Accessible functors and inaccessible cardinals

Universes for category theory (arXiv:1304.5227)

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- Unfortunately, there is no guarantee that a universal object in M_n remains universal in M_{n+1}.
- Indeed, it is well-known that powersets need not be preserved when passing from one model of set theory to another: this is implied by the theory of forcing.

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- When can we be sure that enlarging the universe does *not* change limits, adjoints, Kan extensions etc. in the concrete categories we wish to study?
- For $\mathbf{Set}$ this is easy: we have explicit constructions for limits and colimits. The general strategy will be to reduce the problem to the case where explicit constructions are available.

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Example. The empty set is a pre-universe, and with very mild assumptions, so is the set $\mathbf{HF}$ of all hereditarily finite sets.

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- Moreover, in Mac Lane set theory, if $\mathbf{U}$ is any non-empty pre-universe, then there exists a strongly inaccessible cardinal κ such that the members of $\mathbf{U}$ are all the sets of rank $< \kappa$.
- Of course, the existence of universes (in ZFC or its subsystems) is independent of the axioms of ZFC.

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- More generally, given a regular cardinal κ, by **κ-small category** we mean a category $\mathbb{C}$ such that $\operatorname{mor} \mathbb{C}$ has cardinality $< \kappa$.

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- The category of $\mathbf{U}$-small fields is a $\aleph_0$-accessible $\mathbf{U}$-category that is not locally $\aleph_0$-presentable.

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Definition. The **free** $(\kappa, \mathbf{U})$**-ind-completion of** $\mathcal{C}$ is the category $\mathbf{Ind}_\mathbf{U}^\kappa(\mathcal{C})$.

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Definition. Let $F \dashv G$ and $F' \dashv G'$ be adjunctions and consider a mated pair of natural transformations:

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\mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\
{\scriptstyle F}\downarrow & \swArrow_{\alpha} & \downarrow{\scriptstyle F'} \\
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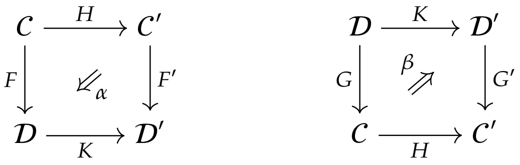

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Morally, the left (resp. right) Beck–Chevalley condition says that H and K preserve the left (resp. right) adjoint of G (resp. F).

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Example. If $\mathbf{Set}$ is the category of $\mathbf{U}$-sets and $\mathbf{Set}^+$ is the category of $\mathbf{U}^+$-sets, then the inclusion $\mathbf{Set} \hookrightarrow \mathbf{Set}^+$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-extension.

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3. *If $B : \mathcal{J} \to \mathcal{C}$ is any diagram (not necessarily $\mathbf{U}$-small) and $\mathcal{C}$ has a limit for B, then i preserves this limit.*

Fundamental results

Example. Let $\mathbb{B}$ be a $\mathbf{U}$-small category in which idempotents split. Then the $(\kappa, \mathbf{U})$-accessible functor $\mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B}) \to \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathbb{B})$ obtained by extending the embedding $\gamma^+ : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathbb{B})$ along $\gamma : \mathbb{B} \to \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$ is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension.

Proposition. *All examples of $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions are (up to equivalence) of the above form.*

Proposition. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

1. *$\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category if and only if $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.*
2. *The functor $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful.*
3. *If $B : \mathcal{J} \to \mathcal{C}$ is any diagram (not necessarily $\mathbf{U}$-small) and $\mathcal{C}$ has a limit for B, then i preserves this limit.*

Remark. Conversely, any fully faithful functor $i : \mathcal{C} \to \mathcal{C}^+$ satisfying the bulleted conditions on the previous slide must be $(\kappa, \mathbf{U})$-accessible.

Stability of accessible adjoint functors

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- $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category.
- $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.

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- $\mathcal{C}$ is a locally κ-presentable $\mathbf{U}$-category.
- $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.
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- $\mathcal{D}$ is a locally λ-presentable $\mathbf{U}$-category.
- $\mathcal{C}^+$ is a locally κ-presentable $\mathbf{U}^+$-category.
- $\mathcal{D}^+$ is a locally λ-presentable $\mathbf{U}^+$-category.
- $i : \mathcal{C} \to \mathcal{C}^+$ and $j : \mathcal{D} \to \mathcal{D}^+$ are fully faithful functors.

Stability of accessible adjoint functors

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Theorem. *Consider the following strictly commutative diagrams:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{i} & \mathcal{C}^+ \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F^+} \\ \mathcal{D} & \xrightarrow[j]{} & \mathcal{D}^+ \end{array} \qquad \begin{array}{ccc} \mathcal{D} & \xrightarrow{j} & \mathcal{D}^+ \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G^+} \\ \mathcal{C} & \xrightarrow[i]{} & \mathcal{C}^+ \end{array}$$

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1. *Given the diagram on the left, if both F and F^+ have right adjoints and i is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension,*

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1. *Given the diagram on the left, if both F and F^+ have right adjoints and i is a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension, then the diagram satisfies the right Beck–Chevalley condition.*

Stability of accessible adjoint functors

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\end{array}
\qquad\qquad
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{\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G^{+}} \\
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\end{array}
$$

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Stability of accessible adjoint functors

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2. *Given the diagram on the right, if G is $(\lambda, \mathbf{U})$-accessible,*

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Stability of accessible adjoint functors

Outline of proof.

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- Deduce that $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extensions preserve colimits for *all* $\mathbf{U}$-small diagrams, not just κ-filtered ones.

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- Finally, prove the theorem itself.

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- Finally, prove the theorem itself. Again, we use the explicit constructions given in the proof of the accessible adjoint functor theorem. ■

Further properties of accessible extensions

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Theorem. *Let* $i : \mathcal{C} \to \mathcal{C}^+$ *be a* $(\kappa, \mathbf{U}, \mathbf{U}^+)$*-accessible extension and let* $\mathcal{C}$ *be a locally* κ*-presentable* $\mathbf{U}$*-category.*

Further properties of accessible extensions

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1. *If* λ *is a regular cardinal in* $\mathbf{U}$ *and* $\kappa \leq \lambda$*,*

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Theorem. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category.*

1. *If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \leq \lambda$, then $i : \mathcal{C} \to \mathcal{C}^+$ is also a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*

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1. *If λ is a regular cardinal in $\mathbf{U}$ and $\kappa \leq \lambda$, then $i : \mathcal{C} \to \mathcal{C}^+$ is also a $(\lambda, \mathbf{U}, \mathbf{U}^+)$-accessible extension.*
2. *If μ is the cardinality of $\mathbf{U}$, then $i : \mathcal{C} \to \mathcal{C}^+$ factors through the inclusion $\mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+) \hookrightarrow \mathcal{C}^+$ as functor $\mathcal{C} \to \mathbf{K}_\mu^{\mathbf{U}^+}(\mathcal{C}^+)$ that is (fully faithful and) essentially surjective on objects.*

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3. *The* $(\mu, \mathbf{U}^+)$*-accessible functor* $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}(\mathcal{C}) \to \mathcal{C}^+$ *induced by* $i : \mathcal{C} \to \mathcal{C}^+$ *is fully faithful and essentially surjective on objects.*

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Corollary. *If $\mathbb{B}$ is a κ-cocomplete $\mathbf{U}$-small category and μ is the cardinality of $\mathbf{U}$,*

Further properties of accessible extensions

Theorem. *Let $i : \mathcal{C} \to \mathcal{C}^+$ be a $(\kappa, \mathbf{U}, \mathbf{U}^+)$-accessible extension and let $\mathcal{C}$ be a locally κ-presentable $\mathbf{U}$-category.*

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3. *The $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}(\mathcal{C}) \to \mathcal{C}^+$ induced by $i : \mathcal{C} \to \mathcal{C}^+$ is fully faithful and essentially surjective on objects.*

Corollary. *If $\mathbb{B}$ is a κ-cocomplete $\mathbf{U}$-small category and μ is the cardinality of $\mathbf{U}$, then the canonical $(\mu, \mathbf{U}^+)$-accessible functor $\mathbf{Ind}_{\mathbf{U}^+}^{\mu}(\mathbf{Ind}_{\mathbf{U}}^{\kappa}(\mathbb{B})) \to \mathbf{Ind}_{\mathbf{U}^+}^{\kappa}(\mathbb{B})$ is fully faithful and essentially surjective on objects.*

Combinatorial model structures

Combinatorial model structures

Let $\mathbb{B}$ be a κ-cocomplete $\mathbf{U}$-small category, let $\mathcal{M} = \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$, and let $\mathcal{I}$ and $\mathcal{J}$ be $\mathbf{U}$-sets.

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- The inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^+$ preserves the functorial factorisations constructed by the small object arguments of Quillen [1967] and of Garner [2009].

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- The inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^{+}$ preserves the functorial factorisations constructed by the small object arguments of Quillen [1967] and of Garner [2009].
- Now define $\mathcal{W}^{+}$ to be the collection of all morphisms in $\mathcal{M}^{+}$ of the form $q \circ j$ where j is a $\mathcal{J}$-cofibration and q is an $\mathcal{I}$-injective morphism.

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- As soon as we know that $\mathcal{W}^+$ has the 2-out-of-3 property in $\mathcal{M}^+$, we would have a cofibrantly generated model structure on $\mathcal{M}^+$ extending the model structure on $\mathcal{M}$.

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Let $\mathbb{B}$ be a κ-cocomplete $\mathbf{U}$-small category, let $\mathcal{M} = \mathbf{Ind}^{\kappa}_{\mathbf{U}}(\mathbb{B})$, and let $\mathcal{I}$ and $\mathcal{J}$ be $\mathbf{U}$-sets. Suppose $\mathcal{I}$ and $\mathcal{J}$ cofibrantly generate a Quillen model structure on $\mathcal{M}$. Do $\mathcal{I}$ and $\mathcal{J}$ also cofibrantly generate a Quillen model structure on $\mathcal{M}^+ = \mathbf{Ind}^{\kappa}_{\mathbf{U}^+}(\mathbb{B})$?

- The inclusion $\mathcal{M} \hookrightarrow \mathcal{M}^+$ preserves the functorial factorisations constructed by the small object arguments of Quillen [1967] and of Garner [2009].
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- As soon as we know that $\mathcal{W}^+$ has the 2-out-of-3 property in $\mathcal{M}^+$, we would have a cofibrantly generated model structure on $\mathcal{M}^+$ extending the model structure on $\mathcal{M}$. But why should $\mathcal{W}^+$ have the 2-out-of-3 property?

Weaker notions of universe

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Let us say that $\mathbf{U}$ is a **weak universe** if it satisfies axioms 1–3 and 5 in the definition of 'universe' *plus* the following axiom:

Weaker notions of universe

Let us say that $\mathbf{U}$ is a **weak universe** if it satisfies axioms 1–3 and 5 in the definition of 'universe' *plus* the following axiom:

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- If things still work in this context, it would afford an adequate framework for applying category-theoretic methods to study category theory, *without* needing any large cardinals.

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