Weak categories following Segal

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Abstract

In [1968] and [1974], Segal introduced special cases of what is now known as the 'Segal condition' for simplicial spaces, which can be interpreted as a theory of weak categories where composition is defined only up to a system of coherent homotopies, or equivalently, as a theory of categories weakly enriched over simplicial sets. This idea has now been studied and elaborated upon by Dwyer, Kan and Smith [1989], Tamsamani [1996], Hirschowitz and Simpson [1998], Rezk [2001], Pellissier [2002], and others, resulting in a general theory of weakly enriched categories.

1 The Segal condition

Given a set A, there is a simplicial set $\operatorname{cosk}_0(A)$ whose n-simplices are the $(n+1)$-tuples of elements of A, with face and degeneracy operators induced by the appropriate projections. We write $\mathbf{\Delta}_A$ for the category of simplices of $\operatorname{cosk}_0(A)$, i.e. the comma category $(\Delta^\bullet \downarrow \operatorname{cosk}_0(A))$. Of course, if A is a singleton, then $\mathbf{\Delta}_A$ is isomorphic to the usual simplex category $\mathbf{\Delta}$.

Definition 1.1. Let n be a natural number. A **principal edge** of $[n]$ is an inclusion of the form $\{i-1, i\} \hookrightarrow [n]$ in $\mathbf{\Delta}$, where $1 \le i \le n$. More generally, if $(a_0, \ldots, a_n)$ is an object in $\mathbf{\Delta}_A$, then a **principal edge** of $(a_0, \ldots, a_n)$ is a morphism whose underlying simplicial operator is a principal edge of $[n]$.

Now, let $\mathcal{M}$ be a model category such as **sSet** or **Top**, let X be a functor ${\mathbf{\Delta}_A}^{\mathrm{op}} \to \mathcal{M}$, and let $(a_0, \ldots, a_n)$ be an object in $\mathbf{\Delta}_A$. The principal edges of $(a_0, \ldots, a_n)$ induce, for $1 < i < n$, a commutative diagram of the following form:

$$\begin{tikzcd} X(a_0, \ldots, a_n) \arrow[r] \arrow[d] & X(a_i, a_{i+1}) \arrow[d] \\ X(a_{i-1}, a_i) \arrow[r] & X(a_i) \end{tikzcd}$$

Definition 1.2. For $n \geq 2$, if $\mathrm{M}(X; a_0, \dots, a_n)$ is the limit of the diagram below in $\mathcal{M}$,

$$\begin{tikzcd}
X(a_0, a_1) \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & X(a_{n-1}, a_n) \arrow[dl] \\
& X(a_1) & & \cdots & & X(a_{n-1}) &
\end{tikzcd}$$

then there is an induced morphism $X(a_0, \dots, a_n) \to \mathrm{M}(X; a_0, \dots, a_n)$, called the **Segal map**. We say X satisfies the **Segal condition** if the Segal maps are weak equivalences for all $(n+1)$-tuples $(a_0, \dots, a_n)$ (where $n \geq 2$) of elements of A, and we say X is **unital** if $X(a_0)$ is a terminal object in $\mathcal{M}$ for all elements a_0 of A.

Remark 1.3. Note that the above definition uses ordinary limits instead of homotopy limits!

Proposition 1.4 (Grothendieck)**.** *Let X be a simplicial set. The following are equivalent:*

(i) *X is isomorphic to the nerve of a (small) category.*

(ii) *X satisfies the Segal conditions as a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$, where $\mathbf{Set}$ is given the minimal model structure.*

(iii) *The unique morphism $X \to 1$ is right orthogonal with respect to all inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ ($n \geq 2$, $0 < k < n$).*

2 Bisimplicial sets

Definition 2.1. A **bisimplicial set** is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{sSet}$, where $\mathbf{sSet}$ is the category of simplicial sets, and a **morphism of bisimplicial sets** is a natural transformation of functors. We write $\mathbf{ssSet}$ for the category of bisimplicial sets.

Remark 2.2. There is a full embedding $\mathrm{K} : \mathbf{sSet} \to \mathbf{ssSet}$ that sends a simplicial set X to the bisimplicial set $\mathrm{K}(X)$, where $\mathrm{K}(X)_n = X$ for all n, with trivial face and degeneracy operators. We can make $\mathbf{ssSet}$ into an $\mathbf{sSet}$-enriched category such that the following adjunction formula holds:

$$\left[X, \underline{\mathbf{ssSet}}(Y, Z)\right]_{\mathbf{sSet}} \cong \underline{\mathbf{ssSet}}(\mathrm{K}(X) \times Y, Z)$$

(The LHS denotes the exponential object in $\mathbf{sSet}$.)

The n-th space functor $(-)_n : \mathbf{ssSet} \to \mathbf{sSet}$ is then represented by a bisimplicial set F^n, where F^n_k is the discrete simplicial set $\mathbf{\Delta}([k], [n])$. (This is *not* isomorphic to $\mathrm{K}(\Delta^n)$!) We write ∂F^n for the largest subobject of F^n that does not contain the element $\mathrm{id}_{[n]}$.

Definition 2.3. A morphism $f : X \to Y$ in **ssSet** is a **Reedy fibration** (resp. **trivial Reedy fibration**) if and only if, for each natural number n, the relative matching map $\underline{\mathbf{ssSet}}(F^n, X) \to \underline{\mathbf{ssSet}}(F^n, Y) \times_{\underline{\mathbf{ssSet}}(\partial F^n, Y)} \underline{\mathbf{ssSet}}(\partial F^n, X)$ induced by the commutative diagram in **sSet** shown below

$$\begin{tikzcd} \underline{\mathbf{ssSet}}(F^n, X) \arrow[r] \arrow[d] & \underline{\mathbf{ssSet}}(\partial F^n, X) \arrow[d] \\ \underline{\mathbf{ssSet}}(F^n, Y) \arrow[r] & \underline{\mathbf{ssSet}}(\partial F^n, Y) \end{tikzcd}$$

is a Kan fibration (resp. trivial Kan fibration) in **sSet**.

Remark 2.4. A **Reedy-fibrant bisimplicial set** is therefore a bisimplicial set X such that the morphism $\underline{\mathbf{ssSet}}(F^n, X) \to \underline{\mathbf{ssSet}}(\partial F^n, X)$ is a Kan fibration for all natural numbers n. In particular, X_0 must be a Kan complex.

Proposition 2.5. *There is a cofibrantly-generated model structure on* **ssSet**, *called the* **Reedy model structure**, *with the following properties:*

- *The cofibrations and weak equivalences are determined levelwise.*
- *The fibrations (resp. trivial fibrations) are the Reedy fibrations (resp. trivial Reedy fibrations).*

The Reedy model structure is moreover compatible with the previously-defined **sSet**-*enrichment of* **ssSet**, *i.e. it satisfies axiom* SM7.

Proof. See Theorem 15.8.7 in [Hirschhorn, 2003]. □

3 Segal spaces

It is well-known that the functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{sSet}$ that sends a (small) category $\mathbb{C}$ to its nerve $\mathrm{N}(\mathbb{C})$ is fully faithful; however, it is not homotopically well-behaved. For instance, if $\mathbb{C}$ has a terminal object, then $\mathrm{N}(\mathbb{C})$ is contractible. To remedy this, Rezk [2001] proposed the following notion:

Definition 3.1. The **classification diagram** of a (small) relative category $\mathbb{C}$ is the bisimplicial set $\mathbf{N}(\mathbb{C})$ defined by $\mathbf{N}(\mathbb{C})_n = \mathrm{N}(\mathrm{weq}\,[\min\,[n], \mathbb{C}]_{\mathrm{h}})$.

Remark 3.2. More explicitly, if we think of the product category $[n] \times [k]$ as a grid with n columns and k rows, then the k-simplices of the n-th level of $\mathbf{N}(\mathbb{C})$ can be identified with the functors $[n] \times [k] \to \mathbb{C}$ where the vertical arrows are sent to weak equivalences in $\mathbb{C}$.

Example 3.3. The **discrete nerve** of a (small) category $\mathbb{C}$ is the bisimplicial set $\mathbf{N}(\min \mathbb{C})$; note that the n-th level of $\mathbf{N}(\min \mathbb{C})$ is the discrete simplicial set $\mathrm{N}(\mathbb{C})_n$. In particular, $\mathbf{N}(\min [n])$ is isomorphic to F^n.

Thus, we obtain a full embedding $\mathbf{N}(\min -) : \mathbf{Cat} \to \mathbf{ssSet}$. Unfortunately, it is also not homotopically well-behaved: equivalent categories can have discrete nerves that are not weakly equivalent. Instead, we should consider the following:

Definition 3.4. The **classifying diagram** of a (small) category $\mathbb{C}$ is the classification diagram of the minimal homotopical category $\min^+ \mathbb{C}$.

Remark 3.5. Writing $\mathbf{I}[k]$ for the groupoid obtained by freely inverting all morphisms in $[k]$, we see that k-simplices in the n-th level of $\mathbf{N}(\min^+ \mathbb{C})$ correspond to functors $[n] \times \mathbf{I}[k] \to \mathbb{C}$. One can also show that $\mathbf{N}(\min^+ \mathbb{C})$ satisfies the strict Segal condition, i.e. that all the Segal maps are isomorphisms.

Theorem 3.6. *The functor* $\mathbf{N}(\min^+ -) : \mathbf{Cat} \to \mathbf{ssSet}$ *is fully faithful, preserves finite products and exponential objects, and moreover sends equivalences in* **Cat** *to weak equivalences in* **ssSet**.

Proof. This is Theorem 3.6 in [Rezk, 2001]. □

Lemma 3.7. *For all (small) categories* $\mathbb{C}$, $\mathbf{N}(\min^+ \mathbb{C})$ *is a Reedy-fibrant bisimplicial set.*

Proof. See Lemma 3.8 in [Rezk, 2001] □

It should therefore be reasonable to say that the following is a candidate definition for weak categories:

Definition 3.8. A **Segal space** is a Reedy-fibrant bisimplicial set that satisfies the Segal condition.

Corollary 3.9. *The classifying diagram of a (small) category is a Segal space.* ■

Remark 3.10. Let G^n be the **spine** of F^n, that is, the smallest subobject of F^n containing all the principal edges $F^1 \to F^n$. It is not hard to show that G^n is the colimit of the evident diagram

$$\begin{tikzcd}
F^1 \arrow[dr, "d^0"'] & & \cdots \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & F^1 \arrow[dl, "d^1"] \\
& F^0 & & \cdots & & F^0 &
\end{tikzcd}$$

and so the Segal maps for a bisimplicial set X are (isomorphic to) the morphisms $\underline{\mathbf{ssSet}}(F^n, X) \to \underline{\mathbf{ssSet}}(G^n, X)$ induced by the inclusions $G^n \hookrightarrow F^n$. If X is Reedy-fibrant, then axiom SM7 implies the Segal maps are Kan fibrations.

Thus, a Reedy-fibrant bisimplicial set X satisfies the Segal condition if and only if all the Segal maps are trivial Kan fibrations. Moreover, the face operators $d_0, d_1 : X_1 \to X_0$ are both Kan fibrations if X is Reedy-fibrant, so the ordinary limit in the definition of the Segal condition is already a *homotopy* limit.

Theorem 3.11 (Rezk)**.** *There exists a cartesian simplicial model structure on* **ssSet***, called the* **injective model structure for Segal spaces***, with the following properties:*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are the Segal spaces.*
- *The weak equivalences are the morphisms $f : X \to Y$ such that the morphism $\underline{\mathbf{ssSet}}(f, Z) : \underline{\mathbf{ssSet}}(Y, Z) \to \underline{\mathbf{ssSet}}(X, Z)$ is a weak homotopy equivalence in* **sSet** *for all Segal spaces Z.*
- *This model structure is a left Bousfield localisation of the Reedy model structure on* **ssSet***.*

Proof. See Theorem 7.1 in [Rezk, 2001]. □

4 Segal categories

The notion of Segal category was implicitly introduced by Dwyer, Kan and Smith [1989] and studied by Schwänzl and Vogt [1992] under the name of 'Δ-category'. In this section, we follow [Simpson, 2012, Ch. 10].

Definition 4.1. Let A be a set and let $\mathcal{M}$ be a (suitable) model category.[1] An **$\mathcal{M}$-precategory with object-set** A is a functor $X : {\mathbf{\Delta}_A}^{\mathrm{op}} \to \mathcal{M}$ that is unital. A **weak $\mathcal{M}$-category with object-set** A is an $\mathcal{M}$-precategory that additionally satisfies the Segal condition. We write $\mathbf{PC}(\mathcal{M}, A)$ for the full subcategory of $[{\mathbf{\Delta}_A}^{\mathrm{op}}, \mathcal{M}]$ spanned by the $\mathcal{M}$-precategories. If $\mathcal{M} = \mathbf{sSet}$ (with the Kan–Quillen model structure), then we say **Segal precategory** and **Segal category** instead of '$\mathcal{M}$-precategory' and 'weak $\mathcal{M}$-category'.

Definition 4.2. Let X be a Segal precategory with object-set A. An **object** of X is an element of A. Given a pair (a, b) of objects in X, an **arrow** or

[1] Simpson [2012, § 7.3] requires that $\mathcal{M}$ be a tractable, left proper, cartesian model category.

morphism $a \to b$ in X is a vertex of $X(a,b)$. Two arrows $a \to b$ are **homotopic** if they are in the same path component of $X(a,b)$. Given an element a in X, the **identity** on a is the vertex $s_0(*)$ in $X(a,a)$, where $*$ is the unique vertex of $X(a)$. Given arrows $f : a \to b$ and $g : b \to c$ in X, a **composition** is a lift of the vertex (f,g) along the Segal map $X(a,b,c) \to X(a,b) \times X(b,c)$; the **result** of a composition is its image under the face operator $d_1 : X(a,b,c) \to X(a,c)$.

Remark 4.3. If X is a Segal category, then the result of any two compositions of (f,g) are homotopic. More generally, by applying the connected components functor $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ to the hom-spaces $X(a,b)$, we obtain a (small) category $\tau_{\leq 1} X$ with well-defined composition, strict identities, strict associativity, etc.

Definition 4.4. Let X be a Segal category. An **internal equivalence** in X is an arrow $f : a \to b$ in X such that (the class of) f has a two-sided inverse in $\tau_{\leq 1} X$.

Proposition 4.5. *Let X be a Segal space, let A be the set of vertices X_0, and define a Segal precategory Y with object-set A by taking $Y(a_0, \dots, a_n)$ to be the fibre of $X_n \to (X_0)^{\times(n+1)}$ over $(a_0, \dots, a_n)$:*

$$\begin{tikzcd} Y(a_0, \dots, a_n) \arrow[r] \arrow[d] & X_n \arrow[d] \\ \{(a_0, \dots, a_n)\} \arrow[r, hook] & (X_0)^{\times(n+1)} \end{tikzcd}$$

Then Y is a Segal category with object-set A.

Proof. In fact, the Segal maps $Y(a_0, \dots, a_n) \to Y(a_0, a_1) \times \cdots \times Y(a_{n-1}, a_n)$ are trivial Kan fibrations, because the class of trivial Kan fibrations is closed under pullback. (See paragraph 5.1 in [Rezk, 2001].) ■

Proposition 4.6. *Let Y be a Segal precategory with object-set A, and define a bisimplicial set X by taking*

$$X_n = \coprod_{(a_0, \dots, a_n) \in A^{\times(n+1)}} Y(a_0, \dots, a_n)$$

with the evident face and degeneracy operators. Then:

(i) *X satisfies the Segal condition (but need not be Reedy fibrant) if Y is a Segal category.*

(ii) *X_0 is isomorphic to the discrete simplicial set A.*

(iii) *$X \mapsto Y$ extends to a full embedding $\mathbf{PC}(\mathbf{sSet}, A) \to \mathbf{ssSet}_{/K(\operatorname{cosk}_0(A))}$.*

Proof. (i). The class of trivial cofibrations is closed under coproducts, as is the class of trivial Kan fibrations; thus, the Segal maps for X are weak homotopy equivalences because Y satisfies the Segal condition.

(ii). Obvious.

(iii). It is clear how to define the functor $\mathbf{PC}(\mathbf{sSet}, A) \to \mathbf{ssSet}_{/\mathrm{K}(A)}$ on morphisms; that it is fully faithful amounts to the observation that we can decompose morphisms in $\mathbf{ssSet}_{/\mathrm{K}(\mathrm{cosk}_0(A))}$ fibrewise. ■

For each map $f : A \to B$ in **Set**, there is a functor $f : \mathbf{\Delta}_A \to \mathbf{\Delta}_B$ obtained by postcomposing by $\mathrm{cosk}_0(f) : \mathrm{cosk}_0(A) \to \mathrm{cosk}_0(B)$ in **sSet**, and this in turn defines a functor $f^* : \mathbf{PC}(\mathcal{M}, B) \to \mathbf{PC}(\mathcal{M}, A)$ by precomposition.

Proposition 4.7.

(i) *The assignment $f \mapsto f^*$ is a strict contravariant functor from* **Set** *to the category of locally small categories.*

(ii) *If Y is a weak $\mathcal{M}$-category with object set B, then $f^* Y$ is a weak $\mathcal{M}$-category with object set A.*

(iii) *If $\mathcal{M}$ is a locally presentable category, then each $f^* : \mathbf{PC}(\mathcal{M}, B) \to \mathbf{PC}(\mathcal{M}, A)$ has a left adjoint, $f_! : \mathbf{PC}(\mathcal{M}, A) \to \mathbf{PC}(\mathcal{M}, B)$.*

Proof. (i) and (ii). Obvious.

(iii). See [Simpson, 2012, § 10.3]. □

Thus, by applying the Grothendieck construction to the functor $f \mapsto f^*$, we obtain a split Grothendieck fibration $\mathrm{ob} : \mathbf{PC}(\mathcal{M}) \to \mathbf{Set}$, and ob is additionally a Grothendieck opfibration. The category $\mathbf{PC}(\mathcal{M})$ is the category of all small $\mathcal{M}$-precategories, without restriction on the object sets.

In the special case $\mathcal{M} = \mathbf{sSet}$, the previously-described full embeddings $\mathbf{PC}(\mathbf{sSet}, A) \to \mathbf{ssSet}_{/\mathrm{K}(A)}$ can be collated to obtain a commutative diagram

$$\begin{tikzcd} \mathbf{PC}(\mathbf{sSet}) \arrow[r] \arrow[d, "\mathrm{ob}"'] & {[\mathbb{2}, \mathbf{ssSet}]} \arrow[d, "\mathrm{codom}"] \\ \mathbf{Set} \arrow[r, "\mathrm{K}(\mathrm{cosk}_0(-))"'] & \mathbf{ssSet} \end{tikzcd}$$

where the functor $\mathbf{PC}(\mathbf{sSet}) \to [\mathbb{2}, \mathbf{ssSet}]$ preserves cartesian morphisms. Thus, by postcomposing with $\mathrm{dom} : [\mathbb{2}, \mathbf{ssSet}] \to \mathbf{ssSet}$, we obtain a full embedding $I : \mathbf{PC}(\mathbf{sSet}) \to \mathbf{ssSet}$.

Definition 4.8. Let X and Y be Segal *categories*. A **functor** $X \to Y$ is a morphism in $\mathbf{PC}(\mathbf{sSet})$. A **Dwyer–Kan equivalence** $X \to Y$ is a functor satisfying these conditions:

- For every pair (a, b) of objects in X, the hom-space morphism $f_{(a,b)} : X(a, b) \to Y((\operatorname{ob} f)(a), (\operatorname{ob} f)(b))$ is a weak homotopy equivalence.
- The induced functor $\tau_{\leq 1} f : \tau_{\leq 1} X \to \tau_{\leq 1} Y$ is (fully faithful and) essentially surjective on objects.

Theorem 4.9 (Hirschowitz and Simpson)**.** *There exists a tractable, left proper, cartesian model structure on* $\mathbf{PC}(\mathbf{sSet})$, *called the* **injective model structure for Segal categories**, *with the following properties:*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are Reedy-fibrant Segal categories (i.e. a Segal category X such that IX is a Reedy-fibrant bisimplicial set).*
- *The weak equivalences between fibrant objects are the Dwyer–Kan equivalences.*

Proof. See Theorem 5.1 and Corollary 5.13 in [Bergner, 2007a] for the main claims, see Theorem 19.3.2 in [Simpson, 2012] for the cartesian property, and see Définition 2.1 and Théorème 2.3 in [Hirschowitz and Simpson, 1998] for the last point. □

5 Complete Segal spaces

We return now to Rezk's theory.

Definition 5.1. Let X be a Segal space. Objects, arrows, internal equivalences, etc. in X are defined to be the corresponding things in the associated Segal category. The **space of internal equivalences** in X is the largest simplicial subset $X_{\mathrm{eq}} \subseteq X_1$ whose vertices are the internal equivalences in X.

It is clear from the definition that the property of being an internal equivalence is an invariant of the path-components of the hom-space $X(a, b)$, but more is true in a Segal space:

Proposition 5.2. *Let X be a Segal space.*

(i) *If f and g are two vertices of X_1 that are in the same path-component, and f is an internal equivalence in X, then g is also an internal equivalence.*

(ii) *X_{eq} is a union of path-components of X_1.*

(iii) *The degeneracy operator $s_0 : X_0 \to X_1$ factors through $X_{\mathrm{eq}} \hookrightarrow X_1$.*

Proof. See Lemma 5.8 in [Rezk, 2001]. ■

Definition 5.3. A **complete Segal space**, or **Rezk category**, is a Segal space X such that the morphism $X_0 \to X_{\mathrm{eq}}$ is a weak homotopy equivalence.

Proposition 5.4. *The classifying diagram of a (small) category is a complete Segal space.*

Proof. Let $X = \mathbf{N}(\min^+ \mathbb{C})$. We have already seen that X is a Segal space. Clearly, X_{eq} is the simplicial subset of $\mathrm{N}(\mathrm{iso}\,[[1], \mathbb{C}])$ corresponding to the full subcategory of $[[1], \mathbb{C}]$ spanned by the isomorphisms in $\mathbb{C}$, and we know that the canonical embedding $\mathrm{iso}\,\mathbb{C} \to \mathrm{iso}\,[\mathbf{I}[1], \mathbb{C}]$ is an equivalence of categories; thus the morphism $X_0 \to X_{\mathrm{eq}}$ is a weak homotopy equivalence. ■

Theorem 5.5 (Rezk)**.** *There exists a cartesian simplicial model structure on* **ssSet**, *called the* **model structure for complete Segal spaces**, *with the following properties:*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are the complete Segal spaces.*
- *The weak equivalences are the morphisms $f : X \to Y$ such that the morphism $\underline{\mathbf{ssSet}}(f, Z) : \underline{\mathbf{ssSet}}(Y, Z) \to \underline{\mathbf{ssSet}}(X, Z)$ is a weak homotopy equivalence in* **sSet** *for all complete Segal spaces Z.*
- *This model structure is a left Bousfield localisation of the Reedy model structure on* **ssSet**.

Proof. See Theorem 7.2 in [Rezk, 2001]. □

Definition 5.6. Let X and Y be Segal spaces. A **Dwyer–Kan equivalence** $X \to Y$ is a morphism in **ssSet** such that the corresponding functor between the associated Segal categories is a Dwyer–Kan equivalence.

Theorem 5.7. *Let $f : X \to Y$ be a morphism between Segal spaces. The following are equivalent:*

(i) *f is a Dwyer–Kan equivalence.*

(ii) *f is a weak equivalence in the model structure for* complete *Segal spaces.*

Moreover, if X and Y are both complete Segal spaces, then the following statement is equivalent to the previous ones:

(iii) *f is a levelwise weak equivalence.*

Proof. See Proposition 7.6 and Theorem 7.7 in [Rezk, 2001]. □

Corollary 5.8. *The model structure for complete Segal spaces is a localisation of the injective model structure for Segal spaces.* ■

The next result suggests the sense in which the theory of complete Segal spaces is a "homotopy theory of homotopy theory".

Theorem 5.9. *Let $\underline{\mathcal{M}}$ be a simplicial model category with underlying relative category $\mathcal{M}$. Then (after passing to a larger universe), a Reedy fibrant replacement X of the classification diagram $\mathbf{N}(\mathcal{M})$ is a complete Segal space, and moreover, for all pairs (a, b) of objects in $\mathcal{M}$, the hom-spaces $X(a, b)$ and $\underline{\mathcal{M}}(a, b)$ have the same homotopy type.*

Proof. See Theorem 8.3 in [Rezk, 2001]. □

6 Comparison theorems

Theorem 6.1 (Bergner). *Let $\mathbf{SeCat}_{\mathrm{inj}}$ be the category $\mathbf{PC}(\mathbf{sSet})$ equipped with the injective model structure for Segal categories, and let* **CSS** *be the category* **ssSet** *equipped with the model structure for complete Segal spaces.*

(i) *The embedding $I : \mathbf{PC}(\mathbf{sSet}) \to \mathbf{ssSet}$ has a right adjoint R.*

(ii) *The adjunction $I \dashv R : \mathbf{CSS} \to \mathbf{SeCat}_{\mathrm{inj}}$ is a Quillen equivalence.*

Proof. (i). See Proposition 6.1 in [Bergner, 2007a].

(ii). See Theorem 6.3 in [Bergner, 2007a]. □

Theorem 6.2 (Joyal). *There exists a tractable, left proper, cartesian (but not simplicial!) model structure on* **sSet***, called the* **Joyal model structure** *or the* **model structure for quasicategories***, with the following properties:*

- *The cofibrations are the monomorphisms.*
- *The fibrant objects are the quasicategories.*
- *The Kan–Quillen model structure is a left Bousfield localisation of this one.*

Proof. See Theorem 1.9 and Proposition 1.15 in [Joyal and Tierney, 2007] for the statements, and see Theorem 6.12 and Proposition 6.15 in [Joyal, 2008] for proofs. ■

Theorem 6.3 (Joyal and Tierney)**.** *Let* **Qcat** *be the category* **sSet** *equipped with the Joyal model structure, let* $p_1^* : \mathbf{sSet} \to \mathbf{ssSet}$ *be the functor that sends a simplicial set* X *to the bisimplicial set* $p_1^* X$*, where* $(p_1^* X)_n$ *is the discrete simplicial set* X_n*, and let* $i_1^* : \mathbf{ssSet} \to \mathbf{sSet}$ *be the functor that sends a bisimplicial set* Y *to the simplicial set whose* n*-simplices are the vertices of* Y_n*.*

(i) *There is an adjunction* $p_1^* \dashv i_1^* : \mathbf{ssSet} \to \mathbf{sSet}$*.*

(ii) *This adjunction is a Quillen equivalence between* **CSS** *and* **Qcat***.*

Proof. See Theorem 4.11 in [Joyal and Tierney, 2007]. ■

By putting together the two comparison theorems, **Qcat** and $\mathbf{SeCat}_{\mathrm{inj}}$ are Quillen equivalent, but one can also obtain a direct equivalence:

Theorem 6.4 (Toën's Conjecture B.1)**.** *Let* $j^* : \mathbf{PC}(\mathbf{sSet}) \to \mathbf{sSet}$ *be the functor that sends a Segal precategory* Y *to the simplicial set whose* n*-simplices are the vertices of* $(IY)_n$*, i.e. let* $j^* = i_1^* I$*.*

(i) *The functor* $j^* : \mathbf{PC}(\mathbf{sSet}) \to \mathbf{sSet}$ *has a left adjoint* q^**.*

(ii) *The adjunction* $q^* \dashv j^* : \mathbf{SeCat}_{\mathrm{inj}} \to \mathbf{Qcat}$ *is a Quillen equivalence.*

Proof. See Conjecture B.1 in [Toën, 2003] and Theorem 5.6 in [Joyal and Tierney, 2007]. □

Theorem 6.5 (Bergner)**.** *Let* **SCat** *be the category of small* **sSet***-enriched categories. There exists a cofibrantly-generated, right proper model structure on* **SCat***, called the* **Bergner model structure for simplicial categories***, with the following properties:*

- *The weak equivalences are the Dwyer–Kan equivalences.*
- *The fibrations are the simplicial functors* $f : \underline{\mathbb{C}} \to \underline{\mathbb{D}}$ *such that*
  - *for every pair* (a, b) *of objects in* $\mathbb{C}$*, the hom-space morphism* $f_{(a,b)} : \underline{\mathbb{C}}(a, b) \to \underline{\mathbb{D}}(f(a), f(b))$ *is a Kan fibration, and*
  - *for every object* a *in* $\mathbb{C}$ *and any internal equivalence* $h : f(a) \to d$ *in* $\mathbb{D}$*, there exist an internal equivalence* $k : a \to b$ *in* $\mathbb{C}$ *such that* $f(k) = h$*.*

Proof. See Theorem 1.1 in [Bergner, 2007b]. □

Theorem 6.6 (Bergner)**.** *There exists a cofibrantly-generated model structure on* $\mathbf{PC}(\mathbf{ssSet})$*, denoted by* $\mathbf{SeCat}_{\mathrm{proj}}$*, with the following properties:*

- *The trivial adjunction* $\mathrm{id} \dashv \mathrm{id} : \mathbf{SeCat}_{\mathrm{inj}} \to \mathbf{SeCat}_{\mathrm{proj}}$ *is a Quillen equivalence.*
- *The inclusion* $R : \mathbf{SCat} \hookrightarrow \mathbf{PC}(\mathbf{sSet})$ *has a left adjoint* F*.*
- *The adjunction* $F \dashv R : \mathbf{SeCat}_{\mathrm{proj}} \to \mathbf{SCat}$ *is a Quillen equivalence.*

Proof. (i). See Theorem 7.5 in [Bergner, 2007a].

(ii). Apply Lemma 5.6 in [Bergner, 2006].

(iii). See Theorem 8.6 in [Bergner, 2007a]. □

Theorem 6.7 (Joyal and Lurie)**.** *Let* $\underline{\mathrm{S}([n])}$ *be the standard resolution (in the sense of Dwyer and Kan [1980]) of the preorder category* $[n]$*, and let* $\mathrm{N^h} : \mathbf{SCat} \to \mathbf{sSet}$ *be the homotopy coherent nerve functor of Cordier [1982, p. 20], i.e. the one sending a (small)* **sSet***-category* $\underline{\mathbb{C}}$ *to the simplicial set whose* n*-simplices are the* **sSet***-functors* $\underline{\mathrm{S}([n])} \to \underline{\mathbb{C}}$*.*

(i) *The functor* $\mathrm{N^h} : \mathbf{SCat} \to \mathbf{sSet}$ *has a left adjoint,* $\mathfrak{C}$*.*

(ii) *The adjunction* $\mathfrak{C} \dashv \mathrm{N^h} : \mathbf{SCat} \to \mathbf{Qcat}$ *is a Quillen equivalence.*

Proof. (i). **SCat** is cocomplete, so the left adjoint may be constructed by left Kan extension.

(ii). See Theorems 2.2.5.1 and 2.4.6.1 in [HTT]. □

Theorem 6.8 (Barwick and Kan)**.** *Let* **RelCat** *be the category of small relative categories.*

(i) *There exists an adjunction* $K_\xi \dashv N_\xi : \mathbf{RelCat} \to \mathbf{ssSet}$ *with the following property: for every left Bousfield localisation of the Reedy model structure on* **ssSet***, there exists a cofibrantly-generated left proper model structure on* **RelCat** *making the adjunction a Quillen equivalence.*

(ii) *Moreover, there exists a natural Reedy weak equivalence* $\mathbf{N} \Rightarrow N_\xi$*, where* $\mathbf{N}$ *is the classification diagram functor, so a relative functor* $f : \mathbb{C} \to \mathbb{D}$ *is a weak equivalence in the transferred model structure on* **RelCat** *if and only if the morphism* $\mathbf{N}(f) : \mathbf{N}(\mathbb{C}) \to \mathbf{N}(\mathbb{D})$ *is a weak equivalence in the localised model structure on* **ssSet***.*

(iii) *In particular, there exists a cofibrantly-generated left proper model structure on* **RelCat** *that is Quillen-equivalent via the above adjunction to the model structure for complete Segal spaces.*

Proof. (i). See Theorem 6.1 in [Barwick and Kan, 2012].

(ii). See Lemma 5.4 in [Barwick and Kan, 2012].

(iii). Immediately follows from claim (i), because the model structure for complete Segal spaces is a left Bousfield localisation of the Reedy model structure on **ssSet**. □

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