The functor of points

Z.L. Low

29 January 2015

Abstract

The functor of points is a way of defining schemes without going through locally ringed spaces. The essential idea is to replace points in the classical sense with generalised elements, i.e. morphisms whose domain need not be point-like in any sense, de-emphasising the special role of prime ideals and fields in the traditional approach.

As usual in algebraic geometry, all rings under consideration are commutative unless otherwise stated. For simplicity, we work with **CRing**, the category of rings, but it should be clear that we could equally well work with the category of R-algebras for any ring R. We could also replace **CRing** with the category of *finitely generated* R-algebras for any ring R, but then we will only get schemes locally of finite type over R instead of general schemes over R.

The functor of points approach to scheme theory is used in only a handful of textbooks, such as [Demazure and Gabriel, 1970] and [Jantzen, 2003]. Aside from some basic category theory and some basic commutative algebra, I aim to give a self-contained introduction in these notes.

One technical point that must be mentioned is that the category $[\mathbf{CRing}, \mathbf{Set}]$ is not locally small; indeed, it is so large that we cannot work with it properly even in von Neumann–Gödel–Bernays set theory (NBG). To use the terminology of Adámek, Herrlich and Strecker [ACC], the collections of objects and morphisms in $[\mathbf{CRing}, \mathbf{Set}]$ are conglomerates, not classes. However, we can get around this by replacing $[\mathbf{CRing}, \mathbf{Set}]$ with the full subcategory of *accessible* functors $\mathbf{CRing} \to \mathbf{Set}$, which is then locally small but still closed under limits and colimits for small diagrams. Of course, one then has to check that all the functors $\mathbf{CRing} \to \mathbf{Set}$ we work with are indeed accessible, but we will not do this.

1 The Zariski topology

Definition 1.1. A **Zariski-local epimorphism** in $[\mathbf{CRing}, \mathbf{Set}]$ is a morphism $\alpha : F \to G$ satisfying the following condition:

- For any ring A and any element y of $G(A)$, there exist elements $a_1, \ldots, a_n$ of A (where n is possibly zero) and elements x_i of $F\left(A\left[a_i^{-1}\right]\right)$ (for $1 \le i \le n$) such that $\alpha_{A[a_i^{-1}]}\left(x_i\right) = G\left(\gamma_i\right)(y)$ (for $1 \le i \le n$), where $\gamma_i : A \to A\left[a_i^{-1}\right]$ is the universal ring homomorphism that inverts a_i, and

$$a_1 b_1 + \cdots + a_n b_n = 1$$

for some elements $b_1, \ldots, b_n$ of A.

REMARK 1.2. If $\alpha : F \to G$ is a Zariski-local epimorphism and A is a field, then $\alpha_A : F(A) \to G(A)$ is surjective (because every non-zero element of A is invertible).

Lemma 1.3. *Let A be a ring, let $\hbar^A = \mathbf{CRing}(A, -)$, and let $U \subseteq \hbar^A$ be a subfunctor. The following are equivalent:*

(i) *The inclusion $U \hookrightarrow \hbar^A$ is a Zariski-local epimorphism.*

(ii) *There exist elements $a_1, \ldots, a_n$ of A such that $\gamma_i \in U\left(A\left[a_i^{-1}\right]\right)$, where $\gamma_i : A \to A\left[a_i^{-1}\right]$ is the universal ring homomorphism that inverts a_i, and*

$$a_1 b_1 + \cdots + a_n b_n = 1$$

for some elements $b_1, \ldots, b_n$ of A.

Proof. (i) ⇒ (ii). Apply the definition to the element id_A of $\hbar^A(A)$.

(ii) ⇒ (i). Let R be a ring and let f be an element of $\hbar^A(R)$, i.e. a ring homomorphism $A \to R$. Then, for $r_i = f\left(a_i\right)$, there is a commutative diagram in **CRing** of the form below,

$$\begin{array}{ccc} A & \xrightarrow{\gamma_i} & A\left[a_i^{-1}\right] \\ {\scriptstyle f}\downarrow & & \downarrow \\ R & \xrightarrow[\gamma_i']{} & R\left[r_i^{-1}\right] \end{array}$$

where $\gamma_i' : R \to R\left[r_i^{-1}\right]$ is the localising homomorphism. Thus, for each i, $\gamma_i' \circ f \in U\left(R\left[r_i^{-1}\right]\right)$; moreover,

$$r_1 f\left(b_1\right) + \cdots + r_n f\left(b_n\right) = 1$$

so we are done. ■

Proposition 1.4.

(i) *The class of Zariski-local epimorphisms in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under pullback, i.e. for any pullback diagram in* $[\mathbf{CRing}, \mathbf{Set}]$, *say*

$$\begin{array}{ccc} F' & \longrightarrow & F \\ {\scriptstyle \alpha'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G' & \longrightarrow & G \end{array}$$

if $\alpha : F \to G$ *is a Zariski-local epimorphism, then so is* $\alpha' : F' \to G'$.

(ii) *Let* $\alpha : F \to G$ *and* $\beta : G \to H$ *be morphisms in* $[\mathbf{CRing}, \mathbf{Set}]$. *If* $\beta \circ \alpha : F \to H$ *is a Zariski-local epimorphism, then so is* $\beta : G \to H$.

(iii) *The class of Zariski-local epimorphisms in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under composition.*

Proof. These are straightforward exercises. ◊

Corollary 1.5. *For each ring* A, *let* $J(A)$ *be the collection of subfunctors* $U \subseteq \hbar^A$ *such that the inclusion is a Zariski-local epimorphism. Then* J *is a Grothendieck topology on* $\mathbf{CRing}^{\mathrm{op}}$, *called the* **Zariski topology**. ■

Definition 1.6.

- A **Zariski-covering sieve** on a ring A is a subfunctor $U \subseteq \hbar^A$ such that the inclusion $U \hookrightarrow \hbar^A$ is a Zariski-local epimorphism.

- A **Zariski sheaf** is a functor $F : \mathbf{CRing} \to \mathbf{Set}$ that satisfies the sheaf condition with respect to the Zariski topology on $\mathbf{CRing}^{\mathrm{op}}$, i.e. for any ring A and any Zariski-covering sieve $U \subseteq \hbar^A$, the induced map

$$[\mathbf{CRing}, \mathbf{Set}]\left(\hbar^A, F\right) \to [\mathbf{CRing}, \mathbf{Set}](U, F)$$

is a bijection.

Proposition 1.7. *The Zariski topology on* **CRing**$^{\mathrm{op}}$ *is subcanonical, i.e. for any ring A, the representable functor $\hbar^A$ is a Zariski sheaf.*

Proof. Let R be a ring, let $a_1, \dots, a_n, b_1, \dots, b_n$ be elements of R such that

$$a_1 b_1 + \cdots + a_n b_n = 1$$

and let $a_{i,j} = a_i a_j$. We will show that the following diagram in **CRing** is an equaliser diagram,

$$R \longrightarrow \prod_{i=1}^{n} R\left[a_i^{-1}\right] \rightrightarrows \prod_{i=1}^{n} \prod_{j=1}^{n} R\left[a_{i,j}^{-1}\right]$$

where the arrows are induced by the various localising homomorphisms.

To begin, we verify that the homomorphism $R \to \prod_{i=1}^{n} R\left[a_i^{-1}\right]$ is injective. Let c be an element of R such that $c = 0$ in each $R\left[a_i^{-1}\right]$, i.e. $a_i^{k_i} c = 0$ in R for some natural numbers $k_1, \dots, k_n$. By considering the equation below,

$$\left(a_1 b_1 + \cdots + a_n b_n\right)^{k_1 + \cdots + k_n} = 1$$

we see that there exist $b'_1, \dots, b'_n$ in R such that

$$a_1^{k_1} b'_1 + \cdots + a_n^{k_n} b'_n = 1$$

and thus,

$$c = \left(a_1^{k_1} b'_1 + \cdots + a_n^{k_n} b'_n\right) c = 0$$

in R. Hence, the kernel of $R \to \prod_{i=1}^{n} R\left[a_i^{-1}\right]$ is trivial, as required.

Now, suppose we have $c_1, \dots, c_n$ in R and natural numbers $k_1, \dots, k_n$ such that $c_i a_i^{-k_i} = c_j a_j^{-k_j}$ in each $R\left[a_{i,j}^{-1}\right]$, i.e.

$$a_{i,j}^{m_{i,j}} a_j^{k_j} c_i = a_{i,j}^{m_{i,j}} a_i^{k_i} c_j$$

in R for some natural number $m_{i,j}$, or equivalently:

$$a_j^{m_{i,j} + k_j} \left(a_i^{m_{i,j}} c_i\right) = a_i^{m_{i,j} + k_i} \left(a_j^{m_{i,j}} c_j\right)$$

Let $m = \max\left\{m_{i,j}\right\}$. There exist $b'_1, \dots, b'_n$ in R such that

$$a_1^{m + k_1} b'_1 + \cdots + a_n^{m + k_n} b'_n = 1$$

and thus:

$$a_i^{m+k_i}\left(a_1{}^m b_1' c_1 + \cdots + a_n{}^m b_n' c_n\right) = a_i{}^m \left(a_1{}^{m+k_1} b_1' + \cdots + a_n{}^{m+k_n} b_n'\right) c_i = a_i{}^m c_i$$

i.e. $a_1{}^m b_1' c_1 + \cdots + a_n{}^m b_n' c_n = c_i a_i{}^{-k_i}$ in $R\left[a_i{}^{-1}\right]$. Thus, the diagram in question is indeed an equaliser diagram.

The above implies that $\hbar^A$ satisfies the sheaf condition with respect to the sieve on R generated by $\left\{R\left[a_1{}^{-1}\right] \leftarrow R, \ldots, R\left[a_n{}^{-1}\right] \leftarrow R\right\}$. Lemma 1.3 says that the Zariski topology is generated by such sieves, so a standard argument shows that $\hbar^A$ is indeed a Zariski sheaf. ∎

Lemma 1.8. *Let $\alpha : F \to G$ be a morphism in $\left[\mathbf{CRing}, \mathbf{Set}\right]$ with the following property:*

- *In any pullback diagram in $\left[\mathbf{CRing}, \mathbf{Set}\right]$ of the form below,*

$$\begin{array}{ccc} V & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ W & \longrightarrow & G \end{array}$$

  if W is a representable functor, then V is a Zariski sheaf.

Under this hypothesis, if G is a Zariski sheaf, then F is also a Zariski sheaf.

Proof. Let A be a ring and let $U \subseteq \hbar^A$ be a Zariski-covering sieve. Suppose G is a Zariski sheaf; we wish to show that F is also a Zariski sheaf. Let $\varphi : U \to F$ be a morphism in $\left[\mathbf{CRing}, \mathbf{Set}\right]$. By hypothesis, there is a unique morphism $\psi : \hbar^A \to G$ in $\left[\mathbf{CRing}, \mathbf{Set}\right]$ making the following diagram commute,

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & F \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \hbar^A & \xrightarrow[\psi]{} & G \end{array}$$

so we get a commutative diagram of the form below,

$$\begin{array}{ccccc} U & \xrightarrow{\theta} & V & \xrightarrow{\pi} & F \\ \downarrow & & {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \hbar^A & \xrightarrow[\mathrm{id}]{} & \hbar^A & \xrightarrow[\psi]{} & G \end{array}$$

where the right square is a pullback square and $\pi \circ \theta = \varphi$. Since V is a Zariski sheaf, there is a unique morphism $\sigma : \hbar^A \to V$ extending $\theta : U \to V$ along $U \hookrightarrow \hbar^A$. Thus, $\pi \circ \sigma : \hbar^A \to F$ extends $\varphi : U \to F$ along $U \hookrightarrow \hbar^A$. Moreover, $\rho \circ \sigma : \hbar^A \to \hbar^A$ extends $U \hookrightarrow \hbar^A$ along itself, so by lemma 1.3, we must have $\rho \circ \sigma = \mathrm{id}_{\hbar^A}$. We deduce that the map

$$[\mathbf{CRing}, \mathbf{Set}]\left(\hbar^A, F\right) \to [\mathbf{CRing}, \mathbf{Set}](U, F)$$

induced by $U \hookrightarrow \hbar^A$ is a surjection.

Now, suppose $\bar{\varphi} : \hbar^A \to F$ is any morphism extending $\varphi : U \to F$ along $U \hookrightarrow \hbar^A$. Since G is a Zariski sheaf, it follows that $\alpha \circ \bar{\varphi} = \psi$, so there is a unique morphism $\tau : \hbar^A \to V$ such that $\pi \circ \tau = \bar{\varphi}$ and $\rho \circ \tau = \mathrm{id}_{\hbar^A}$. By considering the restriction of $\tau : \hbar^A \to V$ along $U \hookrightarrow \hbar^A$ and using the hypothesis that V is a Zariski sheaf, we may deduce that $\tau = \sigma$, hence $\bar{\varphi} = \pi \circ \sigma$. Thus, the map

$$[\mathbf{CRing}, \mathbf{Set}]\left(\hbar^A, F\right) \to [\mathbf{CRing}, \mathbf{Set}](U, F)$$

induced by $U \hookrightarrow \hbar^A$ is an injection, and this completes the proof. ■

2 Open and closed immersions

Definition 2.1. An **affine morphism** in $[\mathbf{CRing}, \mathbf{Set}]$ is a morphism $\alpha : F \to G$ with the following property:

- In any pullback diagram in $[\mathbf{CRing}, \mathbf{Set}]$ of the form below,

$$\begin{array}{ccc} V & \longrightarrow & F \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ W & \longrightarrow & G \end{array}$$

  if W is a representable functor, then V is also a representable functor.

Example 2.2. If F and G are both representable functors $\mathbf{CRing} \to \mathbf{Set}$, then every morphism $F \to G$ is affine, because $\mathbf{CRing}$ has pushouts and the Yoneda embedding sends pushouts in $\mathbf{CRing}$ to pullbacks in $[\mathbf{CRing}, \mathbf{Set}]$.

Proposition 2.3.

(i) *The class of affine morphisms in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under pullback.*

(ii) *The class of affine morphisms in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under composition.*

Proof. This is a straightforward exercise. (Use the pullback pasting lemma.) ◊

Proposition 2.4. *Let* $\alpha : F \to G$ *be an affine morphism in* $[\mathbf{CRing}, \mathbf{Set}]$.

(i) *If* G *is a representable functor, then so is* F.

(ii) *If* G *is a Zariski sheaf, then so is* F.

Proof. (i). Immediate from the definition.

(ii). Use lemma 1.8. ■

Definition 2.5. A **closed immersion** in $[\mathbf{CRing}, \mathbf{Set}]$ is a morphism $\alpha : F \to G$ with the following property:

- For every ring A and every morphism $\psi : \hbar^A \to G$, there is a pullback diagram in $[\mathbf{CRing}, \mathbf{Set}]$ of the form below,

$$\begin{array}{ccc} \hbar^B & \longrightarrow & F \\ {\scriptstyle \hbar^f}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \hbar^A & \underset{\psi}{\longrightarrow} & G \end{array}$$

  where $f : A \to B$ is a surjective ring homomorphism.

A **closed subfunctor** of $G : \mathbf{CRing} \to \mathbf{Set}$ is a subfunctor $F \subseteq G$ such that the inclusion $F \hookrightarrow G$ is a closed immersion.

Example 2.6. For any ring homomorphism $f : A \to B$, $\hbar^f : \hbar^B \to \hbar^A$ is a closed immersion in $[\mathbf{CRing}, \mathbf{Set}]$ if and only if $f : A \to B$ is surjective (because the class of surjective ring homomorphisms is closed under pushout in **CRing**).

Proposition 2.7.

(i) *The class of closed immersions in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under pullback.*

(ii) *The class of closed immersions in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under composition.*

Proof. This is a straightforward exercise. (Use the pullback pasting lemma.) ◊

Proposition 2.8.

(i) *Every closed immersion in* $[\mathbf{CRing}, \mathbf{Set}]$ *is an affine morphism.*

(ii) *Every closed immersion in* $[\mathbf{CRing}, \mathbf{Set}]$ *is a monomorphism.*

Proof. (i). Immediate from the definition.

(ii). Apply the Yoneda lemma. ■

Definition 2.9. Let $f : A \to B$ be a surjective ring homomorphism. The subfunctor $\mathrm{D}(f) \subseteq \hbar^A$ is defined as follows:

- Given a ring homomorphism $g : A \to R$, $g \in \mathrm{D}(f)(R)$ if and only if, for any commutative square in **CRing** of the form below,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

  we have $S \cong \{0\}$.

Lemma 2.10. *Let* $f : A \to B$ *and* $g : A \to R$ *be ring homomorphisms. If* $f : A \to B$ *is surjective, then the following are equivalent:*

(i) $g \in \mathrm{D}(f)(R)$.

(ii) *The following diagram is a pushout square in* **CRing***:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \longrightarrow & \{0\} \end{array}$$

(iii) *There exist elements* $a_1, \ldots, a_n$ *of* A *and elements* $r_1, \ldots, r_n$ *of* R *such that*

$$g(a_1)r_1 + \cdots + g(a_n)r_n = 1$$

and $f(a_i) = 0$ *(for* $1 \leq i \leq n$*).*

Proof. (i) ⇒ (ii). This follows from the fact that every ring homomorphism $\{0\} \to T$ is an isomorphism.

(ii) ⇒ (iii). If the following diagram is a pushout square in **CRing**,

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \longrightarrow & \{0\} \end{array}$$

then (by considering the explicit construction of pushouts in **CRing**) the ideal of B generated by the image of $\ker f \subseteq A$ must be the unit ideal.

(iii) ⇒ (i). Consider a commutative diagram in **CRing** of the form below:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ R & \xrightarrow[h]{} & S \end{array}$$

Then,

$$1 = h\big(g(a_1)r_1 + \cdots + g(a_n)r_n\big) = k\big(f(a_1)\big)h(r_1) + \cdots + k\big(f(a_n)\big)h(r_n) = 0$$

so indeed $S \cong \{0\}$. ∎

Corollary 2.11. *Let A be a ring, let a be an element of A, let $f : A \to A/(a)$ be the quotient homomorphism, and let $\gamma : A \to A[a^{-1}]$ be the universal ring homomorphism inverting a. Then there is a unique isomorphism $\hbar^{A[a^{-1}]} \to \mathrm{D}(f)$ making the following diagram in $[\mathbf{CRing}, \mathbf{Set}]$ commute:*

$$\begin{array}{ccc} \hbar^{A[a^{-1}]} & \overset{\cong}{\dashrightarrow} & \mathrm{D}(f) \\ {\scriptstyle \hbar^{\gamma}}\downarrow & & \downarrow \\ \hbar^{A} & = & \hbar^{A} \end{array}$$

Proof. Apply lemma 2.10. ∎

Definition 2.12. A **open immersion** in $[\mathbf{CRing}, \mathbf{Set}]$ is a morphism $\alpha : F \to G$ with the following property:

- For every ring A and every morphism $\psi : \hbar^A \to G$, there is a pullback diagram in $[\mathbf{CRing}, \mathbf{Set}]$ of the form below,

$$\begin{array}{ccc} \mathrm{D}(f) & \longrightarrow & F \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ \hbar^A & \underset{\psi}{\longrightarrow} & G \end{array}$$

  where $f : A \to B$ is some surjective ring homomorphism.

An **open subfunctor** of $G : \mathbf{CRing} \to \mathbf{Set}$ is a subfunctor $F \subseteq G$ such that the inclusion $F \hookrightarrow G$ is an open immersion.

Proposition 2.13. *Every open immersion in* $[\mathbf{CRing}, \mathbf{Set}]$ *is a monomorphism.*

Proof. Apply the Yoneda lemma. ■

Proposition 2.14. *The class of open immersions in* $[\mathbf{CRing}, \mathbf{Set}]$ *is closed under pullback.*

Proof. This is a straightforward exercise. (Use the pullback pasting lemma.) ◇

Corollary 2.15. *Let* $\alpha : F \to G$ *and* $\beta : G \to H$ *be morphisms in* $[\mathbf{CRing}, \mathbf{Set}]$. *If* $\beta : G \to H$ *is a monomorphism and* $\beta \circ \alpha : F \to H$ *is an open immersion, then* $\alpha : F \to G$ *is also an open immersion.*

Proof. Since $\beta : G \to H$ is a monomorphism, the diagram below is a pullback square in $[\mathbf{CRing}, \mathbf{Set}]$,

$$\begin{array}{ccc} F & = & F \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta\circ\alpha} \\ G & \underset{\beta}{\longrightarrow} & H \end{array}$$

so the claim is a special case of proposition 2.14. ■

Lemma 2.16. *Let* $f : A \to B$ *be a surjective ring homomorphism. Then* $\mathrm{D}(f)$ *is an open subfunctor of* $\hbar^A$.

Proof. Let $g : A \to R$ be any ring homomorphism. Suppose we have the following pushout diagram in **CRing**:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ R & \xrightarrow[h]{} & S \end{array}$$

Then $h : R \to S$ is also a surjective ring homomorphism, and it is straightforward to verify that the following diagram is a pullback square in $[\mathbf{CRing}, \mathbf{Set}]$:

$$\begin{array}{ccc} \mathrm{D}(h) & \longrightarrow & \mathrm{D}(f) \\ \downarrow & & \downarrow \\ \hbar^S & \xrightarrow[\hbar^g]{} & \hbar^R \end{array}$$

Thus, $\mathrm{D}(f) \hookrightarrow \hbar^R$ is indeed an open immersion. ■

REMARK 2.17. In contrast to closed immersions, open immersions are not necessarily affine morphisms. Indeed, if $f : \mathbb{Z}[x, y] \to \mathbb{Z}$ is any (necessarily surjective) ring homomorphism, then $\mathrm{D}(f)$ is not a representable functor **CRing** → **Set**.

3 Schemes

Definition 3.1. A **scheme** is a Zariski sheaf X : **CRing** → **Set** for which there is a set $\mathfrak{U}$ satisfying the following conditions:

- Each element of $\mathfrak{U}$ is a pair (A, x) where A is a ring and x is an element of $X(A)$.
- For each element (A, x) of $\mathfrak{U}$, the corresponding morphism $\hbar^A \to X$ is an open immersion.
- The induced morphism $\coprod_{(A,x)\in\mathfrak{U}} \hbar^A \to X$ is a Zariski-local epimorphism.

An **atlas** for X is such a set $\mathfrak{U}$.

REMARK 3.2. The above definition is equivalent to Définition 3.11 in [Demazure and Gabriel, 1970], but not literally the same. Instead of using the notion of a Zariski-local epimorphism, they require that, for every field k, the induced map $\coprod_{(A,x)\in\mathfrak{U}} \hbar^A(k) \to X(k)$ is a surjection.

Example 3.3. It is clear that every representable functor **CRing** → **Set** is a scheme. It is therefore no abuse of language to say define an **affine scheme** to be a representable functor.

Proposition 3.4. *Let $f : A \to B$ be a surjective ring homomorphism. Then* $\mathrm{D}(f)$ *is a Zariski sheaf.*

Proof. Let R be a ring, let U be a Zariski-covering sieve on R, and let $s : U \to \mathrm{D}(f)$ be a morphism. We must show that there is a unique morphism $\hbar^R \to \mathrm{D}(f)$ extending $s : U \to \mathrm{D}(f)$ along $U \hookrightarrow \hbar^A$.

Firstly, by proposition 1.7 and the Yoneda lemma, there is a unique ring homomorphism $g : A \to R$ making the following diagram in $[\mathbf{CRing}, \mathbf{Set}]$ commute:

$$\begin{array}{ccc} U & \xrightarrow{s} & \mathrm{D}(f) \\ \big\downarrow & & \big\downarrow \\ \hbar^R & \xrightarrow[\hbar^g]{} & \hbar^A \end{array}$$

It thus suffices to show that $g \in \mathrm{D}(f)(R)$. Consider a pushout square in **CRing** of the form below:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ R & \xrightarrow[h]{} & S \end{array}$$

We then have the following pullback square in $[\mathbf{CRing}, \mathbf{Set}]$:

$$\begin{array}{ccc} \mathrm{D}(h) & \longrightarrow & \mathrm{D}(f) \\ \big\downarrow & & \big\downarrow \\ \hbar^R & \xrightarrow[\hbar^g]{} & \hbar^A \end{array}$$

In particular, we have $U \subseteq \mathrm{D}(h) \subseteq \hbar^R$. On the other hand, let V be the subfunctor of $\hbar^S$ making the diagram below a pullback square in $[\mathbf{CRing}, \mathbf{Set}]$:

$$\begin{array}{ccc} V & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ \hbar^S & \xrightarrow[\hbar^h]{} & \hbar^R \end{array}$$

By proposition 1.4, V is a Zariski-covering sieve on S. Thus, for any ring homomorphism $t : S \to T$ such that $t \in V(T)$, we must have $t \circ h \in \mathrm{D}(h)$, and therefore $T \cong \{0\}$. Hence, V is either empty or generated by $\{\{0\} \leftarrow S\}$, so lemma 1.3 implies $S \cong \{0\}$. The claim follows, by lemma 2.10. ■

Corollary 3.5. *Let $\alpha : F \to G$ be an open immersion in $[\mathbf{CRing}, \mathbf{Set}]$. If G is a Zariski sheaf, then so is F.*

Proof. Apply lemma 1.8 and proposition 3.4. ■

Proposition 3.6. *Let $f : A \to B$ be a surjective ring homomorphism. For each element a in A, let $\gamma_a : A \to A[a^{-1}]$ be the universal ring homomorphism inverting a, and define $\mathfrak{U}$ as follows:*

$$\mathfrak{U} = \left\{ \left(A[a^{-1}], \gamma_a\right) \;\middle|\; a \in \ker f \right\}$$

(i) *For $a \in \ker f$, the morphism $\mathfrak{h}^{\gamma_a} : \mathfrak{h}^{A[a^{-1}]} \to \mathfrak{h}^A$ factors through the inclusion $\mathrm{D}(f) \hookrightarrow \mathfrak{h}^A$ as an open immersion $\mathfrak{h}^{A[a^{-1}]} \to \mathrm{D}(f)$.*

(ii) *The induced morphism*

$$\coprod_{(A[a^{-1}], \gamma_a)} \mathfrak{h}^{A[a^{-1}]} \to \mathrm{D}(f)$$

is a Zariski-local epimorphism.

(iii) *$\mathrm{D}(f)$ is a scheme with atlas $\mathfrak{U}$.*

Proof. (i). By lemma 2.10, if $a \in \ker f$, then the localising homomorphism $\gamma_a : A \to A[a^{-1}]$ is an element of $\mathrm{D}(f)\left(A[a^{-1}]\right)$. Thus, by the Yoneda lemma, $\mathfrak{h}^{\gamma_a} : \mathfrak{h}^{A[a^{-1}]} \to \mathfrak{h}^A$ factors through $\mathrm{D}(f) \hookrightarrow \mathfrak{h}^A$. Moreover, by corollary 2.11 and lemma 2.16, $\mathfrak{h}^{\gamma_a} : \mathfrak{h}^{A[a^{-1}]} \to \mathfrak{h}^A$ is an open immersion, so by corollary 2.15, the morphism $\mathfrak{h}^{A[a^{-1}]} \to \mathrm{D}(f)$ corresponding to $\gamma_a \in \mathrm{D}(f)\left(A[a^{-1}]\right)$ is indeed an open immersion.

(ii). Let $g : A \to R$ be any ring homomorphism such that $g \in \mathrm{D}(f)(R)$. Then, for $a \in \ker f$, we have the following pushout diagram in **CRing**,

$$\begin{array}{ccc} A & \xrightarrow{\gamma_a} & A[a^{-1}] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g[a^{-1}]} \\ R & \xrightarrow[\gamma_{g(a)}]{} & R[g(a)^{-1}] \end{array}$$

so $\gamma_{g(a)} \circ g : A \to R\big[g(a)^{-1}\big]$ is in $\mathrm{D}(f)\big(R\big[g(a)^{-1}\big]\big)$. Moreover, there exist elements $a_1, \ldots, a_n$ of $\ker f$ and elements $r_1, \ldots, r_n$ of R such that

$$g(a_1)r_1 + \cdots + g(a_n)r_n = 1$$

and we may conclude that $\coprod_{(A[a^{-1}],\gamma_a)} h^{A[a^{-1}]} \to \mathrm{D}(f)$ is indeed a Zariski-local epimorphism. This completes the proof that $\mathfrak{U}$ is an atlas for $\mathrm{D}(f)$.

(iii). By proposition 3.4, $\mathrm{D}(f)$ is a Zariski sheaf, and the above shows $\mathfrak{U}$ is an atlas for $\mathrm{D}(f)$, so we are done. ■

Lemma 3.7. *Let A be a ring and let $U \subseteq h^A$ be a subfunctor. For each element a of A, let $\gamma_a : A \to A\big[a^{-1}\big]$ be the universal ring homomorphism inverting a, and define I as follows:*

$$I = \big\{a \in A \,\big|\, \gamma_a \in U\big(A\big[a^{-1}\big]\big)\big\}$$

If U is a Zariski sheaf, then:

(i) *I is an ideal of A.*

(ii) *Let a be an element of A and let n be a natural number. If $a^n \in I$, then $a \in I$.*

(iii) *Let $\mathfrak{a}$ be an ideal of A and let $f : A \to A/\mathfrak{a}$ be the quotient homomorphism. Then $\mathrm{D}(f) \subseteq U$ if and only if $\mathfrak{a} \subseteq I$.*

Proof. (i). Since $\emptyset$ is a Zariski-covering sieve on $\{0\}$ and $A\big[0^{-1}\big] \cong \{0\}$, we must have $0 \in I$. Moreover, if $a \in A$ and $b \in I$, then the following diagram in **CRing** commutes,

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle\gamma_b}\downarrow & & \downarrow{\scriptstyle\gamma_{ab}} \\ A\big[b^{-1}\big] & \longrightarrow & A\big[(ab)^{-1}\big] \end{array}$$

so we must have $ab \in I$ as well. Finally, if $a \in I$, $b \in I$, and $c = a + b$, then by lemma 1.3, $\big\{A\big[c^{-1}\big]\big[a^{-1}\big] \leftarrow A\big[c^{-1}\big], A\big[c^{-1}\big]\big[b^{-1}\big] \leftarrow A\big[c^{-1}\big]\big\}$ generates a Zariski-covering sieve on $A\big[c^{-1}\big]$; but the following diagrams in **CRing** commute,

$$\begin{array}{ccc} A & \xrightarrow{\gamma_a} & A\big[a^{-1}\big] \\ {\scriptstyle\gamma_c}\downarrow & & \downarrow \\ A\big[c^{-1}\big] & \longrightarrow & A\big[(ca)^{-1}\big] \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\gamma_b} & A\big[b^{-1}\big] \\ {\scriptstyle\gamma_c}\downarrow & & \downarrow \\ A\big[c^{-1}\big] & \longrightarrow & A\big[(cb)^{-1}\big] \end{array}$$

and $ca \in I$ and $cb \in I$ by the earlier argument, so $c \in I$ because U is a Zariski sheaf.

(ii). Clearly, a is invertible in $A[a^{-n}]$, and it is not hard to see that the induced homomorphism $A\left[a^{-1}\right] \to A[a^{-n}]$ is an isomorphism. Thus, if $a^n \in I$, then $a \in I$ as well.

(iii). Let $a \in \mathfrak{a}$. We have $\gamma_a \in \mathrm{D}(f)\left(A\left[a^{-1}\right]\right)$, so if $\mathrm{D}(f) \subseteq U$, then $a \in I$.

Conversely, suppose $\mathfrak{a} \subseteq I$. For $a \in \mathfrak{a}$, we have $\gamma_a \in U\left(A\left[a^{-1}\right]\right)$, and by proposition 3.6, the induced morphism

$$\coprod_{a \in \mathfrak{a}} h^{A[a^{-1}]} \to \mathrm{D}(f)$$

is a Zariski-local epimorphism; but U is a Zariski sheaf, so $\mathrm{D}(f) \subseteq U$. ■

Lemma 3.8. *Let A be a ring and let $U \subseteq V \subseteq h^A$ be subfunctors. If the inclusions $U \hookrightarrow V$ and $V \hookrightarrow h^A$ are open immersions, then $U \hookrightarrow h^A$ is also an open immersion.*

Proof. For each element a of A, let $\gamma_a : A \to A\left[a^{-1}\right]$ be universal ring homomorphism inverting a, and define I and J as follows:

$$I = \left\{a \in A \,\middle|\, \gamma_a \in U\left(A\left[a^{-1}\right]\right)\right\}$$
$$J = \left\{a \in A \,\middle|\, \gamma_a \in V\left(A\left[a^{-1}\right]\right)\right\}$$

Clearly, $I \subseteq J$. Note that U and V are Zariski sheaves (by proposition 1.7 and corollary 3.5), so I and J are ideals of A, by lemma 3.7. Moreover, if $f : A \to A/I$ and $g : A \to A/J$ are the quotient homomorphisms, then we have $\mathrm{D}(g) = V$ (by proposition 3.6) and $\mathrm{D}(f) \subseteq U$. We will show that $\mathrm{D}(f) = U$; the claim then follows, by lemma 2.16.

For $a \in I$, let $U_a \subseteq h^{A[a^{-1}]}$ be the subfunctor making the following diagram a pullback square in $[\mathbf{CRing}, \mathbf{Set}]$,

$$\begin{array}{ccc} U_a & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ h^{A[a^{-1}]} & \longrightarrow & V \end{array}$$

where the bottom horizontal arrow corresponds to $\gamma_a \in V\left(A\left[a^{-1}\right]\right)$. We then have a pullback square in $\left[\mathbf{CRing}, \mathbf{Set}\right]$ of the form below,

$$\begin{array}{ccc} \coprod_{a \in J} U_a & \longrightarrow & U \\ \big\downarrow & & \big\downarrow \\ \coprod_{a \in J} \mathcal{h}^{A\left[a^{-1}\right]} & \longrightarrow & V \end{array}$$

and since the bottom horizontal arrow is a Zariski-local epimorphism, the top horizontal arrow is also a Zariski-local epimorphism (by proposition 1.4). Let J_a be defined as follows:

$$J_a = \left\{ b \in A\left[a^{-1}\right] \mid \gamma_b \in U_a\left(A\left[a^{-1}\right]\left[b^{-1}\right]\right) \right\}$$

Since $U_a \hookrightarrow \mathcal{h}^{A\left[a^{-1}\right]}$ is an open immersion (by proposition 2.14), we know that the induced morphism $\coprod_{b \in J_a} \mathcal{h}^{A\left[a^{-1}\right]\left[b^{-1}\right]} \to U_a$ is a Zariski-local epimorphism. Thus, the composite

$$\coprod_{a \in I} \coprod_{b \in J_a} \mathcal{h}^{A\left[a^{-1}\right]\left[b^{-1}\right]} \longrightarrow \coprod_{a \in I} U_a \longrightarrow U$$

is a Zariski-local epimorphism. Moreover, for any element a of A and any element b of $A\left[a^{-1}\right]$, there exist an element c of A and an isomorphism $A\left[c^{-1}\right] \to A\left[a^{-1}\right]\left[b^{-1}\right]$ making the diagram in **CRing** shown below commute,

$$\begin{array}{ccc} A & \xrightarrow{\gamma_c} & A\left[c^{-1}\right] \\ {\scriptstyle \gamma_a}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\ A\left[a^{-1}\right] & \xrightarrow[\gamma_b]{} & A\left[a^{-1}\right]\left[b^{-1}\right] \end{array}$$

namely ac' for any element c' of A such that $b = \gamma_a(c')\gamma_a(a)^{-n}$ for some natural number n. It follows that the induced morphism $\coprod_{a \in J} \mathcal{h}^{A\left[a^{-1}\right]} \to U$ is a Zariski-local epimorphism, and therefore $\mathrm{D}(f) = U$, as claimed. ■

Proposition 3.9. *The class of open immersions in* $\left[\mathbf{CRing}, \mathbf{Set}\right]$ *is closed under composition.*

Proof. Use the pullback pasting lemma to reduce to lemma 3.8. ■

Lemma 3.10. *Let $\alpha : X \to Y$ be a morphism in $[\mathbf{CRing}, \mathbf{Set}]$. Suppose Y is a scheme and $\mathfrak{V}$ is an atlas for Y with the following property:*

- *For each element (A, y) of $\mathfrak{V}$, let $U_{(A,y)}$ be the subfunctor of X making the diagram below a pullback square in $[\mathbf{CRing}, \mathbf{Set}]$,*

$$\begin{array}{ccc} U_{(A,y)} & \hookrightarrow & X \\ \downarrow & & \downarrow \alpha \\ \hbar^A & \longrightarrow & Y \end{array}$$

  where the bottom horizontal arrow corresponds to $y \in Y(A)$. Then $U_{(A,y)}$ is a scheme.

Under this hypothesis, if X is a Zariski sheaf, then X is also a scheme.

Proof. For each element (A, y) of $\mathfrak{V}$, let $\mathfrak{U}_{(A,y)}$ be an atlas for the scheme $U_{(A,y)}$. Let $\mathfrak{U} = \bigcup_{(A,y)\in\mathfrak{V}} \mathfrak{V}_{(A,y)}$. We will show that $\mathfrak{U}$ is an atlas for X.

Let (A, y) be an element of $\mathfrak{V}$. By proposition 2.14, $U_{(A,y)}$ is an open subfunctor of X, so by proposition 3.9, for each element (B, x) of $\mathfrak{U}_{(A,y)}$, the morphism $\hbar^B \to X$ corresponding to $x \in U_{(A,y)}(B) \subseteq X(B)$ is an open immersion.

By definition, each $\coprod_{(B,x)\in\mathfrak{U}_{(A,y)}} \hbar^B \to U_{(A,y)}$ is a Zariski-local epimorphism, and by proposition 1.4, $\coprod_{(A,y)\in\mathfrak{V}} U_{(A,y)} \to X$ is also a Zariski-local epimorphism. Thus, $\coprod_{(B,x)\in\mathfrak{U}} \hbar^B \to X$ is a Zariski-local epimorphism. Thus, $\mathfrak{U}$ is indeed an atlas for X. ■

Proposition 3.11. *Let Y be a scheme and let $\alpha : X \to Y$ be a morphism in $[\mathbf{CRing}, \mathbf{Set}]$.*

(i) *If $\alpha : X \to Y$ is an affine morphism, then X is a scheme.*

(ii) *If $\alpha : X \to Y$ is a closed immersion, then X is a scheme.*

(iii) *If $\alpha : X \to Y$ is an open immersion, then X is a scheme.*

open immersion . If Y is a scheme, then so is X.

Proof. In each case, we will use lemma 3.10 to deduce that X is a scheme.

(i). By proposition 2.4, X is a Zariski sheaf, and by proposition 2.3, the hypotheses of the lemma are satisfied (for any atlas of Y whatsoever).

(ii). By proposition 2.8, this is a special case of (i).

(iii). By corollary 3.5, X is a Zariski sheaf, and by propositions 2.14 and 3.6, the hypotheses of the lemma are satisfied (for any atlas of Y whatsoever). ■

References

Adámek, Jiří, Horst Herrlich and George E. Strecker

[ACC] *Abstract and concrete categories. The joy of cats*. Reprints in Theory and Applications of Categories 17. Reprint of the 1990 original [Wiley, New York; MR1051419]. 2006. 507 pp.

Demazure, Michel and Pierre Gabriel

[1970] *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs*. With an appendix, *Corps de classes local*, by Michiel Hazewinkel. Paris: Masson & Cie, Éditeur, 1970. xxvi+700.

Jantzen, Jens Carsten

[2003] *Representations of algebraic groups*. Second. Mathematical Surveys and Monographs 107. American Mathematical Society, Providence, RI, 2003. xiv+576. ISBN: 0-8218-3527-0.