The heart of a combinatorial model category

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- A combinatorial model category is a model for locally presentable $(\infty, 1)$-categories.

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- Joyal and Lurie have proved the analogous theorem for locally presentable $(\infty, 1)$-categories.
- Moreover, every locally presentable $(\infty, 1)$-category is modelled by some combinatorial model category.
- The question: Is every combinatorial model category freely generated by a small model category, and in what sense?

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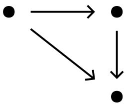

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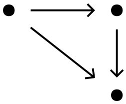

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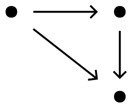

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  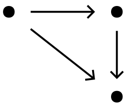

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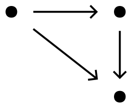

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A **cofibrantly generated model structure** is a model structure, say $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, where the weak factorisation systems $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are cofibrantly generated.

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A **combinatorial model category** is a locally presentable category equipped with a cofibrantly generated model structure.

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What further assumption do we need on λ to deduce that $\mathcal{M}$ admits a model structure cofibrantly generated by $\mathcal{I}$ and $\mathcal{I}'$?

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Example. Let $\mathcal{M}$ be the category of countable simplicial sets. Then $\mathcal{M}$, equipped with the restriction of the usual Kan–Quillen model structure on $\mathbf{sSet}$, is an $(\aleph_0, \aleph_1)$-compact model category.

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- $\mathbf{K}_{\lambda}(\mathcal{M})$ is closed under finite limits in $\mathcal{M}$.

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Example. The category of simplicial sets, equipped with the usual Kan–Quillen model structure, is a strongly $(\aleph_0, \aleph_1)$-combinatorial model category.

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Proposition. *For any combinatorial model category $\mathcal{M}$, there exist regular cardinals κ and λ such that $\mathcal{M}$ is a strongly (κ, λ)-combinatorial model category.*

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It can be shown that the model structure on $\mathcal{M}$ we seek must have $\mathcal{F}$ as its fibrations, $\mathcal{F}'$ as its trivial fibrations, and $\mathcal{W}$ as its weak equivalences. Let us show that such a model structure exists.

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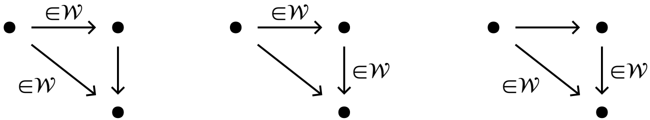

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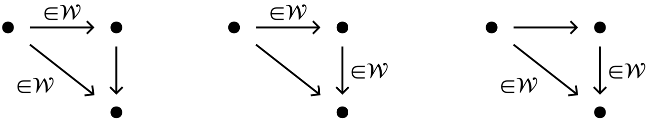

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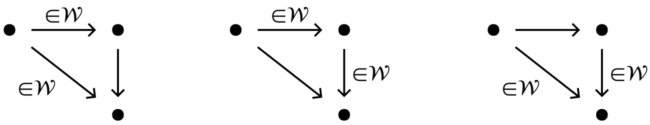

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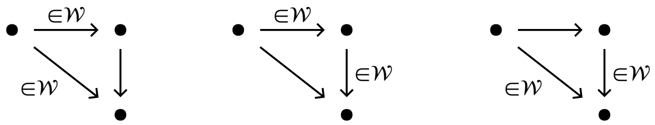

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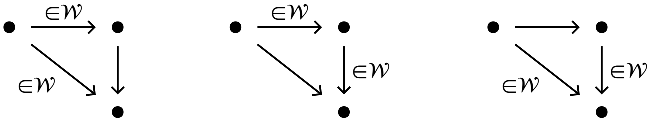

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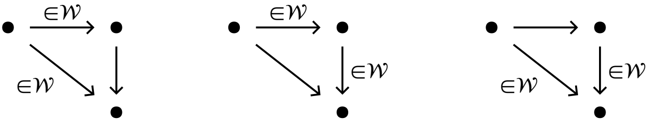

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