Proposition. *Let $\mathcal{J}$ be a small category. For each locally small category C, there is a functor $\mathcal{H}om : [\mathcal{J}, C]^{\mathrm{op}} \times [\mathcal{J}, C] \to [\mathcal{J}, \mathbf{Set}]$ such that*

$$[\mathcal{J}, \mathbf{Set}]\big(1, \mathcal{H}om(A, B)\big) \cong [\mathcal{J}, C](A, B)$$

naturally in A and B, and this makes $[\mathcal{J}, C]$ into a $[\mathcal{J}, \mathbf{Set}]$-enriched category.

Proof. Define $\mathcal{H}om : [\mathcal{J}, C]^{\mathrm{op}} \times [\mathcal{J}, C] \to [\mathcal{J}, \mathbf{Set}]$ by the following end formula:

$$\mathcal{H}om(A, B) = \int_{k:\mathcal{J}} \mathbf{Set}(\mathcal{J}(-, k), C(Ak, Bk))$$

This is clearly functorial. Concretely, an element of α in $\big(\mathcal{H}om(A, B)\big)(j)$ is the same thing as a $\coprod_{k \in \mathrm{ob}\,\mathcal{J}} \mathcal{J}(j, k)$-indexed family of morphisms $\alpha_f : Ak \to Bk$ in C, such that for any two arrows $f : j \to k$, $g : k \to l$ in $\mathcal{J}$, the diagram below commutes:

$$\begin{array}{ccc} Ak & \xrightarrow{\alpha_f} & Bk \\ {\scriptstyle Ag}\downarrow & & \downarrow{\scriptstyle Bg} \\ Al & \xrightarrow[\alpha_{g\circ f}]{} & Bl \end{array}$$

Decomposing an element β of $\big(\mathcal{H}om(B, C)\big)(j)$ the same way, we obtain a commutative rectangle

$$\begin{array}{ccccc} Ak & \xrightarrow{\alpha_f} & Bk & \xrightarrow{\beta_f} & Ck \\ {\scriptstyle Ag}\downarrow & & {\scriptstyle Bg}\downarrow & & \downarrow{\scriptstyle Cg} \\ Al & \xrightarrow[\alpha_{g\circ f}]{} & Bl & \xrightarrow[\beta_{g\circ f}]{} & Cl \end{array}$$

and thus we have an element of $\big(\mathcal{H}om(A, C)\big)(j)$. This is certainly natural in j: given $h : j \to i$, $h \cdot \alpha$ is the element in $\big(\mathcal{H}om(A, B)\big)(i)$ corresponding to the family $(h \cdot \alpha)_e = \alpha_{e \circ h}$. Clearly, this is the required associative composition with identity.

It remains to be shown that a natural transformation $1 \Rightarrow \mathcal{H}om(A, B)$ is the same thing as a natural transformation $A \Rightarrow B$. Given a natural transformation $\alpha : A \Rightarrow B$, we define an element α^j of $\big(\mathcal{H}om(A, B)\big)(j)$ for each object j in $\mathcal{J}$ as follows: given $f : j \to k$, we set

$$\left(\alpha^j\right)_f = \alpha_k$$

and naturality of α_k makes the diagram

$$\begin{array}{ccc} Ak & \xrightarrow{(\alpha^j)_f} & Bk \\ {\scriptstyle Ag}\downarrow & & \downarrow{\scriptstyle Bg} \\ Al & \xrightarrow[(\alpha^j)_{g\circ f}]{} & Bl \end{array}$$

commute for every arrow $g : k \to l$ in $\mathcal{J}$. By construction $h \cdot \alpha^j = \alpha^i$ for all arrows $h : j \to i$ in $\mathcal{J}$, so this defines a natural transformation $1 \Rightarrow \mathcal{H}om(A, B)$. Conversely, given a family of elements α^j such that $h \cdot \alpha^j = \alpha^i$ for all $h : j \to i$, we discover

$$\left(\alpha^j\right)_f = \left(\alpha^j\right)_{\mathrm{id}_k \circ f} = \left(f \cdot \alpha^k\right)_{\mathrm{id}_k} = \left(\alpha^k\right)_{\mathrm{id}_k}$$

for all arrows $f : j \to k$ in $\mathcal{J}$, so we get a natural transformation $\alpha : A \Rightarrow B$ by setting $\alpha_k = (\alpha^k)_{\mathrm{id}_k}$. This establishes the natural bijection in the statement of the proposition. ■