d_1$, so the universal property of $(A, \alpha) \otimes_{\mathbb{T}} (B, \beta)$ as a coequaliser gives a unique homomorphism $\tilde{f} : (A, \alpha) \otimes_{\mathbb{T}} (B, \beta) \to (C, \gamma)$ such that $\tilde{f} \circ p = \gamma \circ Tf$. Note that

$$\tilde{f} \circ p \circ \eta_{A \otimes B} = \gamma \circ Tf \circ \eta_{A \otimes B} = \gamma \circ \eta_C \circ f = f$$

so the mapping $f \mapsto \tilde{f}$ is injective.

On the other hand, if $\tilde{f} : (A, \alpha) \otimes_{\mathbb{T}} (B, \beta) \to (C, \gamma)$ is a given $\mathbb{T}$-module homomorphism, then we can obtain a morphism $f : A \otimes B \to C$ in $\mathcal{C}$ by setting

$$f = \tilde{f} \circ p \circ \eta_{A \otimes B}$$

and we claim this is a bihomomorphism $(A, \alpha) \otimes (B, \beta) \to (C, \gamma)$. Indeed,

$$\begin{aligned} f \circ (\mathrm{id}_A \otimes \beta) &= \tilde{f} \circ p \circ \eta_{A \otimes B} \circ (\mathrm{id}_A \otimes \beta) \\ &= \tilde{f} \circ p \circ \eta_{A \otimes B} \circ (\alpha \otimes \beta) \circ (\eta_A \otimes \mathrm{id}_{TB}) \\ &= \tilde{f} \circ p \circ \sigma_{A,B} \circ (\alpha \otimes \mathrm{id}_{TB}) \circ (\eta_A \otimes \mathrm{id}_{TB}) \\ &= \tilde{f} \circ p \circ \sigma_{A,B} \end{aligned}$$

by the earlier calculation, but we also have

$$\begin{aligned} \gamma \circ Tf \circ \sigma_{A,B} &= \gamma \circ T\tilde{f} \circ Tp \circ T\eta_{A \otimes B} \circ \sigma_{A,B} \\ &= \tilde{f} \circ p \circ \mu_{A \otimes B} \circ T\eta_{A \otimes B} \circ \sigma_{A,B} \\ &= \tilde{f} \circ p \circ \sigma_{A,B} \end{aligned}$$

A similar calculation shows

$$f \circ (\alpha \otimes \mathrm{id}_B) = \tilde{f} \circ p \circ \sigma'_{A,B} = \gamma \circ Tf \circ \sigma'_{A,B}$$

as required for a bihomomorphism $(A, \alpha) \otimes (B, \beta) \to (C, \gamma)$. However,

$$\begin{aligned} \gamma \circ Tf &= \gamma \circ T\tilde{f} \circ Tp \circ T\eta_{A \otimes B} \\ &= \tilde{f} \circ p \circ \mu_{A \otimes B} \circ T\eta_{A \otimes B} \\ &= \tilde{f} \circ p \end{aligned}$$

so by uniqueness this construction is the inverse of the previous one. This establishes a bijection

$$\mathcal{C}^{\mathbb{T}}((A, \alpha) \otimes_{\mathbb{T}} (B, \beta), (C, \gamma)) \cong \mathrm{Multi}_{\mathbb{T}}((A, \alpha), (B, \beta); (C, \gamma))$$

and it is clearly natural in (C, γ). In particular, $p \circ \eta_{A \otimes B}$ is the universal bihomomorphism $(A, \alpha) \otimes (B, \beta) \to (A, \alpha) \otimes_{\mathbb{T}} (B, \beta)$. The right hand side of the displayed bijection is functorial in (A, α) and (B, β), so standard argument[2] shows there is a unique way of making $((A, \alpha), (B, \beta)) \mapsto (A, \alpha) \otimes_{\mathbb{T}} (B, \beta)$ into a functor such that the bijection is natural in (A, α) and (B, β) as well. ■

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[2] See, for example, [CWM, Ch. IV, § 7, Thm 3].

Remark 3.3.18. Let I be the monoidal unit of C. Though one might expect the free $\mathbb{T}$-module (TI, μ_I) to be the tensor unit of $C^{\mathbb{T}}$, the "non-commutativity" of $\mathbb{T}$ is an obstruction to proving this claim. Indeed, let $\mathbb{T}$ be the free group monad on **Set**, and consider the tensor product $G \otimes_{\mathbb{T}} H$. Given any two elements g, g' of G and h, h' of H, we have

$$((gg') \otimes h) \cdot ((gg') \otimes h') = (gg') \otimes (hh') = (g \otimes (hh')) \cdot (g' \otimes (hh'))$$

in $G \otimes_{\mathbb{T}} H$, hence, expanding the both sides,

$$(g \otimes h) \cdot (g' \otimes h) \cdot (g \otimes h') \cdot (g' \otimes h') = (g \otimes h) \cdot (g \otimes h') \cdot (g' \otimes h) \cdot (g' \otimes h')$$

but $G \otimes_{\mathbb{T}} H$ is a group, so we may cancel common terms on both sides of the equation to obtain

$$(g' \otimes h) \cdot (g \otimes h') = (g \otimes h') \cdot (g' \otimes h)$$

which implies $G \otimes_{\mathbb{T}} H \cong G^{\mathrm{ab}} \otimes_{\mathbb{Z}} H^{\mathrm{ab}}$, where G^{ab} and H^{ab} are the abelianisations of G and H, respectively. Thus, there are no candidates for the tensor unit.

4 Commutative monads

Let $\mathbb{T}$ be a one-sorted finitary algebraic theory, and let u and v be terms in n and m variables, respectively. How may we combine u and v to obtain a new term in nm variables $z_{1,1}, \ldots, z_{n,m}$? There are two canonical ways of doing this: we may either go "by rows" and obtain

$$u(v(z_{1,1}, \ldots, z_{1,m}), \ldots, v(z_{n,1}, \ldots, z_{n,m}))$$

or we may go "by columns" and obtain

$$v(u(z_{1,1}, \ldots, z_{n,1}), \ldots, u(z_{1,m}, \ldots, z_{n,m}))$$

but these are *a priori* not the same. $\mathbb{T}$ is said to be **commutative** just if these two constructions always yield provably equal terms.

More generally, we may make the following definition:

Definition 3.4.1. A **commutative monad** on a symmetric monoidal category C is a strong monad $\mathbb{T} = (T, \eta, \mu, \sigma)$ on C such that the diagram below commutes for all objects X and Y in C,

$$\begin{tikzcd}
& T(X\otimes TY) \arrow[r, "T\sigma_{X,Y}"] & T^2(X\otimes Y) \arrow[dr, "\mu_{X\otimes Y}"] & \\
TX\otimes TY \arrow[ur, "\sigma'_{X,TY}"] \arrow[dr, "\sigma_{TX,Y}"'] & & & T(X\otimes Y) \\
& T(TX\otimes Y) \arrow[r, "T\sigma'_{X,Y}"'] & T^2(X\otimes Y) \arrow[ur, "\mu_{X\otimes Y}"'] &
\end{tikzcd}$$

where σ' is the natural transformation defined in lemma 3.3.14.

Lemma 3.4.2. *Let $\mathbb{T}$ be a strong monad on a symmetric monoidal category C. The equation*

$$\mu_{X\otimes Y}\circ T\sigma_{X,Y}\circ\sigma'_{X,TY}\circ(u\otimes v) = \mu_{X\otimes Y}\circ T\sigma'_{X,Y}\circ\sigma_{TX,Y}\circ(u\otimes v) \tag{\dagger}$$

holds for all morphisms $u : I \to TX$ and $v : I \to TY$ in C if and only if the following diagram commutes for all homomorphisms $\bar{u} : (TI, \mu_I) \to (TX, \mu_X)$ and $\bar{v} : (TI, \mu_I) \to (TY, \mu_Y)$ in $C^{\mathbb{T}}$:

(⋆)

$$\begin{tikzcd}
TI \arrow[r, "\bar{u}"] \arrow[d, "\bar{v}"'] & TX \arrow[r, "T\rho_X^{-1}"] & T(X\otimes I) \arrow[dd, dashed, "\tilde{v}"'] & X\otimes TI \arrow[l, "\sigma_{X,I}"'] \arrow[dd, "\mathrm{id}_X\otimes\bar{v}"] \\
TY \arrow[d, "T\lambda_Y^{-1}"'] & & & \\
T(I\otimes Y) \arrow[rr, dashed, "\tilde{u}"] & & T(X\otimes Y) & X\otimes TY \arrow[l, "\sigma_{X,Y}"] \\
TI\otimes Y \arrow[u, "\sigma'_{I,Y}"] \arrow[rr, "\bar{u}\otimes\mathrm{id}_Y"'] & & TX\otimes Y \arrow[u, "\sigma'_{X,Y}"'] &
\end{tikzcd}$$

Here, the dashed arrows are the unique homomorphisms making the respective rectangles commute, as constructed in lemma 3.3.6.

Proof. The free–forgetful adjunction implies that the maps $\bar{u} \mapsto \bar{u}\circ\eta_I$ and $\bar{v} \mapsto \bar{v}\circ\eta_I$ are bijections

$$C^{\mathbb{T}}((TI,\mu_I),(TX,\mu_X)) \xrightarrow{\sim} C(I,TX) \qquad C^{\mathbb{T}}((TI,\mu_I),(TY,\mu_Y)) \xrightarrow{\sim} C(I,TY)$$

so we henceforth identify u and v with $\bar{u}\circ\eta_I$ and $\bar{v}\circ\eta_I$, respectively. By the axioms for σ, we have

$$\begin{aligned}
\sigma_{TX,Y}\circ(u\otimes v)\circ\lambda_I^{-1} &= \sigma_{TX,Y}\circ(u\otimes\mathrm{id}_{TY})\circ(\mathrm{id}_I\otimes v)\circ\lambda_I^{-1}\\
&= T(u\otimes\mathrm{id}_Y)\circ\sigma_{I,Y}\circ(\mathrm{id}_I\otimes v)\circ\lambda_I^{-1}\\
&= T(u\otimes\mathrm{id}_Y)\circ\sigma_{I,Y}\circ\lambda_{TY}^{-1}\circ v\\
&= T\Big((u\otimes\mathrm{id}_Y)\circ\lambda_Y^{-1}\Big)\circ v
\end{aligned}$$

and by symmetry we also have

$$\sigma'_{X,TY}\circ(u\otimes v)\circ\rho_I^{-1} = T\Big((\mathrm{id}_X\otimes v)\circ\rho_X^{-1}\Big)\circ u$$

thus:

$$\begin{aligned}
\tilde{v}\circ T\rho_X^{-1}\circ\bar{u}\circ\eta_I &= \mu_{X\otimes Y}\circ T\Big(\sigma_{X,Y}\circ(\mathrm{id}_X\otimes v)\circ\rho_X^{-1}\Big)\circ u\\
&= \mu_{X\otimes Y}\circ T\sigma_{X,Y}\circ\sigma'_{X,TY}\circ(u\otimes v)\circ\rho_I^{-1}
\end{aligned}$$

$$\begin{aligned}
\tilde{u}\circ T\lambda_Y^{-1}\circ\bar{v}\circ\eta_I &= \mu_{X\otimes Y}\circ T\Big(\sigma'_{X,Y}\circ(u\otimes\mathrm{id}_Y)\circ\lambda_Y^{-1}\Big)\circ v\\
&= \mu_{X\otimes Y}\circ T\sigma'_{X,Y}\circ\sigma_{TX,Y}\circ(u\otimes v)\circ\lambda_I^{-1}
\end{aligned}$$

The coherence theorem (3.1.6) implies $\lambda_I=\rho_I$, and $\bar{u},\bar{v},T\rho_X^{-1},T\lambda_Y^{-1},\tilde{u},\tilde{v}$ are all homomorphisms, so the diagram ($\star$) commutes if and only if the equation (†) holds, as claimed. ■

Proposition 3.4.3. *Let $\mathbb{T}=(T,\eta,\mu,\sigma)$ be a strong monad on a monoidal category C.*

(i) *If $\mathbb{T}$ is commutative, then the diagram ($\star$) commutes for all $\mathbb{T}$-module homomorphisms $\bar{u}:(TI,\mu_I)\to(TX,\mu_X)$ and $\bar{v}:(TI,\mu_I)\to(TY,\mu_Y)$.*

(ii) *Assume C satisfies any* one *of the following conditions:*

  (a) *For any parallel pair $f,g:A\otimes B\to C$ in C, if $f\circ(a\otimes b)=g\circ(a\otimes b)$ for* all *morphisms $a:I\to A$ and $b:I\to B$ in C, then $f=g$.*

  (b) *C is the category of (R,R)-bimodules for a commutative ring R, with R as the monoidal unit and $\otimes_R$ as the monoidal product.*

  (c) *C is a cartesian monoidal category and for any parallel pair $f,g:Z\to W$ in C, if $f\circ z=g\circ z$ for all morphisms $z:1\to Z$ in C, then $f=g$.*

Under this hypothesis, $\mathbb{T}$ is commutative if and only if the diagram $(\star)$ commutes for all $\bar{u}$ and $\bar{v}$.

Proof. (i). If $\mathbb{T}$ is commutative, then the equation (†) is automatically valid for all morphisms $u : I \to TX$, $v : I \to TY$, so the diagram $(\star)$ commutes by the previous lemma.

(ii). Under hypothesis (a), the validity of (†) for all u and v implies

$$\mu_{X\otimes Y} \circ T\sigma_{X,Y} \circ \sigma'_{X,TY} = \mu_{X\otimes Y} \circ T\sigma'_{X,Y} \circ \sigma_{TX,Y}$$

exactly as required to make $\mathbb{T}$ commutative.

Since elementary tensors $a \otimes b$ generate the (R,R)-bimodule $A \otimes_R B$, case (b) reduces to case (a).

In case (c), every morphism $1 \to TX \times TY$ is of the form $\langle u, v\rangle$ for a unique pair of morphisms $u : 1 \to TX$, $v : 1 \to TY$, and $1 \cong 1 \times 1$, so this also reduces to case (a). ■

Now, let us consider what commutativity of $(\star)$ means when $\mathbb{T} = (T, \eta, \mu)$ is a monad on **Set**. Let (A, α) be a $\mathbb{T}$-module, and consider the representable presheaf $\mathcal{C}^{\mathbb{T}}(-,(A,\alpha))$. If we identify the two maps

$$\mathcal{C}^{\mathbb{T}}(\bar{u},(A,\alpha)) : \mathcal{C}^{\mathbb{T}}\big((TX,\mu_X),(A,\alpha)\big) \to \mathcal{C}^{\mathbb{T}}\big((T1,\mu_1),(A,\alpha)\big)$$
$$\mathcal{C}^{\mathbb{T}}(\bar{v},(A,\alpha)) : \mathcal{C}^{\mathbb{T}}\big((TY,\mu_Y),(A,\alpha)\big) \to \mathcal{C}^{\mathbb{T}}\big((T1,\mu_1),(A,\alpha)\big)$$

with $u_{(A,\alpha)} : A^X \to A$ and $v_{(A,\alpha)} : A^Y \to A$, respectively, then we find that the diagram $(\star)$ commutes if and only if the diagram below commutes:

$$\begin{tikzcd}
A^{X\times Y} \arrow[r, "\cong"] \arrow[d, "\cong"'] & (A^X)^Y \arrow[r, "(u_{(A,\alpha)})^Y"] & A^Y \arrow[dd, "v_{(A,\alpha)}"] \\
(A^Y)^X \arrow[d, "(v_{(A,\alpha)})^X"'] & & \\
A^X \arrow[rr, "u_{(A,\alpha)}"'] & & A
\end{tikzcd}$$

Thus, when $\mathbb{T}$ is the free model monad for an equational theory over a one-sorted finitary algebraic signature, we can use the fact that filtered colimits preserve finite products to show that $\mathbb{T}$ is a commutative monad if and only if the associated theory is commutative in the sense explained in the introduction of this section.

Example 3.4.4. Here are some commutative algebraic theories:

- The theory of commutative monoids.
- The theory of A-modules for any commutative ring A.
- The theory of sets equipped with an action of a commutative monoid.

The third example is a special case of the following proposition:

Proposition 3.4.5. *Let (M,e,m) be an algebra in a symmetric monoidal category C, and let $\mathbb{T}=((-)\otimes M,\eta,\mu,\sigma)$ be the associated strong monad constructed in proposition 3.3.9. The algebra (M,e,m) is commutative (in the sense that $m\circ\gamma_{M,M}=m$) if and only if the monad $\mathbb{T}$ is commutative.*

Proof. By coherence theorem (3.1.11), we may assume C is a symmetric *strict* monoidal category without loss of generality. Under this assumption, $\mathbb{T}$ is commutative if and only if the equation below holds:

$$\begin{aligned}&(\mathrm{id}_{X\otimes Y}\otimes m)\circ\left(\gamma_{Y\otimes M,X}\otimes\mathrm{id}_M\right)\circ\gamma_{X\otimes M,Y\otimes M}\\&\qquad=(\mathrm{id}_{X\otimes Y}\otimes m)\circ\left(\gamma_{Y,X}\otimes\mathrm{id}_M\otimes\mathrm{id}_M\right)\circ\left(\gamma_{X\otimes M,Y}\otimes\mathrm{id}_M\right)\end{aligned}$$

The hexagon axiom implies that

$$\left(\gamma_{Y,X}\otimes\mathrm{id}_M\right)\circ\gamma_{X\otimes M,Y}=\mathrm{id}_X\otimes\gamma_{Y,M}$$

and coherence in general gives

$$\left(\gamma_{Y\otimes M,X}\otimes\mathrm{id}_M\right)\circ\gamma_{X\otimes M,Y\otimes M}=\left(\mathrm{id}_{X\otimes Y}\otimes\gamma_{M,M}\right)\circ\left(\mathrm{id}_X\otimes\gamma_{Y,M}\otimes\mathrm{id}_M\right)$$

so we determine that $\mathbb{T}$ is commutative if and only if the equation

$$\mathrm{id}_{X\otimes Y}\otimes\left(m\circ\gamma_{M,M}\right)=\mathrm{id}_{X\otimes Y}\otimes m$$

holds, and this is clearly equivalent to the commutativity of (M,e,m). ■

Remark 3.4.6. The commutativity condition is closer to an interchange law than commutativity in the traditional sense of being invariant under permutations. For example, constants are always invariant under permutation of their arguments (because they have none!), but the interchange law implies a commutative theory has *at most* one constant. In particular, the theory of commutative rings is *not* a commutative theory.

Now, suppose we have two binary operations $\bullet$ and $\circ$ with common unit e in a commutative theory. The interchange law applied to $\bullet$ and $\circ$ yields

$$(x \circ y) \bullet (z \circ w) = (x \bullet z) \circ (y \bullet w)$$

and the **Eckmann–Hilton argument** shows that

$$x \bullet w = x \circ w \qquad y \bullet z = z \circ y \qquad (x \circ y) \bullet w = x \circ (y \bullet w)$$

so $\circ$ and $\bullet$ turn out to be the same associative commutative binary operation! As a corollary, we see that the theory of groups is not a commutative theory, even though the theory of *abelian* groups is.

We now prove the claims made the beginning of the chapter. The first result, due to Linton [1966] and Kock [1971], holds if the base category $\mathcal{C}$ has equalisers. Before we prove their theorem, it will be useful to show that $\mathcal{C}^{\mathbb{T}}$ is cotensored over $\mathcal{C}$—at least in the weak sense described below:

Proposition 3.4.7. *Let $\mathbb{T} = (T, \eta, \mu, \sigma)$ be a strong monad on a symmetric monoidal closed category $\mathcal{C}$. For each object X in $\mathcal{C}$ and each $\mathbb{T}$-module (A, α), there is a $\mathbb{T}$-module $X \pitchfork (A, \alpha)$ whose underlying object is $\mathcal{H}om(X, A)$, and the assignment $(X, (A, \alpha)) \mapsto X \pitchfork (A, \alpha)$ defines a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathbb{T}} \to \mathcal{C}^{\mathbb{T}}$.*

Proof. Define $q_{X,(A,\alpha)} : T\mathcal{H}om(X, A) \to \mathcal{H}om(X, A)$ to be the following morphism in $\mathcal{C}$:

$$q_{X,(A,\alpha)} = \mathcal{H}om\Big(X, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X}\Big) \circ \lambda_{T\mathcal{H}om(X,A),X}$$

Here, $\lambda_{Y,Z} : Y \to \mathcal{H}om(Z, Y \otimes Z)$ and $\mathrm{ev}_{Y,Z} : \mathcal{H}om(Y,Z) \otimes Y \to Z$ are, respectively, the unit and counit of the tensor–hom adjunction in $\mathcal{C}$. We claim $q_{X,(A,\alpha)}$ is a $\mathbb{T}$-module structure on $\mathcal{H}om(X, A)$. Indeed, by the $\mathbb{T}$-module axioms and lemma 3.3.14,

$$\begin{aligned}
&q_{X,(A,\alpha)} \circ \eta_{\mathcal{H}om(X,A)} \\
&\qquad = \mathcal{H}om\Big(X, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X}\Big) \circ \lambda_{T\mathcal{H}om(X,A),X} \circ \eta_{\mathcal{H}om(X,A)} \\
&\qquad = \mathcal{H}om\Big(X, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X} \circ (\eta_{\mathcal{H}om(X,A)} \otimes \mathrm{id}_X)\Big) \circ \lambda_{\mathcal{H}om(X,A),X} \\
&\qquad = \mathcal{H}om(X, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \eta_{\mathcal{H}om(X,A) \otimes X}) \circ \lambda_{\mathcal{H}om(X,A),X} \\
&\qquad = \mathcal{H}om(X, \alpha \circ \eta_A \circ \mathrm{ev}_{X,A}) \circ \lambda_{T\mathcal{H}om(X,A),X} \\
&\qquad = \mathcal{H}om(X, \mathrm{ev}_{X,A}) \circ \lambda_{\mathcal{H}om(X,A),X} = \mathrm{id}_{\mathcal{H}om(X,A)}
\end{aligned}$$

and also, by extranaturality of λ,

$$\begin{aligned}
&q_{X,(A,\alpha)} \circ \mu_{\mathcal{Hom}(X,A)} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}\Big) \circ \lambda_{T\mathcal{Hom}(X,A),X} \circ \mu_{\mathcal{Hom}(X,A)} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X} \circ \big(\mu_{\mathcal{Hom}(X,A)} \otimes \mathrm{id}_X\big)\Big) \circ \lambda_{T^2\mathcal{Hom}(X,A),X} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \mu_{\mathcal{Hom}(X,A)\otimes X} \circ T\sigma'_{\mathcal{Hom}(X,A),X} \circ \sigma'_{T\mathcal{Hom}(X,A),X}\Big)
\end{aligned}$$

but we have

$$\sigma'_{\mathcal{Hom}(X,A),X} \circ \big(\mu_{\mathcal{Hom}(X,A)} \otimes \mathrm{id}_X\big) = \mu_{\mathcal{Hom}(X,A)\otimes X} \circ T\sigma'_{\mathcal{Hom}(X,A),X} \circ \sigma'_{T\mathcal{Hom}(X,A),X}$$

and moreover:

$$\begin{aligned}
\alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \mu_{\mathcal{Hom}(X,A)\otimes X} \circ T\sigma'_{\mathcal{Hom}(X,A),X} &= \alpha \circ \mu_A \circ T\Big(T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}\Big) \\
&= \alpha \circ T\Big(\alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}\Big)
\end{aligned}$$

On the other hand, the right triangle identity for λ and ev implies

$$\mathrm{ev}_{X,A} \circ \big(q_{X,(A,\alpha)} \otimes \mathrm{id}_X\big) = \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}$$

hence,

$$\alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \mu_{\mathcal{Hom}(X,A)\otimes X} \circ T\sigma'_{\mathcal{Hom}(X,A),X} = \alpha \circ T\big(\mathrm{ev}_{X,A} \circ \big(q_{X,(A,\alpha)} \otimes \mathrm{id}_X\big)\big)$$

and therefore

$$\begin{aligned}
&q_{X,(A,\alpha)} \circ Tq_{X,(A,\alpha)} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}\Big) \circ \lambda_{T\mathcal{Hom}(X,A),X} \circ Tq_{X,(A,\alpha)} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A}\big) \circ \sigma'_{\mathcal{Hom}(X,A),X} \circ \big(Tq_{X,(A,\alpha)} \otimes \mathrm{id}_X\big)\Big) \circ \lambda_{T^2\mathcal{Hom}(X,A),X} \\
&= \mathcal{Hom}\Big(X, \alpha \circ T\big(\mathrm{ev}_{X,A} \circ \big(q_{X,(A,\alpha)} \otimes \mathrm{id}_X\big)\big) \circ \sigma'_{\mathcal{Hom}(X,A),X}\Big) \circ \lambda_{T^2\mathcal{Hom}(X,A),X} \\
&= q_{X,(A,\alpha)} \circ \mu_{\mathcal{Hom}(X,A)}
\end{aligned}$$

as required of a $\mathbb{T}$-module structure.

We define $X \pitchfork (A,\alpha)$ to be $\mathcal{Hom}(X,A)$ equipped with the $\mathbb{T}$-module structure $q_{X,(A,\alpha)}$. It is clear that $q_{X,(A,\alpha)}$ is natural in (A,α). It is also natural in X:

indeed, given a morphism $f : Y \to X$ in $\mathcal{C}$, we have

$$\begin{aligned}
&\mathcal{H}om(f,A) \circ q_{X,(A,\alpha)} \\
&= \mathcal{H}om(f,A) \circ \mathcal{H}om\Big(X, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X}\Big) \circ \lambda_{T\mathcal{H}om(X,A),X} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X}\Big) \circ \mathcal{H}om(f, T\mathcal{H}om(X,A) \otimes X) \circ \lambda_{T\mathcal{H}om(X,A),X} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,A),X} \circ (\mathrm{id}_{T\mathcal{H}om(X,A)} \otimes f)\Big) \circ \lambda_{T\mathcal{H}om(X,A),Y} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{X,A} \circ (\mathrm{id}_{\mathcal{H}om(X,A)} \otimes f)) \circ \sigma'_{\mathcal{H}om(X,A),Y}\Big) \circ \lambda_{T\mathcal{H}om(X,A),Y} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{Y,A} \circ (\mathcal{H}om(f,A) \otimes \mathrm{id}_Y)) \circ \sigma'_{\mathcal{H}om(X,A),Y}\Big) \circ \lambda_{T\mathcal{H}om(X,A),Y} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{Y,A}) \circ \sigma'_{\mathcal{H}om(Y,A),Y} \circ (T\mathcal{H}om(f,A) \otimes \mathrm{id}_Y)\Big) \circ \lambda_{T\mathcal{H}om(X,A),Y} \\
&= \mathcal{H}om\Big(Y, \alpha \circ T(\mathrm{ev}_{Y,A}) \circ \sigma'_{\mathcal{H}om(Y,A),Y}\Big) \circ \lambda_{T\mathcal{H}om(Y,A),Y} \circ T\mathcal{H}om(f,A) \\
&= q_{Y,(A,\alpha)} \circ T\mathcal{H}om(f,A)
\end{aligned}$$

by naturality of σ' and extranaturality of λ and ev. It follows that the assignment $(X,(A,\alpha)) \mapsto X \pitchfork (A,\alpha)$ is functorial in X and in (A,α). ■

Remark 3.4.8. The proposition simultaneously generalises two well-known facts:

- If $\mathbb{T}$ is a monad on **Set**, then $\mathbf{Set}^{\mathbb{T}}$ has all small products and is therefore cotensored over **Set**.
- If $\mathbb{T}$ is an equational theory over a one-sorted finitary algebraic signature, then $\mathbf{Set}(X,A)$ is a $\mathbb{T}$-model under pointwise operations whenever A is a $\mathbb{T}$-model.

Theorem 3.4.9 (Kock–Linton)**.** *Let $\mathbb{T} = (T,\eta,\mu,\sigma)$ be a strong monad on a symmetric monoidal closed category $\mathcal{V}$. Suppose $\mathcal{V}$ has equalisers for all parallel pairs.*

(i) *$\mathcal{V}^{\mathbb{T}}$ can be given the structure of a $\mathcal{V}$-category so that $F^{\mathbb{T}} : \mathcal{V} \to \mathcal{V}^{\mathbb{T}}$ and $U^{\mathbb{T}} : \mathcal{V}^{\mathbb{T}} \to \mathcal{V}$ become $\mathcal{V}$-functors.*

(ii) *For $X \pitchfork (B,\beta)$ as constructed in the previous proposition, we have bijections*

$$\mathcal{V}(X, \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))) \cong \mathcal{V}^{\mathbb{T}}((A,\alpha), X \pitchfork (B,\beta))$$

that are natural in X, (A,α), and (B,β).

(iii) *If $\mathbb{T}$ is commutative, then there is a functor $\mathcal{Hom}_{\mathbb{T}} : (\mathcal{V}^{\mathbb{T}})^{\mathrm{op}} \times \mathcal{V}^{\mathbb{T}} \to \mathcal{V}^{\mathbb{T}}$ with bijections*

$$\mathcal{V}^{\mathbb{T}}\big((C,\gamma),\mathcal{Hom}_{\mathbb{T}}((A,\alpha),(B,\beta))\big) \cong \mathrm{Multi}_{\mathbb{T}}((C,\gamma),(A,\alpha);(B,\beta))$$

that are natural in (A,α), (B,β), and (C,γ).

Proof. $T : \mathcal{V} \to \mathcal{V}$ is a $\mathcal{V}$-functor by proposition 3.2.17, and we write $t_{X,Y}$ for its component morphisms $\mathcal{Hom}(X,Y) \to \mathcal{Hom}(TX,TY)$ in $\mathcal{V}$. Let (A,α) and (B,β) be $\mathbb{T}$-modules. We define morphisms $d^0, d^1 : \mathcal{Hom}(A,B) \to \mathcal{Hom}(TA,B)$ by

$$\begin{aligned} d^0 &= \mathcal{Hom}(TA,\beta) \circ t_{A,B} \\ d^1 &= \mathcal{Hom}(\alpha,B) \end{aligned}$$

and we construct the following equaliser in $\mathcal{V}$:

$$\begin{tikzcd} \mathcal{Hom}_{\mathbb{T}}((A,\alpha),(B,\beta)) \arrow[r, "u_{(A,\alpha),(B,\beta)}"] & \mathcal{Hom}(A,B) \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] & \mathcal{Hom}(TA,B) \end{tikzcd}$$

We can show that $\mathcal{V}^{\mathbb{T}}$ can be given the structure of a $\mathcal{V}$-category by formalising in diagrammatic form the usual argument showing that the class of $\mathbb{T}$-module homomorphisms is closed under composition in $\mathcal{V}$. The morphisms $u_{(A,\alpha),(B,\beta)} : \mathcal{Hom}_{\mathbb{T}}((A,\alpha),(B,\beta)) \to \mathcal{Hom}(A,B)$ are then seen to be the components of the $\mathcal{V}$-functor $U^{\mathbb{T}} : \mathcal{V}^{\mathbb{T}} \to \mathcal{V}$.

Proposition 3.2.19 implies $\mu : T^2 \Rightarrow T$ is a $\mathcal{V}$-natural transformation, i.e. the diagram below commutes for all X and Y in $\mathcal{V}$:

$$\begin{tikzcd} & \mathcal{Hom}(T^2X,T^2Y) \arrow[dr, "\mathcal{Hom}(T^2X,\mu_Y)"] & \\ \mathcal{Hom}(X,Y) \arrow[ur, "t_{TX,TY} \circ t_{X,Y}"] \arrow[dr, "t_{X,Y}"'] & & \mathcal{Hom}(T^2X,TY) \\ & \mathcal{Hom}(TX,TY) \arrow[ur, "\mathcal{Hom}(\mu_X,TY)"'] & \end{tikzcd}$$

The universal property of $\mathcal{Hom}_{\mathbb{T}}((TX,\mu_X),(TY,\mu_Y))$ as an equaliser then gives us a factorisation of $t_{X,Y} : \mathcal{Hom}(X,Y) \to \mathcal{Hom}(TX,TY)$ through $u_{(TX,\mu_X),(TY,\mu_Y)}$, and it follows that $F^{\mathbb{T}} : \mathcal{V} \to \mathcal{V}^{\mathbb{T}}$ is also a $\mathcal{V}$-functor.

Recalling proposition 3.2.15, we see that these diagrams commute:

$$\begin{tikzcd}
\mathcal{V}(X, \mathcal{H}om(A, B)) \arrow[r, "\mathcal{V}(X, \mathcal{H}om(\alpha, B))"] \arrow[d, "\cong"'] & \mathcal{V}(X, \mathcal{H}om(TA, B)) \arrow[d, "\cong"] \\
\mathcal{V}(A, \mathcal{H}om(X, B)) \arrow[r, "\mathcal{V}(\alpha, \mathcal{H}om(X, B))"'] & \mathcal{V}(TA, \mathcal{H}om(X, B))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{V}(X, \mathcal{H}om(TA, TB)) \arrow[r, "\mathcal{V}(X, \mathcal{H}om(TA, \beta))"] \arrow[d, "\cong"'] & \mathcal{V}(X, \mathcal{H}om(TA, B)) \arrow[d, "\cong"] \\
\mathcal{V}(TA, \mathcal{H}om(X, TB)) \arrow[r, "\mathcal{V}(TA, \mathcal{H}om(X, \beta))"'] & \mathcal{V}(TA, \mathcal{H}om(X, B))
\end{tikzcd}$$

On the other hand, the diagram below

$$\begin{tikzcd}
\mathcal{V}(X, \mathcal{H}om(A, B)) \arrow[r, "\mathcal{V}(X, t_{A,B})"] \arrow[d, "\cong"'] & \mathcal{V}(X, \mathcal{H}om(TA, TB)) \arrow[d, "\cong"] \\
\mathcal{V}(A, \mathcal{H}om(X, B)) \arrow[r] & \mathcal{V}(TA, \mathcal{H}om(X, TB))
\end{tikzcd}$$

commutes by extranaturality, where the unlabelled arrow is the morphism

$$\mathcal{V}\Big(TA, \mathcal{H}om\Big(X, T(\mathrm{ev}_{X,A}) \circ \sigma'_{\mathcal{H}om(X,B),X}\Big) \circ \lambda_{T\mathcal{H}om(X,B),X}\Big)$$

as in the previous proposition, so the following equaliser diagram in **Set**

$$\mathcal{V}(X, \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))) \longrightarrow \mathcal{V}(X, \mathcal{H}om(A, B)) \rightrightarrows \mathcal{V}(X, \mathcal{H}om(TA, B))$$

is isomorphic to the equaliser diagram

$$\mathcal{V}^{\mathbb{T}}((A,\alpha), X \pitchfork (B,\beta)) \longrightarrow \mathcal{V}(A, \mathcal{H}om(X, B)) \rightrightarrows \mathcal{V}(A, \mathcal{H}om(X, TB))$$

and this yields the required natural bijections:

$$\mathcal{V}(X, \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))) \cong \mathcal{V}^{\mathbb{T}}((A,\alpha), X \pitchfork (B,\beta))$$

Now, suppose $\mathbb{T}$ is a commutative monad. We must show that each hom-object $\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$ can be given a natural $\mathbb{T}$-module structure. Consider the morphism $T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \to \mathcal{H}om(A, B)$ defined below:

$$\mathcal{H}om\Big(A, \beta \circ T(\mathrm{ev}_{A,B}) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ (Tu_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A)\Big) \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}$$

Here, $\lambda_{X,Y} : X \to \mathcal{H}om(Y, X \otimes Y)$ and $\mathrm{ev}_{X,Y} : \mathcal{H}om(X, Y) \otimes X \to Y$ are, respectively, the unit and counit of the tensor–hom adjunction in $\mathcal{V}$. We wish to show that it factorises through $u_{(A,\alpha),(B,\beta)}$. By extranaturality of λ, postcomposing with $\mathcal{H}om(\alpha, B)$ gives

$$\mathcal{H}om\Big(TA, \beta \circ T(\mathrm{ev}_{A,B}) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ (Tu_{(A,\alpha),(B,\beta)} \otimes \alpha)\Big) \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}$$

and we also have

$$\begin{aligned}
&T(\mathrm{ev}_{A,B}) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ (Tu_{(A,\alpha),(B,\beta)} \otimes \alpha) \\
&\qquad = T(\mathrm{ev}_{A,B} \circ (u_{(A,\alpha),(B,\beta)} \otimes \alpha)) \circ \sigma'_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA} \\
&\qquad = T(\mathrm{ev}_{TA,B} \circ ((\mathcal{H}om(\alpha, B) \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA})) \circ \sigma'_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}
\end{aligned}$$

where in the last equation we have used extranaturality of ev, but by definition

$$\mathcal{H}om(\alpha, B) \circ u_{(A,\alpha),(B,\beta)} = \mathcal{H}om(TA, \beta) \circ t_{A,B} \circ u_{(A,\alpha),(B,\beta)}$$

and so:

$$\begin{aligned}
&\mathrm{ev}_{TA,B} \circ ((\mathcal{H}om(\alpha, B) \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA}) \\
&\qquad = \mathrm{ev}_{TA,B} \circ ((\mathcal{H}om(TA, \beta) \circ t_{A,B} \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA}) \\
&\qquad = \beta \circ \mathrm{ev}_{TA,TB} \circ ((t_{A,B} \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA})
\end{aligned}$$

From the proof of proposition 3.2.17, we know that

$$T(\mathrm{ev}_{A,B}) \circ \sigma_{\mathcal{H}om(A,B),A} = \mathrm{ev}_{TA,TB} \circ (t_{A,B} \otimes \mathrm{id}_{TA})$$

therefore:

$$\begin{aligned}
&\mathrm{ev}_{TA,B} \circ ((\mathcal{H}om(\alpha, B) \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA}) \\
&\qquad = \beta \circ \mathrm{ev}_{TA,TB} \circ ((t_{A,B} \circ u_{(A,\alpha),(B,\beta)}) \otimes \mathrm{id}_{TA}) \\
&\qquad = \beta \circ T(\mathrm{ev}_{A,B}) \circ \sigma_{\mathcal{H}om(A,B),A} \circ (u_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_{TA}) \\
&\qquad = \beta \circ T(\mathrm{ev}_{A,B} \circ (u_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A)) \circ \sigma_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}
\end{aligned}$$

Thus, by putting the above calculations together, we have shown that the morphism $T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \to \mathcal{H}om(TA, B)$ we constructed is equal to

$$\begin{aligned}
&\mathcal{H}om(TA, \beta \circ T(\beta \circ T(\mathrm{ev}_{A,B} \circ (u_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A)))) \\
&\qquad \circ \mathcal{H}om\Big(TA, T\sigma_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A} \circ \sigma'_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}\Big) \\
&\qquad\qquad \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}
\end{aligned}$$

but $\beta \circ T\beta = \beta \circ \mu_B$ by the $\mathbb{T}$-module axioms, and μ is a natural transformation, so this is in turn equal to

$$\begin{aligned}&\mathcal{H}om\big(TA, \beta \circ T\big(\mathrm{ev}_{A,B} \circ \big(u_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\big)\big)\\ &\quad\circ \mathcal{H}om\Big(TA, \mu_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))\otimes A} \circ T\sigma_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A} \circ \sigma'_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}\Big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}\end{aligned}$$

and $\mathbb{T}$ is a commutative monad, so we may replace the middle term and get

$$\begin{aligned}&\mathcal{H}om\big(TA, \beta \circ T\big(\mathrm{ev}_{A,B} \circ \big(u_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\big)\big)\\ &\quad\circ \mathcal{H}om\Big(TA, \mu_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))\otimes A} \circ T\sigma'_{\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A} \circ \sigma_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}\Big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}\end{aligned}$$

which, by the naturality of μ and σ', is equal to

$$\begin{aligned}&\mathcal{H}om\Big(TA, \beta \circ T\Big(\beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(Tu_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\Big)\Big)\\ &\qquad\qquad\qquad\qquad\circ \mathcal{H}om\big(TA, \sigma_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}\big) \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),TA}\end{aligned}$$

and thence to

$$\begin{aligned}&\mathcal{H}om\Big(TA, \beta \circ T\Big(\beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(Tu_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\Big)\Big)\\ &\qquad\qquad\qquad\qquad\qquad\circ t_{A,T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))\otimes A} \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}\end{aligned}$$

because we have $\mathcal{H}om(TY, \sigma_{X,Y}) \circ \lambda_{X,TY} = t_{Y,X\otimes Y} \circ \lambda_{X,Y}$ from the proof of proposition 3.2.17. Since t is a natural transformation, the above is also equal to

$$\begin{aligned}&\mathcal{H}om(TA, \beta) \circ t_{A,B}\\ &\qquad\circ \mathcal{H}om\Big(TA, \beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(Tu_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\Big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}\end{aligned}$$

and therefore the morphism $T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \to \mathcal{H}om(A,B)$ that we started with factors through $u_{(A,\alpha),(B,\beta)}$, i.e. we have a morphism

$$p_{(A,\alpha),(B,\beta)} : T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \to \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$$

such that $u_{(A,\alpha),(B,\beta)} \circ p_{(A,\alpha),(B,\beta)}$ is equal to:

$$\mathcal{H}om\Big(A, \beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(Tu_{(A,\alpha),(B,\beta)} \otimes \mathrm{id}_A\big)\Big) \circ \lambda_{T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)),A}$$

It is clear that $p_{(A,\alpha),(B,\beta)}$ so defined is natural in (A,α) and (B,β), but we still have to show that it makes $\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$ into a $\mathbb{T}$-module. By the construction of $p_{(A,\alpha),(B,\beta)}$, we have the following commutative diagram in $\mathcal{V}$:

$$\begin{tikzcd}
T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \arrow[r, "Tu_{(A,\alpha),(B,\beta)}"] \arrow[d, "p_{(A,\alpha),(B,\beta)}"'] & T\mathcal{H}om(A,B) \arrow[d, "q_{A,(B,\beta)}"] \\
\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \arrow[r, "u_{(A,\alpha),(B,\beta)}"'] & \mathcal{H}om(A,B)
\end{tikzcd}$$

Here, $q_{A,(B,\beta)}$ is the $\mathbb{T}$-module structure constructed in the proof of the previous proposition. Since $u_{(A,\alpha),(B,\beta)}$ is a (regular) monomorphism, an argument similar to the one used in proposition 1.2.5 shows that $p_{(A,\alpha),\big(B,\beta\big)}$ is also a $\mathbb{T}$-module structure. It is not hard to show that it is natural in (A,α) and $\big(B,\beta\big)$, so we obtain the required functor $\mathcal{H}om_{\mathbb{T}} : \left(\mathcal{V}^{\mathbb{T}}\right)^{\mathrm{op}} \times \mathcal{V}^{\mathbb{T}} \to \mathcal{V}^{\mathbb{T}}$.

Consider a $\mathbb{T}$-module homomorphism $\tilde{f} : (C,\gamma) \to \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$. It corresponds to a morphism $f : C \otimes A \to B$ by the tensor–hom adjunction in $\mathcal{V}$, and we claim f is a bihomomorphism $(C,\gamma)\otimes(A,\alpha) \to (B,\beta)$. Indeed, note that

$$\begin{aligned}
f \circ \big(\gamma \otimes \mathrm{id}_A\big) &= \mathrm{ev}_{A,B} \circ \Big(\big(u_{(A,\alpha),(B,\beta)} \circ \tilde{f} \circ \gamma\big) \otimes \mathrm{id}_A\Big) \\
&= \mathrm{ev}_{A,B} \circ \Big(\big(u_{(A,\alpha),(B,\beta)} \circ p_{(A,\alpha),(B,\beta)} \circ T\tilde{f}\big) \otimes \mathrm{id}_A\Big) \\
&= \mathrm{ev}_{A,B} \circ \Big(\big(q_{A,(B,\beta)} \circ T\tilde{f}\big) \otimes \mathrm{id}_A\Big) \\
&= \beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(T\tilde{f} \otimes \mathrm{id}_A\big) \\
&= \beta \circ T\Big(\mathrm{ev}_{A,B} \circ \big(\tilde{f} \otimes \mathrm{id}_A\big)\Big) \circ \sigma'_{C,A} \\
&= \beta \circ f \circ \sigma'_{C,A}
\end{aligned}$$

so f is homomorphic in the first variable, and

$$\begin{aligned}
f \circ (\mathrm{id}_C \otimes \alpha) &= \mathrm{ev}_{A,B} \circ \Big(\big(u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \alpha\Big) \\
&= \mathrm{ev}_{TA,B} \circ \Big(\big(\mathcal{H}om(\alpha, B) \circ u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \mathrm{id}_{TA}\Big) \\
&= \mathrm{ev}_{TA,B} \circ \Big(\big(\mathcal{H}om\big(TA, \beta\big) \circ t_{A,B} \circ u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \mathrm{id}_{TA}\Big) \\
&= \beta \circ \mathrm{ev}_{TA,TB} \circ \Big(\big(t_{A,B} \circ u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \mathrm{id}_{TA}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \beta \circ T\big(\mathrm{ev}_{A,B}\big) \circ \sigma_{\mathcal{H}om(A,B),A} \circ \Big(\big(u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \mathrm{id}_{TA}\Big) \\
&= \beta \circ T\Big(\mathrm{ev}_{A,B} \circ \Big(\big(u_{(A,\alpha),(B,\beta)} \circ \tilde{f}\big) \otimes \mathrm{id}_{A}\Big)\Big) \circ \sigma_{C,A} \\
&= \beta \circ Tf \circ \sigma_{C,A}
\end{aligned}$$

so f is homomorphic in the second variable as well.

Conversely, suppose $f : (C,\gamma) \otimes (A,\alpha) \to (B,\beta)$ is a bihomomorphism. Then, its adjoint transpose is a morphism $\hat{f} : C \to \mathcal{H}om(A,B)$ in $\mathcal{V}$, and we have

$$\begin{aligned}
\mathcal{H}om(\alpha, B) \circ \hat{f} &= \mathcal{H}om(\alpha, B) \circ \mathcal{H}om\big(A, f\big) \circ \lambda_{C,A} \\
&= \mathcal{H}om\big(TA, f\big) \circ \mathcal{H}om(\alpha, C \otimes A) \circ \lambda_{C,A} \\
&= \mathcal{H}om\big(TA, f \circ (\mathrm{id}_C \otimes \alpha)\big) \circ \lambda_{C,TA} \\
&= \mathcal{H}om\big(TA, \beta \circ Tf \circ \sigma_{C,A}\big) \circ \lambda_{C,TA} \\
&= \mathcal{H}om\big(TA, \beta \circ Tf\big) \circ t_{A,C\otimes A} \circ \lambda_{C,A} \\
&= \mathcal{H}om\big(TA, \beta\big) \circ t_{A,B} \circ \mathcal{H}om\big(A, f\big) \circ \lambda_{C,A} \\
&= \mathcal{H}om\big(TA, \beta\big) \circ t_{A,B} \circ \hat{f}
\end{aligned}$$

so, by the universal property of $\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$ as an equaliser, $\hat{f}$ must factor as $\hat{f} = u_{(A,\alpha),(B,\beta)} \circ \tilde{f}$ for a unique morphism $\tilde{f} : C \to \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$ in $\mathcal{V}$. We claim it is a homomorphism $(C,\gamma) \to \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))$. Since $u_{(A,\alpha),(B,\beta)}$ is a (regular) monomorphism in $\mathcal{V}$ and also a homomorphism in $\mathcal{V}^{\mathbb{T}}$, it is enough to show that the outer rectangle in the diagram below commutes:

$$\begin{tikzcd}
TC \arrow[r, "T\tilde{f}"] \arrow[d, "\gamma"'] & T\mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \arrow[r, "Tu_{(A,\alpha),(B,\beta)}"] \arrow[d, "p_{(A,\alpha),(B,\beta)}"'] & T\mathcal{H}om(A,B) \arrow[d, "q_{A,(B,\beta)}"] \\
C \arrow[r, "\tilde{f}"'] & \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta)) \arrow[r, "u_{(A,\alpha),(B,\beta)}"'] & \mathcal{H}om(A,B)
\end{tikzcd}$$

It is not hard to verify that

$$\begin{aligned}
\hat{f} \circ \gamma &= \mathcal{H}om\big(A, f\big) \circ \lambda_{C,A} \circ \gamma \\
&= \mathcal{H}om\big(A, f \circ (\gamma \otimes \mathrm{id}_A)\big) \circ \lambda_{TC,A} \\
&= \mathcal{H}om\Big(A, \beta \circ Tf \circ \sigma'_{C,A}\Big) \circ \lambda_{TC,A} \\
&= \mathcal{H}om\Big(A, \beta \circ T\Big(\mathrm{ev}_{A,B} \circ \Big(\hat{f} \otimes \mathrm{id}_A\Big)\Big) \circ \sigma'_{C,A}\Big) \circ \lambda_{TC,A}
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{H}om\Big(A, \beta \circ T(\mathrm{ev}_{A,B}) \circ \sigma'_{\mathcal{H}om(A,B),A} \circ \big(T\hat{f} \otimes \mathrm{id}_A\big)\Big) \circ \lambda_{TC,A} \\
&= \mathcal{H}om\Big(A, \beta \circ T(\mathrm{ev}_{A,B}) \circ \sigma'_{\mathcal{H}om(A,B),A}\Big) \circ \lambda_{T\mathcal{H}om(A,B),A} \circ T\hat{f} \\
&= q_{A,(B,\beta)} \circ T\hat{f}
\end{aligned}$$

and so $\tilde{f}$ is indeed a $\mathbb{T}$-module homomorphism. These two constructions are clearly mutually inverse and give a bijection

$$\mathcal{V}^{\mathbb{T}}\left((C,\gamma), \mathcal{H}om_{\mathbb{T}}((A,\alpha),(B,\beta))\right) \cong \mathrm{Multi}_{\mathbb{T}}((C,\gamma),(A,\alpha);(B,\beta))$$

that is natural in (A,α), (B,β), and (C,γ), as required. ■

Of course, having a $\mathbb{T}$-module structure on the hom-object is only interesting when it is also an internal hom in the sense of a (symmetric) monoidal closed category. As usual, we need a little technical lemma:

Lemma 3.4.10. *Let $\mathcal{V}$ be a locally small symmetric monoidal closed category, and let C be a $\mathcal{V}$-category whose underlying category C_0 is locally small. If there is a functor $(-)\pitchfork(-) : \mathcal{V}^{\mathrm{op}} \times C_0 \to C_0$ for which there are bijections*

$$\mathcal{V}(X, C(A,B)) \cong C_0(A, X \pitchfork B)$$

natural in X, A, and B, then $C(-,B)$ maps colimits in C_0 to limits in $\mathcal{V}$.

Proof. Let $A : \mathcal{J} \to C$ be a diagram in C, and let $\lambda : A \Rightarrow \Delta C$ be a colimiting cocone in C. Since $C(-,B) : C^{\mathrm{op}} \to \mathbf{Set}$ maps colimits to limits by definition, for each X in $\mathcal{V}$ and each B in C, $C_0(\lambda, X \pitchfork B) : \Delta C_0(C, X \pitchfork B) \Rightarrow C_0(A, X \pitchfork B)$ is a limiting cone in **Set**. Naturality of the bijections in the statement of the lemma then imply $\mathcal{V}(X, C(\lambda, B)) : \Delta\mathcal{V}(X, C(C,B)) \Rightarrow \mathcal{V}(X, C(A,B))$ is also a limiting cone in **Set** for all X in $\mathcal{V}$, so $C(\lambda, B) : \Delta C(C,B) \Rightarrow C(A,B)$ must be a limiting cone in $\mathcal{V}$. ■

Theorem 3.4.11. *Let $\mathbb{T} = (T, \eta, \mu, \sigma)$ be a commutative monad on a symmetric monoidal closed category $\mathcal{V}$. If $\mathcal{V}$ has equalisers and $\mathcal{V}^{\mathbb{T}}$ has coequalisers for reflexive pairs, and the forgetful functor $U^{\mathbb{T}} : \mathcal{V}^{\mathbb{T}} \to \mathcal{V}$ sends regular epimorphisms to epimorphisms, then $\mathcal{V}^{\mathbb{T}}$ is a symmetric monoidal closed category under the tensor product with tensor unit (TI, μ_I), where I is the monoidal unit of C and the internal hom is as constructed in the Kock–Linton theorem.*

Proof. Half of work toward this theorem was done in theorem 3.3.17, where we constructed the tensor product and in theorem 3.4.9 where we constructed (what will be) the internal hom. Since $\otimes$ is symmetric, we immediately obtain a natural isomorphism

$$\mathrm{Multi}_{\mathbb{T}}((A,\alpha),(B,\beta);-) \cong \mathrm{Multi}_{\mathbb{T}}((B,\beta),(A,\alpha);-)$$

and thus a natural isomorphism $\mathsf{\gamma}_{(A,\alpha),(B,\beta)} : (A,\alpha) \otimes_{\mathbb{T}} (B,\beta) \to (B,\beta) \otimes_{\mathbb{T}} (A,\alpha)$. We shall now show that there are natural isomorphisms

$$\begin{aligned} &\mathrm{Multi}_{\mathbb{T}}((A,\alpha) \otimes_{\mathbb{T}} (B,\beta),(C,\gamma);-) \\ &\qquad\qquad \cong \mathrm{Multi}_{\mathbb{T}}((A,\alpha),(B,\beta),(C,\gamma);-) \\ &\qquad\qquad\qquad\qquad \cong \mathrm{Multi}_{\mathbb{T}}((A,\alpha),(B,\beta) \otimes_{\mathbb{T}} (C,\gamma);-) \end{aligned}$$

induced by the coherence isomorphisms $\mathsf{a}_{A,B,C}$ and $\mathsf{\gamma}_{A,B}$ in $\mathcal{V}$. Suppose we have a multihomomorphism of the following type:

$$f : (A,\alpha) \otimes ((B,\beta) \otimes (C,\gamma)) \to (D,\delta)$$

In particular, the equation

$$f \circ (\alpha \otimes \mathrm{id}_{B\otimes C}) = \delta \circ Tf \circ \sigma'_{A,B\otimes C}$$

holds, so the diagram below commmutes by naturality of σ,

$$\begin{tikzcd}
TA \otimes T(B \otimes C) \arrow[r, "\sigma_{TA,B\otimes C}"] \arrow[dd, "\alpha \otimes \mathrm{id}_{T(B\otimes C)}"'] & T(TA \otimes (B \otimes C)) \arrow[r, "T\sigma'_{A,B\otimes C}"] \arrow[dd, "T(\alpha \otimes \mathrm{id}_{B\otimes C})"'] & T^2(A \otimes (B \otimes C)) \arrow[d, "T^2 f"] \\
 & & T^2 D \arrow[d, "T\delta"] \\
A \otimes T(B \otimes C) \arrow[r, "\sigma_{A,B\otimes C}"'] & T(A \otimes (B \otimes C)) \arrow[r, "Tf"'] & TD
\end{tikzcd}$$

and the commutativity of $\mathbb{T}$ makes this diagram commute as well,

$$\begin{tikzcd}
TA \otimes T(B \otimes C) \arrow[r, "\sigma'_{A,T(B\otimes C)}"] \arrow[d, "\sigma_{TA,B\otimes C}"'] & T(A \otimes T(B \otimes C)) \arrow[d, "T\sigma_{A,B\otimes C}"] & & \\
T(TA \otimes (B \otimes C)) \arrow[d, "T\sigma'_{A,B\otimes C}"'] & T^2(A \otimes (B \otimes C)) \arrow[r, "T^2 f"] \arrow[d, "\mu_{A\otimes(B\otimes C)}"] & T^2 D \arrow[r, "T\delta"] \arrow[d, "\mu_D"] & TD \arrow[d, "\delta"] \\
T^2(A \otimes (B \otimes C)) \arrow[r, "\mu_{A\otimes(B\otimes C)}"'] & T(A \otimes (B \otimes C)) \arrow[r, "Tf"'] & TD \arrow[r, "\delta"'] & D
\end{tikzcd}$$

so we may deduce the following equation:

$$\delta \circ T\left(\delta \circ Tf \circ \sigma_{A,B\otimes C}\right) \circ \sigma'_{A,T(B\otimes C)} = \delta \circ Tf \circ \sigma_{A,B\otimes C} \circ \left(\alpha \otimes \mathrm{id}_{T(B\otimes C)}\right)$$

Under the tensor–hom adjunction for $\mathcal{V}$, $f : A \otimes (B \otimes C) \to D$ naturally corresponds to a morphism $\hat{f} : A \to \mathcal{H}om(B \otimes C, D)$ such that

$$\mathcal{H}om\left(\sigma'_{B,C}, \delta\right) \circ t_{T(B\otimes C),TD} \circ \hat{f} = \mathcal{H}om(\beta \otimes \mathrm{id}_C, D) \circ \hat{f}$$
$$\mathcal{H}om(\sigma_{B,C}, \delta) \circ t_{T(B\otimes C),TD} \circ \hat{f} = \mathcal{H}om(\mathrm{id}_B \otimes \gamma, D) \circ \hat{f}$$

and internalising the calculations of lemma 3.3.16 yields the equation below:

$$\begin{aligned} &\mathcal{H}om(\sigma_{B,C} \circ (\beta \otimes \mathrm{id}_{TC}), \delta) \circ t_{B\otimes C,D} \circ \hat{f} \\ &\qquad = \mathcal{H}om(\beta \otimes \gamma, D) \circ \hat{f} \\ &\qquad\qquad = \mathcal{H}om(\sigma_{B,C} \circ (\mathrm{id}_{TB} \otimes \gamma), \delta) \circ t_{B\otimes C,D} \circ \hat{f} \end{aligned}$$

We then use the $\mathcal{V}$-naturality of μ to derive the equation

$$\begin{aligned} &\mathcal{H}om_{\mathbb{T}}(\mu_{B\otimes C} \circ T(\sigma_{B,C} \circ (\beta \otimes \mathrm{id}_{TC})), \delta) \circ f_{B\otimes C,D} \circ \hat{f} \\ &\qquad = \mathcal{H}om_{\mathbb{T}}(\mu_{B\otimes C} \circ T(\sigma_{B,C} \circ (\mathrm{id}_{TB} \otimes \gamma)), \delta) \circ f_{B\otimes C,D} \circ \hat{f} \end{aligned}$$

where $f_{X,Y} : \mathcal{H}om(X, Y) \to \mathcal{H}om_{\mathbb{T}}(F^{\mathbb{T}}X, F^{\mathbb{T}}Y)$ is the $\mathcal{V}$-enrichment of $F^{\mathbb{T}}$. The preceding lemma says $\mathcal{H}om_{\mathbb{T}}$ sends coequalisers in the first variable to equalisers, so there is a unique morphism $\hat{g} : A \to \mathcal{H}om_{\mathbb{T}}((B,\beta) \otimes_{\mathbb{T}} (C,\gamma), (D,\delta))$ such that

$$\mathcal{H}om_{\mathbb{T}}(p_{(B,\beta),(C,\gamma)}, (D,\delta)) \circ \hat{g} = \mathcal{H}om_{\mathbb{T}}\left(F^{\mathbb{T}}(B \otimes C), \delta\right) \circ f_{B\otimes C,D} \circ \hat{f}$$

where the homomorphism $p_{(B,\beta),(C,\gamma)} : F^{\mathbb{T}}(B \otimes C) \to (B,\beta) \otimes_{\mathbb{T}} (C,\gamma)$ is the coequaliser of $\mu_{B\otimes C} \circ T(\sigma_{B,C} \circ (\beta \otimes \mathrm{id}_{TC}))$ and $\mu_{B\otimes C} \circ T(\sigma_{B,C} \circ (\mathrm{id}_{TB} \otimes \gamma))$. Writing $g : A \otimes (B \otimes_{\mathbb{T}} C) \to D$ for the transpose of $u_{F^{\mathbb{T}}(TB\otimes TC),(D,\delta)} \circ \hat{g}$, we get

$$g \circ (\mathrm{id}_A \otimes p) = \delta \circ Tf \circ \sigma_{A,B\otimes C}$$

from the definition of $\hat{g}$, and g is a $\mathbb{T}$-module homomorphism in the second variable since $\hat{g}$ is an A-indexed family of $\mathbb{T}$-module homomorphisms. The homomorphism $p_{(B,\beta),(C,\gamma)}$ is a regular epimorphism in $\mathcal{V}^{\mathbb{T}}$ by definition, so it follows from the hypotheses of the theorem that the underlying morphism in $\mathcal{V}$ is an epimorphism. Because $TA \otimes (-)$ has a right adjoint, $\mathrm{id}_A \otimes p$ is also an epimorphism in $\mathcal{V}$, so to prove that the equation

$$\delta \circ Tg \circ \sigma'_{A,(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)} = g \circ \big(\alpha \otimes \mathrm{id}_{(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)}\big)$$

is valid, it is enough to show that the equation

$$\delta \circ Tg \circ \sigma'_{A,(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)} \circ \big(\mathrm{id}_{TA} \otimes p\big) = g \circ \big(\alpha \otimes \mathrm{id}_{(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)}\big) \circ \big(\mathrm{id}_{TA} \otimes p\big)$$

holds, but the earlier calculation implies

$$\begin{aligned}
\delta \circ Tg \circ \sigma'_{A,(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)} \circ \big(\mathrm{id}_{TA} \otimes p\big) &= \delta \circ T\big(g \circ (\mathrm{id}_A \otimes p)\big) \circ \sigma'_{A,T(B\otimes C)} \\
&= \delta \circ T\big(\delta \circ Tf \circ \sigma_{A,B\otimes C}\big) \circ \sigma'_{A,T(B\otimes C)} \\
&= \delta \circ Tf \circ \sigma_{A,B\otimes C} \circ \big(\alpha \otimes \mathrm{id}_{T(B\otimes C)}\big) \\
&= g \circ \big(\mathrm{id}_A \otimes p\big) \circ \big(\alpha \otimes \mathrm{id}_{T(B\otimes C)}\big) \\
&= g \circ \big(\alpha \otimes \mathrm{id}_{(B,\beta)\otimes_{\mathbb{T}}(C,\gamma)}\big) \circ \big(\mathrm{id}_A \otimes p\big)
\end{aligned}$$

and so g is a bihomomorphism of $\mathbb{T}$-modules, and hence corresponds to a unique homomorphism $\tilde{f} : (A,\alpha) \otimes_{\mathbb{T}} ((B,\beta) \otimes_{\mathbb{T}} (C,\gamma)) \to (D,\delta)$. Conversely, every such homomorphism $\tilde{f}$ must come from a unique multihomomorphism $\tilde{f} : (A,\alpha) \otimes ((B,\beta) \otimes (C,\gamma)) \to (D,\delta)$. Using the associativity of $\otimes$ in $\mathcal{V}$ and a symmetric argument, we obtain the required natural isomorphisms $\gamma_{(A,\alpha),(B,\beta),(C,\gamma)} : (A,\alpha) \otimes_{\mathbb{T}} ((B,\beta) \otimes_{\mathbb{T}} (C,\gamma)) \xrightarrow{\sim} ((A,\alpha) \otimes_{\mathbb{T}} (B,\beta)) \otimes_{\mathbb{T}} (C,\gamma)$, and the coherence axioms are satisfied in $\mathcal{V}^{\mathbb{T}}$ because the corresponding coherence axioms are satisfied in $\mathcal{V}$.

Let (A,α), (B,β), and (C,γ) be $\mathbb{T}$-modules. To show that (TI,μ_I) is a tensor unit, it is enough to show that there are bijections

$$\begin{aligned}
\mathrm{Multi}_{\mathbb{T}}((A,\alpha),(TI,\mu_I);(C,\gamma)) &\cong \mathcal{V}^{\mathbb{T}}((A,\alpha),(C,\gamma)) \\
\mathrm{Multi}_{\mathbb{T}}((TI,\mu_I),(B,\beta);(C,\gamma)) &\cong \mathcal{V}^{\mathbb{T}}((B,\beta),(C,\gamma))
\end{aligned}$$

that are natural in (A,α), (B,β), and (C,γ). Suppose $f : (A,\alpha) \otimes (TI,\mu_I) \to (C,\gamma)$ is a bihomomorphism. The diagrams below commute,

$$\begin{tikzcd}
TA \arrow[r, "\alpha"] \arrow[d, "\rho_{TA}^{-1}"'] & A \arrow[d, "\rho_A^{-1}"] \\
TA \otimes I \arrow[r, "\alpha \otimes \mathrm{id}_I"] \arrow[d, "\mathrm{id}_{TA} \otimes \eta_I"'] & A \otimes I \arrow[d, "\mathrm{id}_A \otimes \eta_I"] \\
TA \otimes TI \arrow[r, "\alpha \otimes \mathrm{id}_{TI}"'] & A \otimes TI
\end{tikzcd}$$

$$\begin{tikzcd}
TA \arrow[d, "\rho^{-1}_{TA}"'] \arrow[dr, "T\rho^{-1}_{A}"] & & \\
TA\otimes I \arrow[r, "\sigma'_{A,I}"] \arrow[d, "\mathrm{id}_{TA}\otimes\eta_I"'] & T(A\otimes I) \arrow[d, "T(\mathrm{id}_A\otimes\eta_I)"] & \\
TA\otimes TI \arrow[r, "\sigma'_{A,TI}"] \arrow[d, "\alpha\otimes\mathrm{id}_{TI}"'] & T(A\otimes TI) \arrow[r, "Tf"] & TC \arrow[d, "\gamma"] \\
A\otimes TI \arrow[rr, "f"'] & & C
\end{tikzcd}$$

so the morphism $\bar{f} : A \to C$ defined by

$$\bar{f} = f \circ \big(\mathrm{id}_A \otimes \eta_I\big) \circ \rho^{-1}_A$$

is in fact a homomorphism $(A,\alpha) \to (C,\gamma)$. Similarly, given a bihomomorphism $g : (TI,\mu_I) \otimes (B,\beta) \to (C,\gamma)$,

$$\bar{g} = g \circ \big(\eta_I \otimes \mathrm{id}_B\big) \circ \lambda^{-1}_B$$

defines a homomorphism $(B,\beta) \to (C,\gamma)$.

Conversely, suppose $\bar{f} : (A,\alpha) \to (C,\gamma)$ is a homomorphism. The following diagram commutes:

$$\begin{tikzcd}
TA\otimes TI \arrow[r, "\sigma_{TA,I}"] \arrow[d, "\alpha\otimes\mathrm{id}_{TI}"'] & T(TA\otimes I) \arrow[r, "T\rho_{TA}"] \arrow[d, "T(\alpha\otimes\mathrm{id}_I)"'] & T^2A \arrow[r, "\mu_A"] \arrow[d, "T\alpha"'] & TA \arrow[d, "\alpha"] \\
A\otimes TI \arrow[r, "\sigma_{A,I}"] \arrow[d, "\bar{f}\otimes\mathrm{id}_{TI}"'] & T(A\otimes I) \arrow[r, "T\rho_A"] \arrow[d, "T\big(\bar{f}\otimes\mathrm{id}_I\big)"'] & TA \arrow[r, "\alpha"] \arrow[d, "T\bar{f}"'] & A \arrow[d, "\bar{f}"] \\
C\otimes TI \arrow[r, "\sigma_{C,I}"'] & T(C\otimes I) \arrow[r, "T\rho_C"'] & TC \arrow[r, "\gamma"'] & C
\end{tikzcd}$$

Consider the morphism $f : A \otimes TI \to C$ defined by this formula:

$$f = \bar{f} \circ \alpha \circ T\rho_A \circ \sigma_{A,I}$$

We claim it is a bihomomorphism $(A,\alpha) \otimes (TI,\mu_I) \to (C,\gamma)$. Indeed, the commutativity of the preceding diagram says

$$f \circ (\alpha \otimes \mathrm{id}_{TI}) = \bar{f} \circ \alpha \circ \mu_A \circ T\rho_{TA} \circ \sigma_{TA,I}$$

but we also have

$$\begin{aligned}
\gamma \circ Tf \circ \sigma'_{A,TI} &= \gamma \circ T\big(\bar{f} \circ \alpha \circ T\rho_A \circ \sigma_{A,I}\big) \circ \sigma'_{A,TI} \\
&= \bar{f} \circ \alpha \circ T\big(\alpha \circ T\rho_A \circ \sigma_{A,I}\big) \circ \sigma'_{A,TI} \\
&= \bar{f} \circ \alpha \circ \mu_A \circ T^2\rho_A \circ T\sigma_{A,I} \circ \sigma'_{A,TI} \\
&= \bar{f} \circ \alpha \circ T\rho_A \circ \big(\mu_{A\otimes I} \circ T\sigma_{A,I} \circ \sigma'_{A,TI}\big) \\
&= \bar{f} \circ \alpha \circ T\rho_A \circ \big(\mu_{A\otimes I} \circ T\sigma'_{A,I} \circ \sigma_{TA,I}\big) \\
&= \bar{f} \circ \alpha \circ \mu_A \circ T\big(T\rho_A \circ \sigma'_{A,I}\big) \circ \sigma_{TA,I} \\
&= \bar{f} \circ \alpha \circ \mu_A \circ T\rho_{TA} \circ \sigma_{TA,I}
\end{aligned}$$

by the axioms for σ and σ', and this diagram also commutes,

$$\begin{tikzcd}
A\otimes T^2I \arrow[r, "\sigma_{A,TI}"] \arrow[d, "\mathrm{id}_A\otimes\mu_I"'] & T(A\otimes TI) \arrow[r, "T\sigma_{A,I}"] & T^2(A\otimes I) \arrow[r, "T^2\rho_A"] \arrow[d, "\mu_{A\otimes I}"'] & T^2A \arrow[r, "T\alpha"] \arrow[d, "\mu_A"'] & TA \arrow[d, "\alpha"] \\
A\otimes TI \arrow[rr, "\sigma_{A,I}"] \arrow[d, "\bar{f}\otimes\mathrm{id}_{TI}"'] & & T(A\otimes I) \arrow[r, "T\rho_A"] \arrow[d, "T\left(\bar{f}\otimes\mathrm{id}_I\right)"'] & TA \arrow[r, "\alpha"] \arrow[d, "T\bar{f}"'] & A \arrow[d, "\bar{f}"] \\
C\otimes TI \arrow[rr, "\sigma_{C,I}"'] & & T(C\otimes I) \arrow[r, "T\rho_C"'] & TC \arrow[r, "\gamma"'] & C
\end{tikzcd}$$

therefore,

$$\begin{aligned}
f \circ \big(\mathrm{id}_A \otimes \mu_I\big) &= \bar{f} \circ \alpha \circ T\alpha \circ T^2\rho_A \circ T\sigma_{A,I} \circ \sigma_{A,TI} \\
&= \gamma \circ T\big(\bar{f} \circ \alpha \circ T\rho_A \circ \sigma_{A,I}\big) \circ \sigma_{A,TI} \\
&= \gamma \circ Tf \circ \sigma_{A,TI}
\end{aligned}$$

as required of a bihomomorphism. A similar argument shows that the formula

$$g = \bar{g} \circ \beta \circ T\lambda_B \circ \sigma'_{I,B}$$

defines a bihomomorphism $(TI,\mu_I)\otimes(B,\beta) \to (C,\gamma)$ when $\bar{g} : (B,\beta) \to (C,\gamma)$ is a homomorphism. We note that

$$\begin{aligned}
f \circ \big(\mathrm{id}_A \otimes \eta_I\big) \circ \rho_A^{-1} &= \bar{f} \circ \alpha \circ T\rho_A \circ \sigma_{A,I} \circ \big(\mathrm{id}_A \otimes \eta_I\big) \circ \rho_A^{-1} \\
&= \bar{f} \circ \alpha \circ T\rho_A \circ \eta_{A\otimes I} \circ \rho_A^{-1} \\
&= \bar{f} \circ \big(\alpha \circ \eta_A\big) \circ \big(\rho_A \circ \rho_A^{-1}\big) \\
&= \bar{f}
\end{aligned}$$

and similarly

$$g \circ (\eta_I \otimes \mathrm{id}_B) \circ \lambda_B^{-1} = \bar{g}$$

so $\bar{f}$ and $\bar{g}$ can be recovered from f and g.

On the other hand, given a bihomomorphism f, we have

$$\begin{aligned} \bar{f} \circ \alpha \circ T\rho_A \circ \sigma_{A,I} &= \gamma \circ T\left(\bar{f} \circ \rho_A\right) \circ \sigma_{A,I} \\ &= \gamma \circ T\left(f \circ (\mathrm{id}_A \otimes \eta_I) \circ \rho_A^{-1} \circ \rho_A\right) \circ \sigma_{A,I} \\ &= \gamma \circ T(f \circ (\mathrm{id}_A \otimes \eta_I)) \circ \sigma_{A,I} \\ &= \gamma \circ Tf \circ \sigma_{A,TI} \circ (\mathrm{id}_A \otimes T\eta_I) \\ &= f \circ (\mathrm{id}_A \otimes \mu_I) \circ (\mathrm{id}_A \otimes T\eta_I) \\ &= f \end{aligned}$$

and similarly, given a bihomomorphism g, we have

$$\bar{g} \circ \beta \circ T\lambda_A \circ \sigma'_{I,B} = g$$

so f and g can be recovered from $\bar{f}$ and $\bar{g}$. We conclude that the two constructions are mutually inverse, and this gives us the required natural bijection. Note that the induced unitors in $\mathcal{V}^{\mathbb{T}}$ satisfy the coherence and compatibility axioms because the unitors in $\mathcal{V}$ do. ■

Corollary 3.4.12. *If* $\mathbb{T}$ *is a commutative monad on* **Set**, *then* $\mathbf{Set}^{\mathbb{T}}$ *is a symmetric monoidal closed category, and the composite functor*

$$U^{\mathbb{T}} \circ \mathcal{H}om_{\mathbb{T}} : \left(\mathbf{Set}^{\mathbb{T}}\right)^{\mathrm{op}} \times \mathbf{Set}^{\mathbb{T}} \to \mathbf{Set}$$

is simply the ordinary hom functor for $\mathbf{Set}^{\mathbb{T}}$.

Proof. **Set** certainly has equalisers, and theorem 1.3.14 says $\mathbf{Set}^{\mathbb{T}}$ has coequalisers, so $\mathbf{Set}^{\mathbb{T}}$ is a symmetric monoidal closed category by the above theorem. The claim regarding $U^{\mathbb{T}} \circ \mathcal{H}om_{\mathbb{T}}$ is a straightforward consequence of the construction of $\mathcal{H}om_{\mathbb{T}}$ we gave. ■

Remark 3.4.13. Let $\mathbb{T}$ be a monad on **Set**. It is entirely possible for $\mathbf{Set}^{\mathbb{T}}$ to be a symmetric monoidal closed category even when $\mathbb{T}$ is *not* commutative. For example, if $\mathbb{T}$ is the free G-set monad for some group G, then $\mathbf{Set}^{\mathbb{T}} \cong \mathbf{Set}^{G}$ is a presheaf topos, and presheaf toposes are always cartesian closed categories. Explicitly, the exponential object in this case is the set

$$[X \to Y] = \{f : G \times X \to Y \,|\, f(g, g \cdot x) = g \cdot f(e, x) \text{ for all } g \text{ in } G\}$$

equipped with the G-action $h \cdot f = f(- \cdot h, -)$. Note that the underlying set of $[X \to Y]$ is *not* the set of G-equivariant maps $X \to Y$ in general! Rather, by the exponential adjunction, we have

$$\mathbf{Set}^G(1, [X \to Y]) \cong \mathbf{Set}^G(1 \times X, Y) \cong \mathbf{Set}^G(X, Y)$$

so the set of G-equivariant maps $X \to Y$ is merely the set of G-invariant elements of $[X \to Y]$.

Nonetheless, if G is an *abelian* group, then the theorem applies and we get *another* symmetric monoidal closed structure on $\mathbf{Set}^G$ in which the underlying set of $\mathcal{H}om_G(X, Y)$ is indeed the set of G-equivariant maps $X \to Y$. These two symmetric monoidal closed structures coincide if and only if G is the trivial group.

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