Toposes as localic and spatial groupoids

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The purpose of this talk is expository: I will try explain the construction of these groupoids and some of the ingredients that go into the proof.

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Example. For any small category $\mathcal{C}$, the category $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ is a Grothendieck topos; in particular, $\mathbf{Set}$ is a Grothendieck topos.

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Henceforth we shall use the word 'topos' for the elementary notion.

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Proof. See Theorem 1.4.7 and Remark 1.4.8 in [Johnstone, 2002b, Part C]. □

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$$\begin{tikzcd}
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A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & d_2^* d_1^* A \arrow[dl, "\cong"'] \arrow[dr, "d_2^*\theta"] & \\
d_1^* d_1^* A \arrow[d, "d_1^*\theta"'] & & d_2^* d_0^* A \arrow[d, "\cong"] \\
d_1^* d_0^* A \arrow[dr, "\cong"'] & & d_0^* d_1^* A \arrow[dl, "d_0^*\theta"] \\
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$$\begin{tikzcd} \mathbf{Sh}(G_0) \arrow[r, shift left=2, "d_0^*"] \arrow[r, shift right=2, "d_1^*"'] & \mathbf{Sh}(G_1) \arrow[l, "s_0^*" description] \arrow[r, shift left=2, "d_0^*"] \arrow[r, "d_1^*" description] \arrow[r, shift right=2, "d_2^*"'] & \mathbf{Sh}(G_2) \end{tikzcd}$$

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Proof. See Corollary 3.4.12 in [Johnstone, 2002a, Part B], or Proposition 3.4 in [Moerdijk, 1988]. ∎

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\mathcal{S}_{/Y} \arrow[r, "p^*_{/Y}"] \arrow[d, "f^*"'] & \mathcal{E}_{/p^*Y} \arrow[d, "(p^*f)^*"] \\
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Example. If X is a topological space, then $\mathbf{Sh}(X)$ is a connected (resp. locally connected) topos if and only if X is a connected (resp. locally connected) topological space.

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Proof. See [Moerdijk, 1985, § 3], or Theorem 5.1.1 in [Johnstone, 2002b, Part C]. □

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- Thus, (by Diaconescu's theorem) we may think of $[\mathbb{P}_K, \mathbf{Set}]$ as the classifying topos for partial equivalence relations on K.
- The **generic quotient** is the functor $Q_K : \mathbb{P}_K \to \mathbf{FinSet}$ that sends a partial equivalence relation on K to its set of equivalence classes.

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Now consider the following bicategorical pullback square in $\mathfrak{Top}$:

$$\begin{tikzcd} \mathcal{E} \arrow[r, "c"] \arrow[d, "p"'] & {[\mathbb{P}, \mathbf{Set}]} \arrow[d, "q"] \\ \mathcal{S} \arrow[r, "b"'] & \mathbf{Set}[\mathbb{O}] \end{tikzcd}$$

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Corollary. *If $\mathcal{S}$ is a Grothendieck topos, then $\mathcal{S}$ is equivalent to the category of equivariant sheaves on a localic groupoid.*

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Proof. Let $p : \mathcal{E} \to \mathcal{S}$ be a localic, connected, and locally connected geometric morphism, with $\mathcal{E}$ a localic topos. Now take iterated pullbacks in $\mathfrak{BTop}_{\mathbf{Set}}$:

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Proof. See Lemma 2.2.11 in [Johnstone, 2002b, Part C]. □

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and topologise G_0 by declaring each such to be a basic open subset.

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Proof. See [Butz and Moerdijk, 1998]. $\square$

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