

Cocycles in categories of fibrant objects

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- ▶ [Chris Kapulkin and Karol Szumiło](#). *Quasicategories of frames of cofibration categories*. 29th June 2015. [arXiv: 1506.08681](#)

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What is this mapping space?

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- ▶ The outermost face operators delete a row of vertical arrows.
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- ▶ The degeneracy operators insert a row of identity arrows.

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This terminology is due to Jardine.

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Can we do better?

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In that situation, an old result of Dwyer and Kan says that the mapping spaces are homotopy equivalent to the nerves of the categories of cocycles.

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- ▶ Moreover, the data (p, v) and j are homotopically unique, i.e. the space of such choices is contractible.

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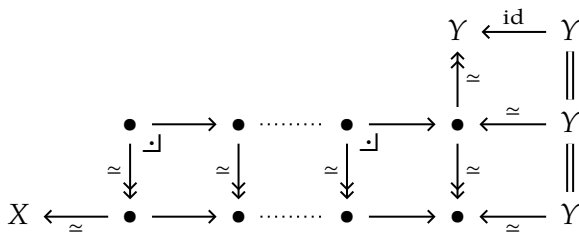
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$$\begin{array}{ccccccc}
 & & & & & & \begin{array}{ccc} \Upsilon & \xleftarrow{\text{id}} & \Upsilon \\ \uparrow \cong & & \parallel \\ \bullet & \xleftarrow{\cong} & \Upsilon \\ \downarrow \cong & & \parallel \\ \bullet & \xleftarrow{\cong} & \Upsilon \end{array} \\
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 & & & & & & \begin{array}{ccc} & \xleftarrow{\cong} & \Upsilon \end{array}
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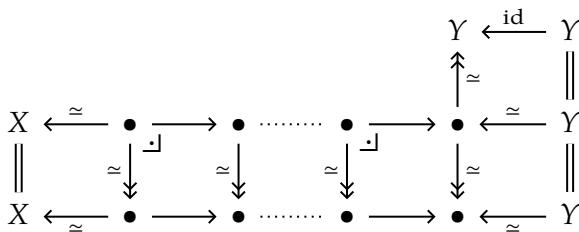
Calculus of fractions

$$\begin{array}{ccccccc}
 & & & & & \gamma & \xleftarrow{\text{id}} & \gamma \\
 & & & & & \uparrow \simeq & & \parallel \\
 & & & & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & \gamma \\
 & & & & \downarrow \simeq & \lrcorner & \downarrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & \gamma
 \end{array}$$

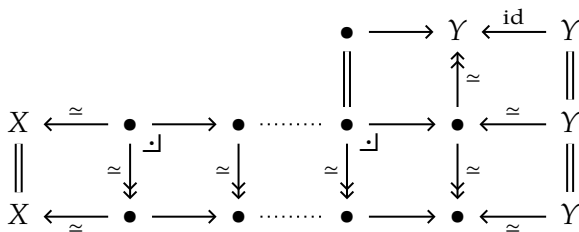
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$$\begin{array}{ccccccc}
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & Y & \xleftarrow{\text{id}} & Y \\
 \parallel & & \parallel & & \parallel & & \parallel & & \uparrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y \\
 \parallel & & \downarrow \simeq \lrcorner & & \downarrow \simeq \lrcorner & & \downarrow \simeq \lrcorner & & \downarrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y
 \end{array}$$

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 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y \\
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 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y
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The first step is well-defined up to a contractible space of choices.

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 \parallel & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y
 \end{array}$$

The first step is well-defined up to a contractible space of choices.
All the other steps are functorial.

Calculus of fractions

$$\begin{array}{ccccccc}
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & Y & \xleftarrow{\text{id}} & Y \\
 \parallel & & \parallel & & \parallel & & \parallel & & \uparrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y \\
 \parallel & & \downarrow \simeq & \lrcorner & \downarrow \simeq & & \downarrow \simeq & \lrcorner & \downarrow \simeq & & \parallel \\
 X & \xleftarrow{\simeq} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & \bullet & \xleftarrow{\simeq} & Y
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$$\begin{array}{ccccccc}
 X & \xleftarrow{\cong} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & Y & \xleftarrow{\text{id}} & Y \\
 \parallel & & \parallel & & \parallel & & \parallel & & \uparrow q & & \parallel \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet & \xrightarrow{\tilde{f}_k} & \bullet & \xleftarrow{u} & Y \\
 \parallel & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \downarrow v & & \parallel \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{f_1} & \bullet & \cdots & \bullet & \xrightarrow{f_k} & \bullet & \xleftarrow{w} & Y
 \end{array}$$

Calculus of fractions

Assume $w = \text{id}_Y$.

$$\begin{array}{ccccccc}
 X & \xleftarrow{\cong} & \bullet & \longrightarrow & \bullet & \cdots & \bullet & \longrightarrow & Y & \xleftarrow{\text{id}} & Y \\
 \parallel & & \parallel & & \parallel & & \parallel & & \uparrow q & & \parallel \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet & \xrightarrow{\tilde{f}_k} & \bullet & \xleftarrow{u} & Y \\
 \parallel & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \downarrow v & & \parallel \\
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Calculus of fractions

Assume $v \circ u = \text{id}_Y$.

$$\begin{array}{ccccccc}
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet & \xrightarrow{\tilde{f}_k} & \bullet & \xrightarrow{q} & Y & \xleftarrow{\text{id}} & Y \\
 \parallel & & \cong \downarrow & \lrcorner & \cong \downarrow & & \cong \downarrow & \lrcorner & \downarrow v & & \parallel & & \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{f_1} & \bullet & \cdots & \bullet & \xrightarrow{f_k} & \bullet & \xrightarrow{u} & Y & & \\
 & & & & & & & & & & & &
 \end{array}$$

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$$\begin{array}{ccccccc}
 & & & & & & Y \xrightarrow{\text{id}} Y \\
 & & & & & & \downarrow u \quad \parallel \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet \xrightarrow{\tilde{f}_k} \bullet \xrightarrow{q} Y \\
 \parallel & & \downarrow \cong \lrcorner & & \downarrow \cong \lrcorner & & \downarrow v \\
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 & & & & & & & & \downarrow u & & \parallel \\
 X & \xleftarrow{\cong} & \bullet & \xrightarrow{\tilde{f}_1} & \bullet & \cdots & \bullet & \xrightarrow{\tilde{f}_k} & \bullet & \xrightarrow{q} & Y \\
 \parallel & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \cong \downarrow \lrcorner & & \downarrow v & & \\
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All of the above steps are functorial.

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All of the above steps are functorial.

These two procedures lie at the heart of the proof that categories of fibrant objects admit a homotopical calculus of right fractions.

Revisiting cocycles

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Question. What is the connection between cocycles in the sense above and cocycles in cohomology?

Answer. The Verdier hypercovering theorem.

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In fact, we can replace $\mathcal{W}_{/X}$ with the full subcategory \mathcal{Q}_X spanned by the trivial fibrations $\tilde{X} \rightarrow X$.

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Thus, hypercovers are generalisations of open covers.

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$$\begin{aligned} H^n(X; \mathcal{A}) &\cong \mathrm{Hom}_{\mathrm{Ho} \, \mathbf{sSh}(X)}(1_X, \mathbf{K}(\mathcal{A}, n)) \\ &\cong \pi_0(\mathbf{RHom}_{\mathbf{sSh}(X)}(1_X, \mathbf{K}(\mathcal{A}, n))) \end{aligned}$$

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 &\cong \underline{\lim}_{\underline{\longrightarrow}_{Q^{\mathrm{op}}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathcal{U}, \mathbf{K}(\mathcal{A}, n)) \\
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 &\cong \varinjlim_{\mathcal{Q}^{\mathrm{op}}} H^n(\mathrm{Hom}(\mathbf{C}(\mathcal{U}), \mathcal{A}))
 \end{aligned}$$

This is basically the Verdier hypercovering theorem.