Cocycles in categories of fibrant objects

(arXiv:1502.03925)

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- Chris Kapulkin and Karol Szumiło. *Quasicategories of frames of cofibration categories*. 29th June 2015. arXiv: 1506.08681

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- The degeneracy operators insert a row of identity arrows.

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  This terminology is due to Jardine.

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Can we do better?

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In that situation, an old result of Dwyer and Kan says that the mapping spaces are homotopy equivalent to the nerves of the categories of cocycles.

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  Indeed, using Brown's factorisation lemma, we just factor $\langle f, w \rangle : \tilde{X} \to Y \times X$ as a weak equivalence followed by a fibration.
- Moreover, the data (p, v) and j are homotopically unique, i.e. the space of such choices is contractible.

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$$
\begin{tikzcd}
 & & & & Y & Y \arrow[l, "\mathrm{id}"'] \\
 & & & & \bullet \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] & Y \arrow[l, "\simeq"'] \arrow[u, equal] \arrow[d, equal] \\
X & \bullet \arrow[l, "\simeq"] \arrow[r] & \bullet \arrow[r, dotted, no head] & \bullet \arrow[r] & \bullet & Y \arrow[l, "\simeq"]
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\begin{tikzcd}
& & & & Y & Y \arrow[l, "\mathrm{id}"'] \\
& & & \bullet \arrow[r] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] & Y \arrow[l, "\simeq"'] \arrow[u, equal] \\
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 & \bullet \arrow[r] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, two heads, "\simeq"'] & \cdots & \bullet \arrow[r] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[u, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] & Y \arrow[l, "\simeq"'] \arrow[d, equal] \\
X & \bullet \arrow[l, "\simeq"] \arrow[r] & \bullet & \cdots & \bullet \arrow[r] & \bullet & Y \arrow[l, "\simeq"]
\end{tikzcd}
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\begin{tikzcd}
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X & \bullet \arrow[l, "\simeq"] \arrow[r] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r] & \bullet & Y \arrow[l, "\simeq"]
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The first step is well-defined up to a contractible space of choices.
All the other steps are functorial.

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X \arrow[d, equal] & \bullet \arrow[l, "\simeq"'] \arrow[r] \arrow[d, equal] & \bullet \arrow[r, dotted, no head] \arrow[d, equal] & \bullet \arrow[r] \arrow[d, equal] & Y & Y \arrow[l, "\mathrm{id}"'] \arrow[d, equal] \\
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X \arrow[u, equal] \arrow[d, equal] & \bullet \arrow[l, "\simeq"'] \arrow[u, equal] \arrow[r, "\tilde{f}_1"] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[u, equal] \arrow[r, phantom, "\cdots"] \arrow[d, two heads, "\simeq"'] & \bullet \arrow[u, equal] \arrow[r, "\tilde{f}_k"] \arrow[d, two heads, "\simeq"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \bullet \arrow[u, two heads, "q"'] \arrow[d, two heads, "v"] & Y \arrow[l, "u"'] \arrow[u, equal] \arrow[d, equal] \\
X & \bullet \arrow[l, "\simeq"] \arrow[r, "f_1"'] & \bullet \arrow[r, phantom, "\cdots"] & \bullet \arrow[r, "f_k"'] & \bullet & Y \arrow[l, "w"]
\end{tikzcd}
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Calculus of fractions

Assume $w = \mathrm{id}_Y$.

$$
\begin{tikzcd}
X \arrow[d, equal] & \bullet \arrow[l, "\simeq"'] \arrow[r] \arrow[d, equal] & \bullet \arrow[r, phantom, "\cdots\cdots"] \arrow[d, equal] & \bullet \arrow[r] \arrow[d, equal] & Y & Y \arrow[l, "\mathrm{id}"'] \arrow[d, equal] \\
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$$
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All of the above steps are functorial.

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All of the above steps are functorial.
These two procedures lie at the heart of the proof that categories of fibrant objects admit a homotopical calculus of right fractions.

Revisiting cocycles

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Question. What is the connection between cocycles in the sense above and cocycles in cohomology?
Answer. The Verdier hypercovering theorem.

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In fact, we can replace $\mathcal{W}_{/X}$ with the full subcategory $\mathcal{Q}_X$ spanned by the trivial fibrations $\tilde{X} \to X$.

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Thus, hypercovers are generalisations of open covers.

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$$\begin{aligned} \mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n)) \\ &\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n))\big) \end{aligned}$$

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\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}\left(1_X, \mathrm{K}(\mathscr{A},n)\right) \\
&\cong \pi_0\left(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n))\right) \\
&\cong \pi_0\left(\underrightarrow{\mathrm{holim}}_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n))\right)
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\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A}, n)) \\
&\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A}, n))\big) \\
&\cong \pi_0\Big(\operatorname{ho}\varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A}, n))\Big) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A}, n))
\end{aligned}
$$

The Verdier hypercovering theorem

$$\begin{aligned}
\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A}, n)) \\
&\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A}, n))\big) \\
&\cong \pi_0\Big(\operatorname{ho}\varinjlim_{\mathcal{Q}^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A}, n))\Big) \\
&\cong \varinjlim_{\mathcal{Q}^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A}, n)) \\
&\cong \varinjlim_{\mathcal{Q}^{\mathrm{op}}} \mathrm{H}_0(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \Sigma^n \mathscr{A}))
\end{aligned}$$

The Verdier hypercovering theorem

$$
\begin{aligned}
\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n)) \\
&\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n))\big) \\
&\cong \pi_0\Big(\mathrm{ho}\varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n))\Big) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n)) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \Sigma^n\mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\Sigma^n\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \mathscr{A}))
\end{aligned}
$$

The Verdier hypercovering theorem

$$
\begin{aligned}
\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n)) \\
&\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n))\big) \\
&\cong \pi_0\Big(\mathrm{hol}\varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n))\Big) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n)) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \Sigma^n \mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\Sigma^n \underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_{-n}(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \mathscr{A}))
\end{aligned}
$$

The Verdier hypercovering theorem

$$\begin{aligned}
\mathrm{H}^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n)) \\
&\cong \pi_0\big(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, \mathrm{K}(\mathscr{A},n))\big) \\
&\cong \pi_0\Big(\mathrm{ho}\varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n))\Big) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, \mathrm{K}(\mathscr{A},n)) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \Sigma^n\mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_0(\Sigma^n\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}_{-n}(\underline{\mathrm{Hom}}(\mathrm{C}(\mathscr{U}), \mathscr{A})) \\
&\cong \varinjlim_{Q^{\mathrm{op}}} \mathrm{H}^n(\mathrm{Hom}(\mathrm{C}(\mathscr{U}), \mathscr{A}))
\end{aligned}$$

The Verdier hypercovering theorem

$$
\begin{aligned}
H^n(X;\mathscr{A}) &\cong \mathrm{Hom}_{\mathrm{Ho}\,\mathbf{sSh}(X)}(1_X, K(\mathscr{A},n)) \\
&\cong \pi_0\left(\mathbf{R}\mathrm{Hom}_{\mathbf{sSh}(X)}(1_X, K(\mathscr{A},n))\right) \\
&\cong \pi_0\left(\mathrm{ho}\varinjlim_{Q^{op}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, K(\mathscr{A},n))\right) \\
&\cong \varinjlim_{Q^{op}} \mathrm{Hom}_{\mathbf{sSh}(X)}(\mathscr{U}, K(\mathscr{A},n)) \\
&\cong \varinjlim_{Q^{op}} H_0(\underline{\mathrm{Hom}}(C(\mathscr{U}), \Sigma^n \mathscr{A})) \\
&\cong \varinjlim_{Q^{op}} H_0(\Sigma^n \underline{\mathrm{Hom}}(C(\mathscr{U}), \mathscr{A})) \\
&\cong \varinjlim_{Q^{op}} H_{-n}(\underline{\mathrm{Hom}}(C(\mathscr{U}), \mathscr{A})) \\
&\cong \varinjlim_{Q^{op}} H^n(\mathrm{Hom}(C(\mathscr{U}), \mathscr{A}))
\end{aligned}
$$

This is basically the Verdier hypercovering theorem.