What is … a topos?

Z.L. Low

17 January 2013

Abstract

The words 'topos theory' seem to strike unnecessary fear in the hearts of mathematicians. I shall try to defuse some of this apprehension by explaining what a topos is in simple terms and why they are interesting. We will see three aspects of the elephant in this talk: toposes as mathematical universes, toposes as generalised spaces, and toposes as theories.

0 Historical overview

The notion of topos was originally conceived by Grothendieck as part of his programme to attack the Weil conjectures via the construction of a so-called 'Weil cohomology theory'. The results of this research was eventually collected in [SGA 4], but can be traced back as early as [G–S, July 18, 1959]; the idea behind étale topology goes back even further and is mentioned in Grothendieck's address at the International Congress of Mathematicians in 1958.[1] Roughly speaking, a topos in the sense of Grothendieck is the category of sheaves on a kind of generalised space whose "points" may have non-trivial automorphisms. In fact, topos cohomology is almost literally a fusion of Galois cohomology and classical sheaf cohomology.[2] Moreover, just knowing the topos of sheaves on a sufficiently nice (e.g. Hausdorff) space is enough to recover that space; and for locally contractible spaces, invariants such as homotopy groups can be computed

[1] See [Grothendieck, 1960].

[2] More precisely, every Grothendieck topos is equivalent to the topos of equivariant sheaves on a localic groupoid; see [Joyal and Tierney, 1984].

directly from the topos—in principle even without knowing what the unit interval is! Consequently Grothendieck and his followers advocated the view that a topos *is* a space, albeit one that could be very strange.

However that is at best only one-half of the story. Around the same time, Lawvere was looking for an elementary axiomatisation of the category of sets—and in particular, set theory without a global membership relation. It is worth remarking that even Mac Lane was sceptical about this endeavour; he said, "Bill, you can't do that. Elements are absolutely essential to set theory."[3] Lawvere's formulation of the 'elementary theory of the category of sets' (ETCS) was published in [1964],[4] the very same year that the theorem of Giraud [1964] characterising Grothendieck toposes was published. This set the stage for Lawvere and Tierney's research on 'axiomatic sheaf theory', which produced elementary axioms for a class of categories including Grothendieck toposes. Lawvere [1971] announced these axioms for what we know today as 'elementary toposes' and 'local operators' at the International Congress of Mathematicians in 1970, and the rest is, as they say, history.

Almost. I promised to also say something about 'toposes as theories', and the origins of this idea seems somewhat more obscure than the others. The first concrete example of a classifying topos for a non-trivial logical theory was perhaps the Zariski topos; its universal property, and an analogous one for the étale topos, appeared in the thesis of Hakim [1972]—but the phrase '*topos classifiant*' does not! Rather, that was probably due to Giraud [1971, Ch. VIII], who showed that (for example) the category of objects equipped with the action of an internal group G classifies G-torsors. The English phrase 'classifying topos' was also the title of a [1972] paper of his.

Another example of a *relative* classifying topos appeared in the lectures of Wraith [1975, § 9], namely the relative object classifier. The key theorem for this work was due to Diaconescu [1973, 1975]. In the same book, Johnstone [1975] gave a construction for relative classifying toposes for finitely-presented finitary algebraic theories; then finally, Bénabou [1975], Tierney [1976], and others showed that relative classifying toposes exist for all geometric theories, a fragment of infinitary first-order logic due to Joyal [unpublished], Makkai [1976], and Reyes [1974]. In this light Diaconescu's theorem is seen as a converse: every bounded topos classifies models for some geometric theory.

[3] See [Mac Lane, 1988, § 15].

[4] — and later republished in [2005].

1 A mathematical universe

Definition 1.1. A category $\mathcal{S}$ is a model of the **elementary theory of the category of sets** (ETCS) if it satisfies the following axioms:

1. $\mathcal{S}$ has all finite limits and colimits; more precisely, $\mathcal{S}$ is required to have an initial object 0, a terminal object 1, binary coproducts, pullbacks, and coequalisers.

2. $\mathcal{S}$ is **cartesian closed**, i.e. for each object Y in $\mathcal{S}$ the endofunctor on $\mathcal{S}$ defined by $X \mapsto X \times Y$ has a right adjoint, denoted by $Z \mapsto Z^Y$.

3. $\mathcal{S}$ has a **natural numbers object**, i.e. an object N in $\mathcal{S}$ equipped with a pair morphisms, $z : 1 \to N$ and $s : N \to N$, such that for any morphisms $z_X : 1 \to X$ and $s_X : X \to X$, there is a unique morphism $f : N \to X$ such that $f \circ z = z_X$ and $f \circ s = s_X \circ f$.

4. $\mathcal{S}$ is **well-pointed**, i.e. for any pair of morphisms $f, g : X \to Y$ in $\mathcal{S}$, if $f \circ x = g \circ x$ for all morphisms $x : 1 \to X$, then $f = g$.

5. $\mathcal{S}$ satisfies the **axiom of choice**, in the sense that all epimorphisms in $\mathcal{S}$ split.

6. $\mathcal{S}$ is **non-degenerate**, i.e. the canonical morphism $0 \to 1$ is not an isomorphism.

7. $\mathcal{S}$ has a **subobject classifier**, i.e. a monomorphism $\top : 1 \rightarrowtail \Omega$ such that, for every monomorphism $m : X \rightarrowtail Y$ in $\mathcal{S}$, there exists a unique morphism $\chi_m : Y \to \Omega$ fitting into a pullback square of the form below:

$$\begin{array}{ccc} X & \longrightarrow & 1 \\ {\scriptstyle m}\big\downarrow & & \big\downarrow{\scriptstyle \top} \\ Y & \underset{\chi_m}{\dashrightarrow} & \Omega \end{array}$$

Axioms 1–5 above are exactly the same axioms that Lawvere gave in [1964], and the axioms regarding non-degeneracy and the existence of a subobject classifier replace the original axioms 6–8. It is now known that ETCS is essentially the same as RZC (or ZBQC in the notation of Mathias [2001]), a weak form of Zermelo set theory that only has Δ_0-separation; accordingly there are many models of ETCS.

Example 1.2. Let V be a model of Zermelo–Fraenkel set theory with choice (ZFC). Then the category of sets and functions in V is a model of ETCS. Moreover, inside V, the category of sets and functions in $V_{\omega+\omega}$ is also a model of ETCS.

Now, what does ETCS have to do with toposes?

Definition 1.3. An **elementary topos** is a category that satisfies axioms 1, 2, and 7 of ETCS.

This was precisely the definition that Lawvere and Tierney settled on in 1970. Thus we may think of an elementary topos as being a category of generalised sets; conversely, an elementary topos with some additional properties will look a lot like the category of sets as we normally understand it. The following reformulation may be more convincing than mere talk:

Theorem 1.4. *A category $\mathcal{E}$ is an elementary topos if and only if it satisfies these conditions:*

- *$\mathcal{E}$ has finite limits.*
- *$\mathcal{E}$ has* **power objects**, *i.e. for every object X in $\mathcal{E}$ there exist an object PX and a monomorphism $[\in]_X \rightarrowtail X \times PX$ such that for each monomophism $R \rightarrowtail X \times Y$ there is a unique morphism $r : Y \to PX$ fitting into a pullback diagram of the form below:*

$$\begin{array}{ccc} R & \longrightarrow & [\in]_X \\ \downarrow & & \downarrow \\ X \times Y & \xrightarrow[\mathrm{id}_X \times r]{} & X \times PX \end{array}$$

Proof. See Corollaries A2.2.9 and A2.3.4 in [Johnstone, 2002], or Theorem 1 in [ML–M, Ch. IV, § 2] plus Corollary 4 in [ML–M, Ch. IV, § 5]. □

Example 1.5. Let V be a model of Zermelo–Fraenkel set theory (ZF), possibly without choice. Then the category of sets and functions in V is a non-degenerate well-pointed elementary topos with a natural numbers object, and the full subcategory of *finite* sets is a well-pointed elementary topos with choice.

Lawvere often emphasises that a topos is a category of “variable sets”, so we should see at least one an example of this before moving on.

Example 1.6. Let G be any discrete group and let $\mathbb{B}G$ be G considered as a one-object category. If $\mathcal{E}$ is any elementary topos, then $[\mathbb{B}G, \mathcal{E}]$, the category of G-objects in $\mathcal{E}$ and G-equivariant morphisms, is also an elementary topos (with limits, colimits, exponentials, and subobject classifier all as in $\mathcal{E}$).

So far this is still fairly tame. Here is a glimpse of something more general:

Definition 1.7. Let $\mathbb{C}$ be a small category. A **presheaf** on $\mathbb{C}$ is a functor of the form $\mathbb{C}^{\mathrm{op}} \to \mathbf{Set}$, and a morphism of presheaves is a natural transformation of functors.

Proposition 1.8. *For any small category $\mathbb{C}$, the presheaf category $\left[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}\right]$ is a locally small elementary topos with coproducts for all small families of objects.*

Proof. See [ML–M, Ch. I, §§ 4–6]. □

If we think of Ω as being the object of truth values, then this already implies that the internal logic of an elementary topos can be very far from boolean! In general, Ω will only be a Heyting algebra—reflecting the fact that the internal logic of an elementary topos is intuitionistic. This can be regarded as a higher-order generalisation of Kripke’s frame semantics for intuitionistic logic.

2 A generalised space

Definition 2.1. Let X be a space.[1] By **presheaf** on X we mean a presheaf on $\mathrm{Ouv}(X)$, where $\mathrm{Ouv}(X)$ is the category of open subspaces of X. We write $\mathbf{Psh}(X)$ for the category of presheaves on X.

If F is a presheaf on X and $s \in F(U)$, then for each open subspace $U' \subseteq U$ we write $s|_{U'}$ for the image of s under the map $F(U) \to F(U')$ induced by the inclusion. A **sheaf** on X is a presheaf F with the following property:

- Given an open cover $\mathfrak{U}$ of an open subspace $V \subseteq X$ and any family of elements $\left(s_U \mid U \in \mathfrak{U}\right)$ such that $s_U \in F(U)$ for all U in $\mathfrak{U}$ and $s_U\big|_{U \cap U'} = s_{U'}\big|_{U \cap U'}$ for all pairs (U, U'), there exists a unique element s in $F(V)$ such that $s|_U = s_U$.

[1] Throughout this section, ‘space’ can mean either topological space or locale.

The category of sheaves on X is the full subcategory $\mathbf{Sh}(X)$ of $\mathbf{Psh}(X)$ spanned by the sheaves on X.

Example 2.2. Let $p : E \to X$ be a continuous map. The **sheaf of sections** of p defined to be $U \mapsto \{ s : U \to E \mid p \circ s = i_U \}$, where U varies over the open subspaces of X and $i_U : U \hookrightarrow X$ is the inclusion. It can be shown using the *espace étalé* construction that every sheaf on X is isomorphic to the sheaf of sections of some local homeomorphism.

Theorem 2.3. *For any space X:*

(i) $\mathbf{Sh}(X)$ *is a topos.*

(ii) *The inclusion* $\mathbf{Sh}(X) \hookrightarrow \mathbf{Psh}(X)$ *has a left adjoint that preserves finite limits.*

(iii) *The category of subobjects of* 1 *in* $\mathbf{Sh}(X)$ *is equivalent to* $\mathrm{Ouv}(X)$.

In particular, if X is either a locale or a sober topological space, then X is determined up to isomorphism by $\mathbf{Sh}(X)$.

Proof. See [ML–M, Ch. II, §§ 5 and 8]. □

Rather than give the original, delicate definition of Grothendieck topos now, let me instead give a concise one motivated by the above observation.

Definition 2.4. A **Grothendieck topos** is a category $\mathcal{E}$ for which there exists a small category $\mathbb{C}$ and a fully faithful functor $j_* : \mathcal{E} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$, such that j_* has a left adjoint j^* that preserves finite limits. Such a choice of $\mathbb{C}$ and j_* is called a **site of definition** for $\mathcal{E}$.

Example 2.5. Obviously, $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is itself a Grothendieck topos, and the earlier theorem says that $\mathbf{Sh}(X)$ is a Grothendieck topos.

However, unlike $\mathbf{Sh}(X)$, Grothendieck toposes in general do not have a canonical site of definition. This is an essential feature of topos theory and underlies the “bridges” technique of Caramello [2010].

There is an elephant in the room that we should address as soon as possible: are these two notions of topos compatible?

Proposition 2.6. *Every Grothendieck topos is an elementary topos.*

Proof. See [ML–M, Ch. IV, §§ 6 and 7]. □

Now, recall the dictum of category theory that says that morphisms are just as important as objects. If toposes are to truly generalise (nice) spaces, then there should be some kind of full embedding of the category of (nice) spaces in the 2-category of toposes. But what *is* a morphism of toposes?

Definition 2.7. Let $\mathcal{E}$ and $\mathcal{F}$ be elementary toposes. A **geometric morphism** $f : \mathcal{E} \to \mathcal{F}$ is an adjunction $f^* \dashv f_* : \mathcal{E} \to \mathcal{F}$ such that $f^* : \mathcal{F} \to \mathcal{E}$ that preserves finite limits. We call f^* the **inverse image functor** and f_* the **direct image functor**. A **transformation** of geometric morphisms $f \Rightarrow g$ is a natural transformation of the inverse image functors $f^* \Rightarrow g^*$.

Of course, the above definition is not completely arbitrary and is motivated by various considerations.

Example 2.8. If $\mathbb{C}$ and $j_* : \mathcal{E} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ constitute a site of definition for $\mathcal{E}$, then j_* together with j^* determine a geometric morphism $j : \mathcal{E} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$.[2]

Theorem 2.9. *Let X and Y be spaces.*

(i) *For each continuous map $f : X \to Y$, there is a geometric morphism $f : \mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ such that*

$$f_* F(V) = F\left(f^{-1} V\right)$$

for all F in $\mathbf{Sh}(X)$ and all open subspaces $V \subseteq Y$.

(ii) *The passage from continuous maps to geometric morphisms is pseudo-functorial.*

(iii) *If X and Y are locales, then each geometric morphism $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$ is isomorphic to one induced by a unique continuous map $X \to Y$.*

In particular, if X and Y are sober topological spaces, then this construction gives a bijection between continuous maps $X \to Y$ and isomorphism classes of *geometric morphisms $\mathbf{Sh}(X) \to \mathbf{Sh}(Y)$.*

Proof. See [ML–M, Ch. II, § 9] and [Johnstone, 2002, § C1.4]. □

[2] This is vacuous with our definition of Grothendieck topos, but requires hard work if one instead uses any of the more standard definitions!

3 A geometric theory

The notion of classifying topos bears a strong resemblance to the notion of classifying space in homotopy theory. In order to define this notion, we need to know what a 'theory' is, and for our purposes the following will suffice:

Definition 3.1. Let $\mathfrak{BTop}$ be the 2-category of Grothendieck toposes, geometric morphisms, and their transformations, and let $\mathfrak{Cat}$ be the 2-category of locally small categories, functors, and natural transformations. A **theory** is a pseudofunctor $\mathbb{T} : \mathfrak{BTop}^{\mathrm{op}} \to \mathfrak{Cat}$. A **classifying topos** for a theory $\mathbb{T}$ is a Grothendieck topos $\mathbf{Set}[\mathbb{T}]$ equipped with equivalences $\mathbb{T}(\mathcal{E}) \simeq \mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}])$ that are pseudonatural in $\mathcal{E}$.

The idea is that, whatever a theory $\mathbb{T}$ *really* is, it should be something that has a category of models $\mathbb{T}(\mathcal{E})$ in any Grothendieck topos $\mathcal{E}$, and we restrict our attention to the theories $\mathbb{T}$ such that models of $\mathbb{T}$ can be pulled back along geometric morphisms. The functor $\mathbb{T}(f)$ is usually induced by the inverse image functor f^*, so one often writes f^* instead of $\mathbb{T}(f)$.

Proposition 3.2. *Let $\mathbb{T}$ be a theory with a classifying topos $\mathbf{Set}[\mathbb{T}]$. Then there exists an object G in $\mathbb{T}(\mathbf{Set}[\mathbb{T}])$ with the following universal property:*

- *For each Grothendieck topos $\mathcal{E}$, the functor $\mathbf{Geom}(\mathcal{E}, \mathbf{Set}[\mathbb{T}]) \to \mathbb{T}(E)$ defined by $f \mapsto f^*(G)$ is fully faithful and essentially surjective on objects.*

In particular, for each object A in $\mathbb{T}(\mathcal{E})$, there exists a geometric morphism $\chi_A : \mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ such that $A \cong \chi_A^*(G)$, and χ_A is unique up to (not necessarily unique) isomorphism. Such an object G is said to be a **universal model** *of the theory $\mathbb{T}$.*

Proof. Use the 2-dimensional Yoneda lemma. ■

It is not at all clear whether even the theory of objects (as embodied by the forgetful 2-functor $\mathfrak{BTop}^{\mathrm{op}} \to \mathfrak{Cat}$) has a classifying topos. To that end we require a theorem.

Theorem 3.3 (Diaconescu). *Let $\mathcal{E}$ be a Grothendieck topos. If $\mathbb{C}$ is a small category with finite limits and $\hbar_\bullet : \mathbb{C} \to [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is the Yoneda embedding, then the functor $f \mapsto f^* \hbar_\bullet$ is fully faithful and essentially surjective from $\mathbf{Geom}(\mathcal{E}, [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}])$ onto the category of functors $\mathbb{C} \to \mathcal{E}$ that preserve finite limits.*

Proof. This is, strictly speaking, a special case of what Diaconescu proved. See [ML–M, Ch. VII, §§ 6–9], or [Johnstone, 2002, § B3.2]. □

Corollary 3.4. *Let* **FinSet** *be the category of (hereditarily) finite sets. The topos* [**FinSet**, **Set**] *is a classifying topos for the theory of objects. We say it is the* **object classifier**.

Proof. Any functor $A : \mathbf{FinSet}^{\mathrm{op}} \to \mathcal{E}$ that preserves finite limits is determined up to unique isomorphism by $A(1)$. ■

More generally:

Proposition 3.5. *Let $\mathbb{T}$ be a finitary algebraic theory, i.e. a first-order logical theory over a finitary signature containing no relation symbols and axiomatised by a set of equations. Let $\mathcal{A}$ be (a small skeleton of) the category of all finitely-presentable models of $\mathbb{T}$. Then $[\mathcal{A}, \mathbf{Set}]$ is a classifying topos for $\mathbb{T}$.*

Proof. See Theorem B3.1.1 in [Johnstone, 2002]. □

What about more general theories? It is a fact that that every theory in a certain fragment of infinitary first-order logic, called **geometric logic**, has a classifying topos. Essentially, given a geometric theory $\mathbb{T}$, one constructs a small category $\mathbb{C}$ based on the syntactic properties of $\mathbb{T}$; the axioms of $\mathbb{T}$ are then translated into a "forcing topology" defining a subtopos of $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$. Geometric logic is also just expressive enough to be able to axiomatise the theory of geometric morphisms into any fixed Grothendieck topos $\mathcal{G}$: given a site of definition $(\mathbb{C}, j_*)$, the "topology" on $\mathbb{C}$ induced by j_* may be regarded as an extension of the theory of finite-limit-preserving functors with domain $\mathbb{C}$.

Thus, every Grothendieck topos can be considered as a geometric theory modulo pseudonatural equivalence of the categories of models. In particular, one might ask what the Zariski topos of an affine scheme $\mathrm{Spec}(A)$ classifies.

Theorem 3.6 (Hakim–Wraith)**.** *Let A be a commutative ring.*

(i) *The* petit *Zariski topos of* $\mathrm{Spec}(A)$*, i.e.* $\mathbf{Sh}(\mathrm{Spec}(A))$*, is a classifying topos for the theory of local rings of fractions of A.*

(ii) *The* gros *Zariski topos of* $\mathrm{Spec}(A)$ *is a classifying topos for the theory of local A-algebras, i.e. the theory of commutative A-algebras plus the following axioms:*

- $0 \neq 1$.
- $(\exists y.\, x \cdot y = 1) \vee (\exists y.\, (1 + x) \cdot y = 1)$.

Proof. (i). See [Wraith, 1979, § *The Zariski spectrum*].

(ii). The case $A = \mathbb{Z}$ is Example D3.1.11(a) in [Johnstone, 2002] and also covered in [ML–M, Ch. VIII, § 6]. The general case is essentially the same. □

References

Artin, Michael, Alexander Grothendieck and Jean-Louis Verdier

[SGA 4a] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics 269. Berlin: Springer-Verlag, 1972. xix+525. ISBN: 3-540-05896-6; Artin, Michael, Alexander Grothendieck and Jean-Louis Verdier

[SGA 4b] *Théorie des topos et cohomologie étale des schémas. Tome 2*. Lecture Notes in Mathematics 270. Berlin: Springer-Verlag, 1972. iv+418. ISBN: 3-540-06012-X; Artin, Michael, Alexander Grothendieck and Jean-Louis Verdier

[SGA 4c] *Théorie des topos et cohomologie étale des schémas. Tome 3*. Lecture Notes in Mathematics 305. Berlin: Springer-Verlag, 1973. vi+640. ISBN: 3-549-06118-5.

Bénabou, Jean

[1975] 'Théories relatives à un corpus'. In: *C. R. Acad. Sci. Paris Sér. A-B* 281.20 (1975), Ai, A831–A834.

Caramello, Olivia

[2010] *The unification of mathematics via topos theory*. 20th June 2010. arXiv: 1006.3930v1 [math.CT].

Colmez, Pierre and Jean-Pierre Serre

Eds. [G–S] *Grothendieck–Serre correspondence*. Bilingual. Providence, RI: American Mathematical Society, 2004. x+288+288. ISBN: 0-8218-3424-X.

Diaconescu, Radu

[1973] 'Change of base for some toposes'. PhD thesis. Dalhousie University, 1973.

[1975] 'Change of base for toposes with generators'. In: *J. Pure Appl. Algebra* 6.3 (1975), pp. 191–218. ISSN: 0022-4049.

Giraud, Jean

[1964] 'Analysis situs'. In: *Séminaire Bourbaki, 1962/63. Fasc. 3, No. 256*. Paris: Secrétariat mathématique, 1964, p. 11.

[1971] *Cohomologie non abélienne*. Die Grundlehren der mathematischen Wissenschaften, Band 179. Berlin: Springer-Verlag, 1971, pp. ix+467.

[1972] 'Classifying topos'. In: *Toposes, algebraic geometry and logic (Conf., Dalhousie Univ., Halifax, N.S., 1971)*. Lecture Notes in Mathematics 274. Berlin: Springer, 1972, pp. 43–56.

Grothendieck, Alexander

[1960] 'The cohomology theory of abstract algebraic varieties'. In: *Proc. Internat. Congress Math. (Edinburgh, 1958)*. New York: Cambridge Univ. Press, 1960, pp. 103–118.

Hakim, Monique

[1972] *Topos annelés et schémas relatifs*. Ergebnisse der Mathematik und ihrer Grenzgebiete 64. Berlin: Springer-Verlag, 1972. vi+160.

Johnstone, Peter T.

[1975] 'Internal categories and classification theorems'. In: *Model theory and topoi (Conf., Bangor, 1973)*. Berlin: Springer, 1975, 103–113. Lecture Notes in Math., Vol. 445.

[2002] *Sketches of an elephant: a topos theory compendium. Vol. 1*. Oxford Logic Guides 43. New York: The Clarendon Press Oxford University Press, 2002. xxii+468+71. ISBN: 0-19-853425-6; Johnstone, Peter T. *Sketches of an elephant: a topos theory compendium. Vol. 2*. Oxford Logic Guides 44. Oxford: The Clarendon Press Oxford University Press, 2002. i–xxii, 469–1089 and I1–I71. ISBN: 0-19-851598-7.

Joyal, André and Myles Tierney

[1984] 'An extension of the Galois theory of Grothendieck'. In: *Mem. Amer. Math. Soc.* 51.309 (1984), pp. vii+71. ISSN: 0065-9266.

Lawvere, F. William

[1964] 'An elementary theory of the category of sets'. In: *Proc. Nat. Acad. Sci. U.S.A.* 52 (1964), pp. 1506–1511. ISSN: 0027-8424.

[1971] 'Quantifiers and sheaves'. In: *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*. Paris: Gauthier-Villars, 1971, pp. 329–334.

[2005] 'An elementary theory of the category of sets (long version) with commentary'. In: *Repr. Theory Appl. Categ.* 11 (2005). Reprinted and expanded from Lawvere (1964), with comments by the author and Colin McLarty, pp. 1–35.

Mac Lane, Saunders

[1988] 'Concepts and categories in perspective'. In: *A century of mathematics in America, Part I.* Vol. 1. Hist. Math. Providence, RI: Amer. Math. Soc., 1988, pp. 323–365.

Mac Lane, Saunders and Ieke Moerdijk

[ML–M] *Sheaves in geometry and logic. A first introduction to topos theory.* Corrected. Universitext. New York: Springer-Verlag, 1992. xii+629. isbn: 0-387-97710-4.

Makkai, Michael and Gonzalo E. Reyes

[1976] 'Model-theoretical methods in the theory of topoi and related categories'. In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 24.6 (1976), pp. 379–392. issn: 0001-4117.

Mathias, Adrian R. D.

[2001] 'The strength of Mac Lane set theory'. In: *Ann. Pure Appl. Logic* 110.1-3 (2001), pp. 107–234. issn: 0168-0072. doi: 10.1016/S0168-0072(00)00031-2.

Reyes, Gonzalo E.

[1974] 'From sheaves to logic'. In: *Studies in algebraic logic.* Math. Assoc. Amer., Washington, D.C., 1974, 143–204. Studies in Math., Vol. 9.

Tierney, Myles

[1976] 'Forcing topologies and classifying topoi'. In: *Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg).* New York: Academic Press, 1976, pp. 211–219.

Wraith, Gavin C.

[1975] 'Lectures on elementary topoi'. In: *Model theory and topoi (Conf., Bangor, 1973).* Berlin: Springer, 1975, 114–206. Lecture Notes in Math., Vol. 445.

[1979] 'Generic Galois theory of local rings'. In: *Applications of sheaves (Proc. Res. Sympos. Appl. Sheaf Theory to Logic, Algebra and Anal., Univ. Durham, Durham, 1977).* Lecture Notes in Math. 753. Berlin: Springer, 1979, pp. 739–767.