Bicartesian closed categories and logic

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1 Introduction

Category theory and logic are two of the cornerstones of mathematical foundations, the others being set theory and model theory, yet at first sight there is no obvious connection between the two subjects. This is, of course, an illusion. There is a deep and rich connection between category theory and mathematical logic, touching theoretical computer science. The purpose of this talk is to illustrate a small fragment of this correspondence and perhaps give a glimpse into the field of 'categorical logic'.

For further reading in categorical logic, I suggest Lambek and Scott [1988], and for further reading in proof theory and type theory, I suggest Girard, Taylor and Lafont [1989]. The standard reference for category theory is Mac Lane [1998], but Awodey [2010] (up to Chapter 7) is more than sufficient for our purposes.

2 Bicartesian closed categories

First, we recall the definition of category.

Definition 2.1. A **category** $\mathscr{C}$ consists of the following data:

- a class of **objects**, $\operatorname{ob}\mathscr{C}$
- for each pair of objects (A, B), a class of **arrows**, $\mathscr{C}(A, B)$
- for each object A, an **identity arrow** id_A in $\mathscr{C}(A, A)$
- for each triple of objects (A, B, C), a **composition** operator $\circ : \mathscr{C}(B, C) \times \mathscr{C}(A, B) \to \mathscr{C}(A, C)$

These data are moreover required to satisfy the following laws:

- For all f in $\mathscr{C}(A, B)$,

$$f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$$

- If $f \in \mathscr{C}(A, B)$, $g \in \mathscr{C}(B, C)$, and $h \in \mathscr{C}(C, D)$, then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

When the category $\mathscr{C}$ is clear from context, we may write $f : A \to B$ to mean that $f \in \mathscr{C}(A, B)$.

Example 2.2. The category of all sets, denoted by **Set**, has as its objects every set, and its arrows are just functions between sets.

One of the most pervasive concepts in category theory is the concept of a **universal property**. I will not attempt to give a definition here, but I will give some important examples now.

Definition 2.3. An **initial object** of a category $\mathscr{C}$ is an object 0 with the following universal property: for every object A of $\mathscr{C}$, there is a unique arrow $\Box_A : 0 \to A$.

Another important concept is that of duality. Informally, given some notion in a category, its dual notion is obtained by reversing all the arrows. For example, the dual of an initial object is a terminal object. Explicitly,

Definition 2.4. A **terminal object** of a category $\mathscr{C}$ is an object 1 with the following universal property: for every object A of $\mathscr{C}$, there is a unique arrow $\bigcirc_A : A \to 1$.

We will also need to recall the notions of product and coproduct in a category.

Definition 2.5. Let $\mathscr{C}$ be a category, and let (A, B) be a pair of objects in $\mathscr{C}$. Their **product** consists of the following data:

- an object, $A \times B$
- a pair of projection arrows, $\pi_1^{A,B} : A \times B \to A$ and $\pi_2^{A,B} : A \times B \to B$

These data are required to have the following universal property: for each pair of arrows (f, g), where $f \in \mathscr{C}(C, A)$ and $g \in \mathscr{C}(C, B)$, there is a unique arrow $\langle f, g \rangle : C \to A \times B$ such that

$$\pi_1^{A,B} \circ \langle f, g \rangle = f \qquad \pi_2^{A,B} \circ \langle f, g \rangle = g$$

Definition 2.6. The **coproduct** is the notion dual to the product. Explicitly, the coproduct of a pair of objects (A, B) consists of the following data:

- an object, $A + B$
- a pair of insertion arrows, $\iota_1^{A,B} : A \to A + B$ and $\iota_2^{A,B} : B \to A + B$.

These data are required to have the following universal property: for each pair of arrows (f, g) where $f \in \mathscr{C}(A, C)$ and $g \in \mathscr{C}(B, C)$, there is a unique arrow $[f, g] : A + B \to C$ such that

$$[f, g] \circ \iota_1^{A,B} = f \qquad [f, g] \circ \iota_2^{A,B} = g$$

Example 2.7. In the category **Set**, the empty set $\varnothing$ is an initial object, and *any* singleton set is a terminal object. The product of two sets is simply their cartesian product, and the coproduct is the disjoint union.

Finally, we need a notion akin to that of a function space.

Definition 2.8. Let $\mathscr{C}$ be a category with all binary products. An object B is **exponentiable** just if there are data

- for each object C, an object C^B and an arrow $\varepsilon_{B,C} : C^B \times B \to C$
- for every pair of objects (A, C), a bijection

$$\lambda_{A,B,C} : \mathscr{C}(A \times B, C) \to \mathscr{C}\left(A, C^B\right)$$

satisfying the following conditions:

- For every object C,
$$\lambda_{C^B,B,C}(\varepsilon_{B,C}) = \mathrm{id}_{C^B}$$
- If $f \in \mathscr{C}(A', A)$ and $g \in \mathscr{C}(C, C')$, then for every h in $\mathscr{C}(A \times B, C)$,
$$\lambda_{A',B,C'}(g \circ h \circ (f \times \mathrm{id}_B)) = g^B \circ \lambda_{A,B,C}(h) \circ f$$
where $f \times \mathrm{id}_B$ is the arrow $\langle f \circ \pi_1, \pi_2 \rangle : A' \times B \to A \times B$ and g^B is the arrow $\lambda_{C^B,B,C'}(g \circ \varepsilon_{B,C}) : C^B \to (C')^B$

We may sometimes write $(B \to C)$ instead of C^B.

The map $\lambda_{A,B,C}$ is sometimes referred to as the **currying** map, since it takes an arrow of type $A \times B \to C$ to an arrow of type $A \to (B \to C)$ in a natural and bijective manner.

Example 2.9. In the category **Set**, every set B is exponentiable, with the set C^B being the usual set of all functions $B \to C$.

We are at last able to define the main object of interest of this talk:

Definition 2.10. A **bicartesian closed category** is a category $\mathscr{C}$ with

- an initial object and a terminal object
- a product $A \times B$ and a coproduct $A + B$, for all pairs of objects (A, B)
- all objects exponentiable

Example 2.11. The category **Set** is a bicartesian closed category.

Example 2.12. The trivial category with only one object is a bicartesian closed category.

3 Sequent calculus

And now, to the logical side of things. The astute auditor has probably noticed that there appears to be a formal similarity between the universal properties of the notions defined in the preceding section and the rules of inference for the various logical connectives. We now sharpen this formal similarity by fixing a formal deduction calculus for propositional logic.

Definition 3.1. A **context** is a finite (and possibly empty) list of propositions. In particular, if Γ is a context and p is a proposition, we denote by Γ, p the context obtained by appending p to the list Γ.

Definition 3.2. A **sequent** is a string of the form $\Gamma \vdash p$, where Γ is a context and p is a proposition. The intended interpretation of a valid sequent $\Gamma \vdash p$ is that p is provable from Γ.

Of course, we should probably say what a proposition is. First of all, suppose we are given some atomic propositions $\bot, \top, p_1, p_2, \ldots$; from these we may build up compound propositions using the logical connectives $\wedge$, $\vee$, and $\Rightarrow$ Explicitly, if p and q are propositions, then so are $p \wedge q$, $p \vee q$, and $p \Rightarrow q$. Of course, we would like to know how these compound propositions are related to one another, and for this, we need some **rules of inference**:

- **Conjunction introduction**: If Γ is a context and p and q are propositions, if the sequents $\Gamma \vdash p$ and $\Gamma \vdash q$ are valid, so is the sequent $\Gamma \vdash p \wedge q$. We write this as follows:

$$\frac{\Gamma \vdash p \qquad \Gamma \vdash q}{\Gamma \vdash p \wedge q}$$

- **Conjunction elimination**: If Γ is a context and p, q, r are propositions,

$$\frac{\Gamma, p \vdash r}{\Gamma, p \wedge q \vdash r} \qquad\qquad \frac{\Gamma, q \vdash r}{\Gamma, p \wedge q \vdash r}$$

- **Disjunction introduction**: If Γ is a context and p, q are propositions,

$$\frac{\Gamma \vdash p}{\Gamma \vdash p \vee q} \qquad\qquad \frac{\Gamma \vdash q}{\Gamma \vdash p \vee q}$$

- **Disjunction elimination**: If Γ is a context and p, q, r are propositions,

$$\frac{\Gamma, p \vdash r \qquad \Gamma, q \vdash r}{\Gamma, p \vee q \vdash r}$$

- **Conditional proof**: If Γ is a context and p and q are propositions,

$$\frac{\Gamma, p \vdash q}{\Gamma \vdash p \Rightarrow q}$$

- **Modus ponens**: If Γ is a context and p, q, r are propositions,

$$\frac{\Gamma \vdash p \qquad \Gamma, q \vdash r}{\Gamma, p \Rightarrow q \vdash r}$$

- **Ex falso quodlibet**: If p is any proposition,

$$\frac{}{\bot \vdash p}$$

- **Tautology introduction**: If Γ is any context,

$$\frac{}{\Gamma \vdash \top}$$

- **Identity axiom**: If p is any proposition,

$$\frac{}{p \vdash p}$$

- **Cut rule**: If Γ is a context and p and q are propositions,

$$\frac{\Gamma \vdash p \qquad \Gamma, p \vdash q}{\Gamma \vdash q}$$

- There are also some structural rules which I will omit.

4 The Brouwer–Heyting–Kolmogorov category

It should now be quite clear that there ought to be a correspondence between the logical connectives $\bot, \top, \wedge, \vee, \Rightarrow$ and the operations of a bicartesian closed category $0, 1, \times, +, \rightarrow$. For example, the universal property of $A \times B$ can be represented schematically as

$$\frac{f : C \to A \qquad g : C \to B}{\langle f, g \rangle : C \to A \times B}$$

This looks suspiciously like the rule of conjunction introduction

$$\frac{\Gamma \vdash p \qquad \Gamma \vdash q}{\Gamma \vdash p \wedge q}$$

and similar observations can be made about $\vee$ and $+$, $\Rightarrow$ and exponentiation, $\bot$ and the initial object 0, and $\top$ and the terminal object 1.

Thus, one is led to wonder if there is some bicartesian closed category $\mathscr{C}$ in which propositions are objects and the $0, 1, \times, +, \rightarrow$ are precisely the logical connectives $\bot, \top, \wedge, \vee, \Rightarrow$. The answer turns out to be yes. In fact, there are at least two ways of constructing such a category: one way is to construct the Lindenbaum–Tarski algebra, which yields a preorder category where there is at most one arrow between any two objects. However, a much more interesting approach is to construct the Brouwer–Heyting–Kolmogorov category, where the arrows between two propositions are precisely the (equivalence classes of) formal proofs of one from the other.

But what *is* a formal proof, and when are they equivalent? Well, the trick is that we can rig the definition so that what we want becomes true. First of all, we need to algebraise our notion of bicartesian closed category.

Definition 4.1. An **algebraic bicartesian closed category** is a bicartesian closed category $\mathscr{C}$ together with the following algebraic data:

- a chosen initial object 0 and a chosen terminal object 1
- for every pair of objects (A, B), a chosen product $A \times B$, together with projections $\pi_1^{A,B}, \pi_2^{A,B}$, and a chosen coproduct $A + B$, together with insertions $\iota_1^{A,B}, \iota_2^{A,B}$
- for every pair of objects (B, C), a chosen exponential object C^B, together with an arrow $\varepsilon_{B,C}$ and natural bijections $\lambda_{A,B,C}$ for each A

Note that the algebraic structure on a bicartesian closed category need not be unique: there are very many possible algebraic structures on **Set**, for example. Once we have fixed such data, the category becomes a model of the essentially algebraic theory of bicartesian closed categories. As usual, there is a notion of homomorphism between two models:

Definition 4.2. A **strict bicartesian closed functor** is a functor between two algebraic bicartesian closed categories which respects the algebraic data.

So, for example, if $F : \mathscr{C} \to \mathscr{C}'$ is a strict bicartesian closed functor, then for any pair of objects (A, B) in $\mathscr{C}$, we have $F(A \times B) = F(A) \times F(B)$ as objects of $\mathscr{C}'$. Note that we have *equality*, and not just an isomorphism! (In other words, strictness is an 'evil' notion.) Similarly, we have $F\left(\pi_1^{A,B}\right) = \pi_1^{F(A),F(B)}$ as arrows in $\mathscr{C}'$.

There is also a natural notion of a free model:

Definition 4.3. An algebraic bicartesian closed category $\mathscr{C}$ is **freely generated** by S, $S \subseteq \operatorname{ob}\mathscr{C}$, just when the following holds: for every algebraic bicartesian closed category $\mathscr{C}'$ and every map $F : S \to \operatorname{ob}\mathscr{C}'$, there is a unique strict bicartesian closed functor $\overline{F} : \mathscr{C} \to \mathscr{C}'$ with $\overline{F}(A) = F(A)$ for each object A in S.

We define the Brouwer–Heyting–Kolmogorov category **BHK** of a propositional theory to be the free (algebraic) bicartesian closed category generated by the set of atomic propositions of the theory. As one would expect, every arrow in such a freely-generated category can be expressed using some combination of

$$\circ, \mathrm{id}, \Box, \bigcirc, \langle -, - \rangle, \pi_1, \pi_2, [-, -], \iota_1, \iota_2, \lambda, \varepsilon$$

and each of these operations corresponds precisely with one of the rules of inferences in the sequent calculus. Conversely, every valid derivation of a sequent of the form $p \vdash q$ can be translated into an arrow $p \to q$ constructed using these operations. Thus, we are led to define a formal proof of q assuming p to be an arrow $p \to q$ in **BHK**, and we say two derivations are equivalent if they correspond to the same arrow in this category.

Example 4.4. Let p and q be atomic propositions. Consider the following two derivations of the sequent $p \wedge q \vdash p \vee q$:

$$\begin{array}{ccc} p \vdash p & \qquad & q \vdash q \\ p \wedge q \vdash p & & p \wedge q \vdash q \\ p \wedge q \vdash p \vee q & & p \wedge q \vdash p \vee q \end{array}$$

The left derivation corresponds to the composite

$$p \wedge q \xrightarrow{\pi_1} p \xrightarrow{\iota_1} p \vee q$$

while the right derivation corresponds to the composite

$$p \wedge q \xrightarrow{\pi_2} q \xrightarrow{\iota_2} p \vee q$$

Now, it seems obvious that we *should* have

$$\iota_1 \circ \pi_1 \neq \iota_2 \circ \pi_2$$

and this is indeed the case: since **BHK** is freely generated by the atomic propositions, once we fix an algebraic structure on **Set**, there is a unique strict bicartesian closed functor $F : \mathbf{BHK} \to \mathbf{Set}$ with $F(p) = \{0\}$ and $F(q) = \{1\}$. It is then clear that

$$F(\iota_1) \circ F(\pi_1) \neq F(\iota_2) \circ F(\pi_2)$$

since the two maps have distinct images in $\{0, 1\}$. Thus, the original arrows in **BHK** must have been distinct.

Of course, merely defining **BHK** to be a free bicartesian closed category does not prove that we have one: after all, it could well be the case that our requirements are somehow self-contradictory. Nonetheless, it is possible to construct **BHK** explicitly by using results about normal forms of natural deduction proofs. (For details about normalisation, see Girard, Taylor and Lafont [1989].)

5 The law of excluded middle

Perhaps the most remarkable property of the category of natural deduction proofs is that there *can* be genuinely different arrows $p \to q$: in other words, the category is able to distinguish between (some) proofs of q from p. For example, an arrow $\top \to (p \vee q)$ corresponds to either an arrow $\top \to p$ or an arrow $\top \to q$, and cannot correspond to both at once: this is the disjunction property of intuitionistic logic. However, there is no free lunch: something must have been sacrificed in order for us to have this remarkable property. If one contemplates incompleteness for a few moments, one immediately realises that what we have given up must be the law of excluded middle: for, if q is an undecidable proposition, then we have neither a proof of q nor a proof of $\neg q$, but we certainly do have a proof of $q \vee \neg q$.

Recall that $\neg p$ is simply an abbreviation for $p \Rightarrow \bot$. It is a simple exercise in formal logic to show that the **law of excluded middle**

$$\frac{}{\Gamma \vdash p \vee \neg p}$$

is equivalent to the rule of **double negation elimination**

$$\frac{\Gamma \vdash \neg\neg p}{\Gamma \vdash p}$$

so one is led to wonder, what happens if we require our bicartesian closed category $\mathscr{C}$ to have an isomorphism $\neg\neg p \xrightarrow{\sim} p$? For this, we require a little lemma, originally due to Joyal.

Lemma 5.1. *Let $\mathscr{C}$ be a cartesian closed category (i.e. a category with a terminal object 1, binary products, and all objects exponentiable) with an initial object 0. Then, for any object A, there is* at most *one arrow $f : A \to 0$.*

Proof. This proof follows Freyd [1972, Prop. 1.12]. First, observe that

$$\mathscr{C}(0 \times A, B) \cong \mathscr{C}(0, (A \to B))$$

so there is a unique arrow $0 \times A \to B$ for every object B in $\mathscr{C}$, i.e. $0 \times A$ is also an initial object. Now, suppose we have an arrow $f : A \to 0$. Then, we have an arrow $\langle f, \mathrm{id}_A \rangle : A \to 0 \times A$, and $\pi_2 \circ \langle f, \mathrm{id}_A \rangle = \mathrm{id}_A$, while $\langle f, \mathrm{id}_A \rangle \circ \pi_2 = \mathrm{id}_{0 \times A}$ since $0 \times A$ is initial. So $A \cong 0$, and f is the unique arrow $A \to 0$. ∎

Theorem 5.2. *Let $\mathscr{C}$ be a cartesian closed category with an initial object 0. If for every object A we have an isomorphism $((A \to 0) \to 0) \xrightarrow{\sim} A$, then $\mathscr{C}$ is a preorder category, i.e. $\mathscr{C}$ has at most one arrow between any two objects.*

Proof. By definition of the exponential object $(C \to 0)$,

$$\mathscr{C}(B, (C \to 0)) \cong \mathscr{C}(B \times C, 0)$$

so the above lemma implies there is at most one arrow $B \to (C \to 0)$ for any two objects B and C. But if we have $A \cong ((A \to 0) \to 0)$, then there is at most one arrow $B \to A$ for any object B, so we have a preorder category $\mathscr{C}$. ■

So, unfortunately, there is no hope of constructing a bicartesian closed category which adequately captures the proof theory of classical propositional logic: the theorem above implies that adding double negation elimination to our requirements causes the category of proofs to lose the ability to distinguish between different derivations $p \to q$. One might argue that at least we do not have an outright contradiction, but in fact we have gained nothing new: this collapsed category is equivalent to a well-understood boolean algebra, namely the free boolean algebra generated by the atomic propositions. Thus, it would appear that in order to have a good theory, we have to restrict our attention to intuitionistic logic.

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