Generalising the functor of points approach

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- What additional structure on $\mathcal{E}$ and/or $\mathcal{C}$ do we need to answer these questions?

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- We obtain a uniform procedure for building $\mathcal{E}$ from $\mathcal{C}$.

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- However, $\mathcal{C}$ is not always closed under "open" subobjects in $\mathcal{E}$.

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[The manifolds we get this way are possibly non-Hausdorff and non-second-countable.]

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A **pseudocomplement** of $f : X \to Y$ is a representation of $\mathrm{D}(f)$.

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- If $f : X \to Y$ is a member of $\mathcal{P}$, then $\mathrm{D}(f)$ is a representable presheaf on $\mathcal{C}$.

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Definition. A **sheaf** on $\mathcal{C}$ is a presheaf on $\mathcal{C}$ that satisfies the sheaf condition with respect to all basic open covers.

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Theorem. *The class of open subsheaves of any sheaf A on $\mathcal{C}$ is a (possibly non-small) subframe of the (possibly non-small) frame of subsheaves of A.*

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Example. Any representable sheaf admits an atlas.

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[As before, manifolds are possibly non-Hausdorff and non-second-countable.]

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Corollary. *A sheaf on $\mathcal{E}$ is a charted space if and only if it is a representable sheaf on $\mathcal{E}$.*