Logic in a topos

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Abstract

A topos is a mathematical universe unto itself in the following sense: any finitary logical theory can be interpreted in a topos. In particular, associated with each topos is an intuitionistic higher-order type theory, namely the Mitchell–Bénabou language of that topos; and associated with this theory is a forcing interpretation, called Kripke–Joyal semantics. Once all the groundwork is done, this machinery enables us to reason naturally about objects in a topos as if they were sets of elements.

For the purposes of this talk, by 'topos' what is meant is 'elementary topos'. I will assume that the audience is familiar with standard concepts from propositional and first-order logic. The main references for this topic are [Johnstone, 2002, §§ D1.1, D1.2, and D4.1], [Lambek and Scott, 1988, Part II], and [ML–M, Ch. VI].

1 Higher-order type theory

Before we can really do any logic we must define the formal system we want to work with. Let me stress that everything in this section is purely symbolic and is *at present* without meaning.

We take as our primitives the logical constant $\bot$, the binary connectives $\wedge$, $\vee$, and $\Rightarrow$, and the quantifiers $\forall$ and $\exists$; and we treat $\neg\varphi$ as an abbreviation for $\varphi \Rightarrow \bot$ and $\top$ as an abbreviation for $\neg\bot$. Our logic is many-sorted: so each variable has a type and different variables may have different types; the same goes for constant symbols, function symbols, and relation symbols.

What distinguishes type theory from ordinary logic is the presence of type-forming operations. A **higher-order signature** Σ consists of various data, but first and foremost, there is a collection of *sorts* in Σ. The collection of types in Σ is generated inductively:

- There is a **unit type**, denoted by 1.
- Every sort is a type.
- Given two types A and B, there is a **product type** $A \times B$.
- Given two types A and B, there is a **function type** $[A \to B]$.
- Given a type A, there is a **power type** PA.

The presence of the last two are what makes this signature 'higher-order'. The other data of Σ are a collection of constant symbols, function symbols, and relation symbols; and to each function symbol f in Σ is assigned a type $A \to B$, where A and B are types in Σ in the sense defined above, while to each constant symbol and each relation symbol is assigned a single type in Σ.

The well-formed formulae are defined recursively using the usual rules starting from these atomic formulae:

- If t is a term of type A and R is a relation symbol of type A, then $R(t)$ is an atomic formula with the same free variables as t.
- If s and t are terms of type A, then $s =_A t$ is an atomic formula with free variables $\mathrm{FV}(s) \cup \mathrm{FV}(t)$.
- If s is a term of type A and t is a term of type PA, then $s \in_A t$ is an atomic formula with free variables $\mathrm{FV}(s) \cup \mathrm{FV}(t)$.

Since well-formed formulae are in particular well-typed, we may omit the subscripts. Terms are also defined recursively:

- There is a distinguished term of type 1, denoted by $*$, and it has no free variables.
- Every constant of type A is a term of type A without free variables.
- Every variable of type A is a term of type A whose only free variable is the variable itself.

- If f is a function symbol of type $A \to B$ and t is a term of type A, then $f(t)$ is a term of type B with the same free variables as t.
- If s is a term of type A with free variables $\mathrm{FV}(s)$ and t is a term of type B with free variables $\mathrm{FV}(t)$, then $\langle s, t\rangle$ is a term of type $A \times B$ with free variables $\mathrm{FV}(s) \cup \mathrm{FV}(t)$.
- If u is a term of type $A \times B$, then $\mathsf{left}(u)$ and $\mathsf{right}(u)$ are terms of type A and B, respectively, and have the same free variables as u.
- If t is a term of type B and z is a variable of type A, then $\lambda z : A. t$ is a term of type $[A \to B]$ with free variables $\mathrm{FV}(t) \setminus \{z\}$.
- If f is a term of type $[A \to B]$ and t is a term of type A, then $\mathsf{ev}(f, t)$ is a term of type B with free variables $\mathrm{FV}(f) \cup \mathrm{FV}(t)$. We may abbreviate this as $f(t)$ when convenient.
- If θ is a formula and z is a variable of type A, then $\{z : A \mid \theta\}$ is a term of type PA with free variables $\mathrm{FV}(\varphi) \setminus \{z\}$.

It is worth noting that the last rule forces terms and formulae to be defined by *mutual* recursion. Finally, we say that a **suitable context** for a term or formula is an ordered list $\vec{x} = \left(x_1, \ldots, x_n\right)$ of variables (without repetitions and annotated with types) such that every free variable of that term or formula appears in $\vec{x}$.

2 Categorical semantics

Now suppose we have a higher-order signature Σ. What do we mean by an interpretation of Σ? Let $\mathcal{E}$ be a topos. An interpretation M consists of the following data:

- For each sort A in Σ, a chosen object MA in $\mathcal{E}$.
- $M1$ is defined to be the terminal object 1 in $\mathcal{E}$.
- If MA and MB have been defined, then $M(A \times B)$ is defined to be the product $MA \times MB$ in $\mathcal{E}$, and $M[A \to B]$ is defined to be the exponential object MB^{MA}.

- If MA has been defined, then $M(PA)$ is defined to be the power object $P(MA)$.
- For each constant symbol a of type A, a chosen morphism $Ma : 1 \to MA$ in $\mathcal{E}$.
- For each function symbol f of type $A \to B$, a chosen morphism $Mf : MA \to MB$ in $\mathcal{E}$.
- For each relation symbol R of type A, a chosen subobject $MR \rightarrowtail MA$.

Given an interpretation M, the interpretation of a term t of type B in a suitable context $\vec{x}$ is a morphism $[\![\vec{x}.\, t]\!]_M : MA_1 \times \cdots \times MA_n \to MB$, where $A_1, \ldots, A_n$ are, respectively, the types of $x_1, \ldots, x_n$; on the other hand, the interpretation of a formula φ in a suitable context $\vec{x}$ is a subobject $[\![\vec{x}.\, \varphi]\!]_M \rightarrowtail MA_1 \times \cdots \times MA_n$. We define these interpretations by recursion:

- $[\![\vec{x}.\, *]\!]_M$ is defined to be the unique morphism $MA_1 \times \cdots \times MA_n \to 1$.
- If a is a constant of type A, then $[\![\vec{x}.\, a]\!]_M$ is defined to be the composite $Ma \circ [\![\vec{x}.\, *]\!]_M$.
- $[\![\vec{x}.\, x_i]\!]_M$ is defined to be the i-th projection $MA_1 \times \cdots \times MA_n \to MA_i$.
- Given a term t of type A and a function symbol f of type $A \to B$, if $[\![\vec{x}.\, t]\!]_M$ has been defined, then $[\![\vec{x}.\, f(t)]\!]_M$ is defined to be the composite $Mf \circ [\![\vec{x}.\, t]\!]_M$.
- If $[\![\vec{x}.\, s]\!]_M$ and $[\![\vec{x}.\, t]\!]_M$ have been defined, then $[\![\vec{x}.\, \langle s, t\rangle]\!]_M$ is defined to be $\left\langle [\![\vec{x}.\, s]\!]_M, [\![\vec{x}.\, t]\!]_M \right\rangle$.
- If u is a term of type $A \times B$ and $[\![\vec{x}.\, u]\!]_M$ has been defined, then $[\![\vec{x}.\, \mathsf{left}(u)]\!]_M$ is $\pi_1 \circ [\![\vec{x}.\, u]\!]_M$ and $[\![\vec{x}.\, \mathsf{right}(u)]\!]_M$ is $\pi_2 \circ [\![\vec{x}.\, u]\!]_M$, where π_1 and π_2 are the first and second projections respectively.
- Given a term t of type B and a variable z of type A not appearing in the list $\vec{x}$, if $\vec{x}, z$ is a suitable context for t and $[\![\vec{x}, z.\, t]\!]_M$ has been defined, then $[\![\vec{x}.\, \lambda z : A.\, t]\!]_M : MA_1 \times \cdots \times MA_n \to MB^{MA}$ is defined to be the exponential transpose of $[\![\vec{x}, z.\, t]\!]_M : MA_1 \times \cdots \times MA_n \times MA \to MB$.

- Given a term f of type $[A \to B]$ and a term t of type A, if $[\![\vec{x}.\, f]\!]_M$ and $[\![\vec{x}.\, t]\!]_M$ have been defined, then $[\![\vec{x}.\, \mathsf{ev}(f,t)]\!]_M$ is defined to be the composite $\mathrm{ev} \circ \big\langle [\![\vec{x}.\, f]\!]_M, [\![\vec{x}.\, t]\!]_M \big\rangle$.

- If R is a relation symbol of type B and t is a term of type B, then $[\![\vec{x}.\, R(t)]\!]_M$ is defined by the pullback square shown below:

$$\begin{array}{ccc} [\![\vec{x}.\, R(t)]\!]_M & \longrightarrow & MR \\ \downarrow & & \downarrow \\ MA_1 \times \cdots \times MA_n & \xrightarrow[{[\![\vec{x}.t]\!]_M}]{} & MB \end{array}$$

- If s and t are terms of type B and $[\![\vec{x}.\, s]\!]_M$ and $[\![\vec{x}.\, t]\!]_M$ have been defined, then $[\![\vec{x}.\, s =_B t]\!]_M$ is defined to be the equaliser of $[\![\vec{x}.\, s]\!]_M$ and $[\![\vec{x}.\, t]\!]_M$.

- If s is a term of type B and t is a term of type PB and both $[\![\vec{x}.\, s]\!]_M$ and $[\![\vec{x}.\, t]\!]_M$ have been defined, then $[\![\vec{x}.\, s \in_B t]\!]_M$ is defined by the pullback square shown below:

$$\begin{array}{ccc} [\![\vec{x}.\, s \in_B t]\!]_M & \longrightarrow & [\in]_{MB} \\ \downarrow & & \downarrow \\ MA_1 \times \cdots \times MA_n & \xrightarrow[{\langle [\![\vec{x}.s]\!]_M, [\![\vec{x}.t]\!]_M \rangle}]{} & MB \times P(MB) \end{array}$$

  Here $[\in]_{MB} \rightarrowtail MB \times P(MB)$ is the universal binary relation.

- Given a formula θ and a variable z of type A, if $z, \vec{x}$ is a suitable context for θ and $[\![z, \vec{x}.\, \varphi]\!]_M$ has been defined, then $[\![\vec{x}.\, \{z : A \mid \theta\}]\!]_M$ is defined to be the unique morphism r making the diagram below a pullback square:

$$\begin{array}{ccc} [\![z, \vec{x}.\, \theta]\!]_M & \longrightarrow & [\in]_{MA} \\ \downarrow & & \downarrow \\ MA \times MA_1 \times \cdots \times MA_n & \xrightarrow[{\mathrm{id}_{MA} \times r}]{} & MA \times P(MA) \end{array}$$

We must also give interpretations for the logical constants, connectives, and quantifiers. For this we will need the following fact:

Proposition. *For any object X in a topos $\mathcal{E}$, the category of subobjects* $\mathrm{Sub}(X)$ *is equivalent to a Heyting algebra; and for any morphism $f : X \to Y$ in $\mathcal{E}$, the pullback functor $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$.*

In particular, $\mathrm{Sub}(X)$ has a bottom element 0, binary meets $\cap$, binary joins $\cup$, and for each subobject X', $(-) \cap X'$ has a right adjoint, namely $X' \Rightarrow (-)$. Thus, we may define the following interpretations:

$$\begin{aligned} [\![\vec{x}.\,\bot]\!]_M &= 0 & [\![\vec{x}.\,\varphi \wedge \psi]\!]_M &= [\![\vec{x}.\,\varphi]\!]_M \cap [\![\vec{x}.\,\psi]\!]_M \\ [\![\vec{x}.\,\varphi \Rightarrow \psi]\!]_M &= [\![\vec{x}.\,\varphi]\!]_M \Rightarrow [\![\vec{x}.\,\psi]\!]_M & [\![\vec{x}.\,\varphi \vee \psi]\!]_M &= [\![\vec{x}.\,\varphi]\!]_M \cup [\![\vec{x}.\,\psi]\!]_M \end{aligned}$$

Also, if $\vec{x}, z$ is a suitable context for the formula θ, we define

$$\begin{aligned} [\![\vec{x}.\,\forall z : A.\,\theta]\!]_M &= \forall_\pi [\![\vec{x}, z.\,\theta]\!]_M \\ [\![\vec{x}.\,\exists z : A.\,\theta]\!]_M &= \exists_\pi [\![\vec{x}, z.\,\theta]\!]_M \end{aligned}$$

where $\pi : MA_1 \times \cdots \times MA_n \times MA \to MA$ is the projection.

Now at last we can say what it means for the interpretation M to **satisfy** a formula φ: this happens precisely if, for a suitable context $\vec{x}$, $[\![\vec{x}.\,\varphi]\!]_M$ is the top element of $\mathrm{Sub}\big(MA_1 \times \cdots \times MA_n\big)$. We write this symbolically as $M \vDash_{\vec{x}} \varphi$.

3 Mitchell–Bénabou language

A topos $\mathcal{E}$ gives rise to a higher-order signature in a fairly straightforward way: the basic sorts are the objects in $\mathcal{E}$, the function symbols are the morphisms in $\mathcal{E}$, and the relation symbols are the monomorphisms in $\mathcal{E}$. This is called the **Mitchell–Bénabou language** of $\mathcal{E}$, and it has a tautological interpretation in $\mathcal{E}$.

Given a formula φ in the Mitchell–Bénabou language and a suitable context $\vec{x}$, let us write $\mathcal{E} \vDash_{\vec{x}} \varphi$ if the tautological interpretation satisfies φ. To demonstrate the utility of the Mitchell–Bénabou language, here are some examples:

- A morphism $f : A \to B$ in $\mathcal{E}$ is a monomorphism if and only if

  $$\mathcal{E} \vDash_{x,y} (f(x) = f(y)) \Rightarrow (x = y)$$

  where x and y are variables of type A.

- A morphism $f : A \to B$ in $\mathcal{E}$ is an epimorphism if and only if

  $$\mathcal{E} \vDash_y \exists x : A.\, f(x) = y$$

  where y is a variable of type B.

- A subobject $R \rightarrowtail A$ is an initial object if and only if

  $$\mathcal{E} \vDash \neg(\exists x : A.\, x \in r)$$

  where x is a variable of type A and $r : 1 \to PA$ is the exponential transpose of the classifying morphism of R.

- An object N equipped with morphisms $o : 1 \to N$ and $s : N \to N$ is a natural numbers object if and only if

  $$\mathcal{E} \vDash_x \neg(s(x) = o) \qquad \text{and}$$
  $$\mathcal{E} \vDash_{x,y} (s(x) = s(y)) \Rightarrow (x = y) \qquad \text{and}$$
  $$\mathcal{E} \vDash_z ((o \in z) \wedge \forall x : N.\, (x \in z) \Rightarrow (s(x) \in z)) \Rightarrow (\forall x : N.\, x \in z)$$

  where x and y are variables of type N and z is a variable of type PN; in other words, (N, o, s) is a NNO if and only if it satisfies the axioms of second-order arithmetic!

4 Kripke–Joyal semantics

However, at times categorical semantics is cumbersome and it is difficult to directly verify whether some formula is satisfied or not. This is especially a problem when one is trying to relate conditions expressed in the Mitchell–Bénabou language to external conditions, which might be expressed diagrammatically or in terms of hom-sets etc. In this case it can be more helpful to use the forcing interpretation afforded by **Kripke–Joyal semantics**.

Definition 4.1. Let $\mathcal{E}$ be a topos and let φ be a formula in the Mitchell–Bénabou language of $\mathcal{E}$. Suppose $\vec{x}$ is a suitable context for the formula φ, where x_i has type A_i. A **generalised element** of type $A_1 \times \cdots \times A_n$ is any morphism in $\mathcal{E}$ with codomain $A_1 \times \cdots \times A_n$. Given a generalised element $\vec{u} : U \to A_1 \times \cdots \times A_n$, we write $U \Vdash \varphi[\vec{u}/\vec{x}]$ to mean that $\vec{u}$ factors through $[\![\vec{x}.\, \varphi]\!]_M$; and when this happens we say that U **forces** $\varphi[\vec{u}/\vec{x}]$, or that $\varphi[\vec{u}/\vec{x}]$ **holds at stage** U.

Using the fact that a topos is a regular category in which every epimorphism is extremal, one may obtain this useful result:

Proposition 4.2.

(i) *If $h : V \to U$ is any morphism and $U \Vdash \varphi[\vec{u}/\vec{x}]$, then $V \Vdash \varphi[\vec{u} \circ h/\vec{x}]$.*

(ii) *If $h : V \to U$ is an epimorphism and $V \Vdash \varphi[\vec{u} \circ h/\vec{x}]$, then $U \Vdash \varphi[\vec{u}/\vec{x}]$.*

(iii) *If $\vec{u}$ is an epimorphism, then $U \Vdash \varphi[\vec{u}/\vec{x}]$ if and only if $\mathcal{E} \vDash_{\vec{x}} \varphi$.*

In general, however, we may not have enough epimorphisms, so we should try to amplify the above proposition.

Definition 4.3. A **separating family** for a category $\mathcal{E}$ is a family $\mathcal{C}$ of objects with the following property:

- For any parallel pair $f, g : X \to Y$ in $\mathcal{E}$, if $f \circ u = g \circ u$ for all morphisms $u : U \to X$ where U is in $\mathcal{C}$, then $f = g$.

Lemma 4.4. *Let $\mathcal{C}$ be a separating family for a topos $\mathcal{E}$ and let $X' \rightarrowtail X$ be a monomorphism in $\mathcal{E}$. If every morphism $u : U \to X$ with U in $\mathcal{C}$ factors through X', then $X' \rightarrowtail X$ is an isomorphism.*

Proof. Let $\chi : X \to \Omega$ be the classifying morphism for $X' \rightarrowtail X$. If $u : U \to X$ factors through X', then we have a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{!} & 1 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi]{} & \Omega \end{array}$$

and so if this holds for all morphisms $u : U \to X$ with U in $\mathcal{C}$, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{!} & 1 \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \top} \\ X & \xrightarrow[\chi]{} & \Omega \end{array}$$

must commute; but this implies $X' \rightarrowtail X$ is an isomorphism. ■

Corollary 4.5. *If $\mathcal{C}$ is a separating family for $\mathcal{E}$, then $\mathcal{E} \vDash_{\vec{x}} \varphi$ if and only if $U \Vdash \varphi[\vec{u}/\vec{x}]$ for all generalised elements $\vec{x} : U \to A_1 \times \cdots \times A_n$ with U in $\mathcal{C}$.*

However, the true force of Kripke–Joyal semantics is in the theorem below:

Theorem 4.6 (Joyal)**.** *Let $\mathcal{E}$ be a topos. Let s and t be terms in the Mitchell–Bénabou language of $\mathcal{E}$ such that $\vec{x}$ is a suitable context for both s and t, where x_i is of type A_i, and let $\vec{u} : U \to A_1 \times \cdots \times A_n$ be a generalised element.*

(i) $U \Vdash \top$ *always.*

(ii) $U \Vdash \bot$ *if and only if U is an initial object in $\mathcal{E}$.*

(iii) $U \Vdash (s = t)[\vec{u}/\vec{x}]$ *if and only if* $[\![\vec{x}.\, s]\!]_M \circ \vec{u} = [\![\vec{x}.\, t]\!]_M \circ \vec{u}$, *provided s and t have the same type.*

(iv) $U \Vdash (s \in t)[\vec{u}/\vec{x}]$ *if and only if* $\left\langle [\![\vec{x}.\, s]\!]_M \circ \vec{u}, [\![\vec{x}.\, t]\!]_M \circ \vec{u} \right\rangle$ *factors through* $[\in]_B \rightarrowtail B \times PB$, *where s is of type B and t is of type PB.*

Now, let φ and ψ be formulae in the Mitchell–Bénabou language of $\mathcal{E}$ such that $\vec{x}$ is a suitable context for both φ and ψ.

(v) $U \Vdash (\varphi \wedge \psi)[\vec{u}/\vec{x}]$ *if and only if* $U \Vdash \varphi[\vec{u}/\vec{x}]$ *and* $U \Vdash \psi[\vec{u}/\vec{x}]$.

(vi) $U \Vdash (\varphi \vee \psi)[\vec{u}/\vec{x}]$ *if and only if there exist morphisms* $h : V \to U$ *and* $k : W \to U$ *such that* $[h, k] : V \amalg W \to U$ *is an epimorphism,* $V \Vdash \varphi[\vec{u} \circ h \,/\, \vec{x}]$, *and* $W \Vdash \psi[\vec{v} \circ k \,/\, \vec{x}]$.

(vii) $U \Vdash (\varphi \Rightarrow \psi)[\vec{u}/\vec{x}]$ *if and only if, for all morphisms* $h : V \to U$, $V \Vdash \varphi[\vec{u} \circ h \,/\, \vec{x}]$ *implies* $V \Vdash \psi[\vec{u} \circ h \,/\, \vec{x}]$.

(viii) $U \Vdash (\neg\varphi)[\vec{u}/\vec{x}]$ *if and only if, for all morphisms* $h : V \to U$, $V \Vdash \varphi[\vec{u} \circ h \,/\, \vec{x}]$ *implies V is an initial object in $\mathcal{E}$.*

Finally, let θ be a formula such that $\vec{x}, z$ is a suitable context for φ, where z is of type A.

(ix) $U \Vdash (\forall z : A.\, \theta)[\vec{u}/\vec{x}]$ *if and only if, for all morphisms* $h : V \to U$ *and all generalised elements* $v : V \to A$, $V \Vdash \theta\left[\vec{u} \circ h, v \,/\, \vec{x}, z\right]$.

(x) $U \Vdash (\exists z : A.\, \theta)[\vec{u}/\vec{x}]$ *if and only if there exist an epimorphism* $h : V \to U$ *and a generalised element* $v : V \to A$ *such that* $V \Vdash \theta\left[\vec{u} \circ h, v \,/\, \vec{x}, z\right]$.

We may then specialise to the case where $\mathcal{E}$ is the topos of sheaves on a space X and combine Joyal's theorem with the earlier corollary, taking the Yoneda embedding of $\mathrm{Ouv}(X)$ as our separating family $\mathcal{C}$.

Recalling the Yoneda lemma, if U is an open subspace of X, and F is a sheaf on X, then sheaf morphisms $U \to F$ are in natural bijection with elements of $F(U)$. This gives, for example, the following interpretation of the existential quantifier in $\mathbf{Sh}(X)$: given an element $\vec{u}$ of $A_1(U) \times \cdots \times A_n(U)$, $U \Vdash (\exists z : A.\, \theta)[\vec{u}/\vec{x}]$ if and only if there exist an open cover $\mathfrak{U}$ of U and elements v_i of $F\left(V_i\right)$ for each V_i in $\mathfrak{U}$ such that $V_i \Vdash \theta\left[\vec{u}|_{V_i}, v_i \,/\, \vec{x}, z\right]$. In short, $\exists$ in $\mathbf{Sh}(X)$ only means local existence! Similarly, $\vee$ in $\mathbf{Sh}(X)$ only means local disjunction. This picture is one that should always be kept in mind when working with toposes.

One special case of this is when $\mathcal{E}$ is the presheaf topos on a preorder $\mathbb{P}$. (It can be shown that $\left[\mathbb{P}^{\mathrm{op}}, \mathbf{Set}\right]$ is equivalent to $\mathbf{Sh}(X)$ for a certain locale X.) Modulo Gödel–Gentzen double negation translation, Kripke–Joyal semantics reduces to something that set theorists should find similar to forcing over $\mathbb{P}$.

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