Another viewpoint on cartesian theories

Zhen Lin Low

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

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- Cartesian theories have an underlying algebraic theory, but this disappears after passing to the syntactic category.
- For example, it is possible for different algebraic theories to generate cartesian theories that have equivalent syntactic categories.
- Thus, what we would like to do is to augment the notion of (many-sorted) Lawvere theory so it can encode cartesian theories.

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- However, cartesian theories are problematic: $\exists$ cannot be applied to arbitrary formulae. So we have to start from scratch and build up to cartesian theories.

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Remark. This defines an enrichment of $\mathbf{E}(\Omega)$ in preordered sets.

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(ii) $F : \mathbf{E}(\Omega) \to \mathcal{C}$ *factors through the quotient functor* $\mathbf{E}(\Omega) \to \mathbf{Leib}(\Omega)$.

In other words, $\mathbf{Leib}(\Omega)$ is (isomorphic to) the category obtained by freely inverting the Leibniz equivalences in $\mathbf{E}(\Omega)$.

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Moreover, $\mathbf{Leib}(\Omega)$ *is the homotopy category.*

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Definition. The **category of loci** is the category $\mathbf{Lc}(\Omega)$ obtained by freely inverting the topological equivalences in $\mathbf{E}(\Omega)$.

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In other words, the natural map $\Omega(X) \to \mathrm{Sub}_{\mathbf{Lc}(\Omega)}((X, \top_X))$ is an isomorphism.

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- Is every topological equivalence in $\mathbf{E}(\Omega)$ already a Leibniz equivalence?
- This happens if every monomorphism in $\mathbf{Leib}(\Omega)$ is a regular monomorphism, e.g. when $\mathbb{T}$ is the theory of abelian groups. What happens otherwise?